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Mathematical Analysis of Selected Problems for Complex Fluids

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Title: Mathematical Analysis of Selected Problems for Complex Fluids

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Abstract: We study long-time and large-data existence theory of selected recently developed fluid mechanics models suitable for describing the mechanical behavior of materials with complex microstructure. In the first part of this work we focus on the Bingham type models for granular materials with the activation parameter (a critical value for the magnitude of the stress) dependent on the internal pore pressure. Our motivation comes from recent research concerning the implicitly constituted materials and also from an interesting paper by Chupin and Mathé [1], where the existence of weak solutions to the given problem was proved only in two spatial dimensions. Here we consider slightly different model (than in [1]) that we are able to derive from the basic governing equations of the theory of mixtures and we extend the existence result to three spatial dimensions. In the second part of this work we are concerned with fast developing field of viscoelastic materials. We study long-time and large-data existence of viscoelastic rate-type fluid models of higher order as they represent the simplest models suitable for describing the mechanical behavior of viscoelastic materials with complex microstructure. We are not aware of any long-time and large-data existence results for such models. Motivated by a study by Masmoudi [2], where the proof of the existence of weak solutions to the Giesekus model was briefly sketched, we prove the existence of weak solutions to the second order model, which can be written as a mixture of two Giesekus models, in two spatial dimensions.

Keywords: Bingham model, Burgers model, Giesekus model, existence theory, weak solution

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1. Introduction

This work is devoted to selected recently developed fluid mechanics models suitable for describing the mechanical behavior of materials with complex microstructure. Our aim is to justify the use of these models, more precisely to put the models into the context of the theory of interacting continua and to provide the robust mathematical analysis for them. We focus especially on the long-time and large-data existence theory for two particular models.

The first model is a generalization of the Bingham type model introduced in [1]. The model gives certain insight into the mechanical behavior of granular water saturated materials and into the physics of flows in porous media. Here, we present the results from the articles [3] and [4]. In [3] we provide the derivation of the model from the basic thermodynamical principles of interacting continua, and we provide the proof of the existence of weak solutions to the corresponding initial and boundary value problem in three spatial dimensions. (In [4] we consider more general model than in [3] and we prove the existence result under weaker assumptions for the data of the problem.) Let us note that in [1] the existence of weak solutions is proved only in two spatial dimensions.

The second model is a generalized version of the model introduced in [2]. We consider a viscoelastic rate type fluid model of the second order. The model is capable of capturing two different relaxation mechanisms, hence it represents one of the simplest models suitable for modeling the flows of viscoelastic materials with complex microstructure. Let us note that the model analyzed in [2] (the Giesekus model) is just a first order model. Here, we present the results from the preprint [5], i.e. we provide the proof of the existence of weak solutions to the model that can be written as a mixture of two Giesekus models in *two* spatial dimensions.

In all cases we prove the existence of weak solutions $[\boldsymbol{v}, p, \mathbb{S}]$ to the following system of equations supposed to be satisfied in $Q_T := (0,T) \times \Omega$, where $T \in (0,\infty)$ is a given number and $\Omega \subset \mathbb{R}^d$ is a flow domain (i.e. bounded, open and connected set), $d = 2$ or 3

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \operatorname{div}(2\nu\mathbb{D}) &= \operatorname{div} \mathbb{S} + \boldsymbol{f}, \\ \mathbb{G}(\mathbb{S}, \mathbb{D}, \nabla \boldsymbol{v}, h) &= \mathbb{O}. \end{aligned}$$

Here and in what follows, ∂_t denotes the partial time derivative, $\operatorname{div} \boldsymbol{h}$ denotes the divergence of the vector field $\boldsymbol{h}$ with respect to the spatial variables, i.e. $\operatorname{div} \boldsymbol{h} := \sum_{j=1}^d \partial x_j h_j$. Next, we define $\mathbb{D}$ as $\mathbb{D} := \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$, where the symbol ∇ stands for the gradient with respect to the spatial variables and for any tensor $\mathbb{A} \in \mathbb{R}^{d\times d}$ the symbol $\mathbb{A}^T$ denotes the transpose of $\mathbb{A}$. Finally 2ν is a given constant, $\boldsymbol{f}$ is a given vector-valued function, $\mathbb{G}$ is a given generally nonlinear continuous tensor-valued function, and h is an additional parameter.

The long-time and large-data existence theory is a fundamental and starting point of the development of the robust PDE analysis for the given initial and boundary value problem. We find the concept of weak solutions as a suitable concept of solutions. It is a concept that might be well defined for $[\boldsymbol{v}, p, \mathbb{S}]$ satisfying the apriori estimates provided by the corresponding system of equations (the apriori estimates also indicate, in which function spaces we shall search for the weak solution) and the numerical methods such as mixed finite element or spectral methods are based on this concept of solutions.

In all models, which we study, in two spatial dimensions we can use the solution $\boldsymbol{v}$ as a test function in the evolutionary equation for $\boldsymbol{v}$ (the so called energy *equality* holds true). However, in three spatial dimensions we are not allowed to do so. As one may expect, this is the crucial difference between the analysis in the twodimensional and threedimensional setting. Having the possibility to test the equation for $\boldsymbol{v}$ by the solution itself, it is much easier, for example, to show the convergence of nonlinear terms (acting in suitably introduced approximations to the problem in interest) to the proper functions.

Let us note that for the first model (the Bingham type model that we study in Chapter 2) and its generalization (studied in Chapter 3), the function $\mathbb{G}$ is monotone with respect to $\mathbb{S}$ and $\mathbb{D}$, i.e. for all couples $[\mathbb{S}_1, \mathbb{D}_1]$, $[\mathbb{S}_2, \mathbb{D}_2]$ satisfying $\mathbb{G}(\mathbb{S}_1, \mathbb{D}_1) = \mathbb{G}(\mathbb{S}_2, \mathbb{D}_2) = \mathbb{O}$, we have (the symbol " : " stands for the scalar product of two tensors)

$$(\mathbb{S}_1 - \mathbb{S}_2) : (\mathbb{D}_1 - \mathbb{D}_2) \geq 0.$$

Once we have the introduced monotonicity, we can relatively simply overcome the lack of the energy equality in three spatial dimensions, introducing standard truncations of the velocity fields whose applications generate the weak L^1 limits of nonlinear (uniformly bounded in $L^1(Q_T)$) terms in the biting sense (e.g. L^∞ truncation, see [6], [7], or [8], or parabolic Lipschitz truncation, see [9] or [10]), and then proceeding in the limit of the nonlinear term $\mathbb{G}(\mathbb{S}, \mathbb{D})$ in virtue of [11]. We use the introduced attitude for example in the relations (2.4.59), (2.4.60), their consequence (2.4.62) and in the rest of the proof of Theorem 2.2.1 including the use of Proposition 2.3.3 in Chapter 2.

For the viscoelastic rate-type fluid model (studied in Chapter 4) we do not know whether the function $\mathbb{G}$ is monotone with respect to $\mathbb{S}$ and $\mathbb{D}$, and, as a consequence, the lack of the energy equality in three spatial dimensions cannot be overcome directly by the procedure described above. This is the main reason why we restrict ourselves just to the two dimensional setting. However, the existence theory in the three dimensional setting is a subject of our current research.

In the rest of the Introduction, we introduce the general model for mixtures, from which all models of our interest are derived under physically reasonable simplifying assumptions. Then we briefly describe the physical background and the derivation of the particular models (the derivation of the models can be also found in [3] and [5]). Last, we introduce the basic notation needed in the analysis of these models performed in forthcoming chapters.

General model

We come from the general model for an N component mixture ($N \in \mathbb{N}$), where we distinguish the densities and the velocities of each component and we consider the energy and the entropy for the whole mixture. The resulting system of equations (see e.g. [12] or [13]), supposed to be satisfied in $Q_T := (0, T) \times \Omega$, where $T \in (0, \infty)$ is arbitrary, $\Omega \subset \mathbb{R}^d$ is a flow domain (i.e. bounded, open and connected set), $d = 2, 3$, reads as follows ($\alpha = 1, ..., N$)

$$\begin{aligned}
\partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \boldsymbol{v}_\alpha) &= m_\alpha, \\
\partial_t(\rho_\alpha \boldsymbol{v}_\alpha) + \operatorname{div}(\rho_\alpha(\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha)) &= \rho_\alpha \boldsymbol{f} + m_\alpha \boldsymbol{v}_\alpha + \boldsymbol{I}_\alpha, \\
\mathbb{T}_\alpha &= (\mathbb{T}_\alpha)^T, \\
\partial_t\left(\rho\left(e + \frac{|\boldsymbol{v}|^2}{2}\right)\right) + \operatorname{div}\left(\rho\left(e + \frac{|\boldsymbol{v}|^2}{2}\right)\boldsymbol{v}\right) &= \operatorname{div}(\mathbb{T}\boldsymbol{v} - \boldsymbol{q}_e) + \rho \boldsymbol{f} \cdot \boldsymbol{v} + \rho r, \\
\rho \dot{\eta} + \operatorname{div} \boldsymbol{q}_\eta &= \xi
\end{aligned}$$

such that m_α and $\boldsymbol{I}_\alpha$ satisfy

$$\begin{aligned}
\sum_{\alpha=1}^{N} m_\alpha &= 0, \\
\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{v}_\alpha + \boldsymbol{I}_\alpha &= 0.
\end{aligned}$$

Here, ρ_α, $\boldsymbol{v}_\alpha$ and $\mathbb{T}_\alpha$ ($\alpha = 1, ..., N$) stand for the densities, velocities and the Cauchy stress tensors (respectively) of the individual components of the mixture, $\mathbb{T}$ denotes the Cauchy stress tensor for the whole mixture. Next, the terms m_α, $m_\alpha \boldsymbol{v}_\alpha$ and $m_\alpha \boldsymbol{v}_\alpha + \boldsymbol{I}_\alpha$ represent the transport of mass, the consequent transport of linear momenta and the total transport of linear momenta (respectively) of the individual components inside the mixture. Finally, $\boldsymbol{f}$ denotes the density of the external forces (for example the gravity), e stands for the internal energy, $\boldsymbol{q}_e$ stands for the heat flux, r for the outer energy sources (e.g. radiation), $\boldsymbol{q}_\eta$ for the entropy flux and ξ for the rate of entropy production, all these quantities are related to the whole mixture. In order to close the system we have to introduce the so called constitutive relations between the static variables, i.e. the densities, the velocities and the internal energy, and the other quantities, i.e. the Cauchy stress tensors, the energy and entropy fluxes, the entropy and the rate of entropy production. The constitutive relations can be fully determined once we have the constitutive relations for two scalar quantities, for example for the rate of entropy production and for the Helmholtz free energy Ψ defined as $\Psi := e - \theta\eta$, where θ denotes the temperature.

1.1 Bingham type model for water saturated granular materials

The mechanical behavior of water saturated geological materials such as soils or sands is known to involve the notion of the so-called effective pressure, introduced in 1920 by Terzaghi [14]. We are concerned with internal flows of the materials with the value of the activation stress (once the stress exceeds this value, the material starts to flow) dependent on the effective pressure. The effective pressure is defined as the difference between the mean normal stress in the medium and the pressure in

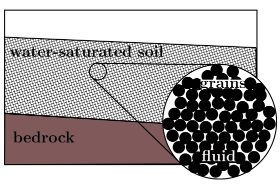

Figure 1.1: Sketch of the typical problem geometry and zoom into the structure of the material composed of a graular unconsolidated solid filled with an interstitial fluid.

the interstitial fluid - *pore pressure.* Water saturated geological materials are mixtures composed of an unconsolidated granular solid material and an interstitial pore space occupied by a fluid, see Fig. 1.1. With this picture in mind the total stress exerted on any control surface in such a medium comprises two contributions, namely the stress transmitted by the fluid and the stress transmitted by the granular solid. Mechanical loading or unloading of such a saturated material due to external forcing leads to redistribution of the stresses between the two constituents, which in general can be a rather complex process. Despite its complexity, several general observations can be made. First, if during the process the stress in the granular material increases, for example by reducing the pore pressure while keeping the total loading constant, the granular structure compactifies and becomes more rigid. A textbook example of this process is the beach sandcastle stabilization, when the fluid flowing out of the wet sand stabilizes the sand by "sticking" the sand grains closer together (here also capillary phenomena play a significant role). Second, as an opposite extreme, it may happen that during some processes the pressurized interstitial fluid bears almost the whole mechanical load exerted on the system, which leads to effective mechanical decoupling of the solid grains and the so called "liquefaction" can occur.

We develop a mathematical theory for a model that can be viewed as a simple toy model for the process of pore-pressure activated flows of saturated granular materials decribed above. We give up the ambition to model the actual process of liquefaction of real-world geological materials such as soils, since compared to what is presented here, this would require much more involved modeling of the activation yield criteria for such materials and of their rheological properties after the activation. However, we believe that even the strongly simplified setting presented here provides certain qualitative insight into the physics of pore-pressure activated flows and may even have some relevance to the problems of static liquefaction (see [15]) or enhanced oil recovery.[1] All this in our view justifies to study the associated initial and boundary value problems in terms of the mathematical well-posedness.

The model developed here is obtained within the context of the theory of interacting continua initiated by Truesdell [19], [20] (see also the review articles by Bowen [21], Atkin and Craine [22], and the numerous appendices in the book on rational thermodynamics by Truesdell [23], and the books by Samohýl [24], Rajagopal and Tao [12]). Within this framework, we are concerned with the flow of a mixture composed of two components, one representing the unconsolidated granular material flowing once the activation criterion is met, and the second fluid being Newtonian, representing the pore-space fluid. The Darcy-type flow of the pore fluid relative to the second material is considered, driven by the pore pressure gradient and gravity. This flow accomodates the pore presssure. Once the pore pressure reaches a certain threshold, the "granular" material starts to flow.

[1] In enhanced oil recovery steam or carbon dioxide is injected to reclaim oil that remains after initial extraction (see [16] for a discussion of enhanced oil recovery and [17], [18] for modeling and numerical studies). The recovery takes place after a pressure builds-up in the porous substrate containing the remnant oil and the oil starts to flow. Before the flow takes place we do have the steam/carbon dioxide being pumped into the porous rock and this flow is governed by some Darcy-like equation. The pressures involved are quite high and the material properties like the viscosity of the fluid would be pressure dependent, and at such high pressures the porous rock would undergo some deformation, these two effects are being ignored. We are also not modeling the porous rock as an individual constituent, as the considered mixture is constituted by steam and oil.

Derivation of the model

We refer to quantities related to the granular material (flowing after an activation criterion takes place) as *solid* (denoted by subscript "s") and to the interstitial fluid simply as *fluid* (denoted by subscript "f").

Based on the theory of multi-component materials (see e.g. [12], [13], from where we have the General model introduced above), we first formulate the individual mass and momentum balances for both components (constituents). Restraining ourselves to a purely mechanical setting, for simplicity, we do not need to formulate the balance equations for energy and entropy.

All the equations bellow used in the derivation of our model are supposed to be satisfied in $Q_T := (0,T)\times\Omega$, where $T \in (0,\infty)$ and $\Omega \subset \mathbb{R}^3$ is a flow domain (bounded, open and connected set).

The balance equations for mass read as follows:

$$\frac{\partial(\phi\rho_{\mathrm{f}}^{\mathrm{m}})}{\partial t} + \mathrm{div}(\phi\rho_{\mathrm{f}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{f}}) = 0 \ , \tag{1.1.1a}$$

$$\frac{\partial((1-\phi)\rho_{\mathrm{s}}^{\mathrm{m}})}{\partial t} + \mathrm{div}((1-\phi)\rho_{\mathrm{s}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{s}}) = 0 \ . \tag{1.1.1b}$$

Here, $\rho_{\mathrm{f}}^{\mathrm{m}}$ and $\rho_{\mathrm{s}}^{\mathrm{m}}$ denote the material (true) densities of the fluid and the solid, ϕ denotes the volume fraction of the fluid (equal to the porosity of the granular solid in the saturated case considered here), and $\boldsymbol{v}_{\mathrm{f}}$ and $\boldsymbol{v}_{\mathrm{s}}$ denote the velocities of the constituents, respectively. The zero on the right-hand side of equations (1.1.1) expresses the fact that we do not consider any mass transfer between the constituents.

The balance equations for linear momenta of these two constituents take the form

$$\frac{\partial(\phi\rho_{\mathrm{f}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{f}})}{\partial t} + \mathrm{div}(\phi\rho_{\mathrm{f}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{f}}\otimes\boldsymbol{v}_{\mathrm{f}}) = \mathrm{div}\,\mathbb{T}_{\mathrm{f}} + \phi\rho_{\mathrm{f}}^{\mathrm{m}}\boldsymbol{f} + \mathbf{I} \ , \tag{1.1.2a}$$

$$\frac{\partial((1-\phi)\rho_{\mathrm{s}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{s}})}{\partial t} + \mathrm{div}((1-\phi)\rho_{\mathrm{s}}^{\mathrm{m}}\boldsymbol{v}_{\mathrm{s}}\otimes\boldsymbol{v}_{\mathrm{s}}) = \mathrm{div}\,\mathbb{T}_{\mathrm{s}} + (1-\phi)\rho_{\mathrm{s}}^{\mathrm{m}}\boldsymbol{f} - \mathbf{I} \ , \tag{1.1.2b}$$

where, for $\boldsymbol{u},\boldsymbol{w} \in \mathbb{R}^3$, the symbol $\boldsymbol{u}\otimes\boldsymbol{w}$ denotes the tensor of components $(\boldsymbol{u}\otimes\boldsymbol{w})_{ij} := u_i w_j$ with $i,j = 1,2,3$, while $\mathbb{T}_{\mathrm{f}}$ and $\mathbb{T}_{\mathrm{s}}$ stand for the fluid and solid Cauchy stresses, respectively, both of which are assumed to be symmetric (i.e. $\mathbb{T}_{\mathrm{f}}{=}\mathbb{T}_{\mathrm{f}}^{\mathrm{T}}$, $\mathbb{T}_{\mathrm{s}}{=}\mathbb{T}_{\mathrm{s}}^{\mathrm{T}}$). The quantity $\mathbf{I}$ represents the interaction force between the constituents. The interaction nature of this force is reflected by the fact that it appears with plus sign in one equation and with minus in the other. Finally $\boldsymbol{f}$ is the body force (same for both constituents, typically this is the gravity acceleration vector). In reality, both constituents are compressible, i.e. both material densities $\rho_{\mathrm{s}}^{\mathrm{m}}$ and $\rho_{\mathrm{f}}^{\mathrm{m}}$ must be specified by a corresponding state equation. In the isothermal setting considered here, such relation would take the form of dependence on the material stress state of the particular constituent. Since the dominant compressibility effect in the context of real-world geological materials is not related to the changes of material densities, but rather to the changes in porosity in reaction to the applied loading (see [25], chapter 4), we neglect the former effect by setting

$$\rho_{\mathrm{f}}^{\mathrm{m}} = \mathrm{const}_{\mathrm{f}} \quad \text{and} \quad \rho_{\mathrm{s}}^{\mathrm{m}} = \mathrm{const}_{\mathrm{s}}. \tag{1.1.3}$$

Dividing now (1.1.1a) by $\rho_{\mathrm{f}}^{\mathrm{m}}$ and (1.1.1b) by $\rho_{\mathrm{s}}^{\mathrm{m}}$ and summing the resulting equations, we obtain

$$\mathrm{div}\,\boldsymbol{v}_{\mathrm{s}} = -\,\mathrm{div}(\phi(\boldsymbol{v}_{\mathrm{f}}-\boldsymbol{v}_{\mathrm{s}})) \ . \tag{1.1.4a}$$

Inserting this relation into (1.1.1b) (divided by ρ_s^m) yields the evolutionary equation for the porosity

$$\frac{\partial\phi}{\partial t} + \boldsymbol{v}_s \cdot \nabla\phi = -(1-\phi)\,\mathrm{div}\,(\phi(\boldsymbol{v}_f - \boldsymbol{v}_s)) \ . \tag{1.1.4b}$$

Under the assumptions (1.1.3), the system of equations (1.1.4) is equivalent to the system (1.1.1).

The balance equations for linear momentum (1.1.2) are reformulated in terms of an equivalent system, where the balance equation for linear momentum of the solid is replaced by the balance equation for linear momentum of the mixture as a whole. Thus, using (1.1.3), we get

$$\rho_f^m \left(\frac{\partial(\phi \boldsymbol{v}_f)}{\partial t} + \mathrm{div}(\phi \boldsymbol{v}_f \otimes \boldsymbol{v}_f) \right) = \mathrm{div}\,\mathbb{T}_f + \phi\rho_f^m \boldsymbol{f} + \mathbf{I} \ , \tag{1.1.5a}$$

$$\begin{aligned} \rho_s^m \left(\frac{\partial \boldsymbol{v}_s}{\partial t} + \mathrm{div}\,(\boldsymbol{v}_s \otimes \boldsymbol{v}_s) \right) &= \mathrm{div}\,\mathbb{T} + \rho \boldsymbol{f} - \frac{\partial}{\partial t}\,(\phi(\rho_f^m \boldsymbol{v}_f - \rho_s^m \boldsymbol{v}_s)) \\ &\quad - \mathrm{div}(\phi(\rho_f^m \boldsymbol{v}_f \otimes \boldsymbol{v}_f - \rho_s^m \boldsymbol{v}_s \otimes \boldsymbol{v}_s)) \ , \end{aligned} \tag{1.1.5b}$$

where, in the second equation, we introduced the total Cauchy stress $\mathbb{T}$ and the total density ρ by

$$\mathbb{T} := \mathbb{T}_s + \mathbb{T}_f, \qquad \rho := \phi\rho_f^m + (1-\phi)\rho_s^m \ = \rho_s^m + \phi(\rho_f^m - \rho_s^m). \tag{1.1.6}$$

Next we introduce several simplifications.

- In the balance of linear momentum for the fluid (1.1.5a), we ignore the inertial forces, i.e. the whole left-hand side of (1.1.5a) is set to zero. We further consider $\mathbb{T}_f$ of the form

  $$\mathbb{T}_f = -p_f^t \phi\, \mathbb{I} \ , \tag{1.1.7}$$

  where p_f^t is the true pressure in the interstitial fluid (pore pressure). Finally, the interaction force $\mathbf{I}$ takes a simple form corresponding to the linear drag

  $$\mathbf{I} = -\alpha(\boldsymbol{v}_f - \boldsymbol{v}_s) + p_f^t \nabla\phi \ , \tag{1.1.8}$$

  where α is the drag coefficient of the form

  $$\alpha := \frac{\phi^2 \mu_f}{k(\phi)} \ , \tag{1.1.9}$$

  μ_f being the dynamic viscosity of the fluid (assumed to be constant for simplicity) and $k(\phi)$ the permeability of the granular material. The presence of the second term on the right-hand side of (1.1.8) is known from multiphase continuum theory as an artefact of the volume averaging technique [26], which must be present to cancel out in the fluid momentum balance with a corresponding term coming from the divergence of eq. (1.1.7). See also [13], where such terms occur from the derivation directly.

- In the balance equation (1.1.5b), we keep the inertial term only on the left-hand side and neglect the last two terms on the right hand side using a rough scaling argument stating that the scale of these terms is at most the scale of the left hand side, multiplied by the scale of porosity, which, in the considered applications, typically does not exceed a few percent. Furthermore, since ϕ is typically below 0.1, we conclude that ρ introduced in (1.1.6) is approximately equal to $\rho_{\mathrm{s}}^{\mathrm{m}}$, so we replace $\rho \boldsymbol{f}$ by $\rho_{\mathrm{s}}^{\mathrm{m}} \boldsymbol{f}$ in (1.1.5b).

With these simplifications, the balance equations (1.1.4) and (1.1.5) take the form

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla \phi = -(1-\phi)\,\mathrm{div}\left(\phi(\boldsymbol{v}_{\mathrm{f}} - \boldsymbol{v}_{\mathrm{s}})\right), \tag{1.1.10a}$$

$$\mathrm{div}\,\boldsymbol{v}_{\mathrm{s}} = -\,\mathrm{div}(\phi(\boldsymbol{v}_{\mathrm{f}} - \boldsymbol{v}_{\mathrm{s}})), \tag{1.1.10b}$$

$$\phi \nabla p_{\mathrm{f}}^{\mathrm{t}} = \phi \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f} - \alpha(\boldsymbol{v}_{\mathrm{f}} - \boldsymbol{v}_{\mathrm{s}}), \tag{1.1.10c}$$

$$\rho_{\mathrm{s}}^{\mathrm{m}} \left(\frac{\partial \boldsymbol{v}_{\mathrm{s}}}{\partial t} + \mathrm{div}(\boldsymbol{v}_{\mathrm{s}} \otimes \boldsymbol{v}_{\mathrm{s}}) \right) = \mathrm{div}\,\mathbb{T} + \rho_{\mathrm{s}}^{\mathrm{m}} \boldsymbol{f}. \tag{1.1.10d}$$

Next, we multiply (1.1.10c) by $\frac{\phi}{\alpha}$, use (1.1.9) and apply divergence to the result. After inserting the outcome of these computations in (1.1.10b) we obtain

$$\mathrm{div}\,\boldsymbol{v}_{\mathrm{s}} = \mathrm{div}\left(\frac{k(\phi)}{\mu_{\mathrm{f}}} (\nabla p_f^t - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}) \right). \tag{1.1.11}$$

As a consequence, (1.1.11) replaces (1.1.10b).

The next assumption states that the porosity of the granular solid can be described by a constitutive relation of the form

$$\phi = \widehat{\phi}(p^{\mathrm{eff}}) \qquad \text{where} \qquad p^{\mathrm{eff}} := p - p_{\mathrm{f}}^{\mathrm{t}}. \tag{1.1.12}$$

The quantity p^{eff}, called *effective pressure* as introduced by Terzaghi [14], is defined as the difference between the total mixture pressure and the fluid (pore) pressure. The quantity p^{eff} is assumed to reflect the part of the loading bore by the granular solid. Inserting the constitutive assumption (1.1.12) in (1.1.10a) and using (1.1.10b) and (1.1.11), we obtain the following evolutionary equation for the effective pressure

$$\frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}p^{\mathrm{eff}}} \left(\frac{\partial p^{\mathrm{eff}}}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p^{\mathrm{eff}} \right) = (1-\phi)\,\mathrm{div}\left(\frac{k(\phi)}{\mu_{\mathrm{f}}} (\nabla p_{\mathrm{f}}^{\mathrm{t}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}) \right). \tag{1.1.13}$$

Setting

$$-\frac{1}{\beta} := \frac{\mathrm{d}\widehat{\phi}}{\mathrm{d}p^{\mathrm{eff}}} \quad \text{with } \beta > 0, \tag{1.1.14}$$

replacing $p_{\mathrm{f}}^{\mathrm{t}}$ in (1.1.11) and (1.1.13) by $p - p^{\mathrm{eff}}$ and splitting the total Cauchy stress $\mathbb{T}$ as

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S}, \tag{1.1.15}$$

we arrive at the following set of governing equations

$$\operatorname{div} \boldsymbol{v}_{\mathrm{s}} = \operatorname{div}\left(\frac{k(\phi)}{\mu_{\mathrm{f}}}(\nabla p - \nabla p^{\mathrm{eff}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f})\right) , \tag{1.1.16a}$$

$$\rho_{\mathrm{s}}^{\mathrm{m}}\left(\frac{\partial \boldsymbol{v}_{\mathrm{s}}}{\partial t} + \operatorname{div}(\boldsymbol{v}_{\mathrm{s}} \otimes \boldsymbol{v}_{\mathrm{s}})\right) = -\nabla p + \operatorname{div} \mathbb{S} + \rho_{\mathrm{s}}^{\mathrm{m}} \boldsymbol{f} , \tag{1.1.16b}$$

$$\frac{\partial p^{\mathrm{eff}}}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p^{\mathrm{eff}} = -(1-\phi)\beta \operatorname{div}\left(\frac{k(\phi)}{\mu_{\mathrm{f}}}(\nabla p - \nabla p^{\mathrm{eff}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f})\right), \tag{1.1.16c}$$

$$\boldsymbol{v}_{\mathrm{f}} = \boldsymbol{v}_{\mathrm{s}} - \frac{1}{\alpha}\widehat{\phi}(p^{\mathrm{eff}})\Big(\nabla p_{\mathrm{f}}^{\mathrm{t}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}\Big) . \tag{1.1.16d}$$

Since $p_{\mathrm{f}}^{\mathrm{t}} = p - p^{\mathrm{eff}}$, we can view (1.1.16) as the system of partial differenctial equations describing the evolution of p, $\boldsymbol{v}_{\mathrm{s}}$, p^{eff} and $\boldsymbol{v}_{\mathrm{f}}$, where the rheology of the material needs to be specified by providing a constitutive relation for the stress $\mathbb{S}$. We shall assume that the material behaves as very stiff until the threshold is reached at which moment the material starts to flow as a liquid. The simplest constitutive relation for this type of response is characterized by the so-called Bingham fluid [27], where the solid part responses as a perfectly rigid body until the magnitude of the stress exceeds the threshold when the solid flows as a Newtonian fluid. This type of response is usually written in the following way:

$$\begin{cases} |\mathbb{S}| \leq \tau(p^{\mathrm{eff}}) & \text{if and only if } \mathbb{D} = \mathbb{O}, \\ |\mathbb{S}| > \tau(p^{\mathrm{eff}}) & \text{if and only if } \mathbb{S} = \tau(p^{\mathrm{eff}})\frac{\mathbb{D}}{|\mathbb{D}|} + 2\nu_* \mathbb{D}. \end{cases} \tag{1.1.17}$$

Here $\mathbb{D}$ is the symmetric part of the velocity gradient

$$\mathbb{D} := \frac{1}{2}(\nabla \boldsymbol{v}_{\mathrm{s}} + (\nabla \boldsymbol{v}_{\mathrm{s}})^{\mathrm{T}}) ,$$

$\nu_* > 0$ is the viscosity and $\tau(p^{\mathrm{eff}})$ is the threshold, depending on the effective pressure. Typically,

$$\tau(p^{\mathrm{eff}}) = q_0 (p^{\mathrm{eff}})^+, \tag{1.1.18}$$

where q_0 is a constant and the symbol $()^+$ denotes the positive part of a quantity, i.e. $(\psi)^+ := \max(\psi, 0)$.

The activation criterion (1.1.17) is too simple to describe the shear instability of real-world granular materials since it does not take into account any concept of internal friction. In reality, it should be replaced by some form of Mohr-Coulomb criteria, see, e.g., [28]. Similarly, also the fact that the material, which is a mixture of flowing solid particles and fluid (a slurry), is supposed to response, after being activated, as a Navier-Stokes (linear viscous) fluid, is a severe limitation of the model considered here. To overcome this defficiency one would need to incorporate more realistic models used for description of flows of granular materials (capable of exhibiting normal stress differences, etc.). Such models have been developed in [29], [30], see also a review article [31].

Despite these important limitations, the system (1.1.16) – (1.1.18) seems to be a meaningful, physically justified and relatively simple model worth of studying. We however do not further investigate this system here, as our goal is to identify the assumptions that lead to the model analyzed in [1]. Towards this aim, we introduce the following additional assumptions:

- *Pore pressure evolution approximation.* Using the relation $p_{\mathrm{f}}^{\mathrm{t}} = p - p^{\mathrm{eff}}$ we rewrite (1.1.16c) as an evolutionary equation for $p_{\mathrm{f}}^{\mathrm{t}}$

$$\frac{\partial p_{\mathrm{f}}^{\mathrm{t}}}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p_{\mathrm{f}}^{\mathrm{t}} = \frac{\partial p}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p + (1-\phi)\beta \operatorname{div}\left(\frac{k(\phi)}{\mu_{\mathrm{f}}}(\nabla p_{\mathrm{f}}^{\mathrm{t}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f})\right). \tag{1.1.19}$$

  Next we assume that the dominant contribution to the total pressure p in the equation (1.1.19) comes from the hydrostatic part, which may in general depend explicitly on time to include problems with evolving boundary. Consequently, we replace $p(x,t)$ by $p_s(x,t)$ in eq. (1.1.19), where p_s is a given function. Also, we replace $(1-\phi)$ by 1 on the right-hand side of (1.1.19) since, as set above, we are interested in situations where $\phi < 0.1$ and finally, we assume that both the permeability k and the compressibility parameter β are constant. Setting thus

$$K := \frac{\beta k}{\mu_{\mathrm{f}}} \geq 0 \ , \tag{1.1.20}$$

  the equation (1.1.19) simplifies to the form

$$\frac{\partial p_{\mathrm{f}}^{\mathrm{t}}}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p_{\mathrm{f}}^{\mathrm{t}} = K\Delta p_{\mathrm{f}}^{\mathrm{t}} - \operatorname{div}(K\rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}) + \frac{\partial p_s}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p_s. \tag{1.1.21}$$

- *Yield criterion approximation.* Also in the yield criterion, we replace the pressure p in the definition of the effective pressure (1.1.12) by p_s, i.e. instead of (1.1.18), we have

$$\tau(p_{\mathrm{f}}^{\mathrm{t}}) = q_0(p_s - p_{\mathrm{f}}^{\mathrm{t}})^+ \ . \tag{1.1.22}$$

- *Incompressibility.* We ignore the effect of porosity changes in (1.1.16a) by replacing (1.1.16a) with the incompressibility constraint

$$\operatorname{div} \boldsymbol{v}_{\mathrm{s}} = 0 \ . \tag{1.1.23}$$

With the above set of simplifying assumptions, the final reduced system of governing equations reads as follows

$$\operatorname{div} \boldsymbol{v}_{\mathrm{s}} = 0 \ , \tag{1.1.24a}$$

$$\rho_{\mathrm{s}}^{\mathrm{m}} \left(\frac{\partial \boldsymbol{v}_{\mathrm{s}}}{\partial t} + \operatorname{div}(\boldsymbol{v}_{\mathrm{s}} \otimes \boldsymbol{v}_{\mathrm{s}})\right) = -\nabla p + \operatorname{div} \mathbb{S} + \rho_{\mathrm{s}}^{\mathrm{m}} \boldsymbol{f} \ , \tag{1.1.24b}$$

$$\frac{\partial p_{\mathrm{f}}^{\mathrm{t}}}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p_{\mathrm{f}}^{\mathrm{t}} = K\Delta p_{\mathrm{f}}^{\mathrm{t}} - \operatorname{div}(K\rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}) + \frac{\partial p_s}{\partial t} + \boldsymbol{v}_{\mathrm{s}} \cdot \nabla p_s \ , \tag{1.1.24c}$$

where $\mathbb{S}$ satisfies

$$\begin{cases} |\mathbb{S}| \leq \tau(p_{\mathrm{f}}^{\mathrm{t}}) & \text{if and only if } \mathbb{D} = \mathbb{O}, \\ |\mathbb{S}| > \tau(p_{\mathrm{f}}^{\mathrm{t}}) & \text{if and only if } \mathbb{S} = \tau(p_{\mathrm{f}}^{\mathrm{t}}) \frac{\mathbb{D}}{|\mathbb{D}|} + 2\nu_* \mathbb{D}, \end{cases} \quad \text{with} \ \ \tau(p_{\mathrm{f}}^{\mathrm{t}}) = q_*(p_s - p_{\mathrm{f}}^{\mathrm{t}})^+, \tag{1.1.25}$$

and where the velocity $\boldsymbol{v}_{\mathrm{f}}$ is given by

$$\boldsymbol{v}_{\mathrm{f}} = \boldsymbol{v}_{\mathrm{s}} - \frac{1}{\alpha} \widehat{\phi}(p - p_{\mathrm{f}}^{\mathrm{t}}) \left(\nabla p_{\mathrm{f}}^{\mathrm{t}} - \rho_{\mathrm{f}}^{\mathrm{m}} \boldsymbol{f}\right) . \tag{1.1.26}$$

Since $\boldsymbol{v}_{\mathrm{f}}$ does not enter into (1.1.24) and (1.1.25), the equation (1.1.26) describing the evolution of $\boldsymbol{v}_{\mathrm{f}}$ is not considered anymore in what follows (as $\boldsymbol{v}_{\mathrm{f}}$ can be always obtained from equation of Darcy's type (1.1.26) once $\boldsymbol{v}_{\mathrm{s}}$ and $p_{\mathrm{f}}^{\mathrm{t}}$ are known/computed from (1.1.24) and (1.1.25)).

Let us note that, following a recent observation in [10] (see also [32], [33], [34]) the rheological behaviour (1.1.25) can be equivalently written as

$$2\nu_*\mathbb{D} = \frac{(|\mathbb{S}| - \tau(p_{\mathrm{f}}^{\mathrm{t}}))^+}{|\mathbb{S}|}\mathbb{S}, \quad \text{where} \quad \tau(p_{\mathrm{f}}) = q_*(p_s - p_{\mathrm{f}}^{\mathrm{t}})^+. \tag{1.1.27}$$

One of the motivations for analyzing the introduced model is a recent paper by Chupin and Mathé [1], where the authors characterize the tensorial response (1.1.27) through two scalar constraints:

$$|\mathbb{Z}| \le \tau(p_{\mathrm{f}}^{\mathrm{t}}) \;\&\; \mathbb{Z}:\mathbb{D} \ge \tau(p_{\mathrm{f}}^{\mathrm{t}})|\mathbb{D}|, \text{ where } \mathbb{Z} := \mathbb{S} - 2\nu_*\mathbb{D} \;\&\; \tau(p_{\mathrm{f}}^{\mathrm{t}}) = q_*(p_s - p_{\mathrm{f}}^{\mathrm{t}})^+. \tag{1.1.28}$$

The equivalence between (1.1.25), (1.1.27) and (1.1.28) will be proved in Chapter 2.

