FACULTY
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Charles University

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Filip Jankovec

Minimal Taylor Clones on Three Elements

Department of Algebra

Supervisor of the master thesis: doc. Mgr. Libor Barto, Ph.D.
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I would like to thank my supervisor, doc. Mgr. Libor Barto, Ph.D., for a lot of patience, time, and guidance.

Title: Minimal Taylor Clones on Three Elements

Author: Filip Jankovec

Department: Department of Algebra

Supervisor: doc. Mgr. Libor Barto, Ph.D., Department of Algebra

Abstract: Brady have classified all the minimal Taylor algebras on a three-element set up to term equivalence and isomorphism; there are 24 such algebras. The thesis studies the clones of these algebras. For 12 of them, the thesis characterizes operations in the clone and, also, describes the clone by means of relations.

Keywords: operation, relation, compatibility, clone, minimal Taylor algebra

Contents

Introduction

In this thesis, we study so-called minimal Taylor algebras on three elements. These structures have been recently described in an unpublished work Brady [2022]. There are, in total, 24 minimal Taylor algebras on three elements up to isomorphism and term equivalence, and they are listed in the last section of the first chapter. The goal of this thesis is to give a description of clones of some of these algebras. Every clone can be described explicitly by characterizing operations in that clone or by listing a generating set of compatible relations. We provide both of these descriptions for 12 of the minimal Taylor algebras.

Minimal Taylor algebras are significant for several reasons. Taylor algebras can be defined as those that satisfy a nontrivial idempotent Mal'cev condition Taylor [1977]. Since most of the familiar types of algebras, such as groups, rings, or lattices, satisfy some nontrivial idempotent Mal'cev condition, the class of Taylor algebras is rather broad Bergman [2012]. Very informally, one can view Taylor algebras as "structured", while the remaining algebras are "wild" or "unstructured". Minimal Taylor algebras are then those that are the least structured among the structured algebras. Remarkably, every finite algebra contains (in some sense) a minimal Taylor algebra Barto et al. [2021], therefore understanding minimal Taylor algebras would provide useful information about finite structured algebras in general.

Another motivation to study minimal Taylor algebras comes from theoretical computer science. In the well developed theory of so-called fixed template constraint satisfaction problems (CSPs) Barto et al. [2017], one associates to a CSP an algebra in such a way that the complexity of the CSP depends only on the associated algebra. A celebrated theorem by Bulatov [2017] and Zhuk [2020] shows that the CSP is solvable in polynomial time if the associated algebra is a Taylor algebra and, otherwise, the CSP in NP-complete. Minimal Taylor algebras then correspond to the "hardest" CSPs that are solvable in polynomial time. This viewpoint is discussed in Barto et al. [2021], in fact, minimal Taylor algebras are introduced in that paper.

This thesis consists of six chapters. In the first chapter, we give a general introduction to the topic. We give definitions of a clone and a relation clone and show a connection between these two concepts, discovered in Geiger [1968], Bodnarčuk et al. [1969]. We introduce essential operations, which will serve as a useful tool later on. Finally, we introduce Taylor algebras and minimal Taylor algebras, and we give a list of minimal Taylor algebras on two elements and three elements.

In the second chapter, we describe the clones of all minimal Taylor algebras on a two-element set. We prove two more general results, we describe all clones of algebras $\mathbb{Z}_n$ for $n \in \mathbb{N}$, and we also describe clones of all finite algebras with the structure of a semilattice. These results were already known Post [1941], Davey [1996].

The third chapter focuses on nonconservative minimal Taylor algebras and gives a description of two clones of such algebras. Both of these algebras are isomorphic to a subdirect product of two minimal Taylor algebras on two elements.

The next chapter deals with the specific case of minimal Taylor algebras on three elements, where there is an ”absorbing” element, i.e., an element a which, if it is among the arguments, enforces that the result will be a. It turns out that there are three minimal Taylor algebras satisfying these conditions. One of these algebras is a semilattice, and the other two are quite easy to describe.

The fifth chapter deals with the minimal Taylor algebras which have a majority operation as a single basic operation. The main tool used in this chapter is the Baker-Pixley Theorem from Baker and Pixley [1975] which allows us to focus on the description of binary relations.

The last chapter describes the clones of another two minimal Taylor algebras. The operations contained in clones of these algebras rarely return one particular element, which allows us to transform this problem into a simpler problem of studying certain relations compatible with a two-element minimal Taylor algebra.

The main contribution of this thesis is the description of the clones of twelve minimal Taylor algebras on three elements. The relational description of the clone of $\mathbf{T}_1^P$ has been already sketched in [Brady, 2022, Example 1.6.5]. The remaining results in Chapters 3,4,5, and 6 are original contributions.

1. Preliminaries

The main goal of this chapter is to give definitions of a clone, an essential operation and a minimal Taylor algebra. Before we start, let us introduce some basic notation. By the set of natural numbers we understand the set $\mathbb{N} = \{1, 2, 3, \dots\}$. In particular, $0 \notin \mathbb{N}$. We denote the set $\{1, \dots, n\}$ by $[n]$.

1.1 Operations, Relations and Algebras

This section aims to give some basic definitions, like definitions of a relation, an operation, and an algebra. At the end of this section, we define term operations. Let us start with simpler definitions.

Definition 1.1. *Let A be a set and $n \in \mathbb{N}$. An n-*ary operation on *A is a mapping from A^n to A. An n-*ary relation on *A is a subset of A^n.*

Informally, an algebra is a set with some collection of operations. In order to conveniently work with standard constructions, such as subalgebras, products, or isomorphic copies, we present a standard definition via signatures.

Definition 1.2. *A* signature *is a set Σ of symbols together with a mapping*

$$ar : \Sigma \to \mathbb{N}.$$

The symbols in a signature should be viewed as names of operations, and the mapping ar gives arities of operations. Usually, it is allowed for ar to map elements of Σ into $\mathbb{N} \cup \{0\}$. However, in this thesis we disallow nullary operations for convenience.

Definition 1.3. *An* algebra $\mathbf{A}$ of a signature Σ *is a pair (A, F) in which A is a nonempty set and $F = (f^{\mathbf{A}} \mid f \in \Sigma)$ is a family of operations on A, where $f^{\mathbf{A}}$ is an $ar(f)$-ary operation. The set A is called the* universe of $\mathbf{A}$*, and the elements of F are called the* basic operations*.*

We usually denote the universe of an algebra by the same letter, for example, the universe of an algebra $\mathbf{A}$ is A, the universe of an algebra $\mathbf{B}$ is B and so on. Sometimes we want to speak about some substructures or products of structures. This motivates the following two definitions.

Definition 1.4. *Let $\mathbf{A}$ be an algebra of a signature Σ. We say that an algebra $\mathbf{B}$ is a* subalgebra of $\mathbf{A}$ *if it has the same signature Σ, $B \subseteq A$, and, for each $f \in \Sigma$, we have $f^{\mathbf{A}}_{\restriction B^n} = f^{\mathbf{B}}$, where $n = ar(f^{\mathbf{A}})$.*

A subuniverse of $\mathbf{A}$ *is the universe of a subalgebra of $\mathbf{A}$.*

Definition 1.5. *Let $\mathbf{A}_i = (A_i, F_i)$ be algebras of the same signature Σ, where $i \in [n]$. The product of algebras $\mathbf{A}_1, \dots, \mathbf{A}_n$ is the algebra $\prod_{i\in[n]} \mathbf{A}_i$ of the signature Σ with the universe $\prod_{i\in[n]} A_i$ (the standard Cartesian product) and the basic operations are computed coordinate-wise. For $n = 2$ we write $\mathbf{A}_1 \times \mathbf{A}_2$. If $\mathbf{A}_i = \mathbf{A}$ for each $i \in [n]$ we write $\mathbf{A}^n$.*

Now when we have a product algebra, it is natural to define projections.

Definition 1.6. *Let A_i be sets, where $i \in [n]$, R be a subset of $\prod_{i\in[n]} A_i$, and $k \in [n]$. By the* projection of R to the set of coordinates $\{i_1, \ldots, i_k\} \subseteq [n]$ *we understand the set $\pi_{i_1,\ldots,i_k}(R) \subseteq \prod_{j\in\{i_1,\ldots,i_k\}} A_j$ defined as follows.*

$$\pi_{i_1,\ldots,i_k}(R) = \{(x_{i_1}, \ldots, x_{i_k}) \mid (x_1, x_2, \ldots, x_n) \in R\}$$

In this definition we use the letter R for a subset of $\prod_{i\in[n]} A_i$ for the particular reason. Usually, when we will be speaking about projections of some set, that set will be a relation on some set A.

Now we introduce subdirect products. Informally, a subdirect product is a subalgebra of a product which has full projections to each of the coordinates. We give this definition only for a product of two algebras since we do not need a more general version of this definition.

Definition 1.7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras of the same signature Σ. An algebra* $\mathbf{C}$ *is a* subdirect product of $\mathbf{A}$ and $\mathbf{B}$ *if* $\mathbf{C}$ *is a subalgebra of* $\mathbf{A} \times \mathbf{B}$*, $\pi_1(C) = A$, and $\pi_2(C) = B$. We write $\mathbf{C} \leq_{sd} \mathbf{A} \times \mathbf{B}$ or $C \leq_{sd} \mathbf{A} \times \mathbf{B}$.*

In the rest of this section, we give definitions of a term and a term operation. The definitions of a term operation is inductive, starting from very simple operations called projections.

Definition 1.8. *Let A be a set. The* n-ary projection to the i-th coordinate*, where $1 \leq i \leq n$, is the n-ary operation $\pi_i^n : A^n \to A$ defined as follows.*

$$\pi_i^n : (a_1, \ldots, a_n) \mapsto a_i$$

We say that an operation f on A is a projection*, if $f = \pi_i^n$ for some $i, n \in \mathbb{N}$.*

We give a definition of a term. Informally, a term is a meaningful composition of allowed symbols (variables and elements of some signature).

Definition 1.9. *Let X be a set of variables and Σ a signature.*

The set of terms of a signature Σ over X *is the smallest set of (formal) expressions, denoted by $\mathfrak{T}$, satisfying the following conditions.*

1. *$X \subseteq \mathfrak{T}$.*

2. *Assume $t_1, \ldots, t_n \in \mathfrak{T}$, $f \in \Sigma$, and $ar(f) = n$. Then $f(t_1, \ldots, t_n) \in \mathfrak{T}$.*

We will further use the set of variables $X_n = \{x_1, \ldots, x_n\}$. It remains to give a definition of a term operation. Let Σ be a signature. Symbols from the signature Σ are interpreted in any algebra of the signature Σ as basic operations. Clearly, we also can interpret any term of the signature Σ in an algebra of the same signature as a composition of basic operations. This is exactly what the following definition says.

Definition 1.10. *Let* $\mathbf{A}$ *be an algebra of a signature Σ and t be a term of the signature Σ over X_n. We define a* term operation $t^{\mathbf{A}} : A^n \to A$ *recursively.*

1. *Assume $t = x_i$, where $i \in [n]$. Then $t^{\mathbf{A}} = \pi_i^n$.*

2. *Assume $t = f(p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n))$, where $p_1, \ldots, p_n$ are terms of the signature Σ over X_n and $f \in \Sigma$. Then*

$$t^{\mathbf{A}} : (a_1, \ldots, a_n) \mapsto f^{\mathbf{A}}(p_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, p_n^{\mathbf{A}}(a_1, \ldots, a_n)).$$

1.2 Clones

In this section, we give definitions of a clone and a relation clone, and we discuss how these structures are related. This connections was established in Geiger [1968] and Bodnarčuk et al. [1969].

We start with a definition of a clone. Informally, a clone is a collection of operations which is closed under forming new operations by means of meaningful expressions. Using the concept of term operations in Definition 1.10, clones can be introduced as follows.

Definition 1.11. *Let A be a set. We say that a set $\mathcal{C}$ of operations on A is a* clone*, if for each algebra $\mathbf{A}$ whose basic operations are in $\mathcal{C}$, each term operation of $\mathbf{A}$ is in $\mathcal{C}$ as well. For a clone $\mathcal{C}$ we denote the set of all n-ary operations from $\mathcal{C}$ by $\mathcal{C}_n$.*

Another possible way to define a clone is that a clone is a set of operations containing all the projections and closed under so-called generalized composition, see Bergman [2012].

We are working with clones which contain precisely all the term operations of some particular algebra.

Definition 1.12. *Let $\mathbf{A}$ be an algebra. A* clone of $\mathbf{A}$ *is the clone containing exactly all the term operations of $\mathbf{A}$. We denote it by* $\mathrm{Clo}(\mathbf{A})$*. We denote the collection of all n-ary operations from* $\mathrm{Clo}(\mathbf{A})$ *by* $\mathrm{Clo}_n(\mathbf{A})$*.*

Two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same set $A = B$ are called term equivalent *if* $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(\mathbf{B})$*.*

Term equivalent algebras can be considered equal for most purposes since they share many structural properties, e.g., they have the same subuniverses of powers Bergman [2012]. This thesis contributes to the study of algebras up to term-equivalence by giving two descriptions of $\mathrm{Clo}(\mathbf{A})$ for several concrete algebras $\mathbf{A}$.

One way to describe $\mathrm{Clo}(\mathbf{A})$ is to characterize operations that are members of that clone. An alternative way is by means of relations. The crucial concept linking operations and relations is the following.

Definition 1.13. *Let f be an n-ary operation on a set A. We say that a relation $R \subseteq A^m$ is* compatible with f *if for every choice of $a_{i,j} \in A$, $i \in [m]$, $j \in [n]$ the following holds.*

$$\forall i \in [n]\ (a_{1,i}, a_{2,i}, \ldots, a_{m,i}) \in R \Longrightarrow (f(a_{1,1}, \ldots, a_{1,n}), \ldots, f(a_{m,1}, \ldots, a_{m,n})) \in R$$

We say that R is compatible with an algebra $\mathbf{A}$ *if R is compatible with every basic operation of $\mathbf{A}$. If R is compatible with f, we also say that R is* invariant under f*, that f* preserves R*, or that f* is compatible with R*.*

We observe that an n-ary relation R is compatible with an algebra $\mathbf{A}$ if and only if R is a subuniverse of $\mathbf{A}^n$.

We will introduce some notation related to the compatibility of relations with operations.

Definition 1.14. *Let A be a set, $\mathcal{C}$ be a set of operations on A, and $\mathfrak{D}$ be a set of relations on A. We denote the set of all the relations on A, which are compatible with all the operations from $\mathcal{C}$, by* $\mathrm{Inv}(\mathcal{C})$*. Similarly, for an algebra* $\mathbf{A}$ *with a universe A, we denote by* $\mathrm{Inv}(\mathbf{A})$ *the set of all the relations on A, which are compatible with all the basic operations of* $\mathbf{A}$*.*

We denote the set of all the operations on A, which are compatible with all the relations from $\mathfrak{D}$, by $\mathrm{Pol}(\mathfrak{D})$*.*

For finitely many relations $R_1, \dots, R_n$, we also write $\mathrm{Inv}(R_1, \dots, R_n)$ instead of $\mathrm{Inv}(\{R_1, \dots, R_n\})$. We call the elements of $\mathrm{Inv}(\mathcal{C})$ *relations invariant under* $\mathcal{C}$ and the elements of $\mathrm{Pol}(\mathfrak{D})$ *polymorphisms of* $\mathcal{D}$. Using this new notation we can state the first important result about clones.

Theorem 1.1 (Geiger [1968], Bodnarčuk et al. [1969])**.** *Let* $\mathbf{A}$ *be a finite algebra. Then*

$$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathrm{Inv}(\mathbf{A})).$$

This result implies that the clone of each finite algebra can be "described" using relations; more precisely, the clone is equal to the set of all polymorphisms of a set of relations, namely $\mathrm{Inv}(\mathbf{A})$. Instead $\mathrm{Inv}(\mathbf{A})$, we can take an (often much smaller) subset $\mathfrak{D} \subseteq \mathrm{Inv}(\mathbf{A})$ so that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathfrak{D})$ still holds. As we shall see, this is the case if all relations in $\mathrm{Inv}(\mathbf{A})$ are pp-definable from $\mathfrak{D}$, in the sense of the following definition.

Definition 1.15. *Let $\mathfrak{D}$ be a set of relations on a set A. We say that a relation R is* primitively positively definable *(*pp-definable *for short)* from $\mathfrak{D}$*, if R can be defined by a first-order formula φ, which uses only the conjunction, the existential quantification, relations from $\mathfrak{D}$, and the equality relation (such formula is called a primitively positive formula). We also say that R is* generated by relations in $\mathfrak{D}$ *in this situation.*

For finitely many relations $R_1, \dots, R_n$, we say that R is pp-definable from $R_1, \dots, R_n$ instead of $\{R_1, \dots, R_n\}$.

To illustrate the definition, we give an example. Let R_1 and R_2 be binary relations. An example of a pp-definable relation from $\{R_1, R_2\}$ is the ternary relation R defined in 1.1.

$$(x, y, z) \in R \iff \exists w \ (w, y) \in R_1 \ \wedge \ x = z \ \wedge \ (y, x) \in R_2 \tag{1.1}$$

We give a definition of a relation clone using pp-definitions. A relation clone can be seen as an analogue of a clone in the world of relations. Instead of operations we have relations and instead of term compositions we have pp-definitions.

Definition 1.16. *A set of relations $\mathfrak{D}$ is a* relation clone *if $\mathfrak{D}$ is closed under pp-definable relations from $\mathfrak{D}$. A relation clone $\mathfrak{D}$ is* generated by a set of relations $\mathfrak{F}$ *if $\mathfrak{F} \subseteq \mathfrak{D}$ and every relation in $\mathfrak{D}$ is generated by relations from $\mathfrak{F}$.*

Now when relation clones are introduced, we can state another important result which connects clones with relational clones.

Theorem 1.2 (Geiger [1968], Bodnarčuk et al. [1969])**.** *Let A be a finite set. Let* $\mathbf{A}$ *be an algebra (whose universe is A) and $\mathfrak{D}$ be a set of relations on A. Then* $\mathrm{Inv}(\mathbf{A})$ *is a relation clone on A and* $\mathrm{Pol}(\mathfrak{D})$ *is a clone on A. Moreover,* $\mathrm{Inv}(\mathrm{Pol}(\mathfrak{D}))$ *is the relation clone generated by $\mathfrak{D}$.*

In particular, the set of relations compatible with an algebra $\mathbf{A}$ is closed under pp-definitions. Another consequence of the previous theorem is the following corollary. Both facts will be used without explicit reference.

Corollary 1.3. *Let A be a set. Let $\mathfrak{D}$ be a set of relations on A and $\mathbf{A}$ be an algebra with the universe A. Then* $\mathrm{Inv}(\mathbf{A})$ *is generated by $\mathfrak{D}$ if and only if* $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathfrak{D})$.

Proof. If $\mathrm{Inv}(\mathbf{A})$ is generated by $\mathfrak{D}$, by applying Pol we get $\mathrm{Pol}(\mathrm{Inv}(\mathbf{A})) = \mathrm{Pol}(\mathcal{D})$, because pp-definitions preserve compatibility. Since we have $\mathrm{Pol}(\mathrm{Inv}(\mathbf{A})) = \mathrm{Clo}(\mathbf{A})$ by Theorem 1.1, we get $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathcal{D})$.

If $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathcal{D})$ then by applying Inv we get $\mathrm{Inv}(\mathrm{Clo}(\mathbf{A})) = \mathrm{Inv}(\mathrm{Pol}(\mathcal{D}))$. Since $\mathrm{Inv}(\mathrm{Clo}(\mathbf{A})) = \mathrm{Inv}(\mathbf{A})$ we get $\mathrm{Inv}(\mathbf{A}) = \mathrm{Inv}(\mathrm{Pol}(\mathcal{D}))$. By Theorem 1.2 this exactly tells that $\mathcal{D}$ generates the relation clone $\mathrm{Inv}(\mathbf{A})$. $\square$

Before we finish this section, we define idempotent clones since all the clones we meet in this thesis are idempotent.

Definition 1.17. *Let f be an operation on a set A. We say that f is* idempotent *if $f(a, \dots, a) = a$ for all $a \in A$. A clone $\mathcal{C}$ is* idempotent *if $\mathcal{C}$ contains only idempotent operations. An algebra $\mathbf{A}$ is* idempotent *if all the basic operations of $\mathbf{A}$ are idempotent.*

For $a \in A$ we denote the unary relation $C_a = \{a\}$. We call such relations *singleton unary relations*. We observe that an operation f on A is idempotent if and only if f is compatible with all the singleton unary relations C_a for every $a \in A$. It follows that $\mathbf{A}$ is idempotent if and only if $\mathrm{Clo}(\mathbf{A})$ is idempotent.

Many of the algebras we study satisfy a stronger condition, called conservativity.

Definition 1.18. *Let f be an operation on a set A. We say that f is* conservative *if $f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$ for all $a_1, \dots, a_n \in A$. A clone $\mathcal{C}$ is* conservative *if $\mathcal{C}$ contains only conservative operations. An algebra $\mathbf{A}$ is* conservative *if all the basic operations of $\mathbf{A}$ are conservative.*

Note that an operation on A is conservative if and only if it preserves all the unary relations on A.

1.3 Essential Operations

In this section we give definitions of an essential operation and an essential part of a clone. We start with a definition of an essential coordinate. To understand the concept of essential coordinates, consider the operation π^2_2. This operation depends only on the second coordinate while the first coordinate is irrelevant. The relevant coordinates are called *essential*, the others are called *inessential*.

Definition 1.19. *Let A be a set and $i \in [n]$. For an n-ary operation f on A we say that the i-th coordinate of f is* essential *if there exist $a_j, b, c \in A$, where $j \in [n] \setminus \{i\}$, such that*

$$f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n).$$

We say that the i-th coordinate is inessential *if the i-th coordinate is not essential. An n-ary operation f is* essential *if every coordinate of f is essential.*

Sometimes, it is easier to describe essential operations in a clone rather than all operations. We introduce the following notation.

Definition 1.20. *Let* $\mathbf{A}$ *be an algebra. We define*

$$\mathrm{EssClo}(\mathbf{A}) = \{f \in \mathrm{Clo}(\mathbf{A}) \mid \ f \ \textit{is an essential operation} \ \}.$$

We denote the set of all n*-ary operations from* $\mathrm{EssClo}(\mathbf{A})$ *by* $\mathrm{EssClo}_n(\mathbf{A})$*. Similarly, for a clone* $\mathcal{C}$ *on a set* A *we define*

$$\mathrm{Ess}(\mathcal{C}) = \{f \in \mathcal{C} \mid \ f \ \textit{is an essential operation} \ \}.$$

We denote the set of all n*-ary operations from* $\mathrm{Ess}(\mathcal{C})$ *by* $\mathrm{Ess}_n(\mathcal{C})$*.*

Imagine that we can describe all essential operations of a particular clone. The question is, can we describe, in such a case, all the clone operations? The answer is given below. However, before we provide the answer, we give a definition of a minor.

Definition 1.21. *Let* f *be an* m*-ary operation and* $\alpha : [m] \to [n]$ *be a mapping. The* n*-ary operation* f^α *defined by*

$$f^\alpha : (a_1, \dots, a_n) \mapsto f(a_{\alpha(1)}, \dots, a_{\alpha(m)})$$

is called the minor of f determined by α*.*

The minors can provide a permutation and merging of variables and allow us to introduce dummy coordinates. In spite of the fact that this notion allows many interesting uses, we only use it in the following lemma. Now we can state how essential operations in a clone determine the whole clone.

