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Evaluation of the work of Pavel Ruzicka

This letter is my evaluation of the work submitted by Dr. Pavel Ruzicka to be considered for the qualification of 'venium docendi'.

The work submitted spans the years 2006–2018 and covers his research in lattice theory, ring theory, infinitary combinatorics, and general algebra.

I will give some general remarks, then comment on each chapter, then write a short summary.

General remarks. The habilitation thesis is organized around the theme of finding representations of distributive algebraic lattices, or showing that representations do not exist. Specific questions include: Which lattices are representable as the congruence lattice of a lattice? Which lattices are representable as the ideal lattice of a regular ring? Which lattices are representable as the lattice of compact subgroups of a directed abelian group? Each of the 6 chapters focuses on a paper published by the author (with coauthors in two cases).

Chapter 1. Lifting of distributive lattices by locally matricial algebras

This chapter is based on

[66] P. Ruzicka, *Liftings of distributive lattices by locally matricial algebras with respect to the* Id_c *functor*, Algebra Universalis 55 (2006), 239–257.

In fact, this chapter is really based on an earlier work of Ruzicka: in 2004, he proved that every distributive lattice is isomorphic as a join semilattice to the semilattice of finitely generated ideals of a locally matricial algebra. This earlier paper extended still earlier work of George Bergman and Fred Wehrung.

The advance of [66] over Ruzicka's earlier paper is to show that the representation can be made functorial, provided the domain category is the category of bounded distributive lattices under embeddings. At the end of the paper, some easy examples are given which show the necessity of restricting to this category where all morphisms are embeddings.

Chapter 2. Distributive congruence lattices of congruence-permutable algebras

This chapter is based on

[68] P. Ruzicka, J. Tuma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, J. Algebra 311 (2007), 96–116.

This chapter concerns the problem of representing an algebraic distributive lattice as the congruence lattice of an algebra with permuting congruences.

The main negative result is that it is impossible to represent some lattices this way, in particular it is impossible to represent in this way the congruence lattice of the $\aleph_2$-generated free lattice from any nondistributive lattice variety. The argument is an adaptation of earlier work of Ploscica, Tuma, and Wehrung.

The main positive results are that it is possible to represent any algebraic distributive lattice with at most $\aleph_1$ compact elements as the congruence lattice of a group or module.

Chapter 3. Free trees and the optimal bound in Wehrungs solution of the Congruence Lattice Problem

This chapter is based on

[67] P. Ruzicka, *Free trees and the optimal bound in Wehrungs theorem*, Fund. Math. 198 (2008), 217–228.

This paper, which perfects a major result of Fred Wehrung, is probably Ruzicka's most famous paper so far.

Fred Wehrung spent many years trying to solve Dilworth's Problem of determining whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice. Andras Huhn showed in 1989 that every distributive algebraic lattice with $\aleph_1$ compact elements is representable. Employing Kuratowski's Free Set Theorem, Wehrung showed in 2007 that there is a distributive algebraic lattice with $\aleph_{\omega+1}$ compact elements that is not representable. There is a big gap between $\aleph_1$ compact elements and $\aleph_{\omega+1}$ compact elements.

Developing and using refined combinatorial arguments (a "Free Tree Theorem" in place of the Free Set Theorem), Ruzicka managed to obtain the optimal bound for congruence lattice representations of lattices: there is a distributive algebraic lattice with $\aleph_2$-many compact elements that is not representable.

Chapter 4. Countable chains of distributive lattices and dimension groups

This chapter is based on

[65] P. Ruzicka, *Countable chains of distributive lattices as maximal semilattice quotients of positive cones of dimension groups*, Comment. Math. Univ. Carolin. 47 (2006), 11–20.

This chapter concerns the realization of distributive join semilattices as Grothendieck groups of locally matricial algebras over a field.

Specifically, the paper gives a negative answer to a question of Fred Wehrung by constructing a countable chain of Boolean semilattices, with all inclusion maps preserving the join and the bounds, whose union is not representable as the maximal semilattice quotient of the positive cone of a dimension group. This is a technical result related to the problem of determining which distributive join semilattices are representable as the semilattice of ideals of a regular ring.

Chapter 5. Construction and realization of some wild refinement monoids

This chapter is based on

[62] P. Ruzicka, *On the construction and the realization of wild monoids*, Archivum Mathematicum , Vol. 54 (2018), No. 1, 33–64

Let R be a unital ring, and, for an R-module A, let $[A]$ denote its isomorphism type. The set $V(R)$ of isomorphism types of finitely generated projective R-modules forms an additive monoid with unit $0 := [0]$ under the operation $[A] + [B] := [A \oplus B]$. This is an object of K-theory.

It is clear from the definitions that $a + b = 0$ in $V(R)$ implies $a = b = 0$. One says that a monoid is "conical" if it has this property. It is also clear from the definitions that $[R]$ is an order-unit of $V(R)$. If R is a regular ring, then it can be shown that the monoid $V(R)$ satisfies the Riesz refinement property, hence $V(R)$ is a refinement monoid.

This paper is concerned with developing techniques to compute $V(R)$ from R, with constructing new refinement monoids, and with showing that some of the new refinement monoids are representable as $V(R)$ for some regular ring.

Chapter 6. Boolean ranges of Banaschewski functions

This chapter is based on

[50] S. Mokris and P. Ruzicka, *On Boolean ranges of Banaschewski functions*, Algebra universalis March 2018, 79:15.

For this paper, a "Banaschewski function" on a lattice is an order-reversing complementation.

The problems considered here, first asked by Fred Wehrung, are: (1) Is every maximal Boolean sublattice of a countable complemented modular lattice the range of an order-reversing complementation? (2) Are any two maximal Boolean sublattices of a countable complemented modular lattice isomorphic?

Mokris and Ruzicka give negative answers to both questions. They construct a countably infinite, complemented, modular lattice which has two maximal, nonisomorphic, Boolean sublattices, and they prove that one of the sublattices is the range of a order-reversing complementation, while the other is not. They also prove that their lattice is embeddable in the subspace lattice of a vector space.

Summary. The work of Pavel Ruzicka on representations of distributive lattices is original and deep. In his work he demonstrates a powerful and creative imagination. He has established himself as an independent researcher with a broad perspective on his subject and superior technical abilities. This thesis in particular, is a work of extremely high quality.

I recommend granting Dr. Ruzicka the qualification of venium docendi in the area of Mathematics. I support his appointment to Associate Professor without reservation.

Sincerely Yours,

Keith Kearnes
Professor