SCHOOL OF THEORETICAL & APPLIED SCIENCE
505 Ramapo Valley Road, Mahwah, NJ 07430-1680
Phone (201) 684-7734 Fax (201) 684-7637
www.ramapo.edu

TO: Jiri Rosicky and committee

FROM: James Gillespie (Professor of Mathematics, Ramapo College)

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RE: Report on Jan's Saroch's Habilitation Thesis "Constructions and deconstructions of locally well-behaved complex objects"

Dear Committee,

The papers making up Jan Saroch's Habilitation thesis are an excellent representation of his contributions to homological algebra and representation theory. First, let me say that I had already been familiar with some of Saroch's work as he has developed a reputation in the field as a leading mathematician who has proved some very important theorems in our field of (set theoretic) homological algebra. In particular, I was already very familiar with his contributions to Gorenstein homological algebra which I will comment on further below.

Saroch's work is heavily set theoretic, utilizing techniques from model theory. One reason he has seen further than others and proved some very important theorems in our field is precisely because he has brought in and applied these new techniques. He has not only mastered these (difficult) techniques but he has been at the forefront in the last decade in pushing the limits of these techniques and applying them to our field with his coauthors. In other words, he is adept at applying these techniques to real mathematical problems that have been attempted by many mathematicians and bringing complete and satisfactory answers to several open questions.

For example, the single authored article *On the non-existence of right almost split maps* was rightfully published in Invent. Math. It gives a beautiful answer to a classic and easily stated question of M. Auslander concerning the almost split sequences that have become essential to Auslander-Reiten theory. Auslander showed that a right almost split map f : B →C with C finitely presented must have End_R(C) (its endomorphism ring), local. Auslander asked the question of wanting a precise description of the R-modules that can possibly appear as the codomain of a right almost split map. Jan Saroch answers this in the paper, showing that a right almost split map f : B →C MUST have C finitely presented (and End_R(C) local). Moreover, it then follows that the kernel of f must be a pure-injective module. Although the question, and its answer, are natural enough, the solution is entirely nontrivial. Saroch's solution is the result of a deep analysis of an instance of a combinatorial construction called a tree module which serves as a test module for the splitting of a right almost split map f : B → C (where C is a k-presented module for some infinite cardinal k). Along the way to constructing the technical machinery, he also proves a very nice characterization of cotorsion modules over countable rings R with R^{\omega} flat and Mittag-Leffler in Theorem 1.3.2. This answers a question about cotorsion abelian groups posed by George M. Bergman.

As another example, in *Approximations and Mittag-Leffler conditions – the tools*, Saroch develops new tools combining relative Mittag-Leffler conditions with set theoretic homological algebra and proves two major results. First, that the class of all flat Mittag Leffler R-modules is precovering if and only if R is right perfect. This is a great improvement to an earlier result that assumed the ring was countable. Second, the tools he develops allow him to prove that if (A,B) is a cotorsion pair and B is closed under direct limits, then (A,B) is of countable type, (and that B is a definable class). This too is an extension of an earlier result known as the Countable Telescope Conjecture. Moreover, in the second paper *Approximations and Mittag-Leffler conditions – the applications* Saroch and coauthors Lidia Hügel and Jan Trlifaj take a big step in the validity of a famous conjecture of Edgar Enochs (see Theorem 3.5.2). Enochs conjectured that if a class A of R-modules is a covering then it is closed under direct limits. Using tools developed by Saroch, they show this to be true when the class A fits into a cotorsion pair (A,B) with B closed under direct limits. This applies to any tilting cotorsion pair!

The aspect of Jan Saroch's work that I was already most familiar with before doing the current review of his habilitation thesis, and which has personally impressed me the most, are his contributions to Gorenstein homological algebra with coauthor Jan Stovicek. The ideas in this field go back to (Buschweitz, Auslander, and even Tate) but in more recent years Edgar Enochs and many coauthors gave the modern definitions of Gorenstein projective, Gorenstein injective, and Gorenstein flat modules. Over Gorenstein rings these modules provide a perfect relative homological algebra known as Gorenstien homological algebra. In particular, just like the usual idea of taking projective (or injective or flat) resolutions of an R-module leads to the usual theory of homological algebra we can take Gorenstein projective (or injective or flat) resolutions which leads to a rich theory with other applications. It has been known for some time that over Noetherian rings we have Gorenstein injective resolutions, and over coherent rings we have Gorenstein flat resolutions. The basic question of Gorenstein homological algebra has always been: For what rings R do we have Gorenstein injective (or projective or flat) resolutions.