1.2 Viscoelastic rate-type fluid model

Viscoelastic rate-type fluid models involving the stress and its observer-invariant time derivatives of higher order represent the simplest fluid mechanics models suitable to describe the behaviour of materials with complex microstructure. This is due to the fact that higher order viscoelastic rate-type fluid models are capable of capturing several different relaxation mechanisms (as well as other non-Newtonian phenomena). Geomaterials such as asphalt, biomaterials such as vitreous in the eye, synthetic rubbers such as styrene butadiene rubber, are examples of materials that are described by the viscoelastic rate-type fluid models of the second order. Here we refer to Monismith, Secor [35], Narayan et al. [36], Málek, Rajagopal, Tůma [37], Sharif-Kashani et al. [38], Řehoř et al. [39] for experimental data and for corroboration this data using higher order viscoelastic rate type fluid models.

Derivation of the model

A standard model belonging to the category of viscoelastic rate type fluids of the second order is the model due to Burgers, see [40], where Burgers developed a one-dimensional model. The d-dimensional form ($d \geq 2$) of the Burgers model, considered in the time-space cylinder $Q_T := (0,T) \times \Omega$, where $T \in (0,\infty)$ and Ω is an open, bounded and connected set, takes the form:

$$\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0, && (1.2.1)\\
\rho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) &= \operatorname{div} \mathbb{T} + \rho \boldsymbol{f}, && (1.2.2)\\
\mathbb{T} &= -p\mathbb{I} + 2\nu\mathbb{D} + \mathbb{S}, && (1.2.3)\\
\overset{\triangledown\triangledown}{\mathbb{S}} + \alpha_1 \overset{\triangledown}{\mathbb{S}} + \alpha_0 \mathbb{S} &= \beta_1 \overset{\triangledown}{\mathbb{D}} + \beta_0 \mathbb{D}. && (1.2.4)
\end{aligned}$$

Here again, $\boldsymbol{v} = (v_1, \ldots, v_d)$ denotes the velocity, $\mathbb{D} := \frac{1}{2}\left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right)$ is the symmetric part of the velocity gradient $\nabla \boldsymbol{v}$, $\mathbb{T}$ is the Cauchy stress tensor, $\mathbb{I}$ is the identity tensor and p (often called the pressure) is a scalar quantity associated with the fact that the fluid is incompressible, i.e. with the constraint (1.2.1). A given vector-valued function $\boldsymbol{f}$ represents the density of external forces acting on the body, the parameter $\rho > 0$ stands for the (constant) density, 2ν, α_0, α_1, β_0, β_1 are positive material coefficients. Finally, for any $\mathbb{A} : Q_T \to \mathbb{R}^{d\times d}$, the nonlinear differential operators $\overset{\triangledown}{\mathbb{A}}$ and $\overset{\triangledown\triangledown}{\mathbb{A}}$ stand for

$$\overset{\triangledown}{\mathbb{A}} := \partial_t \mathbb{A} + \sum_{j=1}^{d} v_j \partial_{x_j} \mathbb{A} - \nabla \boldsymbol{v} \mathbb{A} - \mathbb{A}(\nabla \boldsymbol{v})^T \quad \text{and} \quad \overset{\triangledown\triangledown}{\mathbb{A}} := \overset{\triangledown}{\overset{\triangledown}{\mathbb{A}}}. \tag{1.2.5}$$

Setting $\mathbb{S} = \mathbb{O}$ in (1.2.1)–(1.2.4), the governing equations reduce to the incompressible Navier-Stokes system. If $\mathbb{S}$ is involved, the additional tensorial equation (1.2.4) is nonlinear (with the nonlinearities containing $\nabla \boldsymbol{v}$ in virtue of (1.2.5)), contains the time derivatives of $\mathbb{S}$ of the second order, and does not involve a diffusion term. Consequently, without having any additional insight it may be a nontrivial task to achieve a priori estimates for the unknowns $\boldsymbol{v}$ and $\mathbb{S}$ controlled by data of the problem. It is thus beneficial to observe, see [37] for details, that the system (1.2.1)–(1.2.4) follows from the following set of equations satisfied in Q_T:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.2.6}$$

$$\rho \left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) - \operatorname{div} \mathbb{T} - \rho \boldsymbol{f} = \boldsymbol{0}, \qquad \rho > 0, \tag{1.2.7}$$

$$\overset{\triangledown}{\mathbb{B}}_i + \frac{1}{\tau_i}(\mathbb{B}_i - \mathbb{I}) = \mathbb{O}, \qquad \tau_i > 0 \ \ (i = 1, 2), \tag{1.2.8}$$

$$-p\mathbb{I} + 2\nu \mathbb{D} + \sum_{i=1}^{2} G_i(\mathbb{B}_i - \mathbb{I}) = \mathbb{T}, \qquad \nu, G_1, G_2 > 0 \tag{1.2.9}$$

provided that we define

$$\mathbb{S} := \sum_{i=1}^{2} G_i(\mathbb{B}_i - \mathbb{I})$$

and set

$$\alpha_1 := \frac{\tau_1 + \tau_2}{\tau_1 \tau_2}, \quad \alpha_0 := \frac{1}{\tau_1 \tau_2}, \quad \beta_1 := \frac{2}{G_1 + G_2}, \quad \beta_0 := 2\left(\frac{G_1}{\tau_1} + \frac{G_2}{\tau_2}\right).$$

In fact, this viewpoint reflects the physical underpinnings of the model, see [40, 41, 42]. Note that (1.2.6)–(1.2.9) reduces to the standard viscoelastic Oldroyd-B model (see [43]) if either $\mathbb{B}_1$ (or G_1) or $\mathbb{B}_2$ (or G_2) vanish. Saying differently, due to its equivalent description (1.2.6)-(1.2.9), one can view the original model (1.2.1)–(1.2.4) as a mixture of two Oldroyd-B models of the first order.

Carrying on a thermodynamical approach developed by Rajagopal and Srinivasa [44] for modeling of responses (constitutive relations) of viscoelastic fluids, see also [45, 46, 41, 47] for extension and further details, Málek, Rajagopal and Tůma [42] have recently developed a physically well sounded hierarchy of viscoelastic rate-type fluid models capturing two different relaxation mechanisms. This class of models is described by the following set of equations (fulfilled in Q_T):

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.2.10}$$

$$\rho \left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) - \operatorname{div} \mathbb{T} - \rho \boldsymbol{f} = \boldsymbol{0}, \quad \rho > 0, \tag{1.2.11}$$

$$\overset{\triangledown}{\mathbb{B}}_i + \frac{1}{\tau_i}(\mathbb{B}_i^{2-\lambda_i} - \mathbb{B}_i^{1-\lambda_i}) = \mathbb{O}, \quad \tau_i = \frac{G_i}{\nu_i} > 0, \ \lambda_i \in \mathbb{R} \ (i = 1, 2), \tag{1.2.12}$$

$$-p\mathbb{I} + 2\nu \mathbb{D} + \sum_{i=1}^{2} G_i(\mathbb{B}_i - \mathbb{I}) = \mathbb{T}, \quad 2\nu, G_1, G_2 > 0, \tag{1.2.13}$$

where $\mathbb{B}_i$ are supposed to be of the form

$$\mathbb{B}_i = \mathbb{F}_i \mathbb{F}_i^T, \quad \mathbb{F}_i \in \mathbb{R}^{d\times d}, \ \det \mathbb{F}_i > 0 \text{ in } Q_T \ (i = 1, 2). \tag{1.2.14}$$

Here, $\mathbb{F}_i$, $i = 1, 2$, denote the deformation tensors representing elastic parts of the overall responses associated with two different relaxation mechanisms of the material, see for example [42] for details and physical interpretation. Notice that (1.2.10)–(1.2.13) coincides with (1.2.6)–(1.2.9) if we set $\lambda_1 = \lambda_2 = 1$.

As a starting point of the development of the long-time and large-data existence theory for the system (1.2.10)–(1.2.14) completed with suitable initial and boundary conditions and carrying on the study by Masmoudi [2], where the couple of original ideas for proving the existence of weak solutions to the system (1.2.10)–(1.2.14) with $G_1 = 1$, $G_2 = 0$ and $\lambda_1 = 0$ were brought (and we are not aware of any other results concerning a robust PDE analysis of the system (1.2.10)–(1.2.14)), we study the system (1.2.10)–(1.2.14) with $\lambda_1 = \lambda_2 = 0$ and $G_1, G_2 > 0$ arbitrary (in this work just in two spatial dimensions).

1.3 Basic notation

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geq 2$) be a domain (i.e. bounded open connected set) and let us assume that its boundary $\partial\Omega$ is Lipschitz. For given $T \in (0, \infty)$, we set $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$. Next, the operator "$\cdot$" denotes the scalar product of two vectors, the operator " $:$ " denotes the scalar product of two tensors. The operator "$\otimes$" denotes the tensor product of two vectors, i.e. $(\boldsymbol{b} \otimes \boldsymbol{z})_{ij} := b_i z_j$. For a matrix $\mathbb{A} = \{A_{ij}\}_{i,j=1}^d$ and a vector $\boldsymbol{b} = (b_1, \dots, b_d)$ we define the third order tensor $\mathbb{A} \otimes \boldsymbol{b} = \{(\mathbb{A} \otimes \boldsymbol{b})_{ijk}\}_{i,j,k=1}^d$ as

$$(\mathbb{A} \otimes \boldsymbol{b})_{ijk} := A_{ij} b_k.$$

For a matrix-valued function $\mathbb{A} = (A_{ij})_{i,j=1}^d$ and a vector-valued function $\boldsymbol{b} = (b_1, ..., b_d)$ we define the operator div acting on the third order tensor $\mathbb{A} \otimes \boldsymbol{b}$ as

$$\operatorname{div}(\mathbb{A} \otimes \boldsymbol{b}) := \sum_{j=1}^d \partial_{x_j}(b_j \mathbb{A}).$$

The symbol $|\cdot|$ denotes the Euclidean norm of a vector, the Frobenius norm of a tensor, or the Lebesgue measure of the given measurable subset of $\mathbb{R}^d$, $d \in \mathbb{N}$.

For $q \in [1, \infty]$ the symbol $\|\cdot\|_q$ stands for the norm in the usual Lebesgue space $L^q(\Omega)$ (or in its multidimensional variant $(L^q(\Omega))^d$, $(L^q(\Omega))^{d\times d}$, etc.), while the symbol $\|\cdot\|_{1,q}$ stands for the norm in the usual Sobolev space $W^{1,q}(\Omega)$ (or in its multidimensional variant $(W^{1,q}(\Omega))^d$, $(W^{1,q}(\Omega))^{d\times d}$, etc.). The symbol $\mathcal{M}(\overline{Q_T})$ stands for the space of the Radon measures defined on the closure of Q_T. If X is a Banach space, then X^* denotes its dual space. The dualities between Banach spaces and their duals are denoted as $\langle \cdot, \cdot \rangle$. If X is a Banach space, then $L^q(0, T; X)$ is the corresponding Bochner space, $C([0, T]; X)$ is the space of functions continuous in $[0, T]$ with values in X, $C_{weak}([0, T]; X)$ is the space of functions weakly continuous in $[0, T]$ with values in X. For an open set $O \subset \mathbb{R}^d$, $C_c^\infty(O)$ is the space of smooth functions compactly supported in O, $L^q_{loc}(O)$ is the space of functions, whose q-power is locally integrable over O.

Then we introduce the spaces relevant to our problem. For any $q \in [1, \infty)$ we set

$$\begin{aligned} W_0^{1,q}(\Omega) &:= \{u \in W^{1,q}(\Omega); u = 0 \text{ on } \partial\Omega\}, \\ W_{\mathbf{0},\mathrm{div}}^{1,q} &:= \{\boldsymbol{u} \in (W^{1,q}(\Omega))^d; \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega; \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}, \end{aligned}$$

and we equip the spaces with the norms (due to the Poincaré inequality)

$$\|u\|_{W_0^{1,q}(\Omega)} := \|\nabla u\|_q, \quad \|\boldsymbol{u}\|_{W_{\mathbf{0},\mathrm{div}}^{1,q}} := \|\nabla \boldsymbol{u}\|_q.$$

Next, we define

$$\begin{aligned} L_{\boldsymbol{n},\mathrm{div}}^q &:= \overline{\{\boldsymbol{u} \in (C_c^\infty(\Omega))^d; \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}}^{\|\cdot\|_q}, \\ W_{\boldsymbol{n}}^{1,q} &:= \{\boldsymbol{u} \in (W^{1,q}(\Omega))^d; \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ W_{\boldsymbol{n},\mathrm{div}}^{1,q} &:= \{\boldsymbol{u} \in (W^{1,q}(\Omega))^d; \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega; \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}, \end{aligned}$$

where $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^d$ denotes the outer unit normal vector. The latter spaces are equipped with the norms

$$\|\boldsymbol{u}\|_{W_{\boldsymbol{n},\mathrm{div}}^{1,q}} = \|\boldsymbol{u}\|_{W_{\boldsymbol{n}}^{1,q}} := \|\boldsymbol{u}\|_{1,q}, \quad \|\boldsymbol{u}\|_{L_{\boldsymbol{n},\mathrm{div}}^q} := \|\boldsymbol{u}\|_q.$$

Also, we use the notation $W_{\boldsymbol{n},\mathrm{div}}^{-1,q} := (W_{\boldsymbol{n},\mathrm{div}}^{1,q})^*$ and $W_{\boldsymbol{n}}^{-1,q} := (W_{\boldsymbol{n}}^{1,q})^*$. Let us mention that if we assume $\Omega \in C^{1,1}$, then the following Helmholtz decomposition holds true

$$W_{\boldsymbol{n}}^{1,2} = W_{\boldsymbol{n},\mathrm{div}}^{1,2} \oplus \{\nabla\varphi; \varphi \in W^{2,2}(\Omega), \nabla\varphi \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

Note that such decomposition is not valid for $(W_0^{1,2}(\Omega))^d$.

If no misunderstanding can occur, we write the integrals over time and space without the symbols $\mathrm{d}t$, $\mathrm{d}\boldsymbol{x}$, for example, if $g = g(t, \boldsymbol{x})$ is a given function defined in Q_T, we write $\int_{Q_T} g$ instead of $\int_{Q_T} g \; \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$. We denote the positive constants of uniform bounds, whose exact values are not essential for our aims, as C, $\tilde{C}$, $\hat{C}$, $\overline{C}$, C^*, their values can change throughout the text.

2. Analysis of unsteady flows of pore pressure activated Bingham fluids

We consider internal flows of the model represented by (1.1.24), (1.1.27), when the whole boundary is impermeable and we study the problem with stick-slip boundary conditions (that can be, similarly as the tensor $\mathbb{S}$, see Section 1.1, equivalently written as an implicit constitutive equation on the boundary and characterized by two inequalities). Stick-slip (or threshold slip) states that the velocity does not slip until the amplitude of the tangent part of the normal traction on the boundary exceeds a certain critical value. This boundary condition, which is physically relevant to the pore pressure activated fluids considered in the bulk, includes Navier's slip and (perfect) slip boundary conditions as special cases. We establish the long-time and large-data existence of the corresponding weak solutions; see Section 2.2 for the formulation of the main result and Section 2.4 for its proof. We exploit the characterization of the implicit constitutive equations by two scalar constraints, both in the bulk and on the boundary, as a tool to show that the limit object of suitable approximative sequences fulfils these constitutive equations as well, see Proposition 2.3.3 proved in Section 2.3, where we also introduce the approximations and study their properties.

2.1 Formulations of the problem

Let $\Omega \subset \mathbb{R}^3$ be a flow domain (i.e. bounded, open and connected set) with Lipschitz boundary $\partial\Omega$, let $T \in (0,\infty)$, $Q_T := (0,T) \times \Omega$ and $\Sigma_T := (0,T) \times \partial\Omega$. The symbol $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^3$ denotes the outer unit normal vector, while for any vector $\boldsymbol{z}$ defined on $\partial\Omega$ we set $\boldsymbol{z}_\tau := \boldsymbol{z} - (\boldsymbol{z} \cdot \boldsymbol{n})\boldsymbol{n}$ representing the projection of $\boldsymbol{z}$ to the tangent plane.

We consider unsteady flows of a homogeneous incompressible non-Newtonian fluid of a Bingham type with a variable threshold, described in Section 1.1, see (1.1.24)-(1.1.25), where, as we will show, (1.1.25) can be replaced by (1.1.27) or by (1.1.28). In what follows, we slightly change the notation and write $\boldsymbol{v}$ instead of $\boldsymbol{v}_{\mathrm{s}}$, ϱ_* instead of $\rho_{\mathrm{s}}^{\mathrm{m}}$ and p_{f} instead of $p_{\mathrm{f}}^{\mathrm{t}}$. We also set $g := \frac{\partial p_s}{\partial t} - \operatorname{div}(K\rho_{\mathrm{f}}^{\mathrm{m}}\boldsymbol{f})$.

We are thus interested in solving the following problem. For given $\varrho_*, \nu_*, q_* \in (0,\infty)$, and for given functions $\boldsymbol{f} : Q_T \to \mathbb{R}^3$, $g, p_s : Q_T \to \mathbb{R}$, we look for $\boldsymbol{v} : Q_T \to \mathbb{R}^3$, $p, p_{\mathrm{f}} : Q_T \to \mathbb{R}$ and $\mathbb{S} : Q_T \to \mathbb{R}^{3\times3}$ satisfying in Q_T (we denote $\mathbb{D} := \frac{\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T}{2}$)

$$\begin{aligned} &\operatorname{div}\boldsymbol{v} = 0,\\ &\varrho_*\left(\partial_t\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\right) - \operatorname{div}\mathbb{S} + \nabla p = \varrho_*\boldsymbol{f},\\ &\mathbb{G}(\mathbb{S},\mathbb{D},p_{\mathrm{f}}) = \mathbb{O},\\ &\partial_t p_{\mathrm{f}} + \boldsymbol{v}\cdot\nabla p_{\mathrm{f}} - K\Delta p_{\mathrm{f}} = g + \boldsymbol{v}\cdot\nabla p_s, \end{aligned} \tag{2.1.1}$$

where $\mathbb{G}$ is a continuous function defined on $\mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}$ through

$$\mathbb{G}(\mathbb{S},\mathbb{D},p_{\mathrm{f}}) = \frac{\left(|\mathbb{S}| - q_*(p_s - p_{\mathrm{f}})^+\right)^+}{|\mathbb{S}|}\mathbb{S} - 2\nu_*\mathbb{D}. \tag{2.1.2}$$

In addition, the unknown functions $(\boldsymbol{v}, p, p_{\mathrm{f}}, \mathbb{S})$ are required, for given $\sigma_*, \gamma_* \in [0, \infty)$ and $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^3$ (such that $\operatorname{div} \boldsymbol{v}_0 = 0$ in Ω) and $p_0 : \Omega \to \mathbb{R}$, to fulfil the following initial and boundary conditions:

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \quad \text{and} \quad p_{\mathrm{f}}(0, \cdot) = p_0 \text{ in } \Omega, \tag{2.1.3}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \nabla p_{\mathrm{f}} \cdot \boldsymbol{n} = 0 \;\text{ on }\; \Sigma_T, \tag{2.1.4}$$

and

$$\gamma_* \boldsymbol{v}_\tau = \frac{(|\boldsymbol{s}| - s_*)^+}{|\boldsymbol{s}|} \boldsymbol{s} \;\text{ on } \Sigma_T, \tag{2.1.5}$$

where $s_* \in (0, \infty)$. The conditions in (2.1.4) state that the boundary is impermeable, while (2.1.5) characterizes the result of the interaction of the fluid and the boundary along the boundary. Here

$$\boldsymbol{s} := -(\mathbb{T}\boldsymbol{n})_\tau = -(\mathbb{S}\boldsymbol{n})_\tau.$$

Note that (2.1.5), usually written as

$$\begin{cases} |\boldsymbol{s}| \le s_* & \text{if and only if } \boldsymbol{v}_\tau = \boldsymbol{0}, \\ |\boldsymbol{s}| > s_* & \text{if and only if } \boldsymbol{s} = s_* \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau|} + \gamma_* \boldsymbol{v}_\tau, \end{cases} \tag{2.1.6}$$

describes the stick-slip (or threshold slip) and includes, as special cases, Navier's slip condition by taking $s_* = 0$ and $\gamma_* > 0$, and perfect slip condition if $s_* = 0$ and $\gamma_* = 0$. Note that the no-slip condition is obtained by letting either $s_* \to +\infty$ or $\gamma_* \to +\infty$.

Let us recall that one of the motivations for this work is a recent paper by Chupin and Mathé [1] where the authors characterize the tensorial response (2.1.2) through two scalar constraints introduced in (1.1.28). In fact, Chupin and Mathé [1] consider the second constraint with the equality sign in their existence result concerning planar flows, but then they incorrectly argue when performing the limit in the constitutive equation (see Step 2 (a) in [1]). This difficulty can be overcome easily if the inequality is used here instead of the equality, as shown in the proof of Theorem 2.2.1 in Section 2.4 below.

Before we prove that (2.1.2) and (1.1.28) are equivalent, we provide analogously a condition that characterizes (2.1.5). It takes the form

$$|\boldsymbol{z}| \le s_* \quad \text{and} \quad \boldsymbol{z} \cdot \boldsymbol{v}_\tau \ge s_* |\boldsymbol{v}_\tau|, \quad \text{where } \boldsymbol{z} := \boldsymbol{s} - \gamma_* \boldsymbol{v}_\tau. \tag{2.1.7}$$

Next, we prove the following statement.

Proposition 2.1.1. The following equivalences hold:

(a) (1.1.25) $\iff$ (2.1.2) $\iff$ (1.1.28);

(b) (2.1.6) $\iff$ (2.1.5) $\iff$ (2.1.7).

Proof. The equivalence (1.1.25) $\iff$ (2.1.2) is simple. We prove that (2.1.2) is equivalent to (1.1.28). Let us first assume that $(\mathbb{S}, \mathbb{D}, p_{\mathrm{f}})$ fulfil (2.1.2). If $\mathbb{D} = \mathbb{O}$ then $|\mathbb{S}| \le \tau(p_f)$ and $\mathbb{Z} = \mathbb{S}$ and (1.1.28) holds. If $\mathbb{D} \ne \mathbb{O}$, then $|\mathbb{S}| > \tau(p_f)$, and the formula (2.1.2) implies

$$\mathbb{S} - 2\nu_* \mathbb{D} = \tau(p_f) \frac{\mathbb{S}}{|\mathbb{S}|}. \tag{2.1.8}$$

Hence $\mathbb{Z} := \mathbb{S} - 2\nu_* \mathbb{D}$ fulfils $|\mathbb{Z}| = \tau(p_{\mathrm{f}})$. Next, by taking the modulus of (2.1.8) it follows

$$|\mathbb{S}| - \tau(p_{\mathrm{f}}) = 2\nu_* |\mathbb{D}|.$$

Inserting this back to (2.1.2), we get

$$\frac{\mathbb{S}}{|\mathbb{S}|} = \frac{\mathbb{D}}{|\mathbb{D}|}.$$

Employing this in (2.1.8), we obtain first

$$\mathbb{Z} = \mathbb{S} - 2\nu_* \mathbb{D} = \tau(p_f) \frac{\mathbb{D}}{|\mathbb{D}|}$$

and then, after taking the scalar product with $\mathbb{D}$,

$$\mathbb{Z} : \mathbb{D} = \tau(p_{\mathrm{f}}) |\mathbb{D}|,$$

which is the second assertion in (1.1.28).

Next, we assume that $(\mathbb{S}, \mathbb{D}, p_{\mathrm{f}})$ fulfil (1.1.28). Then, if $\mathbb{D} \neq \mathbb{O}$,

$$\tau(p_{\mathrm{f}}) |\mathbb{D}| \leq \mathbb{Z} : \mathbb{D} \leq |\mathbb{Z}||\mathbb{D}| \leq \tau(p_{\mathrm{f}}) |\mathbb{D}|,$$

which implies

$$\mathbb{Z} : \mathbb{D} = \tau(p_{\mathrm{f}}) |\mathbb{D}| \tag{2.1.9}$$

as well as the equality in the Cauchy-Schwarz inequality. Then necessarilly

$$\mathbb{Z} = a\mathbb{D}.$$

Inserting this structure in (2.1.9) we obtain

$$\tau(p_{\mathrm{f}}) |\mathbb{D}| = a |\mathbb{D}|^2.$$

Hence $\mathbb{Z} = \tau(p_{\mathrm{f}}) \frac{\mathbb{D}}{|\mathbb{D}|}$ and

$$\mathbb{S} = 2\nu_* \mathbb{D} + \tau(p_{\mathrm{f}}) \frac{\mathbb{D}}{|\mathbb{D}|}. \tag{2.1.10}$$

Also, we have

$$|\mathbb{S}| = \left(\frac{2\nu_* |\mathbb{D}| + \tau(p_{\mathrm{f}})}{|\mathbb{D}|} \right) |\mathbb{D}| = 2\nu_* |\mathbb{D}| + \tau(p_{\mathrm{f}}),$$

which implies (as $\mathbb{D} \neq \mathbb{O}$)

$$|\mathbb{S}| - \tau(p_{\mathrm{f}}) > 0, \text{ and also } \frac{\mathbb{S}}{|\mathbb{S}|} = \frac{\mathbb{D}}{|\mathbb{D}|}.$$

This together with (2.1.10) implies (2.1.2) for $\mathbb{D} \neq \mathbb{O}$. If $\mathbb{D} = \mathbb{O}$ then, by (1.1.28), $\mathbb{S} = \mathbb{Z}$ and $|\mathbb{S}| \leq \tau(p_{\mathrm{f}})$ and (2.1.2) holds. The proof of the equivalence of (2.1.2) and (1.1.28) is complete.

The proof of the statement (b) is done in the same manner. □

2.2 Definition of weak solution and main result

In the rest of this chapter, the symbol $\mathbb{D}\boldsymbol{\varphi}$ stands for the symmetric part of the gradient of a vector-valued function $\boldsymbol{\varphi}$, i.e. $\mathbb{D}\boldsymbol{\varphi} := \frac{\nabla\boldsymbol{\varphi}+(\nabla\boldsymbol{\varphi})^T}{2}$. In what follows, we also set for simplicity and without loss of any generality

$$\varrho_* = 2\nu_* = \gamma_* = K = q_* = 1.$$

Definition 1 (Definition of weak solution)**.** *Let $s_* > 0$,*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}, p_0 \in L^\infty(\Omega),\ \boldsymbol{f} \in L^2(0,T;W^{-1,2}_{\boldsymbol{n}}), \tag{2.2.1}$$

and let one of the following requirements be satisfied

$$p_s \in L^\infty(Q_T), p_s(0) \in L^\infty(\Omega), g \in L^q(Q_T), \partial_t p_s - \Delta p_s \in L^q(Q_T) \quad \text{with } q > \frac{5}{2}, \tag{2.2.2}$$

$$p_s \in L^q(0,T;W^{1,q}(\Omega)) \text{ with } q > 10 \text{ and } g \in L^q(Q_T) \quad \text{with } q > \frac{5}{2}. \tag{2.2.3}$$

We say that $(\boldsymbol{v}, p_\mathrm{f}, p, \mathbb{S}, \boldsymbol{s})$ is a weak solution to the problem (2.1.1)-(2.1.5) *if*

$$\begin{aligned}
&\boldsymbol{v} \in L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{\mathbf{n},\mathrm{div}}), \quad \partial_t\boldsymbol{v} \in (L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3)^*,\\
&p_\mathrm{f} \in L^\infty(Q_T) \cap L^2(0,T;W^{1,2}(\Omega)), \quad \partial_t p_\mathrm{f} \in L^2(0,T;(W^{1,2}(\Omega))^*),\\
&p = p_1 + p_2, \text{ where } p_1 \in L^2(Q_T) \text{ and } p_2 \in L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega)),\\
&\mathbb{S} \in (L^2(Q_T))^{3\times 3}, \quad \boldsymbol{s} \in (L^2(\Sigma_T))^3,
\end{aligned}$$

if, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{n}}$, $z \in W^{1,2}(\Omega)$ and almost all $t \in (0,T)$, the following identities hold true

$$\begin{aligned}
\langle\partial_t\boldsymbol{v}, \boldsymbol{w}\rangle + \int_\Omega (\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{w} + \mathbb{S} : \mathbb{D}\boldsymbol{w}) + \int_{\partial\Omega} \boldsymbol{s}\cdot\boldsymbol{w}_{\boldsymbol{\tau}} &= \langle\boldsymbol{f}, \boldsymbol{w}\rangle\\
&+ \int_\Omega (p_1 \,\mathrm{div}\,\boldsymbol{w} - \nabla p_2\cdot\boldsymbol{w}),
\end{aligned} \tag{2.2.4}$$

$$\langle\partial_t p_\mathrm{f}, z\rangle - \int_\Omega (p_\mathrm{f}\boldsymbol{v}\cdot\nabla z - \nabla p_\mathrm{f}\cdot\nabla z) = \int_\Omega (gz - p_s\boldsymbol{v}\cdot\nabla z), \tag{2.2.5}$$

if the following constitutive relations are satisfied

$$\mathbb{D}\boldsymbol{v} = \frac{(|\mathbb{S}| - \tau(p_\mathrm{f}))^+}{|\mathbb{S}|}\mathbb{S}, \text{ where } \tau(p_\mathrm{f}) = (p_s - p_\mathrm{f})^+ \quad \text{a.e. in } Q_T, \tag{2.2.6}$$

$$\boldsymbol{v}_\tau = \frac{(|\boldsymbol{s}| - s_*)^+}{|\boldsymbol{s}|}\boldsymbol{s} \quad \text{a.e. on } \Sigma_T, \tag{2.2.7}$$

and if the initial conditions $\boldsymbol{v}_0$, p_0 are attained in the sense

$$\lim_{t\to 0+} (\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|p_\mathrm{f}(t) - p_0\|_2) = 0. \tag{2.2.8}$$

The aim of this chapter is to prove the following theorem.

Theorem 2.2.1 (Main Theorem)**.** *For any* $\Omega \in C^{1,1}$*,* $T \in (0, \infty)$ *and for arbitrary* $\boldsymbol{v}_0, p_0, p_s$*,* $\boldsymbol{f}$ *fulfilling* (2.2.1) *and for arbitrary* g *and* p_s *fulfilling either* (2.2.2) *or* (2.2.3)*, there exists a weak solution to the problem* (2.1.1)–(2.1.5) *in the sense of Definition 1.*

Remark. We wish to emphasize that due to Proposition 2.1.1, the tensorial constitutive equation $\mathbb{D}\boldsymbol{v} = \frac{(|\mathbb{S}|-\tau(p_{\mathrm{f}}))^+}{|\mathbb{S}|}\mathbb{S}$ in Q_T as well as the vectorial equation $\boldsymbol{v}_\tau = \frac{(|\boldsymbol{s}|-s_*)^+}{|\boldsymbol{s}|}\boldsymbol{s}$ on Σ_T can be replaced by any of its equivalent forms. It is in particular interesting that the tensorial equations can be characterized by two (scalar) inequalities.

Note that Theorem 2.2.1 presents the existence result to a supercritical problem; this is a problem where the solution itself is not an admissible test function in the weak formulation of the governing equations. Indeed, in our case $\boldsymbol{v}$ belongs to $(L^{\frac{10}{3}}(Q_T))^3$, however, admissible test functions have to be from $(L^5(Q_T))^3$ due to the fact that $\mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = \sum_{r=1}^{3} v_r \frac{\partial \boldsymbol{v}}{\partial x_r}$ and ∇p_2 belong to $(L^{\frac{5}{4}}(Q_T))^3$. This is the reason why we cannot involve in our analysis such tools as the energy equality, used in the analysis of planar time-dependent flows in Chupin and Mathé [1] or the higher differentiability techniques used in [48] and [49], also in the analysis of two-dimensional unsteady flows of the Bingham fluids and in the analysis of steady flows in three dimensions in [50]. Neither can we incorporate the tools of calculus of variations suitable for Stokes-type problems (see for example [51] and the references therein). On the other hand, we intentionally aim at avoiding tools such as multivalued calculus or variational inequalities [52] in our analysis, see [53], or [54] (which is considered however in a different context). In our opinion, the concept of solution (expressed in terms of identities) considered here is stronger, its large-data existence can be proved and has some other advantages. For example, it forms the foundation for a direct application of mixed finite element (or spectral) methods. In order to identify the non-linear constitutive equation pointwise in the considered domain $(0, T) \times \Omega$ when taking the limit from the approximative problem to the original one, and in order to overcome difficulties connected with the low integrability of $\boldsymbol{v}$, we incorporate the so-called L^∞-truncation method. This method replaces $\boldsymbol{v}^n - \boldsymbol{v}$, where $\{\boldsymbol{v}^n\}_{n=1}^{+\infty}$ is solution to a suitably constructed approximative problem, by a truncated function that coincides with $\boldsymbol{v}^n - \boldsymbol{v}$ on a large set and the measure of the complementary set can be made arbitrarily small uniformly with respect to n. Although the origin of the method goes back to elliptic problems with an L^1-right-hand side (see [11], [55] and [56]), we refer here mainly to its development for evolutionary problems in fluid mechanics, see [6], [7], [8]. The result by Wolf [8] similarly as those by Solonnikov (see [57] and [58]) and Koch and Solonnikov [59] concerning the properties of evolutionary Stokes-like systems with no-slip boundary conditions indicate the difficulties connected with the impossibility to establish the integrability of the pressure p for generalizations of the Navier-Stokes equations (with variable viscosity) in three spatial dimensions. This is why we treat the stick-slip boundary conditions in this study. It reveals that the analysis of the three-dimensional evolutionary supercritical problems associated with the stick-slip boundary conditions differs remarkably from the analysis of analogous problems connected with the no-slip boundary conditions. We refer to [33] for a detailed discussion of this issue noting that Theorem 2.2.1 guarantees that $p \in L^1(Q_T)$. We remark that the integrability of the pressure is important in the analysis of problems with the viscosity dependent on the temperature (see [60], [34] or [61]) or the viscosity dependent on the pressure (see [7] or [62]), but it is also an interesting mathematical question itself.

2.3 Approximations

Before introducing the approximations, we recall that in this section we set

$$\mathbb{Z} = \mathbb{S} - \mathbb{D}\boldsymbol{v},$$

and analogously we can also define

$$\boldsymbol{z} := \boldsymbol{s} - \boldsymbol{v}_\tau.$$

For any $n \in \mathbb{N}$, let $G_n : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $G_n(u) = 1$ if $|u| \leq n$, $G_n(u) = 0$ if $|u| \geq 2n$ and $|G_n'| \leq \frac{2}{n}$. We consider the following approximative system of equations satisfied in Q_T:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{2.3.1}$$

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) G_n(|\boldsymbol{v}|^2) - \operatorname{div} \mathbb{D}\boldsymbol{v} - \operatorname{div} \mathbb{Z} + \nabla p = \boldsymbol{f}, \tag{2.3.2}$$

$$\partial_t p_{\mathrm{f}} + \boldsymbol{v} \cdot \nabla p_{\mathrm{f}} - \Delta p_{\mathrm{f}} = g + \boldsymbol{v} \cdot \nabla p_s, \tag{2.3.3}$$

where $\mathbb{Z}$ and $\boldsymbol{z}$ fulfill

$$\mathbb{Z} = \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v}) := (p_s - p_{\mathrm{f}})^+ \frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}| + \frac{1}{n}} \text{ in } Q_T, \tag{2.3.4}$$

$$\boldsymbol{z} = \zeta_n(\boldsymbol{v}_\tau) := s_* \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau| + \frac{1}{n}} \text{ on } \Sigma_T, \tag{2.3.5}$$

and the initial conditions are attained in the sense

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \nabla p_{\mathrm{f}} \cdot \boldsymbol{n} = 0 \text{ on } \Sigma_T, \tag{2.3.6}$$

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \quad \text{and} \quad p_{\mathrm{f}}(0, \cdot) = p_0 \text{ in } \Omega. \tag{2.3.7}$$

It is not difficult to check that if $\mathbb{Z} = \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D})$ and $\hat{\mathbb{Z}} = \mathcal{Z}_n(p_{\mathrm{f}}, \hat{\mathbb{D}})$, then

$$(\mathbb{Z} - \hat{\mathbb{Z}}) : (\mathbb{D} - \hat{\mathbb{D}}) \geq \frac{(p_s - p_{\mathrm{f}})^+}{n} \frac{(|\mathbb{D}| - |\hat{\mathbb{D}}|)^2}{\left(|\mathbb{D}| + \frac{1}{n}\right)\left(|\hat{\mathbb{D}}| + \frac{1}{n}\right)} \geq 0. \tag{2.3.8}$$

A similar monotone property holds for $\boldsymbol{z} = \zeta_n(\boldsymbol{v}_\tau)$.