Lemma 1.4. *Let* $\mathbf{A}$ *be an arbitrary idempotent algebra (with universe* A*) such that* $|A| \geq 2$*. Then*

$$\mathrm{Clo}_n(\mathbf{A}) = \{f^\alpha \mid f \in \mathrm{EssClo}_m(\mathbf{A});\ m \in [n];\ \alpha : [m] \to [n]; \alpha \ \textit{is injective}\}. \quad (1.2)$$

Proof. Denote the right hand side of Equation 1.2 by S. The inclusion $S \subseteq \mathrm{Clo}(\mathbf{A})$ is clear since $\mathrm{Clo}(\mathbf{A})$ is closed under minors.

To show the other inclusion, fix some $g \in \mathrm{Clo}_n(\mathbf{A})$ and let $J_g = \{i_1, \dots, i_m\}$ be the set of essential coordinates of g. We show that J_f is nonempty. Because $\mathbf{A}$ is idempotent, $\mathrm{Clo}(\mathbf{A})$ is also idempotent and thus g is idempotent. Let us pick $a, b \in A$ such that $a \neq b$. Then we have $g(a, \dots, a) = a$ and $g(b, \dots, b) = b$. This tells us that J_g in nonempty, otherwise we could change all the arguments from $(a, \dots, a)$ to $(b, \dots, b)$, one by one, and we would get $g(b, \dots, b) = a$ and so $a = b$.

Since J_f is nonempty, we can define an operation f as follows:

$$f : (a_{i_1}, \dots, a_{i_m}) \mapsto g(a_1, \dots, a_n),$$

where a_i are chosen arbitrary from A for $i \notin J_g$. The operation f is well defined, since $g(a_1, \dots, a_n) = g(b_1, \dots, b_n)$, if $a_i = b_i$ for each $i \in J_g$, where $a_j, b_j \in A$ for $j \in [n]$. Clearly, $f \in \mathrm{EssClo}_m(\mathbf{A})$. Define $\alpha : [m] \to [n]$ by $\alpha : j \mapsto i_j$. Obviously, such α is injective. We have $f^\alpha = g$, and the proof is complete.

□

From the previous lemma it follows that every clone $\mathcal{C}$ is determined by $\mathrm{Ess}(\mathcal{C})$. Hence, we get the following useful corollary.

Corollary 1.5. *Let $\mathcal{C}, \mathcal{D}$ be clones on a set A. If $\mathrm{Ess}(\mathcal{C}) \subseteq \mathrm{Ess}(\mathcal{D})$, we have $\mathcal{C} \subseteq \mathcal{D}$.*

We explain why we care about essential operations. Consider the algebra $\mathbf{A}$ with universe $A = \{0, 1\}$ and a single binary operation $\min^2$ that returns the minimum of the two arguments with respect to the ordering $0 \leq 1$. Clearly, if one of the arguments is 0, we get 0 as the result. With a bit of effort, it can be shown that the same holds for any essential operation in the clone generated by $\mathbf{A}$. However, this obviously cannot hold for any inessential operation. In the next definition, we introduce a relation which somehow witnesses this property of essential operations in $\mathbf{A}$.

Definition 1.22. *Let $S \subseteq A$ be sets. We define a ternary relation T_S on A as follows:*

$$(x, y, z) \in T_S \iff x = y \ \vee \ z \in S.$$

The next lemma says that the relation T_S indeed satisfies a property similar to the property of $\{0\}$ discussed in the last paragraph.

Lemma 1.6. *Let $S \subseteq A$ be sets. Let f be an n-ary essential operation on A compatible with the relation T_S. Then for all $a_1, \ldots, a_n \in A$ it holds*

$$\exists i \in [n] \ a_i \in S \implies f(a_1, \ldots, a_n) \in S.$$

Proof. Let us have $a_1, \ldots, a_n \in A$ and assume there is $i \in [n]$ such that $a_i \in S$. Because f is an essential operation, for each $i \in [n]$ there are $b_j, a, b, c, d, \in A$, where $j \in [n] \setminus \{i\}$, such that

$$f(b_1, \ldots, b_{i-1}, c, b_{i+1}, \ldots, b_n) = a, \ f(b_1, \ldots, b_{i-1}, d, b_{i+1}, \ldots, b_n) = \ b \text{ and } a \neq b.$$

Since for all $j \in [n] \setminus \{i\}$ it holds $(b_j, b_j, a_j) \in T_S$ and also $(c, d, a_i) \in T_S$, it follows, by compatibility of f with T_S, that $(a, b, f(a_1, \ldots, a_n)) \in T_S$. Since $a \neq b$, we have $f(a_1, \ldots, a_n) \in S$.

□

1.4 Minimal Taylor Algebras

In this section, we introduce Taylor algebras. We also define minimal Taylor algebras and list all the minimal Taylor algebras on two elements up to an isomorphism and term equivalence.

Definition 1.23. *Let A be a set and f be an n-ary operation on A. We say that f is a* Taylor operation *if for each coordinate $i \in [n]$ there are variables $z_{i,j}, w_{i,j}, x, y$ (not necessary distinct), where $j \in [n] \setminus \{i\}$, such that $x \neq y$ and for each mapping $\varphi : \{z_{i,j}, w_{i,j}, x, y \mid j \in [n] \setminus \{i\}\} \to A$ we have*

$$f(\varphi(z_{i,1}), \ldots, \varphi(x), \ldots, \varphi(z_{i,n})) = f(\varphi(w_{i,1}), \ldots, \varphi(y), \ldots, \varphi(w_{i,n})).$$

A clone is a Taylor clone *if it is idempotent and contains some Taylor operation.*

The definition above says that a Taylor operation is an operation satisfying nontrivial identities of the form

$$t(\dots,*,x,*,\dots) \approx t(\dots,*,y,*,\dots)$$

for each coordinate, where $*$ stands for arbitrary variables. For equivalent formulations of this definition and some context we refer to Taylor [1977], Bergman [2012], Barto et al. [2017]. The paper Barto et al. [2021] introduces and studies minimal Taylor algebras in the following sense.

Definition 1.24. *A Taylor clone $\mathcal{C}$ is a* minimal Taylor clone *if there is no Taylor clone $\mathcal{D}$ such that $\mathcal{D} \subsetneq \mathcal{C}$. An algebra $\mathbf{A}$ is a* minimal Taylor algebra *if* $\mathrm{Clo}(\mathbf{A})$ *is a minimal Taylor clone.*

As mentioned earlier in this chapter, we give a list of minimal Taylor algebras on a two-element set up to isomorphism and term equivalence. However, before we do so, we give a couple of definitions to be able more effectively describe these algebras.

Definition 1.25. *Let f be a ternary idempotent operation on a set A. We say that f is a* majority *operation, if for each $a, b \in A$ we have*

$$f(a,a,b) = f(a,b,a) = f(b,a,a) = a.$$

We say that f is a minority *operation if for each $a, b \in A$ we have*

$$f(a,a,b) = f(a,b,a) = f(b,a,a) = b.$$

Note that on a two-element set $\{a,b\}$, there is exactly one majority and one minority operation. We denote the majority operation on $\{a,b\}$ by $\mathrm{maj}_{a,b}$ and the minority operation on $\{a,b\}$ by $\mathrm{aff}_{a,b}$. (Here aff stands for the affine operation. We discuss more general affine operations in the next chapter.) If there is no danger of confusion we will write maj instead of $\mathrm{maj}_{a,b}$ and aff instead of $\mathrm{aff}_{a,b}$. Before we continue, we need one more definition.

Definition 1.26. *Let $\{a,b\}$ be a two-element set. We denote by $\min^n_{a\leq b}$ the n-ary operation on $\{a,b\}$ that returns the minimum of the arguments with respect to the ordering $a \leq b$.*

Now we give the complete list of minimal Taylor algebras on a set $\{a,b\}$ up to isomorphism and term equivalence. This list, as well as the description of the clones of these algebras, follows from the complete description of all clones on a two-element set by Post [1941]. There are only three such algebras. Two of these algebras have a single ternary basic operation, the third one has a single binary operation.

They are:

1. $\mathbf{M}^{a,b}$ with the operation $\mathrm{maj}_{a,b}$.

2. $\mathbb{Z}_2^{a,b}$ with the operation $\mathrm{aff}_{a,b}$.

3. $\mathbf{L}^{a,b}$ with the operation $\min^2_{a\leq b}$.

In the case $\{a,b\} = \{0,1\}$, we usually write $\mathbf{M}, \mathbb{Z}_2, \mathbf{L}$ instead of $\mathbf{M}^{0,1}, \mathbb{Z}_2^{0,1}$ and $\mathbf{L}^{0,1}$. We often call these algebras *the majority, affine, or semilattice algebra*, respectively. Sometimes it is also useful to work with the algebra $\mathbf{L}_3^{a,b}$ with the universe $\{a,b\}$ and with the single basic operation $\min^3_{a\leq b}$. This algebra is term equivalent to $\mathbf{L}^{a,b}$.

In the next chapter, we describe the clones of all these algebras.

1.5 List of Minimal Taylor Algebras on Three Elements

Here we summarize the mentioned result from Brady [2022]. Altogether there are 24 minimal Taylor algebras on the three-element set $\{0,1,2\}$ up to an isomorphism and term equivalence. All of these algebras consist of a single basic operation t, which is binary or ternary. Five of these algebras are not conservative, the remaining ones are conservative.

1.5.1 Nonconservative Algebras

For all of these algebras, the operation t is idempotent and symmetric (i.e., the result does not depend on the order of the arguments). Three of them are determined by the following table.

Algebra	$t_{\restriction\{0,1\}}$	$t_{\restriction\{0,2\}}$	$t(1,1,2)$	$t(1,2,2)$	$t(0,1,2)$
$\mathbf{T}_1^N$	maj	$\min^3_{0<2}$	1	0	0
$\mathbf{T}_2^N$	aff	$\min^3_{0<2}$	0	1	1
$\mathbf{T}_3^N$	maj	aff	2	0	2

The forth algebra $\mathbf{T}_4^N$ is the semilattice algebra with binary operation $\inf^2$ determined by the ordering $0 \leq 1$, $0 \leq 2$ (see Section 2.3 for definitions). The fifth algebra $\mathbf{T}_5^N$ is the affine algebra $\mathbb{Z}_3$ (see Section 2.2).

The clones of algebras $\mathbf{T}_1^N$ and $\mathbf{T}_2^N$ are described in Chapter 3. The clone of $\mathbf{T}_4^N$ is described in Section 2.3. The clone of $\mathbf{T}_5^N$ is described in Section 2.2. A relational description of the clone of $\mathbf{T}_3^N$ is sketched in [Brady, 2022, Example 2.2.1], a description of the operations in this clone seems open.

1.5.2 Conservative Algebras with a Binary Symmetric Term

There are precisely two such algebras.

Algebra	$t_{\restriction\{0,1\}}$	$t_{\restriction\{1,2\}}$	$t_{\restriction\{0,2\}}$
$\mathbf{T}_1^S$	$\min^2_{0<1}$	$\min^2_{1<2}$	$\min^2_{2<0}$
$\mathbf{T}_2^S$	$\min^2_{0<1}$	$\min^2_{1<2}$	$\min^2_{0<2}$

$\mathbf{T}_1^S$ is the so-called Rock-Paper-Scissors algebra described in, e.g., [Brady, 2022, Section 3.1]. The clone of algebra $\mathbf{T}_2^S$ is described in Section 2.3.

1.5.3 Conservative Algebras without a Symmetric Binary or Cyclic Ternary Operation

A ternary operation t on A is cyclic if

$$t(x, y, z) = t(y, z, x) = t(z, x, y)$$

for each $x, y, z \in A$. There are two algebras with a ternary operation t such that $t(x, y, z) = x$ if $\{x, y, z\} = \{0, 1, 2\}$. It can be shown that clones of these two algebras do not contain a symmetric binary or a cyclic ternary operation.

Algebra	$t_{\restriction\{0,1\}}$	$t_{\restriction\{1,2\}}$	$t_{\restriction\{0,2\}}$
$\mathbf{T}_1^P$	maj	maj	maj
$\mathbf{T}_2^P$	aff	aff	aff

The clone of $\mathbf{T}_1^P$ is described in Section 5.2. A relational description of clone of $\mathbf{T}_2^P$ in sketched in [Brady, 2022, Example 1.7.2].

1.5.4 Conservative Algebras with a Ternary Cyclic Operation

There are sixteen conservative minimal Taylor algebras with a ternary cyclic operation t. One of them is term equivalent to the algebra $\mathbf{T}_2^S$, the others are listed below.

Algebra	$t_{\restriction\{0,1\}}$	$t_{\restriction\{1,2\}}$	$t_{\restriction\{0,2\}}$	$t(0, 1, 2)$	$t(0, 2, 1)$
$\mathbf{T}_1^C$	$\min^3_{0<1}$	$\min^3_{1<2}$	maj	0	0
$\mathbf{T}_2^C$	$\min^3_{0<1}$	maj	$\min^3_{0<2}$	0	0
$\mathbf{T}_3^C$	maj	$\min^3_{1<2}$	$\min^3_{0<2}$	0	1
$\mathbf{T}_4^C$	$\min^3_{0<1}$	maj	maj	0	0
$\mathbf{T}_5^C$	$\min^3_{0<1}$	aff	$\min^3_{0<2}$	0	0
$\mathbf{T}_6^C$	$\min^3_{0<1}$	$\min^3_{1<2}$	aff	0	0
$\mathbf{T}_7^C$	aff	$\min^3_{1<2}$	$\min^3_{0<2}$	0	1
$\mathbf{T}_8^C$	$\min^3_{0<1}$	aff	aff	2	2
$\mathbf{T}_9^C$	$\min^3_{0<1}$	aff	maj	0	0
$\mathbf{T}_{10}^C$	$\min^3_{0<1}$	maj	aff	2	2
$\mathbf{T}_{11}^C$	maj	aff	maj	1	2
$\mathbf{T}_{12}^C$	aff	maj	aff	0	0
$\mathbf{T}_{13}^C$	aff	aff	aff	0	0
$\mathbf{T}_{14}^C$	maj	maj	maj	0	0
$\mathbf{T}_{15}^C$	maj	maj	maj	1	2

The clones of $\mathbf{T}_2^C$ and $\mathbf{T}_5^C$ are described in Chapter 4, the clones of $\mathbf{T}_3^C$ and $\mathbf{T}_7^C$ are described in Chapter 6 and the clones of $\mathbf{T}_{14}^C$ and $\mathbf{T}_{15}^C$ are described in Chapter 5.

A relational description of the clone of $\mathbf{T}_{13}^C$ is sketched in [Brady, 2022, Example 1.7.3].

2. Clones on Two Elements

This chapter describes clones and relation clones related to the minimal Taylor algebras on a two-element set. In the first section, we describe the clone of $\mathbf{M}$. The second section gives a description of the clone of affine algebras $\mathbb{Z}_n$ for $n \in \mathbb{N}$, in particular we describe the clones of $\mathbb{Z}_2$ and $\mathbb{Z}_3 = T_5^N$. In the last section, we describe the clones of all algebras which are semilattices. In particular, we describe the clones of $\mathbf{L}$, T_1^N and T_2^S.

The results of this section are mostly folklore and follow from Post [1941], Davey [1996].

2.1 The Majority Clone

The main goal of this section is to describe the clone of $\mathbf{M} = (\{0,1\}, \mathrm{maj})$. To do this, we use an important result which allows us to focus on the binary relations on $\{0,1\}$.

Theorem 2.1 (Baker and Pixley [1975])**.** *Let $\mathcal{C}$ be a clone with a majority operation, R be n-ary relation in* $\mathrm{Inv}(\mathcal{C})$*, and $n \geq 2$. Then*

$$R = \bigwedge_{\substack{i,j \in [n] \\ i \neq j}} \pi_{i,j}(R).$$

In particular, every relation in $\mathrm{Inv}(\mathcal{C})$ *is pp-definable from unary and binary relations in* $\mathrm{Inv}(\mathcal{C})$*.*

We denote $\mathrm{Clo}(\mathbf{M}^{a,b})$ by $\mathcal{M}^{\{a,b\}}$. Instead of $\mathcal{M}^{\{0,1\}}$ we will usually write just $\mathcal{M}$. In the light of the previous theorem, it is enough to determine binary relations on $\{0,1\}$ compatible with maj. It turns out that all the binary relations on $\{0,1\}$ are actually compatible with maj.

Lemma 2.2. *Every binary relation on* $\{0,1\}$ *is compatible with* maj*.*

Proof. Let R be a binary relation on $\{0,1\}$. Pick $a_i, b_i, a, b \in \{0,1\}$, where $i \in [3]$, such that $(a_i, b_i) \in R$ for all $i \in [3]$, $\mathrm{maj}(a_1, a_2, a_3) = a$ and $\mathrm{maj}(b_1, b_2, b_3) = b$. By the definition of maj there are at least two indexes $i, j \in [3]$ such that $a_i = a_j = a$ and there are at least two indexes $k, l \in [3]$ such that $b_k = b_l = b$. Without loss of generality we can assume that $i = k$. Then $(a_i, b_i) = (a, b)$, hence $(a, b) \in R$. This shows R is compatible with maj. □

Because all the unary relations can be pp-defined from the binary relations (as projections), using Theorem 2.1 and Lemma 2.2, we get that $\mathrm{Inv}(\mathbf{M})$ is generated by all the binary relations. Our next goal is to simplify this result and show that $\mathrm{Inv}(\mathbf{M})$ can be generated only by two particular binary relations.

Definition 2.1. *On the two-element set $\{a, b\}$ we define the binary relations*

$$\leq_{a,b} = \{(a,a), (a,b), (b,b)\}$$

and

$$\neq_{a,b} = \{(a,b), (b,a)\}.$$

If there is no danger of confusion, we will write just $\leq$ *and* $\neq$*. We call the operations compatible with* $\neq$ self-dual operations *and the operations compatible with* $\leq$ monotone operations*.*

We show that $\leq$ and $\neq$ allow us to pp-define every binary relation. This gives us that $\mathcal{M}$ consists precisely of monotone and self-dual operations on {0,1}.

Theorem 2.3. Inv($\mathbf{M}$) *is generated by the relations* $\leq$ *and* $\neq$*.*

Proof. We just need to show that we can pp-define every binary relation on {0,1} from $\leq$ and $\neq$.

Clearly, every binary relation on {0,1} can be pp-defined as the intersection of the three element binary relations. One of these relations is $\leq$. The remaining three element relations are $\{(0,0),(1,0),(1,1)\}$, $\{(0,1),(1,0),(1,1)\}$ and $\{(0,0),(1,0),(0,1)\}$. These relations can be pp-defined using $\leq$ and $\neq$ as follows.

$$(x,y) \in \{(0,0),(1,0),(1,1)\} \iff y \leq x$$
$$(x,y) \in \{(0,1),(1,0),(1,1)\} \iff \exists z\ z \leq y \ \wedge \ z \neq x$$
$$(x,y) \in \{(0,0),(1,0),(0,1)\} \iff \exists z\ x \leq z \ \wedge \ z \neq y$$

Therefore, every binary relation on {0,1} can be pp-defined from $\leq$ and $\neq$, thus Inv($\mathbf{M}$) is generated by $\leq$ and $\neq$. ☐

Now we know that $\mathcal{M}$ consists of monotone and self-dual operations on {0,1}. This description is, in some sense, the best we can give. Although, this description is not very explicit. The following two definitions are an attempt to understand these operations more concretely.

Definition 2.2. *Let* S *be a set and* $\mathfrak{F} \subseteq 2^S$*. The collection* $\mathfrak{F}$ *is a* monotone self-dual collection of sets (ms-collection for short) on S *if:*

$$(\forall A, B \in S)\ A \in \mathfrak{F} \ \wedge \ A \subseteq B \implies B \in \mathfrak{F} \text{ (monotonicity)}$$

and

$$(\forall A \in S)\ A \in \mathfrak{F} \iff S \setminus A \notin \mathfrak{F} \text{ (self-duality).}$$

The ms-collection can be seen as a weakened form of an ultrafilter (we just do not require the closure property under finite intersections). We observe that for an arbitrary ms-collection $\mathfrak{F}$ on S, it holds $\emptyset \in \mathfrak{F} \ \vee \ S \in \mathfrak{F}$ by the self-duality. However, $\emptyset \in \mathfrak{F}$ would give $S \in \mathfrak{F}$ by the monotonicity of $\mathfrak{F}$, which would contradict the self-duality of $\mathfrak{F}$. Thus it holds $\emptyset \notin \mathfrak{F}$ and $S \in \mathfrak{F}$ for each ms-collection $\mathfrak{F}$ on S.

Definition 2.3. *For an ms-collection* $\mathfrak{F}$ *on* $[n]$ *we define an* n*-ary operation* $h_{\mathfrak{F}}$ *on* A *as follows.*

$$h_{\mathfrak{F}}(a_1, \ldots, a_n) = \begin{cases} 1 & \{i \in [n] \mid a_i = 1\} \in \mathfrak{F} \\ 0 & \{i \in [n] \mid a_i = 1\} \notin \mathfrak{F} \end{cases}$$

Now we will show that each n-ary monotone and self-dual operation is equal to $h_{\mathfrak{F}}$ for some ms-collection $\mathfrak{F}$ on $[n]$. In fact, there is a bijective correspondence between n-ary monotone self-dual operations on $\{0,1\}$ and ms-collections on $[n]$.

Theorem 2.4. $\mathrm{Clo}_n(\mathbf{M}) = \{h_{\mathfrak{F}} \mid \mathfrak{F}$ *is ms-collection on* $[n]\}$.

Proof. To show the inclusion $\mathrm{Clo}_n(\mathbf{M}) \supseteq \{h_{\mathfrak{F}} \mid \mathfrak{F}$ is an ms-collection on $[n]\}$ it is enough to show that for each ms-collection $\mathfrak{F}$ on $[n]$, the operation $h_{\mathfrak{F}}$ is compatible with $\leq$ and $\neq$. Fix a ms-collection $\mathfrak{F}$ on $[n]$. Let us assume that $h_{\mathfrak{F}}(a_1, \ldots, a_n) = a$ and $h_{\mathfrak{F}}(b_1, \ldots, b_n) = b$ for some $a, b, a_i, b_i \in A$, where $i \in [n]$. First we show the compatibility with $\neq$ and assume $a_i \neq b_i$ for all $i \in [n]$. Denote the set

$$I = \{i \in [n] \mid a_i = 1\}.$$

So $I \in \mathfrak{F}$ iff $a = 1$. Because $a_i \neq b_i$ for all $i \in [n]$, we have

$$\{i \in [n] \mid b_i = 1\} = [n] \setminus I.$$

Thus, by the definition of a ms-collection, precisely one of the sets $I, [n] \setminus I$ is in $\mathfrak{F}$, which tells us $a \neq b$. Hence $h_{\mathfrak{F}}$ is compatible with $\neq$.

To show the compatibility with $\leq$, assume that we have $h_{\mathfrak{F}}(a_1, \ldots, a_n) = a$ and $h_{\mathfrak{F}}(b_1, \ldots, b_n) = b$ for some $a, b, a_i, b_i \in A$, where $i \in [n]$, and $a_i \leq b_i$ for all $i \in [n]$. Denote

$$I = \{i \in [n] \mid a_i = 1\},\ J = \{i \in [n] \mid b_i = 1\}.$$

The conditions $a_i \leq b_i$, where $i \in [n]$, exactly tell us that $I \subseteq J$. Hence by the monotonicity of $\mathfrak{F}$ we have $I \in \mathfrak{F} \implies J \in \mathfrak{F}$ and thus $a = 1 \implies b = 1$. Therefore, we have $a \leq b$ and so $h_{\mathfrak{F}}$ is compatible with $\leq$.

