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HABILITATION THESIS

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On the interplay of Continuum Theory, Topological Dynamics and Descriptive Set Theory

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Contents

0 Preface

This thesis consists of some selected special topics whose common ground is settled in *General Topology* (GT). These topics can be divided into three main areas: *Continuum Theory* (CT), *Topological Dynamics* (TD) and *Descriptive Set Theory* (DST). The aim of the chapters 1-3 is to discuss some specific parts in each of the fields that are related to our results. Especially we want to focus on some recent progress initiated occasionally by our results and also to mention some intriguing open problems. We also describe roughly some proof methods. Of course, there is no aim to give a comprehensive introduction either to CT, TD or DST. In spite of the fact each of the fields is well defined, all the three parts are mutually interconnected, and the interplay of these branches forms fruitful topic as well. Depending on the point of view, some of our papers could be moved to different sections without any doubt.

Over the last ten years my research focus was moving from CT through TD to applications of DST in both CT and TD. I was mainly influenced by Engelking [41] in GT; Nadler [70], Macías [64], van Mill [76], Illanes and Nadler [55] in CT; Katok and Haselblatt [57], Brucks and Bruin [24] and Kůrka [61] in TD; Kechris [58], Gao [44] and Hjorth [49] in DST.

0.1 List of papers

All the ten papers attached to this thesis were published or submitted within the last five years. I decided not to attach my joint papers [47] and [35] (that were also published within the last five years) because they do not fit into the framework of this thesis.

- **[A]** J. Bobok, P. Pyrih, B. Vejnar; Non-cut, shore and non-block points in continua, Glasnik Matematicki 51 (2016), 237-253.
  doi: 10.3336/gm.51.1.14
- **[B]** J. Bobok, P. Pyrih, B. Vejnar; On blockers in continua, Topology and its Applications 202 (2016), 346-355.
  doi: 10.1016/j.topol.2016.01.013
- **[C]** A. Bartoš, R. Marciňa, P. Pyrih, B. Vejnar; Incomparable compactifications of a ray with Peano remainder, Topology and its Applications 208 (2016), 93-105.
  doi: 10.1016/j.topol.2016.05.008
- **[D]** A. Bartoš, J. Bobok, J. van Mill, P. Pyrih, B. Vejnar; Compactifiable classes of compacta, Topology and its Applications 266 (2019), 106836.
  doi: 10.1016/j.topol.2019.106836
- **[E]** B. Vejnar; Every continuous action of a compact group on a uniquely arcwise connected continuum has a fixed point, Journal of Fixed Point Theory and Applications 20 (2018), 69, 9pp.
  doi: 10.1007/s11784-018-0552-3
- **[F]** J. Bobok, P. Pyrih, B. Vejnar; On minimal homeomorphisms on Peano continua, Topology and its Applications 210 (2016), 263-268.
  doi: 10.1016/j.topol.2016.07.022
- **[G]** A. Bartoš, J. Bobok, P. Pyrih, S. Roth, B. Vejnar; Constant slope, entropy and horseshoes for a map on a tame dendrite, Ergodic Theory and Dynamical Systems (2019).
  doi: 10.1017/etds.2019.29
- **[H]** M. Doležal, M. Rmoutil, B. Vejnar, V. Vlasák; Haar meager sets revisited, Journal of Mathematical Analysis and Applications 440 (2016), 922-939.
  doi: 10.1016/j.jmaa.2016.03.065
- **[I]** M. Doležal, B. Vejnar; Classification of the spaces $C_p^*(X)$ within the Borel-Wadge hierarchy for a projective space X, Topology and its Applications 183 (2015), 11-17.
  doi: 10.1016/j.topol.2014.12.021
- **[J]** P. Krupski, B. Vejnar; The complexity of homeomorphism relations on some classes of compacta, arXiv e-prints (2018). (submitted)
  eid: arXiv:1808.08760

These papers can be grouped together according to their main focus into CT [A, B, C, D], TD [E, F, G] and DST [H, I, J].

1 Continuum Theory

The main object of this field is *continuum* that is a compact connected metrizable space. The following classes arise naturally by some sort of simple constructions applied to simple spaces: *Peano continua* (continuous images of $[0, 1]$), *chainable continua* (inverse limits of arcs with continuous bonding maps), *dendrites* (inverse limits of trees with monotone bonding maps), *tree-like continua* (inverse limits of trees) or *compactifications* of the ray $[0, \infty)$. Up to now it seems that all these classes and a lot of others are well understood in general. On the other hand plenty of specific problems are still open, some of them for a very long time [73]. Many of them are related to the topology of the *plane*.

Recently a solution of one such longstanding problem was published. The problem was to describe all *homogeneous* continua in the plane. Of course a point or a circle are such, but there were two other known for several decades: the *pseudoarc* and the *circle of pseudoarcs*. Recently it was proved that there are no others [50]. One can deduce from this how all the homogeneous compacta in the plane look like.

