HABILITATION THESIS

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Representations of Distributive Algebraic Lattices

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To My Parents

Contents

Introduction

All monoids in the thesis are supposed to be commutative. The *stable equivalence* on a monoid $\boldsymbol{M}$, denoted by $\sim_s$, is the least congruence on $\boldsymbol{M}$ such that the quotient $\boldsymbol{M}_s := \boldsymbol{M}/\sim_s$ is cancellative. The congruence is defined by $x \sim_s y$ if there exists $z \in \boldsymbol{M}$ such that $x + z = y + z$, for all $x, y \in \boldsymbol{M}$. The correspondence $\boldsymbol{M} \mapsto \boldsymbol{M}_s$ extends canonically to a functor that we denote by $(-)_s$.

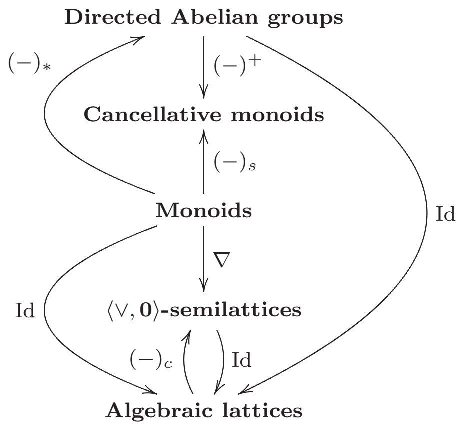

FIGURE 1. Partially ordered Abelian groups, monoids, and algebraic lattices

There is an universal map $(-)_*\colon \boldsymbol{M} \to \boldsymbol{M}_*$ sending monoids to Abelian groups. Moreover the algebraic order on a monoid $\boldsymbol{M}$ induces a partial order on the target Abelian group $\boldsymbol{M}_*$; such that the image of the monoid corresponds to the positive cone of $\boldsymbol{M}_*$. The construction of the partially ordered Abelian group $\boldsymbol{M}_*$ for a given monoid $\boldsymbol{M}$ is an analogy of the construction of the field of fractions of a given commutative ring. We consider the set of formal differences between pairs of elements from $\boldsymbol{M}$ and an equivalence relation, say $\sim_*$, on them. The equivalence is given by $x - y \sim_* z - u$ provided that there is $w \in \boldsymbol{M}$ such that $x + u + w = z + y + w$. The map $(-)_*$ is dermined by $x \mapsto [x - 0]_{\sim_*}$, $x \in \boldsymbol{M}$. Again, the correspondence is canonically functorial. Notice that the partially ordered Abelian group $\boldsymbol{M}_*$ is directed, that is, it is, as a group, generated by the positive cone. It is straightforward to see that this is equivalent to the partial order on $\boldsymbol{M}_*$ being upwards directed.

Let $\boldsymbol{G}^+ := \{p \in \boldsymbol{G} \mid 0 \leq p\}$ denote the positive cone of a partially ordered Abelian group $\boldsymbol{G}$. Observing that an order preserving homomorphism $\boldsymbol{G} \to \boldsymbol{H}$ maps the positive cone $\boldsymbol{G}^+$ of $\boldsymbol{G}$ into the positive cone $\boldsymbol{H}^+$ of $\boldsymbol{H}$, we see that there is a functor $(-)^+$ from the category of partially ordered Abelian groups to monoids. Moreover, the composition $(-)^+ \circ (-)_*$ is naturally equivalent to the functor $\sim_s$.

We denote by $\asymp$ the least congruences on $\boldsymbol{M}$ such that $\boldsymbol{M}/\!\asymp$ is a $\langle\vee,\mathbf{0}\rangle$-semilattice and we set $\nabla(\boldsymbol{M}) := \boldsymbol{M}/\!\asymp$. As in the previous cases, the correspondence $\boldsymbol{M} \to \nabla(\boldsymbol{M})$ extends a functor.

The ideal lattice $\mathrm{Id}(\boldsymbol{\mathcal{S}})$ of a $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$ is an algebraic lattice and, conversely, compact elements of an algebraic lattice $\boldsymbol{\mathcal{L}}$ form a $\langle\vee,\mathbf{0}\rangle$-semilattice, denoted by $\boldsymbol{\mathcal{L}}_c$. Both the correspondences extend to functors that are inverse to each other (up to obvious natural equivalences).

Here are more *ideal-type* functors to consider. Firstly, the functor that assigns to a monoid $\boldsymbol{M}$ the algebraic lattice $\mathrm{Id}(\boldsymbol{M})$ of all o-ideals of $\boldsymbol{M}$. Secondly, the functor $\boldsymbol{G} \mapsto \mathrm{Id}(\boldsymbol{G}^+)$ which assigns to a directed Abelian group the algebraic lattice of all convex subgroups of $\boldsymbol{G}$.