To show the inclusion $\mathrm{Clo}_n(\mathbf{M}) \subseteq \{h_{\mathfrak{F}} \mid \mathfrak{F}$ is an ms-collection on $[n]\}$, we have to show for any operation $f \in \mathrm{Clo}_n(\mathbf{M})$ that the collection of sets

$$\mathfrak{F}_f = \{I \subseteq [n] \mid \{i \in [n] \mid a_i = 1\} = I \implies f(a_1, \ldots, a_n) = 1\}$$

is an ms-collection. First we check the monotonicity and assume $I \in \mathfrak{F}_f$ and $J \supseteq I$. Let us have $f(a_1, \ldots, a_n) = a$ and $f(b_1, \ldots, b_n) = b$ for some $a, b, a_i, b_i \in A$, where $i \in [n]$, such that

$$I = \{i \in [n] \mid a_i = 1\} \text{ and } J = \{i \in [n] \mid b_i = 1\}.$$

By the assumption $a = 1$. Because $J \supseteq I$ we have $a_i \leq b_i$ for all $i \in [n]$ thus by the compatibility of f with $\leq$ we get $a \leq b$. Because $a = 1$, we have $b = 1$ and thus $J \in \mathfrak{F}_f$. This shows $\mathfrak{F}_f$ is monotone.

Now we check the self-duality of $\mathfrak{F}_f$. Let us fix $I \in [n]$. Let us have $f(a_1, \ldots, a_n) = a$ and $f(b_1, \ldots, b_n) = b$ for some $a, b, a_i, b_i \in A$, where $i \in [n]$, such that

$$I = \{i \in [n] \mid a_i = 1\} \text{ and } [n] \setminus I = \{i \in [n] \mid b_i = 1\}.$$

Because f is compatible with $\neq$ and we have $a_i \neq b_i$ for all $i \in [n]$, we get $a \neq b$. So $a = 1 \iff b \neq 1$, which exactly means $I \in \mathfrak{F}_f \iff [n] \setminus I \notin \mathfrak{F}_f$. This shows the self-duality of $\mathfrak{F}_f$.

Hence $\mathfrak{F}_f$ is ms-collection and $\mathrm{Clo}_n(\mathbf{M}) = \{h_{\mathfrak{F}} \mid \mathfrak{F}$ is ms-collection on $[n]\}$. □

During the proof we proved that for any $f \in \mathrm{Clo}(\mathbf{M}^{a,b})$, the collection

$$\mathfrak{F}_f = \{I \subseteq [n] \mid \{i \in [n] \mid a_i = a\} = I \implies f(a_1, \dots, a_n) = a\}$$

is an ms-collection. We will use this notation in the next chapters.

Before we end this section, we state one more lemma, which will be useful later on.

Lemma 2.5. *Let* $\mathbf{A}$ *be an algebra of signature* Σ *such that* $\mathrm{Clo}(\mathbf{A})$ *contains some majority operation* m*. Then for every operation* $f \in \mathrm{Clo}_n(\mathbf{A})$ *there is a term* T *of signature* Σ *over* X_n*, such that* $T^{\mathbf{M}} = f$ *and each one of the variables from* X_n *occurs in* T*.*

Proof. Let $f \in \mathrm{Clo}_n(\mathbf{A})$. Clearly, there is a term T of signature Σ over X_n such that $T^{\mathbf{M}} = f$. Assume $x_i \in X_n$ is not presented in T. Then pick $j \in [n]$ such that x_j is presented in T. Now replace one occurrence of x_j in T by $\mathrm{maj}(x_j, x_j, x_i)$. Clearly, it still holds that $T^{\mathbf{M}} = f$ and now there is one more variable presented in T. The proof is finished by induction. $\square$

2.2 The Affine Clone

The main goal of this section is to describe the clone of $\mathbb{Z}_2 = (\{0,1\}, \mathrm{aff})$. Instead of describing just $\mathrm{Clo}(\mathbb{Z}_2)$, we choose a more general approach, and we describe clones of all affine algebras $\mathbb{Z}_n$.

Definition 2.4. *Let us have* $n \in \mathbb{N}$*. We define a ternary* affine operation aff^n on $\{0, \dots, n-1\}$ *as follows.*

$$\mathrm{aff}^n : (x, y, z) \mapsto x - y + z \mod n$$

We define an algebra $\mathbb{Z}_n$ *by* $\mathbb{Z}_n = \{\{0, \dots, n-1\}, \mathrm{aff}^n\}$.

Note that this notion is a generalization of the operation aff defined in the previous chapter. All of these algebras have the same signature, we denote this signature by Σ_0. Now we describe $\mathrm{Clo}_m(\mathbb{Z}_n)$. To do so, we use an m-ary operations $\sum_{i=1}^m a_i x_i$ on $\{0, \dots, n-1\}\}$ defined in the obvious way as the polynomial evaluation.

$$\sum_{i=1}^m a_i x_i : (b_1, \dots, b_m) \mapsto \sum_{i=1}^m a_i b_i$$

Theorem 2.6. $\mathrm{Clo}_m(\mathbb{Z}_n) = \{\sum_{i=1}^m a_i x_i \mid \sum_{i=1}^m a_i = 1, a_i \in \{0, \dots, n-1\}\}$.

Proof. Denote $S = \{\sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n a_i = 1, a_i \in \{0, \dots, n-1\}\}$. We first show $\mathrm{Clo}(\mathbb{Z}_n) \subseteq S$. It is easy show that every $f \in \mathrm{Clo}_m(\mathbb{Z}_n)$ is equal to $\sum_{i=1}^n a_i x_i$ for some $a_i \in \{0, \dots, n-1\}$, because $f = T^{\mathbb{Z}_n}$ for some term T of signature Σ_0 over X_n (if x_i is not presented in T, we set $a_i = 0$). Also, $\mathrm{Clo}(\mathbb{Z}_n)$ is idempotent, thus $f(1, 1, \dots, 1) = 1$, which means that $\sum_{i \in I} a_i = 1$.

To show the inclusion $\mathrm{Clo}(\mathbb{Z}_n) \supseteq S$ fix an m-ary operation $f = \sum_{i=1}^m a_i x_i$, $a_i \in \{0, \dots, n-1\}$. Clearly $x - y + x \in \mathrm{Clo}(\mathbb{Z}_n)$. By induction we obtain

$k \cdot x - (k-1) \cdot y \in \mathrm{Clo}(\mathbb{Z}_n)$ for each $k \in \mathbb{N}$. This tells us $f \in \mathrm{Clo}(\mathbb{Z}_n)$ for $m = 2$. If $m \geq 2$, we can write

$$f = (\sum_{i=1}^{m-1} a_i x_i + a_m x_{m-1}) - x_{m-1} + (a_m x_m - (a_m - 1) x_{m-1}).$$

We already know that $(a_m x_m - (a_m - 1)x_{m-1}) \in \mathrm{Clo}(\mathbb{Z}_n)$ and $x_{m-1} \in \mathrm{Clo}(\mathbb{Z}_n)$. This tell us that if $(\sum_{i=1}^{m-1} a_i x_i + a_m x_{m-1}) \in \mathrm{Clo}(\mathbb{Z}_n)$, then $f \in \mathrm{Clo}(\mathbb{Z}_n)$. Therefore, by induction on m, we get $f \in \mathrm{Clo}(\mathbb{Z}_n)$. $\square$

We want to give a relational description of $\mathrm{Clo}(\mathbb{Z}_n)$. Before we do so, we observe that every m-ary relation on $\{0, \ldots, n-1\}$, which is compatible with aff^n, is in fact an affine subspace of $\{0, \ldots, n-1\}^m$ (since being an affine subspace can be defined by $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R \implies \mathbf{a} - \mathbf{b} + \mathbf{c} \in R$). In the next definition, we define a few relations which can be used to describe $\mathrm{Inv}(\mathbb{Z}_n)$.

Definition 2.5. *On the set $\{0, \ldots, (n-1)\}$ we define ternary relations $S^n_{0,1}$ and $S^n_{0,1}$ by*

$$(x, y, z) \in S^n_{0,1} \iff x - y + z = 1 \mod n$$

and

$$(x, y, z) \in S^n_{1,0} \iff x - y + z = 0 \mod n.$$

On the set $\{a, b\}$ we define a ternary relation $S_{a,b}$ by

$$S_{a,b} = \{(a, a, a), (a, b, b), (b, b, a), (b, a, b)\}.$$

Note that

$$(x, y, z) \in S^n_{0,1} \iff \mathrm{aff}^n(x, y, z) = 1$$
$$(x, y, z) \in S^n_{1,0} \iff \mathrm{aff}^n(x, y, z) = 0$$

and

$$(x, y, z) \in S_{a,b} \iff \mathrm{aff}_{a,b}(x, y, z) = b$$

The relation $S_{a,b}$ is a generalization of $S^2_{0,1}$ and $S^2_{1,0}$ since we have $S^2_{0,1} = S_{0,1}$ and $S^2_{1,0} = S_{1,0}$. In this section, we will be mostly using $S^n_{1,0}$. The relation $S_{a,b}$ is used later on, when we want to use the structure of $\mathbb{Z}_2^{a,b}$ for $a, b \notin \{0, 1\}$.

Theorem 2.7. $\mathrm{Inv}(\mathbb{Z}_n)$ *is generated by $S^n_{1,0}$ and C_1.*

Proof. We show $\mathrm{Pol}(S^n_{1,0}, C_1) = \mathrm{Clo}(\mathbb{Z}_n)$. To check the inclusion $\mathrm{Pol}(S^n_{1,0}, C_1) \supseteq \mathrm{Clo}(\mathbb{Z}_n)$ we check the compatibility of $S^n_{1,0}$ and C_1 with aff^n. Firstly, we show that $S^n_{1,0}$ is compatible with aff^n. Let us have $\mathrm{aff}^n(a_1, a_2, a_3) = a$, $\mathrm{aff}^n(b_1, b_2, b_3) = b$ and $\mathrm{aff}^n(c_1, c_2, c_3) = c$, where $a_i, b_i, c_i, a, b, c \in \{0, \ldots, n-1\}$, $i \in [3]$, such that $(a_i, b_i, c_i) \in S^n_{1,0}$ for all $i \in [3]$. Because $a = a_1 - a_2 + a_3$, $b = b_1 - b_2 + b_3$ and $c = c_1 - c_2 + c_3$, clearly

$$a - b + c = a_1 - b_1 + c_1 - a_2 + b_2 - c_2 + a_3 - b_3 + c_3 = 0 - 0 + 0 = 0.$$

This shows $(a, b, c) \in S^n_{1,0}$, thus $S^n_{1,0}$ is compatible with aff^n. The singleton unary relations are trivially compatible with aff^n, because aff^n is an idempotent operation. This gives $\mathrm{Pol}(S^n_{1,0}, C_1) \supseteq \mathrm{Clo}(\mathbb{Z}_n)$.

To show the inclusion $\mathrm{Pol}(S^n_{1,0}, C_1) \subseteq \mathrm{Clo}(\mathbb{Z}_n)$, we denote by $\mathbf{e}_i$ the m-tuple $\mathbf{e}_i = (e_1, \dots, e_n)$ which has 1 on the i-th coordinate and 0 on the other coordinates. We show $f(x_1, \dots, x_m) = \sum_{i=1}^m f(\mathbf{e}_i)x_i$. To verify this, it is enough to check that for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_n^m$ it holds $f(\mathbf{a}) + f(\mathbf{b}) = f(\mathbf{a} + \mathbf{b})$. However, this follows from compatibility with $S^n_{1,0}$, since we have

$$(f(\mathbf{a}), f(\mathbf{a}+\mathbf{b}), f(\mathbf{b})) \in S^n_{1,0} \iff f(\mathbf{a}) + f(\mathbf{b}) = f(\mathbf{a}+\mathbf{b}) \mod n.$$

It only remains to check $\sum_{i=1}^m f(\mathbf{e}_i) = 1$, which immediately follows from the compatibility of f with C_1. $\square$

The following pp-definitions show that we can pp-define $S^n_{0,1}$ from $S^n_{1,0}$ and C_1 and we can pp-define $S^n_{1,0}$ from $S^n_{0,1}$ and C_0.

$$(x, y, z) \in S^n_{1,0} \iff \exists w\ (x, y, w) \in S^n_{0,1} \wedge (w, z, 0) \in S^n_{0,1}.$$

$$(x, y, z) \in S^n_{0,1} \iff \exists w\ (x, y, w) \in S^n_{1,0} \wedge (w, z, 1) \in S^n_{1,0}.$$

Thus, we could dually prove that $\mathrm{Inv}(\mathbb{Z}_n)$ is generated by $S^n_{0,1}$ and C_0. However, the relation $S^n_{1,0}$ is more useful for our purposes.

In this section, we described clones of affine algebras. In particular, we described the clone of $\mathbb{Z}_2$, which is a minimal Taylor algebra on $\{0, 1\}$, and the clone of $\mathbb{Z}_3$, which is a minimal Taylor algebra on $\{0, 1, 2\}$.

Before the end of this section, we state the following lemma, which will be useful in the next chapters. This lemma is very similar to Lemma 2.5.

Lemma 2.8. *Let $\mathbf{A}$ be an algebra of signature Σ, which has a minority term operation m. For every operation $f \in \mathrm{Clo}_n(\mathbf{A})$ there is a term T of signature Σ over X_n such that $T^{\mathbf{A}} = f$ and each of the variables from X_n occurs in T.*

Proof. Similarly, as in the proof of Lemma 2.5, we find term T such that $T^{\mathbf{A}} = f$ and for every missing variable x_i find variable x_j, which is in T and replace x_j by $m(x_i, x_i, x_j)$. $\square$

2.3 The Semilattice Clone

The main goal of this section is to describe the clone of the algebra $\mathbf{L}^{0,1} = \{\{0, 1\}, \min^2_{0 \le 1}\}$. This particular algebra is easily describable since it is a semilattice. Recall that a finite set A with an ordering $\le$ is a semilattice, if for each $a, b \in A$ there is $c \in A$, such that c is the infimum of the set $\{a, b\}$.

Definition 2.6. *For an ordering $\le$ on a set A we denote the n-ary operation, which maps n elements to its infimum with respect to the ordering $\le$, by $\inf^n_{\le}$. We say that an algebra $\mathbf{A}$ is a* semilattice *if $\mathbf{A}$ has a single basic operation, which is equal to $\inf^n_{\le}$ for some $2 \le n \in \mathbb{N}$, where $\le$ is some ordering on A. We say that $\mathbf{A}$ is a* semilattice algebra *with ordering $\le$.*

In case of $\mathbf{L}^{0,1}$, the operation $\inf^2_{\le}$ coincides with $\min^2$. In this section, we describe the clones of all the finite algebras which are semilattices. We already know from Davey [1996] that every such an algebra $\mathbf{A}$ has $\mathrm{Inv}(\mathbf{A})$ generated by some ternary relations. We will show which relations we need to generate $\mathrm{Inv}(\mathbf{A})$. Before we start, let us make some preparations. We start with the definition of an upper set and a lower set.

Definition 2.7. *Let A be a set with an ordering $\leq$. For $a \in A$ we define the following sets.*

$$(\uparrow a) = \{b \in A \mid a \leq b\}$$

and

$$(\downarrow a) = \{b \in A \mid b \leq a\}$$

Lemma 2.9. *Let $\mathbf{A} = (A, f)$ be a finite semilattice algebra with ordering $\leq$. Then $T_{(\downarrow a)}$ and $(\uparrow a)$ are compatible with $\mathbf{A}$ for each $a \in A$.*

Proof. Without loss of generality we assume $f = \inf^2_{\leq}$ (since $(A, \inf^n_{\leq})$ and $(A, \inf^2_{\leq})$ have the same clone). Fix $a \in A$. The compatibility of $(\uparrow a)$ with $\inf^2_{\leq}$ immediately follows from the definition of $\inf^2$. We check the compatibility of $T_{(\downarrow a)}$ with $\inf^2_{\leq}$. Let us have $a_i, b_i, c_i \in A$, where $i \in [2]$, such that $(a_1, b_1, c_1), (a_2, b_2, c_2) \in T_{(\downarrow a)}$. We distinguish two cases.

1. There is $i \in [n]$ such that $c_i \in T_{(\downarrow a)}$. In such a case we have $\inf^2_{\leq}(c_1, c_2) \in T_{(\downarrow a)}$.

2. $a_1 = b_1$ and $b_2 = c_2$. In such a case we get $\inf^2_{\leq}(a_1, a_2) = \inf^2_{\leq}(b_1, b_2)$.

Thus we get $T_{(\downarrow a)}$ is compatible with $\mathbf{A}$, which finishes the proof. □

We show that $\mathrm{EssClo}_n(\mathbf{A})$ does not contain any operation except the n-ary infimum operation.

Theorem 2.10. *Let $\mathbf{A} = (A, f)$ be a finite semilattice algebra with ordering $\leq$. Then* $\mathrm{EssClo}_n(\mathbf{A}) = \{\inf^n_{\leq}\}$.

Proof. Without loss of generality, let $f = \inf^2_{\leq}$ and let 0 be the least element of A with respect to $\leq$. Clearly, $\inf^n_{\leq} \in \mathrm{Clo}_n(\mathbf{A})$ since $\inf^n_{\leq}$ is a term operation given by the term $t(x_1, t(x_2, \ldots, t(x_{n-1}, x_n) \ldots))$, where $t^{\mathbf{A}} = \inf^2$. The operation $\inf^n_{\leq}$ is essential since $\inf^n_{\leq}(a, \ldots, a) = a$ for each $a \in A$ and $\inf^n_{\leq}$ returns 0, if we change one coordinate from a to 0. Hence $\inf^n_{\leq} \in \mathrm{EssClo}_n(\mathbf{A})$.

Now let us have $f \in \mathrm{EssClo}_n(\mathbf{A})$ and we show $f = \inf^n_{\leq}$. We pick an arbitrary $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$. Denote $a = \inf^n_{\leq}(\mathbf{a})$. Because f is essential and compatible with $T_{(\downarrow a_i)}$ for each $i \in [n]$, by Lemma 1.6 we have $f(\mathbf{a}) \in T_{(\downarrow a_i)}$ for each $i \in [n]$. This means $f(\mathbf{a}) \leq a$. Because f is compatible with $(\uparrow a)$, we get $f(\mathbf{a}) = a$ and thus $f = \inf^n_{\leq}$. □

Theorem 2.11. *Let $\mathbf{A} = (A, f)$ be a finite semilattice algebra with ordering $\leq$. Then* $\mathrm{Inv}(\mathbf{A})$ *is generated by relations $T_{(\downarrow a)}$ and $(\uparrow a)$, where $a \in A$.*

Proof. Again, without loss of generality assume $f = \inf^n_2$. We show $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{T_{(\downarrow a)}, (\uparrow a) \mid a \in A\})$. From Lemma 2.9 we know $\{T_{(\downarrow a)}, (\uparrow a) \mid a \in A\}$ is a set of compatible relations with $\mathbf{A}$. This gives $\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Pol}(\{T_{(\downarrow a)}, (\uparrow a) \mid a \in A\})$.

To show the inclusion $\mathrm{Clo}(\mathbf{A}) \supseteq \mathrm{Pol}(\{T_{(\downarrow a)}, (\uparrow a) \mid a \in A\})$, it is enough to show that any n-ary essential operation $f \in \mathrm{Pol}(\{T_{(\downarrow a)}, (\uparrow a) \mid a \in A\})$ has to be equal to $\inf^n_{\leq}$. This can be in the same manner as in the proof of Theorem 2.10. □

We observe that $T_A = A^3$, thus we can pp-define this relation trivially. Using Theorem 2.11, we finally describe all three minimal Taylor algebras which are semilattices.

Corollary 2.12. $\mathrm{Inv}(\mathbf{L}^{0,1})$ *is generated by* T_0 *and* C_1.

Corollary 2.13. $\mathrm{Inv}(\mathbf{T}_1^N)$ *is generated by* $T_{\{0,1\}}$, $T_{\{0,2\}}$, C_1 *and* C_2.

Proof. Here we just use that we can pp-define $T_{\{0\}}$ as follows.

$$(x,y,z) \in T_{\{0\}} \iff (x,y,z) \in T_{\{0,1\}} \ \wedge \ (x,y,z) \in T_{\{0,2\}}$$

□

Corollary 2.14. $\mathrm{Inv}(\mathbf{T}_2^S)$ *is generated by* T_0, $T_{\{0,1\}}$, $\{1,2\}$ *and* C_2.

Before we finish this section, we take a look at one interesting relation, which is related to two-element semilattices.

Definition 2.8. *Let* A *be a set and* $a,b \in A$. *Define a ternary relation* $L_{a,b}$ *as follows.*

$$(x,y,z) \in L_{a,b} \iff x \in \{a,b\} \ \wedge \ (x = b \implies y = z)$$

The reason why we care about the relation $L_{a,b}$ is that for an arbitrary algebra $\mathbf{A}$ and $a,b \in A$ such that $\{a,b\}$ is the universe of the semilattice subalgebra of $\mathbf{A}$, $L_{a,b}$ has to be compatible with $\mathbf{A}$.

Lemma 2.15. *Let* $\mathbf{A}$ *be an algebra with* $a,b \in A$, *such that* $\{a,b\}$ *is the universe of a semilattice subalgebra of* $\mathbf{A}$ *with the ordering* $a \leq b$. *Then the ternary relation* $L_{a,b}$ *is compatible with* $\mathrm{Clo}(\mathbf{A})$.

Proof. Let us have f the basic operation of $\mathbf{A}$. Choose $a_i, b_i, c_i \in A$, where $i \in [n]$, such that $(a_i, b_i, c_i) \in L_{a,b}$ for all $i \in [n]$. There are two possibilities:

1. $a_i = b$ for each $i \in [n]$, in which case $b_i = c_i$ for all $i \in [n]$. Thus $f(a_1, \ldots, a_n) = b$ and $f(b_1, \ldots, b_n) = f(c_1, \ldots, c_n)$.
2. There is $i \in [n]$ such that $a_i = a$. In such a case we have $f(a_1, \ldots, a_n) = a$.

This shows $L_{a,b}$ is compatible with $\mathrm{Clo}(\mathbf{A})$.

□

Now we can immediately say for some algebras $\mathbf{A}$ that they are compatible with $L_{a,b}$. We can use this relation to pp-define relations $T_{\{a\}}$ and $T_{A\setminus\{b\}}$. To do this, we need one more relation.

Definition 2.9. *Let* A *be a set. For* $a,b \in A$ *we define a binary relation* $D_{a,b}$ *as follows.*

$$(x,y) \in D_{a,b} \iff x \neq a \ \vee \ y = b$$

It is easy to check that we can pp-define $T_{A\setminus\{b\}}$ and $T_{\{a\}}$ as follows.

$$(x,y,z) \in T_{A\setminus\{b\}} \iff \exists w \ (z,w) \in D_{b,b} \wedge (w,x,y) \in L_{a,b}$$

and

$$(x,y,z) \in T_{\{a\}} \iff \exists w \ (w,z) \in D_{a,a} \wedge (w,x,y) \in L_{a,b}.$$

This gives us the following corollary.

Corollary 2.16. *Let* $\mathbf{A}$ *be an algebra with* $a, b \in A$*, such that* $\{a, b\}$ *is the universe of a semilattice subalgebra of* $\mathbf{A}$ *with an ordering* $a \leq b$*. If the relation* $D_{b,b}$ *is compatible with* $\mathbf{A}$ *then* $T_{A \setminus \{b\}}$ *is compatible with* $\mathbf{A}$*. If the relation* $D_{a,a}$ *is compatible with* $\mathbf{A}$ *then* $T_{\{a\}}$ *is compatible with* $\mathbf{A}$*.*

The reason why we stated this corollary is that it is easier to check compatibility with some binary relation than checking compatibility with a ternary relation. We use this result frequently in the following chapters.

