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Report on the Habilitation Thesis of Mgr. Vítězslav KALA

to

the dean of the Faculty of Mathematics and Physics,
Charles University,
prof. RNDr. Jan Kratochvíl

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To begin with it is worth mentioning that the short 12 pages long presentation of Mgr. Vítězslav KALA's Thesis is very clear and make you feel like having an enthusiastic close look at his 7 published papers hinted at in this Introduction. I would also like to point out that these 7 papers referred have been published in very good mathematic journals during the last five years, which attests to a hard working and successful candidate.

The unifying topic of the 7 published papers listed in the Habilitation is the notion of universal classical positive-definite quadratic form, where classical means that it comes from an integer symmetric square matrix.

So, let $Q(x_1,\cdots,x_r) = \sum_{1\leq i,j\leq n} q_{i,j}x_ix_j \in \mathbb{Z}[x_1,\cdots,x_r]$ be a classical positive-definite quadratic form where the off-diagonal terms $q_{i,j}$ for $i \neq i$ are even, i.e. the quadratic form is associated with a symmetric square matrix in $M_r(\mathbb{Z})$. Then it is said to be universal if it represents every positive integer. For example, Lagrange's four square theorem asserts that $Q(x,y,z,t) = x^2+y^2+z^2+t^2$ is universal, i.e. that every positive integer is the sum of four squares. The most remarkable result in this area is the 1993 Conway and Schneeberger's Theorem according to which a positive-definite quadratic form having integer matrix represents every positive integer up to 15 then it is universal, i.e. represents every positive integer. (Notice that the positive-definite quadratic form $Q(x,y,z,t) = x^2+2y^2+5z^2+5t^2$ represents every positive integer up to 14 but does not represent 14.)

It is then natural but much more difficult to deal with the same questions for positive-definite quadratic forms with coefficient in the ring of algebraic integers $\mathbb{Z}_{\mathbb{K}}$ of a totally real number field $\mathbb{K}$. In that situation of critical importance is the notion of indecomposable totally positive algebraic integers $\alpha \in \mathbb{Z}_{\mathbb{K}}^{+}$, those that cannot be decomposed as a sum $\alpha = \beta + \gamma$ of two totally positive elements $\beta, \gamma \in \mathbb{Z}_{\mathbb{K}}^{+}$. The idea being that if an indecomposable element $\alpha \in \mathbb{Z}_{\mathbb{K}}^{+}$ has a representation by a diagonal positive-definite quadratic form $\alpha = \sum_{i=1}^{r} \alpha_i x_i^2$ with coefficients $\alpha_i \in \mathbb{Z}_{\mathbb{K}}^{+}$, then $\alpha = \alpha_i x_i^2$ for some i. Thus the number of different classes of indecomposables modulo squares gives a lower bound for the rank r. Let $m(\mathbb{K})$ be the minimal rank r of a universal totally positive quadratic form over $\mathbb{K}$.

The first author's paper (in the Math. Proc. Cambridge Philo. Soc. in 2015) is devoted to proving that for every M there are infinitely many real quadratic fields $\mathbb{K}$ such that every classical universal quadratic form over $\mathbb{K}$ has rank greater than M. In his second paper (in the Bull. Aust. Math. Soc. in 2016) the candidate managed to dispense with the restriction that the form be classical. He therefore obtains that for every M there are infinitely many real quadratic fields $\mathbb{K}$ such that $m(\mathbb{K}) \geq M$. Then, in his fourth paper (in Doc. Math. in 2018) he obtains good estimates on the minimal rank $m_{diag}(\mathbb{K})$ of universal totally positive diagonal quadratic forms

During his research the candidate has essentially only dealt with real quadratic number fields where he could use continued fractions to understand the behaviors of the indecomposable totally positive algebraic integers of $\mathbb{K}$. As mentioned in the conclusion of his thesis, the candidate's current research is to better understand this behavior for general totally real number fields.

I would like to also mention that the candidate's work bears on both the algebraic and analytic aspects of number theory. Whereas the first four papers are mainly algebraic, his fifth paper (in the Proc. Edinb. Math. Soc. in 2018) is purely analytical. A lot is known about class numbers of imaginary quadratic fields. For example, (i) their class numbers go to infinity with their discriminants, by the Brauer-Siegel theorem or by the Siegel-Tatuzawa theorem and (ii) all the imaginary quadratic fields of small class numbers are known. In contrast, not much is known about class numbers $h(d)$ of real quadratic number fields $\mathbb{Q}(\sqrt{d})$, where extracting information from the analytic class number formula is difficult since not much is known about the size of the fundamental unit. One way to get around this problem is to focus on families of real quadratic fields of known fundamental unit. A. Dahl and Y. Lamzouri investigated the distribution of $h(d)$ for

$$d \in \mathcal{D}_{ch} := \{d;\ d \text{ squarefree of the form } d = 4m^2 + 1 \text{ for } m \geq 1\}$$

in which case $2m + \sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for $m > 1$. They proved that for x large and $1 \leq \tau \leq \log\log x - 3\log\log\log x$ the numbers of discriminants $d \leq x$ in $\mathcal{D}_{ch}$ such that

$$h(d) \geq 2e^{\gamma} \frac{\sqrt{d}}{\log d} \cdot \tau \text{ or } h(d) \leq 2e^{-\gamma}\zeta(2)\frac{\sqrt{d}}{\log d} \cdot \frac{1}{\tau}$$

are both equal

$$(\#\mathcal{D}_{ch} \cap (1,x]) \cdot \exp\left(-\frac{e^{\tau - C_0}}{\tau}\left(1 + O\left(\frac{1}{\tau}\right)\right)\right),$$

where $C_0 = 0.8187\cdots$.

In the present paper the authors extend this distribution result to a much larger family of real quadratic fields whose fundamental units are likewise known beforehand and are as small as possible. This larger family is the one for which the continued fraction expansion of the canonical generator

$$\omega_d = \frac{\epsilon_d + \sqrt{d}}{1 + \epsilon_d}$$

of the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$ is of the form

$$\omega_d = [k, \overline{u_1, \cdots, u_{s-1}, 2k - \epsilon_d}],$$

where $u_1, \cdots, u_{s-1}$ is a given symmetric sequence of positive integers and $\epsilon_d = 0$ or 1 according as $d \equiv 2, 3 \pmod 4$ or $d \equiv 1 \pmod 4$.

Even though Mgr. Vítězslav KALA is not the only author of some of the 7 papers referred to in his Thesis it is quite clear to me that they all deal with Mgr. Vítězslav KALA's unifying topic and that he has done more than his share while writing them.

In conclusion, I have a very good opinion of the work of Mgr. Vítězslav KALA presented in the present Thesis. The results are original, new, interesting and nicely proved. Moreover he clearly explains that the subject is far from being closed and nice follow-up work should appear soon.

I have gone through the check of originality of the thesis done by the system Turnitin and it is absolutely clear that the thesis represents an original work with minimum overlap with the existing literature.

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