Homogeneity as a concept of CT can be studied in a more general context of transitive group actions which presents a bridge between different disciplines. A useful tool from the field of DST for studying homogeneity is a theorem of Effros in which X is usually assumed to be a continuum and G the full group of its homeomorphisms [14]:

Theorem 1. *Let G be a Polish group acting transitively on a Polish space X. Then the action is microtransitive, i.e. whenever $U \subseteq G$ is a neighborhood of the identinty element and $x \in X$ then Ux is a neighborhood of x.*

As homogeneity seems to be a rather rare property it is natural to study generalized homogeneity. The most simple way is to consider a bound on the number of orbits. A direct application of Theorem 1 to chainable continua was used in [20] when studying the concept of half-homogeneity in order to give a topological characterization of a special continuum (which is called *arc-less arc* in there) built on the idea of the pseudoarc.

1.1 Blocking points

A classical result in CT states that every non-degenerate continuum has at least two *non-cut* points (a cut point is defined by the property that its complement is disconnected) [70, Theorem 6.6]. Using a result of Bing [16] we concluded a more general result in [A] that follows.

Theorem 2. *Every continuum is spanned by its non-block points.*

Consequently every non-degenerate continuum contains at least two non-block points. Notice that a point x is a non-block point of a continuum X if and only if there is an increasing sequence of continua in $X \setminus \{x\}$ whose union is dense. Thus clearly every non-block point is a non-cut point but not vice versa.

Blocking properties of points play an important role in understanding one-dimensional continua. These concepts were inspired by the corresponding properties of vertices and edges in combinatorial graphs and trees. The type of extreme points considered in [A] was recently intensively studied also in the non-metrizable setting [3–6, 12] and with respect to hyperspaces [26, 27, 43].

1.2 Blocking sets

Continua of topological dimension one form a varied world of objects that is difficult to classify. On the other hand there is a reasonable visualisation of these objects since they are all embeddable into $\mathbb{R}^3$. By generalizing the idea of a combinatorial tree one can deal with the classes of *dendrites*, *dendroids* or λ-*dendroids*. Blocking properties of sets form an ingredient that helps to understand connections in these spaces.

Definition 3. Let X be a continuum. For $A, B \in 2^X$ we say that B does not *block* A if $A \cap B = \emptyset$ and the union of all subcontinua of X intersecting A and contained in $X \setminus B$ is dense in X. When B blocks A, we say that B is a *blocker* of A. For a subset $\mathcal{H} \subseteq 2^X$ we use the notation

$$\mathcal{B}(\mathcal{H}) = \{B \in 2^X : \ B \text{ blocks each element of } \mathcal{H}\}.$$

Let us denote by $F(X)$ the system of all finite subsets of X. Illanes and Krupski proved that for a locally connected continuum X it holds that $\mathcal{B}(F(X)) = \mathcal{B}(2^X)$. They asked whether the last equality is in fact a characterization of local connectedness. We provided a negative answer to their question in [B].

Theorem 4. *There exists a non-locally connected λ-dendroid X for which* $\mathcal{B}(F(X)) = \mathcal{B}(2^X)$.

On the other hand we proved that in the realm of *hereditarily decomposable* chainable continua or among *smooth* dendroids the answer to the question of Illanes and Krupski is positive [B]. To this end, *radially convex* metrics turned out to be a useful tool (these are known to exist in every smooth dendroid). Whether the equality $\mathcal{B}(F(X)) = \mathcal{B}(2^X)$ remains true for the pseudoarc is still an open question.

Definition 5. Let X be a continuum. A set $A \subseteq X$ is called a *shore* set if there exist subcontinua $K_n \subseteq X \setminus A$ such that the sequence (K_n) converges to X in 2^X with respect to the Vietoris topology.

Answering a question of [19] we proved that in a smooth dendroid the union of finitely many disjoint closed shore sets is a shore set [B]. This forms a direct continuation of the research initiated by Illanes and subsequently by van Nall [54, 71, 72].

1.3 Incomparable compactifications of the ray

In [C], we study compactifications of the ray (i.e. the space $[0, \infty)$) whose remainder is a fixed Peano continuum X. Such a compactification is called a *spiral over X*. The main result of our paper [C] follows.

Theorem 6. *Let X be a non-degenerate Peano continuum. Then there is a family of continuum many spirals over X each pair of which is incomparable by continuous mappings (i.e. there is no continuous surjection between two distinct elements in the family).*

The paper [C] follows some ideas of [74] where topological properties of spirals over a circle were revisited. Namely, there is a similarity in the general strategy formed by a reduction of a topological problem through analysis and infinite combinatorics to a set theoretical problem of some almost disjoint system. Similar type of construction appeared recently e.g. in a very far context of Ψ-spaces [13]. Identifying big classes of topologically distinct objects has a long history with its origins arguably in the work of Waraszkiewicz [77]. Also this topic was recently investigated by Minc [69] in the context of spirals over arbitrary non-degenerate continua.