3. The Nonconservative Clones

In this chapter, we are dealing with nonconservative minimal Taylor algebras on {0,1,2}. Recall that an algebra $\mathbf{A}$ is *conservative* if every nonempty subset of A is a subuniverse of $\mathbf{A}$. We already described clones of two nonconservative minimal Taylor algebras in the previous chapter, namely the clones of $\mathbf{T}_4^N$ and $\mathbf{T}_5^N$. Here we aim to describe the clones of $\mathbf{T}_1^N$ and $\mathbf{T}_2^N$. In the rest of this chapter we denote $\{0, 1, 2\}$ by A.

3.1 The Clone of the Algebra $\mathbf{T}_1^N$

In this section, we describe the clone of the algebra $\mathbf{T}_1^N$. Recall that $\mathbf{T}_1^N$ has a single ternary basic operation t defined as follows.

$$t(x_1, x_2, x_3) = \begin{cases} 1 & (x_1 = x_2 = 1) \;\vee\; (x_1 = x_3 = 1) \;\vee\; (x_2 = x_3 = 1) \\ 2 & x_1 = x_2 = x_3 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathbf{T}_1^N$ is isomorphic to the subdirect product of $\mathbf{L}_3$ and $\mathbf{M}$ with the universe $\{(0,0),(0,1),(1,0)\}$ via the isomorphism $0 \mapsto (0,0)$, $1 \mapsto (0,1)$, $2 \mapsto (1,0)$.

Throughout this section we denote $\mathbf{T}_1^N$ by $\mathbf{A}$.

Definition 3.1. *We define a unary operation σ on A as follows.*

$$\sigma(x) = \begin{cases} x & x \neq 2 \\ 0 & x = 2 \end{cases}$$

Also, we define a binary relation $R_\sigma = \{(0,0),(1,1),(2,0)\}$.

Obviously, the relation R_σ can be defined as $R_\sigma = \{(a, \sigma(a)) \mid a \in A\}$. The reason why we are care about the operation σ and the relation R_σ is that R_σ is compatible with $\mathbf{A}$.

Lemma 3.1. *R_σ is compatible with $\mathbf{A}$.*

Proof. Let us have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1, a_2, a_3) = a$, $t(b_1, b_2, b_3) = b$ and $(a_i, b_i) \in R_\sigma$ for each $i \in [3]$. We distinguish three cases.

1. $a = 2$. In such a case we have $a_i = 2$ for all $i \in [3]$. This gives $b_i = 0$ for all $i \in [3]$ and we get $b = 0$.

2. $a = 1$. In this case there are $i, j \in [3]$ such that $i \neq j$ and $a_i = a_j = 1$. This implies $b_i = b_j = 1$ and $b = 1$.

3. $a = 0$. Here we show $b \neq 2$ and $b \neq 1$. Clearly, $b \neq 2$, because $b = 2$ implies $b_i = 2$ for all $i \in [3]$ and $2 \notin \pi_2(R_\sigma)$. If $b = 1$ there are $i, j \in [3]$ such that $b_i = b_j = 1$, so $a_i = a_j = 1$, which imply $a = 1$. Thus we have $b = 0$.

This shows the compatibility of R_σ with t. $\square$

The compatibility of R_σ with $\mathbf{A}$ gives us some information about the behaviour of an operation from $\mathrm{Clo}_n(\mathbf{A})$ if we know how it behaves on $\{0,1\}^n$. We prove this in a more general setting.

Lemma 3.2. *Let B be a set and g be an unary operation on B. Let R be a binary operation on B defined as $R = \{(a, g(a) \mid a \in B\}$. Let f be an n-ary operation on B. Then $g(f(a_1,\ldots,a_n)) = f(g(a_1),\ldots,g(a_n))$ for all $a_1,\ldots,a_n \in B$ if and only if f is compatible with R.*

Proof. We pick $a_1,\ldots,a_n,a,b \in B$, where $i \in [n]$, such that $f(a_1,\ldots,a_n) = a$ and $f(g(a_1),\ldots,g(a_n)) = b$. Clearly, we have $(a_i, g(a_i)) \in R$ for all $i \in [n]$. Now it is easy to see that $(a,b) \in R$ for an arbitrary choice of a_i, for all $i \in [n]$, if and only if R is compatible with f. The condition $(a,b) \in R$ means that $b = g(a)$, thus $f(g(a_1),\ldots,g(a_n)) = g(f(a_1,\ldots,a_n))$. $\square$

Since $0,1$ are the fixed points of σ, we get by Lemma 3.2 that $f(a_1,\ldots,a_2) = a$ implies $f(\sigma(a_1),\ldots,\sigma(a_2)) = a$ for $a \in \{0,1\}$ whenever f is compatible with R_σ. This motivates the following definition.

Definition 3.2. *For an n-ary operation $f \in \mathcal{M}$ we define an n-ary operation $\overline{f}$ on A as follows.*

$$\overline{f}(x_1,\ldots,x_n) = \begin{cases} 2 & x_i = 2 \textit{ for all } i \in [n] \\ f(\sigma(x_1),\ldots,\sigma(x_n)) & \textit{otherwise} \end{cases}$$

We show that $\mathrm{EssClo}_n(\mathbf{A})$ consists exactly of these operations. However, before we prove this, we need to ensure that operations in $\mathrm{EssClo}_n(\mathbf{A})$ do not map n-tuples other than $(2,\ldots,2)$ to 2. To do so, we show the compatibility of $T_{\{0,1\}}$ with $\mathbf{A}$.

Lemma 3.3. *The relation $T_{\{0,1\}}$ is compatible with* $\mathbf{A}$.

Proof. Because $\{0,2\}$ is the universe of a semilattice subalgebra, by Corollary 2.16 we just need need to check that t is compatible with $D_{2,2}$. So assume we have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1,a_2,a_3) = a$, $t(b_1,b_2,b_3) = b$ and $(a_i,b_i) \in D_{2,2}$ for each $i \in [3]$. Without loss of generality assume $a = 2$. Then we have $a_i = 2$ for all $i \in [3]$ and so $b_i = 2$ for all $i \in [3]$. This gives $b = 2$ and $(a,b) \in D_{2,2}$. This shows that $D_{2,2}$ is compatible with t and therefore $T_{\{0,1\}}$ is compatible with t. $\square$

We can now describe $\mathrm{EssClo}_n(\mathbf{A})$ using the operations defined above.

Theorem 3.4. $\mathrm{EssClo}_n(\mathbf{A}) = \{\overline{f} \mid f \in \mathcal{M}_n\}$.

Proof. We denote the signature of $\mathbf{A}$ by Σ_0 and $S = \{\overline{f} \mid f \in \mathcal{M}_n\}$. First we show that $S \subseteq \mathrm{EssClo}_n(\mathbf{A})$. Pick $f \in \mathcal{M}_n$ and a term T of signature Σ_0 over X_n such that $T^{\mathbf{M}} = f$. By Lemma 2.5, we can assume that T contains all the variables from X_n.

Now consider the operation $g = T^{\mathbf{A}} \in \mathrm{Clo}(\mathbf{A})$. The operation g is essential, since we have $g(2,\ldots,2)$ from the idempotency of $\mathrm{Clo}(\mathbf{A})$. However, if we change one coordinate from 2 to 1, then we get an element of $\{0,1\}$. This can be shown by induction on the depth of T, because T contains all the variables from X_n. Thus $g \in \mathrm{EssClo}_n(\mathbf{A})$. By the compatibility of g with R_σ, we get that $g(\sigma(a_1),\dots,\sigma(a_n)) = \sigma(g(a_1,\dots,a_n))$. As mentioned earlier, if there is $i \in [n]$ such that $a_i \neq 2$, we get $g(a_1,\dots,a_n) \neq 2$ and thus $g(a_1,\dots,a_n) = \sigma(g(a_1,\dots,a_n)) = g(\sigma(a_1),\dots,\sigma(a_n))$. This shows $g = \overline{f}$ and hence we have $\overline{f} \in \mathrm{EssClo}_n(\mathbf{A})$.

To show the inclusion $S \supseteq \mathrm{EssClo}_n(\mathbf{A})$, we pick $g \in \mathrm{EssClo}_n(\mathbf{A})$. Since $\{0,1\}$ is a subuniverse of $\mathbf{A}$, we have that $g_{\restriction\{0,1\}}$ is an operation on $\{0,1\}$. Because $t_{\restriction\{0,1\}} = \mathrm{maj}$, we have $g_{\restriction\{0,1\}} \in \mathcal{M}$. To show $\overline{g_{\restriction\{0,1\}}} = g$, we just need to check that $g(2,\dots,2) = 2$ (which holds since $\mathrm{Clo}(\mathbf{A})$ is idempotent) and $g(a_1,\dots,a_n) = g(\sigma(a_1),\dots,\sigma(a_n))$ for each tuple $\mathbf{a} = (a_1,\dots,a_n) \in A^n$ such that there is $i \in [n]$ that $a_i \neq 2$. Since g is essential and compatible with $T_{\{0,1\}}$ we have by Lemma 1.6 that $g(a_1 \dots, a_n) \in \{0,1\}$ and by the compatibility of g with R_σ we get $g(a_1,\dots,a_n) = g(\sigma(a_1),\dots,\sigma(a_n))$. This finishes the proof. □

Using Theorem 3.4 it is rather easy to provide generaters of $\mathrm{Inv}(\mathbf{A})$.

Theorem 3.5. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $\neq_{0,1}$, $\leq_{0,1}$, $R_\sigma, T_{\{0,1\}}$, *and* C_2.

Proof. We show $\mathrm{Pol}(\neq_{0,1}, \leq_{0,1}, R_\sigma, T_{\{0,1\}}, C_2) = \mathrm{Clo}(\mathbf{A})$. We check the compatibility of $\neq_{0,1}$, $\leq_{0,1}$, $R_\sigma, T_{\{0,1\}}$ and C_2 with t. We already checked the compatibility of R_σ and $T_{\{0,1\}}$. The compatibility of $\neq_{0,1}$ and $\leq_{0,1}$ follows from the fact that $t_{\restriction\{0,1\}} = \mathrm{maj}$ and the compatibility with C_2 is trivial since $\mathbf{A}$ is an idempotent algebra. This shows the inclusion $\mathrm{Pol}(\neq_{0,1}, \leq_{0,1}, R_\sigma, T_{\{0,1\}}, C_2) \supseteq \mathrm{Clo}(\mathbf{A})$.

We show the other inclusion for essential operations. Let f be an n-ary essential operation compatible with $\neq_{0,1}$, $\leq_{0,1}$, $R_\sigma, T_{\{0,1\}}$ and C_2. First, we observe that the unary relation $\{0,1\}$ is pp-definable from $\leq_{0,1}$ as a projection. Therefore, f is compatible with $\{0,1\}$. This tell us that $f_{\restriction\{0,1\}}$ is an operation on $\{0,1\}$. Since f is compatible with $\leq_{0,1}$ and $\neq_{0,1}$, we get $f_{\restriction\{0,1\}} \in \mathcal{M}$. Now, similarly as in the proof of the previous theorem, we show by the compatibility of f with R_σ, $T_{\{0,1\}}$, and C_2, that $f = \overline{f_{\restriction\{0,1\}}}$ which shows $f \in \mathrm{EssClo}_n(\mathbf{A})$. This completes the proof. □

3.2 The Clone of the Algebra $\mathbf{T}_2^N$

In this section, we analyze the clone of the algebra $\mathbf{T}_2^N$. Recall that $\mathbf{T}_2^N$ has one basic ternary operation t, which can be defined as follows.

$$t(x_1,x_2,x_3) = \begin{cases} 1 & (x_1 = x_2 = x_2 = 1) \ \vee \ (\exists! i \in [3]\ x_i = 1) \\ 2 & x_1 = x_2 = x_3 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathbf{T}_2^N$ is isomorphic to the subdirect product of $\mathbf{L}_3$ and $\mathbb{Z}_2$ with the universe $\{(0,0),(0,1),(1,0)\}$ via the isomorphism $0 \mapsto (0,0)$, $1 \mapsto (0,1)$, $2 \mapsto (1,0)$.

In this section, we denote $\mathbf{T}_2^N$ by $\mathbf{A}$. The theory used in this section is almost the same as in the previous one. The only work which has to be done is to check the compatibility of R_σ and $T_{\{0,1\}}$ with $\mathbf{A}$.

Lemma 3.6. *The relations R_σ and $T_{\{0,1\}}$ are compatible with* $\mathbf{A}$.

Proof. Firstly, we check the compatibility of R_σ with t. Let us have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1, a_2, a_3) = a$, $t(b_1, b_2, b_3) = b$ and $(a_i, b_i) \in R_\sigma$ for all $i \in [3]$. We distinguish three possibilities.

1. $a = 2$. Then $a_i = 2$ for all $i \in [3]$, which gives $b_i = 0$ for all $i \in [3]$. Therefore, we have $b = 0$.

2. $a = 1$. If $a_i = 1$ for all $i \in [3]$, then $b_i = 1$ for all $i \in [3]$ and $b = 1$. If there is exactly one $i \in [3]$ such that $a_i = 1$, then we have $b_i = 1$ and $b_j \neq 1$ for each $j \in [3] \setminus \{i\}$, because $(a_i, 1) \in R_\sigma$ implies $a_i = 1$. Thus we have $b = 1$.

3. $a = 0$. We show $b \neq 2$ and $b \neq 1$. If $b = 2$, then $b_i = 2$ for all $i \in [3]$. However, this is not possible since there is no such $c \in A$ that $(c, 1) \in R$. If $b = 1$ it would mean $b_i = 1$ for all $i \in [n]$ or there is exactly one $i \in [3]$ such that $b_i = 1$. In both of these cases we easily get $a = 1$, which contradicts the assumption that $a = 0$. Therefore, we have $b = 0$.

This shows that R_σ is compatible with t.

Now we check the compatibility of $T_{\{0,1\}}$ with t. Since $\{0, 2\}$ is the universe of a semilattice subalgebra of $\mathbf{A}$, Corollary 2.16 tells us that it is enough to show that t is compatible with $D_{2,2}$. Let us have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1, a_2, a_3) = a$, $t(b_1, b_2, b_3) = b$ and $(a_i, b_i) \in D_{2,2}$ for each $i \in [3]$. Without loss of generality, we can assume that $a = 2$. Then we have $a_i = 2$ for all $i \in [3]$, thus $b_i = 2$ for all $i \in [3]$ and hence $b = 2$. This shows $D_{2,2}$ is compatible with t and thus $T_{0,2}$ is compatible with t.

□

Similarly as in the previous section, we are going to extend operations on $\{0, 1\}$ to operations on $\{0, 1, 2\}$.

Definition 3.3. *For an n-ary operation $f \in \mathrm{Clo}(\mathbb{Z}_2)$ we define an n-ary operation $\overline{f}$ on A as follows.*

$$\overline{f}(x_1, \ldots, x_n) = \begin{cases} 2 & x_i = 2 \textit{ for all } i \in [n] \\ f(\sigma(x_1), \ldots, \sigma(x_n)) & \textit{otherwise} \end{cases}$$

Now we describe $\mathrm{EssClo}(\mathbf{A})$ and $\mathrm{Inv}(\mathbf{A})$ in the same manner as we did in the previous section.

Theorem 3.7. $\mathrm{EssClo}_n(\mathbf{A}) = \{\overline{f} \mid f \in \mathrm{Clo}(\mathbb{Z}_2)\}$.

Proof. The proof is almost the same as the proof of Theorem 3.4. The only differences are that we are using Lemma 2.8 instead of Lemma 2.5 and that we have $t_{\restriction\{0,1\}} = \mathrm{aff}$ instead of $t_{\restriction\{0,1\}} = \mathrm{maj}$. □

Theorem 3.8. $\mathrm{Inv}(\mathbf{A})$ *is generated by $S_{1,0}$, $R_\sigma, T_{\{0,1\}}$, C_1 and C_2.*

Proof. Since $S_{1,0}$ and C_1 is compatible with aff and $t_{\restriction\{0,1\}} = \mathrm{aff}$, we get that $S_{1,0}$ and C_1 are compatible with $\mathbf{A}$. We have that $\{0, 1\}$ is pp-definable from $S_{1,0}$ as a projection. Therefore, for an arbitrary $g \in \mathrm{EssClo}(\mathbf{A})$, we have that $g_{\restriction\{0,1\}}$ is an operation on $\{0, 1\}$ and, by using the compatibility of $S_{1,0}$ and C_1 with $g_{\restriction\{0,1\}}$, we get $g_{\restriction\{0,1\}} \in \mathrm{Clo}(\mathbb{Z}_2)$ The rest of the proof just copies the proof of Theorem 3.5. □

4. The Clones of the Algebras $\mathbf{T}_2^C$ and $\mathbf{T}_5^C$

In this chapter, we want to describe the clones of algebras $\mathbf{T}_2^C$ and $\mathbf{T}_5^C$. These algebras are quite similar. Both of them have a single symmetric ternary basic operation t such that $t_{\restriction\{0,1\}} = \min^3_{0\leq 1}$, $t_{\restriction\{0,2\}} = \min^3_{0\leq 2}$ and $t(0,1,2) = 0$. This tells us that t is maps a tuple to 0 whenever one of its coordinates is 0. This is quite a strong condition, which we will use throughout this chapter.

4.1 The Clone of the Algebra $\mathbf{T}_2^C$

In this section, we describe an algebra $\mathbf{T}_2^C$. Recall that $\mathbf{T}_2^C$ has a single ternary operation t, which is defined as follows.

$$t(x_1, x_2, x_3) = \begin{cases} \mathrm{maj}_{1,2}(x_1, x_2, x_3) & x_1, x_2, x_3 \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

Throughout this section we denote $\{0, 1, 2\}$ by A and $\mathbf{T}_2^C$ by $\mathbf{A}$. Immediately, from the definition of t, we can guess that $\mathrm{Clo}(\mathbf{A})$ consists of monotone self-dual operations (with respect to 1,2), which are returning 0 if they have 0 on the input. Let us make this intuition more formal.

Definition 4.1. *For $f \in \mathcal{M}^{1,2}$ we define an n-ary operation $\overline{f}$ on A as follows.*

$$\overline{f}(x_1, \dots, x_n) = \begin{cases} 0 & \text{there is } i \in [n] \text{ such that } x_i = 0 \\ f(x_1, \dots, x_n) & \text{otherwise} \end{cases}$$

Now using the definition above, we directly show how $\mathrm{EssClo}(\mathbf{A})$ looks.

Theorem 4.1. $\mathrm{EssClo}(\mathbf{A}) = \{\overline{f} \mid f \in \mathcal{M}^{1,2}\}$.

Proof. Let us denote $S = \{\overline{f} \mid f \in \mathcal{M}^{1,2}\}$ and Σ the signature of $\mathbf{A}$. First we check that $S \subseteq \mathrm{EssClo}(\mathbf{A})$. Let us fix $f \in \mathcal{M}^{1,2}_n$. We denote by T a term over X_n of the signature Σ such that $T^{\mathbf{M}^{1,2}} = f$. By Lemma 2.5, we may assume that T contains all the variables of X_n. Now we take $g = T^{\mathbf{A}}$. By the definition, we have $g \in \mathrm{Clo}(\mathbf{A})$. Using the induction on the depth of T, we can easily show that g returns 0 if there is 0 among the arguments (here, we are using that T contains all the variables of X_n). On the arguments not containing 0 we have that g behaves like f, thus we get $g = \overline{f}$ and so $\overline{f} \in \mathrm{Clo}(\mathbf{A})$. Clearly, $\overline{f}$ is essential since $\overline{f}(1, \dots, 1) = 1$ and if change one arbitrary coordinate to 0, we get 0. Therefore, we get $\overline{f} \in \mathrm{EssClo}(\mathbf{A})$.

It remains to show $S \supseteq \mathrm{EssClo}(\mathbf{A})$. We fix $g \in \mathrm{EssClo}(\mathbf{A})$ and T a term of Σ over X_n such that $g = T^{\mathbf{A}}$. Clearly, $g_{\restriction\{1,2\}} \in \mathcal{M}^{1,2}$ since $\{1, 2\}$ is a subuniverse of $\mathbf{A}$. We show that $g = \overline{g_{\restriction\{1,2\}}}$. To see this, it is enough to check that g returns 0 if there is 0 presented in the entry. Pick some term T over X_n of the signature Σ such that $T^{\mathbf{A}} = g$. Since g is essential, every variable of X_n occurs in T, therefore the induction on the depth of T gives as above that g return 0 if 0 is among the arguments. This gives $g = \overline{g_{\restriction\{1,2\}}}$, and so $g \in S$ and the proof is complete. $\square$

In the the last theorem, we proved that operations in $\mathrm{EssClo}(\mathbf{A})$ are determined by two things. Their restrictions to $\{1,2\}$ are monotone self-dual operations, and they return 0 on the remaining inputs. We want to give a description of $\mathrm{Inv}(\mathbf{A})$. The important relations are $\neq_{1,2}$, $\leq_{1,2}$, which tell us that operations in $\mathrm{Clo}(\mathbf{A})$ restricted to $\{1,2\}$ are indeed monotone and self-dual, and T_0, which ensures that essential operations applied to tuples containing 0 return 0.

Theorem 4.2. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $\leq_{1,2}$, $\neq_{1,2}$ *and* $T_{\{0\}}$.

Proof. We show $\mathrm{Pol}(\leq_{1,2}, \neq_{1,2}, T_{\{0\}}) = \mathrm{Clo}(\mathbf{A})$. For the inclusion $\mathrm{Pol}(\leq_{1,2}, \neq_{1,2}, T_{\{0\}}) \supseteq \mathrm{Clo}(\mathbf{A})$, we check the compatibility of $\leq_{1,2}$, $\neq_{1,2}$ and $T_{\{0\}}$ with t. The compatibility with of $\leq_{1,2}$ and $\neq_{1,2}$ with t is obvious since $t_{\restriction\{1,2\}} = \mathrm{maj}_{1,2}$. We need to check the compatibility of $T_{\{0\}}$. By Corollary 2.16, it is enough to check that $D_{0,0}$ is compatible with t (because $\{0,1\}$ is the universe of a semilattice subalgebra with the ordering $0 \leq 1$). Assume $t(a_1,a_2,a_3) = a$, $t(b_1,b_2,b_3) = b$ for some $a_i, a, b_i, b \in A$, where $i \in [3]$, and $(a_i, b_i) \in D_{\{0,0\}}$ for all $i \in [3]$. Without loss of generality we also assume $a = 0$. Then there is $i \in [3]$ such that $a_i = 0$. Therefore, we get $b_i = 0$ and so $b = 0$. This shows $(a,b) \in D_{0,0}$, hence $D_{0,0}$ is compatible with t. Thus $T_{\{0\}}$ is compatible with t and we have $\mathrm{Pol}(\leq_{1,2}, \neq_{1,2}, T_{\{0\}}) \supseteq \mathrm{Clo}(\mathbf{A})$.

Without loss of generality we check the other inclusion $\mathrm{Pol}(\leq_{1,2}, \neq_{1,2}, T_{\{0\}}) \subseteq \mathrm{Clo}(\mathbf{A})$ only for an essential operation f. Observe that the unary relation $\{1,2\}$ is pp-definable as a projection of $\leq_{1,2}$, thus $f_{\{1,2\}}$ is an operation on $\{1,2\}$. Since $f_{\{1,2\}}$ is compatible with $\leq_{1,2}$ and $\neq_{1,2}$, we get $f_{\restriction\{1,2\}} \in \mathcal{M}^{1,2}$. We show that $f = \overline{f_{\restriction\{1,2\}}}$. To do so, we just have to show that if there is 0 among the arguments, then f returns 0. But this follows from the compatibility of f with T_0 by Lemma 1.6. This shows $f = \overline{f_{\restriction\{1,2\}}}$ and hence $f \in \mathrm{Clo}(\mathbf{A})$. □

4.2 The Clone of the Algebra $\mathbf{T}_5^C$

In this section we study the algebra $\mathbf{T}_5^C$. Recall that $\mathbf{T}_5^C$ has a single ternary operation defined as follows.

$$t(x_1,x_2,x_3) = \begin{cases} \mathrm{aff}_{1,2}(x_1,x_2,x_3) & x_1,x_2,x_3 \in \{1,2\} \\ 0 & \text{otherwise} \end{cases}$$

This algebra is almost the same as $\mathbf{T}_2^C$ and we use the same technique to describe its clone.