All the introduced functors are depicted in Figure 1. Note that the diagram of functors is commutative (up to natural equivalences).

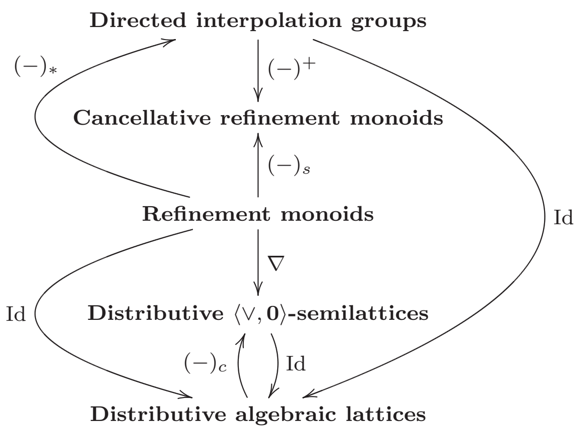

FIGURE 2. Directed interpolation groups, refinement monoids, and distributive algebraic lattices

We will be interested in structures that are mapped by the ideal functor Id to algebraic lattices that are distributive. Starting from the bottom of Figure 2, these are distributive $\langle\vee,\mathbf{0}\rangle$-semilattice (cf. [**15**, Section II.5]). Indeed, a $\langle\vee,\mathbf{0}\rangle$-semilattice is distributive if and only if $\mathrm{Id}(\boldsymbol{\mathcal{S}})$ is an algebraic distributive lattice. Next we consider the class of refinement monoids, i.e, the conical monoids that satisfy the Riesz refinement property. The maximal semilattice quotient $\nabla(\boldsymbol{M})$ of a refinement monoid $\boldsymbol{M}$ is a distributive $\langle\vee,\mathbf{0}\rangle$-semilattice and the lattice $\mathrm{Id}(\boldsymbol{M})$ of all o-ideals of $\boldsymbol{M}$ is distributive (cf. [**14**, lemma 2.4]. Finally, a directed Abelian group $\boldsymbol{G}$ is an interpolation group if and only if the positive cone $\boldsymbol{G}^+$ is a refinement monoid [**10**, Prop. 2.1]. In particular, the lattice $\mathrm{Id}(\boldsymbol{G})$ of all ideals (i.e, convex subgroups) of a directed interpolation group is again distributive.

There are more structures in the picture as we tried to depict in Figure 3. Given a ring $\boldsymbol{R}$, we denote by $\boldsymbol{V}(\boldsymbol{R})$ the monoid of all isomorphism classes

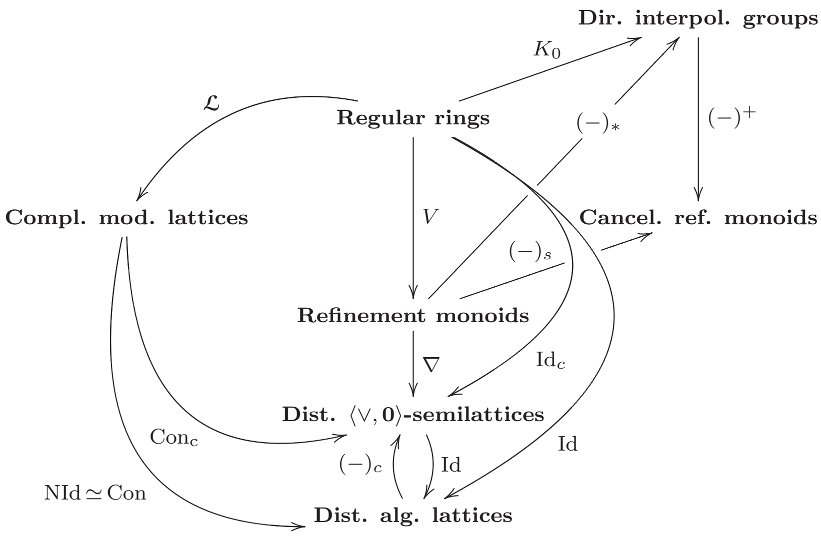

FIGURE 3. Regular rings, refinement monoids, and distributive $\langle\vee,\mathbf{0}\rangle$-semilattices

of finitely generated projective right $\boldsymbol{R}$-modules with addition derived from direct sums. If the ring $\boldsymbol{R}$ is (Von-Neumann) regular, the monoid $\boldsymbol{V}(\boldsymbol{R})$ satisfies the Riesz refinement property (see [**11**, Corollary 2.7]). The partially ordered Abelian group $\boldsymbol{V}(\boldsymbol{R})_*$, denoted by $K_0(\boldsymbol{R})$, is called the *Grothendieck group* of $\boldsymbol{R}$. When we limit ourselves unital rings, it is appropriate to assign to a ring $\boldsymbol{R}$ a partially ordered Abelian group $K_0(\boldsymbol{R})$ with an order-unit corresponding to the isomorphism class $[\,\boldsymbol{R}\,]$ and study the category of partially ordered Abelian groups with order units (cf. [**11**, Chapter 15]). If the ring $\boldsymbol{R}$ is regular, then $K_0(\boldsymbol{R})$ is a directed interpolation group.

