Opponent Report

Title: **Iterative methods for Tichonov regularization with generalized regularization terms**
Author: **Andrea Kučerová**

Summary of the work's content

The focus of this thesis is on the use of Krylov subspace methods with Tichonov regularization for solving ill-posed linear inverse problems which contain noise in the right-hand side. Of particular interest is when generalized regularization terms are used, meaning that the regularization term is a matrix rather than a scalar quantity. A common choice in some applications is to choose this regularization matrix to be a discrete approximation of a differential operator, which serves to penalize non-smooth solutions.

The thesis starts with a thorough background on the solution of inverse problems, and then details Krylov subspace methods, Tichonov regularization, and hybrid approaches which combine the two in a "project and regularize" paradigm. The thesis then presents a series of numerical experiments, using model problems from MATLAB toolboxes, which compare the quality of the solution for various forms of generalized regularization matrices. The tested regularization matrices include those constructed using forward and centered differences for the first derivative in 1D and 2D, second and fourth order approximations of the second derivative in 1D and 2D, and various boundary conditions. The author gives interesting insights obtained from the experimental results.

Evaluation

- **Thesis topic:** The thesis topic is adequate for the Master's thesis level and the student has fulfilled the outlined requirements.
- **Author contribution:** The thesis gives a thorough and clear background on regularization techniques, Krylov subspace methods, and hybrid approaches. The academic contribution is primarily in deriving generalized regularization terms which use higher order approximation of derivatives than those typically used via Taylor expansions, and the subsequent numerical experiments to determine if this has practical benefit in terms of solution quality. There are also a number of interesting additional experiments, both those demonstrating the background material and those providing new insights (for example, an investigation of the impact of a wrong choice of boundary conditions, and a comparison of the behavior when using forward differences versus centered differences).
- **Mathematics level:** The level of mathematics is acceptable for a Master's thesis. The background mathematical material is generally well presented, although there are a few errors. For example:
  - On page 16, in the expression for x_λ, I think there is some mismatch of dimensions. In (1.4), $\Sigma \in \mathbb{R}^{m\times n}$, so $A^T = V\Sigma^T U^T$, not $V\Sigma U^T$.
  - Similar to the above, on page 17, you have "$I = VV^T = U^TU$", but these are identity matrices of different sizes, so this is confusing.
  - On page 30, I do not understand the summation limit notation. For example, in the last displayed equation, you have $\sum_{i-1}^{k}$ and then use i within the summation. Does this mean, e.g., $\sum_{j=i-1}^{k}[MA^TA]^j MA^Tb$?
  - On page 32: In the displayed equation after "$f(t+h)$", the second line should be "$f(t-h)$".
- **Literature and sources:** The sources used are correctly cited, and include a good mix of textbooks, classical articles, and more recent articles.

- **English and writing:** The text too many grammatical and spelling errors to list here. In particular, commas are often used as they are in Czech (separating subordinate clauses and conjunctions) rather than English. This does not, however, hinder the clarity of the text.

Questions and remarks

1. Your numerical experiments use the relative error measure to determine what regularization strategy is "best". How can we determine which regularization strategy is "best" when we don't know the true solution? What do people do in practice?
2. How common is it to have an estimate of $\|e\|$? Are there some techniques for automatically estimating the noise level?
3. You rightfully conclude that the best choice of regularization term is strongly problem dependent, but do your experimental results suggest any particular "rules of thumb"? For instance, how should I choose the order of precision based on my problem?
4. The thesis states (page 18) that computing the SVD is usually too expensive, and this motivates Krylov subspace-based approaches. With the advent of the randomized SVD, is this still true? Is these any work comparing the randomized SVD to iterative approaches? Is there any inherent benefit to using iterative approaches based on Krylov subspace methods?

Conclusion

I recommend accepting the presented thesis as a Master's Thesis.

in Prague, 30.08.2022

Dr. Erin Claire Carson, Ph.D.