Definition 4.2. *For* $f \in \mathrm{Clo}(\mathbb{Z}_2^{1,2})$ *we define an* n*-ary operation* $\overline{f}$ *on* A *as follows.*

$$\overline{f}(x_1,\dots,x_n) = \begin{cases} 0 & \text{there is } i \in [n] \text{ such that } x_i = 0 \\ f(x_1,\dots,x_n) & \text{otherwise} \end{cases}$$

Theorem 4.3. $\mathrm{EssClo}(\mathbf{A}) = \{\overline{f} \mid f \in \mathbb{Z}_2^{1,2}\}$.

Proof. The proof is almost the same in the proof of Theorem 4.1. We just use Lemma 2.8 instead of Lemma 2.5 and that $g_{\restriction\{1,2\}} \in \mathrm{Clo}(\mathbb{Z}_2^{1,2})$ instead of $g_{\restriction\{1,2\}} \in \mathcal{M}^{1,2}$. □

Theorem 4.4. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $S_{1,2}$*,* $T_{\{0\}}$ *and* C_1*.*

Proof. We use that $S_{1,2}$ and C_1 generate $\mathrm{Inv}(\mathbb{Z}_2^{1,2})$ and that $\{1,2\}$ is pp-definable as a projection of $S_{1,2}$. The rest of the proof is the same as the proof of Theorem 4.2. $\square$

5. The Majority Clones on Three Elements

In this chapter, we give a description of clones of all minimal Taylor algebras on three elements which are majority algebras. There are three such algebras: $\mathbf{T}_1^P$, $\mathbf{T}_{14}^C$ and $\mathbf{T}_{15}^C$. These algebras have a single basic operation, which is a majority operation.

5.1 Introduction to Majority Algebras

In this section, we recall Section 2.1, and we introduce some new notation related to majority algebras. As it was already mentioned, a useful tool related to majority algebras is Theorem 2.1, which allows us to describe invariant relations by using only binary relations. In the following sections of this chapter, we always first describe $\mathrm{Inv}(\mathbf{A})$ using the binary relations, and then we explicitly describe elements of $\mathrm{Clo}(\mathbf{A})$.

There is a couple of relations we use throughout this chapter.

Definition 5.1. *Let B be a set. For $a, b \in B$ we define binary relations $O_{a,b}$, $E_{a,b}$, $D_{a,b}$, $D_{a,b}^{-1}$, N_a and N_a^{-1} as*

$$(x, y) \in O_{a,b} \iff x = a \ \vee \ y = b$$

$$(x, y) \in E_{a,b} \iff x \neq a \ \vee \ y \neq b,$$

$$(x, y) \in D_{a,b} \iff x \neq a \ \vee \ y = b,$$

$$(x, y) \in D_{a,b}^{-1} \iff x = a \ \vee \ y \neq b,$$

$$(x, y) \in N_a \iff y \neq a,$$

and

$$(x, y) \in N_a^{-1} \iff x \neq a.$$

Recall that we already saw relation $D_{a,b}$ earlier in this thesis. We can see that the relations $D_{a,b}^{-1}$ and N_a^{-1} can be pp-defined from $D_{b,a}$ and N_a. Therefore, relations $D_{a,b}^{-1}$ and N_a^{-1} are redundant (in the sense of generating relations using pp-definitions). Although, we keep these relations since it simplifies stating some claims later on.

First, let us analyze the relation $O_{a,b}$.

Lemma 5.1. *Let m be a majority operation on a set A and $a, b \in A$. Then the relation $O_{a,b}$ is compatible with m.*

Proof. Let us have $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $m(c_1, c_2, c_3) = c$ and $m(d_1, d_2, d_3) = d$. Assume we have $(c_i, d_i) \in O_{a,b}$ for all $i \in [3]$. Then there are $i, j \in [3]$ such that $c_i = c_j = a$ or $d_i = d_j = b$. In the first case we have (by definition of majority operation) that $c = a$, in the second case we have $d = b$. Anyway, we have $(c, d) \in O_{a,b}$. This shows $O_{a,b}$ is compatible with m. $\square$

We observe that N_a can be pp-defined as follows.

$$(x, y) \in N_a \iff \exists z\ (y, z) \in O_{b,1} \ \wedge\ (z, y) \in O_{2,c},$$

where $\{a, b, c\} = \{0, 1, 2\}$. This tells us that N_a and N_a^{-1} are compatible with all the majority operations.

Although $O_{a,b}$ and N_a are always compatible with a majority clone, the same not holds for $D_{a,b}$ and $E_{a,b}$. However, these relations are still compatible in many cases. To understand these relations, we first give a definition of weak points of majority operations.

Definition 5.2. *Let m be a majority operation on a set A. We say $a \in A$ is a* weak point *of m if*

$$m(x, y, z) = a \iff x = y = a \ \vee\ y = z = a \ \vee\ x = z = a.$$

Lemma 5.2. *Let m be a majority operation on a set A and $a, b \in A$. Then*

1. *If a is a weak point of m, then $D_{a,b}$ is compatible with m.*
2. *If a, b are weak points of m, then $E_{a,b}$ is compatible with m.*

Proof. Let a be a weak point of m and let us have $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $m(c_1, c_2, c_3) = c$ and $m(d_1, d_2, d_3) = d$. Assume $(c_i, d_i) \in D_{a,b}$ for all $i \in [3]$. We want to show $(c, d) \in D_{a,b}$. Without loss of generality we can assume $c = a$. Because a is a weak point of m, there are $i, j \in [3]$ such that $i \neq j$ and $c_i = c_j = a$. The conditions $(c_i, d_i), (c_j, d_j) \in D_{a,b}$ give $d_i = d_j = b$, so by the definition of a majority operation it follows $d = b$ and $(c, d) \in D_{a,b}$. Hence $D_{a,b}$ is compatible with t.

Let a, b be a weak points of m and let us have $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $m(c_1, c_2, c_3) = c$ and $m(d_1, d_2, d_3) = d$. Assume $(c, d) \notin E_{a,b}$. We want to show that there is $i \in [3]$ such that $(c_i, d_i) \notin E_{a,b}$. Because $(c, d) \notin E_{a,b}$, we have $c = a$ and $d = b$. As a, b are weak points of m, there are $i, j, k, l \in [3]$ such that $i \neq j$, $k \neq l$, $c_i = c_j = a$ and $d_k = d_l = b$. Without loss of generality we may assume $i = k$. Thus we have $c_i = a$ and $d_i = d_k = b$, so $(c_i, d_i) \notin E_{a,b}$. This shows that $E_{a,b}$ is compatible with m. ☐

Before we end this section, we define two more relations.

Definition 5.3. *For $A = \{a, b, c\}$ we define binary relations $P_{a,b}$ and $P_{a,b,c}$ on A as follows.*

$$P_{a,b} = \{(a, b), (b, a), (c, c)\}$$
$$P_{a,b,c} = \{(a, b), (b, c), (c, a)\}$$

By Lemma 3.2, an algebra $\mathbf{A}$ with universe $\{a, b, c\}$ is compatible with $P_{a,b}$ (resp. $P_{a,b,c}$) if and only if the permutation, which is represented by the cycle (a, b) (resp. (a, b, c)), is an automorphism of $\mathbf{A}$. It is easy to decide whether some permutation is an automorphism, therefore the compatibility of relations $P_{a,b}$ and $P_{a,b,c}$ with some algebra is easy to check. Here we recall that any permutation on $\{a, b, c\}$ can be composed from permutations (a, b) and (a, b, c). Since we will be dealing with algebras with universe $\{0, 1, 2\}$, we denote for the rest of this chapter $\{0, 1, 2\}$ by A.

5.2 The Clone of the Algebra $\mathbf{T}_1^P$

In this section, we describe the clone of the algebra $\mathbf{T}_1^P$. Recall that $\mathbf{T}_1^P$ has only one ternary basic operation m, which can be defined as follows.

$$m(x,y,z) = \begin{cases} \mathrm{maj}_{a,b}(x,y,z) & \{x,y,z\} \in \{a,b\} \\ x & \{x,y,z\} = \{0,1,2\} \end{cases}$$

In the rest of this section, we denote $\mathbf{T}_1^P$ by $\mathbf{A}$. We immediately see that m is a majority operation and that every permutation on A is an automorphism of $\mathbf{A}$. Therefore $\mathbf{A}$ is compatible with $P_{a,b}$, $P_{a,b,c}$ and $O_{a,b}$ for all $a,b \in A$.

As was already mentioned in the previous section, we start by describing all the binary relations which are compatible with $\mathbf{A}$. We will use the following lemma.

Lemma 5.3. *Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ with universe B, $R \leq_{sd} \mathbf{A} \times \mathbf{B}$, $A = \{a_1, a_2, a_3\}$, and $b \in B$. If $(a_1, b), (a_2, b) \in R$, then $(a_3, b) \in R$.*

Proof. Because R is a subdirect product of $\mathbf{A}$ and $\mathbf{B}$, there is $c \in B$ such that $(a_3, c) \in R$. Because $m(a_3, a_1, a_2) = a_3$ and $m(c, b, b) = b$, using the compatibility of R with m we get $(a_3, b) \in R$. □

Throughout the rest of this section, we use Lemma 5.3 and its version for $R \leq_{sd} \mathbf{B} \times \mathbf{A}$ all the time. Now we take a look at binary subdirect relations which are compatible with $\mathbf{A}$.

Lemma 5.4. *Let R be a nontrivial binary relation (i.e. $R \neq A^2$) from* $\mathrm{Inv}(\mathbf{A})$ *and $R \leq_{sd} A \times A$. Then $R = O_{a,b}$, $R = P_{a,b}$ or $R = P_{a,b,c}$ for some $a, b, c \in A$.*

Proof. Let us have some nontrivial binary relation $R \in \mathrm{Inv}(\mathbf{A})$ such that $R \leq_{sd} A^2$ and $(a_1, b_1) \notin R$ for some $a_1, b_1 \in A$. By Lemma 5.3, there has to be $a_2 \in A \setminus \{a_1\}$ and $b_2 \in A \setminus \{b_1\}$ such that $(a_2, b_1), (a_1, b_2) \notin A$. Set $A = \{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$. Using Lemma 5.3 again, we get $(a_2, b_2) \notin R \ \vee \ (a_2, b_3) \notin R$ and $(a_2, b_2) \notin R \ \vee \ (a_3, b_2) \notin R$.

So there are two possibilities:

1. $(a_2, b_2) \notin R$. In this case we know that $R \in A^2 \setminus (\{a_1, a_2\} \times \{b_1, b_2\})$, thus $R \subseteq O_{a_3, b_3}$. Because R is subdirect, it follows that $(a_1, b_3), (a_2, b_3)$, (a_3, b_1), $(a_3, b_2) \in R$. Once again using Lemma 5.3 it follows $(a_3, b_3) \in R$. Thus $R = O_{a,b}$ for some $a, b \in A$.

2. $(a_3, b_2), (a_2, b_3) \notin R$. In this case we can use Lemma 5.3 again and we get $(a_1, b_3) \notin R \ \vee \ (a_3, b_3) \notin R$ and $(a_3, b_1) \notin R \ \vee \ (a_3, b_3) \notin R$. The case $(a_3, b_1), (a_1, b_3) \notin R$ implies $(a_1, b_1), (a_1, b_2), (a_1, b_3) \notin R$, thus R is not a subdirect product. So we may assume $(a_3, b_3) \notin R$. It follows $R \subseteq \{(a_1, b_3), (a_2, b_2), (a_3, b_1)\}$. Because every subdirect product of A^2 has to have at least three elements, we get $R = \{(a_1, b_3), (a_2, b_2), (a_3, b_1)\}$ and thus $R = P_{a,b}$ or $P_{a,b,c}$ for some $a, b, c \in A$.

This shows $R = O_{a,b}$, $R = P_{a,b}$ or $P_{a,b,c}$ for some $a, b, c \in A$. □

We describe the remaining binary relations in $\mathrm{Inv}(\mathbf{A})$.

Lemma 5.5. *Every binary relation in* $\mathrm{Inv}(\mathbf{A})$ *is pp-definable from the set*

$$\{O_{a,b}, P_{a,b}, P_{a,b,c} \mid a, b, c \in A\}.$$

Proof. From Lemma 5.4 we already know that the statement holds for relations which are subdirect in $\mathbf{A}^2$. We fist show that any binary relation in $\mathrm{Inv}(\mathbf{A})$ is pp-definable from the set $\{O_{a,b}, P_{a,b}, P_{a,b,c} \mid a, b, c \in A\}$ and unary relations. Let us consider the remaining cases.

1. $R \leq_{sd} \mathbf{A} \times \mathbf{B}$, where $\mathbf{B} \lneq \mathbf{A}$. Then there has to be $b_1 \in B$ and a_1, a_2 such that $a_1 \neq a_2$ and $(a_1, b_1), (a_2, b_1) \in R$. Denote $A = \{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$, and $b_3 \in A \setminus B$. From Lemma 5.3 it follows that $(a_3, b_1) \in R$. Now observe that if $(a_1, b_2), (a_2, b_2) \in R$, then Lemma 5.3 gives $(a_3, b_2) \in R$ and $R = A \times \{b_1, b_2\}$. Thus we can have one of the following relations: $A \times \{b_1, b_2\}$, $A \times \{b_1\}$ and $(A \times \{b_1\}) \cup \{(a_i, b_2)\}$ for some $i \in [3]$. The first two relations are pp-definable using the unary relations. The third one is pp-definable as

$$(x, y) \in (A \times \{b_1\}) \cup \{(a_i, b_2)\} \iff ((x, y) \in O_{a_i,b_1}) \wedge (y \in \{b_1, b_2\}).$$

2. $R \leq_{sd} \mathbf{B} \times \mathbf{A}$, where $\mathbf{B} \lneq \mathbf{A}$. This case is analogous to the previous one.

3. $R \leq_{sd} \mathbf{B}_1 \times \mathbf{B}_2$, $|R| \geq 3$, $\mathbf{B}_1, \mathbf{B}_2 \lneq \mathbf{A}$, $B_1 = \{a_1, a_2\}$ and $B_2 = \{b_1, b_2\}$. In such a case we have $R = B_1 \times B_2$ or $R = (B_1 \times B_2) \setminus \{(a_i, b_j)\}$, for some $i, j \in [2]$. The first relation is pp-definable from unary relations, the second one can be pp-defined as

$$(x, y) \in (B_1 \times B_2) \setminus \{(a_i, b_j)\} \iff ((x, y) \in B_1 \times B_2) \ \wedge \ ((x, y) \in O_{a_k,b_l}),$$

where $k \in [2] \setminus \{i\}$ and $l \in [2] \setminus \{j\}$.

4. $R = \{(a_1, b_1), (a_2, b_2)\}$ for some $a_1, a_2, b_1, b_2 \in A$. Then R can be pp-defined as

$$(x, y) \in \{(a_1, b_1), (a_2, b_2)\} \iff (x, y) \in O_{a_1,b_2} \ \wedge \ (x, y) \in O_{a_2,b_1}.$$

5. $R = \{a, b\}$ for some $a, b \in A$. This relation is obviously pp-definable using only unary relations.

Now it remains to check that we can pp-define an arbitrary unary relation from the set $\{O_{a,b}, P_{a,b}, P_{a,b,c} \mid a, b, c \in A\}$. It is enough to show that any two element unary relation can be pp-defined from such a set. Clearly, any two element unary relation can be pp-defined as a projection of some two element binary relation and we can pp-define all the two element binary relations only from the set $\{O_{a,b} \mid a, b \in A\}$. Therefore, we can pp-define any unary relation from the set $\{O_{a,b} \mid a, b \in A\}$. This completes the proof. $\square$

We can pp-define $O_{a,b}$ for any $a, b \in A$ just from $O_{0,0}$, $P_{0,1}$, and $P_{0,1,2}$. Since $\mathrm{Inv}(\mathbf{A})$ is generated by binary relations, we get the following result.

Theorem 5.6. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $O_{0,0}, P_{0,1}, P_{0,1,2}$.

When we have a description of $\mathrm{Inv}(\mathbf{A})$, we describe the elements of $\mathrm{Clo}(\mathbf{A})$. Not surprisingly, we use ms-collections.

Theorem 5.7. $\mathrm{Clo}_n(\mathbf{A})$ *consists of such operations f which are compatible with all the permutation relations on A (i.e the relations $P_{a,b}$, $P_{a,b,c}$, where $a,b,c \in A$) and there is some ms-collection $\mathfrak{F}$ on $[n]$ such that*

$$(\forall a, a_1, \dots, a_n \in A)\ \ \{i \in [n] \mid a_i = a\} \in \mathfrak{F} \implies f(a_1, \dots, a_n) = a. \tag{5.1}$$

Proof. Denote S the set of n-ary operations which are compatible with all the permutation relations and which satisfies Equation 5.1 for some ms-collection on $[n]$. To show $S \subseteq \mathrm{Clo}(\mathbf{A})$ we only need to check that every $f \in S$ is compatible with $O_{0,0}$. So let us fix $f \in S$, $\mathfrak{F}$ ms-collection, such that Equation 5.1 holds for f and assume we have $a_1, b_1, \dots, a_n, b_n \in A$ such that $(a_i, b_i) \in O_{0,0}$ for all $i \in [n]$. Fix $a, b \in A$ such that $f(a_1, \dots, a_n) = a$ and $f(b_1, \dots, b_n) = b$. Without loss of generality assume $a \neq 0$. Then, by Equation 5.1, we know that $\{i \in [n] \mid a_i = 0\} \notin \mathfrak{F}$ so by self-duality of $\mathfrak{F}$ we have $\{i \in [n] \mid a_i \neq 0\} \in \mathfrak{F}$. Using $(a_i, b_i) \in O_{0,0}$ we get $\{i \in [n] \mid b_i = 0\} \supseteq \{i \in [n] \mid a_i \neq 0\}$ and by monotonicity of $\mathfrak{F}$ we get $\{i \in [n] \mid b_i = 0\} \in \mathfrak{F}$. Thus $b = 0$ and $(a, b) \in O_{0,0}$. This shows that f is compatible with $O_{0,0}$, so $S \subseteq \mathrm{Clo}_n(\mathbf{A})$.

We show the other inclusion. Fix $f \in \mathrm{Clo}_n(\mathbf{A})$. Because $\{0,1\}$ is a subuniverse of A and $t_{\restriction\{0,1\}} = \mathrm{maj}_{0,1}$, it follows that $f_{\restriction\{0,1\}} \in \mathcal{M}^{0,1}$ and so $\mathfrak{F}_{f_{\restriction\{0,1\}}}$ is an ms-collection on $[n]$. We show

$$\forall a_1, \dots, a_n \in A\ \ \{i \in [n] \mid a_i = 0\} \in \mathfrak{F}_{f_{\restriction\{0,1\}}} \implies f(a_1, \dots, a_n) = 0,$$

the rest of Equation 5.1 follows from the compatibility of f with the permutation relations. Let us have $a_1, \dots, a_n \in A$ such that $\{i \in [n] \mid a_i = 0\} \in \mathfrak{F}_{f_{\restriction\{0,1\}}}$. We define $b_i \in \{0,1\}$ for all $i \in [n]$ as follows.

$$b_i = \begin{cases} 1 & a_i = 0 \\ 0 & a_i \neq 0 \end{cases}$$

So we have $(a_i, b_i) \in O_{0,0}$ for all $i \in [n]$. We have $\{i \in [n] \mid b_i = 1\} \in \mathfrak{F}_{f_{\restriction\{0,1\}}}$, thus by the definition of $\mathfrak{F}_{f_{\restriction\{0,1\}}}$ we have $f(b_1, \dots, b_n) = 1$. Because f is compatible with $O_{0,0}$ and $(a_i, b_i) \in O_{0,0}$ for all $i \in [n]$, it follows that $(f(a_1, \dots, a_n), 1) \in O_{0,0}$, hence $f(a_1, \dots, a_n) = 0$. Therefore $S \supseteq \mathrm{Clo}(\mathbf{A})$, which completes the proof.

□

5.3 The Clone of the Algebra $\mathbf{T}_{14}^C$

In this section, we describe the clone of the algebra $\mathbf{T}_{14}^C$. Recall that the algebra $\mathbf{T}_{14}^C$ has a single ternary operation defined as follows.

$$m(x,y,z) = \begin{cases} \mathrm{maj}_{a,b}(x,y,z) & \{x,y,z\} = \{a,b\}\ \ a,b \in \{0,1,2\} \\ 0 & \{x,y,z\} = \{0,1,2\} \end{cases}$$

For the rest of this section, denote $\mathbf{T}_{14}^C$ by $\mathbf{A}$. We immediately see that m is a majority operation with the weak points 1 and 2 and that the permutation given by the cycle $(1,2)$ is an automorphism. This tells us that $\mathbf{A}$ is compatible with relations $E_{a,b}$, $D_{a,b}$, $O_{a,b}$, N_0, and $P_{1,2}$, where $a,b \in \{1,2\}$. Because every relation in a majority algebra can be pp-defined from binary relations, we, similarly as in the previous section, describe all the binary relations compatible with $\mathbf{A}$. Before we start, we make one observation.

Lemma 5.8. *Let R be a binary relation in* $\mathrm{Inv}(\mathbf{A})$. *Assume $(a,0) \in R$ and $(b,0) \notin R$ (resp. $(0,a) \in R$ and $(0,b) \notin R$) for some $a,b \in A$. Then $(1,b) \notin R$ or $(2,b) \notin R$ (resp. $(b,1) \notin R$ or $(b,2) \notin R$).*

Proof. We have $m(1,2,0) = 0$, $m(b,b,a) = b$, $(0,a) \in R$ and $(0,b) \notin R$. Since R is the compatible with m, we get $(1,b) \notin R$ or $(2,b) \notin R$. The rest of the claim follows by symmetry. $\square$

Now we describe all the binary relations in $\mathrm{Inv}(\mathbf{A})$.

Lemma 5.9. *Let R be a binary relation compatible with m. Then*

$$R = \bigwedge_{\substack{E_{a,b} \supseteq R \\ a,b \in \{1,2\}}} E_{a,b} \wedge \bigwedge_{\substack{D_{c,d} \supseteq R \\ c,d \in \{1,2\}}} D_{c,d} \wedge \bigwedge_{\substack{D^{-1}_{e,f} \supseteq R \\ e,f \in \{1,2\}}} D^{-1}_{e,f} \wedge \bigwedge_{\substack{O_{g,h} \supseteq R \\ g,h \in \{1,2\}}} O_{g,h} \wedge \bigwedge_{\substack{S \supseteq R \\ S \in \{N_0, N_0^{-1}\}}} S. \quad (5.2)$$

Proof. Denote by S the right hand side of Equation 5.2. Let $x,y \in A$. If $(x,y) \in R$, then obviously $(x,y) \in S$. Assume now that $(x,y) \notin R$. In order to show that $(x,y) \notin S$, we distinguish four cases.

1. $x,y \in \{1,2\}$. In this case $R \subseteq E_{x,y}$ and $(x,y) \notin E_{x,y}$, thus $(x,y) \notin S$.

2. $x \in \{1,2\}$ and $y = 0$. If $(a,0) \notin R$ for each $a \in A$, we get $R \subseteq N_0$ and thus $(x,y) \notin S$. Otherwise, Lemma 5.8 gives that there is $d \in \{1,2\}$ such that $(x,d) \notin R$. Since $(x,d),(x,0) \notin R$, then we get $D_{x,c} \supseteq R$ for $c \in A \setminus \{0,d\}$. We get $(x,0) \notin D_{x,c}$, thus $(x,0) = (x,y) \notin S$.

