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Modules over string algebras

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In Prague 7. 5. 2019 Jakub Löwit

I would like to dedicate this work to all the people I have shared something with. A special thanks go to my supervisor Jan Šťovíček, for all his free time he has been ready to spend on speaking about mathematics with me. Another special thanks go to Denisa Zrubecká for her support and understanding.

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Abstract: The aim of this thesis is to investigate the categories of modules over the so called string algebras. In particular, we try to understand the cotorsion pairs in these categories, which boils down to understanding the decompositions of extensions of such modules. For string algebras with some oriented tree for the underlying quiver, we describe some classes given by these cotorsion pairs in terms of purely combinatorial closure properties. For any string algebras, the combinatorics appears to be similar, althought more complicated.

Keywords: module categories, extension, decomposition, string algebra, cotorsion pair, gentle algebra, string module

Contents

Introduction

The string algebras form for several reasons an interesting class of K-algebras. From the point of view of representation theory, they form important examples of non-commutative finite-dimensional algebras. Because of their elementary nature, the string algebras can be examined by combinatorial and topological means, as well as by methods of homological algebra. On the other hand, this doesn't mean that the description of their modules is easy – despite of the progress that has been made over the years, their module categories are not yet fully understood. The string algebras are complicated enough to display many important concepts, and it appears that some questions about them have surprisingly topological flavour. From the point of view of an ordinary mathematician, string algebras can be found at many other places – they are for instance of some importance in physics.

In this thesis, we try to understand some aspects of the category of finite-dimensional modules mod$R$ over a string algebra R. We shall try to describe the classes in mod$R$ given by the cotorsion pairs in Mod$R$. We will be able to do this at least for some particular string algebras. Althought this description leads to other questions, we will stop there.

In Chapter 1, we summarize the necessary terminology and state some important results concerning indecomposable modules over string algebras and cotorsion pairs in general. This will allow us to reduce the desired description to the understanding of decompositions of extensions into indecompodable modules.

In Chapter 2, we focus on *tree algbras*, which are especially well-behaved string algebras, whose underlying quiver is only an oriented tree. First we give some space to the study of extensions of pairs of indecomposable modules and to their decompositions. Then we turn to establishing combinatorial closure properties, which determine the intersection of left classes of cotorsion pairs in Mod$R$ with mod$R$.

We use Chapter 3 to outline the relevance of results of Chapter 2 concerning their possible generalization to the case of any string algebras.

1. String algebras and their modules

The aim of this section is to fix the necessary notation, introduce the string algebras and state some long-known results about them.

1.1 Preliminaries and notation

Throughout the thesis, we work with left modules – the multiplication by the elements of the ring is composed from right to left, in the same way as we compose functions. Let K be a field. For a K-algebra R, we denote by $\text{mod}R$ the category of finite dimensional left modules over R and by $\text{Mod}R$ the category of all left modules over R. By $\text{ind}R$ we denote the set of representatives of isoclasses of indecomposable finite dimensional modules over R. In the same sense we usually don't distinguish between isomorphic objects.

First we recapitulate some basic terminology of representation theory when finite dimensional algebras are concerned. For a detailed treatment of these topics see for instance Assem et al. [2006], Schiffler [2014] or [Auslander et al., 1997].

A quiver $Q = (Q^0, Q^1)$ is an oriented multigraph, i.e Q^0 is a set of vertices and Q^1 is a set of arrows, each arrow $\alpha \in Q^1$ having a unique source and unique target in Q^0. We use the superscripts instead of subscripts because of the necessity to work with more quivers at once. To say that α is an arrow from u to v we write $\alpha : u \to v$. Our quivers can have multiple edges or loops. We shall be mainly interrested in finite quivers.

A *path* in a quiver is a sequence of oriented consecutive arrows joining two vertices of Q. In particular, we regard every single vertex as a path of lenght zero. In this context we write Q^k for the set of paths of length k. Given a quiver Q and a field K, we have its *path algebra* KQ, whose elements are formal linear combinations of oriented paths in Q over K with pointwise addition and multiplication extending the concatenation of paths. The *arrow ideal* I_a of KQ is the ideal generated by all the arrows in KQ. It is clearly a two sided ideal which consists of all the linear combinations of paths of length at least one.

From now on, let Q be a finite quiver. A two sided ideal $I \leq KQ$ is called *admissible*, if $I_a^m \subseteq I \subseteq I_a^2$ for some natural number m. The algebra KQ/I is then called a *bound quiver algebra*. Since Q is finite and $I_a^m \subseteq I$, this algebra is finite dimensional. Bound quiver algebras form an important class of algebras – when $K = \overline{K}$, every finite dimensional K-algebra is actually Morita equivalent to some KQ/I for a suitable finite quiver Q and admissible ideal I (by [Assem et al., 2006, Corollary 6.10 and Theorem 3.7]). If $R = KQ/I$ is a bound quiver algebra and $M \in \text{mod } R$, we write $M_v = v \cdot M$. By the correspondence between modules and linear representations, we then have $M = \bigoplus_{v \in Q^0} M_v$ as a vector space. We call these M_v the *vertex subspaces*.

We shall be interrested in some special classes of bound quiver algebras. At this point we introduce the necessary terminology following Schröer [2016].

Definition. *A bound quiver algebra KQ/I is called monomial, if I is generated by paths.*

Definition. *A special biserial algebra is a bound quiver algebra KQ/I with I satisfying the conditions*

- *for every $v \in Q^0$ there are at most two arrows begining at v and at most two arrows ending at v,*

- *for every arrow $\alpha \in Q^1$ there is at most one arrow $\beta \in Q^1$ with $\alpha\beta \notin I$ and at most one arrow $\gamma \in Q^1$ with $\gamma\alpha \notin I$.*

Definition. *A string algebra is a monomial algebra, which is special biserial.*

The string algebras and their module categories will be the main object of our study. Still, in order to simplify things, we will often work with even smaller classes of algebras.

Definition. *A gentle algebra is a string algebra KQ/I which moreover satisfies*

- *the ideal I is generated by paths of length 2,*

- *for any $\alpha \in Q^1$ there is at most one $\beta \in Q_1$ with $0 \neq \alpha\beta \in I$ and at most one arrow $\gamma \in Q^1$ with $0 \neq \gamma\alpha \in I$, the equlities considered in KQ.*

1.2 Indecomposable modules

Let K be a field and $R = KQ/I$ a string algebra. When working in mod$R$, the Krull-Schmidt theorem ([Assem et al., 2006, Theorem 4.10]) applies, so every $M \in$ mod$R$ has a unique decomposition into indecomposable finite dimensional modules.

The classification of ind$R$ is already known for some time; the first steps were supposedly made by Gelfand and Ponomarev [1968] and have been later generalized by Butler and Ringel [1987] into a complete classification. We now formulate this result. What follows is summarized in almost every article treating string algebras, for instance in Laking [2016].

1.2.1 Strings and string modules

Take a string algebra $R = KQ/I$. For every arrow $\alpha : u \to v$ of Q^1 we introduce a formal inverse $\alpha^{-1} : v \to u$. These inverses form a set Q^{-1}. For every word w over $Q^1 \cup Q^{-1}$, we can form its formal inverse word. A *string* s over R is a word $s = \alpha_n^{\epsilon_n} \dots \alpha_2^{\epsilon_2}\alpha_1^{\epsilon_1}$ over $Q^1 \cup Q^{-1}$, such that

- s is a walk in the undirected verison of Q,
  so $\forall i = 1, 2, \dots n-1$, the target of $\alpha_i^{\epsilon_i}$ equals to the source of $\alpha_{i+1}^{\epsilon_{i+1}}$,

- s is a reduced walk,
  so there are no neighbouring arrows which are mutually inverse,

- s doesn't break any relation,
  ie. no relation from I is a subword of s or its inverse.

For every vertex $v \in Q^0$ we define its trivial string, coresponding to the trivial walk beginning and ending in v.

If t and s are mutually inverse strings, we regard them as the same. We shall denote the set of strings over R up to this equivalence by $\mathrm{St}(R)$.

Luckily, there are other ways of defining strings – and we shall adopt the following one. It can be convenient to view s as a sequnce of arrows and vertices between them. Thus a string s is in fact a linear quiver together with a map of quivers $\eta : s \to Q$, such that no neighbouring arrows of s with the same source or target map to the same arrow of Q^1 and no linear subquiver of s maps to a relation from I. We then write s^0 for the set of vertices of the string s, s^1 for the set of arrows of s, and so on. Every string has precisely two *outer* vertices, the rest of its vertices are *inner*.

For any string $s \in \mathrm{St}(R)$ we define the string module $M(s) \in \mathrm{mod}R$ as follows. For every $u \in s^0$ we introduce an element e_u and we let $(e_u)_{u \in s^0}$ be a formal basis of a vector space $M(s)$ over K. Now it suffices to define the action of arrows of Q^1, which in turn uniquely determines the action of all elements of KQ on $M(s)$. So for any $\alpha \in Q^1$ and any e_u, we define $\alpha e_u = e_v$ if there is an (necessarily unique) arrow $\gamma \in s^1$ with $\gamma : u \to v$, $\eta(\gamma) = \alpha$. In all other cases we define $\alpha e_u = 0$.

Less formaly, we just put a copy of the one dimensional vector space K to every vertex of s and define the action by identities on all the arrows of s. By the construction, $M(s)$ is a well defined KQ module. By the defnition of strings, the action respects all relations from I, so $M(s)$ is in fact a well defined KQ/I module. The basis $(e_u)_{u \in s^0}$ will be called the *vertex basis* of $M(s)$, althought it is not unique.

1.2.2 Bands and band modules

Again, bands can be defined a bit painfully as words in letters form $Q^1 \cup Q^{-1}$ up to some equivalence which satisfy a few properties. So, a band b over R is a nonempty word $s = \beta_n^{\epsilon_n} \dots \beta_2^{\epsilon_2} \beta_1^{\epsilon_1}$ over $Q^1 \cup Q^{-1}$, such that

- b is a cyclic walk in the undirected verison of Q,
  ie. $\forall i = 1, 2, \dots n$ modulo n, the target of $\beta_i^{\epsilon_i}$ equals to the source of $\beta_{i+1}^{\epsilon_{i+1}}$,
- b is reduced,
  so there are no neighbouring arrows which are mutually inverse,
- b doesn't break any relation,
  no relation from I can be a subword of any power of a or its inverse.
- b is primitive,
  ie. it cannot be written as a proper power of another word w.