Proposition 2.3.1. Let $n \in \mathbb{N}$ be fixed and $s_* > 0$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $p_0 \in L^2(\Omega)$, $\boldsymbol{f} \in L^2(0,T; W^{-1,2}_{\boldsymbol{n}})$, $g \in L^2(Q_T)$ and $p_s \in L^5(Q_T)$, then there exists a weak solution to the problem (2.3.1)–(2.3.7), i.e. there exists a quadruple $(\boldsymbol{v}^n, p_{\mathrm{f}}^n, \mathbb{Z}^n, \boldsymbol{z}^n)$ such that

$$\boldsymbol{v}^n \in L^\infty(0,T; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{n},\mathrm{div}}), \quad \partial_t \boldsymbol{v}^n \in L^2(0,T; W^{-1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{2.3.9}$$

$$p_{\mathrm{f}}^n \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \quad \partial_t p_{\mathrm{f}}^n \in L^{\frac{4}{3}}(0,T; (W^{1,2}(\Omega))^*), \tag{2.3.10}$$

$$\mathbb{Z}^n \in (L^{\frac{10}{3}}(Q_T))^{3\times 3}, \quad \boldsymbol{z}^n \in (L^\infty(\Sigma_T))^3, \tag{2.3.11}$$

for all $\boldsymbol{w} \in L^2(0,T; W^{1,2}_{\boldsymbol{n},\mathrm{div}})$ and $z \in L^4(0,T; W^{1,2}(\Omega))$, it holds

$$\begin{aligned}\textstyle\int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w} \rangle + \int_{Q_T} \mathbb{D}\boldsymbol{v}^n : \mathbb{D}\boldsymbol{w} + \mathbb{Z}^n : \mathbb{D}\boldsymbol{w} + G_n(|\boldsymbol{v}^n|^2) \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \boldsymbol{w} \\ + \int_{\Sigma_T} (\boldsymbol{z}^n + \boldsymbol{v}^n_\tau) \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = \int_0^T \langle \boldsymbol{f}, \boldsymbol{w} \rangle,\end{aligned} \tag{2.3.12}$$

$$\int_0^T \langle \partial_t p_{\mathrm{f}}^n, z \rangle - \int_{Q_T} (p_{\mathrm{f}}^n \boldsymbol{v}^n \cdot \nabla z - \nabla p_{\mathrm{f}}^n \cdot \nabla z) = \int_{Q_T} (gz - p_s \boldsymbol{v}^n \cdot \nabla z), \tag{2.3.13}$$

where

$$\mathbb{Z}^n = \mathcal{Z}_n(p_{\rm f}^n, \mathbb{D}\boldsymbol{v}^n) \text{ a.e. in } Q_T, \tag{2.3.14}$$

$$\boldsymbol{z}^n = \zeta_n(\boldsymbol{v}_\tau^n) \text{ a.e. in } \Sigma_T, \tag{2.3.15}$$

and the initial conditions $\boldsymbol{v}_0$, p_0 are attained in the sense

$$\lim_{t\to 0+} \left(\|\boldsymbol{v}^n(t) - \boldsymbol{v}_0\|_2 + \|p_{\rm f}^n(t) - p_0\|_2\right) = 0. \tag{2.3.16}$$

Proof. Due to the presence of G_n that truncates the convective term and the properties of the approximations $\mathcal{Z}_n$ and ζ_n introduced above, the proof of the existence of weak solutions to the problem (2.3.1)–(2.3.7) is a variant of the standard monotone operator technique (see [63], [64] or [7]). To be more specific, we briefly outline the proof using the Galerkin method. Since n is fixed, we write $(\boldsymbol{v}, p_{\rm f}, \mathbb{Z}, \boldsymbol{z})$ instead of $(\boldsymbol{v}^n, p_{\rm f}^n, \mathbb{Z}^n, \boldsymbol{z}^n)$ in the proof.

Step 1. Galerkin system. Let $\{\boldsymbol{w}^i\}_{i\in\mathbb{N}}$ be an orthogonal basis in $W^{1,2}_{\mathbf{n},\rm div}$ consisting of eigenfunctions of the Stokes operator subject to $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_\tau = \mathbf{0}$ on Σ_T. Let analogously $\{z_j\}_{j\in\mathbb{N}}$ be an orthogonal basis in $W^{1,2}(\Omega)$ consisting of eigenfunctions of the Laplace operator subject to the relevant homogeneous boundary conditions. Then the local in time existence of

$$\boldsymbol{v}^m(t,\boldsymbol{x}) := \sum_{r=1}^m c_r^m(t)\boldsymbol{w}^r(\boldsymbol{x}), \quad p_f^m(t,\boldsymbol{x}) := \sum_{r=1}^m d_r^m(t) z^r(\boldsymbol{x}), \tag{2.3.17}$$

satisfying for all $r = 1, ..., m$ that

$$\begin{aligned}\int_\Omega \partial_t \boldsymbol{v}^m \cdot \boldsymbol{w}^r + \int_\Omega \Big(\mathbb{D}\boldsymbol{v}^m : \mathbb{D}\boldsymbol{w}^r + \operatorname{div}(\boldsymbol{v}^m \otimes \boldsymbol{v}^m) G(|\boldsymbol{v}^m|^2)\cdot \boldsymbol{w}^r\Big)\\ + \int_\Omega \mathcal{Z}_n(p_{\rm f}^m, \mathbb{D}\boldsymbol{v}^m) : \mathbb{D}\boldsymbol{w}^r + \int_{\partial\Omega} (\boldsymbol{v}_\tau^m + \zeta_n(\boldsymbol{v}_\tau^m))\cdot \boldsymbol{w}^r = \langle \boldsymbol{f}, \boldsymbol{w}^r\rangle,\end{aligned} \tag{2.3.18}$$

and

$$\int_\Omega \partial_t p_f^m z^r - \int_\Omega \Big(p_f^m \boldsymbol{v}^m \cdot \nabla z^r - \nabla p_f^m \cdot \nabla z^r\Big) = \int_\Omega (g z^r - p_s \boldsymbol{v}^m \cdot \nabla z^r), \tag{2.3.19}$$

together with the corresponding initial conditions $\boldsymbol{v}_0^m$ and p_0^m, obtained by projecting $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\rm div}$ onto the span of $\{\boldsymbol{w}^1, \dots, \boldsymbol{w}^m\}$ and $p_0 \in L^2(\Omega)$ onto the span of $\{z^1, \dots, z^m\}$, follows from the Caratheodory theory for systems of ordinary differential equations (see Appendix of [65] for details).

Global in time existence is, as usual, a consequence of the uniform estimates which we show next.

Step 2. Uniform estimates. Multiplying (2.3.18) by $c_r^m(t)$ and (2.3.19) by $d_r^m(t)$ and taking the sum over r from 1 to m, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m\|_2^2 + \|\mathbb{D}\boldsymbol{v}^m\|_2^2 + \int_\Omega \mathcal{Z}_n(p_{\rm f}^m, \mathbb{D}\boldsymbol{v}^m) : \mathbb{D}\boldsymbol{v}^m + \|\boldsymbol{v}_\tau^m(t)\|_{2,\partial\Omega}^2\\ + \int_{\partial\Omega} \zeta_n(\boldsymbol{v}_\tau^m)\cdot \boldsymbol{v}_\tau^m = \langle \boldsymbol{f}, \boldsymbol{v}^m\rangle,\end{aligned} \tag{2.3.20}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_{\rm f}^m(t)\|_2^2 + \|\nabla p_{\rm f}^m(t)\|_2^2 = \int_\Omega (g p_{\rm f}^m - p_s \boldsymbol{v}^m \cdot \nabla p_{\rm f}^m). \tag{2.3.21}$$

By Korn's and Young's inequalities (see for example [7, Lemma 1.11] for details), using also the fact that the last two terms at the right-hand side of (2.3.20) are non-negative, one concludes from (2.3.20) that

$$\begin{aligned}\sup_{t\in[0,T]}\|\boldsymbol{v}^m(t)\|_2^2+\int_{Q_T}\left(|\mathbb{D}\boldsymbol{v}^m|^2+|\boldsymbol{v}^m|^{\frac{10}{3}}\right)+\int_{\Sigma_T}|\boldsymbol{v}^m_\tau|^2\\ \le C(\|\boldsymbol{f}\|^2_{L^2(0,T;W_{\boldsymbol{n}}^{-1,2})}+\|\boldsymbol{v}_0\|_2^2)=:C(\boldsymbol{f},\boldsymbol{v}_0),\end{aligned}\tag{2.3.22}$$

where we also used the interpolation inequality

$$\|z\|_{\frac{10}{3}}\le\|z\|_2^{\frac25}\|z\|_6^{\frac35}\le C\|z\|_2^{\frac25}\|z\|_{1,2}^{\frac35}.\tag{2.3.23}$$

Similarly, using also

$$\int_0^T|(p_s\boldsymbol{v}^m,\nabla p_{\mathrm f}^m)|\le\|p_s\|_{5,Q_T}\|\boldsymbol{v}^m\|_{\frac{10}{3},Q_T}\|\nabla p_{\mathrm f}^m\|_{2,Q_T},$$

one obtains from (2.3.21), using also (2.3.22), that

$$\sup_{t\in[0,T]}\|p_{\mathrm f}^m(t)\|_2^2+\int_{Q_T}|\nabla p_{\mathrm f}^m|^2\le C\|g\|_{2,Q_T}+C(\boldsymbol{f},\boldsymbol{v}_0)\|p_s\|_{5,Q_T}+\|p_0\|_2^2.\tag{2.3.24}$$

By the interpolation inequalities (2.3.23) and

$$\|z\|_4\le\|z\|_2^{\frac14}\|z\|_6^{\frac34}\le C\|z\|_2^{\frac14}\|z\|_{1,2}^{\frac34},\tag{2.3.25}$$

and by the trace inequalities (see [7, Lemma 1.11]), we obtain

$$\sup_m\left(\|p_{\mathrm f}^m\|_{\frac{10}{3},Q_T}+\|\boldsymbol{v}_\tau^m\|_{\frac83,\Sigma_T}\right)<+\infty,\tag{2.3.26}$$

and also

$$\sup_m\left(\int_0^T\left(\|\boldsymbol{v}^m\|_4^{\frac83}+\|p_{\mathrm f}^m\|_4^{\frac83}\right)\right)<+\infty.\tag{2.3.27}$$

It then follows from the explicit formulas for $\mathcal{Z}_n$ and ζ_n that $\mathbb{Z}^m:=\mathcal{Z}_n(p_{\mathrm f}^m,\mathbb{D}\boldsymbol{v}^m)$ and $\boldsymbol{z}^m:=\zeta_n(\boldsymbol{v}_\tau^m)$ fulfil

$$\sup_m\left(\|\mathbb{Z}^m\|_{\frac{10}{3},Q_T}+\|\boldsymbol{z}^m\|_{\infty,\Sigma_T}\right)<+\infty.\tag{2.3.28}$$

Finally, the fact that the projectors

$$W^{1,2}_{\boldsymbol{n},\mathrm{div}}\longmapsto\operatorname{span}\{\boldsymbol{w}^1,\dots,\boldsymbol{w}^m\},\ W^{1,2}(\Omega)\longmapsto\operatorname{span}\{z^1,\dots,z^m\}$$

are continuous and (2.3.27) imply that

$$\sup_m\left(\|\partial_t\boldsymbol{v}^m\|_{L^2(0,T;W^{-1,2}_{\boldsymbol{n},\mathrm{div}})}+\|\partial_t p_{\mathrm f}^m\|_{L^{\frac43}(0,T;(W^{1,2}(\Omega))^*)}\right)<+\infty.\tag{2.3.29}$$

Step 3. Limit. The above uniform estimates imply the existence of $\boldsymbol{v}$, $p_{\mathrm f}$, $\mathbb{Z}$ and $\boldsymbol{z}$ and subsequences of $\{\boldsymbol{v}^m\}$, $\{p_{\mathrm f}^m\}$, $\{\mathbb{Z}^m\}$ and $\{\boldsymbol{z}^m\}$ converging weakly (or *-weakly) to $\boldsymbol{v},p_{\mathrm f},\mathbb{Z}$ and $\boldsymbol{z}$ in the function spaces indicated in Proposition 2.3.1, and fulfilling the following strong convergences (due to Aubin-Lions compactness lemma and its variant, see [7, Lemma 1.12], involving the trace theorem):

$$\boldsymbol{v}^m \to \boldsymbol{v} \text{ a.e. in } Q_T \text{ and strongly in } (L^q(Q_T))^3 \text{ for any } q \in \left[1, \frac{10}{3}\right), \tag{2.3.30}$$

$$p_{\mathrm{f}}^m \to p_{\mathrm{f}} \text{ a.e. in } Q_T \text{ and strongly in } L^q(Q_T) \text{ for any } q \in \left[1, \frac{10}{3}\right), \tag{2.3.31}$$

$$\boldsymbol{v}_\tau^m \to \boldsymbol{v}_\tau \text{ a.e. in } \Sigma_T \text{ and strongly in } (L^q(\Sigma_T))^3 \text{ for any } q \in \left[1, \frac{8}{3}\right). \tag{2.3.32}$$

These weak and strong convergences suffice to show that $\boldsymbol{v}, p_{\mathrm{f}}, \mathbb{Z}$ and $\boldsymbol{z}$ fulfil the weak formulations (2.3.12)–(2.3.13) stated in Proposition 2.3.1.

Since the proof of the attainment of the initial conditions is standard, see e.g. [66], it remains to show that $\mathbb{Z} = \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v})$ and $\boldsymbol{z} = \zeta_n(\boldsymbol{v}_\tau)$.

Step 4. Attainment of the constitutive equations. We first notice that (2.3.32) together with (2.3.28) imply, by Lebesgue's theorem that

$$\boldsymbol{z}^m = \zeta_n(\boldsymbol{v}_\tau^m) \rightharpoonup \zeta_n(\boldsymbol{v}_\tau) \text{ weakly in } (L^2(\Sigma_T))^3.$$

It implies that

$$\boldsymbol{z} = \zeta_n(\boldsymbol{v}_\tau) \text{ a.e. in } \Sigma_T \tag{2.3.33}$$

and

$$\lim_{m\to+\infty} \int_{\Sigma_T} \boldsymbol{z}^m \cdot \boldsymbol{v}_\tau^m = \int_{\Sigma_T} \zeta_n(\boldsymbol{v}_\tau) \cdot \boldsymbol{v}_\tau. \tag{2.3.34}$$

Next, integrating (2.3.20) over $(0, T)$ and taking limsup of the resulting identity, we obtain, using the above convergences and the weak lower semicontinuity of the L^2-norm, that

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \int_{Q_T} |\mathbb{D}\boldsymbol{v}|^2 + \int_{\Sigma_T} |\boldsymbol{v}_\tau|^2 + \int_{\Sigma_T} \zeta_n(\boldsymbol{v}_\tau) \cdot \boldsymbol{v}_\tau \\ &+ \limsup_{m\to+\infty} \int_{Q_T} \mathbb{Z}^m : \mathbb{D}\boldsymbol{v}^m \le \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2. \end{aligned} \tag{2.3.35}$$

On the other hand, taking $\boldsymbol{w} = \boldsymbol{v}$ in the established weak formulation of the equation for $\boldsymbol{v}$, we get, using also (2.3.33),

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \int_{Q_T} |\mathbb{D}\boldsymbol{v}|^2 + \int_{\Sigma_T} |\boldsymbol{v}_\tau|^2 + \int_{\Sigma_T} \zeta_n(\boldsymbol{v}_\tau) \cdot \boldsymbol{v}_\tau \\ &+ \int_{Q_T} \mathbb{Z} : \mathbb{D}\boldsymbol{v} = \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \frac{1}{2}\|\boldsymbol{v}_0\|_2^2. \end{aligned} \tag{2.3.36}$$

Comparing (2.3.35) with (2.3.36), we conclude that

$$\limsup_{m\to+\infty} \int_{Q_T} \mathbb{Z}^m : \mathbb{D}\boldsymbol{v}^m \le \int_{Q_T} \mathbb{Z} : \mathbb{D}\boldsymbol{v}. \tag{2.3.37}$$

Finally, it follows from (2.3.8) that

$$0 \le \int_{Q_T} (\mathcal{Z}_n(p_{\mathrm{f}}^m, \mathbb{D}\boldsymbol{v}^m) - \mathcal{Z}_n(p_{\mathrm{f}}^m, \mathbb{A})) : (\mathbb{D}\boldsymbol{v}^m - \mathbb{A}) \quad \text{for all } \mathbb{A} \in (L^2(Q_T))^{3\times 3}. \tag{2.3.38}$$

Since, by (2.3.31),

$$\begin{aligned}&\mathcal{Z}_n(p_{\mathrm{f}}^m, \mathbb{A}) := (p_s - p_{\mathrm{f}}^m)^+ \frac{\mathbb{A}}{|\mathbb{A}|+\frac{1}{n}}\\ &\to (p_s - pf)^+ \frac{\mathbb{A}}{|\mathbb{A}|+\frac{1}{n}} =: \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{A}) \text{ strongly in } (L^2(Q_T))^{3\times 3}\end{aligned}$$

and

$$\mathbb{D}\boldsymbol{v}^m \rightharpoonup \mathbb{D}\boldsymbol{v} \quad \text{weakly in } (L^2(Q_T))^{3\times 3},$$

we conclude from (2.3.38) and (2.3.37) that

$$0 \le \int_{Q_T} (\mathbb{Z} - \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{A})) : (\mathbb{D}\boldsymbol{v} - \mathbb{A}) \quad \text{for all } \mathbb{A} \in (L^2(Q_T))^{3\times 3}. \tag{2.3.39}$$

The choice $\mathbb{A} = \mathbb{D}\boldsymbol{v} \pm \lambda\mathbb{B}$ for $\mathbb{B} \in (L^2(Q_T))^{3\times 3}$ arbitrary and $\lambda > 0$, leads to

$$0 \le \pm \int_{Q_T} (\mathbb{Z} - \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v} \pm \lambda\mathbb{B})) : \mathbb{B} \quad \text{for all } \mathbb{B} \in (L^2(Q_T))^{3\times 3}.$$

Letting $\lambda \to 0^+$, we obtain

$$0 = \int_{Q_T} (\mathbb{Z} - \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v})) : \mathbb{B} \quad \text{for all } \mathbb{B} \in (L^2(Q_T))^{3\times 3},$$

which implies $\mathbb{Z} = \mathcal{Z}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v})$ a.e. in Q_T.
The proof of Proposition 2.3.1 is complete. □

Proposition 2.3.2. Let all the assumptions in Proposition 2.3.1 be satisfied. In addition, assume that $p_0 \in L^\infty(\Omega)$ and one of the following requirements holds true:

$$p_s(0) \in L^\infty(\Omega),\ p_s \in L^\infty(Q_T) \ \text{ and } g, \partial_t p_s - \Delta p_s \in L^q(Q_T) \ \text{ with } q > \frac{5}{2}, \tag{2.3.40}$$

$$p_s \in L^q(0,T; W^{1,q}(\Omega)) \text{ with } q > 10 \text{ and } g \in L^q(Q_T) \ \text{ with } q > \frac{5}{2}, \tag{2.3.41}$$

then, for each $n \in \mathbb{N}$, there exists a weak solution to the problem (2.3.1)–(2.3.7) in the sense of Proposition 2.3.1 satisfying $p_{\mathrm{f}}^n \in L^\infty(Q_T)$. In fact,

$$\sup_n \|p_{\mathrm{f}}^n\|_{\infty,Q_T} < +\infty. \tag{2.3.42}$$

Consequently, we have $\partial_t p_{\mathrm{f}}^n \in L^2(0,T;(W^{1,2}(\Omega))^*)$ and (2.3.13) holds true for all z belonging to $L^2(0,T;W^{1,2}(\Omega))$.

Proof. In what follows we shall prove explicitly that $p_{\mathrm{f}} := p_{\mathrm{f}}^n \in L^\infty(Q_T)$ for every fixed $n \in \mathbb{N}$ using the Moser iteration technique. Then, since (throughout the proof) any constant of uniform bounds does not depend on n, we immediately obtain (2.3.42).

By the interpolation inequality (2.3.23), it follows that

$$\int_0^T \|z\|_{\frac{10}{3}}^{\frac{10}{3}} \le C \left(\sup_{t\in[0,T]} \|z\|_2\right)^{\frac{4}{3}} \int_0^T \|z\|_{1,2}^2. \tag{2.3.43}$$

Consequently,

$$p_{\mathrm{f}} \in L^{\frac{10}{3}}(Q_T) \ \text{ and } \ \boldsymbol{v} \in (L^{\frac{10}{3}}(Q_T))^3. \tag{2.3.44}$$

Let us first consider the case given by (2.3.40). Then, once we set $h := -(\partial_t p_s - \Delta p_s)$, $G := g + h$ and $P := p_{\mathrm{f}} - p_s$, we can rewrite the equation (2.3.3) as

$$\partial_t P + \boldsymbol{v} \cdot \nabla P - \Delta P = G \quad \text{with } G \in L^q(Q_T) \text{ and } q > \frac{5}{2}. \tag{2.3.45}$$

For $s > 2$ and $m \in \mathbb{N}$ consider $|P_m|^{s-2}P_m$ with $P_m := T_m(P)$ as test function in the weak formulation of (2.3.45). Here $T_m : \mathbb{R} \to \mathbb{R}$ is defined through $T_m(z) = z$ if $|z| \leq m$ and $T_m(z) = m \operatorname{sgn} z$ if $|z| > m$. Note that $|P_m|^{s-2}P_m$ is an admissible test function. After integrating by parts and employing $\operatorname{div} \boldsymbol{v} = 0$, we get

$$\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\|P_m\|_s^s + (s-1)\int_\Omega |\nabla P_m|^2 |P_m|^{s-2} \leq \int_\Omega |G||P_m|^{s-1}. \tag{2.3.46}$$

Next, integrating with respect to the time, straightforward computations imply

$$\||P_m(t)|^{\frac{s}{2}}\|_2^2 + \frac{4s(s-1)}{s^2}\|\nabla |P_m|^{\frac{s}{2}}\|_{2,Q_T}^2 \leq s\int_{Q_T} |G||P_m|^{s-1} + \|P(0)\|_s^s =: A. \tag{2.3.47}$$

Since $\frac{4s(s-1)}{s^2} > 1$, it follows from (2.3.47) that

$$\sup_{t\in[0,T]} \||P_m(t)|^{\frac{s}{2}}\|_2 \leq A^{\frac{1}{2}}, \quad \int_0^T \|\nabla |P_m(t)|^{\frac{s}{2}}\|_2^2 \leq A. \tag{2.3.48}$$

Using (2.3.23) and (2.3.43) with $z = |P_m|^{\frac{s}{2}}$, and combining the result with (2.3.48), we obtain

$$\begin{aligned} \int_0^T \|P_m(t)\|_{\frac{5s}{3}}^{\frac{5s}{3}} &= \int_0^T \||P_m(t)|^{\frac{s}{2}}\|_{\frac{10}{3}}^{\frac{10}{3}} \\ &\leq C\left(\sup_{t\in[0,T]} \||P_m(t)|^{\frac{s}{2}}\|_2\right)^{\frac{4}{3}} \int_0^T \|\nabla |P_m(t)|^{\frac{s}{2}}\|_2^2 \leq CA^{\frac{5}{3}}. \end{aligned} \tag{2.3.49}$$

The definition of A and (2.3.49) then leads to

$$\|P_m\|_{\frac{5s}{3},Q_T} \leq s^{\frac{1}{s}} C^{\frac{3}{5s}} \|G\|_{q,Q_T}^{\frac{1}{s}} \|P_m\|_{q'(s-1),Q_T}^{\frac{s-1}{s}} + C^{\frac{3}{5s}}\|P(0)\|_\infty. \tag{2.3.50}$$

We can introduce the following iteration scheme. Setting

$$s_0 := \frac{10}{3}, \ \frac{q}{q-1}(\tilde{s}_i - 1) := s_i \ \text{ and } s_{i+1} := \frac{5}{3}\tilde{s}_i, \tag{2.3.51}$$

which leads to $\tilde{s}_i = \frac{q-1}{q}s_i + 1$ and hence

$$s_{i+1} = \frac{5}{3}\frac{q-1}{q}s_i + \frac{5}{3}, \ \tilde{s}_{i+1} = \frac{5}{3}\frac{q-1}{q}\tilde{s}_i + 1, \tag{2.3.52}$$

we obtain

$$\|P_m\|_{s_{i+1},Q_T} \leq \tilde{s}_i^{\frac{1}{\tilde{s}_i}} C^{\frac{3}{5\tilde{s}_i}} \|G\|_{q,Q_T}^{\frac{1}{\tilde{s}_i}} \|P_m\|_{s_i,Q_T}^{\frac{\tilde{s}_i-1}{\tilde{s}_i}} + C^{\frac{3}{5\tilde{s}_i}}\|P(0)\|_\infty.$$

Noticing that

$$\frac{5}{3}\frac{q-1}{q} > 1 \iff q > \frac{5}{2},$$

we observe that $s_i \to +\infty$ as $i \to +\infty$. By iteration, we get

$$\begin{aligned}
&\|P_m\|_{s_{i+1},Q_T} \\
\leq\, & C^{\sum_{j=0}^{i} \frac{3}{5\tilde{s}_j} \prod_{h=j+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \prod_{j=0}^{i} \tilde{s}_j^{\frac{1}{\tilde{s}_j} \prod_{h=j+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \|G\|_q^{\sum_{j=0}^{i} \frac{1}{\tilde{s}_j} \prod_{h=j+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \|P\|_{s_0}^{\prod_{j=0}^{i} \frac{\tilde{s}_j-1}{\tilde{s}_j}} \\
&+ \sum_{j=1}^{i} C^{\sum_{r=j}^{i} \frac{3}{5\tilde{s}_r} \prod_{h=r+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \prod_{k=j+1}^{i} \tilde{s}_k^{\frac{1}{\tilde{s}_k} \prod_{h=k+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \\
&\cdot \|G\|_q^{\sum_{k=j+1}^{i} \frac{1}{\tilde{s}_k} \prod_{h=k+1}^{i} \frac{\tilde{s}_h-1}{\tilde{s}_h}} \|P(0)\|_\infty^{\prod_{k=j+1}^{i} \frac{\tilde{s}_k-1}{\tilde{s}_k}}.
\end{aligned} \tag{2.3.53}$$

Next, we use

$$\begin{aligned}
&\|G\|_{q,Q_T} \leq C_1 \ \text{ where } C_1 := \max\{1, \|G\|_{q,Q_T}\}, \\
&\|P(0)\|_\infty \leq C_2 \ \text{ where } C_2 := \max\{1, \|P(0)\|_\infty\}.
\end{aligned}$$

Since $\frac{\tilde{s}_h-1}{\tilde{s}_h} \leq 1$, we notice that all products can be bounded by 1. Consequently, (2.3.53) leads to (assuming that $C \geq 1$)

$$\begin{aligned}
\|P_m\|_{s_{i+1},Q_T} \leq\, & C^{\sum_{j=0}^{i} \frac{3}{5\tilde{s}_j}} e^{\sum_{j=0}^{i} \frac{\ln \tilde{s}_j}{\tilde{s}_j}} C_1^{\sum_{j=0}^{i} \frac{1}{\tilde{s}_j}} \max\{1, \|P_m\|_{s_0,Q_T}\} \\
&+ \sum_{j=0}^{i} C^{\sum_{r=j}^{i} \frac{3}{5\tilde{s}_r}} e^{\sum_{r=j+1}^{i} \frac{\ln \tilde{s}_r}{\tilde{s}_r}} C_1^{\sum_{r=j+1}^{i} \frac{1}{\tilde{s}_r}} C_2.
\end{aligned} \tag{2.3.54}$$

Note that the right-hand side is independent of m as well as n. Taking the limit as $i \to +\infty$, since $s_{i+1} \to +\infty$, by the convergence of the sums due to the d'Alambert criterion, we conclude that

$$P_m \in L^\infty(Q_T) \text{ for all } m \in \mathbb{N} \quad \Rightarrow P \in L^\infty(Q_T).$$

From the relation $p_{\mathrm{f}} = P + p_s$ and since $p_s \in L^\infty(Q_T)$, it finally follows that

$$p_{\mathrm{f}} \in L^\infty(Q_T).$$

On the other hand, assuming (2.3.41) we first observe that if $p_s \in L^q(0,T;W^{1,q}(\Omega))$ with $q > 10$ and $\boldsymbol{v} \in L^{\frac{10}{3}}(Q_T)$, then $\boldsymbol{v} \cdot \nabla p_s \in L^\ell(Q_T)$ with $\ell > \frac{5}{2}$. Consequently, (2.3.41) implies that $g + \boldsymbol{v} \cdot \nabla p_s \in L^\ell(Q_T)$ with $\ell > \frac{5}{2}$. Then, we conclude exactly as in the case given by (2.3.40) that

$$p_{\mathrm{f}} \in L^\infty(Q_T).$$

□

The following lemma regards the attainment of the constitutive equations.

Proposition 2.3.3 (Convergence Lemma). Let $U \subset Q_T$ be an arbitrary measurable bounded set and let $\{\mathbb{Z}^n\}_{n=1}^{+\infty}$, $\{\mathbb{D}^n\}_{n=1}^{+\infty}$ and $\{p_{\rm f}^n\}_{n=1}^{+\infty}$ be such that

$$\mathbb{Z}^n = \tau(p_{\rm f}^n)\frac{\mathbb{D}^n}{|\mathbb{D}^n| + \frac{1}{n}} \text{ with } \tau(p_{\rm f}^n) = q_*(p_s - p_{\rm f}^n)^+, \tag{2.3.55}$$

$$\sup_{n\in\mathbb{N}} \|p_{\rm f}^n\|_\infty < +\infty, \tag{2.3.56}$$

$$\mathbb{Z}^n \rightharpoonup \mathbb{Z} \text{ weakly in } (L^2(U))^{3\times 3}, \tag{2.3.57}$$

$$\mathbb{D}^n \rightharpoonup \mathbb{D} \text{ weakly in } (L^2(U))^{3\times 3}, \tag{2.3.58}$$

$$p_{\rm f}^n \to p_{\rm f} \text{ strongly in } L^2(U) \text{ and a.e. in } U, \tag{2.3.59}$$

$$\limsup_{n\to\infty} \int_U \mathbb{Z}^n : \mathbb{D}^n \le \int_U \mathbb{Z} : \mathbb{D}, \tag{2.3.60}$$

then, setting $\mathbb{S} = \mathbb{Z} + \mathbb{D}$,

$$\mathbb{D} = \frac{(|\mathbb{S}| - \tau(p_{\rm f}))^+}{|\mathbb{S}|}\mathbb{S} \text{ a.e. in } U. \tag{2.3.61}$$

Proof. We split the proof into three steps. Using the fact that (2.3.61) is, by Proposition 2.1.1, equivalent to (1.1.28) (with $2\nu_* = q_* = 1$), we first show that $|\mathbb{Z}| \le \tau(p_{\rm f})$. Then in order to verify that $\mathbb{Z} : \mathbb{D} \ge \tau(p_{\rm f})|\mathbb{D}|$ in the third step, we show that $\mathbb{Z}^n : \mathbb{D}^n \rightharpoonup \mathbb{Z} : \mathbb{D}$ weakly in $L^1(U)$, which is the second part of the proof.

Step 1. For all $n \in \mathbb{N}$, by (2.3.55), $\mathbb{Z}^n = \tau(p_{\rm f}^n)\frac{\mathbb{D}^n}{|\mathbb{D}^n|+\frac{1}{n}}$ and thus $|\mathbb{Z}^n| \le \tau(p_{\rm f}^n)$. For any subset $\omega \subset U$ it holds

$$\int_\omega |\mathbb{Z}^n| \le \int_\omega \tau(p_{\rm f}^n). \tag{2.3.62}$$

Since $\tau(\cdot)$ is Lipschitz, (2.3.59) implies that

$$\tau(p_{\rm f}^n) \to \tau(p_{\rm f}) \text{ strongly in } L^2(U) \tag{2.3.63}$$

and

$$\tau(p_{\rm f}^n) \to \tau(p_{\rm f}) \text{ a.e. in } U. \tag{2.3.64}$$

By virtue of (2.3.62), (2.3.64) and the lower semicontinuity of $\int_\omega |\mathbb{Z}^n|$ with respect to the weak convergence in $L^1(\omega)$ (which follows from (2.3.57) since U is bounded), we get

$$\|\mathbb{Z}\|_{L^1(\omega)} \le \|\tau(p_{\rm f})\|_{L^1(\omega)} \text{ for all } \omega \subset U. \tag{2.3.65}$$

Lebesgue's differentiation theorem then implies

$$|\mathbb{Z}| \le \tau(p_{\rm f}) \text{ a.e. in } U. \tag{2.3.66}$$

Step 2. In order to establish that

$$\mathbb{Z}^n : \mathbb{D}^n \rightharpoonup \mathbb{Z} : \mathbb{D} \text{ weakly in } L^1(U) \tag{2.3.67}$$

we set

$$\widehat{\mathbb{Z}}^n := \tau(p_{\rm f}^n)\frac{\mathbb{D}}{|\mathbb{D}| + \frac{1}{n}}, \tag{2.3.68}$$

and

$$\widehat{\mathbb{Z}} := \begin{cases} \tau(p_{\mathrm{f}})\frac{\mathbb{D}}{|\mathbb{D}|} & \text{if } \mathbb{D} \neq \mathbb{O}, \\ \mathbb{O} & \text{otherwise.} \end{cases} \tag{2.3.69}$$

Thanks to (2.3.64) we have that $\widehat{\mathbb{Z}^n} \to \widehat{\mathbb{Z}}$ almost everywhere in Q_T, and since $\widehat{\mathbb{Z}}^n$ is essentially bounded (because of (2.3.77)), Lebesgue's Convergence Theorem yields

$$\widehat{\mathbb{Z}}^n \to \widehat{\mathbb{Z}} \;\text{ strongly in } (L^2(U))^{3\times 3}. \tag{2.3.70}$$

Employing (2.3.57) and (2.3.60) and the convergences (2.3.70) and (2.3.58), we get

$$\limsup_{n\to\infty} \int_U (\mathbb{Z}^n - \widehat{\mathbb{Z}}^n) : (\mathbb{D}^n - \mathbb{D}) \le 0. \tag{2.3.71}$$

But due to (2.3.68), the monotone property (2.3.8) yields

$$(\mathbb{Z}^n - \widehat{\mathbb{Z}}^n) : (\mathbb{D}^n - \mathbb{D}) \ge 0 \text{ a.e. in } U.$$

This together with (2.3.71) implies that

$$(\mathbb{Z}^n - \widehat{\mathbb{Z}}^n) : (\mathbb{D}^n - \mathbb{D}) \to 0 \text{ strongly in } L^1(U),$$

and thus surely

$$(\mathbb{Z}^n - \widehat{\mathbb{Z}}^n) : (\mathbb{D}^n - \mathbb{D}) \rightharpoonup 0 \text{ weakly in } L^1(U). \tag{2.3.72}$$

Since the strong convergence (2.3.70) and weak convergence (2.3.58) imply that

$$\widehat{\mathbb{Z}}^n : (\mathbb{D}^n - \mathbb{D}) \rightharpoonup 0 \text{ weakly in } L^1(U), \tag{2.3.73}$$

so (2.3.72) yields

$$\mathbb{Z}^n : (\mathbb{D}^n - \mathbb{D}) \rightharpoonup 0 \text{ weakly in } L^1(U). \tag{2.3.74}$$

Finally employing (2.3.57) in (2.3.74) we conclude

$$\mathbb{Z}^n : \mathbb{D}^n \rightharpoonup \mathbb{Z} : \mathbb{D} \text{ weakly in } L^1(U). \tag{2.3.75}$$

Step 3. It remains to show $\mathbb{Z} : \mathbb{D} \ge \tau(p_{\mathrm{f}})|\mathbb{D}|$. First we note that

$$|\tau(p_{\mathrm{f}}^n)|\mathbb{D}^n| - \mathbb{Z}^n : \mathbb{D}^n| = \tau(p_{\mathrm{f}}^n)\frac{1}{n}\frac{|\mathbb{D}^n|}{|\mathbb{D}^n| + \frac{1}{n}}. \tag{2.3.76}$$

Since τ is a Lipschitz function, (2.3.56) gives

$$\|\tau(p_{\mathrm{f}}^n)\|_\infty \le C \text{ uniformly in } \; n. \tag{2.3.77}$$

Then the right hand side of (2.3.76) is essentially bounded by $\frac{C}{n}$ and thus

$$|\tau(p_{\mathrm{f}}^n)|\mathbb{D}^n| - \mathbb{Z}^n : \mathbb{D}^n| \to 0 \text{ in } L^\infty(U), \tag{2.3.78}$$

which implies, for all $\varphi \in L^\infty(U)$, that

$$\lim_{n\to+\infty} \int_U \varphi(\tau(p_{\mathrm{f}}^n)|\mathbb{D}^n| - \mathbb{Z}^n : \mathbb{D}^n) = 0. \tag{2.3.79}$$

Moreover, from (2.3.58) and (2.3.63) we get, for all $\varphi \in L^\infty(U)$, that

$$\varphi\tau(p_{\mathrm{f}}^n)\mathbb{D}^n \rightharpoonup \varphi\tau(p_{\mathrm{f}})\mathbb{D} \text{ in } L^1(U) \tag{2.3.80}$$

and the weak lower semicontinuity of the L^1-norm implies that, for all $\varphi \in L^\infty(U)$ such that $\varphi \geq 0$,

$$\int_U \varphi\tau(p_{\mathrm{f}})|\mathbb{D}| \leq \liminf_{n\to+\infty} \int_U \varphi\tau(p_{\mathrm{f}}^n)|\mathbb{D}^n|. \tag{2.3.81}$$

Using (2.3.67) together with (2.3.79) and (2.3.81), we obtain

$$\int_U \varphi(\tau(p_{\mathrm{f}})|\mathbb{D}| - \mathbb{Z} : \mathbb{D}) \leq \liminf_{n\to+\infty} \int_U \varphi(\tau(p_{\mathrm{f}}^n)|\mathbb{D}^n| - \mathbb{Z}^n : \mathbb{D}^n) = 0 \tag{2.3.82}$$

for any non-negative $\varphi \in L^\infty(U)$. Hence

$$\mathbb{Z} : \mathbb{D} \geq \tau(p_{\mathrm{f}})|\mathbb{D}| \text{ a.e. in } U,$$

which is $(1.1.28)_2$.

