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MĀNOA

To: Faculty of Mathematics and Physics, Charles University

From: J. B. Nation, University of Hawai'i

Re: Habilitation Thesis of Pavel Růžička

17 -04- 2018

I read this thesis with great interest. I knew the statements of some of the theorems contained herein, and used this opportunity to study the details of the arguments. There is some very nice mathematics here.

Broadly speaking, the thesis centers on negative results and counterexamples. A dominant theme is the interplay between infinite combinatorics and algebraic structures. This can be expressed naturally in the language of category theory. Rather than trying to summarize them all, let me briefly highlight a selection of the results that are particularly important for lattice theory and universal algebra.

In the 1950s, Bjarni Jónsson asked whether every Arguesian lattice could be represented as a lattice of permuting equivalence relations. As someone who worked on this and related problems, I can fairly state that none of us really expected this to be the case. But at the same time, none of us would have remotely considered looking for a distributive counterexample. That is exactly what the arguments of Chapter 2 provide: a distributive lattice that has no representation as a lattice of permuting equivalence relations.

It follows, of course, that such a lattice cannot be embedded into the lattice of normal subgroups of a group, or the lattice of submodules of a module, or indeed into the congruence lattice of any congruence-permutable algebra. We have known other restrictions on these classes for some time, but a *distributive* example is surely amazing.

In the 1940s, R. P. Dilworth asked whether every distributive algebraic lattice could be represented as the congruence lattice of a lattice. This was proved when the lattice had at most $\aleph_1$ compact elements, and Friedrich Wehrung eventually found a distributive algebraic lattice with $\aleph_{\omega+1}$ compact elements that is not isomorphic to the congruence lattice of a lattice. In Chapter 3 of the thesis, Růžička refines Wehrung's arguments to show that there are distributive algebraic lattices with $\aleph_2$ compact elements that are not so representable. The refinements come in the combinatorics involving use of Kuratowski's free set theorem, and are subtle indeed. A variation in the argument shows that these lattices are not isomorphic to the congruence lattice of an algebra with operations including a semilattice join with a largest element.

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The first step in coordinatizing complemented modular lattices is to obtain a large 3-frame or 4-frame. To this end, Bernhard Banaschewski showed that the lattice of subspaces of a vector space has an order-reversing map that sends each element to one of its complements, and Wehrung proved that every countable complemented modular lattice admits such a function with Boolean range. In Chapter 6, Růžička constructs lattices where this function is not well-behaved. He constructs a countable complemented modular lattice with two non-isomorphic maximal Boolean sublattices. one of which is the range of Banaschewski function, and the other of which is not. Moreover, the lattice is coordinatizable, but contains no large 3-frame.

There are other nice results in the thesis; the samples above all relate directly to classical problems that I have worked on in the course of my career. In addition. the construction and realization of wild type monoids in Chapter 6 represents progress in our understanding of refinement monoids.

Most of the results in this thesis are extensions and refinements of previously known work. But they are *substantial* extensions. They show a considerable depth of understanding of the problems, ingenuity in constructions, and a mastery of the tools of the trade, including algebra, combinatorics and category theory. In each case new ideas were required to surmount old difficulties, and valuable results were obtained. These are important contributions.

I recommend that the habilitation thesis be approved, and that Dr. Pavel Růžička be appointed to the faculty as an associate professor.