1.4 Compactifiable classes of compacta

Motivated by a paper of Minc [69] we introduced several new notions in [D] and we studied their properties systematically.

Definition 7. Let $\mathcal{C}$ be a class of compact metrizable spaces. We say that $\mathcal{C}$ is *compactifiable* if there is a compact metrizable space $K \subseteq X \times Y$ such that every space in $\mathcal{C}$ is homeomorphic to some vertical section $K_x = \{y \in Y : (x, y) \in K\}$ and vice versa. Similarly we say that a class $\mathcal{C}$ is *strongly compactifiable* if there is a compact subset $\mathcal{K}$ of the hyperspace of the Hilbert cube, such that every element of $\mathcal{C}$ is homeomorphic to some element of $\mathcal{K}$ and vice versa.

We proved in [D] that there is a surprisingly close connection of these notions to *analytic* sets and also to some low *Borel classes*. This provides an interesting interplay between CT and DST.

Theorem 8. *Let $\mathcal{C}$ be a class of metrizable compacta. If $\mathcal{C}$ is compactifiable, then the set of all compacta in $[0,1]^{\mathbb{N}}$ that are homeomorphic to some member of $\mathcal{C}$ is analytic. On the other hand if $\mathcal{C}$ is an analytic set (as a subset of the hyperspace of the Hilbert cube) and contains all Peano continua, then $\mathcal{C}$ is compactifiable.*

Especially, by the theorem above, the class of all Peano continua is compactifiable, since it is known to be Borel. In spite of the fact that it is easy to prove that strongly compactifiable classes are compactifiable, it is not known whether the converse implication holds. Possible class that distinguishes these notions could be the class of Peano continua which is known to be compactifiable but not known to be strongly compactifiable (Question 4.20 in [D]).

2 Topological Dynamics

The central concept of TD is (classical topological) *dynamical system*. A dynamical system is a compact metrizable space with a continuous map of that space into itself. Two dynamical systems are called *conjugate* if there is a homeomorphism of the underlying spaces that preserves the dynamics. The most basic dynamical property of a point is being *fixed*. By Brouwer's fixed point theorem such points are always available when dealing with continuous selfmaps of an n-dimensional cell and one can easily deduce that consequently every *absolute retract* has the fixed point property as well.

A complete opposite to a dynamical system with a fixed point is a *minimal system*, that is, a dynamical system in which the (forward) orbit of every point is dense. A canonical example is an irrational rotation of the circle or generally rotations of compact *monothetic groups* (these include e.g. p-adic integers). There are many examples on the Cantor set such as the Devaney systems. It is known that minimal homeomorphisms of the Cantor set form a coanalytic and non-Borel set [53], which makes them difficult to handle. Minimal dynamical systems form the smallest building blocks. Indeed, as a consequence of Zorn's lemma every dynamical system contains a minimal subsystem.

The *entropy* of a dynamical system is a quantitative way of describing the complexity of the dynamics. It measures the exponential growth-rate of the number of far orbits. In the case of dynamical systems on a graph, entropy can be expressed using the notion of a *horseshoe* [63], which makes entropy more accessible.

Dynamical systems are naturally generalized to continuous actions of topological groups or semigroups. These naturally occur e.g. as full homeomorphism groups of a compact space acting on themselves. Even the existence of a (common) fixed point is then much more complicated since we need to reflect both the structure of the underlying space as well as algebraic properties of the (semi)group.

2.1 Fixed points of continuous group actions

The existence of fixed points in the realm of continuous selfmaps of low dimensional continua requires some sort of acyclicity. This suggests to consider the classes of chainable continua, dendrites, dendroids, λ-dendroids, uniquely arcwise connected continua and tree-like continua. Surprisingly, tree-like continua do not have the fixed point property [15]. The original example was recently revisited and simplified [48].

There is a fairly general question on what groups or semigroups acting on which spaces always produce a fixed point. There is a huge number of results and one can hardly expect to obtain a full characterization. We contributed to this area in [E] by the following theorem and by a systematic search for results of this kind.

Theorem 9. *Every compact group action on a uniquely arc-wise connected continuum has a fixed point.*

As noted by Boroński, there exists a pair of commuting homeomorphisms of a chainable continuum without a common fixed point [21]. His result is an inverse limit construction of an inverse sequence formed by functions constructed independently by Boyce and Huneke [23,51]. Thus even the abelian group $\mathbb{Z}^2$ can act on a chainable continuum without having a fixed point. On the other hand, every action of $\mathbb{N}$ on a chainable continuum has a fixed point [70].

Invariant measures plays an important ingredient in the proof of Theorem 9. It can be considered as a generalization of a fixed point. A formal similarity of the notation $f[\mu] = \mu$ is not the only reason: more importantly, if the invariant measure is a Dirac measure, then the exceptional point is indeed a fixed point. There is a classical result of Krylov and Bogolyubov that every continuous self-map of a compact metrizable space admits an invariant measure [62]. This was extended soon after to continuous actions of amenable groups (and thus covering all compact groups and all abelian groups). Nowadays, a full characterisation is available: a topological group G is amenable if and only if every continuous action of G on a compact metrizable space admits an invariant measure [45].