This time we don't allow any degenerate bands. We consider bands only up to formal inverses and cyclic rotations. We shall denote the set of bands over R up to this equivalence by $\mathrm{Ba}(R)$.

As in the case of strings, we can rephrase the definition in a nicer way. A band b is nothing more than a cyclic quiver together with a map of quivers $\eta : b \to Q$, such that no neighbouring arrows of b with the same source or target map to the same arrow of Q^1, no linear subquiver of any power of b maps to a relation from I and the function η doesn't have any smaller period than the lengt of b. We again use the notation b^0 for the set of vertices of the band, b^1 for the set of arrows of b, and so on.

At this point it it fair to say that these "quiver" definitions of strings and bands can be pushed even further into the world of topolology. We shall comment on that in Section 3.1.

For technical reasons, we highlight one arrow in every $b \in \mathrm{Ba}(R)$. Given any $b \in \mathrm{Ba}(R)$, $n \in \mathbb{N}$ and an indecomposable automorphism φ of the vector space $V = K^n$, we construct the band module $M_\varphi(b)$ as follows. For every $u \in b^0$ we take a vector space $V_u \cong K^n$. We now define the action of any arrow α on these vector spaces V_u, which in turn uniquely determines the module structure. So, if there exists a non-highlited arrow $\gamma \in b^0$, $\gamma : u \to v$ with $\eta(\gamma) = \alpha$, the arrow $\alpha : V_u \to V_v$ will act as the n-dimensional identity. Further if there is the highlited arrow $\delta \in b^0$, $\delta : u \to v$ with $\eta(\delta) = \alpha$, the arrow $\alpha : V_u \to V_v$ will act as φ. In all other cases α acts as the zero map.

Again it is trivial that $M_\varphi(b)$ becomes a well defined KQ module, and by the definition of bands it is in fact a KQ/I module. The union of the canonical bases of V_u will be again called the *vertex basis* of $M_\varphi(b)$. We shall call the band module $M_\varphi(b)$ *primitive* if the common dimension of subspaces V_u is 1.

1.2.3 The classification

It is rather straightforward to verify that string modules $M(s)$ for $s \in \mathrm{St}(R)$ and band modules $M_\varphi(b)$ for $b \in \mathrm{Ba}(R)$ with φ an indecomposable linear automorpism up to equivalence are pairwise nonisomorphic indecomposable finite dimensional modules over the string algebra R.

The promised classification simply states that these are all of them.

Fact. *For R a string algebra the set ind$R$ consists precisely of string and band modules.*

We will use this result as often as possible. For more details and proof consult Butler and Ringel [1987].

Althought we have defined a notation $M(s)$ and $M_\varphi(b)$ for string and band modules, we will sometimes omit the isomorphism φ from the notation of band modules. When working with many modules at once, we will also dare to use different letters instead of M, for instance N.

1.3 Closed classes of modules

First fix some more notation. Let R be a finite dimensional K-algebra. Again, we shall consider all modules only up to isomorphism. Take the category mod$R$, with indecomposable modules ind$R$. Denote by $\mathcal{P}$ all its projective modules and by $\mathcal{I}$ all its injective modules.

For any class of modules $\mathcal{A}$ in mod$R$, denote $\mathrm{Add}(\mathcal{A})$ the closure of $\mathcal{A}$ on finite direct sums and direct summands in mod$R$. Also denote by $\mathrm{Ext}(\mathcal{A})$ the closure of $\mathcal{A}$ on extensions of pairs of modules. Althought it is more convential not to use capital letters in this context, we will be working inside mod$R$ the entire time, so there should be no confusion.

Given a ring R, there are many important classes of modules in mod$R$ and Mod$R$. In this thesis we will be interrested in the left classes of cotorsion pairs. All the necessary theory concerning these can be found in Göbel and Trlifaj [2012].

Definition. *A pair $(\mathcal{X}, \mathcal{Y})$ of classes in ModR is called a cotorsion pair if*

$$\mathcal{Y} = \mathcal{X}^{\perp_1} = \{Y \in ModR \mid \forall X \in \mathcal{X} : Ext^1_R(X, Y) = 0\},$$

$$\mathcal{X} = {}^{\perp_1}\mathcal{Y} = \{X \in ModR \mid \forall Y \in \mathcal{Y} : Ext^1_R(X, Y) = 0\}.$$

The cotorsion pairs are precisely the pairs of closed classes under the Galois correspondence given by $\perp_1$. A cotorsion pair is determined by any of its two classes. For any classes of modules $\mathcal{A}$, $\mathcal{B}$, we can form the cotorsion pair $({}^{\perp_1}(\mathcal{A}^{\perp_1}), \mathcal{A}^{\perp_1})$ *generated* by $\mathcal{A}$ and the cotorsion pair $({}^{\perp_1}\mathcal{B}, ({}^{\perp_1}\mathcal{B})^{\perp_1})$ *cogenerated* by $\mathcal{B}$. Especially interesting cotorsion pairs are the ones generated only by a subclass of finite dimensional modules $\mathcal{A} \subseteq$ mod$R$. Our aim will be to characterize the classes in mod$R$ induced by left classes of such cotorsion pairs; ie. to find classes of the form ${}^{\perp_1}(\mathcal{A}^{\perp_1}) \cap$ mod$R$ for $\mathcal{A} \subseteq$ mod$R$.

Fact. *For any $\mathcal{A} \subseteq modR$, the class ${}^{\perp_1}(\mathcal{A}^{\perp_1}) \cap modR$ equals to the closure of $\mathcal{A}$ on projective modules, Ext and Add .*

To check that ${}^{\perp_1}(\mathcal{A}^{\perp_1})$ is indeed closed on these is straightforward – since every epimorphism onto a projective module splits, we get $\mathcal{P} \subseteq \mathcal{X}$. The closeness on Add follows by additivity of Ext^1_R, while the closeness on Ext follows from looking on long exact sequences given by the Ext functor. The other direction is the interesting one, the proof can be done for instance via [Göbel and Trlifaj, 2012, Corollary 6.14 and Theorem 7.17].

Motivated by this fact, we denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A} \subseteq$ mod$R$ on Ext and Add. As a consequence of Krull-Schmidt theorem, every such closed set $\overline{\mathcal{A}}$ is determined by its indecomposable modules.

The knowledge of such classes has further consequences concerning the structure of mod$R$. But for the sake of this thesis, we will be concerned only with the description of these. The previous fact reduces this description only to understanding the relationship of extensions and direct summands over a string algebra R, which is the subject of what follows.

2. Tree algebras

2.1 Extensions over tree algebras

The most innocently looking bound quiver algebras are those without any, even unoriented, cycles. Therefore we shall start our study of string algebras there. It would make sense to call a "tree algebra" any algebra of the form KQ/I for Q a tree. Since we are interested only in string algebras, we will use the following definition.

Definition. *A tree algebra is a string algebra KQ/I for Q an oriented tree.*

There are plainly no band modules over tree algebras. Therefore by Section 1.2.3 the only indecomposable finite dimensional modules over a tree algebra KQ/I are the string modules. Since Q is by definition a finite quiver, there are only finitely many strings, so KQ/I is of finite representation type.

2.1.1 Drawings of tree algebras

For any tree algebra $R = KQ/I$ we can partition the arrows in Q^1 into two sets Q^l and Q^r, which will be called *left* and *right* arrows, in such a way that for any $\alpha \in Q^l$ and $\beta \in Q^r$ it holds that $\alpha \cdot \beta = 0 = \beta \cdot \alpha$. For any component of Q, this partition is uniquely determined by the direction of any single arrow. Having chosen such orientation, we can depict KQ/I in such a way that the arrows from Q^l point from upper left to lower right and the arrows from Q^r point from upper right to lower left.

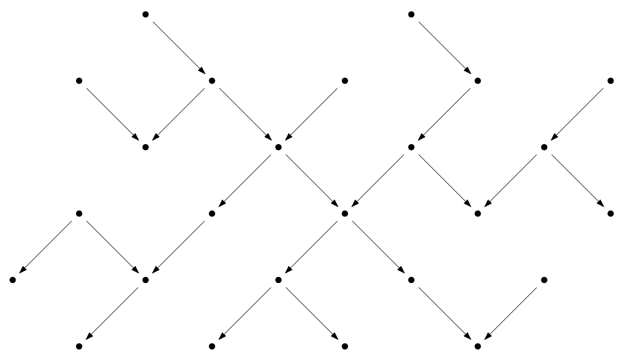

Figure 2.1: A drawing of some tree algebra. Althought this picture fits into a two dimensional grid, it doesn't have to be the case in general, because the regular tree T_2 doesn't fit there.

This shall be further refered to as the *drawing* of KQ/I. We will automatically work with tree algebras with a fixed drawing. Notice that in such a drawing the arrows with nontrivial composition must have the same direction.

For Q a tree, any string is uniquely determined by its two endpoints. By fixing a drawing of Q, we automatically get a left-right orientation of all strings of $\mathrm{St}(R)$. We then denote by $l(s) \in Q^0$ the left end of a string s and by $r(s) \in Q^0$ the right end of a string s.

Notice that when KQ/I is a tree algebra, Q is a subquiver of the infinite regular tree T_2 – the regular tree with all incoming and outcoming degrees equal to 2.

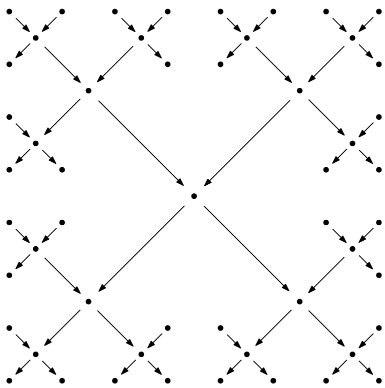

Figure 2.2: Infinite regular tree T_2.

