Habilitation thesis

The continuum function and large cardinal properties

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Foreword

This habilitation thesis has the form of a unifying commentary to a series of author's papers with various co-authors concerning the continuum function in the context of large cardinals and large-cardinal properties (such as the tree property); the papers appear in the Appendix. The unifying commentary provides a self-contained description which starts with a brief review of the original Easton's result concerning the continuum function on regular cardinals in ZFC, continues with a description of the relevant large cardinals and related notions, and then moves on to stating the original results of the author and his co-authors. The text concludes with a section devoted to the discussion of further research.

All unattributed theorems are due to the author and his co-authors. To keep the text in a manageable size, the theorems are often stated without their proofs (which can be found in the referenced papers); however, we attempt to provide basic ideas and introduce concepts which are crucial for the results.

Remark: The papers are included in the form of an Appendix of this thesis.

Contents

1 Introduction

In this thesis we study the behaviour of the *continuum function* $\kappa \mapsto 2^\kappa$ on infinite regular cardinals κ (or, more generally, cardinals which were regular in some inner model).[1] The behaviour of this function on regular cardinals in ZFC is subject to just two simple restrictions (see Definition 2.1 and Theorem 2.2) – a fact shown by Easton in the early 70's [21]. However, once we extend ZFC by additional axioms claiming the existence of large cardinals, or cardinals sharing with large cardinals some combinatorial properties, the behaviour of the continuum function on regular cardinals becomes a much more complex, and more interesting, topic of study.[2]

Apart from folklore results, the study of the continuum function in the presence of large cardinals was initiated by Menas in [69] who studied this topic in the context of supercompact cardinals. The present author focused on the measurable cardinals and the continuum function in his PhD thesis; the PhD project eventually grew into a larger project which addressed other cardinals and large cardinal notions (see for instance [32, 34, 52, 51, 35, 8]). Brent, Gitman, Golshani and Magidor [9, 6, 10, 7, 49] studied the continuum function for further large cardinals. The scope of the project eventually widened to include a specific large-cardinal property which can be applied to successor cardinals as well (i.e. the tree property[3]). The effect of the tree property on the continuum function was studied by the author and his co-authors in [36, 37, 53].

The thesis is structured as follows: In Section 2, we review basic notions and facts which we use in the thesis; the purpose of this section is also to fix notational conventions. In Section 3, we study the continuum function in the presence of several types of large cardinals. In Section 4 we prove some results regarding the continuum function on singular strong limit cardinals. In Section 5, we follow up with results about the continuum function in the presence of the tree property. In Section 6, we discuss further topics which can be explored and mention some open questions.

[1] "A regular cardinal" always means "an infinite regular cardinal" in this thesis.

[2] For a broader discussion of large cardinals and their motivation, their mathematical importance, and also philosophical considerations, see for instance the following. Background and motivation: [59, 60, 82, 83], use in mathematics and logic: [29, 58, 64, 66, 75, 22]; philosophical considerations: [23, 24, 3, 61].

[3] See Section 5.1 for definitions and Definition 2.6 for connections to large cardinals.

2 Preliminaries

We follow the usual notational conventions, see for instance Jech [56]. We list some additional conventions which are not fully standardised in literature. We use Reg to denote the class of infinite regular cardinals. We use the convention that a forcing notion is a partially ordered class $(\mathbb{P}, \leq)$ with the greatest element (denoted $1_{\mathbb{P}}$); a condition p is stronger than q, in symbols $p \leq q$, if it carries more information. If κ is a regular cardinal, we say that a forcing notion $(\mathbb{P}, \leq)$ is κ-closed if there are lower bounds in $\mathbb{P}$ for all decreasing sequences in $\mathbb{P}$ of size less than κ; similarly, $\mathbb{P}$ is κ-distributive if the intersection of less than κ-many open dense sets is dense.

We use the boldface font for forcing notions related to our proofs, such as $\mathbb{P}$ or $\mathbb{R}$, and the regular font for auxiliary facts and lemmas (such as P and R).

2.1 Easton's theorem

Using the method of forcing developed by Cohen [11], Easton [21] proved that in ZFC the continuum function *on infinite regular cardinals* needs to satisfy only two conditions: For all regular $\kappa, \lambda \geq \omega$:

(i) $\kappa < \lambda \to 2^\kappa \leq 2^\lambda$.
(ii) $\kappa < \mathrm{cf}(2^\kappa)$.

In order to formulate Easton's theorem properly, let us introduce the notion of an *Easton function*:

Definition 2.1 *A proper class function F from regular cardinals to cardinals is an* Easton function *if it satisfies for all regular $\kappa, \lambda \geq \omega$:*

(i) $\kappa < \lambda \to F(\kappa) \leq F(\lambda)$.
(ii) $\kappa < \mathrm{cf}(F(\kappa))$.

We say that an Easton function F is realised in V^* (a transitive model extending V and with the same ordinals as V) if regular cardinals in V and V^* coincide and for all regular κ, $F(\kappa) = 2^\kappa$ in V^*.

Theorem 2.2 (Easton) *Assume V satisfies* GCH *and let F be an Easton function definable over V. Then there is a definable cofinality-preserving proper-class forcing notion $\mathbb{P}$ such that if G is $\mathbb{P}$-generic, then in $V[G]$,*

$$2^\kappa = F(\kappa),$$

for all regular κ, i.e. F is realised in $V[G]$.

More general versions of Easton's theorem can be formulated which remove the restriction on the definability of F: for instance if κ is inaccessible and GCH holds below κ, one can identify V with $H(\kappa)$[4] and show that for any Easton function F on regular cardinals below κ which exists in $H(\kappa^+)$ there exists a cofinality-preserving forcing notion $\mathbb{P}$ of size κ such that if G is $\mathbb{P}$-generic over V, then $H(\kappa)[G]$ is a model of ZFC and F is realised in $H(\kappa)[G]$ on regular cardinals below κ.

Remark 2.3 One can extend the notion of an Easton function to all cardinals: both regular and singular. We will not study such extensions here in full generality; however, in Sections 4 and 5, we do provide a few results which discuss the continuum function on singular strong limit cardinals which were regular in some inner model.

2.2 Large cardinals

In this section we briefly review definitions and important facts related to large cardinals which appear later in the text; this section also serves the purpose of fixing the notation. For more details and definitions of other larger cardinals, consult [59] or [56].

Definition 2.4 *Let κ be a regular uncountable cardinal. We say that κ is* inaccessible *if $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$ (this property is called* being a strong limit cardinal*).*

Note that if GCH holds, then κ is inaccessible if and only if κ is a regular and limit cardinal.

A strengthening of inaccessibility is Mahloness.

[4]The set of all x whose transitive closure has size less than κ.

Definition 2.5 *We say that an inaccessible cardinal κ is Mahlo if the set of regular cardinals below κ is stationary in κ.*

One can easily show that if κ is Mahlo, then the set of inaccessible cardinals is stationary below κ.

An important strengthening of Mahloness is weak compactness. There are many equivalent definitions of this notion;[5] we only state two such reformulations which are used later in the text (see [14] for more details).

Recall that if κ is regular, than a κ-tree is a tree of height κ whose levels have size $< \kappa$.

Definition 2.6 *Suppose κ is an inaccessible cardinal. We say κ is* weakly compact *if the following equivalent conditions are satisfied:*

(i) Every κ-tree has a cofinal branch.
(ii) For every transitive set M with $|M| = \kappa$, $\kappa \in M$, and ${}^{<\kappa}M \subseteq M$, there is an elementary embedding $j : M \to N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, and κ is the critical point of j, i.e. the first ordinal moved by j.

Recall that an ultrafilter U on a regular uncountable cardinal κ is called non-principal if it does not contain singletons, and κ-complete if it is closed under intersections of members of U of length less than κ. U is called normal if it is closed under diagonal intersections of length κ: if A_i, $i < \kappa$, are in U, then $\triangle_{i<\kappa} A_i = \{\xi < \kappa \,|\, \xi \in \bigcap_{\zeta<\xi} A_\zeta\}$ is in U, where $\triangle_{i<\kappa} A_i$ is called the diagonal intersection of the A_i's. An ultrafilter on κ is often called *a measure*.

Definition 2.7 *Suppose κ is an inaccessible cardinal. We say that κ is* measurable *if the following equivalent conditions are satisfied:*

(i) There is a non-principal κ-complete ultrafilter U on κ.
(ii) There is an elementary embedding[6] $j : V \to M$, where M is a transitive

[5] Originally, the name of the cardinal was motivated by the study of certain extensions of the first-order logic; in this context, κ is weakly compact if the corresponding infinitary logic is compact. See the relevant sections in [59] or [56].

[6] j is a proper class; thus we should view this definition as taking place in a class set theory, or more technically – but preferably – as a statement expressible in ZFC because the relevant part of j which we need, $j \restriction H(\kappa^+)$, is a set.

class, and κ is the critical point of j, i.e. κ is the first ordinal moved by j.[7]

If κ is measurable, then there exists a normal ultrafilter on κ; normal ultrafilters can be used to classify measurable cardinals according to their "size". The idea goes back to Mitchell who used it to obtain lower bounds for consistency strength of various set-theoretical statements (see for instance [72]). Let U and W be normal ultrafilters on κ; we say that U is less than W in the Mitchell order, and write it as $U \lhd W$, if U is an element of the ultrapower of the universe V by W. One can show that $\lhd$ is a well-founded strict ordering on normal ultrafilters. In particular, we can assign to κ its Mitchell order $o(\kappa)$ which is defined as the height of the ordering $\lhd$. $o(\kappa)$ is less or equal to $(2^\kappa)^+$; hence under GCH, $o(\kappa) \leq \kappa^{++}$ (see [56] for more details). As we will see (see Section 3.3.1), the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ in some inner model is exactly the consistency strength of a measurable cardinal violating GCH.

By strengthening the properties of the elementary embedding in the definition of the measurable cardinal, we get the notion of the $H(\lambda)$-*strong* (or sometimes also called $H(\lambda)$-*hypermeasurable*) cardinal.[8]

Definition 2.8 *Suppose κ is an inaccessible cardinal and λ is a cardinal greater than κ. We say that κ is* $H(\lambda)$-strong *if there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$ and $H(\lambda) \subseteq M$. We say that κ is* strong *if it is $H(\lambda)$-strong for all $\lambda > \kappa$.*

One can show that being measurable is the same as being $H(\kappa^+)$-strong. Although we have not made it a part of our definitions of the measurable and the $H(\lambda)$-strong cardinals, we can always obtain an M with the additional property of being closed under κ-sequences in V: ${}^\kappa M \subseteq M$ whenever the cofinality of λ is greater than κ.

$H(\lambda)$-strong cardinals can be without the loss of generality represented by embeddings having some desirable properties.

[7] If $j : V \to M$ is an elementary embedding, M is often called the *target model* of j.

[8] We divert from the notation in [59] where the strength of the model M is measured in terms of the V-hierarchy, and use the H-hierarchy instead; for instance under GCH, the $\kappa+2$-strong cardinal in [59], where $V_{\kappa+2}$ is required to be a subset of M, corresponds to the $H(\kappa^{++})$-strong cardinal in our notation. Our definition is more suitable for the formulation of the results in this thesis (see for instance Theorem 3.11). Some papers, for instance [12], use yet another notation: κ is $\mathscr{P}_2(\kappa)$-hypermeasurable if it is $\kappa+2$-strong.

Definition 2.9 *Suppose $j : V \to M$ is an elementary embedding with critical point κ. Let $\lambda > \kappa$ be a cardinal with cofinality greater than κ. We say that j is a (κ, λ)-*extender embedding *if the following hold:*

(i) ${}^{\kappa}M \subseteq M$.
(ii) $M = \{j(f)(\alpha) \mid f : \kappa \to V \ \& \ \alpha < \lambda\}$.[9]

One can show under GCH that if κ is $H(\lambda)$-strong for κ and λ as in Definition 2.9, then there exists a (κ, λ)-extender embedding $j : V \to M$ which witnesses the $H(\lambda)$-strength of κ, i.e. apart from (i)–(ii) in Definition 2.9, it also satisfies $H(\lambda) \subseteq M$.