2.2 Rectification of piecewise monotone maps with positive entropy

In 1980's, Milnor and Thurston developed the celebrated *kneading theory*. By this theory one can assign to a piecewise monotone function a matrix (with formal power series in each entry) whose properties capture the dynamical spirit of the function. They also proved that a piecewise monotone transitive map with positive topological entropy can be conjugate to a map with constant slope (which equals to the exponential of the topological entropy) [68]. This last result was later generalized to trees [9] and recently to finite graphs [2]. A similar type of result was obtained by Bobok for countably piecewise monotone maps on the interval [18]. We extended these results in [G] in both directions covering the case for graphs (and in fact all *tame graphs*) as well as the countably piecewise monotone case. To this end we introduced the class of tame graphs.

Let us denote by $E(X)$ the set of all *end points* of a continuum X (i.e. points of order one) and by $B(X)$ the set of all *branch points* (i.e. points of order at least three).

Definition 10. A continuum X is called a *tame graph* if the set $E(X) \cup B(X)$ has countable closure.

This new class of tame graphs includes not only all graphs but also some infinite dendrites, which makes our results more general.

2.3 Minimal maps on continua

The question whether a given compact space admits a minimal homeomorphism turned out to be a very difficult task and thus even singular results in this area are of high importance. Handel constructed a minimal homeomorphism of the pseudo-circle [46]. A minimal noninvertible map on the pseudocircle was constructed by Boroński, Kennedy and Oprocha [22]. On the other hand Kolyada, Snoha and Trofimchuk observed that for every minimal map there is a comeager set of points whose preimages are singletons [59]. Roughly speaking a minimal map is always quite close to a homeomorphism. To the contrary, a continuum with a cut point does not admit a minimal homeomorphism [42]; recently this was extended to mappings instead of homeomorphisms [65].

Auslander conjectured that no non-degenerate non-separating plane continuum admits a minimal map. Even though this is a weakening of the popular and longstanding conjecture that every non-separating plane continuum has the fixed point property, it has not been solved yet.

The complete description of spaces admitting a minimal homeomorphism was given by Blokh, Oversteegen and Tymchatyn within the realm of 2-dimensional manifolds (with or without boundary) [17] and by Balibrea and his coauthors within almost zero-dimensional spaces (i.e. spaces in which the union of degenerate components is dense) [10].

The main result of [F] is a an answer to a question of Artigue [8] dealing with minimal maps. We proved the following theorem.

Theorem 11. *There is a minimal homeomorphism f of a Peano continuum X, a dense G_δ set $E \subseteq X$ and $\varepsilon > 0$ such that for every subcontinuum $K \subseteq X$ which intersects E there is some $n \in \mathbb{N}$ for which $diam(f^n(K)) > \varepsilon$.*

The idea of our proof is quite simple: we consider an irrational rotation of the two-dimensional torus and then we find a suitable extension on the Sierpiński T_2-set (which already is a one-dimensional continuum).

Several properties similar to minimality and transitivity were recently systematically studied in [1].

3 Descriptive Set Theory

Central object of DST is *Polish space* (i.e. a space that is completely metrizable and separable) and its *definable subsets*. The measurable space obtained by taking the σ-algebra of Borel sets of a Polish space (the so-called standard Borel space) is also of interest very often. The absence of topology collapses almost everything. Indeed, up to isomorphism there is only one uncountable standard Borel space and its cardinality is $|\mathbb{R}|$. A σ-ideal is a collection of sets that is closed under countable unions and also closed with respect to subsets. The main examples of σ-ideals in this context are measure zero sets in measure spaces and meager sets in topological spaces.

A *Polish group* is a group endowed with a Polish topology, such that the group operations are continuous. There are many examples occuring e.g. as homeomorphism groups of compact metric spaces. One of the most important problems in this field is the *Topological Vaught conjecture*, which states that a continuous action of a Polish group on a Polish space has either countably many orbits or there are perfectly many of them [14]. A limitation for a possible counterexample was given by Burgess who proved that an analytic equivalence relation on a Polish space has either countably many, ω_1 or perfectly many equivalence classes [25].

3.1 Haar meager sets in Polish groups

In 1972, Christensen defined *Haar null* sets in Polish groups as a generalization of Haar measure zero sets for locally compact Polish groups [32]. This notion turned out to be useful in several applications, e.g. in some differentiability results on real functions [78] or in dynamical systems [52]. Several decades later, Darji established *Haar meager* sets as a topological counterpart to Haar null sets [36].

Definition 12. A set X in an abelian Polish group G is called *Haar meager* if there exists a Borel set $B \supseteq X$ and a continuous map f of a compact metrizable space K to G such that $f^{-1}(x + B)$ is meager for every $x \in G$.