□

2.4 Proof of Theorem 2.2.1

The proof is split into the following five steps.

Step 1. Approximations. From Proposition 2.3.1 and Proposition 2.3.2, we get, for each $n \in \mathbb{N}$, the existence of $(\boldsymbol{v}^n, p_{\mathrm{f}}^n, \mathbb{Z}^n, \boldsymbol{z}^n)$ satisfying, for all $\boldsymbol{w} \in L^2(0,T; W^{1,2}_{\mathbf{n},\mathrm{div}})$ and for all $z \in L^2(0,T; W^{1,2}(\Omega))$, the following identities

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}\rangle + \int_{Q_T} G_n\left(|\boldsymbol{v}^n|^2\right) \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \boldsymbol{w} + \int_{Q_T} (\mathbb{D}\boldsymbol{v}^n + \mathbb{Z}^n) : \mathbb{D}\boldsymbol{w}& \\ + \int_{\Sigma_T} (\boldsymbol{v}_\tau^n + \boldsymbol{z}^n)\cdot \boldsymbol{w}_\tau - \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}\rangle = 0,&\end{aligned} \tag{2.4.1}$$

$$\int_0^T \langle \partial_t p_{\mathrm{f}}^n, z\rangle - \int_{Q_T} (p_{\mathrm{f}}^n \boldsymbol{v}^n \cdot \nabla z - \nabla p_{\mathrm{f}}^n \cdot \nabla z) = \int_{Q_T} (gz - p_s \boldsymbol{v}^n \cdot \nabla z), \tag{2.4.2}$$

and it holds

$$\mathbb{Z}^n = (p_f^n - p_s)^+ \frac{\mathbb{D}\boldsymbol{v}^n}{|\mathbb{D}\boldsymbol{v}^n| + \frac{1}{n}} \text{ a.e. in } Q_T \text{ and } \boldsymbol{z}^n = s_* \frac{\boldsymbol{v}_\tau^n}{|\boldsymbol{v}_\tau^n| + \frac{1}{n}} \text{ a.e. in } \Sigma_T. \tag{2.4.3}$$

Step 2. Reconstruction of the pressure. We set

$$p^n := (-\Delta_N)^{-1} \operatorname{div} \mathbf{h}^n \quad \text{with} \int_\Omega p^n(t,\cdot) = 0, \tag{2.4.4}$$

where $-\Delta_N$ denotes the Laplace operator associated with the homogeneous Neumann boundary conditions and

$$\boldsymbol{h}^n := -\operatorname{div}(\mathbb{D}\boldsymbol{v}^n + \mathbb{Z}^n) + \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|^2) - \boldsymbol{f}, \tag{2.4.5}$$

associated with the boundary condition $\boldsymbol{v}\cdot\boldsymbol{n}=0$ and $\boldsymbol{z}^n = s_*\frac{\boldsymbol{v}_\tau^n}{|\boldsymbol{v}_\tau^n|+\frac{1}{n}}$ on Σ_T. It means that p^n satisfies for all $\varphi\in W^{2,2}(\Omega)$ such that $\nabla\varphi\cdot\boldsymbol{n}=0$ on $\partial\Omega$ and for almost all $t\in[0,T]$

$$\int_\Omega p^n\Delta\varphi = \langle \boldsymbol{h}^n,\nabla\varphi\rangle + \int_{\partial\Omega}(\boldsymbol{v}_\tau^n+\boldsymbol{z}^n)\cdot(\nabla\varphi)_\tau, \tag{2.4.6}$$

whereas

$$\boldsymbol{h}^n\in L^2(0,T;W^{-1,2}_{\boldsymbol{n}}). \tag{2.4.7}$$

Consequently,

$$p^n\in L^2(Q_T). \tag{2.4.8}$$

Since any $\boldsymbol{w}\in W^{1,2}_{\boldsymbol{n}}$ satisfies

$$\boldsymbol{w}=\tilde{\boldsymbol{w}}+\nabla\varphi \text{ where } \tilde{\boldsymbol{w}}\in W^{1,2}_{\boldsymbol{n},\mathrm{div}},\ \varphi\in W^{2,2}(\Omega), \nabla\varphi\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega,$$

we observe that due to (2.4.1) and (2.4.6) we get

$$\begin{aligned}\langle\boldsymbol{h}^n,\boldsymbol{w}\rangle &= \int_\Omega(\mathbb{D}\boldsymbol{v}^n+\mathbb{Z}^n):\mathbb{D}\boldsymbol{w} + \int_\Omega \operatorname{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)G_n(|\boldsymbol{v}^n|^2)\cdot\boldsymbol{w} + \int_{\partial\Omega}(\boldsymbol{v}_\tau^n+\boldsymbol{z}^n)\cdot\boldsymbol{w}_\tau - \langle\boldsymbol{f},\boldsymbol{w}\rangle\\ &= -\langle\partial_t\boldsymbol{v}^n,\tilde{\boldsymbol{w}}\rangle + \textstyle\int_\Omega p^n\Delta\varphi = -\langle\partial_t\boldsymbol{v}^n,\tilde{\boldsymbol{w}}+\nabla\varphi\rangle + \int_\Omega p^n\operatorname{div}(\tilde{\boldsymbol{w}}+\nabla\varphi).\end{aligned}$$

Hence it holds, for all $\boldsymbol{w}\in W^{1,2}_{\boldsymbol{n}}$, that

$$\begin{aligned}&\langle\partial_t\boldsymbol{v}^n,\boldsymbol{w}\rangle + \int_\Omega(\mathbb{D}\boldsymbol{v}^n+\mathbb{Z}^n):\mathbb{D}\boldsymbol{w} + \int_{\partial\Omega}(\boldsymbol{v}^n+\boldsymbol{z}^n)\cdot\boldsymbol{w}_\tau\\ &+\textstyle\int_\Omega\operatorname{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)G_n(|\boldsymbol{v}^n|^2)\cdot\boldsymbol{w} = \int_\Omega p^n\operatorname{div}\boldsymbol{w} + \langle\boldsymbol{f},\boldsymbol{w}\rangle.\end{aligned} \tag{2.4.9}$$

Step 3. Uniform estimates with respect to n and limit as $n\to+\infty$. Taking $\boldsymbol{v}^n$ as test function in (2.4.1) and p_{f}^n in (2.4.2), and proceeding similarly as in the derivation of (2.3.22) and (2.3.26) using also Korn's inequality, we obtain

$$\sup_n\Big(\|\boldsymbol{v}^n\|_{L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}})} + \|\mathbb{D}\boldsymbol{v}^n\|_{2,Q_T} + \|\nabla\boldsymbol{v}^n\|_{2,Q_T} + \|\boldsymbol{v}_\tau^n\|_{2,\Sigma_T}\Big) < +\infty, \tag{2.4.10}$$

$$\sup_n\Big(\|p_{\mathrm{f}}^n\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p_{\mathrm{f}}^n\|_{2,Q_T}\Big) < +\infty, \tag{2.4.11}$$

$$\sup_n\Big(\|\boldsymbol{v}^n\|_{\frac{10}{3},Q_T} + \|\boldsymbol{v}^n\|_{\frac{8}{3},\Sigma_T}\Big) < +\infty. \tag{2.4.12}$$

Also, from Proposition 2.3.2, we have that

$$\sup_n\|p_{\mathrm{f}}^n\|_{\infty,Q_T} < +\infty. \tag{2.4.13}$$

It then follows from (2.4.3) that

$$\sup_n\left(\|\mathbb{Z}^n\|_{\infty,Q_T} + \|\boldsymbol{z}^n\|_{\infty,\Sigma_T}\right) < +\infty. \tag{2.4.14}$$

Since

$$G_n(|\boldsymbol{v}^n|^2)\operatorname{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n) = \sum_{h=1}^3 v_h^n\frac{\partial\boldsymbol{v}^n}{\partial x_h}G_n(|\boldsymbol{v}^n|^2),$$

and $\sup_n \|G_n(|\boldsymbol{v}^n|^2)\|_{\infty,Q_T} \le 1$, it follows from (2.4.10), (2.4.13) and Hölder's inequality that

$$\sup_n \|G_n(|\boldsymbol{v}^n|^2)\,\mathrm{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n)\|_{\frac{5}{4},Q_T} < +\infty. \tag{2.4.15}$$

For further analysis it is suitable to perform the following decomposition of the pressure p^n. Setting

$$\boldsymbol{h}_2^n := G_n(|\boldsymbol{v}^n|^2)\,\mathrm{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n)$$

and

$$p_2^n := (-\Delta_N)^{-1}\,\mathrm{div}\,\boldsymbol{h}_2^n,$$

we conclude from (2.4.15) that

$$\sup_n \|\nabla p_2^n\|_{\frac{5}{4},Q_T} < +\infty. \tag{2.4.16}$$

Furthermore, $\boldsymbol{h}_1^n := \boldsymbol{h}^n - \boldsymbol{h}_2^n$ fulfills $\sup_n \|\boldsymbol{h}_1^n\|_{L^2(0,T;W_{\boldsymbol{n}}^{-1,2})} < +\infty$, consequently $p_1^n := p^n - p_2^n$ satisfies

$$\sup_n \|p_1^n\|_{2,Q_T} < +\infty. \tag{2.4.17}$$

Hence, integrating (2.4.9) over $(0,T)$, we have, for all $\boldsymbol{w} \in L^2(0,T;W_{\boldsymbol{n}}^{1,2}) \cap (L^5(Q_T))^3$, that

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}\rangle = \int_{Q_T} (-\mathbb{Z}^n - \mathbb{D}\boldsymbol{v}^n + p_1^n \mathbb{I}) : \nabla \boldsymbol{w} + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}\rangle - \int_{\Sigma_T} (\boldsymbol{v}_\tau^n + \boldsymbol{z}^n) \cdot \boldsymbol{w}_\tau \\ - \int_{Q_T} \Big(G_n(|\boldsymbol{v}^n|^2)\,\mathrm{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) + \nabla p_2^n\Big) \cdot \boldsymbol{w}.\end{aligned} \tag{2.4.18}$$

The above uniform estimates then imply that

$$\sup_n \|\partial_t \boldsymbol{v}^n\|_{(L^2(0,T;W_{\boldsymbol{n}}^{1,2}) \cap (L^5(Q_T))^3)^*} < +\infty, \tag{2.4.19}$$

and similarly

$$\sup_n \|\partial_t p_{\mathrm{f}}^n\|_{L^2(0,T;(W^{1,2}(\Omega))^*)} < +\infty. \tag{2.4.20}$$

Due to the uniform estimates (2.4.10), (2.4.13), (2.4.14), (2.4.15), (2.4.16), (2.4.17), (2.4.19), (2.4.20), the Aubin-Lions compactness lemma and the compact embedding of the Sobolev spaces into the space of traces, we get the following convergences for subsequences that we do not relabel:

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{2.4.21}$$
$$p_{\mathrm{f}}^n \rightharpoonup p_{\mathrm{f}} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{2.4.22}$$
$$p_{\mathrm{f}}^n \to p_{\mathrm{f}} \text{ strongly in } L^q(Q_T) \text{ for all } q \in \left[1, \frac{10}{3}\right), \tag{2.4.23}$$
$$\partial_t p_{\mathrm{f}}^n \rightharpoonup \partial_t p_{\mathrm{f}} \text{ weakly in } L^2(0,T;(W^{1,2}(\Omega))^*), \tag{2.4.24}$$
$$p_{\mathrm{f}}^n \rightharpoonup^* p_{\mathrm{f}} \text{ weakly}^* \text{ in } L^\infty(Q_T), \tag{2.4.25}$$
$$\mathbb{Z}^n \rightharpoonup \mathbb{Z} \text{ weakly}^* \text{ in } (L^\infty(Q_T))^{3\times 3}, \tag{2.4.26}$$
$$\boldsymbol{z}^n \rightharpoonup \boldsymbol{z} \text{ weakly}^* \text{ in } (L^\infty(\Sigma_T))^3, \tag{2.4.27}$$
$$\boldsymbol{v}^n \to \boldsymbol{v} \text{ a.e. in } Q_T \text{ and strongly in } (L^q(Q_T))^3 \text{ for all } q \in \left[1, \frac{10}{3}\right), \tag{2.4.28}$$
$$\boldsymbol{v}^n_\tau \to \boldsymbol{v}_\tau \text{ a.e. in } \Sigma_T \text{ and strongly in } (L^q(\Sigma_T))^3 \text{ for all } q \in \left[1, \frac{8}{3}\right), \tag{2.4.29}$$
$$p_1^n \rightharpoonup p_1 \text{ weakly in } L^2(Q_T), \tag{2.4.30}$$
$$p_2^n \rightharpoonup p_2 \text{ weakly in } L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega)), \tag{2.4.31}$$
$$\partial_t \boldsymbol{v}^n \rightharpoonup \partial_t \boldsymbol{v} \text{ weakly in } (L^2(0,T;W^{1,2}_{\boldsymbol{n}}(\Omega)) \cap (L^5(Q_T))^3)^*, \tag{2.4.32}$$
$$G_n(|\boldsymbol{v}^n|^2)\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \rightharpoonup \mathbf{g} \text{ weakly in } L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega)). \tag{2.4.33}$$

It is not difficult to observe that due to the fact that

$$\|G_n(|\boldsymbol{v}^n|^2)\|_{\infty,Q_T} \le 1 \text{ and } G_n(|\boldsymbol{v}^n|^2) \to 1 \text{ strongly in } L^q(Q_T) \text{ for all } q \in [1,+\infty),$$

and due to (2.4.21) and (2.4.28) we have

$$\mathbf{g} = \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}).$$

Taking then the limit as $n \to \infty$ in (2.4.18), we obtain, for all $\boldsymbol{w}$ belonging to $L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3$, that

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle + \int_{Q_T} \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{w} + \int_{Q_T} (\mathbb{D}\boldsymbol{v} + \mathbb{Z}) : \mathbb{D}\boldsymbol{w} - \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}\rangle \\ + \int_{\Sigma_T} (\boldsymbol{v}_\tau + \boldsymbol{z})\cdot\boldsymbol{w}_\tau - \int_{Q_T} (p_1 \operatorname{div}\boldsymbol{w} - \nabla p_2 \cdot \boldsymbol{w}) = 0.\end{aligned} \tag{2.4.34}$$

Integrating (2.4.2) with respect the time between 0 and T and taking the limit as $n \to \infty$ we get, for all $z \in L^2(0,T;W^{1,2}(\Omega))$, that

$$\int_0^T \langle \partial_t p_{\mathrm{f}}, z\rangle - \int_{Q_T} (p_{\mathrm{f}}\boldsymbol{v}\cdot\nabla z - \nabla p_{\mathrm{f}}\cdot\nabla z) = \int_{Q_T} (gz - p_s \boldsymbol{v}\cdot\nabla z). \tag{2.4.35}$$

Let us note that from (2.4.34) and (2.4.35) valid for all $\boldsymbol{w} \in L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3$ and $z \in L^2(0,T;W^{1,2}(\Omega))$ we immediately conclude the validity of (2.2.4) and (2.2.5) for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{n}}$, $z \in W^{1,2}(\Omega)$ and almost all $t \in (0,T)$.

Step 4. Attainment of the constitutive equation on the boundary. Since the structure of the constitutive equation on the boundary $(2.4.3)_2$ is simpler than that used in Proposition 2.3.3, we can apply this assertion to this case as well. Indeed, we know that not only

$$\begin{aligned}
&\boldsymbol{z}^n = s_* \frac{\boldsymbol{v}_\tau^n}{|\boldsymbol{v}_\tau^n| + \frac{1}{n}} \quad \text{a.e. on } \Sigma_T,\\
&\boldsymbol{z}^n \rightharpoonup \boldsymbol{z} \text{ weakly in } (L^q(\Sigma_T))^3 \text{ for all } q \in [1, +\infty),\\
&\boldsymbol{v}_\tau^n \rightharpoonup \boldsymbol{v}_\tau \text{ weakly in } (L^{\frac{8}{3}}(\Sigma_T))^3,
\end{aligned}$$

but also

$$\boldsymbol{v}_\tau^n \to \boldsymbol{v}_\tau \text{ strongly in } (L^q(\Sigma_T))^3 \text{ for all } q \in \left[1, \frac{8}{3}\right).$$

Consequently,

$$\lim_{n\to+\infty} \int_{\Sigma_T} \boldsymbol{z}^n \cdot \boldsymbol{v}_\tau^n = \int_{\Sigma_T} \boldsymbol{z} \cdot \boldsymbol{v}_\tau,$$

and by Proposition 2.3.3 we get for $\boldsymbol{s} = \boldsymbol{z} + \boldsymbol{v}_\tau$

$$\boldsymbol{v}_\tau = \frac{(|\boldsymbol{s}| - s_*)^+}{|\boldsymbol{s}|} \boldsymbol{s} \quad \text{a.e. on } \Sigma_T.$$

Step 5. Attainment of the constitutive equation in the bulk. We wish to use Proposition 2.3.3, and we can notice that all of its assumptions (2.3.55)–(2.3.59) are all fulfilled except (2.3.60). To prove it, we have to overcome the difficulty that $\boldsymbol{v}$ is not an admissible test function in (2.4.34). This is why we employ the so-called L^∞-truncation method applied to $\boldsymbol{v}^n - \boldsymbol{v}$. Let $\{\lambda^n\}$, A, B such that $0 < A \le \lambda^n \le B < \infty$, where A, B are independent of n (but sufficiently large) and together with λ^n will be specified later. Consider the truncated velocity difference

$$\boldsymbol{w}^n := T_{\lambda^n}(\boldsymbol{v}^n - \boldsymbol{v}) := (\boldsymbol{v}^n - \boldsymbol{v}) \min\left\{1, \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}\right\}. \tag{2.4.36}$$

Since $\sup_n \|\boldsymbol{w}^n\|_{\infty, Q_T} \le B$, and $\boldsymbol{v}^n \to \boldsymbol{v}$ a.e. in Q_T, Lebesgue's dominated convergence theorem implies that

$$\boldsymbol{w}^n \to \boldsymbol{0} \text{ strongly in } (L^s(Q_T))^3 \text{ for every } s \in [1, \infty), \tag{2.4.37}$$

and similarly, as $\sup_n \|\boldsymbol{w}^n\|_{\infty,\Sigma_T} \le B$, by (2.4.29) and the Lebesgue theorem, we have

$$\boldsymbol{w}_\tau^n \to \boldsymbol{0} \text{ strongly in } (L^2(\Sigma_T))^3. \tag{2.4.38}$$

Since

$$\nabla \boldsymbol{w}^n = \begin{cases} \nabla \boldsymbol{v}^n - \nabla \boldsymbol{v} & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| \le \lambda^n,\\ \dfrac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}(\nabla \boldsymbol{v}^n - \nabla \boldsymbol{v}) - \lambda^n (\boldsymbol{v}^n - \boldsymbol{v}) \otimes \dfrac{(\nabla \boldsymbol{v}^n - \nabla \boldsymbol{v})(\boldsymbol{v}^n - \boldsymbol{v})}{|\boldsymbol{v}^n - \boldsymbol{v}|^3} & \text{otherwise,} \end{cases} \tag{2.4.39}$$

we observe that

$$|\text{div } \boldsymbol{w}^n| \leq \begin{cases} 0 & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| \leq \lambda^n, \\ \dfrac{2\lambda^n(|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|)}{|\boldsymbol{v}^n - \boldsymbol{v}|} & \text{otherwise} \end{cases} \tag{2.4.40}$$

and

$$|\nabla \boldsymbol{w}^n| \leq 2|\nabla \boldsymbol{v}^n - \nabla \boldsymbol{v}|. \tag{2.4.41}$$

Then, due to (2.4.10) and (2.4.41), $\nabla \boldsymbol{w}^n$ is uniformly bounded in $(L^2(Q_T))^{3\times 3}$ and, up to a subsequence, it converges weakly in $(L^2(Q_T))^{3\times 3}$. But employing (2.4.37) it follows that the weak limit has to be zero, i.e.

$$\nabla \boldsymbol{w}^n \rightharpoonup \mathbb{O} \text{ weakly in } (L^2(Q_T))^{3\times 3} \text{ and } \mathbb{D}\boldsymbol{w}^n \rightharpoonup \mathbb{O} \text{ weakly in } (L^2(Q_T))^{3\times 3}. \tag{2.4.42}$$

Inserting $\boldsymbol{w}^n$ in (2.4.18), we get

$$\begin{aligned} &\limsup_{n\to\infty} \int_{Q_T} \mathbb{Z}^n : \nabla \boldsymbol{w}^n - p_1^n \text{div } \boldsymbol{w}^n + \mathbb{D}\boldsymbol{v}^n : \mathbb{D}\boldsymbol{w}^n \\ &= \limsup_{n\to\infty} \Bigg[-\int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}^n \rangle - \int_{Q_T} G\left(|\boldsymbol{v}^n|^2\right) \text{div } (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \boldsymbol{w}^n \\ &+ \int_{\Sigma_T} (\boldsymbol{v}_\tau^n + \boldsymbol{z}^n) \cdot \boldsymbol{w}_\tau^n - \int_{Q_T} \nabla p_2^n \cdot \boldsymbol{w}^n - \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}^n \rangle \Bigg]. \end{aligned} \tag{2.4.43}$$

Now, by virtue of (2.4.37), (2.4.33) and (2.4.31), we observe that

$$\lim_{n\to\infty} \int_{Q_T} \left(G_n\left(|\boldsymbol{v}^n|^2\right) \text{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) + \nabla p_2^n \right) \cdot \boldsymbol{w}^n + \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}^n \rangle = 0, \tag{2.4.44}$$

and by virtue of (2.4.10) and (2.4.38), it holds

$$\lim_{n\to\infty} \int_{\Sigma_T} (\boldsymbol{v}_\tau^n + \boldsymbol{z}^n) \cdot \boldsymbol{w}_\tau^n = 0. \tag{2.4.45}$$

Since $\boldsymbol{w}^n \rightharpoonup \boldsymbol{0}$ weakly in $L^2(0,T;(W^{1,2}(\Omega))^3) \cap (L^5(Q_T))^3$ by (2.4.37) then $\lim_{n\to+\infty} \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}^n \rangle = 0$, thus

$$\liminf_{n\to+\infty} \int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}^n \rangle = \liminf_{n\to+\infty} \int_0^T \langle \partial_t (\boldsymbol{v}^n - \boldsymbol{v}), \boldsymbol{w}^n \rangle. \tag{2.4.46}$$

Moreover,

$$\int_0^T \langle \partial_t (\boldsymbol{v}^n - \boldsymbol{v}), \boldsymbol{w}^n \rangle = \int_0^T \partial_t \left(\frac{|\boldsymbol{v}^n - \boldsymbol{v}|^2}{2} \right) \min\left\{1, \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}\right\} = \int_{Q_T} \partial_t F^n, \tag{2.4.47}$$

where

$$F^n := \begin{cases} \dfrac{|\boldsymbol{v}^n - \boldsymbol{v}|^2}{2} & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| \le \lambda^n, \\ \lambda^n |\boldsymbol{v}^n - \boldsymbol{v}| - \dfrac{(\lambda^n)^2}{2} & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n. \end{cases}$$

Thus from (2.4.47) and since $F^n(0,\cdot) = 0$ almost everywhere in Ω (as $\boldsymbol{v}^n(0,\cdot) = \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0$ a.e. in Ω), we get

$$\int_0^T \langle \partial_t(\boldsymbol{v}^n - \boldsymbol{v}), \boldsymbol{w}^n \rangle = \int_\Omega F^n(T,\cdot), \tag{2.4.48}$$

taking the liminf we finally arrive at

$$\liminf_{n\to+\infty} \int_0^T \langle \partial_t(\boldsymbol{v}^n - \boldsymbol{v}), \boldsymbol{w}^n \rangle = \liminf_{n\to+\infty} \int_\Omega F^n(T,\cdot) \ge 0, \tag{2.4.49}$$

but this is equivalent to

$$\limsup_{n\to\infty} \left[- \int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}^n \rangle \right] \le 0. \tag{2.4.50}$$

Collecting (2.4.44), (2.4.45), (2.4.50), it follows from (2.4.43) that

$$\limsup_{n\to\infty} \int_{Q_T} \mathbb{Z}^n : \mathbb{D}\boldsymbol{w}^n - (p_1^n \operatorname{div} \boldsymbol{w}^n) + \mathbb{D}\boldsymbol{v}^n : \mathbb{D}\boldsymbol{w}^n \le 0. \tag{2.4.51}$$

Since $(\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) : \mathbb{D}\boldsymbol{w}^n \ge 0$ and $\lim_{n\to+\infty} \int_{Q_T} \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{w}^n = 0$, (2.4.40) and (2.4.51) imply that

$$\begin{aligned} \limsup_{n\to\infty} \int_{Q_T} & \mathbb{Z}^n : \mathbb{D}\boldsymbol{w}^n + \mathbb{D}\boldsymbol{v}^n : \mathbb{D}\boldsymbol{w}^n \\ & \le \limsup_{n\to\infty} \int_{Q_T} |p_1^n| |\operatorname{div} \boldsymbol{w}^n| \\ & \le \limsup_{n\to\infty} \int_{\{|\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n\}} \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} |p_1^n| (|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|). \end{aligned} \tag{2.4.52}$$

Let $\overline{\mathbb{Z}} \in (L^{\frac{10}{3}}(Q_T))^{3\times 3}$ be such that

$$\overline{\mathbb{Z}} = \begin{cases} \mathbb{O} & \text{if } \mathbb{D}\boldsymbol{v} = \mathbb{O}, \\ \tau(p_{\mathrm{f}}) \dfrac{\mathbb{D}}{|\mathbb{D}|} & \text{if } \mathbb{D}\boldsymbol{v} \ne \mathbb{O}. \end{cases} \tag{2.4.53}$$

Since $\lim_{n\to+\infty} \int_{Q_T} \overline{\mathbb{Z}} : \mathbb{D}\boldsymbol{w}^n = 0$ thanks to (2.4.42), we arrive at

$$\limsup_{n\to\infty} \int_{Q_T} (\mathbb{Z}^n - \overline{\mathbb{Z}}) : \mathbb{D}\boldsymbol{w}^n \le \limsup_{n\to\infty} \int_{\{|\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n\}} \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} |p_1^n| (|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|). \tag{2.4.54}$$

Splitting the integral on the left-hand side of (2.4.54) into two parts, one integrated over $\{|\boldsymbol{v}^n - \boldsymbol{v}| \le \lambda^n\}$ the other over $\{|\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n\}$, using (2.4.39), and moving the latter to the right-hand side and estimating it by (2.4.41), we get

$$\limsup_{n\to\infty} \int_{\{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n\}} (\mathbb{Z}^n - \overline{\mathbb{Z}}) : \mathbb{D}(\boldsymbol{v}^n - \boldsymbol{v}) \le C \limsup_{n\to\infty} \int_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>\lambda^n\}} \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n, \quad (2.4.55)$$

where

$$I^n := (|p_1^n|^2 + |\nabla \boldsymbol{v}^n|^2 + |\nabla \boldsymbol{v}|^2 + |\mathbb{Z}^n|^2 + |\overline{\mathbb{Z}}|^2) \text{ and } \sup_n \int_{Q_T} I^n < +\infty.$$

Let $N \in \mathbb{N}$ be arbitrary. We fix $A = N$ and $B = N^{N+1}$ and define

$$Q_i^n := \{[t, \boldsymbol{x}] \in Q_T; N^i < |\boldsymbol{v}^n - \boldsymbol{v}| \le N^{i+1}\}, \qquad i = 1, \dots, N.$$

Since

$$\sum_{i=1}^{N} \int_{Q_i^n} I^n \le C_*, \quad (2.4.56)$$

there is, for each $n \in \mathbb{N}$, an index $i_n \in \{1, \dots, N\}$ such that

$$\int_{Q_{i_n}^n} I^n \le \frac{C_*}{N}. \quad (2.4.57)$$

Setting $\lambda^n = N^{i_n}$, the right-hand side of (2.4.55) can be estimated as follows using (2.4.57) and the fact that I^n is uniformly bounded in $L^1(Q_T)$

$$\begin{aligned} &\int_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>N^{i_n}\}} \frac{N^{i_n}}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n \\ &\quad = \int_{\{N^{i_n}<|\boldsymbol{v}^n-\boldsymbol{v}|\le N^{i_n+1}\}} \frac{N^{i_n}}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n + \int_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>N^{i_n+1}\}} \frac{N^{i_n}}{|\boldsymbol{v}^n - \boldsymbol{v}|} I^n \\ &\quad \le \int_{Q_{i_n}^n} I^n + \frac{1}{N} \int_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>N^{i_n+1}\}} I^n \le \frac{C_*}{N}. \end{aligned} \quad (2.4.58)$$

Let

$$W^n := \left(\mathbb{Z}^n - \overline{\mathbb{Z}}\right) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}).$$

Then (2.4.55) and (2.4.58) imply that

$$\begin{aligned} &\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n} W^n \le \frac{C_*}{N} \\ \iff &\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n} |W^n| \le \frac{C_*}{N} + 2 \limsup_{n\to\infty} \int_{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n} (W^n)^- \end{aligned} \quad (2.4.59)$$

Now we show that

$$(W^n)^- \to 0 \text{ strongly in } L^1(Q_T). \quad (2.4.60)$$

Recalling that $\mathbb{Z}^n = \mathcal{Z}^n(p_{\mathrm{f}}^n, \mathbb{D}\boldsymbol{v}^n)$ and incorporating (2.3.8), we get

$$\begin{aligned} W^n =& (\mathbb{Z}^n - \mathcal{Z}^n(p_{\mathrm{f}}^n, \mathbb{D}\boldsymbol{v})) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) + (\mathcal{Z}^n(p_{\mathrm{f}}^n, \mathbb{D}\boldsymbol{v}) - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \\ &\geq (\mathcal{Z}^n(\mathbb{D}\boldsymbol{v}, \tau(p_{\mathrm{f}}^n)) - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}). \end{aligned} \tag{2.4.61}$$

Splitting $Q_T = \{|\mathbb{D}\boldsymbol{v}| = 0\} \cup \{|\mathbb{D}\boldsymbol{v}| > 0\}$, thanks to the definitions of $\mathbb{Z}^n$ and $\overline{\mathbb{Z}}$ and since p_{f}^n converges pointwise, we get

$$\mathcal{Z}^n(\mathbb{D}\boldsymbol{v}, \tau(p_{\mathrm{f}}^n)) \to \overline{\mathbb{Z}} \text{ a.e. in } \; Q_T.$$

Also, independently of n,

$$|\mathcal{Z}^n(\mathbb{D}\boldsymbol{v}, \tau(p_{\mathrm{f}}^n)) - \overline{\mathbb{Z}}| \leq C|\mathbb{D}\boldsymbol{v}|.$$

By the Dominated Convergence Theorem and since $(\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})$ is uniformly bounded in $(L^2(Q_T))^{3\times 3}$, we conclude (2.4.60).
Combining (2.4.59), (2.4.60) and recalling that $A = N \leq \lambda^n$,

$$\limsup_{n\to\infty} \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \leq N} |W^n| \leq \frac{C_*}{N}. \tag{2.4.62}$$

With the help of the Hölder and Chebyshev inequalities, we observe that

$$\begin{aligned} &\int_{Q_T} \sqrt{|W^n|} \leq \int_{|\boldsymbol{v}^n - \boldsymbol{v}| \leq N} \sqrt{|W^n|} + \int_{|\boldsymbol{v}^n - \boldsymbol{v}| > N} \sqrt{|W^n|} \\ &\leq |Q_T|^{\frac{1}{2}} \sqrt{\int_{|\boldsymbol{v}^n - \boldsymbol{v}| \leq N} |W^n|} + \|W^n\|_{L^2(Q_T)}^{\frac{1}{2}} \sqrt{|\{|\boldsymbol{v}^n - \boldsymbol{v}| > N\}|} \leq \frac{C}{\sqrt{N}} \end{aligned} \tag{2.4.63}$$

which implies that for a suitable subsequence,

$$W^n \to 0 \quad \text{ a.e. in } Q_T\,. \tag{2.4.64}$$

Applying Egorov Theorem, one concludes that

$$W^n \to 0 \quad \text{strongly in } L^1(Q_T \setminus E_j)\,,$$

where $E_j \subset Q_T$ are such that $\lim_{j\to\infty} |E_j| = 0$. It follows from the definition of W^n and the weak convergences (2.4.21), (2.4.26) that

$$\limsup_{n\to\infty} \int_{Q_T\setminus E_j} \mathbb{Z}^n : \mathbb{D}\boldsymbol{v}^n = \limsup_{n\to\infty} \int_{Q_T\setminus E_j} \overline{\mathbb{Z}} : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) + \mathbb{Z}^n : \mathbb{D}\boldsymbol{v} = \int_{Q_T\setminus E_j} \mathbb{Z} : \mathbb{D}\boldsymbol{v}.$$

Thus, the assumptions (2.3.55)-(2.3.60) of Proposition 2.3.3 are verified with $U = Q_T \setminus E_j$, for all $j \in \mathbb{N}$. Due to the properties of E_j, we finally conclude, using (2.3.61), that

$$\mathbb{D}\boldsymbol{v} = \frac{(|\mathbb{S}| - \tau(p_{\mathrm{f}}))^+}{|\mathbb{S}|}\mathbb{S}.$$

The proof of Theorem 2.2.1 is complete.

3. Analysis of unsteady flows of pore pressure activated granular materials

3.1 Formulation of the problem

In this chapter we strengthen the results from Chapter 2. More specifically, we investigate the following system of PDEs supposed to be satisfied in $Q_T := (0,T)\times\Omega$, where $T\in(0,\infty)$ and $\Omega\subset\mathbb{R}^3$ is a flow domain, i.e. bounded, open and connected set (with Lipschitz boundary $\partial\Omega$)

$$\operatorname{div}\boldsymbol{v} = 0, \tag{3.1.1a}$$

$$\rho_{\rm s}^{\rm m}\left(\partial_t\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\right) = \operatorname{div}\mathbb{S} - \nabla p + \rho_{\rm s}^{\rm m}\boldsymbol{f}\ , \tag{3.1.1b}$$

$$\partial_t p_{\rm f} + \boldsymbol{v}\cdot\nabla p_{\rm f} = K\Delta p_{\rm f} - \operatorname{div}(K\rho_{\rm f}^{\rm m}\boldsymbol{f}) + \partial_t p_s + \boldsymbol{v}\cdot\nabla p_s\ , \tag{3.1.1c}$$

$$\boldsymbol{v}_{\rm f} = \boldsymbol{v} - \frac{1}{\alpha}\widehat{\phi}(p-p_{\rm f})\left(\nabla p_{\rm f} - \rho_{\rm f}^{\rm m}\boldsymbol{f}\right), \tag{3.1.1d}$$

where $\mathbb{S}$ and $\mathbb{D}\boldsymbol{v} := \frac{\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T}{2}$ satisfy

$$\begin{aligned}
&\mathbb{D}\boldsymbol{v} = \mathbb{O} \Rightarrow\ |\mathbb{S}| \le \tau(p_{\rm f}),\\
&\mathbb{D}\boldsymbol{v} \neq \mathbb{O} \Rightarrow\ \mathbb{S} = \tau(p_{\rm f})\frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|} + 2\nu_*\left(|\mathbb{D}\boldsymbol{v}| - \delta_*\right)^+\frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|}\\
&\qquad\text{with } \tau(p_{\rm f}) := q_*(p_s - p_{\rm f})^+.
\end{aligned} \tag{3.1.1e}$$

The system (3.1.1) coincides with the equations (2.1.1) stated in Chapter 2 (see also [1]) provided that we set $\delta_* = 0$ in (3.1.1e) and we identify the symbols $\boldsymbol{v}$ and $p_{\rm f}$ with $\boldsymbol{v}_{\rm s}$ and $p_{\rm f}^{\rm t}$ used in Chapter 2. Note that $\boldsymbol{v}_{\rm f}$ appears only in (3.1.1d) and can be always obtained a posteriori once $\boldsymbol{v}, p_{\rm f}$ and p are obtained from (3.1.1a)–(3.1.1c) and (3.1.1e). Consequently, in what follows, we consider the system (3.1.1) without the equation (3.1.1d). It is worth observing that the constitutive relation (3.1.1e) can be rewritten in a more compact way as an implicit constitutive relation:

$$\begin{aligned}
&\mathbb{S} = \mathbb{Z} + 2\nu_*\left(|\mathbb{D}\boldsymbol{v}| - \delta_*\right)^+\frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|}\\
&\quad\text{with } \mathbb{Z} \text{ fulfilling } (|\mathbb{Z}| - \tau(p_{\rm f}))^+ + \big||\mathbb{D}\boldsymbol{v}|\mathbb{Z} - \tau(p_{\rm f})\mathbb{D}\boldsymbol{v}\big| = 0.
\end{aligned} \tag{3.1.2}$$

We will exploit formulation (3.1.2) in our analysis. A systematic study of implicit constitutive equations go back to the original works [67] and [68].