We should add that if $\kappa < \lambda$ are as above, it is possible to define a (κ, λ)-*extender* as a sequence of κ-complete ultrafilters on finite subsets of κ together with commutative embeddings which ensure that the direct limit of the ultrapowers taken via these ultrafilters is well-founded. A (κ, λ)-extender has the same relation to a (κ, λ)-extender embedding as an ultrafilter U on κ has to the ultrapower embedding generated by U. We will not go into details here since we only need the notion of a (κ, λ)-extender embedding defined in Definition 2.9. For more details on extenders, consult [59], Section 26.

Remark 2.10 It is instructive to compare (under GCH) the large cardinal strength of two typical assumptions which we use in the thesis (see Section 3.3.1 for more details): (i) κ is measurable with Mitchell order $o(\kappa) = \kappa^{++}$, and (ii) κ is $H(\kappa^{++})$-strong. The assumption (ii) is strictly stronger than (i), but the difference is relatively a small one. Analogously to the Mitchell ordering on normal ultrafilters, one can define Mitchell ordering on extenders discussed in the previous paragraph; under this generalised ordering the assumption (ii) corresponds to the existence of an uncountable regular κ with Mitchell order $o(\kappa) = \kappa^{++} + 1$ on extenders. See Dodd's book [19] for more details.

We close this section by defining the largest large cardinal we use in the thesis:

Definition 2.11 *Suppose $\kappa < \lambda$ are uncountable cardinals, κ regular. We say that κ is λ-*supercompact *if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. A cardinal κ is* supercompact *if it is λ-supercompact for every $\lambda \geq \kappa$.*

[9] M is often defined with respect to finite subsets of λ and not ordinals below λ; the two representations are equivalent.

There are still larger large cardinals which require M to be even closer to V, but we will not discuss them here. Note in this context that by Kunen's result [62], there is a sharp bound on how M can be close to V: in ZFC, the existence of a non-identical elementary embedding $j : V \to V$ is contradictory.

2.3 Forcing

We take the Cohen forcing at α to be defined as follows:

Definition 2.12 *If α is a regular cardinal, and $\beta > 0$ is an ordinal, then the* Cohen forcing at α of length β, *denoted* $\mathrm{Add}(\alpha, \beta)$, *is the set of all partial functions from $\alpha \times \beta$ to 2 of size less than α; the ordering is by reverse inclusion: $p \leq q \leftrightarrow q \subseteq p$.*

Under GCH, $\mathrm{Add}(\alpha, \beta)$ is α-closed and α^+-cc.

Let F be an Easton function. In the thesis, we use variants of the "forward Easton forcing with Cohen forcings" – denoted $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$ – and of the "reverse Easton forcing with Cohen forcings" – denoted $\mathbb{P}_F(\mathrm{Add})$ – to realise F. For the benefit of the reader, we review their definitions now (in the form we need them).

The following forcing notion composed of Cohen forcings was used by Easton [21] to prove Theorem 2.2.

Definition 2.13 (Forward Easton forcing) *Let F be an Easton function. Define*

$$\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add}) = \prod\nolimits_{\alpha \in \mathrm{Reg}} \mathrm{Add}(\alpha, F(\alpha)),$$

where the product has the "Easton support": for every inaccessible α and any condition $p \in \mathbb{P}_F^{\mathrm{product}}$, $\mathrm{dom}(p) \cap \alpha$ is bounded in α.[10]

The following lemma is useful for showing that $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$ preserves all cofinalities if we assume GCH:

Lemma 2.14 (Easton) *Let $\kappa > \omega$ be a regular cardinal. Suppose P is κ-closed, and Q is κ^+-cc. Then the following hold:*

[10] We disregard the fact that we deal with proper classes here. One can make the exposition rigorous as it is done in Jech [56], or alternatively view V as a set-size model in some larger universe.

(i) $P \Vdash$ "*Q is κ-cc.*"
(ii) $Q \Vdash$ "*P is κ-distributive.*"

If F is an Easton function, let C_F be the closed unbounded class of limit cardinals which are the closure points of F: i.e.

$$C_F = \{\alpha \mid \alpha \text{ limit cardinal } \ \& \ (\forall \beta \in \alpha \cap \mathrm{Reg})(F(\beta) < \alpha)\}.$$

We can modify $\mathbb{P}_F^{\text{product}}(\mathrm{Add})$ by turning it into an iteration of products indexed by the elements of C_F (this idea first appeared in Menas [69]). As it turns out, this modification is more appropriate when we wish to preserve large cardinals.

Definition 2.15 (Reverse Easton forcing) *Let F be an Easton function. Define $\mathbb{P}_F(\mathrm{Add})$ as follows: For every pair (α, β) of successive elements of C_F, let us denote*

$$Q_{\alpha,\beta} = \prod\nolimits_{\gamma \in [\alpha,\beta) \cap \mathrm{Reg}} \mathrm{Add}(\gamma, F(\gamma)), \tag{2.1}$$

where $[\alpha, \beta)$ denotes the half-open interval between α and β and the product has the Easton support.
$\mathbb{P}_F(\mathrm{Add})$ is the iteration $(\langle P_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \dot{Q}_\alpha \mid \alpha \in \mathrm{Ord}\rangle)$ with Easton support such that $\dot{Q}_\alpha$ is the canonical name for the trivial forcing whenever α is not in C_F. If α is in C_F, let $\dot{Q}_\alpha$ be a name for the forcing $Q_{\alpha,\beta}$ in V^{P_α}, where β is the successor of α in C_F.

Similarly as for $\mathbb{P}_F^{\text{product}}(\mathrm{Add})$, one can show that under GCH, $\mathbb{P}_F(\mathrm{Add})$ is cofinality preserving and forces the continuum function on regular cardinals to obey F.

Remark 2.16 We use the phrase "a reverse Easton iteration" in a more general sense to denote iterations with the Easton support.

2.4 Liftings of embeddings

In view of the fact that many large cardinals are defined in terms of elementary embeddings, it is useful to have a general technique which allows us to argue that an elementary embedding in the ground model V gives rise to an elementary embedding in a generic extension of V. The following lemma due to Silver is the key tool in this respect:

Lemma 2.17 (Silver) *Assume M and N are transitive models of* ZFC*, $\mathbb{P} \in M$ is a forcing notion, and $j : M \to N$ is an elementary embedding. Let G be $\mathbb{P}$-generic over M, and let H be $j(\mathbb{P})$-generic over N. Then the following are equivalent:*

(i) $(\forall p \in G)(j(p) \in H)$.
(ii) There exists an elementary embedding $j^+ : M[G] \to N[H]$ such that $j^+(G) = H$ and $j^+ \restriction M = j$.

We say that j^+ is a *lifting of j*.

It is a useful fact that Silver's lemma can be generalised to claim that a lifting of a (κ, λ)-extender embedding (see Definition 2.9) is again a (κ, λ)-extender embedding. More details about these concepts can be found in Cummings [14].

The basic use of Silver's lemma can be illustrated as follows: suppose $j : V \to M$ is a (κ, λ)-extender embedding with $\lambda > \kappa$ being a cardinal with cofinality greater than κ, P_κ is a reverse-Easton forcing iteration of length κ which is κ-cc and $P_\kappa \subseteq V_\kappa$. Suppose P_κ forces that $\dot{Q}_\kappa$ is a κ-closed forcing included in $H(\lambda)^{V^{P_\kappa}}$, and $P_\kappa * \dot{Q}_\kappa$ forces that $\dot{Q}$ is κ^+-distributive. Let P denote $P_\kappa * \dot{Q}_\kappa * \dot{Q}$, and let $G = G * g_\kappa * F$ be P-generic. In order to show that κ remains measurable in $V[G]$, it suffices to lift j to $V[G]$ according to Silver's lemma. The argument proceeds by constructing a $j(P)$-generic H over M:

$$H = G * g_\kappa * H_1 * h_2 * H_3,$$

which satisfies the following points:

(a) $G * g_\kappa$ is $j(P)$-generic for the iteration up to $\kappa+1$,[11] and H_1 is $j(P)$-generic for the open interval $(\kappa, j(\kappa))$. In this situation, we can lift j to

$$j : V[G] \to M[G * g_\kappa * H_1].$$[12]

(b) Denoting $Q_\kappa = (\dot{Q}_\kappa)^G$, h_2 is $j(Q_\kappa)$-generic over $M[G * g_\kappa * H_1]$ with j"g_κ included in h_2. Then j can be lifted again to

$$j : V[G * g_\kappa] \to M[G * g_\kappa * H_1 * h_2].$$

[11] The initial segment of $j(P)$ up to $\kappa + 1$ is typically equivalent to $P_\kappa * \dot{Q}_\kappa$ by the assumptions on $P_\kappa * \dot{Q}_\kappa$; sometimes $H(\lambda)$-strength of j is required here.

[12] It is customary to denote the lifted embedding with the same letter j to avoid notational clutter.

(c) The generic H_3 is usually easily generated from F using the extender representation of M and κ^+-distributivity of $\dot{Q}$.

The hardest part is usually to construct h_2 and ensure that it contains $j"g_\kappa$ – the main problem is that a (κ, λ)-extender embedding only ensures closure of M under κ-sequences, while $j"g_\kappa$ is typically larger than κ.[13] Woodin's "surgery argument" is designed to obtain such an h_2 when Q_κ is the Cohen forcing at κ; Sacks forcing at κ used in Theorem 3.11 provides another way of constructing h_2 (see Section 3.3 below for more details).

3 The continuum function and large cardinals

3.1 Preserving large cardinals while realising an Easton function F

Let V be a model where GCH holds.[14] We wish to study the following question:

(Q) Assume Γ is a class of large cardinals which have some property "X". Which Easton functions F are consistent with Γ in the sense that there exists a cofinality-preserving generic extension $V[G]$ of V where F is realised, and all cardinals in Γ retain the property "X" in $V[G]$?

Here, "X" typically stands for a type of large cardinals, such as Mahlo, weakly compact, measurable, etc.

We of course prefer to find answers to (Q) which are close to being optimal in the sense of the original Easton's result: i.e. we wish to identify restrictions imposed on the continuum function by large cardinals, and show they are the only provable restrictions (by showing that all F's which obey them can be realised in some model). The optimality of this type is hard to achieve in general: sometimes it suffices to argue that the given large cardinals do not impose any new restrictions (inaccessible, Mahlo, weakly compact, Ramsey,

[13] It is at this step that Silver used the "master condition argument" (see Section 3.3) when he started with a supercompact cardinal κ: with a suitable degree of supercompactness, $j"g_\kappa$ is a valid condition in $j(Q_\kappa)$, and therefore any generic filter which contains the condition $j"g_\kappa$ suffices.

[14] Assuming GCH with large cardinals is typically without the loss of generality because all known large cardinals are consistent with GCH: see Friedman [30] for the strong cardinals which are widely used in this text.

or Woodin cardinals discussed in Section 3.2), but starting with a measurable cardinal there are new restrictions and it is harder to argue for optimality (see for instance Theorem 3.11 and the discussion which follows for measurable cardinals).

Before we go into details of specific constructions, notice that if F is an Easton function and κ is an inaccessible cardinal, then κ must be *closed under F*, i.e. for all $\mu < \kappa$, $F(\mu) < \kappa$, to remain inaccessible when F is realised.

Remark 3.1 In Section 4, we extend (Q) to certain singular cardinals, and in Section 5 to cardinals satisfying the tree property.

3.2 Large cardinals with no effect on the continuum function

Though it is not the main focus of this thesis, it is instructive to first discuss small large cardinals and their effect on the continuum function. We will see that such cardinals typically put no additional restrictions on the continuum function.