A tree algebra $R = KQ/I$ is not fully determined by its drawing, because the drawing needs not remember the entire ideal I. On the other hand, the drawing remembers something of I – the ideal J generated by all the paths in Q^2 of the form $\sigma\tau$, where the target of τ equals to the source of σ and σ, τ have opposite directions, satisfies $J \subseteq I$. Moreover, KQ/J is already a string algebra.

In fact, such KQ/J is by definition a gentle algebra. On the other hand if KQ/I is a gentle algebra, than there is an appropriate drawing with $I = J$. In this sense the tree algebras which can be fully determined by their drawings are precisely the gentle tree algebras.

In the survey of string algebras it will be sometimes useful to fix some drawing and ignore the relations outside of J, which results into a gentle tree algebra. Fortunately, many results work also for the original algebra, with implicit restrictions given by the rest of I.

2.1.2 Action pictures

It is a basic fact that the modules over KQ/I correspond to K-linear representations of Q which are trivial on I, ie. that the categories $\mathrm{mod}KQ/I$ and $\mathrm{rep}(Q, I)$ are equivalent. We will switch between these two viewpoints without any unnecessary formalism.

Let $R = KQ/I$ be a bound quiver algebra and $M \in \mathrm{mod}R$. Then M can be depicted by the following multigraph with edges labelled by the arrows in Q^1 and numbered by the elements of $K \setminus \{0\}$: Given a basis $E = (e_i \mid i \in I)$ of M over K, take the elements of this basis as vertices. For any $\alpha \in Q^1$ and any e_i we have unique form $\alpha \cdot e_i = \sum_{j \in I} \lambda_{ij} e_j$ for some $\lambda_{ij} \in K$. For all $j \in I$ for which $\lambda_{ij} \neq 0$ we draw an edge from e_i to e_j, then number this edge by λ_{ij} and label it by α.

Doing this for all $\alpha \in Q^1$ and e_i, $i \in I$ results in a decorated multigraph which we shall call the *action picture* of M with respect to the basis E. The edges of this picture will be called the *action arrows* and usually denoted by underlined greek letters. Every action arrow is labelled by the corresponding arrow in Q^1.

On the other hand, having such a multigraph yields an action of the arrows of Q on the vector space spanned by the vertices. If this action respects all the necessary relations, we get a well defined R-module.

The action pictures are handy when working with extensions. Suppose we have an extension X of N by M over a string algebra R. Viewing M and N only as vector spaces over K, it splits. The non obvious part of the extension is determined by the action of the arrows in Q^1 on some basis E^X of X. If we already have a fixed bases E^M of M and E^N of N, we can only take $E^X = E^M \cup E^M$.

The action of Q^1 on E^N remains unchanged, since N is a submodule of X. Because M is a factor of X, the action of Q^1 on M is unchanged apart for some additional nonzero action arrows into N. Put together, the action picture of X with respect to this E^X is obtained by taking the action pictures of M and N and addinng some action arrows from E^M into E^N.

2.1.3 Overlappings of strings

We already know that a string s can be seen as a linear quiver together with a map f quivers $\eta : s \to Q$. But in the case of tree algebras, this η must be always injective, so the strings are just linear subquivers of Q which don't cross any relations. Moreover, every such string is uniquely determined by its two endpoints.

As we said earlier, the string module $M(s)$ has then the vertex basis $(e_v)_{v \in s^0}$. This basis has the property that the corresponding action picture is again just the string s, with all edges numbered by $1 \in K$ and labeled in the obvious way.

Suppose $s, t \in \mathrm{St}(R)$. Then, since Q is a tree, these two strings have a unique maximal common substring $p = s \cap t$. We then call this p the *overlapping* of s and t. If p is empty, we say that s and t have *empty intersection*.

To show the aim of the previous discussion, we give a straightforward lemma which describes the additional action arrows in an extension between string modules. But first we formulate one trivial fact true for all string algebas.

Proposition 1. *Let $R = KQ/I$ be a string algebra, $M \in mod(R)$ and $v \in Q^0$. Take any $m \in M_v$. Let $O_m = \{\alpha \in Q^1 \mid m \notin Ker(\alpha)\} \cup \{\beta \in Q^1 \mid m \in Im(\beta)\}$. Then $|O_m| \leq 2$*

Proof. The arrows of O_m must be incident with the vertex v, so trivially $|O_m| \leq 4$. But if $|O_m| > 2$, there would be two arrows α, $\beta \in O_m$ with $\alpha\beta \neq 0$ and $\alpha\beta \in I$, a contradiction. □

Lemma 2. *Suppose that KQ/I is a tree algebra, s, $t \in St(R)$, $p = s \cap t$. Then the additional action arrows in any extension X of $M(t)$ by $M(s)$ with respect to the union of their vertex bases $E^s = \{e^s_u \mid u \in s^0\}$ and $E^t = \{e^t_v \mid v \in t^0\}$ are among the following.*

(i) If $p \neq \emptyset$, then for every arrow $\alpha \in p^1$, $\alpha : u \to v$, there is one possible additional action arrow $\underline{\alpha} : e^s_u \to e^t_v$.

(ii) If $p \neq \emptyset$ and arrow $\beta \in (s^1 \cup t^1) \setminus p^1$, $\alpha : u \to v$ for $u \in s^0$, $v \in t^0$ is incident with an outer vertex of p, then there is one possible additional action arrow $\underline{\beta} : e^s_u \to e^t_v$.

(iii) If $p = \emptyset$ and there is an arrow $\gamma : u \to v$ such that either $u = r(s)$, $v = l(t)$ or $u = l(s)$, $v = r(t)$, then there is one possible additional action arrow $\underline{\gamma} : e^s_u \to e^t_v$.

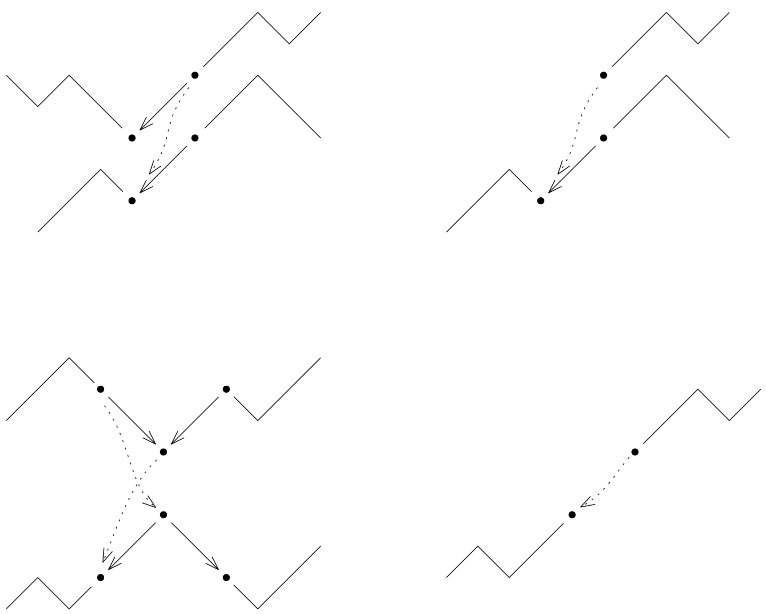

Figure 2.3: The possible additional action arrows, by rows cases (i), (ii), (ii), (iii).

Proof. For the basis of X we have the $E^s = (e^s_u | u \in s^0)$ and $E^t = (e^t_v | v \in t^0)$. Any additional action arrow labeled by $\alpha \in Q^1$, $\alpha : u \to v$ must run from some e^s_u to some e^t_v.

If $p = s \cap t \neq \emptyset$ and $\alpha : u \to v$, $u \in s^0$, $v \in t^0$, then already $\alpha \in s^1 \cup t^1$, because otherwise there would be a cycle in the tree Q. So we get immediately the cases (i) and (ii), depending on whether $\alpha \in p^1$ or $\alpha \notin p^1$.

On the other hand, if $p = \emptyset$, every such $\alpha : u \to v$, $u \in s^0$, $v \in t^0$ lies outside of $s^1 \cup t^1$. All the vertex subspaces X_w, $w \in Q^0$ of the extension X are either empty or one dimensional. Suppose that $\underline{\alpha} : e^s_u \to e^t_v$ was a nontrivial action arrow. If u was an inner vertex of s, then $e^s_u \in M_u$ but $|O_{e^s_u}| \geq 3$ in the notation of Proposition 1, which is a contradiction with this proposition. The same reasoning applies if v was an inner vertx of t, this time for $e^t_v \in M_v$. Hence u is an outer vertex of s and v is an outer vertex of t. Having the left right orientation from the drawing of KQ/I, the arrow α must be either $\alpha : r(s) \to l(t)$ or $\alpha : l(s) \to r(t)$, otherwise the additional action arrow would break a relation from $J \subseteq I$. This yields the case (iii). □

Notice that the case (iii) is disjoint with cases (i), (ii), which can often occur simultaneously. When R is a gentle tree algebra, all the additional arrows from cases (i) and (iii) are always well defined. The additional arrows from case (ii) don't need to be well defined even in the case of gentle tree algebras, but their suitable nontrivial linear combinations are. In the case of non-gentle tree algebras, many of the described additional action arrows can break relations from I, so there are possibly less extensions. But it is the only difference.

2.1.4 Extension decompositions

Our main aim is to characterize extensions of finite dimensional modules over tree algebras in terms of their unique decompositions. Similar characterization for extesions between string modules over gentle algebras was independently obtained by Brüstle et al. [2018]. But we shall use many concepts form this section further.

Definition. *Let R be a K-algebra, N, M be indecomosable modules in mod(R). An extension E of N by M is then called interesting if and only if $E \ncong N \oplus M$.*

To give some context to this defininition, we provide a well-known fact, which asserts that inside mod$R$ the interesting extensions precisely correspond to the non-split ones; nevertheless, the introduced terminology can be convenient.