We complete the system (3.1.1a)–(3.1.1c) and (3.1.2) by considering the following boundary and initial conditions:

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ and } \nabla p_{\rm f}\cdot\boldsymbol{n} = 0 \text{ on } \Sigma_T := (0,T)\times\partial\Omega, \tag{3.1.3a}$$

$$\begin{aligned}
&\boldsymbol{s} = \boldsymbol{z} + \gamma_*\left(|\boldsymbol{v}_\tau| - \beta_*\right)^+\frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau|}\\
&\text{with } \boldsymbol{z} \text{ fulfilling } (|\boldsymbol{z}| - s_*)^+ + \big||\boldsymbol{v}_\tau|\boldsymbol{z} - s_*\boldsymbol{v}_\tau\big| = 0 \text{ on } \Sigma_T,
\end{aligned} \tag{3.1.3b}$$

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 \text{ and } p_{\rm f}(0,\cdot) = p_0 \text{ in } \Omega. \tag{3.1.3c}$$

Here, we used the following notation: $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^3$ stands for the unit outer normal vector, while for any vector $\boldsymbol{z}$ defined on $\partial\Omega$, $\boldsymbol{z}_\tau := \boldsymbol{z} - (\boldsymbol{z} \cdot \boldsymbol{n})\boldsymbol{n}$ denotes the tangential component of $\boldsymbol{z}$, in particular, $\boldsymbol{s} := -(\mathbb{S}\boldsymbol{n})_\tau$, and γ_*, β_*, s_* are non-negative constants. Condition (3.1.3b) describes the shifted stick-slip (or threshold slip) and it is analogous to that for the stress tensor in the bulk (see (3.1.2)). It includes as special cases, the stick-slip by taking $\beta_* = 0$ while $s_*, \gamma_* > 0$, Navier's slip $\boldsymbol{s} = \gamma_* \boldsymbol{v}_\tau$ by taking $s_*, \beta_* = 0$ while $\gamma_* > 0$, and perfect slip $\boldsymbol{s} = \boldsymbol{0}$ by setting $s_* = \gamma_* = 0$. Note that the no-slip condition is obtained by letting either $s_* \to +\infty$ or by setting $\beta_* = 0$ and letting $\gamma_* \to +\infty$.

The main purpose of this chapter is to establish long-time and large-data theory to the initial- and boundary-value problem described by (3.1.1a)–(3.1.1c), (3.1.2), (3.1.3), see Theorem 3.2.1 below. The novelties consist not only in incorporating a more general model with $\delta_* \geq 0$, but more importantly in providing a different proof for more general class of data (particularly for $\boldsymbol{f}$ that is merely L^2-integrable). More precisely, we can avoid using L^∞-estimates for p_f needed in Chapter 2 (see also [3]). Consequently, the main tool for taking the limit in the constitutive equations cannot be applied in the form given in Proposition 2.3.3 (or [3, Proposition 5.3]), but has to be modified in an essential way due to a lower integrability of p_f, but also a more complicated material response.

The novel key tool regarding the attainment of the constitutive equations by the limiting objects is proved separately in Proposition 3.3.1. The key assumption of this proposition, namely (3.3.5) and (3.3.9), call for taking $\boldsymbol{v}^n - \boldsymbol{v}$ as a test function in the weak formulation of the balance of linear momentum. However, $\boldsymbol{v}^n - \boldsymbol{v}$ is not an admissible test function in the setting considered here. This difficulty can be overcome, similarly as in Chapter 2 (see also [3]), by using the L^∞-truncation method, which requires to introduce an integrable pressure, as the truncations $(\boldsymbol{v}^n - \boldsymbol{v})_\infty$ are not divergenceless. Following the approach originally developed in [7] (see also [33]), we overcome such difficulty by considering slipping boundary conditions (3.1.3a)–(3.1.3b). As pointed out in [69], the analysis for unsteady flows changes remarkably when the no–slip condition is considered.

We use the L^∞-truncation method in the proof of Theorem 3.2.1 below. While the truncations $(\boldsymbol{v}^n - \boldsymbol{v})_\infty$ are difficult to make solenoidal, the authors of [9] succeeded to make the Lipschitz approximations $(\boldsymbol{v}^n - \boldsymbol{v})_{1,\infty}$ divergenceless and they thus developed a solenoidal version of the Lipschitz truncation method. This tool allows one to avoid the presence of the pressure in the setting, therefore one may include more general responses as well as boundary conditions. As a matter of fact, we present new results available for systems describing materials that behave after activation $|\mathbb{D}\boldsymbol{v}| > \delta_*$, as a power-law fluid, i.e. the constitutive equation (3.1.2) is replaced by

$$\begin{gathered}\mathbb{S} = \mathbb{Z} + 2\nu_* |\mathbb{D}\boldsymbol{v}|^{q-2} \left(|\mathbb{D}\boldsymbol{v}| - \delta_*\right)^+ \frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|} \\ \text{with } \mathbb{Z} \text{ fulfilling } (|\mathbb{Z}| - \tau(p_\mathrm{f}))^+ + ||\mathbb{D}\boldsymbol{v}|\mathbb{Z} - \tau(p_\mathrm{f})\mathbb{D}\boldsymbol{v}| = 0.\end{gathered} \tag{3.1.4}$$

The available results are presented in Theorem 3.2.2. We are not providing the proof of these results as they can be deduced from the approach used when proving Theorem 3.2.1 and from the methods used recently, for example, in [69]. Note that the latter results are restricted to models (3.1.4) with $q > \frac{6}{5}$ (in three spatial dimensions). Recently, another concept of dissipative solution was introduced in [70] and, its long-time and large-data existence is proved independently on what is the value of q (in particular also for $q \in [1, 6/5]$). In fact in the theory developed in [70] the stress tensor can be merely subdifferential of a convex potential depending on $\mathbb{D}\boldsymbol{v}$, whose growth is at least linear. There are other approaches to analyze the mathematical properties of Bingham fluids (see e.g. [50] and [49]), but they are usually based on regularity techniques requiring smoother data.

3.2 Preliminaries and main results

For the sake of simplicity in the right-hand side of (3.1.1c), which has the form $g := \partial_t p_s - \operatorname{div} \boldsymbol{f}$, we omit the effect of $\partial_t p_s$ as it plays the role of a given external force and it can be easily incorporated into the analysis. We also set without loss of any generality $\rho_{\mathrm{s}}^{\mathrm{m}} = \rho_{\mathrm{f}}^{\mathrm{m}} = K = 2\nu_* = \gamma_* = q_* = 1$, while we assume $\delta_*, s_*, \beta_* \geq 0$. In this chapter again, the symbol $\mathbb{D}\boldsymbol{\varphi}$ stands for the symmetric part of the gradient of a vector-valued function $\boldsymbol{\varphi}$, i.e.

$$\mathbb{D}\boldsymbol{\varphi} := \frac{\nabla\boldsymbol{\varphi} + (\nabla\boldsymbol{\varphi})^T}{2}.$$

We are ready to enunciate the first result, which is the existence of weak solutions to the system (3.1.1a)–(3.1.1c), (3.1.2), (3.1.3), proved in Section 3.5.

Theorem 3.2.1. *For any $\Omega \in C^{1,1}$, $T \in (0, \infty)$ and for any $\boldsymbol{v}_0, p_0, \boldsymbol{f}, p_s$ fulfilling*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}, \;\; p_0 \in L^2(\Omega), \;\; \boldsymbol{f} \in (L^2(Q_T))^3, \;\; p_s \in L^5(Q_T),$$

there exists a quintuplet $(\boldsymbol{v}, p_{\mathrm{f}}, p, \mathbb{S}, \boldsymbol{s})$:

$$\begin{aligned}
&\boldsymbol{v} \in L^\infty(0,T; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\mathbf{n},\mathrm{div}}), \partial_t \boldsymbol{v} \in (L^2(0,T; W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3)^*,\\
&p_{\mathrm{f}} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \;\; \partial_t p_{\mathrm{f}} \in L^{\frac{4}{3}}(0,T; (W^{1,2}(\Omega))^*),\\
&p = p_1 + p_2 \text{ where } p_1 \in L^2(Q_T) \text{ and } p_2 \in L^{\frac{5}{4}}(0,T; W^{1,\frac{5}{4}}(\Omega)),\\
&\mathbb{S} \in (L^2(Q_T))^{3\times 3}, \;\; \boldsymbol{s} \in (L^{\frac{8}{3}}(\Sigma_T))^3,
\end{aligned}$$

satisfying, for all $\boldsymbol{w} \in L^2(0,T; W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3$ and for all $z \in L^4(0,T; W^{1,2}(\Omega))$, the following equations:

$$\begin{aligned}
\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle + \int_{Q_T} \mathbb{S} : \mathbb{D}\boldsymbol{w} - \int_{Q_T} (\boldsymbol{v}\otimes\boldsymbol{v}) : \mathbb{D}\boldsymbol{w} + \int_{\Sigma_T} \boldsymbol{s}\cdot\boldsymbol{w}_{\boldsymbol{\tau}}\\
= \textstyle\int_{Q_T} \boldsymbol{f}\cdot\boldsymbol{w} + \int_{Q_T} p_1 \operatorname{div}\boldsymbol{w} - \int_{Q_T} \nabla p_2\cdot\boldsymbol{w},
\end{aligned} \tag{3.2.1}$$

$$\int_0^T \langle \partial_t p_{\mathrm{f}}, z\rangle - \int_{Q_T} p_{\mathrm{f}}\boldsymbol{v}\cdot\nabla z + \int_{Q_T} \nabla p_{\mathrm{f}}\cdot\nabla z = \int_{Q_T} \boldsymbol{f}\cdot\nabla z - \int_{Q_T} p_s\boldsymbol{v}\cdot\nabla z, \tag{3.2.2}$$

and the following constitutive relations:

$$\begin{aligned}
&\mathbb{S} = \mathbb{Z} + (|\mathbb{D}\boldsymbol{v}| - \delta_*)^+ \frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|}\\
&\quad \textit{with } \mathbb{Z} \textit{ fulfilling } (|\mathbb{Z}| - \tau(p_{\mathrm{f}}))^+ + ||\mathbb{D}\boldsymbol{v}|\mathbb{Z} - \tau(p_{\mathrm{f}})\mathbb{D}\boldsymbol{v}| = 0,\\
&\quad \textit{where } \tau(p_{\mathrm{f}}) = (p_s - p_{\mathrm{f}})^+ \textit{ a.e. in } Q_T,
\end{aligned} \tag{3.2.3}$$

$$\begin{aligned}
&\boldsymbol{s} = \boldsymbol{z} + (|\boldsymbol{v}_\tau| - \beta_*)^+ \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau|}\\
&\quad \textit{with } \boldsymbol{z} \textit{ fulfilling } (|\boldsymbol{z}| - s_*)^+ + ||\boldsymbol{v}_\tau|\boldsymbol{z} - s_*\boldsymbol{v}_\tau| = 0 \textit{ a.e. on } \Sigma_T,
\end{aligned} \tag{3.2.4}$$

and attaining the initial conditions in the following sense:

$$\lim_{t\to 0+} \left(\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2 + \|p_{\mathrm{f}}(t)-p_0\|_2\right) = 0. \tag{3.2.5}$$

The second result concerns the system (3.1.1a)–(3.1.1c), (3.1.3), and (3.1.4).

Theorem 3.2.2. *Let $\Omega \in C^{0,1}$, $T \in (0,\infty)$, and $q > \frac{6}{5}$. Set $m := \max\{2, q'\}$ and $r := \max\left\{q, \frac{5q}{5q-6}\right\}$. For any $\boldsymbol{v}_0, p_0, \boldsymbol{f}, p_s$ fulfilling*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}},\ p_0 \in L^2(\Omega),\ \boldsymbol{f} \in (L^m(Q_T))^3,\ p_s \in L^{\frac{10q}{5q-6}}(Q_T),$$

there exists a quadruplet $(\boldsymbol{v}, p_{\mathrm{f}}, \mathbb{S}, \boldsymbol{s})$:

$$\begin{aligned}
&\boldsymbol{v} \in L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^q(0,T;W^{1,q}_{\mathbf{n},\mathrm{div}}), \partial_t \boldsymbol{v} \in L^{r'}(0,T;(W^{1,r}_{\boldsymbol{n},\mathrm{div}})^*),\\
&p_{\mathrm{f}} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \partial_t p_{\mathrm{f}} \in (L^2(0,T;W^{1,2}(\Omega)) \cap L^{\frac{10q}{5q-6}}(Q_T))^*,\\
&\mathbb{S} \in (L^{q'}(Q_T))^{3\times 3},\ \boldsymbol{s} \in (L^2(\Sigma_T))^3,
\end{aligned}$$

attaining the initial conditions (3.2.5)*, fulfilling, for all $\varphi \in L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}})$ and $z \in L^2(0,T;W^{1,2}(\Omega)) \cap L^{\frac{10q}{5q-6}}(Q_T)$, the following equations:*

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle + \int_{Q_T} \mathbb{S} : \mathbb{D}\boldsymbol{\varphi} - \int_{Q_T} (\boldsymbol{v}\otimes\boldsymbol{v}) : \mathbb{D}\boldsymbol{\varphi} + \int_{\Sigma_T} \boldsymbol{s}\cdot\boldsymbol{w}_{\boldsymbol{\tau}} = \int_{Q_T} \boldsymbol{f}\cdot\boldsymbol{\varphi}, \tag{3.2.6}$$

$$\int_0^T \langle \partial_t p_{\mathrm{f}}, \varphi\rangle + \int_{Q_T} \nabla p_{\mathrm{f}}\cdot\nabla\varphi - \int_{Q_T} p_{\mathrm{f}}\boldsymbol{v}\cdot\nabla\varphi = \int_{Q_T} (\boldsymbol{f} - p_s\boldsymbol{v})\cdot\nabla\varphi, \tag{3.2.7}$$

and the following constitutive relations:

$$\begin{aligned}
&\mathbb{S} = \mathbb{Z} + (|\mathbb{D}\boldsymbol{v}| - \delta_*)^+ |\mathbb{D}\boldsymbol{v}|^{q-2}\frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|}\\
&\quad \text{with } \mathbb{Z} \text{ fulfilling } (|\mathbb{Z}| - \tau(p_{\mathrm{f}}))^+ + ||\mathbb{D}\boldsymbol{v}|\mathbb{Z} - \tau(p_{\mathrm{f}})\mathbb{D}\boldsymbol{v}| = 0,\\
&\quad \text{where } \tau(p_{\mathrm{f}}) = (p_s - p_{\mathrm{f}})^+ \text{ a.e. in } Q_T,
\end{aligned} \tag{3.2.8}$$

$$\begin{aligned}
&\boldsymbol{s} = \boldsymbol{z} + (|\boldsymbol{v}_\tau| - \beta_*)^+ \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau|}\\
&\quad \text{with } \boldsymbol{z} \text{ fulfilling } (|\boldsymbol{z}| - s_*)^+ + ||\boldsymbol{v}_\tau|\boldsymbol{z} - s_*\boldsymbol{v}_\tau| = 0 \text{ a.e. on } \Sigma_T.
\end{aligned} \tag{3.2.9}$$

This result is stated without the proof here. The proof can be however achieved in the spirit of Theorem 3.2.1 by employing a solenoidal version of the Lipschitz-truncation method developed in [9], and by using the approximation scheme presented in [69, Theorem 3.3].

3.3 Attainment of the constitutive equations

In this section, we establish a new scheme how to take the limit in the constitutive equations needed when proving Theorem 3.2.1.

Proposition 3.3.1. Let $U \subset Q_T$ be an arbitrary measurable bounded set and let $\{\mathbb{Z}^n\}_{n=1}^{+\infty}$, $\{\mathbb{D}^n\}_{n=1}^{+\infty}$ and $\{p_{\mathrm{f}}^n\}_{n=1}^{+\infty}$ be sequences such that

$$\mathbb{Z}^n = \tau(p_{\mathrm{f}}^n)\frac{\mathbb{D}^n}{|\mathbb{D}^n| + \frac{1}{n}} \text{ with } \tau(p_{\mathrm{f}}^n) = (p_s - p_{\mathrm{f}}^n)^+ \text{ a.e. in } U, \tag{3.3.1}$$

$$\mathbb{Z}^n \rightharpoonup \mathbb{Z} \text{ weakly in } (L^2(U))^{3\times 3}, \tag{3.3.2}$$

$$\mathbb{D}^n \rightharpoonup \mathbb{D} \text{ weakly in } (L^2(U))^{3\times 3}, \tag{3.3.3}$$

$$p_{\mathrm{f}}^n \to p_{\mathrm{f}} \text{ strongly in } L^2(U) \text{ and a.e. in } U, \tag{3.3.4}$$

$$\limsup_{n\to\infty} \int_U \mathbb{Z}^n : \mathbb{D}^n \le \int_U \mathbb{Z} : \mathbb{D}. \tag{3.3.5}$$

Then

$$(|\mathbb{Z}| - \tau(p_{\mathrm{f}}))^+ + ||\mathbb{D}|\mathbb{Z} - \tau(p_{\mathrm{f}})\mathbb{D}| = 0 \quad \text{a.e. in } U. \tag{3.3.6}$$

In addition, assume that $\{\mathbb{V}^n\}_{n=1}^{+\infty}$ is a sequence such that

$$\mathbb{V}^n = \left(1 - \frac{\delta_*}{|\mathbb{D}^n|}\right)^+ \mathbb{D}^n \quad \text{a.e. in } U, \tag{3.3.7}$$

fulfilling

$$\mathbb{V}^n \rightharpoonup \mathbb{V} \text{ weakly in } (L^2(U))^{3\times 3}, \tag{3.3.8}$$

$$\limsup_{n\to\infty} \int_U \mathbb{V}^n : \mathbb{D}^n \le \int_U \mathbb{V} : \mathbb{D}. \tag{3.3.9}$$

Then

$$\mathbb{V} = \left(1 - \frac{\delta_*}{|\mathbb{D}|}\right)^+ \mathbb{D} \quad \text{a.e. in } U. \tag{3.3.10}$$

Proof. First, note that by virtue of (3.3.4) and the Lipschitz-continuity of τ, it follows that

$$\tau(p_{\mathrm{f}}^n) \to \tau(p_{\mathrm{f}}) \text{ strongly in } L^2(U). \tag{3.3.11}$$

Now, as for any $\mathbb{A} \in (L^2(U))^{3\times 3}$, $\mathbb{A} \neq \mathbb{O}$, it holds

$$\left(\tau(p_{\mathrm{f}})\frac{\mathbb{D}^n}{|\mathbb{D}^n| + \frac{1}{n}} - \tau(p_{\mathrm{f}})\frac{\mathbb{A}}{|\mathbb{A}| + \frac{1}{n}}\right) : (\mathbb{D}^n - \mathbb{A}) \ge 0, \tag{3.3.12}$$

integrating this inequality over U, subtracting and adding $\mathbb{Z}^n$ and using (3.3.1), we get

$$\begin{aligned} &\int_U \left(\tau(p_{\mathrm{f}})\frac{\mathbb{D}^n}{|\mathbb{D}^n| + \frac{1}{n}} - \tau(p_{\mathrm{f}}^n)\frac{\mathbb{D}^n}{|\mathbb{D}^n| + \frac{1}{n}}\right) : (\mathbb{D}^n - \mathbb{A}) \\ &\qquad + \int_U \left(\mathbb{Z}^n - \tau(p_{\mathrm{f}})\frac{\mathbb{A}}{|\mathbb{A}| + \frac{1}{n}}\right) : (\mathbb{D}^n - \mathbb{A}) \ge 0. \end{aligned} \tag{3.3.13}$$

Taking limsup as $n \to \infty$ and employing the facts that the first integral converges to zero due to (3.3.11), the term $\frac{\mathbb{D}^n}{|\mathbb{D}^n|+\frac{1}{n}}$ is uniformly bounded in $(L^\infty(U))^{3\times 3}$ and the sequence $\mathbb{D}^n - \mathbb{A}$ is bounded in $(L^2(U))^{3\times 3}$, we obtain

$$\limsup_{n\to\infty} \int_U \left(\mathbb{Z}^n - \tau(p_\mathrm{f}) \frac{\mathbb{A}}{|\mathbb{A}| + \frac{1}{n}} \right) : (\mathbb{D}^n - \mathbb{A}) \geq 0. \tag{3.3.14}$$

Referring then to the convergences (3.3.2) and (3.3.3) and using also (3.3.5), we conclude that

$$\int_U \left(\mathbb{Z} - \tau(p_\mathrm{f}) \frac{\mathbb{A}}{|\mathbb{A}|} \right) : (\mathbb{D} - \mathbb{A}) \geq 0. \tag{3.3.15}$$

Now, for any $\delta > 0$, $\varepsilon \in (0, \delta)$ and for arbitrary matrices $\mathbb{C}$ and $\mathbb{B}_1$ bounded in $(L^2(U))^{3\times 3}$ and satisfying $|\mathbb{C}| \leq 1$ and $\mathbb{B}_1 \neq \mathbb{O}$, consider

$$\mathbb{A} := \mathbb{B}_1 \, \chi_{\{|\mathbb{D}|=0\}} + (\mathbb{D} - \varepsilon\mathbb{C}) \, \chi_{\{|\mathbb{D}|>\delta\}} + \mathbb{D} \, \chi_{\{0<|\mathbb{D}|\leq\delta\}}.$$

Note that such $\mathbb{A}$'s are non-zero in U. Inserting them into (3.3.15) we obtain

$$-\int_{\{|\mathbb{D}|=0\}} \left(\mathbb{Z} - \tau(p_\mathrm{f}) \frac{\mathbb{B}_1}{|\mathbb{B}_1|} \right) : \mathbb{B}_1 + \varepsilon \int_{\{|\mathbb{D}|>\delta\}} \mathbb{C} : \left(\mathbb{Z} - \tau(p_\mathrm{f}) \frac{\mathbb{D} - \varepsilon\mathbb{C}}{|\mathbb{D} - \varepsilon\mathbb{C}|} \right) \geq 0. \tag{3.3.16}$$

Letting first $\varepsilon \to 0$ in (3.3.16), we observe that

$$\int_{\{|\mathbb{D}|=0\}} \mathbb{Z} : \mathbb{B}_1 \leq \int_{\{|\mathbb{D}|=0\}} \tau(p_f)|\mathbb{B}_1| \tag{3.3.17}$$

for any $\mathbb{B}_1 \neq \mathbb{O}$. Consider, for any $a > 0$ and $\omega \subset U$, the matrix $\mathbb{B}_1$ of the form

$$\mathbb{B}_1 = a \, \mathbb{I} \chi_{\{(U\setminus\omega)\cup\{\mathbb{Z}=\mathbb{O}\}\}} + \frac{\mathbb{Z}}{|\mathbb{Z}|} \chi_{\{\omega\setminus\{\mathbb{Z}=\mathbb{O}\}\}}.$$

It then follows from (3.3.16) that

$$\int_{\{|\mathbb{D}|=0\}\cap\omega\cap\{\mathbb{Z}\neq\mathbb{O}\}} |\mathbb{Z}| \leq \int_{\{|\mathbb{D}|=0\}\cap\omega\cap\{\mathbb{Z}\neq\mathbb{O}\}} \tau(p_\mathrm{f}) + a\, C \int_{(U\setminus\omega)\cup\{\mathbb{Z}=\mathbb{O}\}} (\tau(p_\mathrm{f}) + |\mathbb{Z}|)$$

with C positive constant, which implies letting $a \to 0$

$$\int_{\{|\mathbb{D}|=0\}\cap\omega\cap\{\mathbb{Z}\neq\mathbb{O}\}} |\mathbb{Z}| \leq \int_{\{|\mathbb{D}|=0\}\cap\omega\cap\{\mathbb{Z}\neq\mathbb{O}\}} \tau(p_\mathrm{f}).$$

Since ω is arbitrary, we conclude that

$$|\mathbb{Z}| \leq \tau(p_\mathrm{f}) \text{ on the set } \{|\mathbb{D}| = 0\}. \tag{3.3.18}$$

Next, letting $|\mathbb{B}_1| \to 0$ in (3.3.16), employing (3.3.17), we get

$$\int_{\{|\mathbb{D}|>\delta\}} \mathbb{C} : \left(\mathbb{Z} - \tau(p_\mathrm{f}) \frac{\mathbb{D} - \varepsilon\mathbb{C}}{|\mathbb{D} - \varepsilon\mathbb{C}|} \right) \geq 0,$$

which, after letting $\varepsilon \to 0$, leads to

$$\int_{\{|\mathbb{D}|>\delta\}} \mathbb{C} : \left(\mathbb{Z} - \tau(p_{\mathrm{f}}) \frac{\mathbb{D}}{|\mathbb{D}|}\right) \geq 0.$$

Finally, letting $\delta \to 0$, we get, for arbitrary $\mathbb{C}$,

$$\int_{\{|\mathbb{D}|>0\}} \mathbb{C} : \left(\mathbb{Z} - \tau(p_{\mathrm{f}}) \frac{\mathbb{D}}{|\mathbb{D}|}\right) \geq 0.$$

This implies

$$\mathbb{Z} = \tau(p_{\mathrm{f}}) \frac{\mathbb{D}}{|\mathbb{D}|} \text{ when } |\mathbb{D}| \neq 0. \tag{3.3.19}$$

The latter and (3.3.18) coincide with (3.3.6).

It remains to prove (3.3.10), which however follows from standard Minty's argument. Indeed, by the monotonicity, we have

$$\limsup_{n\to\infty} \int_U \left(\mathbb{V}^n - \mathbb{A}\left(1 - \frac{\delta_*}{|\mathbb{A}|}\right)^+\right) : (\mathbb{D}^n - \mathbb{A}) \geq 0$$

for any $\mathbb{A} \in (L^2(U))^{3\times 3}$. By virtue of (3.3.9) and of convergences (3.3.8) and (3.3.3) we get

$$\int_U \left(\mathbb{V} - \mathbb{A}\left(1 - \frac{\delta_*}{|\mathbb{A}|}\right)^+\right) : (\mathbb{D} - \mathbb{A}) \geq 0.$$

Choosing $\mathbb{A} := \mathbb{D} \pm \varepsilon\mathbb{C}$, with arbitrary $\mathbb{C} \in (L^2(U))^{3\times 3}$ and $\varepsilon > 0$, and after the limit as $\varepsilon \to 0$ we obtain

$$\int_U \mathbb{C} : \left(\mathbb{V} - \mathbb{D}\left(1 - \frac{\delta_*}{|\mathbb{D}|}\right)^+\right) = 0$$

for any $\mathbb{C}$, which implies (3.3.10). □

Note that here we provided a proof of (3.3.6), which is simplified and shorter than the one given in Proposition 2.3.3 (or [3, Proposition 5.3]).

3.4 Approximations

In this section, we prepare all the needed tools in order to prove Theorem 3.2.1. For any $n \in \mathbb{N}$, we introduce the following approximating system supposed to be satisfied in Q_T

$$\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0, \\ &\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) G_n(|\boldsymbol{v}|^2) - \operatorname{div} \mathbb{S} + \nabla p = \boldsymbol{f}, \\ &\partial_t p_{\mathrm{f}} + \boldsymbol{v} \cdot \nabla p_{\mathrm{f}} - \Delta p_{\mathrm{f}} = -\operatorname{div} \boldsymbol{f} + \boldsymbol{v} \cdot \nabla p_s, \end{aligned} \tag{3.4.1}$$

where $G_n : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $G_n(u) = 1$ if $|u| \leq n$, $G_n(u) = 0$ if $|u| \geq 2n$ and $|G_n'| \leq \frac{2}{n}$. Next, we consider the following regularization of the constitutive equations (both in the bulk and on the boundary)

$$\begin{aligned} \mathbb{S} = \mathcal{S}_n(p_\mathrm{f}, \mathbb{D}\boldsymbol{v}) = \mathcal{Z}_n(p_\mathrm{f}, \mathbb{D}\boldsymbol{v}) + \left(1 - \frac{\delta_*}{|\mathbb{D}\boldsymbol{v}|}\right)^+ \mathbb{D}\boldsymbol{v}, \\ \text{where } \mathcal{Z}_n(p_\mathrm{f}, \mathbb{D}\boldsymbol{v}) := \tau(p_\mathrm{f}) \frac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}| + \frac{1}{n}} \text{ with } \tau(p_\mathrm{f}) = (p_s - p_\mathrm{f})^+ \text{ in } Q_T, \end{aligned} \tag{3.4.2}$$

$$\boldsymbol{s} = s_n(\boldsymbol{v}_\tau) = \zeta_n(\boldsymbol{v}_\tau) + \left(1 - \frac{\beta_*}{|\boldsymbol{v}_\tau|}\right)^+ \boldsymbol{v}_\tau, \tag{3.4.3}$$

$$\text{where } \zeta_n(\boldsymbol{v}_\tau) := s_* \frac{\boldsymbol{v}_\tau}{|\boldsymbol{v}_\tau| + \frac{1}{n}} \text{ on } \Sigma_T, \tag{3.4.4}$$

and we complete the problem with the boundary and the initial conditions

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \nabla p_\mathrm{f} \cdot \boldsymbol{n} = 0 \text{ on } \Sigma_T, \tag{3.4.5}$$

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \quad \text{and} \quad p_\mathrm{f}(0, \cdot) = p_0 \text{ in } \Omega. \tag{3.4.6}$$

Note that both mappings $\mathbb{D} \mapsto \mathcal{Z}_n(p_\mathrm{f}, \mathbb{D})$ and $\mathbb{D} \mapsto \left(1 - \frac{\delta_*}{|\mathbb{D}|}\right)^+ \mathbb{D}$ are monotone, i.e.

$$(\mathbb{Z} - \hat{\mathbb{Z}}) : (\mathbb{D} - \hat{\mathbb{D}}) \geq 0 \text{ for any } \mathbb{Z} = \mathcal{Z}_n(p_\mathrm{f}, \mathbb{D}), \hat{\mathbb{Z}} = \mathcal{Z}_n(p_\mathrm{f}, \hat{\mathbb{D}}), \tag{3.4.7}$$

see formula (2.3.8), and

$$(\mathbb{V} - \hat{\mathbb{V}}) : (\mathbb{D} - \hat{\mathbb{D}}) \geq 0 \text{ for any } \mathbb{V} = \left(1 - \frac{\delta_*}{|\mathbb{D}|}\right)^+ \mathbb{D}, \hat{\mathbb{V}} = \left(1 - \frac{\delta_*}{|\hat{\mathbb{D}}|}\right)^+ \hat{\mathbb{D}}, \tag{3.4.8}$$

see Lemma B.1 in [69]. Therefore, due to the presence of the truncation in the convective term and the introduced approximations in the constitutive equations, the existence of weak solutions to the system (3.4.1)–(3.4.6) can be proved through standard techniques of monotone operators, following also the spirit of the proof in Proposition 2.3.1. We enunciate the relevant result below and for the reader's convenience the proof can be found in Appendix A.1.

Proposition 3.4.1. Let $n \in \mathbb{N}$ be fixed. For any

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}, \; p_0 \in L^2(\Omega), \; \boldsymbol{f} \in (L^2(Q_T))^3 \text{ and } p_s \in L^5(Q_T),$$

there exists a weak solution to the problem (3.4.1)–(3.4.6). More precisely, for each $n \in \mathbb{N}$ there is a quadruplet $(\boldsymbol{v}, p_\mathrm{f}, \mathbb{S}, \boldsymbol{s}) := (\boldsymbol{v}^n, p_\mathrm{f}^n, \mathbb{S}^n, \boldsymbol{s}^n)$ such that

$$\boldsymbol{v} \in L^\infty(0, T; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0, T; W^{1,2}_{\boldsymbol{n},\mathrm{div}}), \quad \partial_t \boldsymbol{v} \in L^2(0, T; W^{-1,2}_{\boldsymbol{n},\mathrm{div}}), \tag{3.4.9}$$

$$p_\mathrm{f} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \quad \partial_t p_\mathrm{f} \in L^{\frac{4}{3}}(0, T; (W^{1,2}(\Omega))^*), \tag{3.4.10}$$

$$\mathbb{S} \in (L^2(Q_T))^{3\times 3}, \quad \boldsymbol{s} \in (L^{\frac{8}{3}}(\Sigma_T))^3, \tag{3.4.11}$$

satisfying, for all $\boldsymbol{w} \in L^2(0, T; W^{1,2}_{\boldsymbol{n},\mathrm{div}})$ and $z \in L^4(0, T; W^{1,2}(\Omega))$,

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle + \int_{Q_T} (\mathbb{S} : \mathbb{D}\boldsymbol{w} + G_n(|\boldsymbol{v}|^2) \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \boldsymbol{w}) + \int_{\Sigma_T} \boldsymbol{s} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{w}, \tag{3.4.12}$$

$$\int_0^T \langle \partial_t p_\mathrm{f}, z \rangle - \int_{Q_T} p_\mathrm{f} \boldsymbol{v} \cdot \nabla z + \int_{Q_T} \nabla p_\mathrm{f} \cdot \nabla z = \int_{Q_T} (\boldsymbol{f} \cdot \nabla z - p_s \boldsymbol{v} \cdot \nabla z), \tag{3.4.13}$$

where

$$\mathbb{S} = \mathcal{S}_n(p_{\mathrm{f}}, \mathbb{D}\boldsymbol{v}) \text{ a.e. in } Q_T, \tag{3.4.14}$$

$$\boldsymbol{s} = s_n(\boldsymbol{v}_\tau) \text{ a.e. in } \Sigma_T, \tag{3.4.15}$$

and

$$\lim_{t\to 0^+} \left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|p_{\mathrm{f}}(t) - p_0\|_2\right) = 0. \tag{3.4.16}$$

3.5 Proof of Theorem 3.2.1

The proof is split into the following steps.

Step 1. Approximations. From Proposition 3.4.1 and following the reconstruction of the pressure in the proof of Theorem 2.2.1, Step 2, we get for each $n \in \mathbb{N}$ the existence of $(\boldsymbol{v}^n, p_{\mathrm{f}}^n, p^n, \mathbb{S}^n, \boldsymbol{s}^n)$, with $p^n \in L^2(Q_T)$, satisfying

$$\begin{aligned} \int_0^T \langle \partial_t \boldsymbol{v}^n, \boldsymbol{w}\rangle + \int_{Q_T} (\mathbb{S}^n : \mathbb{D}\boldsymbol{w} + \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) G_n(|\boldsymbol{v}^n|^2) \cdot \boldsymbol{w}) + \int_{\Sigma_T} \boldsymbol{s}^n \cdot \boldsymbol{w}_{\boldsymbol{\tau}} \\ = \int_{Q_T} p^n \operatorname{div} \boldsymbol{w} + \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{w}, \end{aligned} \tag{3.5.1}$$

$$\int_0^T \langle \partial_t p_{\mathrm{f}}^n, z\rangle - \int_{Q_T} (p_{\mathrm{f}}^n \boldsymbol{v}^n) \cdot \nabla z + \int_{Q_T} \nabla p_{\mathrm{f}}^n \cdot \nabla z = \int_{Q_T} (\boldsymbol{f} - p_s \boldsymbol{v}^n) \cdot \nabla z, \tag{3.5.2}$$

with $\mathbb{S}^n, \boldsymbol{s}^n$ fulfilling (3.4.14), (3.4.15) respectively, and satisfying also (3.4.16).