It is easy to check that in order to generalise Easton's theorem to inaccessible cardinals, it suffices to use $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$ over a model of GCH: every inaccessible κ closed under F is forced by $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$ to remain strong limit, and regular because $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$ preserves cofinalities.

Fact 3.2 *Let V satisfy* GCH *and let F be an Easton function. Then in any generic extension $V[G]$ by $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$, every inaccessible κ closed under F is still inaccessible.*

One can formulate a version of the fact for Mahlo cardinals.

Fact 3.3 *Let V satisfy* GCH *and let F be an Easton function. Then in any generic extension $V[G]$ by $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$, every Mahlo κ closed under F is still Mahlo.*

Proof. Let G be $\mathbb{P}_F^{\mathrm{product}}(\mathrm{Add})$-generic and let κ be a Mahlo cardinal in V closed under F. Since the set of inaccessible cardinals I below κ is stationary in κ in V, $C_F \cap I$ is also stationary. It follows by Fact 3.2 that all inaccessible $\alpha \in C_F \cap I$, and also κ, remain inaccessible in $V[G]$. To finish the argument, it suffices to check that $C_F \cap I$ is still stationary in $V[G]$. Factor $\mathbb{P}_F^{\text{product}}(\text{Add})$ into $P_0 \times P_1$ such that P_1 is κ-closed and P_0 is κ-cc: P_0 is defined as $\mathbb{P}_F^{\text{product}}(\text{Add})$, but with the domain of the functions in the product limited to $\kappa \cap \text{Reg}$; similarly, P_1 has the domain limited to $\text{Reg} \setminus \kappa$. One can show that P_1 preserves stationary subsets of κ because it is κ-closed. As P_1 forces by Easton's lemma that P_0 is κ-cc, P_0 preserves stationary subsets over V^{P_1}. Thus $\mathbb{P}_F^{\text{product}}(\text{Add}) = P_0 \times P_1$ preserves stationary subsets of κ, and in particular stationarity of $C_F \cap I$. $\square$

We now turn to weakly compact cardinals. It is easy to find an example where the product-style Easton forcing $\mathbb{P}_F^{\text{product}}(\text{Add})$ destroys the weak compactness of κ over some well-chosen ground model such as L: since $\mathbb{P}_F^{\text{product}}(\text{Add})$ is a product, one can manipulate the forcing and use the fact that adding a new subset $x \subseteq \kappa$ over L with the property that $x \cap \alpha \in L$, for all $\alpha < \kappa$, destroys the weak compactness of κ (see Exercise I5 in Kunen [63]). One therefore uses the reverse Easton forcing $\mathbb{P}_F(\text{Add})$. We give the following Theorem without a proof (see Cody and Gitman [9] for the proof).

Theorem 3.4 (Cody, Gitman) *Let V satisfy* GCH *and let F be an Easton function. Then in any generic extension $V[G]$ by $\mathbb{P}_F(\text{Add})$, every weakly compact κ closed under F is still weakly compact.*

There are more results along these lines which show that the continuum function is not restricted by Ramsey, strongly-Ramsey, or Woodin cardinals; see Cody and Gitman [9] and Cody [7] for more details and definitions.[15]

3.3 Measurable and strong cardinals

3.3.1 Background material

The following simple lemma shows that an Easton function for measurable cardinals requires more restrictions than the ordinary Easton function in Definition 2.1. Hence regarding the Easton's theorem, the situation for measurable cardinals is more complex than for the large cardinals we have mentioned so far (inaccessible, Mahlo, weakly compact, Ramsey, Woodin).

[15] It may be surprising that Woodin cardinals do not restrict the continuum function, but recall that in general a Woodin cardinal is Mahlo, but may not even be weakly compact (however, its consistency strength is greater than that of a strong cardinal). See Kanamori [59] for more details.

Lemma 3.5 *Assume κ is measurable and let U be a normal ultrafilter. If $A = \{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is in U, then $2^\kappa = \kappa^+$.*

Proof. Let $\mathrm{Ult}(V, U)$ be the transitive collapse of the ultrapower via U. By Łos theorem and normality, $A \in U$ implies

$$\mathrm{Ult}(V, U) \models 2^\kappa = \kappa^+.$$

It is easy to check that $H(\kappa^+) = (H(\kappa^+))^{\mathrm{Ult}(V,U)}$. Therefore any bijection $f \in \mathrm{Ult}(V, U)$ between $(\kappa^+)^{\mathrm{Ult}(V,U)}$ and $\mathscr{P}(\kappa)^{\mathrm{Ult}(V,U)}$ is a bijection between κ^+ and $\mathscr{P}(\kappa)$ in V, proving $2^\kappa = \kappa^+$. □

By Lemma 3.5, in order to obtain a model for the sentence "there is a measurable cardinal κ and $2^\kappa = \kappa^{++}$" (let us denote this sentence temporarily ψ), one needs to violate GCH at a large set below κ. The obvious strategy is to use a reverse Easton iteration which forces with $\mathrm{Add}(\alpha, \alpha^{++})$ at every inaccessible cardinal less or equal to κ, and then show that κ is still measurable in the resulting generic extension. Silver made this strategy work starting with a κ^{++}-supercompact κ, using the "master condition" lifting argument which we briefly reviewed in Section 2.4.[16] The upper bound for the consistency strength of ψ identified by Silver was improved in the early 1980's by Woodin (unpublished; see [14] for details) who was able to construct a model of ψ from this assumption:

($*$) There is $j : V \to M$ with critical point κ such that ${}^\kappa M \subseteq M$ and $j(f)(\kappa) = (\kappa^{++})^V$ for some $f : \kappa \to \kappa$.

Using canonical models for sequences of measures, Mitchell [72] showed that the lower bound for the consistency strength of ψ is at least the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$. For a while the relation between ($*$) and $o(\kappa) = \kappa^{++}$ was unclear until finally Gitik [42] proved that the assumption $o(\kappa) = \kappa^{++}$ suffices to construct an embedding ($*$) used by Woodin. Thus by the combination of these results, the failure of GCH at a measurable cardinal is equiconsistent with a measurable κ with $o(\kappa) = \kappa^{++}$.

The results in the previous paragraph generalise to larger gaps at a measurable cardinal κ, i.e. to situations where $\kappa = \aleph_\gamma$ for some γ, and $2^\kappa = \aleph_{\gamma+\alpha}$ for some $\alpha > 2$ (see Gitik [44]).

[16] The supercompactness is used to ensure that $j"g_\kappa$ is an element of the relevant target model, where g_κ is $\mathrm{Add}(\kappa, \kappa^{++})$-generic over the reverse Easton iteration up to κ.

As we discussed in Remark 2.10, there is a difference in the consistency strength between the assumption $(*)$ of Woodin, and the $H(\kappa^{++})$-strong cardinal κ. While this difference is not substantial for the proof of Woodin (as his iteration is non-trivial only at inaccessible cardinals), it starts to be relevant once we wish to control the continuum function on successor cardinals (such as κ^+). We therefore give two constructions for the Easton's theorem for measurable cardinals: In Section 3.3.2 we formulate the result in full generality using the stronger assumption of $H(F(\kappa))$-strength. In Section 3.3.3, we show with more effort that the weaker assumption is sufficient for the special case of an Easton function F which satisfies $F(\alpha) = \alpha^{++}$ for every regular α (we also indicate how to generalise this result to include more functions F).

3.3.2 From $H(F(\kappa))$-strong cardinals

As we mentioned in the previous section, Woodin constructed a model with a measurable cardinal violating GCH using the assumption $(*)$. His forcing construction is more complicated than Silver's because his starting assumption is much weaker. We briefly review Woodin's construction[17] in order to motivate our own proof in [32] (see Theorem 3.11 below). Suppose $(*)$ and GCH hold. Woodin first uses a reverse Easton iteration P_1 which forces with $\mathrm{Add}(\alpha^+, \alpha^{++})$ for every inaccessible $\alpha \leq \kappa$, and then over this generic extension he forces with a reverse Easton iteration P_2 of $\mathrm{Add}(\alpha, \alpha^{++})$ for every inaccessible $\alpha \leq \kappa$. The first iteration P_1 uses the forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$ because of the fact that assuming $2^\kappa = \kappa^+$, $\mathrm{Add}(\kappa^+, \kappa^{++})$ is equivalent to $j_U(\mathrm{Add}(\kappa, \kappa^{++}))$, where j_U is the ultrapower embedding corresponding to the ultrafilter $U = \{X \subseteq \kappa \,|\, \kappa \in j(X)\}$ derived from j; thus P_1 adds a generic filter h_0 for $j_U(\mathrm{Add}(\kappa, \kappa^{++}))$ (over the relevant model), which is then used to obtain a generic filter h for the j-image of $\mathrm{Add}(\kappa, \kappa^{++})$. The argument is finished by manipulating h to h^*[18] so that h^* contains the pointwise image of the generic filter for $\mathrm{Add}(\kappa, \kappa^{++})$ to be able to use Silver's lemma 2.17 (this stage is called the "surgery argument").

Woodin's argument is intriguing, but the construction of h^* is complex and it is not clear how it modifies to an arbitrary Easton function F. In [32] we used the Sacks forcing at an inaccessible cardinal α (as introduced in Kanamori

[17] We omit many details to keep the review in manageable size; see Cummings [12] for the whole proof.

[18] h_2 in the notation in the review of the lifting argument after Silver's lemma in Section 2.4.

[57]) which is more friendly with respect to lifting arguments as showed in Friedman and Thompson [41].

Definition 3.6 *If α is an inaccessible cardinal, then $p \subseteq 2^{<\alpha}$ is a perfect α-tree if the following conditions hold:*

(i) If $s \in p, t \subseteq s$, then $t \in p$;

(ii) If $s_0 \subseteq s_1 \cdots$ is a sequence in p of length less than α, then the union of s_i's belongs to p;

(iii) For every $s \in p$ there is some $s \subseteq t$ such that t is a splitting node, i.e. both $t^\frown 0$ and $t^\frown 1$ belong to p;

(iv) Let $\mathrm{Split}(p)$ denote the set of s in p such that both $s^\frown 0$ and $s^\frown 1$ belong to p. Then for some (unique) closed unbounded set $C(p) \subseteq \alpha$, $\mathrm{Split}(p) = \{s \in p \,|\, \mathrm{length}(s) \in C(p)\}$.

A perfect α-tree is an obvious generalisation of the perfect tree at ω ordered by inclusion; there is only one non-trivial condition, and this concerns the limit levels of the tree: if $s \in p$ is an element at a limit level and the splitting nodes $t \subseteq s$ are unbounded in s, then s must be a splitting node as well (continuous splitting). As α is inaccessible, and consequently every level of p is of size $< \alpha$, the trees obeying (iv) above are dense in the trees having continuous splitting.

Generalised perfect trees can be used to define a natural forcing notion.

Definition 3.7 *The forcing notion $\mathrm{Sacks}(\alpha, 1)$ contains as conditions the perfect α-trees, the ordering is by inclusion, i.e. $p \leq q$ iff $p \subseteq q$. More generally, the forcing notion $\mathrm{Sacks}(\alpha, \beta)$, where $0 < \beta$ is an ordinal number, is the product of length β of the forcing $\mathrm{Sacks}(\alpha, 1)$ with support of size at most α.*

$\mathrm{Sacks}(\alpha, \beta)$ is α-closed, and if $2^\alpha = \alpha^+$, it is also α^{++}-cc. The preservation of α^+ is based on the fact that $\mathrm{Sacks}(\alpha, \beta)$ is closed under *fusion sequences* of length α (see Fact 3.8):

For a condition p in $\mathrm{Sacks}(\alpha, 1)$, let $\langle \alpha_i \,|\, i < \alpha \rangle$ be the increasing enumeration of $C(p)$ and let $\mathrm{Split}_i(p)$ be the set of s in p of length α_i. For $p, q \in \mathrm{Sacks}(\alpha, 1)$ let us write $p \leq_\beta q$ iff $p \leq q$ and $\mathrm{Split}_i(p) = \mathrm{Split}_i(q)$ for $i < \beta$. In the generalisation for the product $\mathrm{Sacks}(\alpha, \beta)$ we write $p \leq_{\beta, X} q$ (where X is some subset of β of size less than α) iff $p \leq q$ (i.e. for all $i < \beta, p(i) \leq q(i)$) and moreover for each $i \in X$, $p(i) \leq_\beta q(i)$.

