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To Professor RNDr. Jan Kratochvìl, CSc., Dean of the Faculty

Report on the Habilitation thesis of Dr. Vítězslav Kala entitled
"Universal quadratic forms over number fields"

This thesis contains interesting and original results in the arithmetic theory of quadratic forms, a rich and classical subject in number theory with roots in the works of such masters as Fermat, Lagrange, Legendre and Gauss. The seven papers that form the basis for this thesis constitute a strong and coherent body of work with clear overarching themes. These papers (references [1] through [7] in the thesis) have all appeared, or been accepted for publication, in well-respected internationally-recognized research journals. The thesis is clearly and succinctly presented, and amply demonstrates that Dr. Kala has a thorough command of the literature in this field, a high degree of technical expertise, and an excellent level of mathematical originality and insight.

By definition, a quadratic form is a homogeneous polynomial of degree 2 in n indeterminates with coefficients in some domain R. One of the fundamental problems in this subject is to describe the elements α of R that are represented by a given form f, in the sense that there exist $a_1, \dots, a_n \in R$ such that $f(a_1, \dots, a_n) = \alpha$. The most classical results on this topic are restricted to the case when R is the ring $\mathbb{Z}$ of rational integers, while work in the latter part of the 19th century provided a complete picture of the theory when R is the field $\mathbb{Q}$ of rational numbers. The 11th of the 23 famous problems posed by Hilbert at the International Congress of Mathematicians in Paris in 1900 was essentially to develop the theory of quadratic forms "with any algebraic numerical coefficients". This theory has subsequently been satisfactorily developed when the coefficient domain is an algebraic number field K (i.e., a finite extension of $\mathbb{Q}$). However, there are still many interesting unsolved problems when the coefficient domain is restricted to be the ring of algebraic integers $\mathcal{O}_K$ in such a field K, and some of the remaining questions there are notoriously challenging, even over $\mathbb{Z}$. The work in this thesis makes a valuable contribution to the understanding of the representation theory of quadratic forms over the rings of integers of totally real algebraic number fields, and opens up new avenues of inquiry regarding such forms.

In order to discuss the results of this thesis in more detail, it is helpful to establish some conventions. Throughout this discussion, K will always denote a totally real algebraic number field, $\mathcal{O}_K$ the ring of algebraic integers of K, and $\mathcal{O}_K^+$ the set of totally positive elements of $\mathcal{O}_K$. The word "form" will always refer here to a totally positive integral quadratic form over $\mathcal{O}_K$. That is, a form with coefficients in $\mathcal{O}_K$ for which every value represented by the form at points of $\mathcal{O}_K^n$ lies in $\mathcal{O}_K^+$. Such a form f is said to be diagonal (or classical, respectively) when all the cross-term coefficients of f equal 0 (lie in $2\mathcal{O}_K$, respectively). Finally, a form is said to be universal if it represents every element of $\mathcal{O}_K^+$.

The systematic search for universal forms over $\mathbb{Z}$ was initiated in a paper of Ramanujan in 1916, and came to fruition in a companion pair of remarkable papers, one by Conway and Schneeberger in 1993, which treats the case of classical forms, and the other by Bhargava and Hanke, which treats the general case and has not yet appeared in print. The understanding of universal forms over the rings of integers of other totally real number fields is much less complete, and progress in this direction has been comparatively recent, with the first definitive result appearing in a paper of Chan, M.-H. Kim and Raghavan in 1996. Following the notation of Dr. Kala's thesis, let $m(K)$ and $m_{diag}(K)$ denote the minimal rank of a universal form over K and a diagonal universal form over K, respectively. A general goal is to understand the behavior of $m(K)$ or $m_{diag}(K)$ as K varies over totally real number fields, or over some tractable subfamily of these fields. A pair of papers by B.M. Kim in 1999 and 2000 provide some partial information on the values of $m_{diag}(K)$ for real quadratic fields. For example, by combining results from these two papers it follows that there exist infinitely many real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ such that $m_{diag}(K) = 8$. Aside from some isolated results on the universality of a few specific forms over specific fields, as surveyed in Dr. Kala's thesis, this was essentially the state of knowledge on this subject prior to the work of Dr. Kala and his co-authors in 2015.

