Report on: Construction and deconstructions of locally well-behaved complex objects

Habilitation Thesis

by Jan Saroch

I know Dr. Jan Šaroch and the mathematics he does since 2004, back in the early years of his Ph D period. I have read with pleasure through his habilitation, which has confirmed me he is a mature mathematician that has done very interesting and deep mathematics, and that has solved relevant mathematical problems.

In the good tradition of the Cech school in algebra, Dr. Saroch master's the mixture of set theoretic methods and logic methods with the purely algebraic ones. His works offers also an excellent part of creativity on techniques and on ideas, and it is of outstanding quality. **With absolutely no doubt I strongly recommend J. Šaroch's habilitation.** I will explain with more detail in the rest of my report why I am convinced that he has already acomplished a lot and that the direction of his research is really promising. **I have also checked the report made by the program Turnitin that has confirmed, as I already knew, that there no evidence at all of scientific misconduct.**

Let me start **contextualizing Šaroch's research**. The theory of *cotorsion pairs* was initiated in the late 1970's by Luigi Salce, just with the idea of making an Ext-functor analogue of the torsion theories. At that time Salce's main result was already showing the close relation of cotorsion pairs with the notion of (pre)-enveloping class and (pre)-covering class.

Finding *approximations* of general modules by modules in a nicer class is a central problem in module theory that goes back to Reinhold Baer in 1940 where he was focusing on abelian groups, in 1953 Ekmann and Schopff proved the existence of injective hulls for any module. Since then many important contributions have been done by Kaplansky, Bass, Auslander and Enochs between the most classical ones.

Tilting/Cotilting theory has its origins in a striking paper of Bersntein, Gelfand and Ponomarev from 1973 giving a new proof of Gabriel's classification of path algebras of finite representation type. The ideas on that paper were first extended by Auslander, Platzeck and Reiten in the late seventies, and the abstraction to artin algebras was made by Brenner and Butler in 1980. In the late 80's and early 90's, the Padova school, notably Riccardo Colpi, and independently Fuller and Colby, initiated the study of tilting modules, and its dual cotilting modules, in a non necessarily finitely generated setting. These ideas have evolved towards a rich and promising theory, as it is nicely explained in the impressive monography by Göbel and Trlifaj (1st edition 2006, 2nd edition 2012).

Just let me remark that it is of special interest the cotorsion pairs associated to tilting or cotilting modules because of their relations with well known open problems like the pure semisimple conjecture and the finitistic dimension conjectures. Such relations were stablished in works by Angeleri-Hügel, and Angeleri-Hügel and Trlifaj, respectively.

Going through the papers of the habilitation, I will explain more about the interesting and difficult mathematics that have been developed around all this theory. The Prague school has had a leading role in all that and, in particular, J. Šaroch has made crucial and important contributions in all this context.

Let me conclude this part of the report, pointing out that the area is very much alive and certainly much more is going to come in the near future. The incorporation of new ideas and techniques and the possibilities opened by widening the setting have raised many new problems and opened connections with other areas. For example, derived categories fit very well with the whole theory and give a new perspective in which torsion and cotorsion pairs are seen together in the setting of t-structures and open connections with Algebraic Geometry. As it was proved by Hovey (2002), cotorsion pairs allow to define exact structures on a module category so all the richness of Quillen's theory and its corresponding connections with topology appear here. Groth and Šťovíček's developments on Grothendieck's derivators (2010's) should also represent an important role in the future developments of the theory.

Let me now briefly summarize what are the **main scientific contributions** of the papers presented in the Habilitation Thesis:

(C.1) Solution of a 40 years old problem posed by Auslander. A strong indication of the quality of the result is that it is published in Inventione Math. which is certainly one of the more prestigious journals in mathematics and where only very outstanding papers are published.

(C.2) Very strong results around what is called *Enoch's conjecture on precovering classes.* The joint results obtained by the author and collaborators are really striking, they do not solve the conjecture but they certainly show that it is true un a number of interesting classes. It is important to mention that the main technical tools to prove these results are in a paper with Šaroch as a single author.

(C.3) Relevant results in a very popular subject as Gorenstein Homological Algebra in a joint paper with J. Š ťovíček, in which they are able to describe a couple of new cotorsion pairs. The number of new ideas and techniques introduced in this paper is impressive, and it is certain that they will have further developments in the future.

I also want to make special mention about the hight quality of the journal where these results are published.

(C.4) Results showing how set theoretical issues enter in deciding whether there could be a dual Baer criteria to test projectivity.