We denote by $\mathcal{L}(\boldsymbol{R})$ the $\langle\vee,\mathbf{0}\rangle$-semilattice of all right finitely generated ideals of a ring $\boldsymbol{R}$. For a regular ring, the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{L}(\boldsymbol{R})$ is closed under finite meets, therefore $\mathcal{L}(\boldsymbol{R})$ forms a lattice [**11**, Theorem 2.3]. Moreover, the lattice $\mathcal{L}(\boldsymbol{R})$ is modular and sectionally complemented (complemented if $\boldsymbol{R}$ is with an unit element).

Congruences of sectionally complemented modular lattices correspond to their neutral ideals (see [**15**, Section III.3.10]). In particular, if $\boldsymbol{R}$ is a regular ring, then the lattice $\mathrm{Con}(\mathcal{L}(\boldsymbol{R}))$ is isomorphic to the lattice $\mathrm{NId}(\mathcal{L}(\boldsymbol{R}))$ of all neutral ideals of $\mathcal{L}(\boldsymbol{R})$. By [**36**, Lemma 4.2], an ideal of the lattice $\mathcal{L}(\boldsymbol{R})$ (for a regular ring $\boldsymbol{R}$) is neutral if and only if it contains with each $a\boldsymbol{R}$ all principal ideals $b\boldsymbol{R}$ with $b\boldsymbol{R} \simeq a\boldsymbol{R}$. It follows that $\mathrm{Con}(\mathcal{L}(\boldsymbol{R})) \simeq \mathrm{NId}(\mathcal{L}(\boldsymbol{R})) \simeq \mathrm{Id}(\boldsymbol{R})$ (see [**36**, Lemma 4.3]), and so, the lattice $\mathrm{Id}(\boldsymbol{R})$ of two-sided ideals of a regular ring $\boldsymbol{R}$ is distributive. Moreover, combining [**36**, Corollary 4.4 and Proposition 4.6] we get the isomorphisms $\mathrm{Con}_c(\mathcal{L}(\boldsymbol{R})) \simeq$

$\nabla(\boldsymbol{V}(\boldsymbol{R})) \simeq \mathrm{Id}_c(\boldsymbol{R})$ of distributive $\langle\vee, \mathbf{0}\rangle$-semilattices, for every regular ring $\boldsymbol{R}$.

We have seen that a distributive algebraic lattice that is isomorphic to the lattice of two-sided ideals of a regular ring is at the same time isomorphic to the congruence lattice of a modular sectionally complemented lattice. This brings a connection with the *Congruence lattice problem*, whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice. The conjecture has an interesting history (see [**41**]) and remained open four over sixty years until the counter-example was found by F. Wehrung [**38**]. We will discuss the Congruence Lattice Problem in detail in Chapter 3.

In this thesis we study various *representation problems*, namely for distributive algebraic lattices (resp. corresponding distributive $\langle\vee, \mathbf{0}\rangle$-semilattices), refinement monoids, or directed Abelian groups. For example, we ask whether a given distributive algebraic lattice (or any algebraic lattice with particular properties) is isomorphic to a lattice of all two sided ideals of a regular ring, respectively, as a lattice of all compact subgroups of a directed Abelian group. We might also restrict to some class of regular rings as, for example, locally matricial algebras, or to some class of directed Abelian groups, for example, dimension groups.

A more complex question is when we seek for a functorial solution, that is, when we ask not only for representing a single object but for lifting particular diagrams. Given a diagram $\Delta\colon \mathbf{J} \to \mathbf{C}$ and a functor $\Psi\colon \mathbf{B} \to \mathbf{C}$, a *lifting* of Δ with respect to Ψ is a functor $\Phi\colon \mathbf{J} \to \mathbf{B}$ such that the composition $\Psi \circ \Phi$ is naturally equivalent to Δ.

The thesis consists of six chapters, each based on a single paper and related to a particular realization or lifting problem.

Chapter 1 is based on the paper [**27**]:

> *Liftings of distributive lattices by locally matricial algebras with respect to the* Id_c *functor*, Algebra Universalis **55** (2006), 239 – 257.