In the paper "Singular compactness and definability for S-cotorsion and Gorenstein modules" written jointly with Jan Stovicek, we received a breakthrough in Gorenstein homological algebra. They show us that over any ring R, we do indeed have Gorenstein injective resolutions and Gorenstein flat resolutions (the projective case is still unknown in general, but is discussed a bit below). In fact, Hovey and I were convinced that finiteness conditions on the ring R (such as R being Noetherian or coherent) were essential to whether or not the ring would admit a nice theory of Gorenstein homological algebra. In order to get the theory to work for all general rings R, we introduced new definitions of Gorenstein AC-injective (and projective, and flat variants), which agree with the usual Gorenstein modules for nice rings. But Saroch and Stovicek showed that for ALL rings R we do still have Gorenstein injective and Gorenstein flat resolutions, without the modified definitions! Concerning the open case of projectives, so far it appears that a modified definition may need to be used. They modify the definition of Gorenstein projective to that of a "PGF module", and show that this class of modules does admit resolutions. They also give strong arguments that the PGF modules are the "correct" definition if one wants the "projectives to be flat": It follows from their work that IF all Gorenstein projectives are Gorenstein flat, then Gorenstein projective resolutions exist as they are then the PGF modules. These are all major contributions to Gorenstein homological algebra. Again, their approach is based on deep set theoretical techniques and infinite combinatorics including singular compactness, stationarity and Mittag-Leffler condition, and almost freeness.

The Introduction of Saroch's thesis includes yet another new and interesting result concerning the open problem of whether or not Gorenstein projective resolutions exist universally. The essential problem is to show that the class GP, of all Gorenstein projectives, is a deconstructible class. It is shown in Theorem 3.3 of the Introduction of the thesis that, under the existence of a strongly compact cardinal $k > |R|$, the class GP is a deconstructible class. This takes us outside of ZFC, so we would like to know if it is necessary to do this. But nevertheless, this is a very nice result.

Finally, I wish to comment on the final article of the thesis, *Test sets for factorization properties of modules,* written jointly with Jan Trlifaj. I find this article very interesting, as it examines the dual of the famous "Baer's criterion" for injectivity. Recall that Baer's criterion says that an R-module J is injective if and only if any homomorphism I → J, from an ideal $I < R$, extends to a homomorphism R → J. More generally, if a set S "generates" a cotorsion pair (X,Y), then it is standard that we can find a set of monomorphisms for which a similar extension property characterizes membership J \in Y. But for such cotorsion pairs it would be nice if there were a set of epimorphisms that can be used to test membership M \in X. In particular, for the easiest case of projective modules, there is no dual to Baer's criterion. I am sure that most homological algebraists have thought about this at some point. It becomes natural to wonder if the existence of such a test set is independent of ZFC. In the article they use the set theoretic Weak Diamond Principle and show that the dual to Baer's criterion is indepdent of ZFC + GCH for the following types of rings: (i) commutative Noetherian ring R of finite Krull dimension, (ii) Iwanaga-Gorenstein rings, and (iii) almost perfect rings. This too is a striking result!

I have reviewed the turnitin analysis on Saroch's thesis. The analysis does not contradict anything that I as an expert in the field already know to be true. That Saroch's thesis represents his original work.

I hope I have made clear the value and contributions of Jan Saroch's work. His work on (de)construction is deep and far reaching, and has been at the forefront of the field in the last decade. These contributions have considerably advanced the field. I personally worked on the same problems in Gorenstein homological algebra, so I can attest to the remarkable results proved by Saroch and Stovicek in the fourth paper of the thesis. In my view, this is a thesis of exceptional quality. Certainly, these contributions are worthy of Saroch holding the rank of full professor at any leading research university, and in particular at Charles University.

James Gillespie
[redacted]
Professor of Mathematics
Ramapo College of New Jersey