Fact. *Let R be a K-algebra and M, $N \in$ mod$R$. Then there exist an interesting extension of N by M if and only if $\mathrm{Ext}^1_R(M,N) \neq 0$.*

Proof. By Yoneda definition of Ext^1_R, it is trivial that if $\mathrm{Ext}^1_R(M,N) = 0$, there is no interesting extension of N by M.

For the other direction, we need that M, N are finite dimensonal. We prove that any exact sequence of the form $0 \to N \xrightarrow{f} N \oplus M \xrightarrow{g} M \to 0$, where the morfisms f, g aren't necessary the canonical ones, in fact splits. Using the $\mathrm{Hom}_R(M,-)$ functor, we get an exact sequence $0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M, N \oplus M) \to \mathrm{Hom}_R(M,M)$ of finite dimensional vector spaces over K. The middle term is moreover (naturally) isomorphic to $\mathrm{Hom}_R(M,N) \oplus \mathrm{Hom}_R(M,M)$. Counting the dimensions, the last morphism must be surjective. Hence there is some $h \in \mathrm{Hom}_R(M, N \oplus M)$ such that $\mathrm{id}_M = gh$, so the original sequence splits as claimed. □

Given a tree algebra $R = KQ/I$, we shall define relations $\geq$ and $>$ on the set St(R). The relation $>$ will be in control of the decompositions of extensions: when working with gentle tree algebras, there will be an interesting extension of string modules $M(t)$ by $M(s)$ if and only if $s > t$. For non-gentle algebras, this will be a necessary condition, the exisstenec of interesting extension depending on relations of I.

Definition. *Let s, $t \in$ St(R) and $p = s \cap t$ their intersection. Denote by α the rightmost arrow of s which lies to the left of $l(p)$, if it exists. If $l(s) = l(p)$, define $\alpha = \emptyset$. Similarly define β to be the rightmost arrow of t which lies to the left of $l(p)$, if it exists, and $\beta = \emptyset$ if $l(t) = l(p)$. Then we define $s \geq_l t$ when one of the following holds:*

(i) $p \neq \emptyset$, while $\alpha = \emptyset$ or $l(p)$ is the target of α, and similarly $\beta = \emptyset$ or $l(p)$ is the source of β,

(ii) $p = \emptyset$ and there is either an arrow $\gamma : r(s) \to l(t)$ or an arrow $\delta : l(s) \to r(t)$.

If moreover $l(s) \neq l(t)$, we write $s >_l t$.
We define $\geq_r$ left-right symmetrically, ie. as $\geq_l$ for the opposite orientation of the drawing of Q.

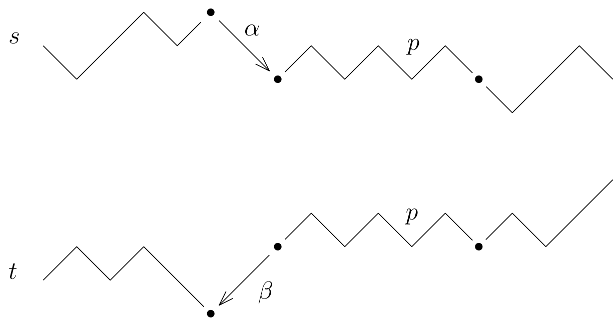

Figure 2.4: Example situation of the definition yielding $s \geq_l t$.

Definition. *Let s, t be strings in St(R). Then we define $s \geq t$ if and only if $s \geq_l t$ and $s \geq_r t$. We write $s > t$ if and only if $s >_l t$ and $s >_r t$.*

Remark. For $>$ to hold we require both $>_l$ and $>_r$ to be strict at the same time. This is strictly aganist the usual convention, but it will clarify what follows.

At this point it is also fair to comment on the properties of these relations. Althought $\geq$ is reflexive and antisymetric, it is usually not transitive. The situation with $>$ is even worse, because of its unconventional definition. Nevertheless it makes sense to think about these relations as if they were orders.

On the other hand, if we look only at the strings from $\mathrm{St}(R)$ containing one fixed vertex v_0, the restricted versions of $\geq_l$ and $\geq_r$ become linear orders. The restriction of $\geq$ is then at least a partial order.

Althought this definiton looks rather technical, it has nice graphical version: $s \geq_l t$ means that there exists some additional action arrow from $M(s)$ to $M(t)$ and the left side of the string t lies beneath the left side of the string s in the drawing of Q. The relation $s > t$ moreover states that the string t is strictly beneath the string s in the drawing of Q on both sides. Here are some examples.

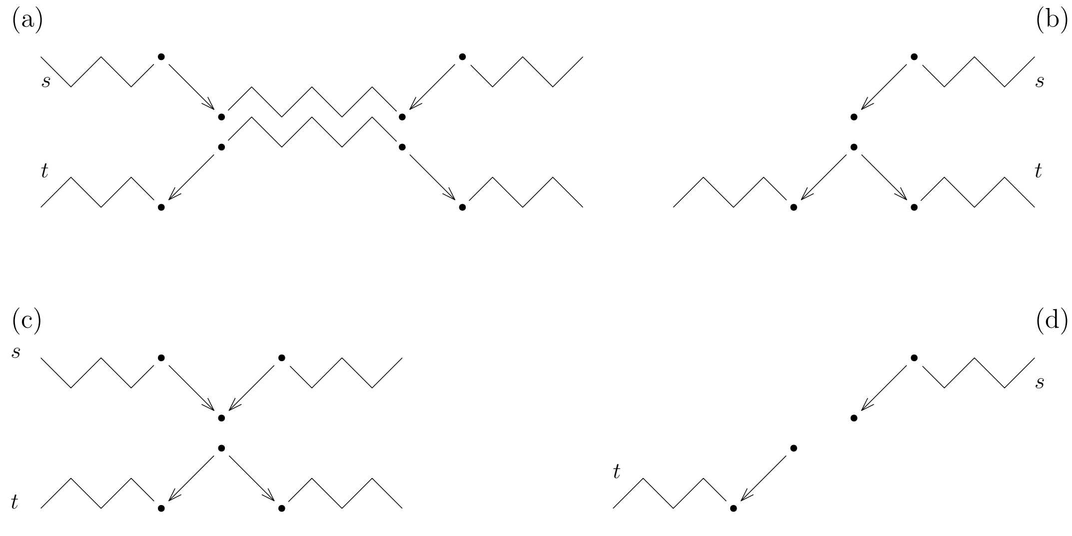

Figure 2.5: Examples of mutual position of pairs of strings. The vertices corresponding to the same vertex of Q are bellow each other. In the examples (a), (b), (c) it always holds $s >_l t$ and $s >_r t$, hence $s > t$. Concerning (d), the relation $s > t$ is by definition equivalent to the existence of an arrow sourcing in the left end of s and targetting in the right end of t.

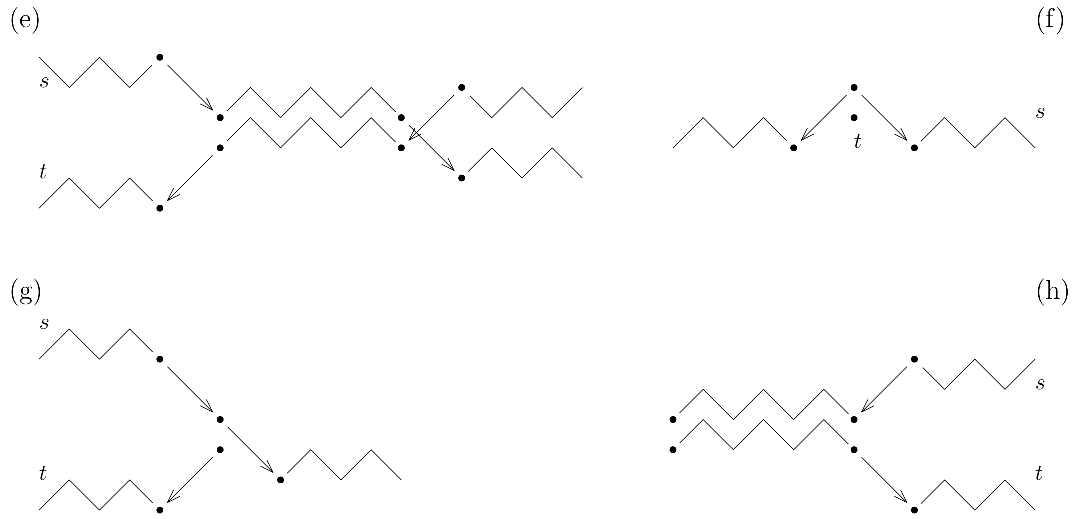

Figure 2.6: More examples. In (e), we have $s >_l t$ but $s <_r t$, so these are incomparable with respect to $\geq$ and $>$. In (f), althought t is somehow degenerate, we have $s < t$. In the example (g) we have by definition $s >_l t$ and $s <_r t$, as in (e). Last of all, (h) satisfies $s \geq_l t$ and $s >_r t$, but on the left sides the ends coincide – so we have $s \geq t$, but the sharp relation $>$ does not hold.

Remark. Our relations $\geq$ and $>$ are connected to the hammock posets introduced by Schröer, which are in control of homomorphisms between the string modules. These hammock posets determine the existence of homomorpisms and their factorizations between string modules or band modules.

Therefore it makes sense that our relations are very similar to the reversed relation of hammock posets – but for instance the pairs of strings $s > t$ with $s \cap t = \emptyset$ don't fit into this context.

In contrast with hammock posets, we do not work with pointed strings. The disadvantage is that $\geq$ and $>$ have worse properties – for instance they are not transitive. On the other hand, we can compare any two strings without considering their mutual position. This wouldn't work for string algebras in general, but in the case of tree algebras it works nicely.

Since we shall compute our exensions directly, we won't use the hammock posets explicitely. Nevertheless these are most relevant. For more about the hammock posets, we refer the reader to the original paper Schröer [1998], for brief introduction see Laking [2016].

Before we finaly get to the heart of the matter, we provide few simple facts and definitions.