Step 2. Uniform estimates with respect to n and limit as $n \to +\infty$. Setting $\boldsymbol{w} := \boldsymbol{v}^n$ in (3.5.1) and $z := p_{\mathrm{f}}^n$ in (3.5.2), following the analogous step as in the proof of Theorem 2.2.1, we obtain

$$\sup_n \left(\|\boldsymbol{v}^n\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|\mathbb{D}\boldsymbol{v}^n\|_{2,Q_T}\right) < +\infty, \tag{3.5.3}$$

$$\sup_n \left(\|p_{\mathrm{f}}^n\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p_{\mathrm{f}}^n\|_{2,Q_T}\right) < +\infty, \tag{3.5.4}$$

$$\sup_n \left(\|\boldsymbol{v}^n\|_{\frac{10}{3},Q_T} + \|p_{\mathrm{f}}^n\|_{\frac{10}{3},Q_T} + \|\boldsymbol{v}^n\|_{\frac{8}{3},\Sigma_T}\right) < +\infty, \tag{3.5.5}$$

$$\sup_n \left(\|\mathbb{Z}^n\|_{\frac{10}{3},Q_T} + \|\mathbb{V}^n\|_{2,Q_T} + \|\boldsymbol{s}^n\|_{\frac{8}{3},\Sigma_T}\right) < +\infty, \tag{3.5.6}$$

where we set $\mathbb{V}^n := \left(1 - \frac{\delta_*}{|\mathbb{D}\boldsymbol{v}^n|}\right)^+ \mathbb{D}\boldsymbol{v}^n$. Consequently, as $\sup_n \|G_n(|\boldsymbol{v}^n|^2)\|_{\infty,Q_T} \le 1$ by employing (3.5.3), (3.5.5) and Korn's inequality, it follows that

$$\sup_n \|G_n(|\boldsymbol{v}^n|^2) \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n)\|_{\frac{5}{4},Q_T} < +\infty. \tag{3.5.7}$$

Now, let us introduce

$$p_2^n := (-\Delta_N)^{-1}\left(G^n(|\boldsymbol{v}^n|^2) \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n)\right), \quad p_1^n := p^n - p_2^n,$$

then

$$\sup_n \left(\|p_2^n\|_{L^{\frac{5}{4}}(0,T;W^{1,\frac{5}{4}}(\Omega))} + \|p_1^n\|_{2,Q_T} \right) < +\infty, \tag{3.5.8}$$

and this implies that

$$\sup_n \|\partial_t \boldsymbol{v}^n\|_{(L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3)^*} < +\infty. \tag{3.5.9}$$

Analogously

$$\sup_n \|\partial_t p_{\mathrm{f}}^n\|_{L^{\frac{4}{3}}(0,T;(W^{1,2}(\Omega))^*)} < +\infty. \tag{3.5.10}$$

Then, there exist subsequences of $\{\boldsymbol{v}^n\}$, $\{p_{\mathrm{f}}^n\}$, $\{\mathbb{Z}^n\}$, $\{\mathbb{V}^n\}$, $\{\boldsymbol{s}^n\}$, $\{p_1^n\}$, $\{p_2^n\}$, which we do not relabel, that converge weakly and *-weakly in the corresponding function spaces. By virtue of the established limits, by the Aubin-Lions compactness lemma and the compact embedding of the Sobolev spaces into the space of traces, we also have

$$p_{\mathrm{f}}^n \to p_{\mathrm{f}} \text{ strongly in } L^q(Q_T) \text{ for all } q \in \left[1, \frac{10}{3}\right), \tag{3.5.11}$$

$$\boldsymbol{v}^n \to \boldsymbol{v} \text{ strongly in } (L^q(Q_T))^3 \text{ for all } q \in \left[1, \frac{10}{3}\right), \tag{3.5.12}$$

$$\boldsymbol{v}_\tau^n \to \boldsymbol{v}_\tau \text{ strongly in } (L^q(\Sigma_T))^3 \text{ for all } q \in \left[1, \frac{8}{3}\right). \tag{3.5.13}$$

Since

$$\|G_n(|\boldsymbol{v}^n|^2)\|_{\infty,Q_T} \le 1 \text{ and } G_n(|\boldsymbol{v}^n|^2) \to 1 \text{ strongly in } L^q(Q_T) \text{ for all } q \in [1, +\infty),$$

it follows from (3.5.12) that

$$G_n(|\boldsymbol{v}^n|^2) \operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \rightharpoonup \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \text{ weakly in } (L^{\frac{5}{4}}(Q_T))^3. \tag{3.5.14}$$

Finally, with the obtained convergences it is standard to prove, for all $\boldsymbol{w}$ belonging to $L^2(0,T;W^{1,2}_{\boldsymbol{n}}) \cap (L^5(Q_T))^3$ and for all $z \in L^4(0,T;W^{1,2}(\Omega))$, that

$$\begin{aligned} \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle + \int_{Q_T} (\mathbb{Z} + \mathbb{V}) : \mathbb{D}\boldsymbol{w} + \int_{Q_T} \boldsymbol{s} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : \mathbb{D}\boldsymbol{w} \\ = \int_{Q_T} p_1 \operatorname{div} \boldsymbol{w} - \int_{Q_T} \nabla p_2 \cdot \boldsymbol{w} + \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{w} \end{aligned} \tag{3.5.15}$$

and

$$\int_0^T \langle \partial_t p_{\mathrm{f}}, z \rangle - \int_{Q_T} p_{\mathrm{f}} \boldsymbol{v} \cdot \nabla z + \int_{Q_T} \nabla p_{\mathrm{f}} \cdot \nabla z = \int_{Q_T} (\boldsymbol{f} - p_s \boldsymbol{v}) \cdot \nabla z. \tag{3.5.16}$$

Step 3. Attainment of the constitutive equations on the boundary. Using that

$$\boldsymbol{s}^n \rightharpoonup \boldsymbol{s} \text{ weakly in } (L^{\frac{8}{3}}(\Sigma_T))^3$$

and (3.5.13), it easily follows

$$\limsup_{n \to +\infty} \int_{\Sigma_T} \boldsymbol{s}^n \cdot \boldsymbol{v}_\tau^n = \int_{\Sigma_T} \boldsymbol{s} \cdot \boldsymbol{v}_\tau.$$

Thus, a suitable adjustment of Proposition 3.3.1 implies that (3.2.4) is fulfilled.

Step 4. Attainment of the constitutive equations in the bulk. In order to employ Proposition 3.3.1 we need to prove the limsup property (3.3.5), but as the solution itself cannot be used as test function in (3.5.15), we follow the strategy as in Chapter 2 (or in [3]) and perform the L^∞-truncation method. To this aim, we introduce

$$\boldsymbol{w}^n := T_{\lambda_n}(\boldsymbol{v}^n - \boldsymbol{v}) := (\boldsymbol{v}^n - \boldsymbol{v}) \min\left\{1, \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}\right\}$$

where $\lambda^n \in [A, B]$ with $0 < A < B < \infty$ will be suitably chosen numbers independent of n, but depending on parameter N tending to $+\infty$, see details below. For the reader's convenience, we recall all the properties of $\boldsymbol{w}^n$ below,

$$\boldsymbol{w}^n \to 0 \text{ strongly in } (L^s(Q_T))^3 \text{ for every } s \in [1, +\infty), \tag{3.5.17}$$

$$\boldsymbol{w}^n \to 0 \text{ strongly in } (L^2(\Sigma_T))^3, \tag{3.5.18}$$

$$\boldsymbol{w}^n \rightharpoonup 0 \text{ weakly in } L^2(0, T; W^{1,2}_{\boldsymbol{n}}), \tag{3.5.19}$$

$$|\mathrm{div}\, \boldsymbol{w}^n| \leq \begin{cases} 0 & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| \leq \lambda^n \\ \dfrac{2\lambda^n \left(|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|\right)}{|\boldsymbol{v}^n - \boldsymbol{v}|} & \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n, \end{cases} \tag{3.5.20}$$

$$\nabla \boldsymbol{w}^n = \begin{cases} \nabla \boldsymbol{v}^n - \nabla \boldsymbol{v} \quad \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| \leq \lambda^n \\ \dfrac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}(\nabla \boldsymbol{v}^n - \nabla \boldsymbol{v}) - \lambda^n (\boldsymbol{v}^n - \boldsymbol{v}) \otimes \dfrac{(\nabla \boldsymbol{v}^n - \nabla \boldsymbol{v})(\boldsymbol{v}^n - \boldsymbol{v})}{|\boldsymbol{v}^n - \boldsymbol{v}|^3} \\ \qquad \text{if } |\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n. \end{cases} \tag{3.5.21}$$

Inserting $\boldsymbol{w}^n$ as test function in (3.5.1), using the properties of $\boldsymbol{w}^n$ we get (cfr. [3])

$$\limsup_{n\to\infty} \int_{Q_T} (\mathbb{Z}^n + \mathbb{V}^n) : \mathbb{D}\boldsymbol{w}^n \leq \limsup_{n\to\infty} \int_{Q_T} |p_1^n| |\,\mathrm{div}\, \boldsymbol{w}^n|. \tag{3.5.22}$$

Now, let us define

$$\overline{\mathbb{Z}} := \begin{cases} \mathbb{O} & \text{if } \mathbb{D}\boldsymbol{v} = \mathbb{O}, \\ \tau(p_{\mathrm{f}}) \dfrac{\mathbb{D}\boldsymbol{v}}{|\mathbb{D}\boldsymbol{v}|} & \text{if } \mathbb{D}\boldsymbol{v} \neq \mathbb{O}. \end{cases} \tag{3.5.23}$$

and

$$\overline{\mathbb{V}} := \left(1 - \frac{\delta_*}{|\mathbb{D}\boldsymbol{v}|}\right)^+ \mathbb{D}\boldsymbol{v}. \tag{3.5.24}$$

Employing (3.5.19) and (3.5.20), formula (3.5.22) can be rewritten as

$$\begin{aligned} &\limsup_{n\to\infty} \int_{Q_T} (\mathbb{Z}^n - \overline{\mathbb{Z}}) : \mathbb{D}\boldsymbol{w}^n + \int_{Q_T} (\mathbb{V}^n - \overline{\mathbb{V}}) : \mathbb{D}\boldsymbol{w}^n \\ &\leq \limsup_{n\to\infty} \int_{\{|\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n\}} |p_1^n| \left(|\nabla \boldsymbol{v}^n| + |\nabla \boldsymbol{v}|\right) \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}. \end{aligned} \tag{3.5.25}$$

Moving the part of the integral on the left-hand side on the set $\{|\boldsymbol{v}^n - \boldsymbol{v}| > \lambda^n\}$ to the right, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n\}} \Big((\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) + (\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \Big) \\ \le C \limsup_{n\to\infty} \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>\lambda^n\}} I^n \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|}\end{aligned} \tag{3.5.26}$$

where

$$I^n := |p_1^n|^2 + |\mathbb{Z}^n|^2 + |\overline{\mathbb{Z}}|^2 + |\mathbb{V}^n|^2 + |\overline{\mathbb{V}}|^2 + |\nabla \boldsymbol{v}^n|^2 + |\nabla \boldsymbol{v}|^2.$$

Note that it holds (see (2.4.60) or formula (6.60) in [3])

$$\Big((\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \Big)^- \to 0 \text{ strongly in } L^1(Q_T), \tag{3.5.27}$$

and analogously the monotonicity implies

$$\Big((\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \Big)^- \to 0 \text{ strongly in } L^1(Q_T). \tag{3.5.28}$$

Let $N \in \mathbb{N}$ and fix $A = N$, $B = N^{N+1}$. For $i \in \{1, ..., N\}$ let us define

$$Q_i^n := \{N^i < |\boldsymbol{v}^n - \boldsymbol{v}| \le N^{i+1}\}.$$

Since

$$\sum_{i=1}^{N} \int\limits_{Q_i^n} I^n \le C^*,$$

for every n there exists $i_n \in \{1, ..., N\}$ such that

$$\int\limits_{Q_{i_n}^n} I^n \le \frac{C^*}{N}.$$

Set $\lambda^n := N^{i_n}$, then it holds

$$\int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>\lambda^n\}} I^n \frac{\lambda^n}{|\boldsymbol{v}^n - \boldsymbol{v}|} = \int\limits_{Q_{i_n}^n} I^n \frac{N^{i_n}}{|\boldsymbol{v}^n - \boldsymbol{v}|} + \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>N^{i_n+1}\}} I^n \frac{N^{i_n}}{|\boldsymbol{v}^n - \boldsymbol{v}|} \le \frac{C^*}{N}$$

where we keep the symbol C^* for a different constant. The latter relation, (3.5.26), (3.5.27) and (3.5.28) give

$$\begin{aligned}\limsup_{n\to+\infty} \Bigg(\int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n\}} |(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})| \\ + \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|\le\lambda^n\}} |(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})| \Bigg) \le \frac{C^*}{N}.\end{aligned} \tag{3.5.29}$$

Using that

$$\begin{aligned}\int\limits_{Q_T} \sqrt{|(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} = \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|\le N\}} \sqrt{|(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} \\ + \int\limits_{\{|\boldsymbol{v}^n-\boldsymbol{v}|>N\}} \sqrt{|(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|}\end{aligned} \tag{3.5.30}$$

and

$$\int\limits_{Q_T} \sqrt{|(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} = \int\limits_{\{|\boldsymbol{v}^n - \boldsymbol{v}| \le N\}} \sqrt{|(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} + \int\limits_{\{|\boldsymbol{v}^n - \boldsymbol{v}| > N\}} \sqrt{|(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|}, \tag{3.5.31}$$

by Hölder's and Chebyshev's inequalities we obtain

$$\limsup_{n \to +\infty} \int\limits_{Q_T} \sqrt{|(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} \le \frac{2\overline{C}}{\sqrt{N}}, \tag{3.5.32}$$

and

$$\limsup_{n \to +\infty} \int\limits_{Q_T} \sqrt{|(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v})|} \le \frac{2\overline{C}}{\sqrt{N}}, \tag{3.5.33}$$

which means, by letting $N \to \infty$, that

$$(\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \to 0 \text{ a.e. in } Q_T,$$

$$(\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \to 0 \text{ a.e. in } Q_T.$$

Egoroff's theorem then gives that for all $\varepsilon > 0$ there exists $U \subset Q_T$, $|Q_T \setminus U| \le \varepsilon$ such that

$$\int\limits_U (\mathbb{Z}^n - \overline{\mathbb{Z}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \to 0, \tag{3.5.34}$$

$$\int\limits_U (\mathbb{V}^n - \overline{\mathbb{V}}) : (\mathbb{D}\boldsymbol{v}^n - \mathbb{D}\boldsymbol{v}) \to 0. \tag{3.5.35}$$

We conclude, thanks to the weak convergences of $\mathbb{Z}^n, \mathbb{V}^n, \mathbb{D}\boldsymbol{v}^n$ respectively to $\mathbb{Z}, \mathbb{V}, \mathbb{D}\boldsymbol{v}$ that

$$\lim_{n \to \infty} \int\limits_U \mathbb{Z}^n : \mathbb{D}\boldsymbol{v}^n = \lim_{n \to \infty} \int\limits_U \mathbb{Z}^n : \mathbb{D}\boldsymbol{v} = \int\limits_U \mathbb{Z} : \mathbb{D}\boldsymbol{v}$$

and

$$\lim_{n \to \infty} \int\limits_U \mathbb{V}^n : \mathbb{D}\boldsymbol{v}^n = \lim_{n \to \infty} \int\limits_U \mathbb{V}^n : \mathbb{D}\boldsymbol{v} = \int\limits_U \mathbb{V} : \mathbb{D}\boldsymbol{v}.$$

Finally, all assumptions of Convergence Lemma 3.4.1 are fulfilled, thus (3.1.2) holds almost everywhere in U. Since $|Q_T \setminus U| \le \varepsilon$ we can let $\varepsilon \to 0$ and obtain that (3.1.2) holds almost everywhere in Q_T. Theorem 3.2.1 is proved.

4. Analysis of planar flows of viscoelastic fluids of Burgers type

4.1 Formulation of the problem

Let us recall from Section 1.2 that we are interested in the system (1.2.10)–(1.2.14). In order to set-up a meaningful initial- and boundary-value problem, we complete the system (1.2.10)–(1.2.14) with the no-slip boundary condition for the velocity and with the initial conditions for $\boldsymbol{v}$, $\mathbb{B}_1$ and $\mathbb{B}_2$, i.e.,

$$\boldsymbol{v} = \mathbf{0} \qquad \text{on } \Sigma_T := (0,T)\times\partial\Omega, \tag{4.1.1}$$

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 \quad \text{and} \quad \mathbb{B}_i(0,\cdot) = \mathbb{B}_{i_0} (i=1,2) \qquad \text{in } \Omega, \tag{4.1.2}$$

where $\boldsymbol{v}_0$ and $\mathbb{B}_{i_0}$ are given functions satisfying suitable compatibility conditions specified later.

Our general goal concerning the problem (1.2.10)–(1.2.14), (4.1.1)– (4.1.2) is to develop a robust PDE analysis for this problem. As a starting point, we aim at developing a sounding existence theory for arbitrary regular enough data (domain, time interval, boundary and initial data, external forces, material coefficients). The results presented here represent the first step towards this goal.

To put our investigation in proper context, we first show what kind of a priori estimates one can expect for the problem (1.2.10)–(1.2.14), (4.1.1)–(4.1.2). For simplicity, let us suppose that $\boldsymbol{f} \equiv \mathbf{0}$ in Q_T. Taking the scalar product of (1.2.11) with $\boldsymbol{v}$, integrating the result over Ω, using the integration by parts, the constraint (1.2.10), the boundary condition (4.1.1) and the symmetry of $\mathbb{T}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\boldsymbol{v}|^2 + \int_\Omega \mathbb{T}:\mathbb{D} = 0. \tag{4.1.3}$$

Next, we take the scalar product of (1.2.12) with $(\mathbb{I}-\mathbb{B}_i^{-1})$ and integrate the result over Ω. Using again (1.2.10) and (4.1.1), and also (1.2.5), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\operatorname{tr}\mathbb{B}_i - d - \ln\det\mathbb{B}_i) + \int_\Omega -2\mathbb{D}:\mathbb{B}_i + \frac{1}{\tau_i}\int_\Omega |\mathbb{B}_i^{\frac{2-\lambda_i}{2}}(\mathbb{I}-\mathbb{B}_i^{-1})|^2 = 0. \tag{4.1.4}$$

Finally, taking the scalar product of (1.2.13) with $\mathbb{D}$ and using the fact that $\operatorname{div}\boldsymbol{v} = 0$, we get

$$2\nu\int_\Omega |\mathbb{D}|^2 + \int_\Omega \sum_{i=1}^2 G_i\mathbb{B}_i:\mathbb{D} = \int_\Omega \mathbb{T}:\mathbb{D}. \tag{4.1.5}$$

Thus, multiplying (4.1.4) by $G_i/2$ followed by taking the sum over $i=1,2$, and adding the result to (4.1.3), using also (4.1.5), we deduce (after integration with respect to time) that

$$\begin{aligned}
\frac{1}{2}\int\limits_{\Omega}&\left(\rho|\boldsymbol{v}(t)|^2+\sum_{i=1}^{2}G_i(\operatorname{tr}\mathbb{B}_i(t)-d-\ln\det\mathbb{B}_i(t))\right)\\
&+\int\limits_0^t\int\limits_{\Omega}\left(2\nu|\mathbb{D}|^2+\sum_{i=1}^{2}\frac{G_i}{2\tau_i}|\mathbb{B}_i^{\frac{2-\lambda_i}{2}}(\mathbb{I}-\mathbb{B}_i^{-1})|^2\right)\\
=\frac{1}{2}\int\limits_{\Omega}&\left(\rho|\boldsymbol{v}_0|^2+\sum_{i=1}^{2}G_i(\operatorname{tr}\mathbb{B}_{i_0}-d-\ln\det\mathbb{B}_{i_0})\right).
\end{aligned} \tag{4.1.6}$$

This identity leads to a priori estimates for the non-negative terms at the left-hand side of (4.1.6) provided that the right-hand side of (4.1.6) is finite, which one achieves by putting the proper requirements on the initial data accordingly. Our aim is to establish the existence of a global weak solution to (1.2.10)–(1.2.14) for a large class of λ_1, λ_2 and we want to perform the analysis based just on the information coming from (4.1.6) since one may not hope, in particular in higher dimensions, for better information.

Let us note that even in the case when $G_2 = 0$ (or $\mathbb{B}_2 \equiv \mathbb{O}$), there are only few studies regarding the long-time and large-data existence theory. Lions and Masmoudi [71] analyzed the system (1.2.10)–(1.2.13) with $\lambda_1 = 1$, $G_2 = 0$, but instead of the convective derivative $\overset{\triangledown}{\mathbb{B}}_1$ in (1.2.12), they considered the term ($\mathbb{B} := \mathbb{B}_1$)

$$\partial_t\mathbb{B}+\sum_{j=1}^{d}\boldsymbol{v}_j\partial_{x_j}\mathbb{B}-\mathbb{W}\mathbb{B}-\mathbb{B}\mathbb{W}^T,$$

where $\mathbb{W} := \frac{1}{2}(\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^T)$. This type of observer-invariant time derivative simplifies the analysis significantly. Its form is however very restrictive from the physical point of view. (Also, this type of time derivative does not come out naturally from the thermodynamical approach presented in [42].) Later on, Masmoudi [2], carrying on some ideas developed in Hu and Lelièvre [72] that are close to the thermodynamical set-up mentioned above, presented the theorem regarding the long-time and large-data existence of weak solutions to the system (1.2.10)–(1.2.13) with $G_2 = 0$ and $\lambda := \lambda_1 = 0$, $\mathbb{B} := \mathbb{B}_1 = \mathbb{F}_1\mathbb{F}_1^T$. This leads to the so called Giesekus model, see [73]. In [2], Masmoudi outlined the proof of the weak sequential stability of hypothetical weak solutions in function spaces coming from a priori estimates. Appropriate approximative problems are however not constructed and consequently the proof of the existence of a weak solution is not given. Despite bringing a couple of original ideas, Masmoudi [2] did not attempt to provide the rigorous mathematical background to the statements connected with the presence of nonlinear terms $\nabla\boldsymbol{v}\mathbb{B}$ in the governing equations, which in our opinion requires some additional work. Further, it seems that the proof of the property $\det\mathbb{F} > 0$ (the requirement (1.2.14)), presented in [2], contains several inconsistencies at some crucial points. We are not aware of any other results for viscoelastic rate-type fluid models fulfilling even the equations (1.2.10)–(1.2.13) with $G_2 = 0$. In particular, the case of the Oldroyd-B model ($\lambda_1 = 1$) remains still open.

As said above, our goal is to develop a robust mathematical theory for (1.2.10)–(1.2.14), (4.1.1)–(4.1.2) for a large range of λ_1, λ_2. However, the only a priori estimate (4.1.6) may not be even sufficient to guarantee the L^1-integrability of all terms in the definition of a weak solution to (1.2.10)–(1.2.13). The most critical terms are $\nabla\boldsymbol{v}\mathbb{B}_i$. Thanks to (4.1.6), $\nabla\boldsymbol{v} \in (L^2(Q_T))^{2\times 2}$. Thus, to make sure that the terms $\nabla\boldsymbol{v}\mathbb{B}_i$ are at least L^1-integrable, we need $\mathbb{B}_i \in (L^2(Q_T))^{2\times 2}$. On the other hand looking at (4.1.6), one may deduce that $\mathbb{B}_i \in (L^{2-\lambda_i}(Q_T))^{2\times 2}$. Hence, we naturally get a restriction on λ_i, namely $\lambda_i \le 0$.

We also restrict ourselves to the analysis of two-dimensional flows. This limitation is needed essentially in the existence proof for the identification of weak limits of $\nabla \boldsymbol{v}\mathbb{B}_i$. Since we work with weak limits in L^1 or weak* limits in the sense of measures, the technique presented in this study requires the validity of the energy *equality* (4.1.3). However, within the context of weak solutions, this is known to be true only in two spatial dimensions. We believe that, in the three-dimensional setting, this crucial step can be overcome using recently developed techniques based on the notion of the limits in biting sense. However, the application of, for example, the L^∞ or the parabolic Lipschitz truncation of the velocity fields is much more complicated than in the case of the Bingham type models. In the case of the Bingham type models studied in Chapters 2 and 3 the dependence between the deviatoric part of the Cauchy stress tensor and the symmetric part of the velocity gradient is monotone, which simplifies the analysis significantly. (Also, in three spatial dimensions we would have to proceed more carefully while proving the property $\det \mathbb{F} > 0$. Here we use the (in)equality $|\mathrm{adj}\ \mathbb{A}| \le |\mathbb{A}|$ valid for all matrices $\mathbb{A} \in \mathbb{R}^{2\times 2}$, which does not hold true in three dimensions, see the inequality (4.5.89) in Subsection 4.5.5.)

To conclude, in this study, we prove *the long-time and large-data existence of weak solutions to* (1.2.10)–(1.2.14), (4.1.1)–(4.1.2) *with* $\lambda_i = 0$, $i = 1, 2$, *in two spatial dimensions.* The precise formulation of this statement is given in the next section.

The rest of this chapter is organized as follows. In Section 4.2 we fix notations needed to define the concept of weak solutions to the studied problem and we formulate the main result. In Section 4.3 we present two assertions used in the existence proof. In Sections 4.4–4.6 we prove the main result for the Giesekus model. In Section 4.7 we complete the proof of the main result by considering the full problem: the mixture/combination of two Giesekus viscoelastic models.

4.2 Formulation of the main result

Let us note that throughout this chapter we use the notation $\mathbb{D} := \frac{\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T}{2}$. For clarity, we start with recalling the problem that we investigate here (achieved by taking $\lambda_1 = \lambda_2 = 0$ in (1.2.10)–(1.2.14), (4.1.1)–(4.1.2) and restricting ourselves to the planar flows, i.e. setting $d = 2$). For given $T \in (0, \infty)$, $\Omega \subset \mathbb{R}^2$ being a domain (open, bounded and connected set), initial data $\boldsymbol{v}_0$, $\mathbb{F}_{1_0}$, $\mathbb{F}_{2_0}$, the external body forces $\boldsymbol{f}$ and material parameters ρ, ν, G_1, G_2, τ_1, τ_2, being all positive, we look for $\boldsymbol{v}$, p, $\mathbb{F}_1$, $\mathbb{F}_2$ satisfying in $Q_T := (0, T) \times \Omega$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.2.1}$$

$$\rho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) + \nabla p - 2\nu \operatorname{div} \mathbb{D} - \sum_{i=1}^{2} G_i \operatorname{div} \mathbb{B}_i = \rho \boldsymbol{f}, \tag{4.2.2}$$

$$\partial_t \mathbb{B}_i + \operatorname{div}\ (\mathbb{B}_i \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{B}_i - \mathbb{B}_i (\nabla \boldsymbol{v})^T + \frac{1}{\tau_i}\left(\mathbb{B}_i^2 - \mathbb{B}_i\right) = \mathbb{O}, \qquad (i = 1, 2) \quad (4.2.3)$$

$$\mathbb{B}_i = \mathbb{F}_i \mathbb{F}_i^T, \quad (i = 1, 2) \quad (4.2.4)$$

$$\det \mathbb{F}_i > 0, \qquad (i = 1, 2) \quad (4.2.5)$$

together with

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T := (0,T) \times \partial\Omega, \tag{4.2.6}$$

and

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \quad \mathbb{F}_i(0,\cdot) = \mathbb{F}_{i_0} \quad \text{and} \quad \mathbb{B}_i(0,\cdot) = \mathbb{B}_{i_0} := \mathbb{F}_{i_0}\mathbb{F}_{i_0}^T \quad (i = 1,2) \quad \text{in } \Omega. \tag{4.2.7}$$

Motivated by the a priori estimate (4.1.6), we introduce the following concept of weak solution.

Definition 2 (weak solution)**.** *Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary, let $T \in (0,\infty)$ be arbitrary. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ and $\mathbb{F}_{i_0} \in (L^2(\Omega))^{d\times d}$ fulfill $\det \mathbb{F}_{i_0} > 0$ almost everywhere in Ω, $\ln\det\mathbb{F}_{i_0} \in L^1(\Omega)$, $i = 1,2$, and $\boldsymbol{f} \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$. A triple $[\boldsymbol{v},\mathbb{F}_1,\mathbb{F}_2]$ is called a weak solution to the problem (4.2.1)–(4.2.7) if $(i = 1,2)$*

$$\boldsymbol{v} \in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}}), \quad \partial_t\boldsymbol{v} \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*), \tag{4.2.8}$$

$$\mathbb{F}_i \in C([0,T];(L^2(\Omega))^{2\times2}) \cap (L^4(Q_T))^{2\times2}, \partial_t\mathbb{F}_i \in L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times2})^*), \tag{4.2.9}$$

$$\det\mathbb{F}_i > 0 \text{ a.e. in } Q_T, \quad \ln\det\mathbb{F}_i \in L^\infty([0,T];L^1(\Omega)), \tag{4.2.10}$$

if for $\mathbb{B}_i := \mathbb{F}_i\mathbb{F}_i^T$ it holds

$$\mathbb{B}_i \in C([0,T];(L^1(\Omega))^{2\times2}) \cap (L^2(Q_T))^{2\times2}, \partial_t\mathbb{B}_i \in L^1(0,T;((W^{1,4}(\Omega))^{2\times2})^*), \tag{4.2.11}$$

if, for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,4}(\Omega))^{2\times2}$ and almost all $t \in (0,T)$, the following identities hold true

$$\rho\langle\partial_t\boldsymbol{v},\boldsymbol{w}\rangle - \rho\int_\Omega(\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} + 2\nu\int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \sum_{i=1}^2\int_\Omega G_i\mathbb{B}_i : \nabla\boldsymbol{w} = \langle\boldsymbol{f},\boldsymbol{w}\rangle, \tag{4.2.12}$$

$$\begin{aligned}\langle\partial_t\mathbb{B}_i,\mathbb{A}\rangle - \int_\Omega(\mathbb{B}_i\otimes\boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega(\nabla\boldsymbol{v}\,\mathbb{B}_i) : \mathbb{A} - \int_\Omega(\mathbb{B}_i(\nabla\boldsymbol{v})^T) : \mathbb{A}& \\ + \int_\Omega \frac{1}{\tau_i}\left(\mathbb{B}_i^2 - \mathbb{B}_i\right) : \mathbb{A} = 0,&\end{aligned} \tag{4.2.13}$$

and if the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_{i_0}$ and $\mathbb{B}_{i_0}$ are attained in the sense

$$\lim_{t\to0+}\left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{F}_i(t) - \mathbb{F}_{i_0}\|_2 + \|\mathbb{B}_i(t) - \mathbb{B}_{i_0}\|_1\right) = 0. \tag{4.2.14}$$

The main result of this chapter is stated in the following theorem.

Theorem 4.2.1. *For arbitrary data of* (4.2.1)–(4.2.7) *satisfying the assumptions in Definition 2, there exists a weak solution to* (4.2.1)–(4.2.7).

The proof of Theorem 4.2.1 is given in Sections 4.4–4.7. In the following Section 4.3, we formulate two assertions, which will be employed in the proof of Theorem 4.2.1. In Sections 4.4–4.6 we provide a detailed proof of Theorem 4.2.1 restricting ourselves to the case $G_1 = 1$ and $G_2 = 0$, i.e. we do not consider the equation for $\mathbb{B}_2$. In Section 4.7, we conclude the result for arbitrary G_1, $G_2 > 0$. Finally for the sake of completeness, in Appendix A.2 we provide an auxiliary result concerning the properties of the evolutionary Stokes systems, which plays an important role in our proof (in order to derive uniform estimates for the pressure); while it is also well understood nowadays.

Also, to simplify the presentation, we set for simplicity the positive constants ρ, 2ν, τ_1, τ_2 to be equal to one and the external forces $\boldsymbol{f}$ to be identically equal to zero. As one may check, if we took ρ, 2ν, τ_1, $\tau_2 > 0$ and $\boldsymbol{f} \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$ arbitrary, the proof of the existence of weak solutions to the system (4.2.1)–(4.2.7) would be made essentially in the same way. We do not set G_1, G_2 to be equal to one since we study the system with two different relaxation mechanisms with different weights.

4.3 Two useful lemmas

In this section we state two lemmas useful in the proof of Theorem 4.2.1, both of them hold true for any $d \in \mathbb{N}$, $d \geq 2$. The first lemma is the Friedrichs lemma on commutators. The second lemma concerns the monotonicity of one special matrix function.

For any $\boldsymbol{z} \in \mathbb{R}^m$, where $m = d$ or $m = d+1$, and $h \in L^1_{loc}(\mathbb{R}^m)$, we set

$$h_\delta(\boldsymbol{z}) := \int_{\mathbb{R}^m} \omega_\delta(\boldsymbol{z}-\boldsymbol{y})h(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{4.3.1}$$

where ω_δ is the standard mollifying kernel supported in the ball of radius $\delta > 0$.

Lemma 4.3.1 (Friedrichs lemma on commutators). Let $m = d$ or $m = d+1$. Let $p, q, r \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $f \in L^p(\mathbb{R}^m)$, $\boldsymbol{g} \in (L^q(\mathbb{R}^m))^d$, $\nabla \boldsymbol{g} \in (L^q(\mathbb{R}^m))^{d\times d}$. Then

$$\|\operatorname{div}(f_\delta \boldsymbol{g}) - \operatorname{div}(f\boldsymbol{g})_\delta\|_{L^r(\mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^m)} \left(\|\boldsymbol{g}\|_{(L^q(\mathbb{R}^m))^d} + \|\nabla \boldsymbol{g}\|_{(L^q(\mathbb{R}^m))^{d\times d}}\right).$$

Moreover, if $r < \infty$, then

$$\operatorname{div}(f_\delta \boldsymbol{g}) - \operatorname{div}(f\boldsymbol{g})_\delta \to 0 \quad \text{strongly in } L^r(\mathbb{R}^m) \text{ as } \delta \to 0_+.$$

Proof. See [74], or [75, Lemma 11.12 and Corollary 11.3]. □

Lemma 4.3.2 (Monotonicity of one matrix function). The function $S : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ defined as

$$S(\mathbb{X}) := \mathbb{X}\mathbb{X}^T\mathbb{X} \quad \text{for all } \mathbb{X} \in \mathbb{R}^{d\times d}$$

is monotone, i.e.

$$(S(\mathbb{X}) - S(\mathbb{Y})) : (\mathbb{X} - \mathbb{Y}) \geq 0 \quad \text{for all } \mathbb{X}, \mathbb{Y} \in \mathbb{R}^{d\times d}. \tag{4.3.2}$$

Proof. In the whole proof the symbol δ_{ij}, where $i, j \in \{1, ..., d\}$, stands for the Kronecker symbol, i.e.

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

The i, j component of any matrix $\mathbb{X} \in \mathbb{R}^{d\times d}$ is denoted either as X_{ij}, or as $(\mathbb{X})_{ij}$. For brevity in the computations the Einstein summation convention is used, i.e. all sum indices are omitted.