Fact 3.8 *The forcing* Sacks(α, β) *satisfies the following* α-fusion property*: Suppose* $p_0 \geq p_1 \geq \dots$ *is a descending sequence in* Sacks(α, β) *of length* α *and suppose in addition that* $p_{i+1} \leq_{i,X_i} p_i$ *for each* i *less than* α*, where* X_i *form an increasing sequence of subsets of* β *of size less than* α *whose union is the union of the supports of* p_i*'s; such a sequence will be called a* fusion sequence*. Then the* p_i*'s have a lower bound in* Sacks(α, β) *(obtained by taking intersections at each component).*

Friedman and Thompson showed in [41] that the fusion property of Sacks forcing can be used to simplify the lifting argument of Woodin when we start with a (κ, λ)-extender embedding for some $\lambda > \kappa$. Let us illustrate the argument with the following prototypical case: Assume GCH and suppose $j : V \to M$ is a (κ, κ^{++})-extender embedding, $(\kappa^{++})^M = \kappa^{++}$, and P is a reverse Easton iteration which forces with Sacks(α, α^{++}) at each inaccessible cardinal $\alpha \leq \kappa$. Then in the notation of Section 2.4 (after Silver's lemma 2.17) one can show that $j"g_\kappa$ (where g_κ is now Sacks(κ, κ^{++})-generic and j is the lifting of the original embedding) generates a generic filter h_2 for Sacks$(j(\kappa), j(\kappa^{++}))$ in the following sense: For $\zeta < \kappa^{++}$ and $\xi = j(\zeta)$, the intersection of the trees $j(p(\zeta))$, $p \in g_\kappa$, determines a tree of height $j(\kappa)$ with exactly two cofinal branches which has a stem of length κ, and splits at κ; both of these branches can be used to define h_2 at the coordinate ξ. For $\xi \in j(\kappa^{++}) \setminus j"(\kappa^{++})$, $j"g_\kappa$ actually determines a single branch at ξ which can be used to define h_2 at ξ. Moreover, the generic filters thus generated by $j"g_\kappa$ are unique for the lifting (once we select the left or the right branch), a fact which was used by Friedman and Magidor to resolve a question regarding the number of normal measures on a measurable cardinal (see [40]).

Although we cannot go into details here, note that the lifting argument in the previous paragraph only works with extender embeddings j: every dense open set in Sacks$(j(\kappa), j(\kappa^{++}))$ which we need to meet to define h_2 is of the form $j(f)(\alpha)$, for some $\alpha < \kappa^{++}$ and some f with domain κ whose range is included in dense open sets in Sacks(κ, κ^{++}). This representation allows one to "diagonalise" over the dense open sets in $\langle f(\xi) \,|\, \xi < \kappa \rangle$ using the κ-fusion property, and this way show that $j"g$ meets $j(f)(\alpha)$ (and hence generates a generic filter h_2 over Sacks$(j(\kappa), j(\kappa^{++}))$.

Let us note that the simplification of the lifting argument discussed in the preceding paragraph is not specific to Sacks forcing; the important property is that of the κ-fusion which is shared by more forcings. For instance in [54], we presented a lifting argument for the generalised Grigorieff forcing at κ.

Remark 3.9 The use of Sacks forcing (and of other forcings with fusion) not only simplifies the lifting argument, but allows one to define forcing constructions which cannot be easily mimicked with the Cohen forcing and Woodin's argument. For instance, one can formulate a lifting argument for the iteration of the Sacks forcing at κ and code some information in the process; this was used in our paper [33] to prove that SCH can *definably* fail at $\aleph_\omega$ in the sense that $2^{\aleph_\omega} = \aleph_{\omega+2}$, $\aleph_\omega$ is strong limit, and the wellordering of the subsets of $\aleph_\omega$ is lightface definable in $H(\aleph_{\omega+1})$ (see Section 4.2 for more details), and in another paper [36] where it was used to show that a certain configuration of the tree property below $\aleph_\omega$ is consistent (see Section 5 for more details). A comprehensive treatment of fusion with respect to preservation of measurability is included in our paper [39].

Returning to our present topic of the Easton's theorem, consider the following variant $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$ of the forcing $\mathbb{P}_F(\mathrm{Add})$ in Definition 2.15:

Instead of $Q_{\alpha,\beta}$ in (2.1) in Definition 2.15, whenever α is an inaccessible closure point of F, set

$$Q_{\alpha,\beta} = \mathrm{Sacks}(\alpha, F(\alpha)) \times \prod_{\beta \in \mathrm{Reg} \cap (\alpha,\beta)} \mathrm{Add}(\beta, F(\beta)).$$

With GCH, the argument for showing cofinality preservation of $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$ is quite standard except for the following lemma which generalises Easton's lemma 2.14 (see [32] for the proof):

Lemma 3.10 *Let α be an inaccessible cardinal and β an ordinal number. Let P be any α^+-closed forcing notion. Then* $\mathrm{Sacks}(\alpha, \beta)$ *forces that P is α^+-distributive.*

We used $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$ to prove the following (Theorem 3.8 in [32]):

Theorem 3.11 *Let F be an Easton function and assume* GCH *holds in the universe. Then* $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$ *preserves cofinalities and realises F. Moreover, whenever G is* $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$*-generic, then the following holds:*

Whenever in V, κ is $H(F(\kappa))$-strong, closed under F and there is $j : V \to M$ witnessing the $H(F(\kappa))$-strength of κ such that

$$j(F)(\kappa) \geq F(\kappa), \tag{3.2}$$

then κ remains measurable in $V[G]$.[19]

Before we discuss the idea of the proof, notice the property (3.2) which captures the degree of reflection which F needs to satisfy for the proof of Theorem 3.11 (with respect to an existentially quantified j): while being included for technical reasons, it seems quite reasonable because if κ is measurable in a cofinality-preserving generic extension W of V where F is the continuum function (and in particular $2^\kappa = F(\kappa)$), then any ultrapower embedding $j_U : W \to \mathrm{Ult}(W, U)$ via a normal ultrafilter U needs to satisfy

$$j_U(F)(\kappa) = (2^\kappa)^{\mathrm{Ult}(W,U)} \geq (2^\kappa)^W = F(\kappa).$$[20]

The idea of the proof of Theorem 3.11 is as follows: let κ be an $H(F(\kappa))$-strong cardinal satisfying (3.2) for some fixed j; without the loss of generality this j is a $(\kappa, F(\kappa))$-extender embedding. By elementarity, $j(\kappa)$ is closed under $j(F)$ and a routine calculation shows

$$F(\kappa) < j(F)(\kappa) < j(\kappa) < F(\kappa)^+.$$

Since $H(F(\kappa))$ is included in M, the generic filter over V for the regular cardinals $\leq F(\kappa)$ can be used after some manipulation to define a generic filter over M for the regular cardinals $\leq F(\kappa)$. There is a problem when $F(\kappa)$ is singular in V, but it is regular in M (a configuration which may occur) – in this case there is no generic filter available over V at $F(\kappa)$, but we need to construct one over M: a technical argument involving a two-dimensional matrix of size $\mathrm{cf}(F(\kappa))^V \times \kappa^+$ of conditions constructed with the help of certain elementary submodels resolves the hardest case when $\kappa^+ < \mathrm{cf}(F(\kappa))^V < F(\kappa) = \mathrm{cf}(F(\kappa))^M$ (Sublemma 3.13 in [32]). Next, lifting the generic for Sacks$(\kappa, F(\kappa))$ is achieved using the properties of the Sacks forcing which we reviewed above. The iteration above κ is lifted straightforwardly

[19] In fact, κ typically retains also some degree of strongness. Compare with Theorem 3.13.

[20] To show that (3.2) is optimal one would need to consider weaker embeddings in the statement of Theorem 3.11 (as we do in Section 3.3.3) and analyse which embeddings in W restrict to embeddings in V. We have not studied this question in detail, but the methods of Hamkins [50] might be a good starting point.

using Lemma 3.10 and the extender representation of M (which guarantees that κ^+-distributive forcing notions lift easily).

In the same paper [32] (Theorem 3.17), a similar argument is used to show that strong cardinals are preserved. Recall that a regular uncountable cardinal is *strong* if it is $H(\lambda)$-strong for every $\lambda > \kappa$. In [69], Menas identified a property of an Easton function – being *locally definable*, see Definition 3.12 – which suffices for preservation of supercompactness. The proof in [69] is based on the master condition argument which is available for supercompact cardinals. In [32], we generalised his result for strong cardinals using the technique available for Sacks forcing.

Definition 3.12 *An Easton function F is said to be* locally definable *if the following condition holds:*

There is a sentence ψ and a formula $\varphi(x, y)$ with two free variables such that ψ is true in V and for all cardinals γ, if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma (F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \tag{3.3}$$

Theorem 3.13 *Let V satisfy* GCH *and let F be a locally definable Easton function. Let Γ be the class of all strong cardinals. Then in any generic extension $V[G]$ by $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$, F is realised and every $\kappa \in \Gamma$ is still strong.*

Note that we do not explicitly require that every strong κ is closed under F; however, this follows from the assumption of local definability.

3.3.3 From the optimal assumptions

By Gitik [44], the optimal large-cardinal assumption for a measurable cardinal cardinal κ with $2^\kappa = \kappa^{+\alpha}$ for $\alpha \geq 2$ is $o(\kappa) = \kappa^{+\alpha}$.[21] As we discussed in Remark 2.10 (where we mentioned the meaning of $o(\kappa) = \xi$ for $\xi > \kappa^{++}$), the large cardinal assumption of $H(\kappa^{+\alpha})$-strength of κ is stronger.

It is therefore a natural question whether Theorem 3.11 can be proved from the optimal large cardinal assumptions. In [34], we showed that this is possible for the prototypical case $2^\kappa = \kappa^{++}$, κ measurable; we also indicated how to

[21] $\kappa^{+\alpha}$ denotes the cardinal $\aleph_{\gamma+\alpha}$ if $\kappa = \aleph_\gamma$.

generalise this result to $2^\kappa = \kappa^{+n}$, $2 \leq n < \omega$, and left it as an open question for infinite gaps.

We now introduce two definitions of embeddings which are relevant for us.

Definition 3.14 *We say that κ is κ^{++}-*tall *if there is an elementary embedding $j : V \to M$ with critical point κ such that ${}^\kappa M \subseteq M$ and $\kappa^{++} < j(\kappa)$.*

The definition of tallness naturally generalises to cardinals $\lambda > \kappa$: κ is λ-tall if there is an embedding $j : V \to M$ with critical point κ, ${}^\kappa M \subseteq M$, and $\lambda < j(\kappa)$.

Definition 3.15 *We say that κ is κ^{++}-*correct *if there is an elementary embedding $j : V \to M$ with critical point κ such that ${}^\kappa M \subseteq M$ and $(\kappa^{++})^M = \kappa^{++}$.*

Under GCH, it is easy to show that we can assume without the loss of generality that the embeddings in Definitions 3.14 and 3.15 are (κ, κ^{++})-extender embeddings.

Clearly, if j is κ^{++}-correct, then it is κ^{++}-tall. It is shown in Gitik [42] that if V satisfies GCH and j is κ^{++}-tall, then there is a generic extension V^* satisfying GCH such that κ is κ^{++}-correct in V^*. Hence the assumption of κ^{++}-correctness has the same consistency strength as the existence of κ with $o(\kappa) = \kappa^{++}$.