3. $y \in \{1,2\}$ and $x = 0$. This case is analogous to the previous one.

4. $x = y = 0$. If $(a,0) \notin R$ (resp. $(0,a) \notin R$) for each $a \in A$, we get $R \subseteq N_0$ (res. $R \subseteq N_0^{-1}$) and thus $(x,y) \notin S$. Otherwise, Lemma 5.8 gives $c \in \{1,2\}$ such that $(c,0) \notin R$. By using Lemma 5.8 more times, we can derive that $(0,d),(c,e),(f,d) \notin R$ for some $d,e,f \in \{1,2\}$.

   If $c = f$ or $d = e$, then we have $(c,d) \notin R$, so $(0,0),(c,0),(0,d),(c,d) \notin R$. Therefore, $R \subseteq O_{g,h}$, where $g \in A \setminus \{0,c\}$ and $h \in A \setminus \{0,d\}$. As $(0,0) \notin O_{g,h}$, it follows that $(0,0) \notin S$.

   Otherwise $c \neq f$ and $d \neq e$ and we may without loss of generality assume $(c,d) \in R$. In this case $(0,e) \notin R$ or $(f,0) \notin R$, because we have $m(0,f,c) = 0$, $m(e,0,d) = 0$, $(0,0) \notin R$ and R is compatible with m. Therefore we can assume without loss of generality $(0,e) \notin R$, hence $(0,0),(0,e),(c,e),(c,0) \notin R$. So we have $R \subseteq O_{f,d}$. Similarly as above we get $(0,0) \notin S$.

$\square$

Now we simplify the result. As it was mentioned earlier, we can pp-define N_0 and N_0^{-1} just using $O_{b,c}$, where $b, c \in \{1, 2\}$. We can pp-define $E_{a,b}$ from $D_{a,1}$ and $D_{b,2}$ as follows.

$$(x, y) \in E_{a,b} \iff \exists z\ (x, z) \in D_{a,1} \ \wedge\ (y, z) \in D_{b,2}$$

Using the relation $P_{1,2}$ we can pp-define $D_{a,1}$, $O_{a,b}$ for $a, b \in \{1, 2\}$ just from $O_{1,1}, D_{1,1}$ and $P_{1,2}$. Since $\mathrm{Inv}(\mathbf{A})$ is generated by binary relations we get the following theorem.

Theorem 5.10. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $D_{1,1}, O_{1,1}$ *and* $P_{1,2}$.

Now we describe the elements of $\mathrm{Clo}(\mathbf{A})$. We will show that there is a one-to-one correspondence between the elements of $\mathrm{Clo}(\mathbf{A})$ and the elements of $\mathcal{M}$. To see this, it is enough to show that there is a one-to-one correspondence between elements of $\mathrm{Clo}_n(\mathbf{A})$ and ms-collections on $[n]$.

Theorem 5.11. $\mathrm{Clo}_n(\mathbf{A})$ *consists of such operations* f *that there is an ms-collection* $\mathfrak{F}$ *satisfying the following.*

$$f(a_1, \dots, a_n) = \begin{cases} 1 & \{i \in [n] \mid a_i = 1\} \in \mathfrak{F} \\ 2 & \{i \in [n] \mid a_i = 2\} \in \mathfrak{F} \\ 0 & \textit{otherwise} \end{cases} \tag{5.3}$$

Proof. Let $\mathfrak{F}$ be an ms-collection. Let f be an n-ary operation defined using the formula 5.3. We show that $f \in \mathrm{Clo}_n(\mathbf{A})$. Clearly, such f is compatible with $P_{1,2}$ by Lemma 3.2. We show that f is compatible with $D_{1,1}$. Let us have $f(a_1, \dots, a_n) = a$ and $f(b_1, \dots, b_n) = b$ for some a_i, b_i, a, b, where $i \in [n]$. Moreover, assume that $(a_i, b_i) \in D_{1,1}$ for all $i \in [n]$. We will show that $(a, b) \in D_{1,1}$. Without loss of generality we can assume $a = 1$. Then $\{i \in [n] \mid a_i = 1\} \in \mathfrak{F}$ and $(a_i, b_i) \in D_{1,1}$ for all $i \in [n]$, thus by the monotonicity of $\mathfrak{F}$ we have $\{i \in [n] \mid b_i = 1\} \in \mathfrak{F}$. Therefore, we have $b = 1$, which shows the compatibility of f with $D_{1,1}$.

It remains to show that f is compatible with $O_{1,1}$. Let us have, once again, $f(a_1, \dots, a_n) = a$ and $f(b_1, \dots, b_n) = b$ for some a_i, b_i, a, b, where $i \in [n]$. Moreover, assume $(a_i, b_i) \in O_{1,1}$ for all $i \in [n]$. Thus $\{i \in [n] \mid a_i = 1\} \cup \{i \in [n] \mid b_i = 1\} = [n]$. By monotonicity and self-duality of $\mathfrak{F}$, it follows that $\{i \in [n] \mid a_i = 1\} \in \mathfrak{F}$ or $\{i \in [n] \mid b_i = 1\} = [n]$, which implies $a = 1$ or $b = 1$. So $(a, b) \in O_{1,1}$ and thus f is compatible with $O_{1,1}$. We showed that $f \in \mathrm{Pol}(P_{1,2}, D_{1,1}, O_{1,1})$, hence by the previous theorem, $f \in \mathrm{Clo}_n(\mathbf{A})$.

Now we show the other inclusion. Assume $f \in \mathrm{Clo}_n(\mathbf{A})$. Because $\{0, 1\}$ and $\{0, 2\}$ are subuniverses of $\mathbf{A}$, it follows that $f_{\restriction\{0,1\}} \in \mathcal{M}^{0,1}$ and $f_{\restriction\{0,2\}} \in \mathcal{M}^{0,2}$. Here observe that $\mathfrak{F}_{f_{\restriction\{0,1\}}} = \mathfrak{F}_{f_{\restriction\{0,2\}}}$ (using the compatibility with $P_{1,2}$). We denote the ms-collection $\mathfrak{F}_{f_{\restriction\{0,1\}}}$ by $\mathfrak{F}$. Let $a_1, \dots, a_n \in A$ and fix $a \in A$ such that $f(a_1, \dots, a_n) = a$. We distinguish three cases.

1. $\{i \in [n] \mid a_i = 1\} \in \mathfrak{F}$. For each $i \in [n]$ take b_i as follows.

$$b_i = \begin{cases} 1 & a_i \neq 1 \\ 0 & a_i = 1 \end{cases}$$

Now we have $(a_i, b_i) \in O_{1,1}$ for all $i \in [n]$. Because $\{i \in [n] \mid b_i = 0\} \in \mathfrak{F}$, we have $f(b_1, \dots, b_n) = f_{\restriction\{0,1\}}(b_1, \dots, b_n) = 0$ by the definition of $\mathfrak{F}$. By compatibility with $O_{1,1}$, we get $(a, b) \in O_{1,1}$ and because $b \neq 0$, it follows that $a = 1$.

2. $\{i \in [n] \mid a_i = 2\} \in \mathfrak{F}$. Similarly as in the previous case, we show that $a = 2$.

3. $\{i \in [n] \mid a_i = 1\}, \{i \in [n] \mid a_i = 2\} \notin \mathfrak{F}$. For each $i \in [n]$, define b_i and c_i as follows.

$$b_i = \begin{cases} 1 & a_i = 1 \\ 0 & a_i \neq 1 \end{cases}$$

and

$$c_i = \begin{cases} 2 & a_i = 2 \\ 0 & a_i \neq 2 \end{cases}$$

Now we have $(a_i, b_i) \in D_{1,1}$ and $(a_i, c_i) \in D_{2,2}$ for all $i \in [n]$. Because $\{i \in [n] \mid b_i = 1\}, \{i \in [n] \mid c_i = 2\} \notin \mathfrak{F}$, it follows that $f(b_1, \dots, b_n) = 0$ and $f(c_1, \dots, c_n) = 0$. From the compatibility with $D_{1,1}$ and $D_{2,2}$, we have $a \neq 1$ and $a \neq 2$, thus $a = 0$.

This shows that f is defined by Equation 5.3, which finishes the proof. □

5.4 The Clone of the Algebra $\mathbf{T}_{15}^{C}$

In this section, we describe the clone of the majority algebra $\mathbf{T}_{15}^{C}$. This algebra has a single basic majority operation m, which is defined as follows.

$$m(x, y, z) = \begin{cases} \mathrm{maj}_{a,b}(x, y, z) & \{x, y, z\} = \{a, b\} \;\; a, b \in \{0, 1, 2\} \\ 1 & (x, y, z) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \\ 2 & (x, y, z) \in \{(2, 1, 0), (1, 0, 2), (0, 2, 1)\} \end{cases}$$

Obviously, $\mathbf{A}$ is a majority algebra with the only weak point 0. Therefore, $\mathbf{A}$ is compatible with $O_{a,b}, N_a, D_{0,a}, E_{0,0}$ for $a, b \in A$. It is easy to see that $\mathbf{A}$ is compatible with $P_{1,2}$. To describe the clone of this particular algebra, we need one more relation.

Definition 5.4. *For $a, b, c \in A$ we define a binary relation $F_{a,b}^{c}$ as follows.*

$$(x, y) \in F_{a,b}^{c} \iff (x = c \iff y = c) \wedge (x \neq a \vee y \neq b)$$

The relation $F_{a,b}^{0}$ is a generalization of the relation $\leq_{b,a}$. Hence it is not surprising that this relation is compatible with $\mathbf{A}$.

Lemma 5.12. *Let $a, b \in \{1, 2\}$. Then $F_{a,b}^{0}$ is compatible with $\mathbf{A}$.*

Proof. We show the proof for $a = b = 1$. The rest follows from the compatibility of $P_{1,2}$ with $\mathbf{A}$. Let us have $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $m(c_1, c_2, c_3) = c$, $m(d_1, d_2, d_3) = d$ and $(c, d) \notin F^0_{1,1}$. We show that there is $i \in [3]$ such that $(c_i, d_i) \notin F^0_{1,1}$.

There are two cases.

1. $c = 0$ and $d \neq 0$ or similarly $c \neq 0$ and $d = 0$. Without loss of generality we assume it is the case that $c = 0$ and $d \neq 0$. Because 0 is a weak point of m, there is $i, j \in [3]$ such that $i \neq j$ and $c_i = c_j = 0$. However, if $(c_i, d_i), (c_j, d_j) \in F^0_{1,1}$, then $d_i = d_j = 0$ and $d = 0$. Thus $(c_i, d_i) \notin F^0_{1,1}$ or $(c_j, d_j) \notin F^0_{1,1}$.

2. $(c, d) = (1, 1)$. We may assume $(c_i, d_i) \neq (1, 1)$ for all $i \in [3]$, so without loss of generality assume $c_1 = 0, c_2 = 1, c_3 = 2$. If $(c_1, d_1) \in F^0_{1,1}$, we have $d_1 = 0$. Because we have $d = 1$, it must hold that $d_2 = 1$ and so $(c_2, d_2) = (1, 1) \notin F^0_{1,1}$.

This shows the compatibility of $F^0_{1,1}$ with $\mathbf{A}$. ☐

We need one more technical lemma.

Lemma 5.13. *Let R be a binary relation on A, which is compatible with $\mathbf{A}$, and $a, b, d \in A$ such that $b \neq 0$. If $(d, b) \in R$ and $(a, b) \notin R$, then there is $c \in A \setminus \{b\}$ such that $(a, c) \notin R$.*

Proof. Assume $b = 1$. We have $m(a, d, a) = a$, $m(0, 1, 2) = 1$, $(a, 1) \notin R$ and $(d, 1) \in R$. Because R is compatible with m, we get $(a, 0) \notin R$ or $(a, 2) \notin R$. This completes the proof for $b = 1$. The case $b = 2$ is analogous. ☐

Obviously, we can state Lemma 5.13 dually, i.e. assuming $a \neq 0$, $(a, d) \in R$ and $(a, b) \notin R$ we can derive there is $c \in A \setminus \{a\}$ such that $(c, b) \notin R$.

The following lemma gives a description of binary relations which are compatible with $\mathbf{A}$.

Lemma 5.14. *Let R be compatible with $\mathbf{A}$. Then R is equal to*

$$\bigwedge_{E_{0,0} \supseteq R} E_{0,0} \wedge \bigwedge_{\substack{D_{0,a} \supseteq R \\ a \in A}} D_{0,a} \wedge \bigwedge_{\substack{D^{-1}_{b,0} \supseteq R \\ b \in A}} D^{-1}_{b,0} \wedge \bigwedge_{\substack{O_{c,d} \supseteq R \\ c,d \in A}} O_{c,d} \wedge \bigwedge_{\substack{F^0_{e,f} \supseteq R \\ e,f \in \{1,2\}}} F^0_{e,f} \wedge \bigwedge_{\substack{N_g \supseteq R \\ g \in \{1,2\}}} N_g \wedge \bigwedge_{\substack{N^{-1}_h \supseteq R \\ h \in \{1,2\}}} N^{-1}_h. \tag{5.4}$$

Proof. We denote the right hand side of Equation 5.4 by S. Clearly, every $(x, y) \in R$ is in S. Now assume $(x, y) \notin R$ and we show $(x, y) \notin S$. There are four possibilities.

1. $x = y = 0$. In this case $E_{0,0} \supseteq R$ and $(0, 0) \notin E_{0,0}$. Therefore, $(0, 0) \notin S$.

2. $x = 0$ and $y \neq 0$. If $(a, y) \notin R$ for each $a \in A$, we have $R \subseteq N_y$ and thus $(x, y) \notin S$. Otherwise, by Lemma 5.13 there is $c \in A \setminus \{y\}$ such that $(0, c) \notin R$. Pick $b \in A \setminus \{y, c\}$. We have $D_{0,b} \supseteq R$ and $(x, y) \notin D_{0,b}$. Hence we have $(x, y) \notin S$.

3. $x \neq 0$ and $y = 0$. Similarly as in the previous case, we show $N_x^{-1} \supseteq R$ or $D_{b,0}^{-1} \supseteq R$ for some $b \in A$ and thus $(x, y) \in S$.

4. $x, y \neq 0$. If we have $(d, y) \notin R$ (resp. $(x, d) \notin R$) for each $d \in A$, we get $R \subseteq N_y$ (resp. $R \subseteq N_x^{-1}$) and hence $(x, y) \notin S$. Otherwise, Lemma 5.13 gives $a, b \in R$ such that $a \neq x, b \neq y$ and $(x, b), (a, y) \notin R$. We distinguish another four cases.

   (a) $a, b \neq 0$. In such a case we have $(x, 0), (0, y) \in R$. We show that from this we can derive $(a, b) \notin R$. For a contradiction, assume $(a, b) \in R$. Then by the compatibility of R with m, we can get from $(x, 0), (0, y), (a, b)$ the tuples $(x, y), (a, b)$ or $(x, b), (a, y)$. Since $(x, y), (x, b) \notin R$, this is not possible. Therefore $(a, b) \notin R$ and so $R \subseteq O_{0,0}$ and $(x, y) \notin S$.

   (b) $a = 0$ and $b \neq 0$. Pick $c \in A \setminus \{x, 0\}$. We have $(c, y), (x, 0) \in R$. Once again, we derive $(0, b) \notin R$, otherwise we would get from $(c, y), (x, 0), (0, b) \in R$ that $(c, y), (x, b) \in R$ or $(c, b), (x, y) \in R$, however this is not possible since $(x, b), (x, y) \notin R$. Therefore, $(a, b) \notin R$ and so $R \subseteq O_{c,0}$ and $(x, y) \notin S$.

   (c) $a \neq 0$ and $b = 0$. This case is analogous to the previous one.

   (d) $a, b = 0$. We have $(x, y), (0, y), (x, 0) \notin R$. Assume $x = y = 1$ (other cases are similar). Then we have $(1, 2), (2, 1) \in R$. We also have $m(0, 2, 1) = 1$ and $m(0, 0, 2) = 0$ and because $(1, 2) \in R$ and $(1, 0) \notin R$ and R is compatible with $\mathbf{A}$, we have $(0, 0) \notin R$ or $(2, 0) \notin R$. Similarly, $(2, 1) \in R$ and $(0, 1) \notin R$, so we have $(0, 0) \notin R$ or $(0, 2) \notin R$. Thus, in the end, we have $(0, 0) \in R$ or $(0, 2), (2, 0) \notin R$. In the case $(0, 0) \notin R$ we have $R \subseteq O_{2,2}$, in the case $(0, 2), (2, 0) \notin R$ we have $R \subseteq F_{x,y}^0$. In both cases, we have $(x, y) \notin S$.

This proves $R = S$. □

Lemma 5.14 tells us that $\mathrm{Inv}(\mathbf{A})$ is generated by $E_{0,0}$, $D_{0,a}$, $O_{b,c}$ and $F_{e,f}^0$, where $a, b, c, d \in A$ and $e, f \in \{1, 2\}$. Here we are once again using the fact that $\mathrm{Inv}(\mathbf{A})$ is generated by binary relations. Using the relation $P_{1,2}$ we can simplify this result and say that $\mathrm{Inv}(\mathbf{A})$ is generated by $E_{0,0}, D_{0,0}, D_{0,1}, O_{0,0}, O_{0,1}, O_{1,1}, F_{1,1}^0$ and $P_{1,2}$. The following pp-definitions tell us that we can generate $\mathrm{Inv}(\mathbf{A})$ using even fewer relations.

$$(x, y) \in O_{0,0} \iff \exists z\ (x, z) \in O_{0,1} \ \wedge\ (z, y) \in O_{2,0}$$
$$(x, y) \in E_{0,0} \iff \exists z\ (x, z) \in D_{0,1} \ \wedge\ (z, y) \in D_{2,0}^{-1}$$
$$(x, y) \in D_{0,1} \iff \exists z\ (x, z) \in F_{1,1}^0 \ \wedge\ (z, y) \in O_{2,1}$$
$$(x, y) \in D_{0,0} \iff \exists z\ (x, z) \in F_{1,1}^0 \ \wedge\ (z, y) \in O_{2,0}$$

Hence we get the following theorem.

Theorem 5.15. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $O_{0,1}, O_{1,1}$, $F_{1,1}^0$ *and* $P_{1,2}$.

Our next goal is to describe the elements of $\mathrm{Clo}(\mathbf{A})$ using ms-collections. However, we find here a more complicated structure, where each set in ms-collection has attached another ms-collection on that set and these ms-collections depend on each other. We call these structures bms-collections.

Definition 5.5. *Let B be a set. By a* big monotone self-dual collection on B (bms-collection on B for short) *we mean an ms-collection $\mathfrak{F}$ on B with ms-collections $\mathfrak{F}_S$ on S, for each $S \in \mathfrak{F}$, such that for all $S, V \in \mathfrak{F}$ satisfying $S \cap V = \emptyset$, we have $V \in \mathfrak{F}_S$.*

Theorem 5.16. *An operation f is in $\mathrm{Clo}(\mathbf{A})$ if and only if there exists a bms-collection $\mathfrak{F}$ on $[n]$ such that we have the following.*

$$f(a_1, \ldots, a_n) = \begin{cases} 0 & \{i \in [n] \mid a_i = 0\} \in \mathfrak{F} \\ 1 & S \in \mathfrak{F} \ \wedge \ \{i \in [n] \mid a_i = 1\} \in \mathfrak{F}_S \\ 2 & S \in \mathfrak{F} \ \wedge \ \{i \in [n] \mid a_i = 2\} \in \mathfrak{F}_S \end{cases}, \tag{5.5}$$

where $S = \{i \in [n] \mid a_i \neq 0\}$.

Proof. Let $f \in \mathrm{Clo}(\mathbf{A})$. Then $f_{\restriction\{0,1\}} \in \mathcal{M}_{0,1}$, since $\{0, 1\}$ is a subuniverse of $\mathbf{A}$. Denote by $\mathfrak{F}$ the ms-collection $\mathfrak{F}_{f_{\restriction\{0,1\}}}$. For each $S \in \mathfrak{F}$ we define an operation $f^S : A^{ms} \to A$ by $f^S(a_1, \ldots, a_{|S|}) = f(b_1, \ldots, b_n)$, where

$$b_i = \begin{cases} 0 & i \notin S \\ a_j & \text{if } i \text{ is the } j\text{-th index in } S \end{cases}$$

Because f is compatible with $P_{1,2}$ and $F^0_{2,1}$, we have for each $S \in \mathfrak{F}$ that f^S is compatible with $\neq_{1,2}$ and $\leq_{1,2}$, thus $f \in \mathcal{M}^{1,2}$. Therefore, $\mathfrak{F}_{f^S}$ is a ms-collection for each $S \in \mathfrak{F}$. By the definition of $\mathfrak{F}_{f^S}$, if $V \in \mathfrak{F}$ and $V \cap S = \emptyset$, we have $V \in \mathfrak{F}_{f^S}$. For each $S \in \mathfrak{F}$, set $\mathfrak{F}_S = \mathfrak{F}_{f^S}$. To check that $\mathfrak{F}$ is an ms-collection, it is enough to show $f(a_1, \ldots, a_n) = 1$ if $\{i \mid a_i = 1\} \in \mathfrak{F}$. Assume $\{i \in [n] \mid a_i = a\} \in \mathfrak{F}$ for some $a \in A$. Define c_i for all $i \in [n]$ as follows.

$$c_i = \begin{cases} 1 & a_i = a \\ 0 & a_i \neq a \end{cases}$$

We have $(a_i, c_i) \in O_{a,0}$, thus by the compatibility of $O_{a,0}$ with $\mathbf{A}$ we have

$$(f(a_1, \ldots, a_n), f(c_1, \ldots, c_n)) \in O_{a,0}.$$

We have $f(c_1, \ldots, c_n) = 1$, because $\{i \in [n] \mid a_i = a\} \subseteq \{i \in [n] \mid c_i = 1\} \in \mathfrak{F}$. Hence we get $f(a_1, \ldots, a_n) = a$. This shows that $\mathfrak{F}$ is an ms-collection.

Now we have to show that Equation 5.5 holds for f. Pick $a_1, \ldots, a_n \in A$. We already showed $\{i \in [n] \mid a_i = 0\} \in \mathfrak{F}$ implies $f(a_1, \ldots, a_n) = 0$. So assume $S \in \mathfrak{F}$ and $\{i \in [n] \mid a_i = b\} \in \mathfrak{F}_S$, where $b \in \{1, 2\}$. By the definition of $\mathfrak{F}_S$, we immediately get $f(a_1, \ldots, a_n) = b$. This shows that every $f \in \mathrm{Clo}(\mathbf{A})$ can be expressed using Equation 5.5 and some bms-collection.

Now assume that we have a bms-collection $\mathfrak{F}$ and f defined as in Equation 5.5. We want to show $f \in \mathrm{Clo}(\mathbf{A})$. We check the compatibility of f with $O_{0,1}, O_{1,1}, P_{1,2}$ and $F^0_{1,1}$. The compatibility with $P_{1,2}$ is clear. To check the compatibility with

$O_{a,b}$ for $a, b \in A$, we take $c_i, d_i, c, d \in A$, where $i \in [n]$, such that $f(c_1, \ldots, c_n) = c$, $f(d_1, \ldots, d_n) = d$ and $(c_i, d_i) \in O_{a,b}$ for all $i \in [n]$. We have $\{i \mid c_i = a\} \cup \{j \mid d_j = b\} = [n]$. By monotonicity and self-duality, we have $\{i \mid c_i = a\} \in \mathfrak{F}$ or $\{j \mid d_j = b\} \in \mathfrak{F}$. This gives $c = a$ or $d = b$, hence $(c, d) \in O_{a,b}$.