As mentioned above, the originality and creativity in the papers presented is very impressive, in topics that are far from easy and with an important number of mathematicians working on them. Because of that I also want to make a list of the original technical contributions that are made in the paper and that certaintly will go beyond their initial scope:

(T.1) Construction of generalized tree modules and its corresponding tree-exact sequence. Specially striking is the construction in the first paper to solve Auslander's problem.

(T.2) Deep understanding of the Mittag-Leffler conditions related with cotorsion pairs, that together with the clever use of tree modules are the main tools in the developments around Enoch's conjecture.

(T.3) Deep understanding of set theory axiomatics and its use in algebraic topics.

(T.4) Development of a version for suitable continuous direct limits of Shelah's singular compactness theorem. This could be seen as a part of T.3, however I think it is relevant enough to sigle it out.

(T.5) Introducing a suggestive notion of cotorsion pair cogenerated by a set, and its relations with cofiltrations.

Now I will proceed to discuss with more details each of the papers:

First Paper: J. Šaroch, On the non-existence of right almost split maps, Invent. Math. 209 (2017), no. 2, 463–479. DOI: 10.1007/s00222-016-0712-2

All the papers presented are dealing with difficult problems and give important and deep results on them. But the solution of the Auslander's problems on codomains of right almost split maps done in this paper is particularly impressive. Even if I am not in favour of the use of bibliographic metrics, the fact that the result was published in a prestigious journal like Inventione Mathematicae, shows its relevance inside mathematics.

Let us explain briefly the context of this problem. As A. Martsinkowsky explains in his review of Šaroch's paper, modern representation theory of Artin algebras, as expounded by M. Auslander and his school, has its origins in the study of functor categories. More precisely, one is looking at additive contravariant (or covariant) functors on the category of finitely presented (= finitely generated in this case) modules with values in abelian groups. A fundamental result of Auslander shows that the original module category can be recovered from the functor category. Thus one is prompted to build a dictionary between functor-theoretic concepts and their module-theoretic counterparts. Almost split sequences, or Auslander-Reiten sequences, appear when interpreting simple functors, and almost split maps are defined patterned on the properties of the right/left map of almost split sequences.

Outside the context of Artin algebras, almost split sequences are not so frequent. Auslander (1986) proved that if C be a finitely presented module then there exists a right almost split map with codomain C if and only if the module C has got local endomorphism ring. It is not difficult to prove that if a module C is the codomain of a right almost split map then it must have local endomorphism ring, but for forty it was an open question whether being finitely presented was really needed. J. Šaroch managed to solve the question in the afirmative.

This is the first instance of application of what Šaroch calls *locally well behaved but complex object*: a tree module. This is a very interesting construction, which is a deep refinement of a construction by Slavik and Trlifaj, based on the idea of decorating a suitable Aronszajn tree with the objects in a prescribed totally ordered direct system. The key fact is that the tree module has an associated *tree-module exact sequence* that is useful to make counting arguments.

Second and Third papers: J. Šaroch, Approximations and Mittag-Leffler conditions–the tools, Israel J. Math. 226 (2018), no. 2, 737–756; L. Angeleri Hügel, J. Šaroch and J. Trlifaj, Approximations and Mittag-Leffler conditions—the applications, Israel J. Math. 226 (2018), 757–780.

Since module categories are, in general, too large the idea of *approximating* general modules by modules in a nicer class is omnipresent in module theory. There are two (not that independent) main authors in the development of the general theory that nowadays is used: M. Auslander, mostly thinking on the case of artin algebras, and E. Enochs and collaborators. In the Auslander language, one talks of right and left (minimal) approximations, and in the Enochs language of pre-envelopes and pre-covers and of envelopes and covers (for the minimal versions). Here, as done by the author, we will follows Enochs terminology.

Paradigmatic examples are the class of injective modules: every module has an injective envelope (that is, a minimal right approximation), but a cellebrated result of Bass states that if all (right) modules have flat covers then the ring must be (right) perfect, equivalently, all flat (right) modules are projective. Let us mention that Bass result could be distilled to prove that if a flat module over a ring has a projective cover, then such module is projective. On the other hand, a cellebrated result by Bican, Enochs and El Bashir, shows that any module has a flat cover. These, examples explain that in module categories enveloping classes and covering classes may have really different behaviour and these poses many interesting quesions.