Definition (Regluing of strings)**.** *Let $R = KQ/I$ be a string algebra. Suppose we have two strings s, $t \in St(R)$. If there exists a string from $l(s)$ to $r(t)$, it is unique. We denote it by $\overleftrightarrow{st}$. In the same manner, the unique string from $r(s)$ to $l(t)$ will be denoted, if it exists, by $\overleftrightarrow{ts}$.*

Proposition 3. *Let $R = KQ/I$ be a string algebra, $X \in modR$, and let Y be a direct summand of X. Take $y \in X$ and $\alpha \in Q^1$. Then*

(1) $\alpha y \in Y$,

(2) if $\exists x \in X$ with $\alpha x = y$, then $\exists x' \in Y$ with $\alpha x' = y$.

Proof. The first fact holds only because Y is a module. For the second one, suppose $X = Y \oplus Z$. Then for $x' = \pi_Y(x) \in Y$ it indeed holds that $\alpha\pi_Y(x) = \pi_Y(\alpha x) = \pi_Y(y) = y$. □

Definition. *If a subset Y of a module X satisfies these two properties from Proposition 3 for any $y \in Y$ and $\alpha \in Q^1$, we say that Y is closed on images and some preimages.*

Such subsets are important, because every direct summand of X must be closed on images and some preimages. In particular, for any element $x \in X$ we can look for a minimal set $G(x)$ of nonzero elements of X which is closed on nonzero images and some preimages and contains x.

Theorem 4. *Let $M(s)$, $M(t)$ be string modules over a gentle tree algebra KQ/J. Then there exist an interesting extension of $M(t)$ by $M(s)$ if and only if $s > t$.*

Proof. Denote by $p = s \cap t$ the intersection of s and t, which can be empty or an overlapping. We already know by Lemma 2 which additional action arrows can appear in the action picture of X with respect to the union of vertex bases of $M(s)$ and $M(t)$.

We begin by decomposing X into the vertex subspaces. Firstly, there are the vertices $v \in s^0 \cup t^0$ such that $v \notin p^0$. For every such vertex the dimension of X_v is 1. The maps between two such vertices are almost always the same as before the extension, either zero or given by the identity 1. The only other eventuality is given by the possible action arrow of the form (iii) of Lemma 2.

Secondly, there are the vertices $v \in p^0$, yielding subspaces $X_v = \text{span}\,(e^s_v, e^t_v)$ of dimension 2. For any arrow $\alpha \in p^1$, $\alpha : u \to v$, there is exactly one possible additional action arrow labeled by α, namely $\underline{\alpha} : e^s_u \to e^t_v$ with some constant $\lambda_{\underline{\alpha}}$. Hence the action of α on X has the form

$$A_\alpha = \begin{pmatrix} 1 & 0 \\ \lambda_{\underline{\alpha}} & 1 \end{pmatrix}$$

with respect to the bases $E_u = (e^s_u, e^t_u)$ of X_u and $E_v = (e^s_v, e^t_v)$ of X_v. The constant $\lambda_{\underline{\alpha}} \in K$ is arbitrary, the case $\lambda_{\underline{\alpha}} = 0$ correspoding to no additional action arrow with label α. So α acts from X_u to X_v as an isomorphism, which is the identity in the first coordinate. This is an important property.

The only arrows we haven't described so far are the ones joining some one dimensional X_u with some two dimensional X_v.

Overlapping case

Suppose that the strings s, t overlap in $p \neq \emptyset$. The possible additional action arrows are given by (i) and (ii) of Lemma 2. We now pick a suitable $x \in M_{l(s)}$ and look for a minimal $G(x)$ closed on nonzero images and some preimages containing x. In almost all cases it is fine to take $x = e^s_{l(s)}$. The only exception will be the case $l(s) = l(p) \neq l(t)$ with an arrow $\beta \in t^1$, $l(p) \to v$ oriented from right to left – if this happens, we take $x = e^s_{l(s)} + \lambda e^t_{l(s)}$ with the unique $\lambda \in K$ such that $\beta x = 0$. This choice will later guarantee that span $G(x)$ is a string module.

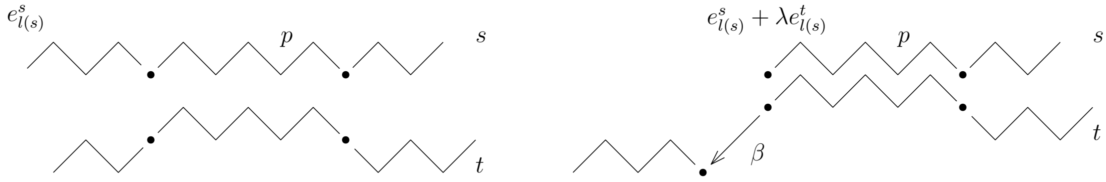

Figure 2.7:

We turn to describing $G(x)$. Plainly $G(x)$ contains all e^s_v for vertices $v \in s_0$ between $l(s)$ and $l(p)$ inclusively, because the corresponding arrows of Q^1 act as one dimensional identities. Notice that $e^s_{l(p)}$ does not interact with the other arrow β incident with $l(p)$ from the left, if such β exists: When $l(s) \neq l(p)$, this is prohibited by the relations in J. If β targets in $l(p)$, it is also easy since there is no additional action arrow labelled by β. Of the last case we have taken care of in advance by redefining $x = e^s_{l(s)} + \lambda e^t_{l(s)}$. All in all, by the choice of x, such arrow does not contribute to $G(x)$.

Now, since all the arrows $\alpha \in p^1$ act as the isomorphisms

$$A_\alpha = \begin{pmatrix} 1 & 0 \\ \lambda_{\underline{\alpha}} & 1 \end{pmatrix},$$

closing on images and some preimages of these yiels for every $v \in p^1$ precisely one element in X_v of the form $e^s_v + \lambda_v e^t_v$, the constants $\lambda_v \in K$ being given by the compositions of the matrices A_α. Finally, we get some $e^s_{r(p)} + \lambda_{r(p)} e^t_{r(p)}$. Taking into account all possible extensions, it is easy to see that this $\lambda_{r(p)} \in K$ can be arbitrary. There are at most two arrows to the right of $r(p)$ incident with it; at most one of them belongs to s and at most one of them to t and by the properties of drawings one of them sources in $r(p)$ and the other one targets in $r(p)$. Denote them $\sigma : r(p) \to u$ and $\tau : w \to r(p)$, respectively.

For a moment, suppose that both σ and τ exist. Then $\sigma\tau \in I$, so $\operatorname{Im} \tau \subseteq \operatorname{Ker} \sigma$. Independently of any additional action arrows labelled by σ or τ, it holds that $\operatorname{rank} \sigma = 1 = \operatorname{rank} \tau$, so in fact $\operatorname{Im} \tau = \operatorname{Ker} \sigma$.

Now, if $\sigma \in s^1$ and $\tau \in t^1$, then there are no additional action arrow with these labels. Hence $\sigma(e^s_{r(p)} + \lambda_{r(p)} e^t_{r(p)}) = \sigma(e^s_{r(p)}) = e^s_u \neq 0$ and $e^s_{r(p)} + \lambda_{r(p)} e^t_{r(p)} \notin \operatorname{Im} \tau$. So only σ contributes to $G(x)$. By closing $G(x)$ on images and some preimages of the remaining arrows of s^1, we get $\operatorname{span} G(x) \cong M(s)$.

On the other hand, if $\tau \in s^1$ and $\sigma \in t^1$, the situation is more interesting and the resulting $G(x)$ truly depends on λ. For one specific $\kappa \in K$ determined by the action of τ, we get $e^s_{r(p)} + \kappa e^t_{r(p)} \in \operatorname{Im} \tau$ and again $\operatorname{span} G(x) \cong M(s)$. But for all other $\lambda \neq \kappa$ this is not the case, so from $\operatorname{Im} \tau = \operatorname{Ker} \sigma$ it follows that $\sigma(e^s_{r(p)} + \lambda_{r(p)} e^t_{r(p)})$ is a nonzero multiple of e^s_u. By closing $G(x)$ on images and some preimages of the remaining arrows of t^1, we get $\operatorname{span} G(x) \cong M(\overleftrightarrow{st})$.

If some of these σ, τ doesn't exist, the above observations about the shape of $G(x)$ follow even more easily (to save work, we can just formally add such arrows over possibly bigger algebra R and then use the same argument).

In the same way, we can take $y \in M_{r(s)}$ and form $G(y)$; by left-right symmetry, this works exactly the same as in the case of x and $G(x)$. Now it is finally time to determine the decomposition of X. Depending on the position of strings s and t, there are two posibilities.

(1) $s >_l t$ and $s >_r t$, equivalently $s > t$

Then $\sigma \in t_1$ and $\tau \in s_1$ and span $G(x)$ is isomorphic either to $M(s)$ or to $M(\overleftrightarrow{st})$ as we have proved. Using the left-right symmetry, we also get that the span of $G(y)$ is isomorphic either to $M(s)$ or to $M(\overleftrightarrow{ts})$.
If span $G(x) \cong M(s)$, the extension splits into $X \cong$ span $G(x) \oplus M(t)$. The same happens if span $G(y) \cong M(s)$.
On the other hand, if span $G(x) \cong M(\overleftrightarrow{st})$ and span $G(y) \cong M(\overleftrightarrow{ts})$, we claim that $X \cong$ span $G(x) \oplus$ span $G(y)$.

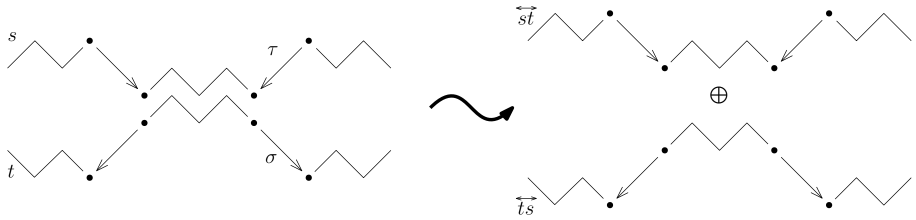

Figure 2.8: The interesting extension in case $s > t$.