It remains to check the compatibility with $F^0_{1,1}$. Take $a_i, b_i, a, b \in A$, where $i \in [n]$, $f(a_1, \ldots, a_n)$, $f(b_1, \ldots, b_n) = b$, and $(a_i, b_i) \in F^0_{1,1}$ for all $i \in [n]$. We distinguish two cases.

1. $a = 1$. Denote $S = \{i \mid a_i \neq 0\}$. We show $b \neq 1$. If $\{i \mid a_i = 1\} \in \mathfrak{F}$, then we have $\{i \mid b_i = 2\} \in \mathfrak{F}$ and $b = 2$. So, we can assume $S \in \mathfrak{F}$ and $\{i \mid a_i = 1\} \in \mathfrak{F}_S$. Therefore, we have $\{i \mid b_i \neq 0\} = S \in F$ and $\{i \mid b_i = 2\} \in \mathfrak{F}_S$. So $b = 2$.

2. $a = 0$. Thus $\{i \mid a_i = 0\} = \{i \mid b_i = 0\} \in \mathfrak{F}$ and $b = 0$.

This shows that f is compatible with $F^0_{1,1}$. Thus $f \in \mathrm{Clo}(\mathbf{A})$. $\square$

6. The Clones of the Algebras $\mathbf{T}_3^C$ and $\mathbf{T}_7^C$

In this chapter, we focus on describing the clones of the algebras $\mathbf{T}_3^C$ and $\mathbf{T}_7^C$. These algebras are similar in one aspect, both of them have a single basic ternary operation t such that $t(a_1, a_2, a_3) = 2$ if and only if $a_i = 2$ for all $i \in [3]$. We take an advantage of this property. Because we are still working with the three-element algebras, we write A instead of $\{0, 1, 2\}$ for the rest of this chapter.

6.1 The Clone of the Algebra $\mathbf{T}_3^C$

In this section, we describe the clone of the algebra $\mathbf{T}_3^C$. Recall that $\mathbf{T}_3^C$ has a single basic ternary operation defined as follows.

$$t(x_1, x_2, x_3) = \begin{cases} \mathrm{maj}_{0,1}(x_1, x_2, x_3) & x_1, x_2, x_3 \in \{0, 1\} \\ \min_{0,2}(x_1, x_2, x_3) & x_1, x_2, x_3 \in \{0, 2\} \\ \min_{1,2}(x_1, x_2, x_3) & x_1, x_2, x_3 \in \{1, 2\} \\ 0 & (x_1, x_2, x_3) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \\ 1 & (x_1, x_2, x_3) \in \{(2, 1, 0), (0, 2, 1), (1, 0, 2)\} \end{cases}$$

For the rest of this section, we denote $\mathbf{T}_3^C$ by $\mathbf{A}$. As in the previous chapter, we immediately see that $\mathbf{A}$ is compatible with $P_{0,1}$. Since $t(a_1, a_2, a_3) = 2$ iff $a_i = 2$ for all $i \in [3]$, we should expect that $T_{\{0,1\}}$ is compatible with $\mathbf{A}$.

Lemma 6.1. *The relation $T_{\{0,1\}}$ is compatible with $\mathbf{A}$.*

Proof. Since $\{0, 2\}$ is the universe of a semilattice subalgebra, by Corollary 2.16 it is enough to check that $D_{2,2}$ is compatible with $\mathbf{A}$. To do so, let us have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1, a_2, a_3) = a, t(b_1, b_2, b_3) = b$ and $(a_i, b_i) \in D_{2,2}$ for all $i \in [3]$. Without loss of generality, we assume $a = 2$. Then $a_i = 2$ for all $i \in [n]$, thus $b_i = 2$ for all $i \in [n]$ and $b = 2$. This shows that $D_{2,2}$ is compatible with $\mathbf{A}$, so $T_{\{0,1\}}$ is compatible with $\mathbf{A}$ as well. $\square$

To successfully describe this algebra, we need to define one more relation.

Definition 6.1. *We define a binary relation G_a for $a \in A$ as follows.*

$$(x, y) \in G_a \iff x = y \ \vee \ y = a$$

Similarly as the relation $F_{a,b}^c$, the relation G_a is a generalization of $\leq_{b,a}$. Since $t_{\restriction\{0,1\}} = \mathrm{maj}$, it is not surprising that these relations for $a, b \in \{0, 1\}$ and $c = 2$ are compatible with $\mathbf{A}$.

Lemma 6.2. *Relations $F_{a,b}^2$ and G_a are compatible with $\mathbf{A}$ for every $a, b \in \{0, 1\}$.*

Proof. First we show that $F_{a,b}^2$ is compatible with $\mathbf{A}$ for every $a, b \in \{0, 1\}$. Let us pick $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $t(c_1, c_2, c_3) = c$, $t(d_1, d_2, d_3) = d$ and $(c_i, d_i) \in F_{a,b}^2$, $i \in [3]$. First assume $c = 2$ and we show $d = 2$. This is easy, because $c = 2$ gives $c_i = 2$ for all $i \in [3]$, thus $d_i = 2$ for all $i \in [3]$ and $d = 2$. Similarly, we can show that $d = 2$ implies $c = 2$. Now assume $c = a$. There is some $i \in [3]$ such that $c_i = a$ and $d_i \in A \setminus \{0, a\}$. Without loss of generality assume $i = 1$.

We distinguish four cases.

1. $c_2, c_3 \in \{0, 1\}$. Then we have $\{d_1, d_2, d_3\} \in \{0, 1\}$. In such a case, we get $d \neq b$ because $t_{\restriction\{0,1\}} = \mathrm{maj}_{0,1}$ and $\mathrm{maj}_{0,1}$ is compatible with every binary relation on $\{0, 1\}$.

2. $\{c_1, c_2, c_3\} = \{0, 1, 2\}$. If $a = 0$, we have $(c_1, c_2, c_3) = (0, 1, 2)$, so $d_1 \in A \setminus \{b, 2\}$. If $b = 0$, we have $d_1 = 1$ and $d_3 = 2$, so $d = 1$. If $b = 1$, we have $d_1 = 0$ and $d_3 = 2$, which gives $d = 2$. In any case, $d \neq b$ and $d \in \{0, 1\}$, thus $(c, d) \in F^2_{a,b}$. The case $a = 1$ is similar.

3. There is $j \in \{2, 3\}$ such that $c_j = c_1 = a$. In such a case, we have $d_j = d_1 \in A \setminus \{b, 2\}$ and so $d \in \{b, 2\}$, which gives $(c, d) \in F^2_{a,b}$.

4. $c_2 = c_3 = 2$. In such a case $d_2 = d_3 = 2$ and $d = d_1 \in \{b, 2\}$, so $(c, d) \in F^2_{a,b}$.

This shows that $F^2_{a,b}$ is compatible with $\mathbf{A}$ for every $a, b \in \{0, 1\}$.

Now we show that G_a is compatible with $\mathbf{A}$ for every $a \in \{0, 1\}$. Pick $c_i, d_i, c, d \in A$, where $i \in [3]$, such that $t(c_1, c_2, c_3) = c$, $t(d_1, d_2, d_3) = d$ and $(c_i, d_i) \in G_a$ for all $i \in [3]$. We again distinguish three cases.

1. $c_i = d_i$ for all $i \in [3]$. In such a case $c = d$, so $(c, d) \in G_a$.

2. $\{d_1, d_2, d_3\} = \{0, 1, 2\}$. Without loss of generality assume $d_1 = a$, so $c_2 = d_2$ and $c_3 = d_3$. If $t(d_1, d_2, d_3) = a$, we have $(c, d) \in D_a$. If $t(d_1, d_2, d_3) \in A \setminus \{a, 2\}$, we have $t(c_1, c_2, c_3) \in A \setminus \{a, 2\}$ and so $c = d$ and $(c, d) \in G_a$.

3. There are $i, j \in [3]$ such that $i \neq j$ and $c_i = d_i = c_j = d_j$. In such a case, we have $c = d$ and so $(c, d) \in G_a$

4. There are $i, j \in [3]$ such that $i \neq j$ and $d_i = d_j = a$. In such a case, we have $d = a$ and so $(c, d) \in G_a$.

This shows that G_a is compatible with $\mathbf{A}$ for $a \in \{0, 1\}$. □

The following lemma is crucial, although technical. Here we take advantage of three facts. Firstly, we use that the n-ary part of a clone can be seen as $|A^n|$-ary relation. Secondly, we observe that if we forbid the entry $(2, \ldots, 2)$ then $\mathrm{Clo}_n(\mathbf{A})$ can be seen as the $|A^n \setminus \{(2, \ldots, 2)\}|$-ary relation on $\{0, 1\}$. The third essential component is realizing that we obtain a relation compatible with $\mathbf{M}$, thus using 2.1 we can restrict our attention to the binary projections.

Lemma 6.3. *An n-ary operation f on A is in $\mathrm{EssClo}_n(\mathbf{A})$ if and only if $f(2, \ldots, 2) = 2$ and for all $a_i, b_i \in \mathbf{A}$, where $i \in [n]$, such that $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \neq (2, \ldots, 2)$, we have*

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in S,$$

where

$$S = \{(a_i, b_i) \mid a_i, b_i \in \{0, 1\};\ i \in [n]\} \\ \cup \{(a_i, b_j) \mid a_i, b_j \in \{0, 1\};\ a_j = 2 \ \vee\ b_i = 2\} \quad (6.1)$$

Proof. Observe that for each $f \in \mathrm{EssClo}_n(\mathbf{A})$ and for each $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = 2$ iff $a_i = 2$ for all $i \in [n]$. This is easy to show using the compatibility of f with C_2, $T_{\{0,1\}}$ and using Lemma 1.6 (compatibility with C_2 follows from the idempotency of $\mathbf{A}$). Thus we only need to describe how $f \in \mathrm{EssClo}_n(\mathbf{A})$ behaves on the other entries than $(2, \ldots, 2)$. For $f \in \mathrm{EssClo}_n(\mathbf{A})$ we denote

$$\overline{f} : A^n \setminus \{(2, \ldots, 2)\} \to \{0, 1\}$$
$$\overline{f} : (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n).$$

Denote $R = \{\overline{f} \mid f \in \mathrm{EssClo}_n(\mathbf{A})\}$. Observe that $R \subseteq \{0, 1\}^{A^n \setminus \{(2,\ldots,2)\}}$. Here we should formally identify the elements of R with the elements of $\{0, 1\}^{|A^n \setminus \{(2,\ldots,2)\}|}$. After this identification, R can be seen as a relation on $\{0, 1\}$. Moreover, R is actually a subuniverse of $\mathcal{M}_{0,1}^{|A^n \setminus \{(2,\ldots,2)\}|}$ (this is easy to check using the fact that $m_{\restriction\{0,1\}} = \mathrm{maj}$). Therefore $R \in \mathrm{Inv}(\mathbf{M})$. Here Theorem 2.1 says that

$$R = \bigwedge_{\substack{\mathbf{a},\mathbf{b} \in A^n \setminus \{(2,\ldots,2)\} \\ \mathbf{a} \neq \mathbf{b}}} \pi_{\mathbf{a},\mathbf{b}}(R).$$

Here, again, we are identifying the elements of $A^n \setminus \{(2, \ldots, 2)\}$ with the elements from $|A^n \setminus \{(2, \ldots, 2)\}|$. To check $\overline{f} \in R$ for $\overline{f} \in A^n \setminus \{(2, \ldots, 2)\}$, it is enough to verify $(f(\mathbf{a}), f(\mathbf{b})) \in \pi_{\mathbf{a},\mathbf{b}}(R)$ for all $\mathbf{a}, \mathbf{b} \in A^n \setminus \{(2, \ldots, 2)\}$.

Now it only remains to describe $\pi_{\mathbf{a},\mathbf{b}}(R)$ for fixed $\mathbf{a}, \mathbf{b} \in A^n \setminus \{(2, \ldots, 2)\}$. Fix $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, where $\mathbf{a}, \mathbf{b} \in A^n \setminus \{(2, \ldots, 2)\}$. We fix $g \in \mathrm{EssClo}_n(\mathbf{A})$ (g can be an arbitrary term operation, given by a term, in which all the variables from X_n occur, thus such g surely exists). For $i, j \in [n]$ take $f_{i,j} \in \mathrm{EssClo}_n(\mathbf{A})$ defined as follows.

$$f_{i,j} = t(\pi_i^n, \pi_i^n, t(\pi_j^n, \pi_j^n, g))$$

Clearly, $f_{i,j}(\mathbf{a}) = a_i$ if $a_i \neq 2$, and $f_{i,j}(\mathbf{a}) = a_j$ if $a_i = 2$ and $a_j \neq 2$. From this, it follows that $\pi_{\mathbf{a},\mathbf{b}}(R) \supseteq S$, where S is defined by Equation 6.1.

We show that $\pi_{\mathbf{a},\mathbf{b}}(R) \subseteq S$. For a contradiction, assume $f(a_1, \ldots, a_n) = a$, $f(b_1, \ldots, b_n) = b$ and $(a, b) \notin S$ for some $f \in \mathrm{EssClo}_n(\mathbf{A})$, $a, b \in \{0, 1\}$ and $a_i, b_i \in A$, where $i \in [n]$. Because $(a, b) \notin S$, we have $\{i \mid a_i = a\} \cap \{i \mid b_i = b\} = \emptyset$ and $(2, b), (a, 2) \neq (a_i, b_i)$ for any $i \in [n]$ (here we are implicitly using that $\mathbf{A}$ is conservative and so there is $i, j \in [n]$ such that $a_i = a$ and $b_j = b$). We define $\overline{a_i} \in \{0, 1\}$ for every $i \in [n]$ as follows.

$$\overline{a_i} = \begin{cases} a_i & a_i \neq 2 \ \vee \ b_i = 2 \\ a & a_i = 2 \ \wedge \ b_i \neq 2 \end{cases}$$

From the compatibility of f with G_a it follows that $f(\overline{a_1}, \ldots, \overline{a_n}) = a$. Similarly, define $\overline{b_i} \in \{0, 1\}$ for every $i \in [n]$ as follows.

$$\overline{b_i} = \begin{cases} b_i & b_i \neq 2 \ \vee \ a_i = 2 \\ b & b_i = 2 \ \wedge \ a_i \neq 2 \end{cases}$$

From the compatibility with G_b, we get $f(\overline{b_1}, \ldots, \overline{b_n}) = b$. We still have $\{i \mid \overline{a_i} = a\} \cap \{i \mid \overline{b_i} = b\} = \emptyset$, since we have $(2, b), (a, 2) \neq (a_i, b_i)$ for all $i \in [n]$. Define c_i for all $i \in [n]$ as follows.

$$c_i = \begin{cases} 2 & \overline{a_i}, \overline{b_i} = 2 \\ 1 & \overline{a_i} = a \\ 0 & \text{otherwise} \end{cases}$$

Now we have $(\overline{a_i}, c_i) \in F^2_{a,0}$ for all $i \in [n]$, thus $f(c_1, \dots, c_n) = 1$. However, we have $(\overline{b_i}, c_i) \in F^2_{b,1}$ for all $i \in [n]$, hence $f(c_1, \dots, c_n) = 0$. This gives the desired contradiction and shows $S = \pi_{\mathbf{a},\mathbf{b}}(R)$. $\square$

If we analyze the proof of Lemma 6.3 more closely, we realize that we proved the following.

Corollary 6.4. *Let f be an n-ary operation on A compatible with $F^2_{a,b}$ and G_a for all $a, b \in \{0, 1\}$, and $f(a_1, \dots, a_n) = 2$ iff $a_i = 2$ for all $i \in [n]$. Then $f \in \mathrm{EssClo}_n(\mathbf{A})$.*

This corollary gives us enough information to describe $\mathrm{Inv}(\mathbf{A})$.

Theorem 6.5. $\mathrm{Inv}(\mathbf{A})$ *is generated by $F^2_{1,1}$, $P_{0,1}$, G_1, C_2, and $T_{\{0,1\}}$.*

Proof. We prove $\mathrm{Pol}(F^2_{1,1}, P_{0,1}, G_1, C_2, T_{\{0,1\}}) = \mathrm{Clo}(\mathbf{A})$. Let us start with the inclusion $\mathrm{Pol}(F^2_{1,1}, P_{0,1}, G_1, C_2, T_{\{0,1\}}) \supseteq \mathrm{Clo}(\mathbf{A})$ by checking the compatibility of $F^2_{1,1}$, $P_{0,1}$, G_1, C_2, and $T_{\{0,1\}}$ with $\mathbf{A}$. We already checked the compatibility of $P_{0,1}$, $T_{\{0,1\}}$, $F^2_{1,1}$, and G_1 with $\mathbf{A}$. The compatibility of C_2 is obvious, since t is an idempotent operation.

Now we show $\mathrm{Pol}(F^2_{1,1}, P_{0,1}, G_1, C_2, T_{\{0,1\}}) \subseteq \mathrm{Clo}(\mathbf{A})$ for essential operations. Let us have an n-ary essential operation $f \in \mathrm{Pol}(F^2_{1,1}, P_{0,1}, G_1, C_2, T_{\{0,1\}})$. By the compatibility with C_2, we have $f(2, \dots, 2) = 2$. By the compatibility with $T_{\{0,1\}}$ and Lemma 1.6, we have $f(a_1, \dots, a_n) = 2$ iff $a_i = 2$ for all $i \in [n]$. Because we can pp-define $F^2_{a,b}$ and G_a for all $a, b \in \{0, 1\}$ just from $P_{0,1}, F^2_{1,1}, G_1$, we can use Corollary 6.4 and we get $f \in \mathrm{EssClo}_n(\mathbf{A})$. This proves $\mathrm{Pol}(F^2_{1,1}, P_{0,1}, G_1, C_2, T_{\{0,1\}}) = \mathrm{Clo}(\mathbf{A})$.

$\square$

Now we can finally describe the elements of $\mathrm{EssClo}(\mathbf{A})$.

Theorem 6.6. $\mathrm{EssClo}_n(\mathbf{A})$ *consist of n-ary operations f on A such that there is a family of ms-collections $\{\mathfrak{F}_I\}_{\emptyset \neq I \subseteq [n]}$ satisfying*

$$V \in \mathfrak{F}_I \implies V \cup (J \setminus I) \in \mathfrak{F}_J \tag{6.2}$$

for each $\emptyset \neq I \subseteq J \subseteq [n]$, and f is given by $f(2, \dots, 2) = 2$ and

$$f(a_1, \dots, a_n) = a \iff \{i \mid a_i = a\} \in \mathfrak{F}_I \text{ for } I = \{i \mid a_i \neq 2\} \tag{6.3}$$

for each $a_1, \dots, a_n \in A$, $a \in \{0, 1\}$.

Proof. First let us have an n-ary operation f with a family of ms-collections $\{\mathfrak{F}_I\}_{\emptyset \neq I \subseteq [n]}$ satisfying Equations 6.3, 6.2 and $f(2, \dots, 2) = 2$. Clearly, such an operation is compatible with C_2. We show that f is compatible with $F^2_{1,1}$, G_1, $T_{\{0,1\}}$ and $P_{0,1}$. Since $\mathfrak{F}_I$ is an ms-collection for each $\emptyset \neq I \subseteq [n]$, we have the compatibility of f with $F^2_{1,2}$ and $P_{0,1}$ (because $F^2_{1,2}$ is equal to $\leq_{0,1} \cup \{(2, 2)\}$ and

$P_{0,1}$ is equal to $\neq_{0,1} \cup \{(2,2)\}$). Since $F^2_{1,1}$ can be pp-defined from $P_{0,1}$ and $F^2_{1,2}$, we get that $F^2_{1,1}$ is compatible with f.

We show the compatibility with G_1. Let us have $a_i, b_i, a, b \in A$, where $i \in [n]$, such that $f(a_1, \dots, a_n) = a$, $f(b_1, \dots, b_n) = b$, and $(a_i, b_i) \in G_1$ for all $i \in [n]$. Denote $I = \{i \mid a_i \neq 2\}$ and $J = \{i \mid b_i \neq 2\}$. We have $I \subseteq J$. We assume $I \neq \emptyset$ (otherwise clearly $b = 2$ or $b = 1$). In such a case $\{i \mid a_i = a\} \in \mathfrak{F}_I$, so $\{i \mid a_i = a\} \cup J \setminus I \in \mathfrak{F}_J$. We clearly have $\{i \mid b_i = a\} \cup J \setminus I \in \mathfrak{F}_J$. Here we distinguish two cases. If $\{i \mid b_i = a\} \notin \mathfrak{F}_J$, we have $a \neq 1$ and so $\{i \mid b_i = 1\} \in \mathfrak{F}_J$ and $b = 1$. If $\{i \mid b_i = a\} \in \mathfrak{F}_J$, we trivially get $b = a$. This shows that f is compatible with G_a.

It is remains to check the compatibility with $T_{\{0,1\}}$. Let us have $a_i, b_i, c_i, a, b, c \in A$, where $i \in [n]$, such that $f(a_1, \dots, a_n) = a$, $f(b_1, \dots, b_n) = b$, $f(c_1, \dots, c_n) = c$, and $(a_i, b_i, c_i) \in T_{\{0,1\}}$ for all $i \in [n]$. Assume $c = 2$. So we have $c_i = 2$ for all $i \in [n]$. Because $(a_i, b_i, c_i) \in T_{\{0,1\}}$ for all $i \in [n]$, we get $a_i = b_i$ for all $i \in [n]$, thus $a = b$ and $(a, b, c) \in T_{\{0,1\}}$. This shows that f is compatible with $T_{\{0,1\}}$, so $f \in \mathrm{Clo}_n(\mathbf{A})$. The operation f is essential, since $f(2, \dots, 2) = 2$ and if we change one coordinate to $a \in \{0,1\}$, then the result is a instead of 2. So $f \in \mathrm{EssClo}_n(\mathbf{A})$.

Now let us have $f \in \mathrm{EssClo}_n(\mathbf{A})$. Using the compatibility with C_2, we get $f(2, \dots, 2) = 2$, and, by the compatibility with $T_{\{0,1\}}$ and Lemma 1.6, we have $f(a_1, \dots, a_n) \in \{0,1\}$ if there is $i \in [n]$ such that $a_i \in \{0,1\}$. For each $S \in [n]$, we define an operation $f^S : A^{|S|} \to A$ by $f^S(a_1, \dots, a_{|S|}) = f(b_1, \dots, b_n)$, where

$$b_i = \begin{cases} 2 & i \notin S \\ a_j & \text{if } i \text{ is the } j\text{-th index in } S \end{cases}$$

Because f is compatible with $P_{0,1}$ and $F^2_{1,1}$, we get $f^S \in \mathbf{M}_{0,1}$ for any $\emptyset \neq S \in [n]$. Because every operation in $\mathbf{M}_{0,1}$ is determined by an ms-collection, for each $S \in [n]$, we can fix an ms-collection $\mathfrak{F}_S$ on S defined as follows.

$$\mathfrak{F}_S = \{V \subseteq S \mid (a_i = 1 \iff i \in V) \wedge (a_i = 2 \iff i \notin S) \implies f(a_1, \dots, a_n) = 1\}.$$

For this family $\{\mathfrak{F}_I\}_{\emptyset \neq I \subseteq [n]}$, Equation 6.3 holds.