We extended the above definition naturally to the nonabelian case by assuming that the preimages of all two-sided translates are meager and we proved that Haar meager sets form a σ-ideal and that they are meager in every Polish group [H]. It is a folklore that the real line (or a locally compact Polish group) can be decomposed into the union of a meager set and a (Haar) measure zero set. We proved the following analogy answering partially a question of Jabłońska [56].

Theorem 13. *Every uncountable abelian Polish group which has a countable local base at the identity element formed by clopen subgroups (e.g. the group $\mathbb{Z}^{\mathbb{N}}$) can be decomposed into the union of a Haar meager set and a Haar null set.*

It is still not known whether such a decomposition can be found in any uncountable Polish group even in the abelian case. Also it is not known whether compact sets in non-locally compact Polish groups are Haar meager. A partial positive answer holds for groups admitting *two-sided invariant* metric.

A lot of attention has been paid to whether the definition is optimal. Recently it was proved that the witnessing function f in Definition 12 cannot be equivalently supposed to be one-to-one [40]. We proved that considering the Borel hull

B in the definition of Haar meager sets is essential, or at least some other sort of definability is needed to get an equivalent notion [H], [38].

Some further recent research includes a common generalisation of Haar meager and Haar null sets, termed Haar-$\mathcal{I}$ sets, with respect to a general σ-ideal $\mathcal{I}$ [11]. It seems that the notion of *generic* Haar meager sets might become of higher importance in the future since it avoids some pathological behavior of Haar meager sets. A recent comprehensive survey paper on Haar meager and Haar null sets can be found in [39].

3.2 Complexity of C_p-spaces

Classification of C_p-spaces with *pointwise topology* turned out to be a difficult problem. For example, it is not even known whether some of the spaces $C_p(2^\omega)$, $C_p([0,1])$ and $C_p([0,1]^2)$ are homeomorphic [66, Question 2.14]. Recently, the first separable metrizable space X was found for which $C_p(X)$ is not homeomorphic to its own square [60]. To this end the rigid *Bernstein set* can be used. This shows how difficult it is to handle the topology of pointwise convergence. The question whether there is a compact metrizable space X with the same property remains unsolved.

Let us consider the set $C_p(X)$ of all continuous real-valued functions on X as a measurable space endowed with the σ-algebra of Borel sets that are generated by the topology of pointwise convergence on $C_p(X)$. In spite of the fact that the topology of $C_p(X)$ is usually non-metrizable, the corresponding measurable space can be considered as a subspace of a standard Borel space for a separable metrizable space X. Indeed, the set $C_p(X)$ can be embedded into $\mathbb{R}^D$ for some countable dense set $D \subseteq X$ and this embedding is known to be measurable [7].

Some sort of classification (with respect to the *Borel-Wadge hierarchy*) of the measurable space $C_p(X)$ is already an accessible task if we consider reasonable assumptions. If X is σ-compact then $C_p(X)$ as well as $C_p^*(X)$ is standard Borel and if X is Σ^1_1 (i.e. analytic) but not σ-compact then $C_p(X)$ as well as $C_p^*(X)$ is Borel-Π^1_1-complete [33], [37]. Joining the results [7] and [I] we obtain the following theorem that assumes the axiom of *projective determinacy* (PD).

Theorem 14. *Suppose PD holds and let X be a separable metrizable projective space which is not in Σ^1_1. Let $n \geq 2$ be the first for which X is in Σ^1_n. Then the measurable space $C_p(X)$ as well as $C_p^*(X)$ is Borel-Π^1_n-complete.*

The assumption of PD is a natural one here. Indeed, the only consequence of PD used in the proof of Theorem 14 is that a subset of a Polish space which is not in Π^1_n is Σ^1_n-hard. It should be noted here that it is not known whether PD is relatively consistent with ZFC.

3.3 Classification of metrizable compacta up to homeomorphism

One of the most basic characteristics of a category of objects is the set theoretical cardinality of the maximal family of non-isomorphic objects (this is what we dealed with in Section 1.3). In this section we deal with a deeper study of complexity (sometimes called *definable cardinality*) expressed by the following definition.

Definition 15. Suppose that X and Y are standard Borel spaces and let E and F be equivalence relations on X and Y respectively. We say that E is *Borel reducible* to F, and we denote this by $E \leq_B F$, if there exists a Borel mapping $f\colon X \to Y$ such that

$$xEx' \iff f(x)Ff(x'),$$

for every $x, x' \in X$. The function f is called a *Borel reduction.* We say that E is *Borel bireducible* with F, and we write $E \sim_B F$, if E is Borel reducible to F and F is Borel reducible to E.