(2) $s \leq_l t$ or $s \leq_r t$
Using the left-right symmetry, wlog assume $s \leq_r t$. Then $\sigma \in s^1$ and $\tau \in t^1$, if they exist, and the span of $G(x)$ is isomorphic to $M(s)$. In this case we claim that $X \cong$ span $G(x) \oplus M(t)$.

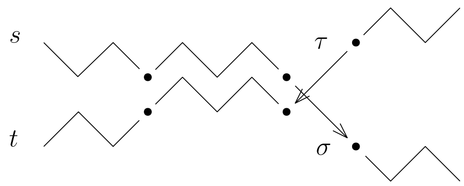

Figure 2.9: Wlog the case $s \leq_r t$, when no interesting extension exists.

In both cases, we have found two well-defined indecomposale submodules A, B and we claim that they provide the decomposition of X. In order to check this, it suffices to show that they provide a direct decomposition as vector spaces. Since X is finite dimensional and dim A + dim B = dim X, it suffices to show that there is no equality of the form $a = b$ for $a \in A$ and $b \in B$, at least one of them nonzero. But if such equality would hold, then the same would be true in X_v for some $v \in Q^0$. So in the end we would have $A \cong B$. But this is obviously not true since the strings corresponding to A and B have different shape (treating separately $s = t$, which trivially results into a split extension).

Empty intersection case
Last of all, we assume that $s \cap t = \emptyset$, but there is an interesting extension of $M(t)$ by $M(s)$. So the case (iii) of Lemma 2 occurs; there exists either an arrow $\gamma : r(s) \to l(t)$ or there exists an arrow $\delta : l(s) \to r(t)$, the only additional action arrow being either $\underline{\gamma} : e^s_{r(s)} \to e^t_{l(t)}$ or $\underline{\delta} : e^s_{l(s)} \to e^t_{r(t)}$, respectively, with some coefficcient $0 \neq \lambda \in K$. So we get (rescaling the elements of E^t by λ) the decomposition $X \cong M(\overleftrightarrow{st})$ or $X \cong M(\overleftrightarrow{ts})$, respectively.

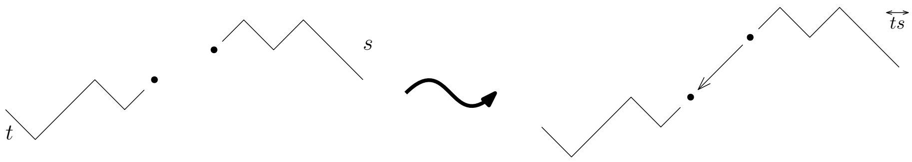

Figure 2.10: The interesting extension in the empty intersection case.

To be absolutely honest, so far we have shown that $s > t$ is a necessary condition for such an interesting extension X to exist. It remains to show that whenever $s > t$, we can truly construct the promised extensions. Because we are working with gentle tree algebras, this indeed works – all additional arrows from (i) and (iii) of Lemma 2 are always well defined, so in these cases it is trivial to construct the desired extensions using just one additional arrow. The only remaining case is when $p = s \cap t$ is a degenerate string, which sometimes forces us to use two additional arrows – but nevertheless it suffices to check this only for few small configurations. Hence we are done. □

Remark. In the proof we have provided an explicit decomposition of the extension into indecomposabe modules, which is then unique by Krull-Schmidt theorem. By the knowledge of indecomposable modules over R , we could have avoided some of the work – there simply aren't any other possibilities how the decompositions could look like.

Remark. Notice that when $s \not> t$, the base change from previous proof only changes the basis of $M(s)$, whereas $M(t)$ remains unchanged. This will be important when working with extensions of other string algebras. This is just another way of stating that whenever $s \not> t$, the extension of $M(t)$ by $M(s)$ always splits.

It is straightforward to generalize Theorem 4 to tree algebras with relations.

Corollary 5. *Let $M(s)$, $M(t)$ be string modules over a tree algebra $R = KQ/I$. Then there exist an interesting extension of $M(t)$ by $M(s)$ if and only if $s > t$ and $\overleftrightarrow{st}$, $\overleftrightarrow{ts}$ are both well defined strings over R. This interesting extension then decomposes as $M(\overleftrightarrow{st}) \oplus M(\overleftrightarrow{ts})$.*

Proof. Follows from the previous Theorem 4. If $s > t$ and both $\overleftrightarrow{st}$, $\overleftrightarrow{ts}$ are in $\mathrm{St}(R)$, then the interesting extension of $M(s)$ by $M(t)$ is well defined and results in $X \cong M(\overleftrightarrow{st}) \oplus M(\overleftrightarrow{ts})$ by the argument of Theorem 4.

On the other hand, the proof of Theorem 4 shows that any interesting extension of $M(s)$ by $M(t)$ over the gentle tree algebra KQ/J decomposes as $X \cong M(\overleftrightarrow{st}) \oplus M(\overleftrightarrow{ts})$, so it does the same over R, hence the underlying strings need to be in $\mathrm{St}(R)$. (The base change of subspaces X_v works always the same, regardless of the relations in I.) □

Remark. When dealing with extensions of decomposable finite dimensional modules, it is possible to give a similar combinatorial characterization (in terms of their decompositions). The combinatorics turns to be very similar to the one just shown, only allowing "multiple jumps between strings". This will be implicitely treated in the following section.

2.2 Closed classes for tree algebras

2.2.1 General observations

For a bound quiver algebra R, the indecomposabe modules of the class $\mathcal{P}$ of projective finite dimensional modules are precisely the submodules of the regular module R generated by the vertices, since their sum is R and they are indecomposable. In the case of string algebras, $\mathcal{P}$ consists of some string modules. Notice that every element of $\mathcal{P}$ is minimal with respect to $>$.

Recall from Section 1.3 that for a subclass $\mathcal{A} \subseteq \mathrm{mod}R$ we shall denote by $\overline{\mathcal{A}}$ the closure of $\mathcal{A}$ on Ext and Add. Ultimately we wish to describe classes of the form $\overline{\mathcal{A} \cup \mathcal{P}}$. We will further denote by $\mathrm{Ext}_{\mathrm{ind}}$ the closure on extensions of pairs of indecomposable modules.

Definition. *We say that a K-algebra R has the extension decomposition property if and only if for any $\mathcal{A} \subseteq \mathrm{mod}R$ it holds that the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ on Ext and Add is the same as the closure of $\mathcal{A}$ only on Ext_{ind} and Add.*

Remark. The extension decomposition property doesn't hold in general for (possibly infinite-dimensional) K-algebras. However, it turns out that it holds for the algebras we are interested in. So it is a sensible thing to ask which algebras have this property.

We now present few propositions which hold for arbitrary string algebras. Althought we must pick words more carefully when proving them, the proofs are almost the same as in the case of tree algebras.

Proposition 6. *Let R by a K-algebra, $\mathcal{A} \subseteq \mathrm{mod}R$. When computing $\overline{\mathcal{A}}$, it suffices to take extensions by indecomposable modules.*

Proof. Suppose that M, $N \in \mathcal{A}$ have decompositions $M = \bigoplus_{i=1}^{m} M_i$ and $N = \bigoplus_{j=1}^{n} N_j$. Consider some extension X of N by M. This extension is given by some action arrows on the union of bases of modules M_i, $i = 1, \ldots, m$ and N_j, $j = 1, \ldots, n$.

In particular if $m \geq 2$, one can view X as an extension of some module X', which appears as an extension of $\bigoplus_{j=1}^{n} N_j)$ by $\bigoplus_{i=1}^{m-1} M_i$, by M_m. Using this observation repeatedly, it is sufficient to close $\mathcal{A}$ only on extensions of the form $\mathrm{Ext}_R^1(M, \bigoplus_{j=1}^{n} N_j)$ for M, $N_j \in \mathrm{ind}R$, as claimed. □

Proposition 7. *Let KQ/I be a string algebra. Suppose that Z is an extension of N by M with $M = \bigoplus_{i=1}^{m} M(g_i)$, $N = \bigoplus_{j=1}^{n} N(h_j)$, $Z = \bigoplus_{k=1}^{z} Z(q_k)$ being the decompositions into string and band modules. Then the endpoints of strigs among $q_1, q_2, \ldots, q_z$ form a subcollection of endpoints of strings among $g_1, g_2, \ldots, g_m$ and $h_1, h_2, \ldots, h_n$.*

Proof. Take an arrow $\alpha \in Q^1$, $\alpha : u \to v$. For each $Z(q_k)$ from $Z = \bigoplus_{k=1}^{z} Z(q_k)$ consider α as its linear endomorphism. Then rank α equals to the number of occurences of α in q_k times the common dimension d_k of the subspaces sitting at vertices of q_k.

But for any $\alpha \in Q^1$, the additional action arrows in the extension Z don't decrease rank α, which must remain the same throught the base change leading to the direct decomposition of Z. So fix a vertex $v \in Q^0$ and focus on the vertex subspace Z_v. There are at most four arrows incident with v, denote them by $\alpha, \beta, \gamma, \delta$. We will compute $r_v = \text{rank } \alpha + \text{rank } \beta + \text{rank } \gamma + \text{rank } \delta$ in Z in two ways (if there are less than four arrows incident with v, we define the remaining ranks formally to be 0).