It remains to check that Equation 6.2 also holds. Let us have $\emptyset \neq I \subseteq J \subseteq [n]$ and $V \in \mathfrak{F}_I$. Let us have $f(a_1, \dots, a_n) = 1$ for $a_i \in A$, where $i \in [n]$ and $I = \{i \mid a_i \neq 2\}$ and $V = \{i \mid a_i = 1\}$. For all $i \in [n]$, we define b_i as follows.

$$b_i = \begin{cases} 1 & i \in J \setminus I \\ a_i & \text{otherwise} \end{cases}$$

Now we have $(a_i, b_i) \in G_1$ for all $i \in [n]$, thus $(1, b) \in G_1$, which means $f(b_1, \dots, b_n) = 1$. This implies $V \cup (J \setminus I) \in \mathfrak{F}_J$. Thus we checked Equation 6.2, which finishes the proof. $\square$

6.2 The Clone of the Algebra $\mathbf{T}_7^C$

In this section we describe the clone of the algebra $\mathbf{T}_7^C$. This algebra has a single basic ternary operation t, defined as follows.

$$t(x_1,x_2,x_3) = \begin{cases} \mathrm{aff}_{0,1}(x_1,x_2,x_3) & x_1,x_2,x_3 \in \{0,1\} \\ \min_{0,2}(x_1,x_2,x_3) & x_1,x_2,x_3 \in \{0,2\} \\ \min_{1,2}(x_1,x_2,x_3) & x_1,x_2,x_3 \in \{1,2\} \\ 0 & (x_1,x_2,x_3) \in \{(0,1,2),(1,2,0),(2,0,1)\} \\ 1 & (x_1,x_2,x_3) \in \{(2,1,0),(0,2,1),(1,0,2)\} \end{cases}$$

For the rest of this section, we write $\mathbf{A}$ instead of $\mathbf{T}_7^C$. We start, as in the previous section, by checking the compatibility of $T_{\{0,1\}}$ with $\mathbf{A}$.

Lemma 6.7. *The relation $T_{\{0,1\}}$ is compatible with $\mathbf{A}$.*

Proof. Because $\{0,2\}$ is a universe of a semilattice subalgebra, by Corollary 2.16 we only have to check that t is compatible with $D_{2,2}$. So let us have $a_i, b_i, a, b \in A$, where $i \in [3]$, such that $t(a_1,a_2,a_3) = a$, $t(b_1,b_2,b_3) = b$, and $(a_i,b_i) \in D_{2,2}$ for all $i \in [3]$. Without loss of generality assume $a = 2$. Then $a_i = 2$ for all $i \in [3]$, thus $b_i = 2$ for all $i \in [3]$, and $b = 2$. Thus $(a,b) \in D_{2,2}$, which shows that t is compatible with $D_{2,2}$. This shows that $T_{\{0,1\}}$ is compatible with $\mathbf{A}$. ☐

In the previous section, we used generalizations of $\leq_{a,b}$ and $\neq_{a,b}$. Here we use a generalization of $S_{a,b}$, which is defined below.

Definition 6.2. *Let $a,b,c \in A$. We define a ternary relation $H^c_{a,b}$ as follows.*

$$(x,y,z) \in H^c_{a,b} \iff ((x,y,z) = (c,c,c)) \ \vee \ ((x,y,z) \in S_{a,b})$$

Indeed, $H^c_{a,b}$ is a generalization of $S_{a,b}$. In particular we have that

$$(x,y,z) \in H^2_{1,0} \iff x,y,z = 2 \ \vee \ (x,y,z \in \{0,1\} \ \wedge \ x+y+z = 0 \mod 2).$$

Now we show the compatibility of $H^2_{1,0}$ with $\mathbf{A}$.

Lemma 6.8. *The relation $H^2_{1,0}$ is compatible with $\mathbf{A}$.*

Proof. We show that $H^2_{1,0}$ is compatible with t. Let us have $a_i, b_i, c_i, a, b, c \in A$, where $i \in [3]$, such that $t(a_1,a_2,a_3) = a$, $t(b_1,b_2,b_3) = b$, $t(c_1,c_2,c_3) = c$, and $(a_i,b_i,c_i) \in H^2_{1,0}$ for all $i \in [3]$. We distinguish four cases.

1. $a_i = 2$ for each $i \in [3]$. In this case, we trivially get $a = b = c = 2$ since $b_i = c_i = 2$ for each $i \in [3]$.

2. $a_i \neq 2$ for each $i \in [3]$. In such a case, we have $(a_i,b_i,c_i) \in S_{1,0}$ for all $i \in [3]$ and thus $(a,b,c) \in S_{1,0}$. Hence $(a,b,c) \in H^2_{1,0}$.

3. There is exactly one $j \in [3]$ such that $a_j = 2$. Without loss of generality assume $j = 1$. If $a_2 = a_3$, $b_2 = b_3$ and $c_2 = c_3$, we easily get $(a,b,c) = (a_2,b_2,c_2) \in H^2_{1,0}$. Without loss of generality assume $a_2 \neq a_3$. Then by the definition of $H^2_{1,0}$, we have $b_2 \neq b_3$ or $c_2 \neq c_3$. Without loss of generality assume $b_2 \neq b_3$. We distinguish two cases.

(a) $a_2 = b_2$ and $a_3 = b_3$. Then we have $c_2 = c_3 = 0$ and thus $a = b \in \{0,1\}$ and $c = 0$. This gives $(a,b,c) \in H^2_{1,0}$.

(b) $a_2 \neq b_2$. In such a case, we have $a_2 = b_3$, $a_3 = b_2$ and $c_1 = c_2 = 1$. Thus we get $a, b \in \{0,1\}$, $a \neq b$ and $c = 1$. This gives $(a,b,c) \in H^2_{1,0}$.

4. There is exactly one $j \in [3]$ such that $a_j \neq 2$. In such a case we have $(a,b,c) = (a_j, b_j, c_j) \in H^2_{1,0}$.

Hence we proved $H^2_{1,0}$ is compatible with t.

□

In the rest of this section, we will be mostly working with n-tuples. Before we start, we need some more effective notation.

Definition 6.3. *Let* $\mathbf{a} = (a_1 \dots, a_n) \in A^n$ *and* $k \in A$. *Denote* $V^k_{\mathbf{a}} = \{i \mid a_i = k\}$.

The definition above allows us to quickly recognize where is some element $a \in A$ presented in a particular n-tuple.

Definition 6.4. *Let* $\mathbf{a} = (a_1 \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in A^n$ *and* $V^2_{\mathbf{a}} = V^2_{\mathbf{b}}$. *Denote* $V = V^2_{\mathbf{a}}$. *By the* sum of $\mathbf{a}$ and $\mathbf{b}$ *we understand the element of* A^n *denoted by*

$$\mathbf{a} \oplus_V \mathbf{b} = (a_1 \oplus_V b_1, \dots, a_n \oplus_V b_n) \in A^n,$$

where $V^2_{\mathbf{a} \oplus_V \mathbf{b}} = V^2_{\mathbf{a}}$ *and for all* $i \notin V^2_{\mathbf{a}}$ *we have* $a_i \oplus_V b_i = a_i + b_i \mod 2$. *For a set* $I = \{i_1, \dots, i_n\}$ *we write just* $\bigoplus_{V, i \in I} \mathbf{a}^i$ *instead of* $\mathbf{a}^{i_1} \oplus_V \mathbf{a}^{i_2} \oplus_V \dots \oplus_V \mathbf{a}^{i_n}$.

We should observe that the last part of this definition makes sense since $\oplus$ is commutative and associative. Although this definition seems technical, it is simple. The collection of all n-tuples from A^n with a fixed set $V^2_{\mathbf{a}}$ is actually a vector space over $\{0,1\}$. In this representation, we ignore coordinates where 2 appears. The addition in this vector space is exactly $\oplus_{V^2_{\mathbf{a}}}$. It is easy to see that $\mathbf{a} \oplus \mathbf{b} = \mathbf{c} \iff (a_i, b_i, c_i) \in H^2_{1,0}$ for each $i \in [n]$ (here by a_i we mean the i-th coordinate of $\mathbf{a}$ and similarly for b_i, c_i).

The following definition introduces notation for vectors in the "canonical basis" of these vector spaces.

Definition 6.5. *Let* $V \subseteq [n]$ *and* $i \in [n] \setminus V$. *Denote by* $e_{V,i}$ *the* n*-tuple* $\mathbf{e} = (e_1, \dots, e_n)$, *where* $e_j \in A$ *for all* $j \in [n]$, $V^2_{\mathbf{e}} = V$, *and* $V^1_{\mathbf{e}} = \{j\}$.

As we discussed earlier, every $\mathbf{a} \in A^n$ can be seen as an element of a vector space with basis $\{e_{V,i_1}, \dots, e_{V,i_k}\}$, where addition is $\oplus_V$ for some $V \subseteq [n]$. Thus $\mathbf{a}$ can be written as the sum of $e_{V,i}$.

We need one more technical definition. This definition is hard to motivate; in essence, it will allow us to describe some particular elements of $\mathrm{Clo}(\mathbf{A})$.

Definition 6.6. *For two* n*-ary operations* f, g *on* A *we define an operation* $f * g$ *as* $t(f,g,g)$. *For* $V \subseteq [n]$ *and* $i, j \in [n] \setminus V$ *we define an* n*-ary operation* $f_{V,i,j}$ *as*

$$\pi^n_{i_1} * (\pi^n_{i_2} *, \dots, (\pi^n_{i_{n-1}} * \pi^n_{i_n})),$$

such that there is $l \in [n]$ *satisfying* $\pi^n_{i_{l+1}} = \pi^n_i$, $\pi_{i^n_{l+2}} = \pi^n_j$, $(i_1, \dots, i_l)$ *is the sequence of naturally ordered indices from* V, *and* $(i_{l+3}, \dots, i_n)$ *is the sequence of naturally ordered indexes from* $[n] \setminus (V \cup \{i,j\})$.

The operations $f_{V,i,j}$ have a special property. They allow us to distinguish basis elements, since for fixed $V \subseteq [n]$ and for all $i, j \in [n]\backslash V$, we have $f_{V,i,j}(e_{V,j}) = 1$ if and only if $i = j$. We observe that this operation is clearly essential and so $f_{V,i,j} \in \mathrm{EssClo}_n(\mathbf{A})$. Now we have everything we need to describe $\mathrm{EssClo}(\mathbf{A})$.

Theorem 6.9. *$f \in \mathrm{EssClo}_n(\mathbf{A})$ if and only if*

$$f(a_1, \ldots, a_n) = 2 \iff a_i = 2 \text{ for all } i \in [n], \tag{6.4}$$

$$f(\mathbf{a}) = \sum_{i \in V_{\mathbf{a}}^1} f(e_{V_{\mathbf{a}}^2,i}) \mod 2 \text{ for each } \mathbf{a} \in A^n \setminus \{(2, \ldots, 2)\} \tag{6.5}$$

and

$$\sum_{j \in [n]\backslash V} f(e_{V,j}) = 1 \mod 2 \text{ for each } V \subsetneq [n]. \tag{6.6}$$

Before we start with the proof, let us describe the main idea behind the proof of the harder implication. We use a similar trick as in the proof of Lemma 6.3 from the previous section. We realize that operations in $\mathrm{EssClo}_n(\mathbf{A})$, if we forbid the input $(2, \ldots, 2)$, are elements of $\{0,1\}^{A^n \backslash \{(2,\ldots,2)\}}$, thus $\mathrm{EssClo}_n(\mathbf{A})$ can be somehow regarded as a relation on $\{0, 1\}$. This relation is compatible with $\mathbb{Z}_2$ and thus we may use the description of the compatible relations with $\mathbb{Z}_2$ given in the second chapter. In the proof, we actually forbid also some inputs other than $(2, \ldots, 2)$. However, the idea of the proof is similar.

Proof. First we show that $f \in \mathrm{EssClo}_n(\mathbf{A})$ satisfies Equations 6.4, 6.5 and 6.6. Clearly, f satisfies Equation 6.4 since, by Lemma 6.7, f is compatible with C_2 and $T_{\{0,1\}}$ (and using Lemma 1.6). We check f satisfies Equation 6.5 as well. First, denote $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n)$, and $e_{V,i} = (e_1, \ldots, e_n)$ for some $V \subseteq [n]$, and assume $\mathbf{a} \oplus e_{V,i} = \mathbf{b}$. This means $(a_i, e_i, b_i) \in H^2_{1,0}$ for all $i \in [n]$. Because $H^2_{1,0}$ is compatible with $\mathbf{A}$, we have $(f(\mathbf{a}), f(e_{V,i}), f(\mathbf{b})) \in H^2_{1,0}$. From $f(e_{V,i}) \neq 2$, we get

$$f(\mathbf{a}) + f(e_{V,i}) = f(\mathbf{b}) \mod 2.$$

Because every $\mathbf{a} \in A^n \setminus \{(2, \ldots, 2)\}$ can be written as a sum of $e_{V,i}$, induction gives Equation 6.5. It remains to check Equation 6.6 for f. Fix $V \subsetneq [n]$. Because $\{1, 2\}$ is a compatible unary relation with $\mathbf{A}$ (since $\{1, 2\}$ is a subuniverse of $\mathbf{A}$), we have, by using Equation 6.4, that $f(\bigoplus_{V,i \in [n]\backslash V} e_{V,i}) = 1$. By using Equation 6.5m we get Equation 6.6. This shows $f \in \mathrm{EssClo}_n(A)$ satisfies equations 6.4, 6.5 and 6.6.

Now we show the other (harder) implication. Denote

$$S = \{e_{V,i} \mid V \subsetneq [n], i \text{ is not the last index in } [n] \setminus V\}.$$

Then each operation $f \in \mathrm{EssClo}_n(\mathbf{A})$ is clearly determined by its values on S (this is exactly what we get from Equations 6.4, 6.5 and 6.6). For each $f \in \mathrm{EssClo}_n(\mathbf{A})$, denote the operation f restricted to S by $\overline{f}$. We denote $W = \{\overline{f} \mid f \in \mathrm{EssClo}_n(\mathbf{A})\}$. Clearly, $W \subseteq \mathbf{A}^S$ and using the Equation 6.4 we get $W \subseteq (\mathbb{Z}_2)^S$. Here we should identify S with $|S|$ and the elements of W with the elements of $A^{|S|}$. After this identification we get that W is $|S|$-ary relation on A. Moreover, W is a subuniverse of $(\mathbb{Z}_2)^S$, this is easy to check. So W is actually compatible with $\mathbb{Z}_2$ and so W has to be an affine subspace of $(\mathbb{Z}_2)^S$. If we show $W = (\mathbb{Z}_2)^S$, the proof will be complete.

Because W is an affine subspace of $(\mathbb{Z}_2)^S$, elements of W can be described as elements of $(\mathbb{Z}_2)^S$ satisfying some set of nontrivial linear equations. Pick one such an equation

$$\sum_{v \in S} a_v \cdot f(v) = b,$$

which holds for some coefficients $a_v, b \in \{0,1\}$ and for each $f \in \mathrm{EssClo}_n(\mathbf{A})$.

We show $a_v = 0$ for all $v \in S$. For a contradiction, pick the largest set $V \subseteq [n]$ such that there is $i \in [n]$ satisfying $v = e_{V,i} \in S$ and $a_v \neq 0$. Denote by j the last index in $[n] \setminus V$. We know

$$\sum_{v \in S} a_v \cdot f_{V,i,j}(v) - \sum_{v \in S} a_v \cdot f_{V,j,i}(v) = 0.$$

Clearly, $f_{V,i,j}(w) = f_{V,j,i}(w)$ for each w with $V_w^2 \not\supseteq V$. By the assumption, we have $a_v = 0$ for $v = e_{U,i}$, where $U \supsetneq V$. Thus we may write

$$\sum_{k \in [n] \setminus (V \cup \{j\})} a_{e_{V,k}} \cdot f_{V,i,j}(e_{V,k}) - \sum_{k \in [n] \setminus (V \cup \{j\})} a_{e_{V,k}} \cdot f_{V,j,i}(e_{V,k}) = 0.$$

Since $f_{V,j,i}(e_{V,k}) = 0$ for all $k \in [n] \setminus (V \cup \{j\})$, and $f_{V,i,j}(e_{V,k}) = 0$ for all $k \in [n] \setminus (V \cup \{j, i\})$, we get

$$a_{e_{V,i}} f_{V,i,j}(e_{V,i}) = a_{e_{V,i}} = 0.$$

This gives the desired contradiction. Thus elements of W do not satisfy any nontrivial equation. This shows $W = (\mathbb{Z}_2)^S$, and the proof is finally complete. □

The following theorem is just a simple consequence of the previous one.

Theorem 6.10. $\mathrm{Inv}(\mathbf{A})$ *is generated by* $T_{\{0,1\}}$, $H^2_{1,0}$, $\{1,2\}$.

Proof. We check that $\mathrm{Pol}(T_{\{0,1\}}, H^2_{1,0}, \{1,2\}) = \mathrm{Clo}(\mathbf{A})$. The inclusion

$$\mathrm{Pol}(T_{\{0,1\}}, H^2_{1,0}, \{1,2\}) \supseteq \mathrm{Clo}(\mathbf{A})$$

follows from the compatibility of $T_{\{0,1\}}$, $H^2_{1,0}$, and $\{1,2\}$ with t.

We already checked the compatibility of $T_{\{0,1\}}$ and $H^2_{1,0}$ with t in Lemma 6.7 and 6.8. The compatibility of $\{1,2\}$ with t follows from the fact that $\{1,2\}$ is a subuniverse of $\mathbf{A}$.

For the other inclusion, assume that we have an essential n-ary operation f compatible with $T_{\{0,1\}}$, $H^2_{1,0}$, and $\{1,2\}$. We show that f satisfies the equations in Theorem 6.9.

Since f is an essential operation compatible with $T_{\{0,1\}}$, by Lemma 1.6 f satisfies Equation 6.4.

We derive the other two equations similarly as in the proof of Theorem 6.9. □

Conclusion

In this thesis, we gave a description of 12 clones of minimal Taylor algebras on three elements. A natural continuation of the thesis would be to describe the remaining clones. The clone of the rock-paper-scissors algebra is already described, e.g., in [Brady, 2022, Section 3.1]. A description of the clones of some other algebras have been also sketched in that work. Namely, $\mathrm{Inv}(\mathbf{T}_3^N)$ in [Brady, 2022, Example 2.2.1], $\mathrm{Inv}(\mathbf{T}_2^P)$ in [Brady, 2022, Example 1.7.2], and $\mathrm{Inv}(\mathbf{T}_{13}^C)$ in [Brady, 2022, Example 1.7.3].

The following basic question remains open: Does every minimal Taylor algebra on three elements have finitely generated $\mathrm{Inv}(\mathbf{A})$? We provided a positive answer for 12 clones. The clone of the rock-paper-scissors algebra described in [Brady, 2022, Section 3.1] is also finitely generated. From Aichinger et al. [2014] and Barto et al. [2021] it follows that every minimal Taylor algebra on three elements, which does not have a semilattice subalgebra, has finitely generated $\mathrm{Inv}(\mathbf{A})$. Thus it only remains to consider the clones of $\mathbf{T}_1^C$, $\mathbf{T}_4^C$, $\mathbf{T}_6^C$, $\mathbf{T}_8^C$, $\mathbf{T}_9^C$, and $\mathbf{T}_{10}^C$.

Bibliography

Erhard Aichinger, Peter Mayr, and Ralph McKenzie. On the number of finite algebraic structures. *J. Eur. Math. Soc. (JEMS)*, 16(8):1673–1686, 2014. ISSN 1435-9855. doi: 10.4171/JEMS/472. URL `https://doi-org.ezproxy.is.cuni.cz/10.4171/JEMS/472`.

Kirby A. Baker and Alden F. Pixley. Polynomial interpolation and the Chinese remainder theorem for algebraic systems. *Math. Z.*, 143(2):165–174, 1975. ISSN 0025-5874. doi: 10.1007/BF01187059. URL `https://doi-org.ezproxy.is.cuni.cz/10.1007/BF01187059`.

L. Barto, Z. Brady, A. Bulatov, M. Kozik, and D. Zhuk. Minimal Taylor Algebras as a Common Framework for the Three Algebraic Approaches to the CSP. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13, Los Alamitos, CA, USA, jul 2021. IEEE Computer Society. doi: 10.1109/LICS52264.2021.9470557. URL `https://doi.ieeecomputersociety.org/10.1109/LICS52264.2021.9470557`.

Libor Barto, Andrei Krokhin, and Ross Willard. Polymorphisms, and how to use them. In *The constraint satisfaction problem: complexity and approximability*, volume 7 of *Dagstuhl Follow-Ups*, pages 1–44. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017.

Clifford Bergman. *Universal algebra*, volume 301 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2012. ISBN 978-1-4398-5129-6. Fundamentals and selected topics.

V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras. I, II. *Kibernetika (Kiev)*, 3(5):1–10, 1969. ISSN 0023-1274.

Z. Brady. Notes on CSPs and Polymorphisms, 2022. URL `https://raw.githubusercontent.com/notzeb/all/master/csp-notes.pdf`. unpublished.

Andrei A. Bulatov. A dichotomy theorem for nonuniform CSPs. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17*, pages 319–330, 2017.

Brian A. Davey. Dualisability in general and endodualisability in particular. In *Logic and algebra (Pontignano, 1994)*, volume 180 of *Lecture Notes in Pure and Appl. Math.*, pages 437–455. Dekker, New York, 1996.

David Geiger. Closed systems of functions and predicates. *Pacific J. Math.*, 27: 95–100, 1968. ISSN 0030-8730. URL `http://projecteuclid.org.ezproxy.is.cuni.cz/euclid.pjm/1102985564`.

E. L. Post. *The Two-Valued Iterative Systems of Mathematical Logic. (AM-5).* Princeton University Press, 1941. ISBN 9780691095707. URL `http://www.jstor.org/stable/j.ctt1bgzb1r`.

Walter Taylor. Varieties obeying homotopy laws. *Canadian J. Math.*, 29(3): 498–527, 1977. ISSN 0008-414X. doi: 10.4153/CJM-1977-054-9. URL `https://doi-org.ezproxy.is.cuni.cz/10.4153/CJM-1977-054-9`.

Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *J. ACM*, 67(5):30:1–30:78, 2020. doi: 10.1145/3402029. URL `https://doi.org/10.1145/3402029`.

List of Used Relations

Let $S \subseteq A$ and $a, b, c \in A$ and $n \in \mathbb{N}$.

$$(x) \in C_a \iff x = a$$
$$(x, y, z) \in T_S \iff x = y \ \vee \ z \in S$$
$$\leq_{a,b} = \{(a, a), (a, b), (b, b)\}$$
$$\neq_{a,b} = \{(a, b), (b, a)\}$$
$$(x, y, z) \in S^n_{0,1} \iff x - y + z = 1 \mod n$$
$$(x, y, z) \in S^n_{1,0} \iff x - y + z = 0 \mod n$$
$$S_{a,b} = \{(a, a, a), (a, b, b), (b, b, a), (b, a, b)\}$$
$$(x, y, z) \in L_{a,b} \iff x \in \{a, b\} \ \wedge \ (x = b \implies y = z)$$
$$R_\sigma = \{(0, 0), (1, 1), (2, 0)\}$$
$$P_{a,b} = \{(a, b), (b, a), (c, c)\}$$
$$P_{a,b,c} = \{(a, b), (b, c), (c, a)\}$$
$$(x, y) \in O_{a,b} \iff x = a \ \vee \ y = b$$
$$(x, y) \in D_{a,b} \iff x \neq a \ \vee \ y = b$$
$$(x, y) \in D^{-1}_{a,b} \iff x = a \ \vee \ y \neq b$$
$$(x, y) \in E_{a,b} \iff x \neq a \ \vee \ y \neq b$$
$$(x, y) \in N_a \iff y \neq a$$
$$(x, y) \in N^{-1}_a \iff x \neq a$$
$$(x, y) \in F^c_{a,b} \iff (x = c \iff y = c) \ \wedge \ (x \neq a \vee y \neq b)$$
$$(x, y) \in G_a \iff x = y \ \vee \ y = a$$
$$(x, y, z) \in H^c_{a,b} \iff ((x, y, z) = (c, c, c)) \ \vee \ ((x, y, z) \in S_{a,b})$$