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One of the open questions in approximation theory is to decide which are the good classes that provide for them. These two papers in the habilitation deal with these type of questions and one of the main ingredients (as mentioned in their title) are the so called Mittag-Leffler conditions on modules. In the first paper, J. Saroch proves the class of all flat Mittag-Leffler right R-modules is not precovering unless R is right perfect, that is flat Mittag-Leffler modules do not behave well with respect to left approximations. Partial versions of the result were known under extra cardinality hypothesis, which the author manages to remove. Again, the key ingredient are the tree modules (now just constructed out of a countable direct limit) and their associated tree-module exact sequences.

Mittag-Leffler modules were introduced by Raynaud and Gruson in the 70's, and the interest on them was renewed because of the role of the relative Mittag-Leffler conditions in proving that all tilting classes are of finite type. The (dual of the) relative Mittag-Leffler conditions appears in a natural way associated to the left class of a cotorsion pair. In the second main results in this paper, they are one of the key ingredients to show that a cotorsion pair with its left class closed under direct limits must be always of countable type. Moreover, in this case, the right class is definable. Let me mention, that this is a striking and neat result, that extends and simplifies many other previous results of this kind (including the results involved in proving that all tilting modules are of finite type), so it is a culmination of the efforts of many authors.

Inspired by the pioneering work of Bass on projective covers (quoted above) Enochs proved that a precovering class of modules that is closed under direct limits is covering. It is an open question whether the converse is true, and the question in known as Enochs conjecture.

The left class of a cotorsion pair generated by a set is always pre-covering and the right class is always pre-enveloping (to this respect they follow the analogy that the left part should be seen as a relative version of projective modules and the right class of injective modules). So cotorsion pairs are a good context for checking Enochs conjecture. In the second paper, it is proved that in a cotorsion pair such that its right hand class is closed under direct limits, then the left hand class is covering if and only if it is also closed under direct limits. Hence, giving a positive answer to Enochs question in this setting. The proof of this result is based on the reduction to the countable case proved in the first paper.

This result has been recently extended to Grothendieck categories by Bazzoni, Šťovíček and Positzelski, but on his side Šaroch has given an interesting further turn to the Enochs conjecture that is described in the habilitation introduction.

Fourth paper: J. Šaroch and J. Šťovíček, Singular compactness and definability for Σ-cotorsion and Gorenstein modules, Selecta Math. New Ser. 26 (2020), Paper No. 23.

When working with perfect cotorsion pairs, the modules in the left hand class are characterized by a filtration property coming from what is known as the Eklof Lemma. But many times what is more interesting is to determine which class of direct limits of modules in the classs are still in the same class, essentially because it is easier to write the module as a direct limit instead of a filtration. For example, what was crucial in the proof that tilting modules are of finite type, was to be able to give this characterization for the case of countable direct systems in terms of Mittag-Leffler conditions.

This paper solves this problem for suitable continuous direct limits. This allows the authors to prove a direct limit version of Shelah's singular compacteness, which in turn is used to prove striking results on problems on Σ-cotorsion modules and on Gorenstein homological algebra.

Here I would like to stress also the quality of the journal where the paper was published. Selecta Math. This journal was founded by Israel Gelfand and has an impressive editorial board.

Fith Paper: J. Saroch and J. Trlifaj, Test sets for factorization properties of modules, Rend. Sem. Mat. Univ. Padova. 144 (2020), 217–238

This paper gives an interesting turn to the theory of cotorsion pairs. To explain it, let us give a couple of general definitions:

Given a monomorphism Ψ in Mod-R, we call a class of modules $\mathcal{C}$ a *factorization class* of Ψ provided that $\mathcal{C} = \{M \in \text{Mod–}R \mid \text{Hom}_R(\Psi, M) \text{ is surjective}\}$.

Cofactorization classes with respect to an epimorphism are defined dually.

The Baer Criteria implies that injective modules are factorization classes. In general, a cotorsion pair is generated by a set if and only if its left hand class is a factorization class. This leads naturally to the notion of cotorsion pair cogenerated by a set, but immediately one steps over the lack of duality in module categories and set theoretical issues appear. For example, in this paper it is proved that for R commutative noetherian of Krull dimension $0 < d < \infty$, the statement: the class of projectives is a cofactorization class, is independent of ZFC + GCH.

For cotorsion pairs generated by a set, there is a nice description of the objects of the left hand class. Moreover, as pointed above, one can even describe which direct limits of objects in the class remain in the class. No such results exist for the right hand class of cotorsion pairs. This paper is not only important by its results, but also because there are a number of ideas that seem to draw a path towards such a description.

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