Reviews on the Habilitation Thesis
Mathematical Analysis of Nonlinear systems describing flows of incompressible fluids

Miroslav Bulíček

Mr. Buliček 's thesis is devoted to the mathematical analysis (the existence and the qualitative properties) of systems of partial differential equations describing the unsteady flow of incompressible homogeneous fluids with complicated rheology. The thesis focus on three general classes of models: purely mechanical setting, heat conducting fluis and the models for turbulence. Mr. Bulíček started with the general description of the model, boundary conditions and the initial condition. He introduced the Cauchy stress tensor $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, where p is the pressure and $\mathbf{S}$ is the part of Cauchy stress tensor. From the second law of thermodynamics the following restriction on the $\mathbf{T}, \mathbf{D}, \mathbf{S}$,

$$\mathbf{T} \cdot \mathbf{D} = \mathbf{S} \cdot \mathbf{D} \geq 0,$$

where D is the symmetrical part of the velocity gradient. The goal of his thesis is to find a such generalize relation between $\mathbf{S}$ and $\mathbf{D}$ that satisfies the second law of thermodynamics for which the global existence of weak solution for large data and secondly uniqueness (regularity) of weak solution can be shown. One possibility is the implicit relation between $\mathbf{S}$ and $\mathbf{D}$, which is explained.

Second chapter of thesis focused on the homogeneous incompressible fluids. Mr. Bulíček introduced the generalized model and explained the differences between the explicit and implicit relations of $\mathbf{D}$ and $\mathbf{S}$. One of the reasons why the implicit relation is more suitable, is the property of so-called pressure thickenning which means that the viscosity depends on the pressure. The implicit relation is defined through the graph and it is given explenation about this relation. After that the weak solution of the problem is define. Moreover also a suitable weak solution in given. Further a short introductory survey of available results is written. One of the main part in Chapter 2 is Theorem 2.1 which concerns the existence of the weak solution. Moreover the generalization of the Minty method is described in Lemma 2.1. Lemma 2.2. gives the extension of the Lipschitz approximation methods from the polynomial growth to a general Young function ψ. This part is included in paper [B1]. Further, Mr. Bulíček discussed the uniqueness and the regularity.

First he gave a short discussion about the existing results and described the differences between the two-dimensional and three dimensional cases. Mr. Bulíček presented novelties which he reached in this direction. Firstly the uniqueness was solved in the whole possible expected range of r, s and moreover results are valid for more general class of boundary conditions than the Navier slip boundary conditions (see paper [B4]). Further he got the optimal time regularity, see paper [B2].

Third chapter focus on the heat conduction fluids. Again author gave the explanation between the explicit and implicit relations, consequences of such relations and very brief survey of existing results. In this chapter author presented two results. In [B3] neglecting the convective term with periodic boundary conditions the L^2 regularity of the full linearized system is proven. In [B6] classical solution for the full nonlinear system is considered. The existence and uniqueness is shown.

Last chapter is devoted to the problem of Kolmogorov model for turbulent flows (see [B5]). Firstly the model is introduced. Further, the formulation of the problem is given together with explanation of the main difficulties, assumptions on the data. In subsection 4.3 main result of the chapter is stated - long time and large data existence for Kolmogorov model.

I have only a few comments to references and state of arts.
To my opinion some of these references should be mention:

- Book related to non-Newtonian case

  Breit, Dominic Existence theory for generalized Newtonian fluids. Mathematics in Science and Engineering. Elsevier/Academic Press, London, 2017. xvii+267 pp. ISBN: 978-0-12-811044-7

- The references about the boundary conditions and the local regularity of pressure

  - G. Q. Chen, Z. Qian, *A study of the Navier-Stokes equations with the kinematic and Navier boundary conditions* (2010)
  - J. Wolf, *On the local pressure of the Navier-Stokes equations and related systems*, (2017)
  - H. Bae, J. Wolf, *Sufficient conditions for local regularity to the generalized Newtonian fluid with shear thinning viscosity.* (2017)

- J. Neustupa, H. Al Baba, *The interior regularity of pressure associated with a weak solution to the Navier-Stokes equations with the Navier-type boundary conditions.* (2018)

- Historical remarks

  - Ladyzhenskaya focused on the nonlinear dependence of the viscosity on the gradient of the velocity field. But by Nečas and his collaborators (under many discussions of J. Nečas with M. Šilhavý and understanding that system must satisfied e.g. the second law of thermodynamics) the dependency **only on the symmetrical part of the gradient** was introduced. Global existence of such type of problems was announced in paper H. Bellout, F. Bloom, J. Nečas: Solutions for incompressible non-Newtonian fluids. C. R. Acad. Sci. Paris Sr. I Math. 317 (1993), and then independently two different proofs were published by Nečas et al.
  - Concerning the first global existence result when the viscosity dependent case on the pressure: It should be mention that only local existence was known for long period. By Nečas's idea and his collaborators in ARMA, 165 (2002) that the viscosity depends not only on pressure but also **on the invariant of velocity field** opened the window to show the global existence result.

The habilitation thesis is written very nicely, very carefully, precisely. I'm fully convinced the results are nice and have very high scientific value. The thesis fulfills all requirements aimed for a habilitation thesis and I fully recommend it in front of the respective committee.

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Prague, February 16, 2020