In the paper we study liftings with respect to the functor Id_c from the category of locally matricial algebras to the category of distributive $\langle\vee, \mathbf{0}\rangle$-semilattices. The problem goes back to [**5**]. In the unpublished notes G. Bergman proved that

- every countable distributive $\langle\vee, \mathbf{0}\rangle$-semilattice,
- every strongly distributive $\langle\vee, \mathbf{0}\rangle$-semilattice (i.e., a $\langle\vee, \mathbf{0}\rangle$-semilattice of all compact elements of the lattice of all hereditary subsets of a poset),

are isomorphic to the $\langle\vee, \mathbf{0}\rangle$-semilattices of all finitely generated two-sided ideals of locally matricial algebras. In [**25**] we developed a new construction and besides reproving the Bergman's results we have realized every distributive $\langle\vee, \mathbf{0}\rangle$-semilattices that is closed under finite meets, and so it forms a distributive lattice, as the $\langle\vee, \mathbf{0}\rangle$-semilattice of all finitely generated two-sided ideals of a locally matricial. In the presented paper [**27**] we simplify the construction from [**25**] and study possibilities of functorial solutions of the problem. We construct

- a simple finite subcategory $\mathbf{D}_{\bullet\!-}$ of the category **DLat** of all distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattices,
- a subcategory $\mathbf{D}_{\lambda}$ of **DLat** corresponding to a partially ordered proper class, which cannot be lifted with respect to the Id_c functor.

On the positive side we prove that every diagram in **DLat** indexed by a partially ordered set and the subcategory $\mathbf{DLat_m}$ of **DLat** whose objects are all distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattices and whose morphisms are $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-embeddings can be lifted with respect to the Id_c functor.

Let us mention some applications of the results:

- The realization of distributive $\langle \vee, \mathbf{0}\rangle$-semilattices closed under finite meets by $\langle \vee, \mathbf{0}\rangle$-semilattices of all finitely generated ideals of locally matricial algebras answers the Γ-invariant realization problem from [**8**]. Given an uncountable cardinal κ we let $\mathcal{B}_\kappa := \mathcal{P}(\kappa)/\operatorname{club}_\kappa$ denote the Boolean algebra of all subsets of κ modulo the filter $\operatorname{club}_\kappa$ generated by all closed unbounded subsets of κ. A $\mathbf{0}$-lattice $\boldsymbol{\mathcal{L}}$ is strongly dense if the poset of its non-zero elements contains a cofinal strictly decreasing chain. The dimension of a strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ is the minimum length of a cofinal strictly decreasing chain in $\boldsymbol{\mathcal{L}}$. Given a strongly dense modular $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ of an uncountable dimension κ with a cofinal strictly decreasing chain $\mathcal{A} = \langle \boldsymbol{a}_\alpha \mid \alpha < \kappa\rangle$, we set

$$E(\mathcal{A}) := \{\alpha < \kappa \mid \exists \beta \in (\alpha, \kappa]\colon \boldsymbol{a}_\alpha \text{ is not complemented over } \boldsymbol{a}_\beta\},$$

  where $\boldsymbol{a}_\alpha$ is complemented over $\boldsymbol{a}_\beta$ if there exists $\boldsymbol{b} \in \boldsymbol{\mathcal{L}}$ such that $\boldsymbol{a}_\alpha \wedge \boldsymbol{b} = \boldsymbol{a}_\beta$ and $\boldsymbol{a}_\alpha \vee \boldsymbol{b} = \mathbf{1}$. The Γ-invariant of the $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ is the block $\overline{E(\mathcal{A})} \in \mathcal{B}_\kappa$. The block does not depend on the choice of the cofinal strictly decreasing chain $\mathcal{A}$ (cf. [**8**]). According to [**8**, Theorem 1.3], there is a distributive strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}_{\overline{E}}$ of dimension κ with a Γ-invariant $\overline{E}$, for every $\overline{E} \in \mathcal{B}_\kappa$. Passing to the ideal lattice $\mathrm{Id}(\boldsymbol{\mathcal{L}}_{\overline{E}})$, we get a distributive algebraic strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice of dimension κ with the Γ-invariant $\overline{E}$. Applying [**25**, Theorem 4.7] or Theorem **??** from Chapter 1, we conclude that the lattice $\mathrm{Id}(\boldsymbol{\mathcal{L}}_{\overline{E}})$ is isomorphic to the lattice of all two-sided of a locally-matricial $\mathbb{k}$-algebra $\boldsymbol{R}$, where the field $\mathbb{k}$ can be chosen arbitrarily. Then $\boldsymbol{S} := \boldsymbol{R} \otimes_{\mathbb{k}} \boldsymbol{R}^{\mathrm{op}}$, where $\boldsymbol{R}^{\mathrm{op}}$ denotes the opposite ring to $\boldsymbol{R}$, is again a locally matricial $\mathbb{k}$-algebra, due to [**8**, Lemma 2.1]. The original $\mathbb{k}$-algebra $\boldsymbol{R}$ is naturally a right $\boldsymbol{S}$-module wia the multiplication given by $a \cdot (b \otimes c) = cab$. Observing that two-sided ideals of the $\mathbb{k}$-algebra $\boldsymbol{R}$ bijectively correspond to submodules of the right $\boldsymbol{S}$-module $\boldsymbol{R}$, we conclude that each algebraic distributive lattice that is realized as the lattice of two-sided ideals of a locally matricial algebra is realized as a submodule lattice of a module over a locally matricial algebra. In particular, all Γ-invariants are realized.