Theorem 4.1 and Corollary 4.2 in [34] imply the following theorem:

Theorem 3.16 *Assume GCH and let F be an Easton function such that $F(\alpha) = \alpha^{++}$ for all regular α. Let κ be a κ^{++}-correct cardinal with a witnessing embedding $j : V \to M$.*[22] *Then there exists a cofinality-preserving forcing $\mathbb{R}$ which realises F and preserves the measurability of κ.*

The forcing $\mathbb{R}$ in the statement of Theorem 3.16 is now more complex than $\mathbb{P}_F(\mathrm{Sacks}, \mathrm{Add})$ because we need to deal with the problem that $H(\kappa^{++})^M$ can be strictly included in $H(\kappa^{++})$. If $H(\kappa^{++}) \setminus H(\kappa^{++})^M$ is non-empty, then $\mathrm{Add}(\kappa^{++}, \kappa^{+4})$ which realises F on κ^{++} is quite different from the Cohen

[22] Note that with our F, every such κ is closed under F and every κ^{++}-correct j satisfies $j(F)(\kappa) \geq F(\kappa)$ required in (3.2) since $j(F)(\kappa) = F(\kappa) = \kappa^{++}$.

forcing at κ^{++} in the sense of M: while it is the case that $\mathrm{Add}(\kappa^{++},1)^M$ is included in $\mathrm{Add}(\kappa^{++},1)$,[23] it is not included as a regular suborder (by a density argument, any $\mathrm{Add}(\kappa^{++},1)$-generic filter g contains an initial segment which is not an element of M). One may attempt to force the generic for $\mathrm{Add}(\kappa^{++},1)^M$ over V but it is not clear whether this forcing preserves cardinals (it may not be κ^{++}-distributive in V). We solved this problem by including a preparatory forcing $\mathrm{Add}(\kappa^{+},\kappa^{++})$ which ensures the κ^{++}-distributivity of $\mathrm{Add}(\kappa^{++},1)^M$ over V (this idea appeared first in [2]).

The argument generalises to all $2 \leq n < \omega$. As we mentioned, by Gitik [44], $2^\kappa = \kappa^{+n}$ is equiconsistent with $o(\kappa) = \kappa^{+n}$. The assumption $o(\kappa) = \kappa^{+n}$ implies the existence of a coherent sequence of length κ^{+n} of $H(\kappa^{+n-1})$-strong extenders at κ (where an extender at κ is $H(\kappa^{+n-1})$-strong if the associated extender ultrapower embedding is $H(\kappa^{+n-1})$-strong). Generalising the construction in Gitik [42], the assumption $o(\kappa) = \kappa^{+n}$ for $2 \leq n < \omega$ implies that there exists a generic extension V^* satisfying GCH and an elementary embedding $j : V^* \to M$ with critical point κ such that:

(i) M is closed under κ-sequences in V^*,
(ii) $H(\kappa^{+n-1})$ of V^* is included in M,
(iii) $(\kappa^{+n})^M = \kappa^{+n}$.

The existence of such a j allows one to generalise Theorem 3.16 for $F(\kappa) = \kappa^{+n}$, $2 \leq n < \omega$. The related problem of infinite gaps, i.e. having Theorem 3.16 with $F(\kappa) = \kappa^{+\alpha}$ for $\omega < \alpha$, was left open but we suspect a variant of the method in [34] might be helpful.

3.4 Supercompact cardinals with a degree of tallness

As we discussed in Section 3.3.2, Sacks forcing at an inaccessible κ behaves well with respect to the lifting of extender embeddings. We used this useful property in Theorems 3.11 and 3.16. Kanamori [57] defines Sacks forcings also at a successor cardinal κ^+ and uses $\diamondsuit_{\kappa^+}$ to show that it preserves cofinalities. As it turns out, even this forcing may be useful in manipulating the continuum function.

[23] It suffices (and is actually necessary by Observation 3.5 in [34]) to deal with $\mathrm{Add}(\kappa^{++},1)^M$ and not $\mathrm{Add}(\kappa^{++},\kappa^{+4})^M$ which is the relevant forcing; an additional argument shows that an $\mathrm{Add}(\kappa^{++},1)^M$-generic filter over V provides an $\mathrm{Add}(\kappa^{++},\kappa^{+4})^M$-generic filter over M (see Claim 3.7 in [34]).

Recall the definition of κ being λ-tall for some cardinal $\lambda > \kappa$ (see Definition 3.14).

Combining the definitions of the supercompact and tall cardinals (of certain degrees), we get the following definition:

Definition 3.17 *Suppose $\kappa < \lambda < \bar{\lambda}$ are cardinals and κ, λ are regular. We say that κ is* λ-supercompact, $\bar{\lambda}$-tall, *if there is an elementary embedding $j : V \to M$ with critical point κ, ${}^{\lambda}M \subseteq M$, and $\bar{\lambda} < j(\kappa)$.*

Assuming GCH, such an embedding is witnessed by an embedding which has similar properties as extender embeddings (see Definition 2.9):

$$j : V \to M = \{j(f)(j''\lambda, \alpha) \,|\, f : P_\kappa(\lambda) \times \kappa \to V \ \&\ \alpha < \bar{\lambda}\}.$$

We showed in [35] the following result for λ-supercompact, $\bar{\lambda}$-cardinals for the paradigmatic case $\bar{\lambda} = \lambda^{++}$:

Theorem 3.18 *Assume* GCH. *Let $\kappa < \lambda$ be regular cardinals and let κ be a λ^{++}-tall λ-supercompact cardinal witnessed by $j : V \to M$. Then over a ground model V which was prepared in a certain way,*[24] *there exists a cofinality-preserving forcing notion $\mathbb{P}$ such that whenever G is $\mathbb{P}$-generic, κ is still λ^{++}-tall λ-supercompact in $V[G]$,* GCH *holds in the interval $[\kappa, \lambda)$ and moreover $2^\lambda = \lambda^{++}$ in $V[G]$.*

Notice that λ is regular, but not necessarily a limit cardinal. For the discussion which follows assume $\lambda = (\lambda')^+$ for some cardinal λ' (not necessarily regular).

Let us provide some brief comments regarding the proof of Theorem 3.18. We used the following modification of Sacks forcing at λ to ensure $2^\lambda = \lambda^{++}$:

Definition 3.19 *Assume* GCH. *For a regular cardinal λ, we say that $p \subseteq {}^{<\lambda}2$ is a* cof ω_1-splitting perfect tree at λ *if p is a tree of height λ closed under initial segments such that:*

(i) For every s in p there is $s' \supseteq s$ in p such that s' splits, where s' splits *if both $s'^\frown 0$ and $s'^\frown 1$ are in p.*

[24] So that it contains the functions f_λ and f_Δ introduced below.

(ii) *If $\langle s_\xi \,|\, \xi < \delta\rangle$ for some limit ordinal $\delta < \lambda$ is an $\subseteq$-increasing chain of nodes in p, then the union $\bigcup_{\xi<\delta} s_\xi$ is also a node in p.*

(iii) *If s is a node in p and s is in ${}^{\delta}2$ for some δ of cofinality ω_1, and moreover the set of nodes $s' \subsetneq s$ which split is unbounded in s, then s splits in p;*

(iv) *If s is a node in p and s is in ${}^{\delta}2$ for some limit δ of cofinality other than ω_1, then s does not split in p.*

Let Sacks$(\lambda, 1)$ be the forcing composed of cof ω_1-splitting perfect trees [25] at λ ordered by inclusion, and let Sacks(λ, λ^{++}) be the λ^{++}-product of Sacks$(\lambda, 1)$ with support of size at most λ. One can define a version of the fusion sequence, and show that Sacks(λ, λ^{++}) has the λ-fusion property. Under GCH, Sacks(λ, λ^{++}) preserves cofinalities; the fact that λ^+ is preserved in the extension is shown using $\Diamond_\lambda$.

The forcing which is used to prove Theorem 3.18 is a reverse Easton iteration $\mathbb{P} * \dot{\text{Sacks}}(\lambda, \lambda^{++})$ of length $\kappa + 1$, where $\mathbb{P}$ prepares for the lifting of $\dot{\text{Sacks}}(\lambda, \lambda^{++})$ using certain auxiliary functions f_λ and f_Δ. These functions can be added using the fast function forcing of Woodin and satisfy the following: let $\pi : \kappa \to \kappa^2$ be some canonical bijection, then $f_\lambda : \kappa \to \kappa$ satisfies:

$$j(f_\lambda)(\kappa) = j(\pi^{-1})(\langle \lambda, \lambda^{++}\rangle).$$

Let Δ denote the supremum of $j``\lambda$, and let $c : P_\kappa(\lambda) \to \lambda$ be a fixed bijection. Then $f_\Delta : \lambda \to \lambda$ satisfies:

$$j(f_\Delta)(\Delta) > j(c)(j``\lambda).$$

In order to lift the embedding j, we generalised Kanamori's construction in [57] of a fusion sequence guided by $\Diamond_\lambda$, and introduce the notion of *rich reduction* (see Definition 2.12 in [35]). Intuitively, instead of choosing a single extension given by $\Diamond_\lambda$ at a certain stage of the construction, we allow up to λ' many extensions (obtained in a certain way): with more extensions, the pointwise image of a generic filter for Sacks(λ, λ^{++}) generates a generic filter for Sacks$(j(\lambda), j(\lambda^{++}))$.

We find it interesting that the lifting argument can be made to work even for the Sacks forcing at a successor cardinal. However, the technical obstacles

[25] The cof ω_1-splitting is used to ensure that the lifting of Sacks(λ, λ^{++}) can be carried out and assumes that $\text{cf}(\lambda') > \omega$; if $\text{cf}(\lambda') = \omega$, the definition must be modified a little with splitting allowed on cof ω (see Remark 2.16 in [35]).

were quite substantial and some questions related to the continuum function and supercompact and tall cardinals (of a certain degree) were left open in our paper [35]. For instance we were not able to modify the construction in the proof of Theorem 3.18 to have the following configuration in the generic extension: $2^\kappa < \lambda$ and $2^\mu = 2^\lambda = \lambda^{++}$ for some regular cardinal μ in the open interval (κ, λ).

Cody and Magidor [10] resolved this open question by finding a generalisation of the Woodin's surgery argument for the Cohen forcing at λ. We used this method later on in our paper [8] to formulate a more general Easton-like theorem for supercompact cardinals of certain tallness.

4 The continuum function and singular cardinals

4.1 The continuum function on former large cardinals

Let us review the forcing $\mathrm{Prk}_U(\kappa)$ introduced by Prikry [74]:

Definition 4.1 *Let κ be a measurable cardinal and U a normal ultrafilter on κ. Conditions in $\mathrm{Prk}_U(\kappa)$ are pairs (s, A), where s is a finite increasing sequence in κ and A is a set in U such that $\max(s) < \min(A)$. (p, A) is stronger than (q, B) if p end-extends q and $p \setminus q$ is in B.*

Prikry forcing $\mathrm{Prk}_U(\kappa)$ preserves all cardinals, does not change V_κ, and introduces a new ω-sequence of ordinals cofinal in κ.

Recall the following formulation of the Singular Cardinal Hypothesis (SCH): whenever κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$.[26]

By results of Gitik and Mitchell, see [72] and [43], the strength of the failure of SCH is exactly the existence of a measurable cardinal α with $o(\alpha) = \alpha^{++}$. The first method of obtaining κ where SCH fails was found by Woodin who used Prikry forcing to singularize a measurable cardinal κ which violates GCH (such a measurable cardinal was obtained by the "surgery argument" which we discussed above (see Sections 2.4 and 3.3.2)).

In [52], we used the model constructed in [32], which realises an Easton function while preserving measurable cardinals, and followed up with an iterated

[26] There are other formulations of SCH; on the closed unbounded class of strong limit cardinals, they are all equivalent.

Prikry forcing to obtain a global pattern of singular strong limit cardinals of cofinality ω which violate SCH. We used two methods of performing the construction: an iteration with full support, and an iteration with Easton support (Easton support is useful for the generalisation in Theorem 4.3).