If $v \notin q_k^0$, then $Z(q_k)$ doesn't contribute to r_v. For any occurence of $v \in q_k^0$ as an inner vertex, $Z(q_k)$ contributes by $2 \cdot d_k$. If q_k is a string, then for every occurence of $v \in q_k^0$ as an outer vertex, $Z(q_k)$ contributes by $d_k = 1$. Summing over all summands, we get

$$r_v = 2 \cdot \dim Z_v - \# \text{ of occurences of } v \text{ as an outer vertex of } q_k.$$

Similarly, we can compute r'_v, the sum of these ranks before the extension. If $v \notin g_i$ resp. $v \notin h_j$, the corresponding module doesn't contribute to r'_v. For every occurence of $v \in g_i^0$ resp. $v \in h_j^0$ as an inner vertex, the module contributes by two times the common dimension of its vertices. Finally, for every occurence $v \in g_i^0$ resp. $v \in h_j^0$ as an outer vertex of a string, it contributes at least by 1. Therefore

$$r'_v = 2 \cdot (\dim M_v + \dim N_v) - \# \text{ of occurences of } v \text{ as an outer vertex of } g_i \text{ or } h_j$$

Since the extension doesn't decrease the ranks, $r_v \geq r'_v$. We also know that $\dim Z_v = \dim M_v + \dim N_v$. Comparing the two results, we are done. □

Corollary 8. *Let Z be an extension of N by M, with M and N having m and n string modules in their unique decompositions. Than Z has at most $m+n$ string modules in its decomposition.*

Proof. Every string has exactly two ends. So the Proposition 7 precisely bounds the number of strings in the decomposition of Z by the number of strings in the decompositions of M and N. □

2.2.2 Closed classes for tree algebras

Take $R = KQ/I$ a tree algebra. In Theorem 4 we have explicitly described the possible decompositions of extensions of pairs of modules from ind$R$. Now we would like to use this description to find the classes $\overline{\mathcal{A}}$. If the tree algebra KQ/I had extension decomposition, we would be done, the closure of any set of string modules being given by the iterative use of the combinatorial closure property from Corollary 5. Luckily, this is the case.

Remark. We can rephrase Proposition 7 to get that if Z is an extension of N by M with $M = \bigoplus_{i=1}^m M(s_i)$, $N = \bigoplus_{j=1}^n N(t_j)$, $Z = \bigoplus_{k=1}^z Z(q_k)$ being the decompositions into string modules, then the left endpoints $(l(q_k) \mid k = 1, \dots z)$ form a subcollection of $(l(s_i) \mid i = 1, \dots m) \cup (l(t_j) \mid j = 1, \dots, n)$ with multiplicities. The same works symmetrically for the right ends.

To prove this modified version, it suffices to fix the vertex u and take the at most two arrows σ, τ incident with u from the left. Then an analogous counting argument for rank σ + rank τ together with $\sigma\tau, \tau\sigma \in I$ yields the result.

Theorem 9. *Every tree algebra $R = KQ/I$ has the extension decomposition property.*

Proof. Because of Proposition 6 it sufficess to check that every indecomposable summand L of an extension X of $\bigoplus_{i=1}^{n} N(t_i)$ by $M(s)$, where s, t_1, t_2, ..., $t_n \in \mathrm{St}(R)$, is in fact already in the closure of $\mathcal{A} = \{M(s)\} \cup \{N(t_i) \mid i = 1, 2, \dots n\}$ on $\mathrm{Ext}_{\mathrm{ind}}$ and Add.

We shall prove this claim by induction. If $n = 0$ then $L = M(s)$. If $n = 1$, then L is a summand of an extension of $N(t_1)$ by $M(s)$. In both cases the claim holds trivially. Further suppose $n \geq 2$.

If some $N(t_j)$ with the canonical inclusion is a direct summand of X, we can rewrite $X = N(t_j) \oplus Y$, so every other direct summand L of X is in fact a direct summands of Y, which is an extension of $\bigoplus_{i \neq j} N(t_i)$ by $M(s)$. Hence L is in the closure of $\mathcal{A}$ on $\mathrm{Ext}_{\mathrm{ind}}$ and Add by the inductive assumption. Since $N(t_j) \in \mathcal{A}$, in this case we are done. Therefore we can (using Corollary 5)further assume $\forall i$ that $s > t_i$.

Now we make the crucial step: we pick some t_j which is minimal with respect to the relation $\geq$ on strings. Moreover, we take t_j with $t_j \cap s = \emptyset$ first. We then for a while forget all modules except for $M(s)$ and $N(t_j)$ and perform the base change from the proof of Theorem 4 on the extension of these two. If it splits, we are done by the observation from preceeding paragraph, so we can assume that this extension C of $N(t_j)$ by $M(s)$ is interesting.

Looking at the action picture, we have thus obtained X as an extension of $\bigoplus_{i \neq j} N(t_i)$ by C. We now use the minimality of t_i to pass to the induction assumption. For clarity, we split the argument into two cases.

<u>C **is indecomposable**</u> (Which corresponds to the case (iii) of Lemma 2.) Then we are almost immediately done, because we know that X is isomorphic to an extension of the sum of $n - 1$ indecomposable modules $N(t_i)$, $i \neq j$ by an indecomposable C. So if all $N(t_i)$ and C were elements of the closure of $\mathcal{A}$ on $\mathrm{Ext}_{\mathrm{ind}}$ and Add, we would be done by the inductive assumption. But the modules $N(t_i)$ are in $\mathcal{A}$ from the beginning, while C is an extension between two indecomposable elements $N(t_j)$ and $M(s)$ from $\mathcal{A}$, so it lies in the closure of $\mathcal{A}$ on $\mathrm{Ext}_{\mathrm{ind}}$ and Add too.

<u>C **is decomposable**</u> (So some of the cases (i), (ii) of Lemma 2 happened.) By the characterization from Corollary 5, $C = M(\overleftrightarrow{st_j}) \oplus M(\overleftrightarrow{t_j s})$ is the decomposition into indecomposables. We can moreover assume that all t_i have a nontrivial intersection with s, otherwise there would be such minimal one and t_j wouldn't be chosen properly. We look now at the action picture of X (with respect to the union of vertex bases) of $M(\overleftrightarrow{st_j})$, $M(\overleftrightarrow{t_j s})$ and $N(t_i)$ for $i \neq j$).

By the minimality of t_j, no string t_i satisfies both $\overleftrightarrow{st_j} > t_i$ and $\overleftrightarrow{t_j s} > t_i$ simultanuosly. This can be seen as follows. From the decomposability of C we know that $s \cap t_j \neq \emptyset$. For any other t_i we already have $t_i \cap s \neq \emptyset$. If moreover $\overleftrightarrow{st_j} > t_i$ and $\overleftrightarrow{t_j s} > t_i$, we actually arrive at $s \cap t_i \cap t_j \neq \emptyset$, so we can take some vertex v_0 from this intersection. But $\geq_l$, $\geq_r$ are linear orders on the set of strings containing v_0. This implies $t_j >_l t_i$ and $t_j >_r t_i$, hence $t_j > t_i$, contradicting the minimality of t_j.

Therefore we can split all the $N(t_i)$ into two collections $\mathcal{B}$ and $\mathcal{C}$, such that the modules from $\mathcal{B}$ have only trivial extensions by $M(\overleftrightarrow{t_j s})$ and the modules from $\mathcal{C}$ have only trivial extensions by $M(\overleftrightarrow{st_j})$.

By a suitable base changes (done in any order accordingly to the proof of

Theorem 4 for these pairs of modules without any nontrivial extension), the action picture splits into two disjoint action pictures with no action arrows between; one of them corresponding to an extension or a direct sum of modules from $\mathcal{B}$ by $M(\overleftrightarrow{st_j})$, the other one corresponding to an extension direct sum of modules from $\mathcal{C}$ by $M(\overleftrightarrow{t_j s})$, the original X being the direct sum of these two extensions. But $\mathcal{B} \cup \mathcal{C} \subseteq \mathcal{A}$ and both $M(\overleftrightarrow{st_j})$, $M(\overleftrightarrow{t_j s})$ are in the closure of $\mathcal{A}$ on $\mathrm{Ext}_{\mathrm{ind}}$ and Add. Since both of the described extensions are extensions of at most $n-1$ indecomposable modules by one indecomposable module, the inductive assumption applies and we are finally done. □

Remark. It is fair to point out one technicality we have ignored in the proof: when forming the decomposition $C = M(\overleftrightarrow{st_j}) \oplus M(\overleftrightarrow{t_j s})$, we should have checked that these are truly well defined strings over R. This is the case; but the work can be avoided by passing to some underlying gentle algebra KQ/J, doing the computations there and then returning to the original R, stating that the computations with vector spaces were independent of the additional relations.

Corollary 10. *Let $R = KQ/I$ be a tree algebra and $\mathcal{A} \subseteq indR$, then the smallest left class of a cotorsion pair containing $\mathcal{A}$ is precisely the closure of $\mathcal{A} \cup \mathcal{P}$ on*

($\star$) If s, t are strings in $\mathcal{A}$ with $s < t$ and $\overleftrightarrow{ts}$, $\overleftrightarrow{st}$ are in $St(R)$, then $\overleftrightarrow{ts}$, $\overleftrightarrow{st} \in \overline{\mathcal{A}}$.

Proof. Previous Theorem 9, definition of extension decomposition property and Corollary 5. □

The proof of Theorem 9 in fact yields a nice algorithm for computing the decomposition of any extension X into indecomposables. Althought we could provide more explicit description, we won't bother to formalize this at the moment. We will rather show how these decompositions look like by an instructive picture.

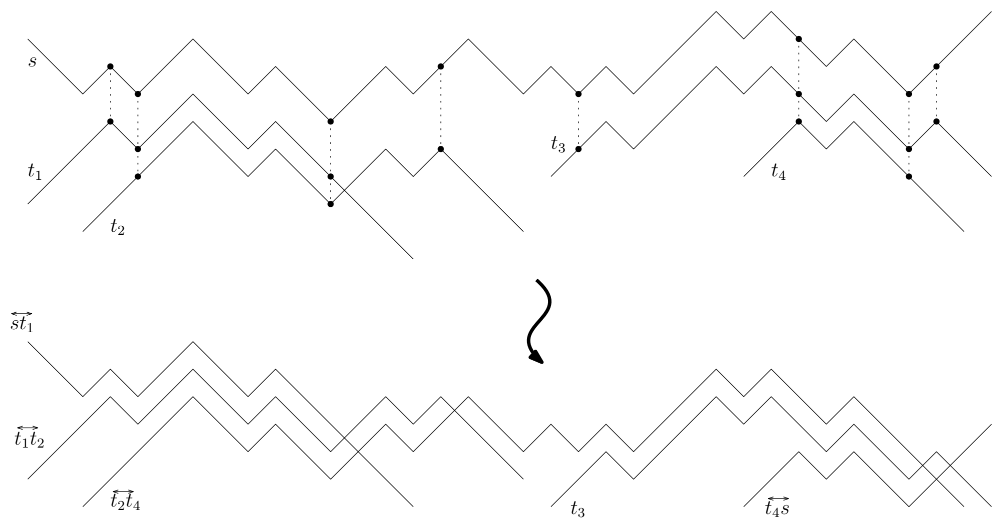

Figure 2.11: The decomposition of extension of $N(t_1) \oplus N(t_2) \oplus N(t_3) \oplus N(t_4)$ by $M(s)$, where all the extensions of $N(t_i)$ by $M(s)$ were interesting.