- The other application of the result is related to the Congruence Lattice Problem. In [**30**] E. T. Schmidt proved that every distributive $\mathbf{0}$-lattice is an image of a generalized Boolean lattice under a distributive $\langle\vee,\mathbf{0}\rangle$-homomorphism, and consequently, it is isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}})$ for a lattice $\boldsymbol{\mathcal{L}}$. Later, in [**32**] (see [**31**] for an earlier weaker result), E. T. Schmidt proved that every finite distributive lattice is the congruence lattice of a complemented modular lattice. Applying our construction, we infer that every distributive $\langle\mathbf{0},\mathbf{1}\rangle$-lattice is isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}}(\boldsymbol{R}))$ for a locally matricial algebra $\boldsymbol{R}$, hence its ideal lattice is representable as the congruence lattice of a complemented modular lattice. The unit element is not essential in the construction, and so we can easily get every distributive $\mathbf{0}$-lattice is isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathrm{Con_c}(\boldsymbol{\mathcal{L}})$ for a sectionally complemented modular lattice $\boldsymbol{\mathcal{L}}$. This gives the result first obtained by P. Pudlák [**22**]. The Pudlák's approach provides a functorial solution and his results are directly (and independently) extended by Theorem **??**.

Let us note that a different approach to the representations of distributive $\mathbf{0}$-lattices as $\mathrm{Id}_c(\boldsymbol{R})$ of locally matricial algebras $\boldsymbol{R}$, similar to the Bergman's constructins [**5**], is in [**20**] by M. Ploščica.

Chapter 2 is based on the paper [**29**]:

> *Distributive congruence lattices of congruence-permutable algebras*, Journal of Algebra **311** (2007), 96 – 116.

The paper is a joint work with Jiří Tůma and Friedrich Wehrung. It closely follows and extends results from [**21**] and [**33**]. In the earlier paper [**36**] F. Wehrung defined the *congruence splitting property* of lattices. The class of congruence splitting lattices (i.e. lattices satisfying the congruence splitting property) is closed under direct limits and it contains all sectionally complemented, all relatively complemented lattices, and all atomistic lattices. The distributive $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}_\kappa$ (for $\kappa \geq \aleph_2$) constructed in [**35**] is not isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattice of all compact congruences of any congruence splitting lattice. Since relatively complemented lattices are congruence splitting, the $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}_\kappa$ (for $\kappa \geq \aleph_2$) is not isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}}(\boldsymbol{R}))$ (and, consequently, to $\mathrm{Id}_c(\boldsymbol{R})$) for any regular ring $\boldsymbol{R}$.

It was in [**36**], where a *uniform refinement property* was used for the first time. This is an infinite system of join-semilattice (or monoid) equations based on the Riesz refinement property that are satisfied for a certain class of join-semilattices, the $\langle\vee,\mathbf{0}\rangle$-semilattices of compact congruences of congruence splitting lattices in this case, and that do not hold for some $\langle\vee,\mathbf{0}\rangle$-semilattice, here $\boldsymbol{\mathcal{S}}_\kappa$. Similar strategy was applied in [**21**], [**33**], and also in our paper [**29**].

The observation that congruence splitting lattices have permutable congruences lays behind [**33**]. Applying a variant of the uniform refinement property, J. Tůma and F. Wehrung proved that $\mathrm{Con_c}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$, where $\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa)$ denotes the free lattice in a non-distributive lattice variety $\mathcal{V}$ with $\kappa \geq \aleph_2$ generators, is not isomorphic to the $\langle \vee, \mathbf{0} \rangle$-semilattice of all compact congruences of any lattice with almost permutable congruences.

In the presented paper we show, using yet another modification of the uniform refinement property, that the $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathrm{Con_c}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is not isomorphic to the $\langle \vee, \mathbf{0} \rangle$-semilattice of all compact congruences of any algebra with almost permutable congruences. In particular, the algebraic distributive lattice $\mathrm{Con}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is isomorphic neither to the normal subgroup lattice of a group, nor to the submodule lattice of a module, nor the lattice of convex subgroups of a lattice-ordered group. These three cases are discussed separately and in the first two of them, the cardinal bound $\aleph_2$ (for the set of compact elements of the algebraic distributive lattice) is proved to be optimal. The negative result is obtained by proving that the algebraic distributive lattice $\mathrm{Con}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is not the range of any distance satisfying the V-condition of type 3/2.