The intuitive meaning of $E \leq_B F$ from the definition above is that once we can *classify* points in Y up to the equivalence relation F, we can also *classify* points in X up to the equivalence relation E. A concrete example of such a result is a consequence of the Banach-Stone theorem [75]:

Proposition 16. *The homeomorphism equivalence relation of compact metrizable spaces is Borel reducible to the isometry equivalence relation of separable Banach spaces.*

Results of this kind are usually stated in such a vague form and some sort of *coding* is needed for a formal statement. In this case one can consider the hyperspace of the Hilbert cube as the set of codes for compact metrizable spaces and the set of all closed subspaces of the universal space $C([0,1])$ space equipped with the Effros Borel structure to formalize Proposition 16. The converse of that result is also true: the isometry equivalence relation of separable Banach spaces is Borel reducible to the homeomorphism equivalence relation of compact metrizable spaces. It should be noted that in spite of the fact that the first reduction is given naturally by assigning the Banach space $C(K)$ of all real valued continuous functions to a given compact metrizable space K, the other one is not known to have a simple natural description and it follows e.g. by a combination of the main results of Zielinski [79] and Melleray [67]. We can conclude that classification of compact metrizable spaces up to homeomorphism is of the same *complexity* as classification of separable Banach spaces up to isometry.

A systematic search for classification results of this kind is a fresh part of DST with many applications. In [J] we were dealing with classification results of some metrizable compacta up to homeomorphism. Among others, we proved the following result there.

Theorem 17. *The homeomorphism equivalence relation of absolute retracts is Borel bireducible to the universal orbit equivalence relation.*

This extends and simplifies recent similar results for continua and locally connected continua by Chang and Gao [30] and Cieśla [34]. We also identified the complexity of a class of one-dimensional spaces.

Theorem 18. *The homeomorphism equivalence relation of metrizable rim-finite continua is classifiable by countable structures (i.e. Borel reducible to the S_∞-universal orbit equivalence relation).*

This extends a similar result for dendrites by Camerlo, Darji and Marcone [28]. Zero-dimensional compacta are known also to be classifiable by countable structures [29]. Thus it was a surprise to observe that the change from continua to compacta increases the complexity, i.e. the homeomorphism equivalence relation of rim-finite compacta is not classifiable by countable structures [J].

Recently Chang and Gao dealt with the homeomorphism equivalence relations for compacta that are embedded into an n-dimensional cell [31]. However, one of the main questions whether there is a Borel reduction that is significantly decreasing topological dimension remains open.

References

[1] Akin, E., Auslander, J., and Nagar, A. Variations on the concept of topological transitivity. *Studia Math. 235*, 3 (2016), 225–249.

[2] Alsedà, L., and Misiurewicz, M. Semiconjugacy to a map of a constant slope. *Discrete Contin. Dyn. Syst. Ser. B 20*, 10 (2015), 3403–3413.

[3] Anderson, D. Shore and non-block points in Hausdorff continua. *Topology Appl. 193* (2015), 152–161.

[4] Anderson, D. A continuum without non-block points. *Topology Appl. 218* (2017), 42–52.

[5] Anderson, D. The shore point existence problem is equivalent to the non-block point existence problem. *Topology Appl. 262* (2019), 1–10.

[6] Anderson, D., and Bankston, P. Extreme points of a continuum. *Topology Appl. 257* (2019), 106–121.

[7] Andretta, A., and Marcone, A. Pointwise convergence and the Wadge hierarchy. *Comment. Math. Univ. Carolin. 42*, 1 (2001), 159–172.

[8] Artigue, A. Minimal expansive systems and spiral points. *Topology Appl. 194* (2015), 166–170.

[9] Baillif, M., and de Carvalho, A. Piecewise linear model for tree maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11*, 12 (2001), 3163–3169.

[10] Balibrea, F., Downarowicz, T., Hric, R., Snoha, L., and Špitalský, V. Almost totally disconnected minimal systems. *Ergodic Theory Dynam. Systems 29*, 3 (2009), 737–766.

[11] Banakh, T., Głab, S., Jabłońska, E., and Swaczyna, J. Haar-$\mathcal{I}$ sets: looking at small sets in Polish groups through compact glasses. *arXiv e-prints* (Mar 2018), arXiv:1803.06712.

[12] Bankston, P. Ultracoproduct continua and their regular subcontinua. *Topology Appl. 210* (2016), 46–62.

[13] Barriga-Acosta, H. A., and Hernández-Hernández, F. $\mathfrak{c}$-many types of a Ψ-space. *Topology Appl. 253* (2019), 1–6.

[14] Becker, H., and Kechris, A. S. *The descriptive set theory of Polish group actions*, vol. 232 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[15] Bellamy, D. P. A tree-like continuum without the fixed-point property. *Houston J. Math. 6*, 1 (1980), 1–13.

[16] Bing, R. H. Some characterizations of arcs and simple closed curves. *Amer. J. Math. 70* (1948), 497–506.

[17] Blokh, A., Oversteegen, L., and Tymchatyn, E. D. On minimal maps of 2-manifolds. *Ergodic Theory Dynam. Systems 25*, 1 (2005), 41–57.

[18] Bobok, J. Semiconjugacy to a map of a constant slope. *Studia Math. 208*, 3 (2012), 213–228.