The following theorem appears as Theorem 3.8 in [52]:

Theorem 4.2 *Assume* GCH *and let $V[G]$ be a generic extension of V from Theorem 3.11 in this thesis. Then there is a cardinal preserving iteration $\mathbb{R}$ of Prikry forcings (either with full support, or Easton support) such that if H is $\mathbb{R}$-generic over $V[G]$, then in $V[G][H]$, every cardinal κ which was $H(F(\kappa))$-strong in V closed under F and with an embedding j witnessing its strength such that $j(F)(\kappa) \geq F(\kappa)$, is now a singular strong limit cardinal of cofinality ω with $2^\kappa = F(\kappa)$.*

Theorem 4.2 provides some information about the possible pattern of cardinals violating SCH. However, the method of the proof is not optimal in the sense that a cardinal κ where SCH fails was first a measurable cardinal κ which violated GCH. By reflection (see Lemma 3.5), this implies that GCH fails unboundedly often below κ – a property which is known not to be required for singular cardinals of countable cofinality. Gitik and Magidor developed the so called *extender-based forcing* to overcome this restriction; using the extender-based forcing it is possible to construct a model with a singular strong limit cardinal κ with countable cofinality which violates GCH, but GCH holds below κ (see Magidor [46] and Gitik [45] for more details).[27]

In order to overcome the restrictions in Theorem 4.2, we continued in [52] by formulating an Easton-supported iteration of extender-based Prikry forcings which can turn a large cardinal κ directly into a singular strong limit cardinal with countable cofinality, without requiring κ to be measurable first. However, since the extender-based Prikry forcing requires some degree of GCH to behave properly, some restrictions on F still remained (item (ii) in Theorem 4.3):

Let us say that an Easton function F is *mild* if all Mahlo cardinals are among the closure points of F and for all Mahlo cardinals α, $F(\alpha^+) = \max(\alpha^{++}, F(\alpha))$.

[27] We will not define the extender-based forcing here, but let us at least say that it requires a (κ, λ)-extender for its definition, for an appropriate λ. Let us also note that one can find a model where GCH fails at a singular strong limit κ of countable cofinality and holds below κ by other methods (see Magidor [67]).

The following theorem appears as Theorem 4.13 in [52]:

Theorem 4.3 *Let V satisfy* GCH *and let F be a mild Easton function. Then there is a cardinal preserving extension V^* of V which realises F and which satisfies the following: If κ is $H(F(\kappa))$-strong in V and satisfies either of the two conditions:*

(i) There an embedding $j : V \to M$ witnessing the $H(F(\kappa))$-strength of κ such that $j(F)(\kappa) \geq F(\kappa)$, or

(ii) There an embedding $j : V \to M$ witnessing the $H(F(\kappa))$-strength of κ such that $j(F)(\kappa) = \kappa^+$,

then κ is a singular strong limit cardinal of cofinality ω in V^* and $2^\kappa = F(\kappa)$.

The idea of the proof is that if (i) holds, then we singularize κ using the Prikry forcing and the method in Theorem 4.2 (the condition (i) says that F violates GCH sufficiently often below κ, modulo some j), and if (ii) holds, we use the extender-based Prikry forcing at κ (the condition (ii) essentially says that F preserves GCH sufficiently often below κ, modulo some j). The condition of mildness appearing in Theorem 4.3 is a technical one and enables one to show that the forcing iteration behaves reasonably (in order to use the option (ii), one needs to show that the iteration preserves the required strength of κ to define the extender based forcing; in this stage we use an argument similar to Gitik, Shelah [48]).

We mention a corollary of Theorem 4.3: if F only toggles GCH on and off in the sense that for every regular α, $F(\alpha) \in \{\alpha^+, \alpha^{++}\}$, then every j satisfies either (i) or (ii), and therefore all $H(F(\kappa))$-strong cardinals for $F(\kappa) = \kappa^{++}$ end up being strong limit cardinals of cofinality ω.

Due to technical obstacles, the paper [52] stops short of discussing singular strong limit cardinals of uncountable cofinality. However, it seems to be an interesting question, not least because the failure of SCH at a cardinal κ of an uncountable cofinality, unlike the case of countable cofinality, does reflect down below κ.[28]

[28] However, the analogy is not perfect: if κ is a singular strong limit cardinal of an uncountable cofinality which violates SCH, then $\{\alpha < \kappa \,|\, 2^\alpha > \alpha^+\}$ is large in the sense that it contains a closed unbounded subset of κ; if κ is measurable and violates GCH, then the corresponding reflection only says – in terms of the closed unbounded filter – that $\{\alpha < \kappa \,|\, 2^\alpha > \alpha^+\}$ must be stationary.

4.2 Definability and the failure of SCH at $\aleph_\omega$

The fact that the Sacks forcing at κ and other forcings with the κ-fusion behave well with respect to lifting of embeddings (see Section 3.3.2) can be used to achieve results which may not be possible with Woodin's surgery argument and the Cohen forcing at κ (as these results seem to require an iteration to code some information). In this section we discuss the following theorems which appear as Theorems 4.1 and 5.1 in our paper [33]:

Theorem 4.4 *(GCH). Starting from an $H(\kappa^{++})$-strong cardinal κ, it is consistent that GCH fails at κ (in fact $2^\kappa = \kappa^{++}$), κ remains measurable, and there is a lightface definable wellorder of $H(\kappa^+)$.*

Theorem 4.5 *(GCH) Starting from an $H(\kappa^{++})$-strong cardinal κ, it is consistent that GCH fails at $\aleph_\omega$ (in fact $2^{\aleph_\omega} = \aleph_{\omega+2}$), $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, and there is a lightface definable wellorder of $H(\aleph_{\omega+1})$.*

Both theorems belong to a program in set theory which tries to obtain many of the nice properties which hold in L-like models by forcing, while violating certain other properties which we may found too restrictive (such as GCH). In our theorems, the nice properties are the definable wellorders of $H(\kappa^+)$ and $H(\aleph_{\omega+1})$, respectively.

The proof starts with an extender model $L[E]$ for an $H(\kappa^{++})$-strong cardinal κ. In this model, we fix definable sequences of certain stationary sets which are selectively killed to code the information about the wellorder. The proof is technically complicated, but let us at least mention where the Sacks forcing comes in: it is used in the form of a perfect-tree coding at α which codes subsets of α^+ for an inaccessible α into a subset of α (see Section 3.2.3 in [33] for more details). When $\alpha = \kappa$ is a critical point of an embedding, the lifting of the perfect-tree coding is an essential part of the argument for the preservation of the measurability of κ in Theorem 4.4.

To obtain Theorem 4.5, we follow up on the model constructed in Theorem 4.4 with the Prikry forcing with collapses which is designed to code the collapses iteratively using again the selective killing of certain mutually stationary sets (see Section 5.1 in [33]).

It is a natural question – along the lines of this thesis – whether we can manipulate the continuum function more freely while keeping the definable wellorder. As it turns out, this is a very difficult question and the methods in [33] do not seem to provide more general results: for one, the perfect-tree coding necessarily forces $2^\kappa = \kappa^{++}$ due to its support of size $\leq \kappa$, and we do not know how to code the wellorder while ensuring $2^\kappa > \kappa^{++}$ and to lift the embedding with critical point κ at the same time. A similar caveat applies for the continuum function below κ (or $\aleph_\omega$): we use a reverse Easton iteration with the subsequent Prikry forcing with collapses to turn the measurable κ into $\aleph_\omega$; this yields unboundedly many failures of GCH below $\aleph_\omega$ in Theorem 4.5.

5 The continuum function and the tree property

5.1 Background material

Suppose κ is a regular uncountable cardinal. Recall that a κ-tree T is a tree of height κ such that all levels of T have size less than κ. We say that κ has the tree property, $\mathsf{TP}(\kappa)$, if every κ-tree has a cofinal branch.[29] A κ-tree which does not have a cofinal branch is called a *κ-Aronszajn tree*. By Definition 2.6, the property that there are no κ-Aronszajn trees characterises weak compactness of κ if we add the assumption of inaccessibility.

A κ^+-Aronszajn tree T with ordering $<_T$ is called *special* if there is a function f from T to κ which is injective on chains in $<_T$. We write $\mathsf{wTP}(\kappa^+)$ (the "weak tree property") if there are no special κ^+-Aronszajn trees. It is known that $\mathsf{wTP}(\kappa^+)$ is equivalent to $\neg\Box^*_\kappa$ (the weak square); see [13] for more details about these concepts.

Note that if T is special, then it is special in all extensions of V which preserve the cardinal κ^+; this is not true for an Aronszajn tree in general: it may get a cofinal branch in an extension of the universe.

By results of Specker [76], whenever $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. In particular, $\neg\mathsf{TP}(\aleph_1)$ which was already proved by Aronszajn (see [65]), and more generally $\neg\mathsf{TP}(\kappa^+)$ whenever κ is inaccessible.

It is a natural question whether there may be a non-inaccessible regular uncountable cardinal κ which has the tree property (or the weak tree property).

[29] By a result of König, every ω-tree has a cofinal branch. However, it is customary to use the notion of the tree property for uncountable cardinals only; thus we do not usually say that ω has the tree property.

By results of Mitchell [71], if κ is regular and $\lambda > \kappa$ is weakly compact, then there is a forcing notion $\mathbb{M}$ which forces $2^\kappa = \kappa^{++} = \lambda$ and $\mathsf{TP}(\lambda)$; in particular, modulo a weakly compact cardinal, $\aleph_2$ can have the tree property (if λ is just Mahlo, we get $\mathsf{wTP}(\aleph_2)$ with the same forcing).[30]

Let us focus on the effect of the tree property on the continuum function. It is a direct consequence of Specker's theorem that for an infinite cardinal μ,

$$\mathsf{TP}(\mu^{++}) \text{ implies } 2^\mu > \mu^+; \tag{5.4}$$

in particular $\mathsf{TP}(\aleph_2)$ decides CH negatively. We may ask whether the (weak) tree property puts more restrictions on the continuum functions apart from (5.4).

Unger showed in [77] that $\mathsf{TP}(\aleph_2)$ is consistent with an arbitrarily high value of 2^ω, answering this question locally for one cardinal. More generally, we can ask: (A) does $\mathsf{TP}(\aleph_2)$ say something about the values of the continuum function on other cardinals such as $\aleph_1$ or $\aleph_2$, or more generally (B) do multiple cardinals with TP put some restriction on the continuum function, and (C) what does $\mathsf{TP}(\mu^{++})$ for a strong limit singular cardinal μ say about the continuum function? We study these questions for the tree property and the weak tree property in the following sections.[31]

5.2 The tree property and the weak tree property below $\aleph_\omega$

Let κ be a regular cardinal.

Let us first discuss the tree property. It is known that the consistency strength of having simultaneously $\mathsf{TP}(\kappa^+)$ and $\mathsf{TP}(\kappa^{++})$ is much larger than the consistency strength of either of these alone: while in the latter case the large cardinal strength is that of a weakly compact cardinal, in the former case the

[30] One can show that if μ is an infinite cardinal and $\mathsf{TP}(\mu^{++})$, then μ^{++} is weakly compact in L so the assumption of the existence of a weakly compact cardinal is optimal. Similarly, if $\mathsf{wTP}(\mu^{++})$ holds, then μ^{++} is Mahlo in L. Note that $\mathsf{wTP}(\mu^+)$, and *a fortiori* $\mathsf{TP}(\mu^+)$, for a singular strong limit μ has a much larger consistency strength as it implies the failure of the weak square at μ.

