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Report on the Habilitation Memoir submitted by Marek Cuth.

This Habilitation memoir presents in a very pedagogical way eight research articles produced by Marek Cuth and some co-authors. These articles focus on the class of Lipschitz-free Banach spaces, which is at this moment a quite lively field of research. Marek Cuth contributed to the recent progress on their analysis in a very significant way. Moreover his work is put here into its right context, and the introduction provides a clear and reader-friendly presentation of the whole theory. Let me know be more precise and give some details on the contents of Marek Cuth's articles.

The article A (with M. Doucha) provides a satisfactory analysis of the Lipschitz-free spaces over ultrametric spaces. If $M$ is ultrametric, then the space $\mathcal{F}(M)$ is isomorphic to $l_1$ and moreover it has a monotone basis. These results improve on a previous work by A. Dalet who showed the isomorphism to $l_1$ under the additional assumption that the ultrametric space is proper – that is, all closed balls are compact – and that such spaces have the metric approximation property, which is weaker than having a monotone basis. The approach and proofs are quite different, and the Cuth-Doucha article uses in particular the representation of a ultrametric space as a subset of a metric tree. Article B (with M. Doucha and P. Wojtaszczyk) concerns as well the structure of Lipschitz-free spaces in full generality. Namely, it is shown that if $M$ is an arbitrary infinite metric space $M$, then the Lipschitz-free space over $M$ contains a complemented copy of $l_1$, by a careful study of all the relevant cases. It is also shown that there exists a compact set $K$, which is actually a convergent sequence, such that $\mathcal{F}(K)$ is not isomorphic to a subspace of $L^1$. The proof relies on the Naor-Schechtman approach, but the authors observe that it was actually anticipated by S. Kisliakov. Finally, it is shown that if $M$ is a subset of a finite-dimensional Euclidean space, then $\mathcal{F}(M)$ is weakly sequentially complete (w.s.c.). The proof of this last result relies in particular on J. Bourgain's theorem stating that the dual of the space of $C^1$-smooth functions on the n-dimensional cube is w.s.c.

The article C (with M. Johanis) improves on some results of the previous article B, and solves a question stated there. Indeed, the first result of B has an isometric version: if $M$ is an infinite metric space, then the space of Lipschitz functions with domain $M$ contains an isometric copy of $l_\infty$ equipped with its natural norm. if $M$ is metric complete and has at least an accumulation point (in other words, $M$ is non discrete) then the free space over $M$ contains a 1-complemented isometric copy of $l_1$. Article D (with O. Kalenda and P. Kaplicky) provides an isometric representation of the space $\mathcal{F}(/Omega)$ when $\Omega$ is a convex open subset of a finite-dimensional Euclidean space $\mathbb{R}^d$. This free-space happens to be isometric to the quotient of the space of all integrable vector fields by the subspace $N$ of divergence-free vector fields.

This representation result is related to some results in a recent paper by N. Lerner and the reviewer, where it is shown in particular that the subspace $N$ is not "nicely placed", which leaves open the interesting question of 1-complementation of the space $\mathcal{F}(\Omega)$ in its bidual. Such complementation problems are investigated in the article E (again with O. Kalenda and P. Kaplicky) where it is shown through a deep analysis of finitely additive measures that the free-space over an n-dimensional normed space is complemented in its bidual. In the Euclidean case, a contractive projection exists. In general, the norm of the projection is controlled by $\sqrt{n}$ but it is not known if this estimate can be improved. This challenging direction of research aims at deciding if the free space over $l_1$ is complemented in its bidual: a positive answer would show that a Banach space which is Lipschitz-isomorphic to $l_1$ is linearly isomorphic to $l_1$.
The article F (with L. Candido and M. Doucha) investigates the existence of isomorphisms between spaces of Lipschitz functions over different metric spaces $M$. Two easy remarks should be stated at this point: first, such a space is non-separable as soon as $M$ is infinite and thus we are led to isomorphisms which are usually not explicit. Second, a rule of thumb is that metric spaces and their "discrete" versions, in Gromov's sense, have isomorphic spaces of Lipschitz functions: a typical example is provided by the real line and the set $\mathbb{Z}$ of integers. In this article, isomorphism is shown to hold for any Carnot group $G$ and the discrete finitely generated torsion-free subgroup $\Gamma$ which is a net in $G$, for $L^p$ and $l^p$ spaces, with similar results in Orlicz spaces, for the Heisenberg group over $\mathbb{R}$ and its counterpart over $\mathbb{Z}$. Note that if the spaces of Lipschitz functions are isomorphic, their preduals (the free spaces) are not: this is already true in the simplest case since of course $L^1$ and $l_1$ are not isomorphic.

The articles G and H (both with F. Albiac, J. Ansorena and M. Doucha) investigate non-locally convex situations, in the spirit of an early work by F. Albiac and N. J. Kalton. Pick any $p$ with $0<p\leq 1$, and say that $d$ is a $p$-metric if $d^p$ is actually a metric. Article G develops a theory of free spaces for $p$-metric spaces which is shown to be somewhat similar to the classical theory. For instance, $p$-metric spaces embed isometrically into the $p$-Banach spaces $\mathcal{F}_p(M)$, whose dual is isomorphic to the (possibly trivial) space of Lipschitz functions on $M$. The Banach envelope of the space $\mathcal{F}_p(M)$ is the usual free space, so for instance the space $\mathcal{F}_p(\mathbb{R})$ is not $L_p$ and it constitutes a new kind of $p$-Banach space. When $N$ is ultrametric, then $\mathcal{F}_p(N)$ is always isomorphic to $l_p$. However, a remarkable difference with the classical case concerns subsets: when $N$ is a subset of a $p$-metric space $M$, then the natural embedding from $\mathcal{F}_p (N)$ into $\mathcal{F}_p(M)$ is usually not an isometry. This answers a question from the Albiac-Kalton article. Finally, article H continues the investigation of the analogies (and differences) between the $p$-metric case and the metric case. It is shown there that $l_p$ embeds into $\mathcal{F}_p(M)$ when $M$ is infinite, but interestingly this cannot be done in general in a complemented way. Moreover, $\mathcal{F}_p([0, 1])$ is shown to have a basis but no unconditional basis and this is the first non-trivial example of such a $p$-Banach space: the Haar system is a basis, but is not unconditional since the Banach envelope $L^1$ has no unconditional basis.

In order to complete my report on M. Cuth's Habilitation Thesis, I wish to say that I have gone through the check of originality of the thesis done by the system Turnitin, and it is absolutely clear that the thesis represents an original work of high quality, with minimum overlap witht the existing litterature.

In my opinion, this Memoir is a nice and deep piece of work, which is certainly sufficient for granting the Habilitation to his author. Marek Cuth is a very active and talented member of the younger generation of researchers in analysis, and an important member of the quite productive Prague community of analysts. I congratulate him for this nice achievement.

In Paris, 5 June 2020,

Gilles Godefroy [redacted]

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