We also study the functorial solution of the problem. We consider the category $\mathcal{D}$ of all surjective distances with morphisms being pairs of one-to-one maps and the forgetful functor Π from $\mathcal{D}$ to the category of $\langle \vee, \mathbf{0} \rangle$-semilattice with $\langle \vee, \mathbf{0} \rangle$-embeddings. On one side, we prove that the restriction of the functor Π to the V-distances of type 2 (i.e, the distances satisfying the V-condition of type 2) has a left inverse. On the other hand we find an unliftable cube by V-ditances of type 3/2. Similar examples are studied in [**33**]. The mysterious connection between sizes of counter-examples for representation problems and dimensions of unliftable cubes was later ingeniously explained by P. Gillibert and F. Wehrung, see [**17**].

Chapter 3 is based on the paper [**28**]:

> *Free trees and the optimal bound in Wehrung's theorem*,
> Fund. Math. **198** (2008), 217 – 228.

Following G. Birkhoff and O. Frink [**6**], the congruence lattice of a lattice is algebraic and due to N. Funayama and T. Nakayama [**9**] it is distributive. In early forties P. Dilworth observed that every finite distributive lattice is representable as a congruence lattice of a finite lattice and conjectured that every algebraic distributive lattice is isomorphic to the congruence lattice of a lattice. The conjecture, named as the *Congruence Lattice Problem*, shortly **CLP**, turned to be a prominent open problem of the lattice theory for over sixty years.

Many partial results was obtained, see [**15**, Appendix C] and the survey paper [**34**] until a counter-example was constructed by F. Wehrung [**38**]. The Wehrung's counter-example has $\aleph_{\omega+1}$ compact elements. In Chapter 3 we improve the size of the counter-example construcitng a distributive $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattice of size $\aleph_2$ such that is not the range of a weakly distributive

$\langle \vee, \mathbf{0} \rangle$-homomorphism from $\operatorname{Con}_c \boldsymbol{A}$ with 1 in its image, for any algebra $\boldsymbol{A}$ with either a congruence-compatible structure of a $\langle \vee, \mathbf{1} \rangle$-semilattice or a congruence-compatible structure of a lattice. In particular, our $\langle \vee, \mathbf{0} \rangle$-semilattice is not isomorphic to the $\langle \vee, \mathbf{0} \rangle$-semilattice of compact congruences of any lattice. Thus we provide a conter-example to **CLP** of the lowest possible cardinality. The main ingredient of our proof is the modification of Kuratowski's Free Set Theorem, which involves what we call *free trees.*

- Chapter 4 is based on the paper [**26**]:

  Countable chains of distributive lattices as maximal semilattice quotients of positive cones of dimension groups, Comment. Math. Univ. Carolin. **47** (2006), 11 – 20.

The Grothendieck group $K_0(\boldsymbol{R})$ of a regular ring $\boldsymbol{R}$ is a directed pre-ordered Abelian group with interpolation. If the ring $\boldsymbol{R}$ is unit-regular, then $K_0(\boldsymbol{R})$ is partially ordered and the positive cone $K_0^+(\boldsymbol{R})$ corresponds to the monoid $\boldsymbol{V}(\boldsymbol{R})$ of isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules.

Recall that a partially ordered Abelian group $\boldsymbol{G}$ is *unperforated* if $np \geq 0$ implies that $p \geq 0$ for all $p \in \boldsymbol{G}$. A dimension group is an unperforated directed partially ordered Abelian group with interpolation. A simplicial directed Abelian group is a free abelian group of a finite rank n with a basis, say, $p_1, \ldots, p_n$ with the positive cone $\mathbb{Z}^+ p_1 \times \cdots \times \mathbb{Z}^+ p_n$. Dimension groups are exactly direct limits of simplicial directed Abelian groups in the category of pre-ordered Abelian groups (with order-preserving group homomorphisms) [**7**, Theorem 2.2].

Let us fix a field $\mathbb{F}$. Locally matricial $\mathbb{F}$-algebras are unit-regular and their Grothendieck groups are dimension groups. Following [**11**, Chapter 15], we call direct limits of countable chains of matricial $\mathbb{F}$-algebras *ultramatricial*, and countable dimension groups *ultrasimplicial*. By [**11**, Theorem 15.24], every ultrasimplicial group appears as the Grothendieck group of an ultramatricial $\mathbb{F}$-algebra and the ultramatricial $\mathbb{F}$-algebra is determined by its Grothendieck group up to the Morita-equivalence [**11**, Corollary 15.27]. The first part of this correspondence extends to dimension groups of size $\aleph_1$, due to [**13**]. In particular, every dimension group of size at most $\aleph_1$ is represented as the Grothendieck group of a locally matricial $\mathbb{F}$-algebra. On the other hand, Grothendieck groups of size $\aleph_1$ do not determine the locally matricial algebras up to the Morita equivalence as in the countable case (see [**11**, Example 15.28]). In [**35**] there is constructed a dimension group of size $\aleph_2$ that is not isomorphic to the Grothendieck group of any regular ring.