[31] We omit the case of $\mathsf{TP}(\mu^+)$ for a singular strong limit μ. The techniques for ensuring $\mathsf{TP}(\mu^+)$ in this context are rather different from the focus of this thesis (these techniques are based on the result of Magidor and Shelah, [68], which says that if μ is a singular limit of strongly compact cardinals, then $\mathsf{TP}(\mu^+)$. See Section 6 and [73] for more details and references).

lower bound is on the level of a Woodin cardinal (see [27]).[32] In contrast, having simultaneously $\mathsf{TP}(\kappa^+)$ and $\mathsf{TP}(\kappa^{+++})$ does not increase the consistency strength: two weakly compact cardinals suffice – a fact which is not completely obvious since the two forcings which achieve the tree property at κ^+ and κ^{+++} overlap. In our paper [36] we showed that if we start with infinitely many weakly compact cardinals, there exists a model with the tree property at every $\aleph_{2n}$, $0 < n < \omega$; if we moreover assume there is an $H(\mu^{++})$-strong cardinal μ, then there is a model where in addition $\neg\mathsf{SCH}$ holds at $\aleph_\omega$ (this requires a more complicated forcing). In [53] we continued to show that any Easton function below $\aleph_\omega$ not outright inconsistent with the tree property at $\aleph_{2n}$, $0 < n < \omega$, can be realised in some model where the tree property holds at each $\aleph_{2n}$, $0 < n < \omega$.

As it turns out, the situation for the weak tree property is quite different: The consistency strength of having $\mathsf{wTP}(\kappa^+)$ and $\mathsf{wTP}(\kappa^{++})$ is just two Mahlo cardinals – a fact which indicates that the tree property is a much stronger concept than the weak tree property. In fact, Unger [78] showed that assuming the existence of ω many Mahlo cardinals, there is a model where the weak tree property holds at every $\aleph_n$, $1 < n < \omega$. In [53], we followed up on the results in [78] and showed that the weak tree property at every $\aleph_n$, $1 < n < \omega$, does not place any restrictions on the continuum function below $\aleph_\omega$ (except for the condition that GCH must fail below $\aleph_\omega$ at every cardinal).

5.2.1 The tree property

In [53], we showed that the continuum function on the cardinals $\aleph_n$, $n < \omega$, can be almost arbitrary (i.e. not outright inconsistent with the tree property) with the tree property holding at every $\aleph_{2n}$, $0 < n < \omega$.

Theorem 5.1 *Start with the* GCH. *Assume there are infinitely many weakly compact cardinals and let $f : \omega \to \omega$ be a non-decreasing function such that $f(2n) \geq 2n + 2$ for every $n < \omega$. Then there is a generic extension $V[G]$ where the tree property holds at each $\aleph_{2n}$, $0 < n < \omega$, and f determines the continuum function in $V[G]$ below $\aleph_\omega$:*

$$2^{\aleph_n} = \aleph_{f(n)}.$$ [33]

[32] In [27], the consistency strength calculation for $\mathsf{TP}(\aleph_2)$ and $\mathsf{TP}(\aleph_3)$ uses the additional assumptions of $2^{\aleph_0} = \aleph_2$ and that every subset of $\aleph_2$ has a sharp. The analogous result for $\mathsf{TP}(\aleph_n)$ for all $1 < n < \omega$ with $\aleph_\omega$ strong limit does not use extra assumptions.

[33] We state the theorem with $f : \omega \to \omega$ because we are naturally interested in the case

The proof of the theorem builds on an argument in our paper [36] which includes this theorem with $f(2n) = 2n+2$ and $f(2n+1) = 2n+2$,[34] $n < \omega$.

Suppose $\kappa_1 < \kappa_2 < \cdots$ are weakly compact cardinals with the limit λ; for convenience let κ_0 denote ω. We first prepare the universe by a reverse Easton iteration $\mathbb{P}$ which adds one Cohen subset to every inaccessible below λ.

Let $\mathbb{M}(\kappa_i, \kappa_{i+1})$ denote the Mitchell forcing (in the presentation from [1]) which collapses κ_{i+1} to the double successor of κ_i while preserving κ_i^+ and forces the tree property at κ_{i+1}. We give the definition for the benefit of the reader:

Definition 5.2 *Conditions in* $\mathbb{M}(\kappa_i, \kappa_{i+1})$ *are pairs* (p, q), *where* p *is in the Cohen forcing* $\mathrm{Add}(\kappa_i, \kappa_{i+1})$, q *is a function with* $\mathrm{dom}(q) \subseteq \kappa_{i+1}$, $|\mathrm{dom}(q)| \leq \kappa_i$, *and for every* $\alpha \in \mathrm{dom}(q)$,

$$1_{\mathrm{Add}(\kappa_i,\alpha)} \Vdash q(\alpha) \in \dot{\mathrm{Add}}(\kappa_i^+, 1)^{V^{\mathrm{Add}(\kappa_i,\alpha)}}.$$

A condition (p, q) *is stronger than* (p', q') *if*

(i) $p \leq p'$,
(ii) $\mathrm{dom}(q) \supseteq \mathrm{dom}(q')$ *and for every* $\alpha \in \mathrm{dom}(q')$, $p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha)$,

where $p \restriction \alpha$ *is the restriction of* p *to* $\mathrm{Add}(\kappa_i, \alpha)$.

In $V^{\mathbb{P}}$, the desired forcing to prove Theorem 5.1 is the product of the product $\prod_{i<\omega} \mathbb{M}(\kappa_i, \kappa_{i+1})$ with full support and the usual product of the Cohen forcings at relevant cardinals to realize the continuum function prescribed by f (let us denote this forcing as $\mathbb{R}$). Thus the whole forcing can be written as

$$\mathbb{P} * (\prod_{i<\omega} \dot{\mathbb{M}}(\kappa_i, \kappa_{i+1}) \times \dot{\mathbb{R}}). \tag{5.5}$$

The fact that $\mathbb{M}(\kappa_i, \kappa_{i+1})$ is a projection of a product of a κ_i^+-closed forcing and $\mathrm{Add}(\kappa_i, \kappa_{i+1})$ (see [1]) makes it possible to argue that the product $\mathbb{R}$ does not destroy the tree property ensured by $\prod_{i<\omega} \mathbb{M}(\kappa_i, \kappa_{i+1})$.

In order to extend the result of Theorem 5.1 to an interval of regular cardinals extending over $\aleph_\omega$, one needs to deal with the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.

where $\aleph_\omega$ is strong limit in the generic extension. The method of the proof in fact does not require that $2^{\aleph_n}$ is below $\aleph_\omega$ for $n < \omega$.

[34] The continuum function which is the direct outcome of the usual method to get the tree property at the prescribed cardinals.

We will focus on $\aleph_{\omega+2}$; by (5.4), $\mathsf{TP}(\aleph_{\omega+2})$ with $\aleph_\omega$ strong limit implies the failure of SCH at $\aleph_\omega$ which means that the product-style strategy of the proof of Theorem 5.1 will not work, i.e. will not ensure $2^{\aleph_\omega} > \aleph_{\omega+1}$ (with $\aleph_\omega$ strong limit). In fact, it seems quite hard to control the continuum function below $\aleph_\omega$ by some $f : \omega \to \omega$ as in Theorem 5.1, while ensuring $2^{\aleph_\omega}$ to be $\aleph_{\omega+m}$ for some prescribed value $1 < m < \omega$ (see [47]), even without ensuring the tree property.

As a first step in this direction, we showed in [36] the following:

Theorem 5.3 *Assume* GCH*,* $\kappa < \lambda$ *are regular cardinals,* λ *is the least weakly compact above* κ*, and* κ *is* $H(\lambda)$*-strong with a witnessing embedding* $j : V \to M$ *with* λ *being the least weakly compact in* M *above* κ*. Then there is a forcing notion which forces the tree property at every* $\aleph_{2n}$*,* $0 < n < \omega$*, with* $2^{\aleph_{2n}} = \aleph_{2n+2}$*, and with the failure of* SCH *at* $\aleph_\omega$*.*

An important step in the proof is the construction of a guiding generic for the Prikry forcing with collapses which collapses κ to become $\aleph_\omega$ (see Gitik [42] for more details about this forcing).[35]

Instead of the usual Levy collapse, we used the iteration of the Sacks forcing for the collapsing part: more concretely, we were able to construct a guiding generic for the forcing $\mathrm{Sacks}^I(\kappa^{++}, j(\kappa))^{M^*}$ (the iteration of the Sacks forcing at κ^{++} of length $j(\kappa)$ with $\leq \kappa$ support) in the sense of a certain measure ultrapower M^*. A similar construction for the Levy collapse starts at the triple successor of κ in M^*, i.e. the guiding generic is constructed for $\mathrm{Coll}(\kappa^{+3}, < j(\kappa))^{M^*}$ (the reason is that under the typical circumstances, $\kappa^{++} = (\kappa^{++})^{M^*}$ while $(\kappa^{+3})^{M^*} < \kappa^{+3}$ which makes the latter construction much easier). The possibility to start collapsing at the second successor ensures the tree property at every second cardinal below $\aleph_\omega$. We used the fusion property of the Sacks forcing at κ^{++} to construct inductively the desired guiding generic.

Remark 5.4 We conjecture that the tree property holds at $\aleph_{\omega+2}$ in the model constructed in [36], but we were not able to verify it. The problem seems to be the lack of closure of $\mathrm{Sacks}^I(\kappa^{++}, j(\kappa))^{M^*}$ (the fusion did

[35] A guiding generic ensures that the choices for the conditions which collapse the cardinals between the points on the Prikry sequence are pairwise compatible (thus ensuring κ^+-cc of the whole forcing).

not seem enough to compensate for the lack of $(\kappa^{+++})^{M^*}$-closure to run the argument of Friedman and Halilović [31] which uses $\mathrm{Coll}(\kappa^{+3}, < j(\kappa))^{M^*})$.

5.2.2 The weak tree property

Unlike the tree property, it is known that the weak tree property at successive cardinals does not incur an increase in the consistency strength. In [53] we generalised a result from Unger [78][36] and showed:

Theorem 5.5 *Start with the* GCH. *Assume there are infinitely many Mahlo cardinals and let $f : \omega \to \omega$ be a non-decreasing function such that $f(n) \geq n+2$, $n < \omega$. Then there is a generic extension $V[G]$ where the weak tree property holds at each $\aleph_n$, $1 < n < \omega$, and f determines the continuum function in $V[G]$ below $\aleph_\omega$:*

$$2^{\aleph_n} = \aleph_{f(n)}.$$ [37]

The proof uses a variant of the Mitchell forcing with an additional parameter which was defined in [78]: for $\kappa < \mu < \lambda$, where κ and μ are regular and λ inaccessible, let us write $\mathbb{M}(\kappa, \mu, \lambda)$ for the variant of the Mitchell forcing which collapses the cardinals in the interval (μ, λ) and forces the tree property at λ (which becomes the successor of μ).[38] For convenience, let $\kappa_0 = \omega$ and $\kappa_1 = \aleph_1$, and let $\kappa_2 < \kappa_3 < \cdots$ be an increasing sequence of Mahlo cardinals. Then the forcing to prove Theorem 5.5 is the product

$$\prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}, \tag{5.6}$$

where $\mathbb{R}$ is the product of the Cohen forcings to realise the given function f. The proof proceeds by showing that the product of the two forcings in (5.6) has nice properties in the sense that $\mathbb{R}$ does not undo the effect of the forcing $\prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ (which by design forces the weak tree property at every $\aleph_n$, $1 < n < \omega$).

[36] The relevant theorem in [78] corresponds to our theorem with the function $f(n) = n+2$ for all $n < \omega$; the result of the product of the respective Mitchell forcings.

[37] We state the theorem with $f : \omega \to \omega$ because we are naturally interested in the case where $\aleph_\omega$ is strong limit in the generic extension. The method of the proof in fact does not require that $2^{\aleph_n}$ is below $\aleph_\omega$ for $n < \omega$.

[38] The definition of $\mathbb{M}(\kappa, \lambda)$ in Definition 5.2 is changed for $\mathbb{M}(\kappa, \mu, \lambda)$ to demand that q has a domain of size $< \mu$ and picks $\mathrm{Add}(\kappa, \alpha)$-names, $\alpha < \lambda$, for conditions in $\mathrm{Add}(\mu, 1)$. Setting $\mu = \kappa^+$ gives the standard forcing $\mathbb{M}(\kappa, \lambda)$.