[19] Bobok, J., Marciňa, R., Pyrih, P., and Vejnar, B. Union of shore sets in a dendroid. *Topology Appl. 161* (2014), 206–214.

[20] Bobok, J., Pyrih, P., and Vejnar, B. Half-homogeneous chainable continua with end points. *Topology Appl. 160*, 9 (2013), 1066–1073.

[21] Boroński, J. P. A note on fixed points of abelian actions in dimension one. *Proc. Amer. Math. Soc. 147*, 4 (2019), 1653–1655.

[22] Boroński, J. P., Kennedy, J., Liu, X.-C., and Oprocha, P. Minimal non-invertible maps on the pseudo-circle. *arXiv e-prints* (Oct 2018), arXiv:1810.07688.

[23] Boyce, W. M. Commuting functions with no common fixed point. *Trans. Amer. Math. Soc. 137* (1969), 77–92.

[24] Brucks, K. M., and Bruin, H. *Topics from one-dimensional dynamics*, vol. 62 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[25] Burgess, J. P. Equivalences generated by families of Borel sets. *Proc. Amer. Math. Soc. 69*, 2 (1978), 323–326.

[26] Camargo, J., Capulín, F., Castañeda Alvarado, E., and Maya, D. Continua whose hyperspace of nonblockers of $\mathcal{F}_1(X)$ is a continuum. *Topology Appl. 262* (2019), 30–40.

[27] Camargo, J., Maya, D., and Ortiz, L. The hyperspace of nonblockers of $\mathcal{F}_1(X)$. *Topology Appl. 251* (2019), 70–81.

[28] Camerlo, R., Darji, U. B., and Marcone, A. Classification problems in continuum theory. *Trans. Amer. Math. Soc. 357*, 11 (2005), 4301–4328.

[29] Camerlo, R., and Gao, S. The completeness of the isomorphism relation for countable Boolean algebras. *Trans. Amer. Math. Soc. 353*, 2 (2001), 491–518.

[30] Chang, C., and Gao, S. The complexity of the classification problem of continua. *Proc. Amer. Math. Soc. 145*, 3 (2017), 1329–1342.

[31] Chang, C., and Gao, S. The Complexity of the Classification Problems of Finite-Dimensional Continua. *arXiv e-prints* (Apr 2019), arXiv:1904.09634.

[32] Christensen, J. P. R. On sets of Haar measure zero in abelian Polish groups. *Israel J. Math. 13* (1972), 255–260 (1973).

[33] Christensen, J. P. R. *Topology and Borel structure*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1974. Descriptive topology and set theory with applications to functional analysis and measure theory, North-Holland Mathematics Studies, Vol. 10. (Notas de Matemática, No. 51).

[34] Cieśla, T. Completeness of the homeomorphism relation of locally connected continua. *Proc. Amer. Math. Soc. 147*, 3 (2019), 1269–1276.

[35] Cúth, M., Kurka, O., and Vejnar, B. Large separated sets of unit vectors in banach spaces of continuous functions. *Colloq. Math 157* (2019), 173–187.

[36] Darji, U. B. On Haar meager sets. *Topology Appl. 160*, 18 (2013), 2396–2400.

[37] Dobrowolski, T., and Marciszewski, W. Classification of function spaces with the pointwise topology determined by a countable dense set. *Fund. Math. 148*, 1 (1995), 35–62.

[38] Doležal, M., and Vlasák, V. Haar meager sets, their hulls, and relationship to compact sets. *J. Math. Anal. Appl. 446*, 1 (2017), 852–863.

[39] Elekes, M., and Nagy, D. Haar null and Haar meager sets: a survey and new results. *arXiv e-prints* (Jun 2016), arXiv:1606.06607.

[40] Elekes, M., Nagy, D., Poór, M., and Vidnyánszky, Z. A Haar meager set that is not strongly Haar meager. *arXiv e-prints* (Jun 2018), arXiv:1806.11524.

[41] Engelking, R. *General topology*, second ed., vol. 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author.

[42] Erdös, P., and Stone, A. H. Some remarks on almost periodic transformations. *Bull. Amer. Math. Soc. 51* (1945), 126–130.

[43] Escobedo, R., Estrada-Obregón, C., and Sánchez-Martínez, J. On hyperspaces of non-cut sets of continua. *Topology Appl. 217* (2017), 97–106.

[44] Gao, S. *Invariant descriptive set theory*, vol. 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[45] Grigorchuk, R., and de la Harpe, P. Amenability and ergodic properties of topological groups: from Bogolyubov onwards. In *Groups, graphs and random walks*, vol. 436 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2017, pp. 215–249.

[46] Handel, M. A pathological area preserving C^∞ diffeomorphism of the plane. *Proc. Amer. Math. Soc. 86*, 1 (1982), 163–168.

[47] Hencl, S., and Vejnar, B. Sobolev homeomorphism that cannot be approximated by diffeomorphisms in $W^{1,1}$. *Arch. Ration. Mech. Anal. 219*, 1 (2016), 183–202.