As depicted in Figure 1, if $\boldsymbol{R}$ is an unit-regular ring, we have the isomorphisms $\operatorname{Id}(K_0(\boldsymbol{R})) \simeq \operatorname{Id}(\boldsymbol{R})$. The question, whether every distributive $\langle \vee, \mathbf{0} \rangle$-semilattice $\boldsymbol{S}$ is isomorphic to $\nabla(\boldsymbol{G}^+)$ for some dimension group $\boldsymbol{G}$ was stated as [**16**, Problem 1]. We solved this problem in [**24**], where we constructed a counter-example of size $\aleph_2$. Since every countable distributive $\langle \vee, \mathbf{0} \rangle$-semilattice $\boldsymbol{S}$ is isomorphic to the maximal semilattice quotient of the positive cone of a dimension group (see [**16**, Theorem 5.2]), only the case of cardinality $\aleph_1$ remained open. This was resolved by F. Wehrung [**37**], who constructed a distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathbb{S}_{\omega_1}$ of size $\aleph_1$ that is not isomorphic to $\nabla(\boldsymbol{M})$ for any Riesz monoid with an order-unit of finite stable rank. This readily implies that the $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathbb{S}_{\omega_1}$ is not realized as the maximal semilattice quotient of the positive cone of any dimension group. As in some previously discussed constructions, he found a variant of the uniform refinement property, here denoted by $\mathbf{URP}_{\mathrm{sr}}$, that holds in any Riesz monoid $\boldsymbol{M}$ with order-unit of finite stable rank but that is not satisfied by $\mathbb{S}_{\omega_1}$.

It follows from [**37**, Corollary 7.2] that every direct limit of a countable sequence of distributive lattices and $\langle\vee,\mathbf{0}\rangle$-homomorphisms satisfies $\mathbf{URP}_{\mathrm{sr}}$ and it was stated as [**37**, Problem 1], whether such a direct limit is isomorphic to $\nabla(\boldsymbol{G}^+)$ for a dimension group $\boldsymbol{G}$. Recall that every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice closed under finite meets is isomorphic to $\mathrm{Id}_c(\boldsymbol{R})$ for a locally-matricial algebra $\boldsymbol{R}$ and consequently to $\nabla(K_0(\boldsymbol{R})^+)$ for the dimension group $K_0(\boldsymbol{R})$ due to [**25**]. In Chapter 4 we give a negative answer to this question by constructing an increasing countable chain of Boolean join-semilattices, with all inclusion maps being $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-homomorphisms, whose union cannot be represented as the maximal semilattice quotient of the positive cone of any dimension group. Furthermore, we construct a similar example with a countable chain of strongly distributive bounded join-semilattices.

Chapter 5 is based on the paper [**23**]:

> *On the construction and the realization of wild monoids*,
> to appear in Archivum Mathematicum (Brno).

Many still open problems about the structure of regular rings have reformulations in terms of the corresponding monoids $\boldsymbol{V}(\boldsymbol{R})$ of isomorphism classe of finitely generated projective right $\boldsymbol{R}$-modules. Let us say that a monoid $\boldsymbol{M}$ is *realizable* (by a regular ring $\boldsymbol{R}$) if $\boldsymbol{M} \simeq \boldsymbol{V}(\boldsymbol{R})$. According to [**11**, Theorem 2.8], all such monoids are refinement monoids. The fundamental problem by K. R. Goodearl [**12**] asks which refinement monoids are realizable. By [**35**] there are non-realizable refinement monoids of cardinality $\aleph_2$ but there is not yet known a non-realizable refinement monoid of size $< \aleph_2$. Particularly interesting question is whether all countable refinement monoids admit realization, indeed, the answer would shed light on a number of related problems regarding regular rings or C^*-algebras.

Some comprehensive positive results were obtained so far, namely the realization of monoids of row finite quivers [**3**, Theorems 4.2 and 4.4] and the realization of finitely generated primitive monoids with all primes free [**2**, Theorem 2.2]. These realizations are obtained via direct limit construction and the monoids can be realized by regular $\mathbb{F}$-algebras over an arbitrary field $\mathbb{F}$. On the other hand there are countable refinement monoids realizable by regular $\mathbb{F}$-algebras over a countable field $\mathbb{F}$ but not over any uncountable field (see [**1**, Sec. 4]).