5.3 The tree property at the double successor of a singular cardinal

In [15], Cummings and Foreman started with a Laver-indestructible supercompact cardinal κ and a weakly compact λ above κ and constructed a generic extension where κ is a strong limit singular cardinal of countable cofinality, $2^\kappa = \lambda = \kappa^{++}$, and the tree property holds at κ^{++}. The construction in [15] features another variant of the Mitchell forcing $\mathbb{M}(\kappa, \lambda)$; we might call it a "Prikry-ised Mitchell forcing", $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$.[39] Recall the definition of the Prikry forcing $\mathrm{Prk}_U(\kappa)$ in Definition 4.1. A condition in $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$ is a pair $((p, r), q)$ such that (p, r) is in $\mathrm{Add}(\kappa, \lambda) * \mathrm{Prk}_{\dot{U}}(\kappa)$, where $\dot{U}$ is forced to be a normal measure in $V^{\mathrm{Add}(\kappa,\lambda)}$. The condition q in $((p, r), q)$ is a function with domain of size at most κ included in a certain unbounded subset of λ such that for every $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha) * \mathrm{Prk}_{\dot{U}_\alpha}(\kappa)$-name for a condition in $\mathrm{Add}(\kappa^+, 1)$ as defined in $V^{\mathrm{Add}(\kappa,\alpha)*\mathrm{Prk}_{\dot{U}_\alpha}(\kappa)}$, where $\dot{U}_\alpha$ is forced to be the restriction of $\dot{U}$ to $V^{\mathrm{Add}(\kappa,\alpha)}$. $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$ simultaneously singularizes κ to have cofinality ω (while preserving all cardinals below and including κ) and forces the tree property at λ.

In [37], we have generalised the construction in [15] to get the following theorem:

Theorem 5.6 *Assume κ is a Laver-indestructible supercompact cardinal and $\lambda > \kappa$ is weakly compact. Assume $\mu > \lambda$ has cofinality larger than κ. Then there is a generic extension V^* where κ is a singular strong limit cardinal of the countable cofinality, $\lambda = \kappa^{++}$, the tree property holds at λ, and $2^\kappa = \mu$.*

In order to prove this theorem, we have defined yet another variant of the Mitchell forcing, which we denote for the purposes of this text by $\mathbb{M}(\kappa, \lambda, \mu)^{\mathrm{Prk}}$. $\mathbb{M}(\kappa, \lambda, \mu)^{\mathrm{Prk}}$ is again a "Prikry-ised Mitchell forcing"; it adds μ many Cohen subsets of κ, while collapsing cardinals in the open interval (κ, λ) and forcing the tree property at $\kappa^{++} = \lambda$.[40]

[39] The notation $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$ does not appear in [15]; we use it here to be consistent with the notation in this thesis.

[40] We will not give a detailed definition here. The forcing is a variant of the forcing $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$ introduced above. The conditions are again pairs $((p, r), q)$ but the Cohen part p now ranges over the conditions in $\mathrm{Add}(\kappa, \mu)$, while q has its domain included in λ; thus only the cardinals in the open interval (κ, λ) are collapsed. In order to make this work, one needs to define appropriate "truncations" of $\dot{U}$ – a normal measure in $V^{\mathrm{Add}(\kappa,\mu)}$ – to stages $\alpha < \lambda$.

Remark 5.7 Our recent results in [38] show that it is possible to avoid having the Prikry part integrated in the Mitchell forcing as in $\mathbb{M}(\kappa, \lambda)^{\mathrm{Prk}}$ or $\mathbb{M}(\kappa, \lambda, \mu)^{\mathrm{Prk}}$; instead, it is possible – and perhaps easier – to place the Prikry forcing after the Mitchell forcing. Using this method we constructed a model where κ is equal to $\aleph_\omega$, the tree property holds at $\aleph_{\omega+2}$, and $2^{\aleph_\omega}$ can be equal to $\aleph_{\omega+n}$ for any desired $2 \le n < \omega$. Furthermore, the construction only requires a strong cardinal of an appropriate degree.

6 Further research and open questions

In this section we discuss possible directions of further research and mention some open questions.

We may regard Section 3 as a general framework which in principle can be applied to any large cardinal we wish to study. In this work, we dealt with inaccessible, Mahlo, weakly compact, measurable, $H(\lambda)$-strong, strong, and supercompact cardinals (with a certain tallness). A related study was carried out for Ramsey, Woodin, Shelah, and rank-to-rank embeddings cardinals, see for instance [10, 6, 7, 9, 49, 18]. In all these cases, showing Easton-like results for certain cardinals allows one to study in detail their properties (for instance regarding their preservation by certain forcing notions) and gauge the strength of their reflection in terms of the effect on the continuum function. Cardinals which still wait for a detailed study along these lines are for instance strongly compact cardinals, Magidor cardinals, subcompact cardinals, Erdős cardinals, and Vopěnka-principle cardinals, and other.[41]

Regarding Section 4, more specifically Theorem 4.3, which deals with the Easton function extended to some singular cardinals (which were large in some inner model), one can naturally ask whether and to what extent it is possible to formulate similar results for strong limit cardinals with uncountable cofinalities. There are methods available which can, at least in principle, be applied to this problem (Magidor forcing, Radin forcing, etc.), but a comprehensive treatment of this topic is missing.[42]

[41] There are some partial results (see for instance [5] or [4]), but a comprehensive treatment seems to be missing.

[42] As discussed in Footnote 28, realising an Easton function on a large cardinal κ which is then changed into a singular cardinal with an uncountable cofinality may not be an optimal approach due to the difference in the reflection between large cardinals and strong limit singular cardinals with uncountable cofinalities (but there is at least some connection

Before we move to discuss the tree property (and similar large-cardinal properties), we should mention that – even if it is not the focus of this thesis – the natural goal of the research discussed in the two previous paragraphs is the realisation of an Easton function on *all cardinals*. It is known that once we wish to control the continuum function on singular cardinals, large cardinals are involved in the background,[43] so the study of Easton functions on large cardinals may prove useful in this respect. For more information, you may consult [28, 12, 70].

Regarding the tree property, note first that if there is a model V^* in which the tree property holds at every regular cardinal greater than $\aleph_1$, then GCH fails everywhere, i.e. at regular and singular cardinals, in V^*.[44] Thus the global tree property may be regarded as a very concrete alternative to GCH. It is still unknown whether such V^* exists (from any large cardinal assumptions), still less whether we could realise in V^* more complicated Easton functions.

Yet, it is an ultimate goal for which the methods discussed in Section 5 might be useful. In particular, it is natural to try to extend Theorem 5.1 to realise an Easton function on every $\aleph_n$, $1 < n < \omega$ (starting with ω-many supercompact cardinals). To obtain in addition $\mathsf{TP}(\aleph_{\omega+2})$ with $\aleph_\omega$ strong limit, one should also ensure the failure of SCH at $\aleph_\omega$. This requires a different approach which has a more "iteration-like" flavour (as in Theorem 5.3); the technique of Unger [79] might prove useful in this respect. More concretely, one can ask whether Theorem 5.1, formulated with all the $\aleph_n$'s, $1 < n < \omega$, can be extended to include a prescribed value at $2^{\aleph_\omega}$, ideally also with $\mathsf{TP}(\aleph_{\omega+2})$ (see the discussion in Section 5.2.1). Another step in this direction is the tackling of the uncountable cofinality; in particular, is it consistent to have $\mathsf{TP}(\aleph_{\omega_1+2})$, $\aleph_{\omega_1}$ strong limit?

With the goal of constructing a model V^* with the tree property at all regular cardinals larger than $\aleph_1$, one also needs to deal with the tree property at the successors of singular cardinals such as $\aleph_{\omega+1}$. This problem has not been the focus of this text, but it is natural to consider successors of singulars as well. For the moment, it is still an open problem whether one can have a strong

unlike the case of singular cardinals with the countable cofinality). Therefore, one might also consider using an extender-based Prikry forcing which cofinalises a large cardinal directly (see [45] for more information).

[43] This follows from the study of core models, more specifically of covering properties of such models, see for instance [72, 20, 43].

[44] Note that if κ is inaccessible, then $\kappa^{<\kappa} = \kappa$, and therefore there is a special κ^+-Aronszajn tree. Thus there may be no inaccessible cardinals in V^*.

limit $\aleph_\omega$ with the tree property holding at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.[45,46]

The study of the tree property is a part of a broader program which investigates compactness and reflection principles of large cardinals, with the aim of identifying those principles which can also hold also on small cardinals.

Among these, the super tree property and the strong tree property are natural strengthenings of the tree property (see [55] for definitions). While these notions have been studied already in 1970s, they were first systematically analysed and applied to small cardinal in Weiss [81] where it is shown that the Mitchell forcing iterated up to a supercompact cardinal yields the super tree property: thus the super tree property can consistently hold at $\aleph_2$. Further results regarding these notions can be found for instance in Fontanella [25]. It is interesting to study what restrictions these stronger notions put on the continuum function in comparison with the tree property.

The reflection property and the approachability property are other examples of compactness principles. Let μ be a successor cardinal. We say that μ^+ has the *reflection property* if every stationary subset of $\mu^+ \cap \mathrm{cof}(< \mu)$ reflects at a point of cofinality μ. We say that μ^+ has the *approachability property* if μ^+ is an element of the approachability ideal $I[\mu^+]$.[47] Cummings, Friedman, Magidor, Rinot and Sinapova showed in [16] that the tree property, the reflection property and the approachability property at μ^+ for $\mu = \aleph_{\omega+1}$ are independent in the sense that all eight Boolean combinations of truth/falsity of these principles is consistent (they use a Laver-indestructible supercompact κ for this result). It is natural to ask what are the connections between these principles if we consider more cardinals at the same time, or ask for other values of the continuum function (in [16], $2^{\aleph_\omega}$ equals $\aleph_{\omega+2}$ if SCH must fail at $\aleph_\omega$).

They are still other possible features of cardinals (such as cardinal invariants, see Cummings and Shelah [17]) which can give rise to Easton-like theorems, providing new and interesting generalisations of the original Easton's result.

[45] By [73], it is known that the tree property can hold in the interval $[\aleph_2, \aleph_{\omega+1}]$; however in the model constructed in [73], the SCH holds at $\aleph_\omega$, preventing an easy generalisation to get the tree property also at $\aleph_{\omega+2}$. Perhaps surprisingly, an analogous question for a strong limit $\aleph_{\omega^2}$ has a positive answer: there is a model where both $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ have the tree property (see [80]).

[46] There are results which are formulated with $\aleph_\omega$ not being strong limit and which do achieve $\mathsf{TP}(\aleph_{\omega+1})$ and $\mathsf{TP}(\aleph_{\omega+2})$ (see [26]). However, having $\aleph_\omega$ strong limit seems to be a more important test case.

[47] See Cummings [13] for the definition of the approachability ideal.

Note: In the Czech language, the name of the author is spelled “Radek Honzík”; however, to avoid typesetting inconsistencies in the English-speaking journals, the name is always stated as “Radek Honzik” in the papers.

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7 Appendix: the original papers

The appendix contains all the papers by the author of this thesis referenced in the unifying commentary. The published papers are included in their final form as they appeared in the respective journals. The submitted papers appear as drafts.

The order of the papers in the appendix is chronological with respect to the date of submission (which may not be the same as the date of publication):

1. Sy-David Friedman and Radek Honzik. Easton's theorem and large cardinals. *Annals of Pure and Applied Logic*, 154(3):191-208, 2008. [32]. doi:10.1016/j.apal.2008.02.001

2. Radek Honzik. Global singularisation and the failure of SCH. *Annals of Pure and Applied Logic*, 161(7):895-915, 2010. [52]. doi:10.1016/j.apal.2009.11.002

3. Radek Honzik. A characterization of lifting generics for Sacks-like forcings. *Acta Universitatis Carolinae, Mathematica et Physica*, 51:49-55, 2010. [51]

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