For all matrices $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{d\times d}$ it holds

$$S(\mathbb{X}) - S(\mathbb{Y}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(S\left(\mathbb{Y} + s(\mathbb{X} - \mathbb{Y})\right)\right)\,\mathrm{d}s \tag{4.3.3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(S\left(\mathbb{Y} + s(\mathbb{X} - \mathbb{Y})\right)\right) = \frac{\partial S\left(\mathbb{K}(s)\right)}{\partial\left(\mathbb{K}(s)\right)_{ab}} L_{ab}, \tag{4.3.4}$$

where $\mathbb{K}(s) := \mathbb{Y} + s(\mathbb{X} - \mathbb{Y})$, $\mathbb{L} := \mathbb{X} - \mathbb{Y}$. Collecting (4.3.3) and (4.3.4), one concludes

$$(S(\mathbb{X}) - S(\mathbb{Y})) : (\mathbb{X} - \mathbb{Y}) = \int_0^1 \frac{\partial\left(S\left(\mathbb{K}(s)\right)\right)_{ij}}{\partial\left(\mathbb{K}(s)\right)_{ab}} L_{ab} L_{ij}\,\mathrm{d}s,$$

thus in order to prove (4.3.2) it suffices to show

$$\frac{\partial\left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} L_{ab} L_{ij} \geq 0 \quad \text{for all } \mathbb{K}, \mathbb{L} \in \mathbb{R}^{d\times d}. \tag{4.3.5}$$

We write

$$\begin{aligned}
\frac{\partial\left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} &= \frac{\partial}{\partial K_{ab}}(K_{im}K_{km}K_{kj}) \\
&= \delta_{ia}\delta_{mb}K_{km}K_{kj} + \delta_{ak}\delta_{mb}K_{im}K_{kj} + \delta_{ak}\delta_{bj}K_{im}K_{km} \\
&= \delta_{ia}K_{kb}K_{kj} + \delta_{ak}K_{ib}K_{kj} + \delta_{bj}K_{im}K_{am},
\end{aligned}$$

and finally

$$\begin{aligned}
\frac{\partial\left(S(\mathbb{K})\right)_{ij}}{\partial K_{ab}} L_{ab}L_{ij} &= \left(\delta_{ia}K_{kb}K_{kj} + \delta_{ak}K_{ib}K_{kj} + \delta_{bj}K_{im}K_{am}\right) L_{ab}L_{ij} \\
&= K_{kb}K_{kj}L_{ib}L_{ij} + K_{ib}K_{kj}L_{kb}L_{ij} + K_{im}K_{am}L_{aj}L_{ij} \\
&= (\mathbb{K}\mathbb{L}^T) : (\mathbb{K}\mathbb{L}^T) + (\mathbb{K}\mathbb{L}^T) : (\mathbb{K}\mathbb{L}^T)^T + (\mathbb{K}^T\mathbb{L}) : (\mathbb{K}^T\mathbb{L}) \\
&\geq \frac{1}{2}\left(|\mathbb{K}\mathbb{L}^T|^2 - |(\mathbb{K}\mathbb{L}^T)^T|^2\right) + |\mathbb{K}^T\mathbb{L}|^2 = |\mathbb{K}^T\mathbb{L}|^2 \geq 0,
\end{aligned}$$

where the first inequality follows from the Young inequality. The lemma is proved. □

4.4 System with evolutionary equation for the tensor $\mathbb{F}$

As introduced above, first we prove Theorem 4.2.1 for $G_1 = 1$ and $G_2 = 0$, and since we do not consider the equation for $\mathbb{B}_2$, we simply abbreviate $\mathbb{B} := \mathbb{B}_1$ and also $\mathbb{F} := \mathbb{F}_1$. Let us recall that we consider the external forces $\boldsymbol{f}$ to be identically equal to zero. Carrying on the ideas developed by Masmoudi [2], we start with the setting containing the evolutionary equation for the tensor $\mathbb{F}$ instead of the evolutionary equation for $\mathbb{B} = \mathbb{F}\mathbb{F}^T$. More specifically, we look for $[\boldsymbol{v}, p, \mathbb{F}]$ satisfying in Q_T the following system of equations:

$$\operatorname{div}\boldsymbol{v} = 0, \tag{4.4.1}$$

$$\partial_t\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) + \nabla p - \operatorname{div}\mathbb{D} - \operatorname{div}\mathbb{F}\mathbb{F}^T = \mathbf{0}, \tag{4.4.2}$$

$$\partial_t\mathbb{F} + \operatorname{div}(\mathbb{F}\otimes\boldsymbol{v}) - (\nabla\boldsymbol{v})\mathbb{F} + \frac{1}{2}(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) = \mathbb{O}, \tag{4.4.3}$$

$$\det\mathbb{F} > 0, \tag{4.4.4}$$

together with the boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Sigma_T, \tag{4.4.5}$$

and with the initial conditions

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 \quad \text{and} \quad \mathbb{F}(0,\cdot) = \mathbb{F}_0 \quad \text{in } \Omega. \tag{4.4.6}$$

Formally, multiplying (4.4.3) by $\mathbb{F}^T$ from right, multiplying the transpose of (4.4.3) by $\mathbb{F}$ from left and summing the results, we obtain the equation

$$\partial_t(\mathbb{F}\mathbb{F}^T) + \operatorname{div}\Big((\mathbb{F}\mathbb{F}^T)\otimes \boldsymbol{v}\Big) - \nabla\boldsymbol{v}(\mathbb{F}\mathbb{F}^T) - (\mathbb{F}\mathbb{F}^T)(\nabla\boldsymbol{v})^T + (\mathbb{F}\mathbb{F}^T)^2 - \mathbb{F}\mathbb{F}^T = \mathbb{O}. \tag{4.4.7}$$

Setting $\mathbb{B} := \mathbb{F}\mathbb{F}^T$, $G_1 = 1$, $G_2 = 0$, the equation (4.4.7) coincides with (4.2.3).

To start with (4.4.1)–(4.4.6) brings several advantages. An obvious but also important one is that the tensor $\mathbb{F}$ has better integrability than $\mathbb{B} = \mathbb{F}\mathbb{F}^T$. Formally, taking the scalar product of (4.4.2) with $\boldsymbol{v}$, and the scalar product of (4.4.3) with $\mathbb{F}$, adding them together and integrating over Q_T, we deduce (for details see the rigorous computations in Appendix A.3) the following a priori estimate (valid for all $t \in (0,T)$):

$$\|\boldsymbol{v}(t)\|_2^2 + \|\mathbb{F}(t)\|_2^2 + \int_0^T \Big(\|\nabla\boldsymbol{v}\|_2^2 + \|\mathbb{F}\|_4^4\Big) \le C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2).$$

Better integrability of $\mathbb{F}$ expands the set of admissible test functions in the corresponding equations, which increases the chance to obtain, for example, weak sequential stability of these solutions, or to make a short proof of the property $\det\mathbb{F} > 0$ (the condition (4.2.5)). Also, once we rigorously proceed from (4.4.3) to (4.4.7), we immediately obtain $\mathbb{B}$ of the form $\mathbb{B} = \mathbb{F}\mathbb{F}^T$ (the condition (4.2.4)), satisfying (4.2.3).

We state the existence result for the system (4.4.1)–(4.4.6) in the following theorem. Let us note that this represents the key result from which we also read the validity of Theorem 4.2.1.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary and let $T \in (0,\infty)$. Let $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_0 \in (L^2(\Omega))^{2\times 2}$, $\det\mathbb{F}_0 > 0$ a.e. in Ω and $\ln\det\mathbb{F}_0 \in L^1(\Omega)$. Then there exists a couple $[\boldsymbol{v},\mathbb{F}]$ fulfilling*

$$\boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}), \quad \partial_t\boldsymbol{v} \in L^2(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*),$$
$$\mathbb{F} \in C([0,T]; (L^2(\Omega))^{2\times 2}) \cap (L^4(Q_T))^{2\times 2}, \quad \partial_t\mathbb{F} \in L^{\frac{4}{3}}\Big(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\Big),$$
$$\det\mathbb{F} > 0 \text{ a.e. in } Q_T,$$

satisfying, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and almost all $t \in (0,T)$,

$$\langle\partial_t\boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla\boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla\boldsymbol{w} + \int_\Omega (\mathbb{F}\mathbb{F}^T) : \nabla\boldsymbol{w} = 0, \tag{4.4.8}$$

$$\langle\partial_t\mathbb{F}, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}\otimes\boldsymbol{v}) : \nabla\mathbb{A} - \int_\Omega ((\nabla\boldsymbol{v})\mathbb{F}) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \mathbb{A} = 0, \tag{4.4.9}$$

and attaining the initial conditions in the sense

$$\lim_{t\to 0+} (\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{F}(t) - \mathbb{F}_0\|_2) = 0. \tag{4.4.10}$$

The following section is devoted to the proof of Theorem 4.4.1. In Subsection 4.5.1, we introduce the parabolic ε-approximation and establish the existence of its weak solution. Then in the rest of Section 4.5 we take the limit $\varepsilon \to 0_+$ and prove Theorem 4.4.1.

4.5 Proof of Theorem 4.4.1

4.5.1 Parabolic ε-approximations

We start with the system approximating (4.4.1)–(4.4.6), where we add to the right-hand side of the equation (4.4.3) the term representing stress diffusions. The system, where $\varepsilon \in (0,1)$ is arbitrary and all the equations are supposed to be satisfied in Q_T, reads as follows:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.5.1}$$

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \operatorname{div} \mathbb{D} - \operatorname{div}(\mathbb{F}\mathbb{F}^T) = \mathbf{0}, \tag{4.5.2}$$

$$\partial_t \mathbb{F} + \operatorname{div}(\mathbb{F} \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{F} + \frac{1}{2}\left(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}\right) = \varepsilon \Delta \mathbb{F}. \tag{4.5.3}$$

The system is completed with the boundary conditions

$$\boldsymbol{v} = \mathbf{0} \quad \text{and} \quad (\nabla \mathbb{F})\boldsymbol{n} = \mathbb{O} \quad \text{on } \Sigma_T, \tag{4.5.4}$$

and with the initial conditions

$$\boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0 \quad \text{and} \quad \mathbb{F}(0,\cdot) = \mathbb{F}_0 \quad \text{in } \Omega. \tag{4.5.5}$$

Let us note that the functions $\boldsymbol{v}_0$, $\mathbb{F}_0$ introduced in (4.5.5) coincide with the functions $\boldsymbol{v}_0$, $\mathbb{F}_0$ introduced in Section 4.2 and Section 4.4.

Let us introduce the advantages of our choice of approximations. First, the presence of the term $\varepsilon\Delta\mathbb{F}$ provides the estimate

$$\varepsilon \|\nabla \mathbb{F}\|_{2,Q_T}^2 \leq C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{4.5.6}$$

The estimate (4.5.6) (together with the uniform bounds of $\mathbb{F}$ and $\partial_t \mathbb{F}$ in appropriate norms and the Aubin-Lions compactness lemma) makes the proof of the existence of weak solutions to (4.5.1)–(4.5.5) relatively simple (the system (4.5.1)–(4.5.3) is close to an advection-diffusion equation). Next, considering the sequence $\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_\varepsilon\}$ of weak solutions to (4.5.1)–(4.5.5) and letting $\varepsilon \to 0_+$, we deduce that the weak limits of the sequences $\{\boldsymbol{v}_\varepsilon\}, \{\mathbb{F}_\varepsilon\}$ are weak solutions to (4.4.1)–(4.4.6) provided that the sequence $\{\mathbb{F}_\varepsilon\}$ is compact in $(L^2(Q_T))^{2\times 2}$. The proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ is the most complicated part of the proof of Theorem 4.4.1. However, with the introduced approximations, it is not much more complicated than the proof of the weak sequential stability of (hypothetical) weak solutions to (4.4.1)–(4.4.6). The only difference is that without the presence of the supplementary stress diffusion term $\varepsilon\Delta\mathbb{F}_\varepsilon$ (it is present in (4.5.3), but not in (4.4.3)) the key relation (4.5.57) proved below would hold true with equality. However, the achieved inequality in (4.5.57) does not complicate further computations.

Since the parabolic approximation is not the main part of the paper and since the existence of solutions to very similar problems in the context of problems of viscoelasticity with stress diffusion were established e.g. in [76, 77, 78, 79] we do not provide the detailed proof here, but the interested reader can find it in Appendix A.3 (see Proposition A.3.1). However, for clarity, we recall that for each $\varepsilon > 0$ there exists a couple $(\boldsymbol{v}_\varepsilon, \mathbb{F}_\varepsilon)$ fulfilling

$$\boldsymbol{v}_\varepsilon \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}), \tag{4.5.7}$$

$$\partial_t \boldsymbol{v}_\varepsilon \in L^2(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*), \tag{4.5.8}$$

$$\mathbb{F}_\varepsilon \in C_{weak}([0,T]; (L^2(\Omega))^{2\times 2}) \cap L^2(0,T; (W^{1,2}(\Omega))^{2\times 2}), \tag{4.5.9}$$

$$\partial_t \mathbb{F}_\varepsilon \in L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right), \tag{4.5.10}$$

and satisfying, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and almost all $t \in (0,T)$ (we denote $\mathbb{D}_\varepsilon := \frac{1}{2}\left(\nabla \boldsymbol{v}_\varepsilon + (\nabla \boldsymbol{v}_\varepsilon)^T\right)$),

$$\langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D}_\varepsilon : \nabla \boldsymbol{w} + \int_\Omega (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : \nabla \boldsymbol{w} = 0, \tag{4.5.11}$$

$$\begin{aligned}\langle \partial_t \mathbb{F}_\varepsilon, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v}_\varepsilon)\mathbb{F}_\varepsilon) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon) : \mathbb{A} \\ + \varepsilon \int_\Omega \nabla \mathbb{F}_\varepsilon : \nabla \mathbb{A} = 0,\end{aligned} \tag{4.5.12}$$

with the initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_0$ fulfilled in the sense

$$\lim_{t\to 0+} (\|\boldsymbol{v}_\varepsilon(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{F}_\varepsilon(t) - \mathbb{F}_0\|_2) = 0. \tag{4.5.13}$$

In addition, we have the following uniform bounds:

$$\begin{aligned}\sup_{t\in(0,T)} \left(\|\boldsymbol{v}_\varepsilon(t)\|_2^2 + \|\mathbb{F}_\varepsilon(t)\|_2^2\right) + \|\nabla \boldsymbol{v}_\varepsilon\|_{2,Q_T}^2 + \|\mathbb{F}_\varepsilon\|_{4,Q_T}^4 + \varepsilon\|\nabla \mathbb{F}_\varepsilon\|_{2,Q_T}^2 \\ \le \tilde{C}(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2),\end{aligned} \tag{4.5.14}$$

$$\|\partial_t \boldsymbol{v}_\varepsilon\|_{L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)} + \|\partial_t \mathbb{F}_\varepsilon\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \le \overline{C}(T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{4.5.15}$$

In the rest of this section, we let $\varepsilon \to 0_+$ and prove Theorem 4.4.1.

4.5.2 Limit $\varepsilon \to 0_+$

The uniform estimates (4.5.14) and (4.5.15) imply the existence of $\boldsymbol{v}$, $\mathbb{F}$ fulfilling the following convergence results (for suitable subsequences of $\{\boldsymbol{v}_\varepsilon\}$, $\{\mathbb{F}_\varepsilon\}$, which we do not relabel):

$$\boldsymbol{v}_\varepsilon \rightharpoonup^* \boldsymbol{v} \quad \text{weakly-* in } L^\infty(0,T; L^2_{\boldsymbol{n},\mathrm{div}}), \tag{4.5.16}$$

$$\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \tag{4.5.17}$$

$$\partial_t \boldsymbol{v}_\varepsilon \rightharpoonup \partial_t \boldsymbol{v} \quad \text{weakly in } L^2\left(0,T; (W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), \tag{4.5.18}$$

$$\mathbb{F}_\varepsilon \rightharpoonup^* \mathbb{F} \quad \text{weakly-* in } L^\infty\left(0,T; (L^2(\Omega))^{2\times 2}\right), \tag{4.5.19}$$

$$\mathbb{F}_\varepsilon \rightharpoonup \mathbb{F} \quad \text{weakly in } (L^4(Q_T))^{2\times 2}, \tag{4.5.20}$$

$$\varepsilon \nabla \mathbb{F}_\varepsilon \to \mathbb{O} \quad \text{strongly in } (L^2(Q_T))^{2\times 2\times 2}, \tag{4.5.21}$$

$$\partial_t \mathbb{F}_\varepsilon \rightharpoonup \partial_t \mathbb{F} \quad \text{weakly in } L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right). \tag{4.5.22}$$

Since $\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\partial_t \boldsymbol{v} \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, $\mathbb{F} \in L^\infty(0,T;(L^2(\Omega))^{2\times 2})$ and $\partial_t \mathbb{F} \in L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)$, the functions $\boldsymbol{v}$, $\mathbb{F}$ satisfy

$$\boldsymbol{v} \in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}}), \tag{4.5.23}$$

$$\mathbb{F} \in C_{weak}\left([0,T];(L^2(\Omega))^{2\times 2}\right). \tag{4.5.24}$$

Next, to identify the nonlinearities involving $\boldsymbol{v}$, we can employ (4.5.17), (4.5.18) and the Aubin–Lions compactness lemma, to get

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } (L^q(Q_T))^2 \text{ for all } q \in [1,4). \tag{4.5.25}$$

Then the weak convergences (4.5.17), (4.5.20) together with the strong convergence (4.5.25) yield

$$\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} \qquad \text{weakly in } (L^2(Q_T))^{2\times 2}, \tag{4.5.26}$$

$$\mathbb{F}_\varepsilon \otimes \boldsymbol{v}_\varepsilon \rightharpoonup \mathbb{F} \otimes \boldsymbol{v} \qquad \text{weakly in } (L^2(Q_T))^{2\times 2\times 2}. \tag{4.5.27}$$

There are however nonlinearities involving $\mathbb{F}$ and $\nabla \boldsymbol{v}$ that cannot be handled by the convergence results (4.5.16)–(4.5.22) and (4.5.25). Using the notation $f(a_\varepsilon) \rightharpoonup^{(*)} \overline{f(a)}$ weakly (or weakly-*) in some space X if $a_\varepsilon \rightharpoonup a$ in some space Y and $\{f(a_\varepsilon)\}$ is uniformly bounded in X, it follows from the above uniform estimates and convergences that (for suitable subsequences)

$$\nabla \boldsymbol{v}_\varepsilon \mathbb{F}_\varepsilon \rightharpoonup \overline{(\nabla \boldsymbol{v})\mathbb{F}} \qquad \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, \tag{4.5.28}$$

$$\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \rightharpoonup \overline{\mathbb{F}\mathbb{F}^T} \qquad \text{weakly in } (L^2(Q_T))^{2\times 2}, \tag{4.5.29}$$

$$|\mathbb{F}_\varepsilon|^2 \rightharpoonup \overline{|\mathbb{F}|^2} \qquad \text{weakly in } L^2(Q_T), \tag{4.5.30}$$

$$\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon \rightharpoonup \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} \qquad \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, \tag{4.5.31}$$

$$|\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T|^2 \rightharpoonup^* \overline{|\mathbb{F}\mathbb{F}^T|^2} \qquad \text{weakly-* in } \mathcal{M}(\overline{Q_T}), \tag{4.5.32}$$

$$\nabla \boldsymbol{v}_\varepsilon \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \rightharpoonup^* \overline{\nabla \boldsymbol{v}\mathbb{F}\mathbb{F}^T} \qquad \text{weakly-* in } (\mathcal{M}(\overline{Q_T}))^{2\times 2}, \tag{4.5.33}$$

$$|\mathbb{D}_\varepsilon|^2 \rightharpoonup^* \overline{|\mathbb{D}|^2} \qquad \text{weakly-* in } \mathcal{M}(\overline{Q_T}). \tag{4.5.34}$$

These convergence results, when applied to (4.5.11) and (4.5.12), suffice to conclude that, for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and almost all $t \in (0,T)$,

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \int_\Omega \overline{\mathbb{F}\mathbb{F}^T} : \nabla \boldsymbol{w} = 0, \tag{4.5.35}$$

$$\langle \partial_t \mathbb{F}, \mathbb{A}\rangle - \int_\Omega (\mathbb{F} \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega \overline{(\nabla \boldsymbol{v})\mathbb{F}} : \mathbb{A} + \frac{1}{2}\int_\Omega (\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} - \mathbb{F}) : \mathbb{A} = 0. \tag{4.5.36}$$

In addition, almost identically as in the preceeding section, we can prove the attainment of the initial conditions (4.4.10) and thus, we omit the details here. It remains to prove the strong continuity of $\mathbb{F}$ in time and to identify the weak limits of nonlinear terms.

4.5.3 Strong continuity of $\mathbb{F}$ in time

The objective of this part is to strengthen (4.5.24), more precisely to prove

$$\mathbb{F} \in C([0,T];(L^2(\Omega))^{2\times 2}). \tag{4.5.37}$$

Recalling that

$$\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \quad \mathbb{F} \in (L^4(Q_T))^{2\times 2}, \\ \overline{\nabla \boldsymbol{v}\mathbb{F}}, \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} \in (L^{\frac{4}{3}}(Q_T))^{2\times 2}, \tag{4.5.38}$$

we extend $\boldsymbol{v}$ by zero and $\mathbb{F}$ by the identity tensor outside of Ω. Let $\delta > 0$ be arbitrary. Then we test (4.5.36) by $\mathbb{A}(\cdot) := \omega_\delta(\boldsymbol{x} - \cdot)\mathbb{I}$, where $\boldsymbol{x} \in \Omega$ is fixed, ω_δ is the standard mollifying kernel with respect to the spatial variable, see (4.3.1), and $\mathbb{I}$ is the identity tensor. Multiplying the result by $\phi(t)\tilde{\mathbb{A}}(\boldsymbol{x})$, where $\phi \in C_c^\infty((0,T))$ and $\tilde{\mathbb{A}} \in (C_c^\infty(\mathbb{R}^2))^{2\times 2}$ are arbitrary and integrating it with respect to $[t, \boldsymbol{x}]$ over $(0,T) \times \mathbb{R}^2$, using also (4.5.38), standard properties of mollification and the du Bois-Reymond lemma, we obtain

$$\partial_t \mathbb{F}_\delta \in L^{\frac{4}{3}}(0,T;(C^1(\overline{\Omega}))^{2\times 2})$$

and

$$\partial_t \mathbb{F}_\delta = -\operatorname{div}(\mathbb{F} \otimes \boldsymbol{v})_\delta + (\overline{\nabla \boldsymbol{v}\mathbb{F}})_\delta - \frac{1}{2}\Big((\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}})_\delta - \mathbb{F}_\delta\Big) \quad \text{a.e. in } Q_T. \tag{4.5.39}$$

Taking the scalar product of (4.5.39) with $2\mathbb{F}_\delta$ and integrating the result over $(t_0, t_1) \times \Omega$, we get

$$\|\mathbb{F}_\delta(t_1)\|_2^2 - \|\mathbb{F}_\delta(t_0)\|_2^2 + 2\int_{t_0}^{t_1}\int_\Omega \Big(\operatorname{div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - (\overline{\nabla \boldsymbol{v}\mathbb{F}})_\delta\Big) : \mathbb{F}_\delta \\ + \int_{t_0}^{t_1}\int_\Omega \Big((\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}})_\delta - \mathbb{F}_\delta\Big) : \mathbb{F}_\delta = 2\int_{t_0}^{t_1}\int_\Omega \mathbb{E}_\delta : \mathbb{F}_\delta, \tag{4.5.40}$$

where

$$\mathbb{E}_\delta := \operatorname{div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - \operatorname{div}(\mathbb{F} \otimes \boldsymbol{v})_\delta.$$

Then we let $\delta \to 0_+$ in (4.5.40). Employing (4.5.38), Lemma 4.3.1 implies

$$\mathbb{E}_\delta(t) \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(\Omega))^{2\times 2} \text{ for a.a. } t \in (0,T) \tag{4.5.41}$$

and (using also the Young inequality)

$$\|\mathbb{E}_\delta(t)\|_{\frac{4}{3}}^{\frac{4}{3}} \le \|\mathbb{F}(t)\|_4^{\frac{4}{3}}\|\boldsymbol{v}(t)\|_{1,2}^{\frac{4}{3}} \le \|\mathbb{F}(t)\|_4^4 + \|\boldsymbol{v}(t)\|_{1,2}^2 \quad \text{for a.a. } t \in (0,T).$$

The application of the Lebesgue dominated convergence theorem on (4.5.41) with the majorant $\|\mathbb{F}\|_4^4 + \|\boldsymbol{v}\|_{1,2}^2$ integrable over $(0,T)$ (the integrability over $(0,T)$ follows from (4.5.38)) then leads to

$$\mathbb{E}_\delta \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}. \tag{4.5.42}$$

Hence letting $\delta \to 0_+$ in (4.5.40), using the integration by parts, (4.5.38), (4.5.42) and standard properties of mollifying, we obtain

$$\|\mathbb{F}(t_1)\|_2^2 - \|\mathbb{F}(t_0)\|_2^2 = 2\int_{t_0}^{t_1}\int_\Omega (\overline{\nabla \boldsymbol{v}\mathbb{F}}) : \mathbb{F} - \int_{t_0}^{t_1}\int_\Omega \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F} + \int_{t_0}^{t_1} \|\mathbb{F}\|_2^2. \tag{4.5.43}$$

By (4.5.38) the terms $|\mathbb{F}|^2$, $\overline{(\nabla \boldsymbol{v})\mathbb{F}} : \mathbb{F}$ and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}$ are integrable over Q_T. Hence it follows from (4.5.43) that

$$\lim_{t_1 \to t_0} \|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(t_0)\|_2^2 \qquad \text{for all } t_0 \in (0,T),$$
$$\lim_{t_1 \to 0+} \|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(0)\|_2^2, \quad \lim_{t_1 \to T-} \|\mathbb{F}(t_1)\|_2^2 = \|\mathbb{F}(T)\|_2^2.$$

These identities combined with (4.5.24) are equivalent to (4.5.37).

4.5.4 Compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$

Our aim is to show that $\overline{\mathbb{F}\mathbb{F}^T} = \mathbb{F}\mathbb{F}^T$, $\overline{\nabla \boldsymbol{v}\mathbb{F}} = \nabla \boldsymbol{v}\mathbb{F}$ and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} = \mathbb{F}\mathbb{F}^T\mathbb{F}$ in (4.5.35) and (4.5.36). Since $\nabla \boldsymbol{v}_\varepsilon \rightharpoonup \nabla \boldsymbol{v}$ weakly in $(L^2(Q_T))^{2\times 2}$ by (4.5.17) and $\mathbb{F}_\varepsilon \rightharpoonup \mathbb{F}$ weakly in $(L^4(Q_T))^{2\times 2}$ by (4.5.20), we see that it is sufficient, for the above identifications, to prove the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$, which is the main result of this section.

Let us start with the observation

$$\lim_{\varepsilon \to 0+} \|\mathbb{F}_\varepsilon - \mathbb{F}\|_{2,Q_T}^2 = \lim_{\varepsilon \to 0+} \int_{Q_T} \left(|\mathbb{F}_\varepsilon|^2 - 2\mathbb{F}_\varepsilon : \mathbb{F} + |\mathbb{F}|^2\right) = \int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right),$$

where the last equality follows from (4.5.20) and (4.5.30). This observation reduces the proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ to proving

$$\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2 \quad \text{a.e. in } Q_T. \tag{4.5.44}$$

Road map of the proof

We first present heuristic arguments how to prove (4.5.44) pointing out the main difficulties. Following the approach outlined by Masmoudi, we *formally* scalarly multiply (4.5.3) by $\mathbb{F}_\varepsilon$ and (4.4.3) by $\mathbb{F}$, make the difference of the resulting identities and let $\varepsilon \to 0_+$. After the calculation, which incorporates, among other things, the use of the evolutionary equations for $\boldsymbol{v}_\varepsilon$ and $\boldsymbol{v}$, we arrive at the inequality formally written as

$$\partial_t(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2) + \operatorname{div}\left((\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\ \boldsymbol{v}\right) \le L\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right), \tag{4.5.45}$$

where L is a fixed function. Let us note that the quantity $\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2$ is nonnegative. The inequality (4.5.45) may seem to be prepared for applying the Gronwall lemma and concluding (4.5.44). However, this conclusion is not straightforward unless $L \in L^\infty(Q_T)$, $\int_\Omega(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)$ belongs to $C([0,T])$ and $\overline{|\mathbb{F}|^2}(0,\cdot) = |\mathbb{F}|^2(0,\cdot)$ almost everywhere in Ω, about which we have no information (we do not even know whether $\overline{|\mathbb{F}|^2}$ is weakly continuous with respect to time). Deriving the inequality (4.5.45) in virtue of the approach described above is also a non-trivial task and requires some new techniques. For example, in order to avoid the obstacles connected with the presence of nonlinear terms $\overline{|\mathbb{F}\mathbb{F}^T|^2}$, coming from (4.5.3) (with $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$) formally multiplied by $\mathbb{F}_\varepsilon$ and limited as $\varepsilon \to 0_+$, and $\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}$, coming from (4.4.3) formally multiplied by $\mathbb{F}$, we show that the difference $\overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}$ is nonnegative in $\mathcal{M}(\overline{Q_T})$ employing the monotonicity of the matrix function $S(\mathbb{X}) := \mathbb{X}\mathbb{X}^T\mathbb{X}$ for all $\mathbb{X} \in \mathbb{R}^{2\times 2}$, introduced in Lemma 4.3.2. Last but not least, in order to obtain a version of the inequality (4.5.45), from which we will be capable of concluding (4.5.44), we need (due to the presence of the terms $\overline{\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)}$, comming from (4.5.3) multiplied by $\mathbb{F}_\varepsilon$ and limited as $\varepsilon \to 0_+$, and $\overline{\nabla \boldsymbol{v}\mathbb{F}} : \mathbb{F}$, comming from (4.4.3) multiplied by $\mathbb{F}$) to use the evolutionary equations for $\boldsymbol{v}_\varepsilon$ and $\boldsymbol{v}$ tested by functions that are not divergence free. This requires to reconstruct the pressures p_ε, p and show the convergence of p_ε to p in a suitable sense. As this is a kind of a more general tool, which might have further applications (as one may check, we can replace $\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T$ and $\overline{\mathbb{F}\mathbb{F}^T}$ acting in Proposition 4.5.1 by any $\mathbb{H}_\varepsilon$ converging weakly to $\mathbb{H}$ in $L^2(0,T;(L^2_{loc}(\Omega))^{2\times 2})$), we introduce the result on the reconstruction of the pressures and their convergence before we provide the proof of (4.5.44) itself.

Reconstruction of the pressures and their convergence

For any $\tilde{\Omega} \in C^{0,1}$, $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$, we use the following notation

$$W_0^{1,2}(\tilde{\Omega}) := \{u \in W^{1,2}(\tilde{\Omega}),\ u = 0 \text{ on } \partial\tilde{\Omega}\},$$
$$\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}} := \{\boldsymbol{u} \in (W_0^{1,2}(\tilde{\Omega}))^2,\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \tilde{\Omega}\}.$$

Proposition 4.5.1. Let $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$, $\tilde{\Omega} \in C^\infty$. Then for every $\varepsilon \in (0,1)$ there exists p_ε of the form $p_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon}$, where

$$p_{1,\varepsilon} \in L^2(0,T;W^{2,2}(\tilde{\Omega})), \tag{4.5.46}$$
$$p_{2,\varepsilon} \in L^2((0,T)\times\tilde{\Omega}), \tag{4.5.47}$$
$$\partial_t\left(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}\right) \in L^2\left(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*\right) \tag{4.5.48}$$

and, for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and almost all $t \in (0,T)$, it holds

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \boldsymbol{w}\rangle = \int_{\tilde{\Omega}} (\mathbb{G}_\varepsilon : \nabla \boldsymbol{w}) + \int_{\tilde{\Omega}} p_{2,\varepsilon} \operatorname{div}\boldsymbol{w}, \tag{4.5.49}$$

where $\mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T$. Next, there exists p of the form $p = p_1 + p_2$, where

$$p_1 \in L^2(0,T;W^{2,2}(\tilde{\Omega})), \tag{4.5.50}$$
$$p_2 \in L^2((0,T)\times\tilde{\Omega}), \tag{4.5.51}$$
$$\partial_t\left(\boldsymbol{v} + \nabla p_1\right) \in L^2\left(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*\right) \tag{4.5.52}$$

and, for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and almost all $t \in (0,T)$, it holds

$$\langle \partial_t(\boldsymbol{v} + \nabla p_1), \boldsymbol{w}\rangle = \int_{\tilde{\Omega}} (\mathbb{G} : \nabla \boldsymbol{w}) + \int_{\tilde{\Omega}} p_2 \operatorname{div}\boldsymbol{w}, \tag{4.5.53}$$

where $\mathbb{G} := (\boldsymbol{v} \otimes \boldsymbol{v}) - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}$. Moreover, we have the following convergence results as $\varepsilon \to 0_+$

$$p_{1,\varepsilon} \to p_1 \qquad \text{strongly in } L^2(0,T;W^{2,2}_{loc}(\tilde{\Omega})), \tag{4.5.54}$$
$$p_{2,\varepsilon} \rightharpoonup p_2 \qquad \text{weakly in } L^2((0,T)\times\tilde{\Omega}). \tag{4.5.55}$$

In addition, the functions $\nabla p_{1,\varepsilon}$ and ∇p_1 belong to $C([0,T];(L^2(\tilde{\Omega}))^2)$, and

$$\nabla p_{1,\varepsilon}(0,\cdot) = \nabla p_1(0,\cdot) \quad \text{a.e. in } \tilde{\Omega}. \tag{4.5.56}$$

Proof. The proof of the first eight statements, i.e. (4.5.46)–(4.5.53), follows almost step by step the proof presented in [80]. However, for the sake of completeness we present the full proof of the proposition in Appendix A.2. □

Rigorous proof of compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times2}$ is split into the following three key steps.

Step 1 - Differential (in)equalities for $\overline{|\mathbb{F}|^2}$ and $|\mathbb{F}|^2$. In this step, we shall show that, for all nonnegative $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$, the following relations hold true:

$$-\int\limits_{Q_T} \overline{|\mathbb{F}|^2}\partial_t\varphi - \int\limits_{\Omega} |\mathbb{F}_0|^2\varphi(0) - \int\limits_{Q_T} \left(\overline{|\mathbb{F}|^2\boldsymbol{v}}\right)\cdot\nabla\varphi - 2\left\langle \overline{\nabla\boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)}, \varphi\right\rangle_{\{\mathcal{M}(\overline{Q_T}),C(\overline{Q_T})\}} + \left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2}, \varphi\right\rangle_{\{\mathcal{M}(\overline{Q_T}),C(\overline{Q_T})\}} - \int\limits_{Q_T} \overline{|\mathbb{F}|^2}\varphi \le 0 \tag{4.5.57}$$

and

$$-\int\limits_{Q_T} |\mathbb{F}|^2\,\partial_t\varphi - \int\limits_{\Omega} |\mathbb{F}_0|^2\varphi(0) - \int\limits_{Q_T} \left(|\mathbb{F}|^2\boldsymbol{v}\right)\cdot\nabla\varphi - 2\int\limits_{Q_T} \overline{\nabla\boldsymbol{v}\mathbb{F}} : (\varphi\mathbb{F}) + \int\limits_{Q_T} \left(\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F} - |\mathbb{F}|^2\right)\varphi = 0. \tag{4.5.58}$$

Let us note that in (4.5.57) and (4.5.58) all differential operators act on the test functions φ. In further computations it enables us to extend all functions acting in the difference between (4.5.57) and (4.5.58) by zero in $(-\infty,0)\times(\mathbb{R}^2\setminus\Omega)$ and to mollify the equations over time and space such that the terms $\int_\Omega(\overline{|\mathbb{F}|^2}(t,\cdot) - |\mathbb{F}|^2(t,\cdot))_\delta$ tend to zero as $\delta\to 0_+$ and $t\to 0_-$. As we will see, this approach eliminates the obstacles connected with the lack of information on the time continuity of $\overline{|\mathbb{F}|^2}$. Moreover, in (4.5.57) there is no more the term containing $\varepsilon\Delta\mathbb{F}_\varepsilon : \mathbb{F}_\varepsilon$, and this is the reason why in (4.5.57) the inequality sign appears.

Step 2 - Differential inequality for $\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2$. In this step, we shall deduce (4.5.45), which in a rigorous form can be written as

$$-\int\limits_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\partial_t\varphi - \int\limits_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\boldsymbol{v}\cdot\nabla\varphi \le \int\limits_{Q_T} L\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi, \tag{4.5.59}$$

where $L \in L^2(Q_T)$ is a fixed function specified below and $\varphi \in C_c^\infty((-\infty,T)\times\Omega)$ nonnegative is arbitrary.

Step 3 - Renormalisation of (4.5.59) **and final conclusion.** In this step, we renormalise (4.5.59) and then finally conclude (4.5.44). Let us note that due to the renormalisation introduced below, we do not need the function L to be in $L^\infty(Q_T)$, it suffices $L\in L^2(Q_T)$. Also, in order to prove (4.5.44), we need to apply the fact that $\mathbb{F}_\varepsilon(0,\cdot) = \mathbb{F}(0,\cdot) = \mathbb{F}_0$ almost everywhere in Ω in an effective way, which we would not be able to do without the renormalisation.