As pointed out by Dr. Kala, it follows from a theorem of Hsia, Kneser and Kitaoka that there exist universal forms over every totally real number field K. However, that result provides no information on the size of $m(K)$ for any K. For example, it leaves open the possibility that $m(K)$ could be bounded from above by some large constant for all K. The ground-breaking papers [1] and [2] of Dr. Kala (the first joint with Blomer) show that this cannot be the case, even within the family of real quadratic fields. Specifically, it is shown that for any real number M, there exist infinitely many real quadratic fields K such that any universal form over $\mathcal{O}_K$ must have rank exceeding M. In [1], it was necessary to impose the restriction to classical forms, and the necessary construction was accomplished with a considerable amount of delicate technical arguments. In the beautiful paper [2], the restriction to classical forms is eliminated, so that the full conclusion $m(K) > M$ is reached. Moreover, many of the technical aspects of the previous paper are replaced by more conceptual arguments. In joint work with Svoboda [6], this result was further extended from quadratic extensions to the case of multiquadratic extensions of any fixed degree.

The following key observation forms the basis for the arguments in these first two papers, and sets the stage for much of the remainder of the work described in the thesis: totally real fields K with large values of $m(K)$ can be constructed by finding K's with many independent additively indecomposable integers. This led, in particular, to a more detailed investigation of such indecomposables undertaken in the papers [3] and [7]. The main reason that most of the progress that has been made here is restricted to the class of real quadratic fields is that for such fields the indecomposables of $\mathcal{O}_K^+$ can be explicitly described and studied in terms of coefficients of a certain periodic continued fraction expansion. In the paper [3], this characterization is exploited in order to give explicit formulas for the norms of indecomposable integers, leading to a counterexample to a conjecture formulated by Jang and B.M. Kim in 2016 regarding a possible refinement for known upper bounds for such norms. A detailed study of the structure of the additive semigroup of $\mathcal{O}_K^+$ in the real quadratic case was undertaken in [7], in joint work with Hejda. The results here are quite interesting, both from the perspective of the abstract theory of semigroups and for their application in proving that this semigroup uniquely determines the real quadratic field and in characterizing the uniquely decomposable elements.

In the paper [4], also written jointly with Blomer, upper and lower bounds for $m_{diag}(\mathbb{Q}(\sqrt{D}))$ are determined in terms of the sums of various coefficients appearing in the periodic continued fraction expansion of $\sqrt{D}$. This gives rise to an asymptotic formula for the sum of these coefficients, which can be viewed as a variation of Kronecker's limit formula for real quadratic fields. This reveals deep connections between special L-values and continued fractions. The results in that paper further suggest an intriguing connection between $m(K)$, $m_{diag}(K)$ and the class number h_K of K. This connection is investigated in fuller detail in the paper [5], written jointly with Dahl, leading to asymptotics for the growth of class numbers within continued fraction families of real quadratic fields (that is, families in which the period length and specified coefficients in the continued fraction expansion are fixed).

In my opinion, the work contained in this thesis represents a substantial contribution that is in the mainstream of an active area of number theory firmly grounded in a rich classical tradition. I am confident that the results contained in this thesis will serve to stimulate future work in this field for many years to come (in fact, this has already proven to be the case for the earlier papers in the collection). While it is perhaps not directly germane to the evaluation of this thesis, it is noteworthy that the results contained therein constitute only a portion of Dr. Kala's research output to date, which spans a wide range of topics in algebra and number theory, and demonstrate that his research program has progressed well beyond the subject of his doctoral work.

I have carefully reviewed the check of originality of the thesis conducted by the Turnitin system, and find that it confirms my conviction that this thesis represents original work with minimal overlap with the existing literature.

In summary, it is my opinion that this work constitutes an Habilitation thesis of high quality, and I recommend that it be defended without major revision.

Sincerely yours,

[redacted]

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