Many positive realization results (in general context) are obtained by direct limit construction from diagrams of finitely generated (or even finite) objects, e.g., every distributive $\langle \vee, \mathbf{0} \rangle$-semilattice is a direct limit of finite distributive $\langle \vee, \mathbf{0} \rangle$-semilattices (cf. [**22**, Fact 4 on p. 100]). This is not the case of refinement monoids. Following [**4**] we call a refinement monoid *time* provided that it is a direct limit of finitely generated refinement monoids and *wild* otherwise. The existence of wild refinement monoids indicates that the Goodeatl's fundamental problem is essentially distinct from the other, seemingly similar, realization problems.

An prominent example of a wild refinement monoid is due to G. Bergman and K. R. Goodearl [**11**, Examples 4.26 and 5.10]. We study the example, develop elementary methods of computing the monoids $\boldsymbol{V}(\boldsymbol{R})$ for directly-finite regular rings $\boldsymbol{R}$, and construct a class of directly finite non-cancellative refinement (therefore wild) monoids realizable by regular algebras over an arbitrary field.

Chapter 6 is based on the paper [**19**]:

> *A maximal Boolean sublattice that is not the range of a Banaschewski function*, to appear in Algebra Universalis.

This paper is a joint work with Samuel Mokriš.

A Banaschewki function on a bounded lattice $\boldsymbol{\mathcal{L}}$ is a map $\beta\colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}}$ such that $\boldsymbol{a} \leq \boldsymbol{b}$ implies $\beta(\boldsymbol{b}) \leq \beta(\boldsymbol{a})$ and $\mathbf{1} = \boldsymbol{a} \oplus \beta(\boldsymbol{a})$, for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{\mathcal{L}}$. The terminology is motivated by the early result of B. Banaschewski that the subspace lattice of a vector space admits such a map. Simultaneously we can define a Banaschewski function on a ring $\boldsymbol{R}$ as a map $f\colon \boldsymbol{R} \to \mathrm{Idem}(\boldsymbol{R})$ such that $a\boldsymbol{R} = f(a)\boldsymbol{R}$ and $a\boldsymbol{R} \subseteq b\boldsymbol{R}$ implies that $f(a) \trianglelefteq f(b)$, for all $a, b \in \boldsymbol{R}$. (Here $e \trianglelefteq f$ means that $e = ef = fe$, for all $e, f \in \mathrm{Idem}(\boldsymbol{R})$.) A connection between these two notions of the Banaschewski function is established by [**39**, Lemma 3.5]: An unital regular ring $\boldsymbol{R}$ admits a Banaschewski function if and only if the complemented modular lattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ does.

A notion replacing Banachewski function for lattices without a maximal element is a *Banaschewski measure* [**39**, Definition 5.5]. Every countable sectionally complemented lattice has a Banaschewski measure due to [**39**, Corollary 5.6].

Yet another notion related to the Banaschewski function and the Banaschewski measure is a Banschewski trace [**39**, Definition 5.1]. In [**39**, Section 6] F. Wehrung discovered a close connection between exitence of Banschewski traces (resp. Banschewski measures) and coorinatizability of sectionally complemented modular lattices. This connection is applied in [**40**] in order to construct a non-cordinatizable sectionally complemented modular lattice of size $\aleph_1$ with a large 4-frame. The example shows that the variant of the *Jónson's coordinatization theorem* that states that sectionally complemented modular lattices $\boldsymbol{\mathcal{L}}$ with large n-frames, for $n \geq 4$, and with a contable cofinal chain is coordinatizable (see [**18**]) does not hold for larger cardinalities.

We study ranges of Banaschewski functions on countable complemented modular lattices. According to [**39**, Theorem 4.1 and Corollary 4.8], a countable complemented modular lattice $\mathcal{L}$ has a Banaschewski function with a Boolean range and all the Boolean ranges of Banschewski functions on the lattice $\mathcal{L}$ are isomorphic maximal Boolean sublattices of $\mathcal{L}$. In [**39**, Problem 2] it is asked whether every maximal Boolean sublattice of a countable complemented modular lattice $\mathcal{L}$ appears as a range of some Banaschewski function and whether the maximal Boolean sublattices of $\mathcal{L}$ are isomorphic. We construct a countable complemented modular lattice $\mathcal{S}$ with two non-isomorphic maximal Boolean sublattices $\mathcal{H}$ and $\mathcal{G}$ and we represent the lattice $\mathcal{H}$ as the range of a Banaschewski function on $\mathcal{S}$. Furthermore, we prove that the lattice $\mathcal{S}$ is coordinatizable, in spite of not containing a 3-frame. We show that the lattices $\mathcal{H}$ and $\mathcal{G}$ correspond to maximal Abelian (regular) subalgebras of the regular algebra $\boldsymbol{S}$ realizing the lattice $\mathcal{S}$.

Body of the thesis

Chapter 1 is based on paper [**27**], Chapter 2 on paper [**29**], Chapter 3 on [**28**], Chapter 4 on [**26**], Chapter 5 on [**23**], and Chapter 6 on paper [**19**].

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