Habilitation Thesis

The continuum function and large cardinal properties

by

Radek Honzik

Opponent's Report written by Arthur W. Apter

The work Honzik presents in his habilitation thesis concerns different behaviors for the continuum function in various models of ZFC. In particular, he and his coauthors have investigated how the continuum function can interact with certain large cardinals, some of its possibilities with certain singular cardinals, and its interactions with different instances of the tree property. Some of the more notable theorems discussed are as follows (where all initial assumptions also include GCH):

1. In Theorem 3.13, Honzik and Friedman show that it is possible to realize suitably definable Easton functions F while preserving all ground model strong cardinals.

2. In Theorem 3.16, Honzik and Friedman use the optimal hypotheses to realize certain very restrictive Easton functions while preserving the measurability of κ.

3. In Theorem 3.18, Honzik and Friedman investigate a GCH pattern that is consistent with κ being both λ-supercompact and λ^{++}-tall, where $\lambda > \kappa$ is a regular cardinal.

4. In Theorem 4.3, Honzik shows that what he calls a "mild Easton function F" can be realized in a model in which $2^\kappa = F(\kappa)$ and $\mathrm{cf}(\kappa) = \omega$. The proof requires in its initial assumptions that κ be $H(F(\kappa))$-strong.

5. In Theorems 4.4 and 4.5, Honzik and Friedman produce models for failures of SCH at singular cardinals λ of cofinality ω, together with lightface definable well-orderings of $H(\lambda^+)$. The proof requires in its initial assumptions the existence of a cardinal κ which is $H(\kappa^{++})$-strong.

6. In Theorems 5.1, 5.3, 5.5, and 5.6, Honzik and his collaborators (Friedman and Stejskalová) investigate various GCH patterns that are consistent with both the tree property and weak tree property.

The ideas Honzik and his collaborators use in the above are sophisticated and ingenious. They require an intricate knowledge of many different topics, e.g., the complexities of interleaving Cohen and Sacks forcing in Easton support iterations, extender based Prikry forcing, extender (i.e., $L[E]$) models for strong cardinals, perfect-tree coding using Sacks forcing, interleaved Prikry forcing with

collapses, diverse notions for forcing the tree and weak tree properties and their interactions with posets realizing different types of Easton functions, etc.

Although the exposition is in general well-written and lucid, there are a few issues I feel need to be addressed (some relatively trivial, but others more serious). These are as follows:

1. On page 5, line 6 of the second paragraph, "Brent" should be "Cody".

2. On page 11, line 2 of Lemma 2.14 contains a serious typo. "κ^+-cc" should be "κ-cc". (The correct statement of Easton's Lemma is found in Fact 5.10 of Cummings' article in the **Handbook of Set Theory**, paper [14] of the bibliography.)

3. On page 24, in Definition 3.14, the definition of κ being κ^{++}-tall is presented. It is then used in Section 3.4 in Theorem 3.18. It is my opinion that a reference to Hamkins' paper "Tall cardinals", *Mathematical Logic Quarterly 55(1)*, 2009, 68–86, in which Hamkins makes a study of tall cardinals, should have been included in Honzik's bibliography and cited by him at this juncture.

4. On page 24, in Definition 3.15, the definition of κ being κ^{++}-correct is given. It is my opinion that a reference to paper [34] of the bibliography should have been included.

5. The Easton functions used in (the generalized version of) Theorem 3.16 on pages 24 and 25 are very restrictive. They require that $2^\delta = \delta^{+n}$ for some fixed $2 \leq n < \omega$. It would have been helpful if Honzik had stated explicitly and clearly how restrictive these Easton functions actually are.

6. On page 28, the definition of the ordering for Prikry forcing given in Definition 4.1 is one of many equivalent definitions. It is my opinion that Honzik should have pointed this out.

7. On page 28, Honzik writes in the second sentence of the last complete paragraph on the page "The first method of obtaining κ where SCH fails was found by Woodin who used Prikry forcing to singularize a measurable cardinal κ which violates GCH ...." This is historically inaccurate. In the third complete paragraph on page 2 of Magidor's seminal paper "On the singular cardinals problem I", *Israel Journal of Mathematics 28(1-2)*, 1977, 1–31, Magidor describes the same method to which Honzik alludes. Although I have no direct evidence of this, it seems quite likely the construction was also known to people such as Jech, Kunen, Menas, Silver, Solovay, etc. in the early to mid 1970s. Honzik needs to revise this sentence to take what Magidor wrote into account (although Honzik's description uses a much smaller large cardinal than the one that would have been used prior to Woodin's work in the 1980s).

8. On page 29, last line of the paragraph immediately following the statement of Theorem 4.2, the reference "Magidor [46]" should be "Gitik and Magidor [46]".

9. On page 30, in both (i) and (ii) of the statement of Theorem 4.3, "There an" should be "There is an".

10. On page 36, in the second paragraph following the statement of Theorem 5.3, I don't understand why the iteration of Sacks forcing described collapses cardinals. It is my opinion that Honzik should have provided some intuition as to why this is the case.

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11. On page 37, in the outline of the proof of Theorem 5.5, it seems as though V first doesn't have to be prepared with an analogue of the iteration of Cohen forcing $\mathbb{P}$ used in the proof of Theorem 5.1. It is my opinion that this should be explicitly noted.

12. On page 38, on lines 1 and 2 of Section 5.3, it is my opinion that a reference to Laver's paper "Making the supercompactness of κ indestructible under κ-directed closed forcing", *Israel Journal of Mathematics 29(4)*, 1978, 385–388, should have been included.

13. On page 40, line 7, it is my opinion that "you may" should be changed to "the reader may".

14. On page 41, line 4, "also hold also" should be either "also hold" or "hold also".

15. On page 45, reference [62] is incorrectly written. The correct citation is *Journal of Symbolic Logic 36*, 1971, 407–413.

16. On page 45, the journal of publication for reference [68] is the *Archive for Mathematical Logic*, not the *Annals of Mathematical Logic*.

Even though I found additional typos and corrections which need to be made, in the interest of brevity, I have restricted myself to those listed above. The ones I feel are the most urgent are (2), (5), (7), (10), and (11).

In summary, I feel that Radek Honzik is a fine, talented young mathematician whose research is technically difficult and extremely interesting. Both by himself and with his coauthors, he has made significant contributions in the general area of large cardinals and forcing which are leading the way to new and important work in set theory (especially in the general area of the preservation of large cardinals while realizing suitably defined Easton functions, a topic in which I am extremely interested and have been pursuing in my own research). As such, I strongly recommend that he be awarded his habilitation degree.

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