To summarize the outputs of this work, for a tree algebra R we have described the closure of any $\mathcal{A} \subseteq \mathrm{mod}R$ on Ext and Add in an elementary way, manipulating with elements of $\mathrm{St}(R)$. These strings are determined only by their endpoints, so they can be represented by a subset of $Q^0 \times Q^0$. The only combinatorial closure property ($\star$) permits to exchange endpoints of some pairs of strings; in $Q^0 \times Q^0$ this is represented by adding the "middle" vertices of rectangles which already have the "upper" and the "lower" vertex. This in turn yields the desired description of classes $\overline{\mathcal{A} \cup \mathcal{P}}$ with $\mathcal{A} \subseteq \mathrm{mod}R$ because the indecomposables of $\mathcal{P}$ can be immediately read from Q and I.

2.2.3 Closed classes for gentle tree algebras

When working with a gentle tree algebra KQ/J, there is a bit nicer restatement of ($\star$) from Corollary 10, using the projectives $\mathcal{P}$. Unfortunately this trick can have problems when we permit more relations – nevertheless it can be often helpful even in the case of arbitrary string algebras.

So start with a gentle tree algebra $R = KQ/I$. Suppose $s, t \in \mathrm{St}(R)$ with $M(s), M(t) \in \mathcal{A}$. We already know by Corollary 10 that if $s > t$ (or vice versa) and $\overleftrightarrow{ts}$, $\overleftrightarrow{st} \in \mathrm{St}(R)$, then $\overleftrightarrow{ts}$, $\overleftrightarrow{st} \in \mathcal{A}$. The case when $l(s) = l(t)$ or $r(s) = r(t)$ is not interesting – there is no interesting extension between $M(t)$ and $M(s)$ and the strings $\overleftrightarrow{st}$, $\overleftrightarrow{ts}$ are only s, t in some order.

Suppose now that $s >_l t$ and $s <_r t$ (or vice versa). In particular, $s \cap t \neq \emptyset$. So take any $u \in p^0$ and consider the projective P_u with underlying string q. Then $s \geq q$ and $t \geq q$. More precisely, we know that $s >_l q$, $s \geq_r q$, $t \geq_l q$ and $t >_r q$. We shall prove that $M(\overleftrightarrow{ts}) \in \overline{\mathcal{A} \cup \mathcal{P}}$.

If $r(s) = r(q)$ or $l(t) = l(q)$, then $\overleftrightarrow{ts} = \overleftrightarrow{tq}$ or $\overleftrightarrow{ts} = \overleftrightarrow{qs}$, respectively. So it suffices to use the interesting extension of P_u by $M(t)$ or $M(s)$. So we can assume $s > q$ and $t > q$. We can now obtain $\overleftrightarrow{ts}$ in two steps. First take the interesting extension of $M(q)$ by $M(t)$, which has $M(\overleftrightarrow{tq})$ as a summand. But $s > \overleftrightarrow{tq}$, so we can then extend $M(\overleftrightarrow{tq})$ by $M(s)$ and get $M(\overleftrightarrow{ts})$ as a summand, hence $M(\overleftrightarrow{ts}) \in \overline{\mathcal{A}}$.

Remark. The auxiliary extension of the projective module P_u can be often used also in the setting of non-gentle tree algebras. For instance when u is an inner peak of p and the desired string $\overleftrightarrow{ts}$ is well-defined, all the necassary strings really exist. But in some degenerate cases there need not be a vertex u with these properties and it can happen that $\overleftrightarrow{ts} \notin \overline{\mathcal{A} \cup \mathcal{P}}$.

We now rephrase the discussion to form a lemma, which formaly staightens the combinatorial closure properties for gentle tree algebras.

Lemma 11. *Let $R = KQ/I$ be a gentle tree algebra. Let $s, t \in St(R)$ with $M(s)$, $M(t) \in \mathcal{A} \cup \mathcal{P}$. When $\overleftrightarrow{ts} \leq s$ or $\overleftrightarrow{ts} \leq t$, then $M(\overleftrightarrow{ts}) \in \overline{\mathcal{A} \cup \mathcal{P}}$.*

Proof. From what we know, this is a straightforward case chase. By left-right symetry, we consider only the first part of the lemma. Take s, $t \in \mathrm{St}(R)$ and suppose $\overleftrightarrow{ts} \in \mathrm{St}(R)$.

If $s \cap t = \emptyset$, then the condition $\overleftrightarrow{st} \leq s$ or $\overleftrightarrow{st} \leq t$ is indeed satisfied and $M(\overleftrightarrow{st}) \in \overline{\mathcal{A}}$.

If $s \cap t \neq \emptyset$, there are two possibilities. The case $s > t$ (or vice versa) is covered by Theorem 4. On the other hand the case $s \geq_l t$ and $s \leq_r t$ (or vice versa) is covered by the discussion above. □

3. Comments and conclusion

There is not enough space in this thesis to proceed further, but nevertheless I would like to casually comment on some related topics.

3.1 String algebras and topology

As promised in the begining, there exist some geometrical viewpoints on string algebras. Althought we won't consider these, I would like to briefly present more topological view on strings and bands, which can be of some help when considering arbitrary string algebras.

Let $R = KQ/I$ be a string algebra. Every string $s \in \mathrm{St}(R)$ is a word yielding a path in Q. This path connects two vertices of Q and doesn't cross any relation from I. Moreover it doesn't have $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ as a subword for any $\alpha \in Q^1$, so it is a reduced word in the corresponding free group. If we now view Q as a topological space, the string s determines a path $\tilde{s} : ([0,1], \{0,1\}) \to (Q, Q_0)$. Working with the words up to cancellation (as with elements of the free group), $\tilde{s}$ is determined uniquely up to homotopy. The word defining the string s is then the unique reduced word defining this path $\tilde{s}$. Conversely, when a path $\tilde{s} : ([0,1], \{0,1\}) \to (Q, Q_0)$ doesn't cross any relation from I, it can be shown that it defines a unique reduced word s giving a well defined string. All in all, strings correspond bijectively to those elements of the fundamental grupoid of Q, which begin and end in vertices and can avoid relations from I.

In the same manner we can treat the bands – a band b determines a loop $\tilde{b} : S^1 \to Q$ uniquely up to homotopy, no power of which crosses any relations from I. Since we demand that the band b is primitive, $\tilde{b}$ is not a proper power of any other element of the fundamental group of Q. Conversely, these topological conditions characterize the elements $\tilde{b}$ of the fundamental group which are given by some band b. The band b then corresponds to the unique reduced word defining $\tilde{b}$. For a given band b, the corresponding band modules are indexed by their dimension m and by some indecomposable automorphisms. This m is intuitively related to b^m.

Having this in mind, we can consider various covering spaces of the quiver Q. The covering maps then induce homomorphisms of the quiver algebras. We can then easily lift the monomial relations to get the *covering algebra*. Modules over this algebra then naturally become modules over the original algebra. Having such a covering space, we can also lift strings and bands.

In particular, we can take the universal covering quiver U of Q. If R was a string algebra, this U is a subtree of the infinite oriented regular tree T_2. Given a, $b \in \mathrm{St}(R) \cup \mathrm{Ba}(R)$ for an arbitrary string algebra R, a and b usually have more common maximal substrings, which can even overlap. But when we lift a and b to a' and b' in U, they again have an unique maximal substring $a' \cap b'$. This corespondence gives a clear way of describing the possible additional action arrows in extensions of $M(b)$ by $M(a)$. Moreover, we can often locally work over U and then project the results back to Q. For instance, if the lifts a', b' of a and b don't satisfy $a' > b'$, we can perform the base change from Theorem 4 which deletes all the action arrows corresponding to this intersection. By projecting this base change to the original modules, we get rid of all the action arrows in the original extension between strings a, b. Hence we can wlog consider only extensions when $a' > b'$.

In this way, it is possible to recover a lot of concepts described earlier in the case of tree algebras for arbitrary string alebras.

3.2 Questions and conjectures

We now take the liberty of posing a few related questions.

Question. *Which K-algebras R have the extension decomposition property?*

The extension decomposition property is formulated only in the language of the category mod$R$, so it is a homological property. Apart for some intristic interest, this property simplifies the description of the classes in mod$R$ given by cotorsion pairs: for instance when an algebra R of finite representation type has the extension decomposition, the search for such classes boils down to calculating finitely many Ext groups.

For this kind of trick, the extension decomposition property is unnecesarilly strong – it suffices to know it only for classes contatining $\mathcal{P}$. Also, althought we have shown extension decomposition property for tree algebras, it was rather a consequence than a method.

Question. *Describe the classes $\overline{\mathcal{A}}$, where $\mathcal{A} \subseteq modR$, over arbitrary string algebra R.*

This should be possible using similar combinatorial means as in the case of tree algebras, although much more technical. In light of Proposition 7 it makes sense to determine the strings in such a class first, because no string can arise from extensions between bands. Having them, it is necessary to determine the primitive bands – the ones which can be obtained by some interesting extension as well as the ones which "just appear". Having these, the rest more or less follows.

This combinatorics should be similar to the one just exhibited – only there can be many "jumps" between strings and bands in single extension. It would be also interesting to check the extension decomposition property for string algebras in general, which appears to hold.

3.3 Conclusion

In conclusion, we have described the extensions and their decompositions in the category mod$R$ for any tree algebra $R = KQ/I$ by purely combinatorial means. Moreover, we have used this to prove that tree algebras posess the extension decomposition property, which enabled us to describe the desired classes induced by cotorsion pairs by one combinatorial closure property ($\star$) on the set of strings St(R). It appears possible to employ the similar ideas for arbitrary string algebras.

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