[48] Hernández-Gutiérrez, R., and Hoehn, L. C. A fixed-point-free map of a tree-like continuum induced by bounded valence maps on trees. *Colloq. Math. 151*, 2 (2018), 305–316.

[49] Hjorth, G. *Classification and orbit equivalence relations*, vol. 75 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2000.

[50] Hoehn, L. C., and Oversteegen, L. G. A complete classification of homogeneous plane continua. *Acta Math. 216*, 2 (2016), 177–216.

[51] Huneke, J. P. On common fixed points of commuting continuous functions on an interval. *Trans. Amer. Math. Soc. 139* (1969), 371–381.

[52] Hunt, B. R., Sauer, T., and Yorke, J. A. Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. *Bull. Amer. Math. Soc. (N.S.) 27*, 2 (1992), 217–238.

[53] Ibarlucía, T., and Melleray, J. Full groups of minimal homeomorphisms and Baire category methods. *Ergodic Theory Dynam. Systems 36*, 2 (2016), 550–573.

[54] Illanes, A. Finite unions of shore sets. *Rend. Circ. Mat. Palermo (2) 50*, 3 (2001), 483–498.

[55] Illanes, A., and Nadler, Jr., S. B. *Hyperspaces*, vol. 216 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1999. Fundamentals and recent advances.

[56] Jabłońska, E. Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. *J. Math. Anal. Appl. 421*, 2 (2015), 1479–1486.

[57] Katok, A., and Hasselblatt, B. *Introduction to the modern theory of dynamical systems*, vol. 54 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[58] Kechris, A. S. *Classical descriptive set theory*, vol. 156 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[59] Kolyada, S., Snoha, L., and Trofimchuk, S. Noninvertible minimal maps. *Fund. Math. 168*, 2 (2001), 141–163.

[60] Krupski, M. a., and Marciszewski, W. A metrizable X with $C_p(X)$ not homeomorphic to $C_p(X) \times C_p(X)$. *Israel J. Math. 214*, 1 (2016), 245–258.

[61] Kůrka, P. *Topological and symbolic dynamics*, vol. 11 of *Cours Spécialisés [Specialized Courses].* Société Mathématique de France, Paris, 2003.

[62] Kryloff, N., and Bogolioùboff, N. Sur les équations de Focker-Planck déduites dans la théorie des perturbations à l'aide d'une méthode basée sur les propriétés spectrales de l'hamiltonien perturbateur. (Application à la mécanique classique et à la mécanique quantique.). *Ann. Chaire Phys. Math. Kiev 4* (1939), 5–157.

[63] Llibre, J., and Misiurewicz, M. Horseshoes, entropy and periods for graph maps. *Topology 32*, 3 (1993), 649–664.

[64] Macías, S. *Topics on continua.* Chapman & Hall/CRC, Boca Raton, FL, 2005.

[65] Mai, J.-H. Minimal sets in compact connected subspaces. *Topology Appl. 158*, 16 (2011), 2216–2220.

[66] Marciszewski, W. Function spaces. In *Recent progress in general topology, II.* North-Holland, Amsterdam, 2002, pp. 345–369.

[67] Melleray, J. Computing the complexity of the relation of isometry between separable Banach spaces. *MLQ Math. Log. Q. 53*, 2 (2007), 128–131.

[68] Milnor, J., and Thurston, W. On iterated maps of the interval. In *Dynamical systems (College Park, MD, 1986–87)*, vol. 1342 of *Lecture Notes in Math.* Springer, Berlin, 1988, pp. 465–563.

[69] Minc, P. $2^{\aleph_0}$ ways of approaching a continuum with $[1, \infty)$. *Topology Appl. 202* (2016), 47–54.

[70] Nadler, Jr., S. B. *Continuum theory*, vol. 158 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1992. An introduction.

[71] Nall, V. C. Centers and shore points in λ-dendroids. *Topology Proc. 31*, 1 (2007), 227–242.

[72] Nall, V. C. Centers and shore points of a dendroid. *Topology Appl. 154*, 10 (2007), 2167–2172.

[73] Pearl, E., Ed. *Open problems in topology. II.* Elsevier B. V., Amsterdam, 2007.

[74] Pyrih, P., and Vejnar, B. Waraszkiewicz spirals revisited. *Fund. Math. 219*, 2 (2012), 97–104.

[75] STONE, M. H. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc. 41*, 3 (1937), 375–481.

[76] VAN MILL, J. *The infinite-dimensional topology of function spaces*, vol. 64 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 2001.

[77] WARASZKIEWICZ, Z. Une famille indenombrable de continus plans dont aucun n'est l'image d'un autre. *Fund. Math. 18* (1932), 118–137.

[78] ZAJÍČEK, L. On differentiability properties of typical continuous functions and Haar null sets. *Proc. Amer. Math. Soc. 134*, 4 (2006), 1143–1151.

[79] ZIELINSKI, J. The complexity of the homeomorphism relation between compact metric spaces. *Adv. Math. 291* (2016), 635–645.