Ad Step 1: Proof of (4.5.57)–(4.5.58)**.** First let us extend $\boldsymbol{v}_\varepsilon$ by zero and $\mathbb{F}_\varepsilon$ continuously with respect to $(W^{1,2}(\mathbb{R}^2))^{2\times2}$ norm outside of Ω. For a fixed $\delta_0 > 0$ let us introduce the notation $\Omega_{\delta_0} := \{\boldsymbol{x}\in\Omega; \mathrm{dist}(\boldsymbol{x},\partial\Omega) > \delta_0\}$. Let $\delta\in(0,\delta_0)$. Throughout Step 1, in order to avoid writing too many lower indices, we denote the standard mollification of a function $g \in L^1_{loc}(\mathbb{R}^2)$ with respect to the spatial variable as g^δ (instead of g_δ). Since $\boldsymbol{v}_\varepsilon$ belongs to $C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}})$ and

$\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$, it follows from (4.5.12) and from standard properties of mollifying that $\partial_t \mathbb{F}^\delta_\varepsilon \in L^{\frac{4}{3}}(0,T;(C^2(\overline{\Omega_{\delta_0}}))^{2\times 2})$, and, almost everywhere in $(0,T)\times\Omega_{\delta_0}$, we have

$$2\partial_t \mathbb{F}^\delta_\varepsilon + 2\,\mathrm{div}\left(\mathbb{F}^\delta_\varepsilon \otimes \boldsymbol{v}_\varepsilon\right) - 2\left(\nabla \boldsymbol{v}_\varepsilon \mathbb{F}_\varepsilon\right)^\delta + \left(\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon\right)^\delta - 2\varepsilon \Delta \mathbb{F}^\delta_\varepsilon = 2\mathbb{E}^\delta_\varepsilon$$

with

$$\mathbb{E}^\delta_\varepsilon := \mathrm{div}(\mathbb{F}^\delta_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \mathrm{div}(\mathbb{F}_\varepsilon \otimes \boldsymbol{v}_\varepsilon)^\delta.$$

Let us multiply this identity scalarly by $\mathbb{F}^\delta_\varepsilon \varphi$, where $\varphi \in C_c((-\infty,T)\times\Omega_{\delta_0})$ nonnegative is arbitrary, and integrate the result over Q_T. Using the integration by parts and the property $\mathrm{div}\,\boldsymbol{v}_\varepsilon = 0$, we get (let us note that $\mathbb{F}^\delta_\varepsilon \in C([0,T];(L^2(\Omega_{\delta_0}))^{2\times 2})$ as $\mathbb{F}^\delta_\varepsilon \in L^4(0,T;(C^2(\overline{\Omega}))^{2\times 2})$ and, as mentioned above, $\partial_t \mathbb{F}^\delta_\varepsilon \in L^{\frac{4}{3}}(0,T;(C^2(\overline{\Omega_{\delta_0}}))^{2\times 2})$)

$$\begin{aligned}
-\int_{Q_T} |\mathbb{F}^\delta_\varepsilon|^2(\partial_t\varphi) - \int_\Omega |\mathbb{F}^\delta_\varepsilon(0)|^2\varphi(0) - \int_{Q_T} (|\mathbb{F}^\delta_\varepsilon|^2 \boldsymbol{v}_\varepsilon)\cdot\nabla\varphi - 2\int_{Q_T} \left(\nabla \boldsymbol{v}_\varepsilon \mathbb{F}_\varepsilon\right)^\delta : (\varphi \mathbb{F}^\delta_\varepsilon) \\
+ \int_{Q_T} \left(\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}_\varepsilon\right)^\delta : (\varphi\mathbb{F}^\delta_\varepsilon) + 2\varepsilon \int_{Q_T} |\nabla \mathbb{F}^\delta_\varepsilon|^2\varphi - |\mathbb{F}^\delta_\varepsilon|^2\Delta\varphi \\
= 2\int_{Q_T} \mathbb{E}^\delta_\varepsilon : (\varphi \mathbb{F}^\delta_\varepsilon).
\end{aligned} \tag{4.5.60}$$

Now let us take the limit $\delta \to 0_+$. Since $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v}_\varepsilon \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, we can use Lemma 4.3.1 to deduce for almost all $t\in(0,T)$

$$\varphi \mathbb{E}^\delta_\varepsilon(t) \to \mathbb{O} \quad \text{strongly in } (L^{\frac{4}{3}}(\Omega))^{2\times 2} \tag{4.5.61}$$

and

$$\|\varphi \mathbb{E}^\delta_\varepsilon(t)\|^{\frac{4}{3}}_{\frac{4}{3}} \le C\|\mathbb{F}_\varepsilon(t)\|_4^{\frac{4}{3}} \|\boldsymbol{v}_\varepsilon(t)\|_{1,2}^{\frac{4}{3}} \le C(\|\mathbb{F}_\varepsilon(t)\|_4^4 + \|\boldsymbol{v}_\varepsilon(t)\|^2_{1,2}), \tag{4.5.62}$$

where the second inequality follows from the Young inequality. Since $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$ and $\boldsymbol{v}_\varepsilon \in L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, we see that for any fixed $\varepsilon > 0$ the right-hand side of (4.5.62) is integrable over $(0,T)$, and thus using (4.5.61) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\delta\to 0+} \int_0^T \|\varphi \mathbb{E}^\delta_\varepsilon\|^{\frac{4}{3}}_{\frac{4}{3}} = 0.$$

Consequently, since $\mathbb{F}_\varepsilon \in (L^4(Q_T))^{2\times 2}$ (hence $\sup_{\delta>0}\|\mathbb{F}^\delta_\varepsilon\|_{4,Q_T} < \infty$), we have that

$$\lim_{\delta\to 0+} \int_{Q_T} \mathbb{E}^\delta_\varepsilon : (\varphi\mathbb{F}^\delta_\varepsilon) = 0. \tag{4.5.63}$$

Letting $\delta \to 0_+$ in (4.5.60), employing (4.5.63), (A.3.29), the fact that $\mathbb{F}_\varepsilon(0) = \mathbb{F}_0$ almost everywhere in Ω, which follows from (A.3.41) with $\mathbb{F}_\varepsilon$ in the role of $\mathbb{F}$, using also standard properties of mollifying and nonnegativity of the term $2\varepsilon\int_{Q_T}|\nabla \mathbb{F}^\delta_\varepsilon|^2\varphi$, we arrive at

$$\begin{aligned}
-\int_{Q_T} |\mathbb{F}_\varepsilon|^2(\partial_t\varphi) - \int_\Omega |\mathbb{F}_0|^2\varphi(0) - \int_{Q_T} (|\mathbb{F}_\varepsilon|^2 \boldsymbol{v}_\varepsilon)\cdot\nabla\varphi - 2\int_{Q_T} \nabla \boldsymbol{v}_\varepsilon : (\varphi \mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T) \\
+ \int_{Q_T} \left(|\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T|^2 - |\mathbb{F}_\varepsilon|^2\right)\varphi \le 2\varepsilon \int_{Q_T} |\mathbb{F}_\varepsilon|^2\Delta\varphi.
\end{aligned} \tag{4.5.64}$$

At this stage, we let $\varepsilon \to 0_+$ in (4.5.64). Taking into account the convergences (4.5.25), (4.5.30), (4.5.32), (4.5.33) and the fact that $\delta_0 > 0$ is arbitrary, we deduce (4.5.57).

To prove (4.5.58), we proceed similarly. Extending $\mathbb{F}$ and $\boldsymbol{v}$ by zero outside of Ω, using very similar arguments as above, (4.5.36) implies that

$$\begin{aligned} -\int_{Q_T} |\mathbb{F}^\delta|^2(\partial_t \varphi) - \int_{\Omega} |\mathbb{F}_0^\delta|^2 \varphi(0) - \int_{Q_T} (|\mathbb{F}^\delta|^2 \boldsymbol{v}) \cdot \nabla\varphi - 2\int_{Q_T} \left(\overline{\nabla \boldsymbol{v} \mathbb{F}}\right)^\delta : (\varphi \mathbb{F}^\delta) \\ + \int_{Q_T} \left((\overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} - \mathbb{F})^\delta : \mathbb{F}^\delta \right) \varphi = 2\int_{Q_T} \mathbb{E}^\delta : (\varphi \mathbb{F}^\delta) \end{aligned} \tag{4.5.65}$$

with

$$\mathbb{E}^\delta := \operatorname{div}(\mathbb{F}^\delta \otimes \boldsymbol{v}) - \operatorname{div}(\mathbb{F} \otimes \boldsymbol{v})^\delta.$$

Using Lemma 4.3.1, the Lebesgue dominated convergence theorem, the properties $\mathbb{F} \in (L^4(Q_T))^{2\times 2}$, $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\mathbb{F}_0 \in (L^2(\Omega))^{2\times 2}$, we let $\delta \to 0_+$ in (4.5.65) and deduce (4.5.58). □

Ad Step 2: Proof of (4.5.59)**.** We shall derive the inequality (4.5.59) from (4.5.57) and (4.5.58) attained in Step 1. Indeed, subtracting (4.5.57) from (4.5.58), we obtain

$$\begin{aligned} -\int_{Q_T} (\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\partial_t \varphi - \int_{Q_T} \left((\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\boldsymbol{v} \right) \cdot \nabla\varphi \le \int_{Q_T} (\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\varphi \\ + 2\left\langle \overline{\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)} - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} \\ - \left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}}. \end{aligned} \tag{4.5.66}$$

Hence, we need to handle the second and the third term on the right-hand side of (4.5.66). To this aim we show that[1]

$$\left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} \ge 0, \tag{4.5.67}$$

$$\left\langle \overline{\nabla \boldsymbol{v} : \mathbb{F}\mathbb{F}^T} - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} \le \int_{Q_T} \tilde{L}(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2)\varphi \tag{4.5.68}$$

with $\tilde{L} \in L^2(Q_T)$ being a fixed function specified below.

We start with proving (4.5.67). The convergence results (4.5.20), (4.5.31) and (4.5.32) imply that

$$\begin{aligned} \left\langle \overline{|\mathbb{F}\mathbb{F}^T|^2} - \overline{\mathbb{F}\mathbb{F}^T\mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}} &= \lim_{\varepsilon \to 0+} \int_{Q_T} \left(|\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T|^2 - (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon) : \mathbb{F} \right) \varphi \\ &= \lim_{\varepsilon \to 0+} \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon) : (\mathbb{F}_\varepsilon - \mathbb{F})\varphi - (\mathbb{F}\mathbb{F}^T\mathbb{F}) : (\mathbb{F}_\varepsilon - \mathbb{F})\varphi \\ &= \lim_{\varepsilon \to 0+} \int_{Q_T} \left((\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T \mathbb{F}_\varepsilon - \mathbb{F}\mathbb{F}^T\mathbb{F}) : (\mathbb{F}_\varepsilon - \mathbb{F}) \right) \varphi \ge 0, \end{aligned}$$

where the last inequality follows from the monotonicity of the matrix function $S(\mathbb{X}) := \mathbb{X}\mathbb{X}^T\mathbb{X}$ for all $\mathbb{X} \in \mathbb{R}^{2\times 2}$, see Lemma 4.3.2.

[1] Note that here is the main reason why we consider only the two dimensional setting. In order to show (4.5.68) one needs to identify I_1 in (4.5.72), which is known only in two dimensions. In the three dimensional setting, one must use a different approach.

The proof of (4.5.68) is more difficult. We decompose the left-hand side of (4.5.68) into $\sum_{j=1}^{3} I_j$, where

$$I_1 := \left\langle \overline{\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)} - \nabla \boldsymbol{v} : \overline{\mathbb{F}\mathbb{F}^T}, \varphi \right\rangle_{\{M(\overline{Q_T}), C(\overline{Q_T})\}} \tag{4.5.69}$$

$$I_2 := \int_{Q_T} \left(\nabla \boldsymbol{v} : \overline{\mathbb{F}\mathbb{F}^T} - \nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T) \right) \varphi, \tag{4.5.70}$$

$$I_3 := \int_{Q_T} \left(\nabla \boldsymbol{v} : (\mathbb{F}\mathbb{F}^T) - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F} \right) \varphi \tag{4.5.71}$$

and treat each term separately.

For I_1, we show, with the help of the limiting passage $\varepsilon \to 0_+$ in (4.5.11), that

$$I_1 = \left\langle |\mathbb{D}|^2 - \overline{|\mathbb{D}|^2}, \varphi \right\rangle_{\{M(\overline{Q_T}), C(\overline{Q_T})\}}. \tag{4.5.72}$$

To prove (4.5.72), we employ Lemma 4.5.1 on the reconstruction of the pressures p_ε, p and their convergence and the convergence results from Subsection 4.5.2. For arbitrary $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$ we can find a smooth set $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$ such that $\varphi \in C_c^\infty((-\infty, T) \times \tilde{\Omega})$. Then for a fixed $t \in (0, T)$ we subtract (4.5.53) tested by $(\boldsymbol{v} + \nabla p_1)\varphi$ from (4.5.49) tested by $(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\varphi$, integrate the result over $(0, T)$, use (4.5.34) and let $\varepsilon \to 0_+$ to obtain

$$I_1 = \lim_{\varepsilon \to 0+} \int_{Q_T} \left(\nabla \boldsymbol{v}_\varepsilon : (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) - \nabla \boldsymbol{v} : (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) \right) \varphi = \lim_{\varepsilon \to 0+} \sum_{j=1}^{9} J_{j,\varepsilon},$$

where

$$J_{1,\varepsilon} := -\int_0^T \langle \partial_t (\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \varphi(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \rangle + \int_0^T \langle \partial_t (\boldsymbol{v} + \nabla p_1), \varphi(\boldsymbol{v} + \nabla p_1) \rangle,$$

$$J_{2,\varepsilon} := \int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\varphi \nabla \boldsymbol{v}_\varepsilon) - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\varphi \nabla \boldsymbol{v}),$$

$$J_{3,\varepsilon} := \int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\boldsymbol{v}_\varepsilon \otimes \nabla \varphi) - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\boldsymbol{v} \otimes \nabla \varphi),$$

$$J_{4,\varepsilon} := \int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\nabla p_{1,\varepsilon} \otimes \nabla \varphi + \varphi \nabla^2 p_{1,\varepsilon}) - \int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\nabla p_1 \otimes \nabla \varphi + \varphi \nabla^2 p_1),$$

$$J_{5,\varepsilon} := \int_{Q_T} \left(|\mathbb{D}|^2 - |\mathbb{D}_\varepsilon|^2 \right) \varphi,$$

$$J_{6,\varepsilon} := -\int_{Q_T} \mathbb{D}_\varepsilon : \left((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \otimes \nabla \varphi + \varphi \nabla^2 p_{1,\varepsilon} \right) + \int_{Q_T} \mathbb{D} : \left((\boldsymbol{v} + \nabla p_1) \otimes \nabla \varphi + \varphi \nabla^2 p_1 \right),$$

$$J_{7,\varepsilon} := \int_{Q_T} p_{2,\varepsilon} \left((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \cdot \nabla \varphi + \varphi \Delta p_{1,\varepsilon} \right) - \int_{Q_T} p_2 \left((\boldsymbol{v} + \nabla p_1) \cdot \nabla \varphi + \varphi \Delta p_1 \right),$$

$$J_{8,\varepsilon} := \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : ((\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \otimes \nabla \varphi) - \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : ((\boldsymbol{v} + \nabla p_1) \otimes \nabla \varphi),$$

$$J_{9,\varepsilon} := \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : (\varphi \nabla^2 p_{1,\varepsilon}) - \int_{Q_T} (\mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T) : (\varphi \nabla^2 p_1).$$

Our aim is to show that all terms $J_{j,\varepsilon}$, $j = 1, ..., 9$, except $J_{5,\varepsilon}$, converge to zero. The terms $J_{3,\varepsilon}$, $J_{4,\varepsilon}$ and $J_{6,\varepsilon}, \dots J_{9,\varepsilon}$ vanish, as $\varepsilon \to 0_+$, due to (4.5.17), (4.5.25), (4.5.26), (4.5.29), (4.5.54) and (4.5.55). In order to treat $J_{1,\varepsilon}$, we use the integration by parts with respect to the time variable, i.e.

$$\int_0^T \langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \varphi(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon})\rangle = -\int_\Omega \frac{|\boldsymbol{v}_\varepsilon(0) + \nabla p_{1,\varepsilon}(0)|^2}{2}\varphi(0) - \int_{Q_T} \frac{|\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}|^2}{2}\partial_t\varphi$$

and

$$\int_0^T \langle -\partial_t(\boldsymbol{v} + \nabla p_1), \varphi(\boldsymbol{v} + \nabla p_1)\rangle = \int_\Omega \frac{|\boldsymbol{v}(0) + \nabla p_1(0)|^2}{2}\varphi(0) + \int_{Q_T} \frac{|\boldsymbol{v} + \nabla p_1|^2}{2}\partial_t\varphi.$$

Since $\boldsymbol{v}_\varepsilon$, $\boldsymbol{v}$ belong to $C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$, $\nabla p_{1,\varepsilon}$, ∇p_1 belong to $C([0,T]; (L^2(\tilde{\Omega}))^2)$, $\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}_0 = \boldsymbol{v}(0)$ and $\nabla p_{1,\varepsilon}(0) = \nabla p_1(0)$ almost everywhere in $\tilde{\Omega}$, we have

$$-\int_\Omega \frac{|\boldsymbol{v}_\varepsilon(0) + \nabla p_{1,\varepsilon}(0)|^2}{2}\varphi(0) + \int_\Omega \frac{|\boldsymbol{v}(0) + \nabla p_1(0)|^2}{2}\varphi(0) = 0.$$

Next, it follows from (4.5.25) and (4.5.54) that

$$\lim_{\varepsilon\to 0+} -\int_{Q_T} \frac{|\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}|^2}{2}\partial_t\varphi + \int_{Q_T} \frac{|\boldsymbol{v} + \nabla p_1|^2}{2}\partial_t\varphi = 0,$$

hence $J_{1,\varepsilon} \to 0$ as $\varepsilon \to 0_+$. It remains to prove that $J_{2,\varepsilon} \to 0$ as $\varepsilon \to 0_+$. Using the integration by parts and the property $\operatorname{div} \boldsymbol{v}_\varepsilon = \operatorname{div} \boldsymbol{v} = 0$ in Q_T, it holds

$$J_{2,\varepsilon} = \frac{1}{2}\int_{Q_T} (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) : (\boldsymbol{v}_\varepsilon \otimes \nabla\varphi) - \frac{1}{2}\int_{Q_T} (\boldsymbol{v} \otimes \boldsymbol{v}) : (\boldsymbol{v} \otimes \nabla\varphi),$$

which converges to zero by (4.5.25) and (4.5.26). Finally, as

$$J_{5,\varepsilon} \to \langle |\mathbb{D}|^2 - \overline{|\mathbb{D}|^2}, \varphi\rangle_{\{\mathcal{M}(\overline{Q_T}), C(\overline{Q_T})\}}$$

as $\varepsilon \to 0_+$ by (4.5.34), the equality (4.5.72) is proved.

The term I_2 (see (4.5.70)) is estimated, using (4.5.20), (4.5.29), (4.5.30) and the Cauchy–Schwartz inequality (together with the inequality $|\mathbb{X}\mathbb{Y}| \leq |\mathbb{X}||\mathbb{Y}|$ for all $\mathbb{X}$, $\mathbb{Y} \in \mathbb{R}^{2\times 2}$) as follows:

$$\begin{aligned} I_2 &= \lim_{\varepsilon\to 0_+} \int_{Q_T} \left(\nabla\boldsymbol{v} : (\mathbb{F}_\varepsilon\mathbb{F}_\varepsilon^T) - \nabla\boldsymbol{v} : (\mathbb{F}\mathbb{F}^T)\right)\varphi \\ &= \lim_{\varepsilon\to 0+} \int_{Q_T} \nabla\boldsymbol{v} : \left((\mathbb{F}_\varepsilon - \mathbb{F})(\mathbb{F}_\varepsilon - \mathbb{F})^T\right)\varphi \leq \lim_{\varepsilon\to 0+} \int_{Q_T} |\nabla\boldsymbol{v}||\mathbb{F}_\varepsilon - \mathbb{F}|^2\varphi \\ &= \int_{Q_T} |\nabla\boldsymbol{v}|\left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi. \end{aligned} \tag{4.5.73}$$

Before we estimate I_3, we show a localised version of the Korn inequality for sequences. More precisely, we observe that (g^δ denotes the standard mollification of $g \in L^1_{loc}(\mathbb{R}^2)$)

$$
\begin{aligned}
\int_{Q_T} |\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}|^2 \varphi &= \lim_{\delta \to 0+} \int_{Q_T} (\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}) \cdot (\nabla \boldsymbol{v}_\varepsilon^\delta - \nabla \boldsymbol{v}^\delta) \varphi \\
&= - \lim_{\delta \to 0+} \int_{Q_T} \varphi (\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \cdot (\Delta \boldsymbol{v}_\varepsilon^\delta - \Delta \boldsymbol{v}^\delta) - \int_{Q_T} (\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}) : ((\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \otimes \nabla \varphi) \\
&= - \lim_{\delta \to 0+} \int_0^T 2\varphi (\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \cdot \operatorname{div}(\mathbb{D}_\varepsilon^\delta - \mathbb{D}^\delta) - \int_{Q_T} (\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}) : ((\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \otimes \nabla \varphi) \\
&= 2 \int_{Q_T} |\mathbb{D}_\varepsilon - \mathbb{D}|^2 \varphi \\
&\quad + \int_{Q_T} 2(\mathbb{D}_\varepsilon - \mathbb{D}) : ((\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \otimes \nabla \varphi) - (\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}) : ((\boldsymbol{v}_\varepsilon - \boldsymbol{v}) \otimes \nabla \varphi),
\end{aligned}
$$

and consequently, with the help of (4.5.17), (4.5.25), we deduce

$$
\lim_{\varepsilon \to 0_+} \int_{Q_T} |\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}|^2 \varphi = 2 \lim_{\varepsilon \to 0_+} \int_{Q_T} |\mathbb{D}_\varepsilon - \mathbb{D}|^2 \varphi. \tag{4.5.74}
$$

Then, the term I_3 (see (4.5.71)) is estimated, employing (4.5.17), (4.5.20), (4.5.30), (4.5.34), (4.5.74) and the Young inequality, as follows:

$$
\begin{aligned}
I_3 &= \int_{Q_T} \left(\nabla \boldsymbol{v} \mathbb{F} - \overline{\nabla \boldsymbol{v} \mathbb{F}} \right) : (\varphi \mathbb{F}) = \lim_{\varepsilon \to 0_+} \int_{Q_T} (\nabla \boldsymbol{v}_\varepsilon (\mathbb{F} - \mathbb{F}_\varepsilon)) : (\varphi \mathbb{F}) \\
&= \lim_{\varepsilon \to 0+} \int_{Q_T} ((\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v})(\mathbb{F} - \mathbb{F}_\varepsilon)) : (\varphi \mathbb{F}) \leq \lim_{\varepsilon \to 0+} \int_{Q_T} |\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}| \, |\mathbb{F}_\varepsilon - \mathbb{F}| \, |\mathbb{F}| \varphi \\
&\leq \lim_{\varepsilon \to 0+} \int_{Q_T} \left(\frac{1}{2} |\nabla \boldsymbol{v}_\varepsilon - \nabla \boldsymbol{v}|^2 + |\mathbb{F}_\varepsilon - \mathbb{F}|^2 |\mathbb{F}|^2 \right) \varphi \\
&= \lim_{\varepsilon \to 0+} \int_{Q_T} \left(|\mathbb{D}_\varepsilon - \mathbb{D}|^2 + |\mathbb{F}_\varepsilon - \mathbb{F}|^2 |\mathbb{F}|^2 \right) \varphi \\
&= \left\langle \overline{|\mathbb{D}|^2} - |\mathbb{D}|^2, \varphi \right\rangle_{\left\{ M(\overline{Q_T}), C(\overline{Q_T}) \right\}} + \int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2 \right) |\mathbb{F}|^2 \varphi.
\end{aligned} \tag{4.5.75}
$$

Summing (4.5.72), (4.5.73) and (4.5.75) and employing the definitions (4.5.69) (4.5.70), (4.5.71), we obtain for all nonnegative $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, that

$$
\left\langle \overline{\nabla \boldsymbol{v} : \mathbb{F} \mathbb{F}^T} - \overline{\nabla \boldsymbol{v} \mathbb{F}} : \mathbb{F}, \varphi \right\rangle_{\left\{ M(\overline{Q_T}), C(\overline{Q_T}) \right\}} = \sum_{j=1}^3 I_j \leq \int_{Q_T} \tilde{L} (\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2) \varphi
$$

with

$$
\tilde{L} := (|\nabla \boldsymbol{v}| + |\mathbb{F}|^2) \in L^2(Q_T),
$$

which is the inequality (4.5.68). Finally, combining (4.5.66)–(4.5.68), we see that (4.5.59) holds with

$$L := (1 + |\nabla \boldsymbol{v}| + |\mathbb{F}|^2) \in L^2(Q_T). \tag{4.5.76}$$

□

Ad Step 3: Proof of compactness of $\{\mathbb{F}_\varepsilon\}$. As introduced above, the last Step 3 consists of renormalising the inequality (4.5.59) achieved in Step 2, proving that the renormalised inequality is valid even for nonnegative smooth test functions φ supported up to the boundary of Ω and concluding by a suitable choice of such φ and of the renormalisation function the result $\overline{|\mathbb{F}|^2} = |\mathbb{F}|^2$ almost everywhere in Q_T, i.e. (4.5.44).

For the rest of the proof, we denote $f := \overline{|\mathbb{F}|^2} - |\mathbb{F}|^2$. Note that f is nonnegative[2],

$$f \in L^2(Q_T) \cap L^\infty(0,T; L^1(\Omega)) \tag{4.5.77}$$

and for lucidity let us rewrite (4.5.59) to

$$-\int_{Q_T} f \partial_t \varphi - \int_{Q_T} f \boldsymbol{v} \cdot \nabla \varphi \le \int_{Q_T} L f \varphi \tag{4.5.78}$$

with L defined in (4.5.76), where $\varphi \in C_c^\infty((-\infty,T) \times \Omega)$ is an arbitrary nonnegative function. Our goal is to show that $f = 0$ almost everywhere in Q_T.

We start the proof with the renormalisation of (4.5.78). It means, we show that, for all nondecreasing $B \in C^1([0,\infty)$ with $B' \in L^\infty((0,\infty))$ and for all $\varphi \in C_c^\infty((-\infty,T) \times \Omega)$ such that $\varphi \ge 0$, it holds

$$-\int_{Q_T} B(f)\partial_t \varphi - \int_{\Omega} B(0)\varphi(0) - \int_{Q_T} B(f)\boldsymbol{v} \cdot \nabla \varphi \le \int_{Q_T} L f B'(f) \varphi. \tag{4.5.79}$$

To prove (4.5.79), we first mollify (4.5.78). Let $\delta_0 > 0$ and $\delta \in (0, \delta_0)$ be arbitrary. We extend $\boldsymbol{v}$, L and f by zero outside of Q_T, and for arbitrary $h \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^2)$ we denote its mollification over the time and the spatial variables as

$$h_\delta(t, \boldsymbol{x}) := \int_{\mathbb{R}\times\mathbb{R}^2} \omega_\delta(t - \cdot, \boldsymbol{x} - \cdot) h(\cdot, \cdot),$$

where ω_δ is the standard mollifying kernel of radius δ. Recall also the definition $\Omega_{\delta_0} := \{\boldsymbol{x} \in \Omega; \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \delta_0\}$. Then it directly follows from (4.5.78) that

$$\partial_t f_\delta + \operatorname{div}(f\boldsymbol{v})_\delta \le (Lf)_\delta \quad \text{a.e. in } (-\infty, T-\delta_0) \times \Omega_{\delta_0}.$$

Let us multiply the above inequality by $B'(f_\delta)$, which is nonnegative and bounded, to get (using the fact that $\operatorname{div} \boldsymbol{v} = 0$)

$$\partial_t B(f_\delta) + \operatorname{div}\left(B(f_\delta)\boldsymbol{v}\right) \le (Lf)_\delta B'(f_\delta) + s_\delta B'(f_\delta) \quad \text{a.e. in } (-\infty, T-\delta_0) \times \Omega_{\delta_0}, \tag{4.5.80}$$

[2] It directly follows from

$$\int_{Q_T} \left(\overline{|\mathbb{F}|^2} - |\mathbb{F}|^2\right)\varphi = \lim_{\varepsilon \to 0+} \int_{Q_T} |\mathbb{F}_\varepsilon - \mathbb{F}|^2 \varphi$$

valid for all nonnegative bounded φ.

where

$$s_\delta := \operatorname{div}(f_\delta \boldsymbol{v}) - \operatorname{div}(f\boldsymbol{v})_\delta.$$

Now, multiplying (4.5.80) by arbitrary nonnegative $\varphi \in C_c^\infty((-\infty, T-\delta_0)\times\Omega_{\delta_0})$, integrating the result over $(-2\delta, T)\times\Omega$, using the fact that $\operatorname{supp} f_\delta \subset (-\delta, T+\delta)\times\mathbb{R}^2$ and the integration by parts, we obtain

$$\begin{aligned}\int_{-2\delta}^{T}\int_\Omega (-B(f_\delta)\partial_t\varphi - B(f_\delta)\boldsymbol{v}\cdot\nabla\varphi) - \int_\Omega B(0)\varphi(-2\delta)\\ \le \int_{-2\delta}^{T}\int_\Omega ((Lf)_\delta B'(f_\delta)\varphi + s_\delta B'(f_\delta)\varphi)\,.\end{aligned} \tag{4.5.81}$$

Finally, we let $\delta \to 0_+$ in (4.5.81). First, since $f \in L^2((-\infty,T)\times\Omega)$ and $\boldsymbol{v}$ belongs to $L^2(-\infty,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, we can use Lemma 4.3.1 and get that $s_\delta \to 0$ strongly in $L^1_{loc}((-\infty,T)\times\Omega)$. Since B' is bounded, it holds

$$\lim_{\delta\to 0+}\int_{-2\delta}^{T}\int_\Omega s_\delta B'(f_\delta)\varphi = 0. \tag{4.5.82}$$

The identification of other limits is standard and directly follows from the properties of the standard mollification and B. Hence, (4.5.79) follows for all nonnegative $\varphi \in C_c^\infty((-\infty,T-\delta_0)\times\Omega_{\delta_0})$. Since $\delta_0>0$ is arbitrary, we deduce the validity of (4.5.79) for all nonnegative $\varphi\in C_c^\infty((-\infty,T)\times\Omega)$.

In the next step, we strengthen (4.5.79). We show that it holds for all nonnegative functions $\varphi\in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$, i.e., we do not require φ being compactly supported in Ω. Hence, let $\psi\in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$ be arbitrary nonnegative function. Since Ω is Lipschitz, we can find a sequence of smooth nonnegative functions $\xi_m \in \mathcal{C}_c^\infty(\Omega)$ fulfilling $0\le\xi_m\le 1$ and satisfying

$$\begin{aligned}\xi_m(\boldsymbol{x}) &= \begin{cases} 0 & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\Omega)\le \dfrac{1}{2m},\\ 1 & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\Omega) > \dfrac{1}{m}, \text{ i.e. if } \boldsymbol{x}\in\Omega_{\frac{1}{m}},\end{cases}\\ |\nabla\xi_m(\boldsymbol{x})| &\le Cm \quad \text{for all } \boldsymbol{x}\in\Omega.\end{aligned} \tag{4.5.83}$$

Next, setting $\varphi := \psi\xi_m$ in (4.5.79), we obtain

$$\begin{aligned}-\int_{Q_T} B(f)(\partial_t\psi)\xi_m - \int_\Omega B(0)\psi(0)\xi_m - \int_{Q_T} B(f)\xi_m(\boldsymbol{v}\cdot\nabla\psi)\\ \le \int_{Q_T} LfB'(f)\psi\xi_m + \int_{Q_T} B(f)\psi(\boldsymbol{v}\cdot\nabla\xi_m).\end{aligned}$$

Finally, we let $m\to\infty$ in the above inequality. In the first four integrals, we can easily identify the limits due to the integrability of the corresponding integrands, (4.5.83) and the fact that $|\Omega\setminus\Omega_{\frac{1}{m}}|\to 0$ as Ω has a Lipschitz boundary, and deduce

$$\begin{aligned}-\int_{Q_T} B(f)(\partial_t\psi) - \int_\Omega B(0)\psi(0) - \int_{Q_T} B(f)(\boldsymbol{v}\cdot\nabla\psi)\\ \le \int_{Q_T} LfB'(f)\psi + \lim_{m\to\infty}\int_{Q_T} B(f)\psi(\boldsymbol{v}\cdot\nabla\xi_m).\end{aligned} \tag{4.5.84}$$

Since ψ is bounded, B is Lipschitz, $f \in L^2(Q_T)$, $\nabla \boldsymbol{v} \in (L^2(Q_T))^{2\times 2}$, $\nabla \xi_m$ is not supported outside of $\Omega \setminus \Omega_{\frac{1}{m}}$ and $|\Omega \setminus \Omega_{\frac{1}{m}}| \to 0$, employing also the Hardy inequality for $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}})$ and the Hölder inequality, we have

$$\begin{aligned}\lim_{m\to\infty} \left| \int_{Q_T} B(f)\psi(\boldsymbol{v}\cdot\nabla\xi_m) \right| &\le \|\psi\|_{L^\infty(Q_T)} \|f\|_{L^2(Q_T)} \lim_{m\to\infty} \left(\int_0^T \int_{\Omega_{\frac{1}{m}}} \left(\frac{|\boldsymbol{v}|}{\mathrm{dist}(\cdot,\partial\Omega)} \right)^2 \right)^{\frac{1}{2}} \\ &\le \|\psi\|_{L^\infty(Q_T)} \|f\|_{L^2(Q_T)} \lim_{m\to\infty} \|\nabla \boldsymbol{v}\|_{\left(L^2((0,T)\times(\Omega\setminus\Omega_{\frac{1}{m}}))\right)^{2\times 2}} = 0.\end{aligned}$$

Consequently, we see that the last term in (4.5.84) vanishes, and thus we conclude (using also the identity $-\psi(0) = \int_0^T \partial_t \psi$) that

$$-\int_{Q_T} (B(f) - B(0))(\partial_t \psi) - \int_{Q_T} B(f)(\boldsymbol{v}\cdot\nabla\psi) \le \int_{Q_T} LfB'(f)\psi \tag{4.5.85}$$

for all nonnegative $\psi \in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$.

Now we are prepared to conclude that $f = 0$ a.e. in Q_T. In (4.5.85) we set $B(s) := \ln(s+\varepsilon)$ and $\psi(t,\boldsymbol{x}) := z(t)$, where $z \in C_c^\infty(-\infty,T)$ is arbitrary nonnegative and in addition we require that it is non-increasing in $(0,T)$, i.e., $\partial_t z(t) \le 0$ in $(0,T)$. Then, we see that the last term on the left-hand side vanishes and we get

$$\int_{Q_T} |\partial_t z| \ln\left(\frac{f+\varepsilon}{\varepsilon}\right) \le \|z\|_{L^\infty((0,T))} \int_{Q_T} |L| \frac{f}{f+\varepsilon} \le \|z\|_{L^\infty((0,T))} \int_{Q_T} |L|. \tag{4.5.86}$$

Since $L \in L^2(Q_T)$, it follows from the above inequality that for all $\delta > 0$, we have

$$\lim_{\varepsilon\to 0_+} \int_0^{T-\delta} \int_\Omega \ln\left(1 + \frac{f}{\varepsilon}\right) < +\infty,$$

but it is possible only if $f = 0$ almost everywhere in Q_T, i.e. only if (4.5.44) is satisfied. The proof of the compactness of $\{\mathbb{F}_\varepsilon\}$ in $(L^2(Q_T))^{2\times 2}$ is finished.

□

4.5.5 Positivity of $\det \mathbb{F}$

In this section we show the nonnegativity of $\det \mathbb{F}$ provided that $\det \mathbb{F}_0 > 0$ almost everywhere in Ω and $\ln \det \mathbb{F}_0 \in L^1(\Omega)$. To justify the use of very special test function, we again mollify the equation (4.4.3) with respect to the spatial variable, i.e., we have that ($\delta > 0$ is arbitrary)

$$\partial_t \mathbb{F}_\delta + \mathrm{div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v}\mathbb{F})_\delta + \frac{1}{2}(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F})_\delta = \mathbb{E}_\delta, \tag{4.5.87}$$

where

$$\mathbb{E}_\delta := \mathrm{div}(\mathbb{F}_\delta \otimes \boldsymbol{v}) - \mathrm{div}(\mathbb{F}\otimes\boldsymbol{v})_\delta.$$

The equation (4.5.87) is satisfied in $(0,T)\times\mathbb{R}^2$, where we use the convention that $\boldsymbol{v}$ is extended by zero and $\mathbb{F}$ by the identity tensor outside of Ω. It also follows from Lemma 4.3.1 that

$$\mathbb{E}_\delta \to \mathbb{O} \text{ strongly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2} \text{ as } \delta \to 0_+. \tag{4.5.88}$$

Next, for arbitrary $\epsilon \in (0,1)$, we define $\mathbb{A}_\epsilon^\delta$ as

$$\mathbb{A}_\epsilon^\delta := \frac{(\det \mathbb{F}_\delta - \epsilon)^+ \mathbb{F}_\delta^{-T}}{\det \mathbb{F}_\delta}.$$

Such a definition is meaningful even for (possibly) singular matrix $\mathbb{F}_\delta$, using the convention $\mathbb{A}_\epsilon^\delta = \mathbb{O}$ whenever $\det \mathbb{F}_\delta \le \epsilon$. In addition, using the algebraic identity (valid whenever $\det \mathbb{F}_\delta \neq 0$)

$$\mathbb{F}_\delta^{-T} = \frac{\text{adj } \mathbb{F}_\delta^T}{\det \mathbb{F}_\delta},$$

we see, thanks to the fact that we consider the two dimensional setting, that[3]

$$|\mathbb{A}_\epsilon^\delta| = \frac{(\det \mathbb{F}_\delta - \epsilon)^+}{\det \mathbb{F}_\delta} \frac{|\text{adj } \mathbb{F}_\delta^T|}{\det \mathbb{F}_\delta} \le \frac{|\mathbb{F}_\delta|}{\epsilon}. \tag{4.5.89}$$

Using (4.5.89), the properties of convolution and the fact that $\mathbb{F} \in (L^4(Q_T))^{2\times 2}$, we see that for $\delta \to 0_+$ it holds

$$\mathbb{A}_\epsilon^\delta \to \mathbb{A}_\epsilon := \frac{(\det \mathbb{F} - \epsilon)^+}{\det \mathbb{F}} \mathbb{F}^{-1} \text{ strongly in } (L^4((0,T)\times\mathbb{R}^2))^{2\times 2}. \tag{4.5.90}$$

Next, we introduce $S_\epsilon(a)$ as a primitive function to $\frac{(a-\epsilon)^+}{a^2}$, i.e.

$$S_\epsilon(a) := \begin{cases} \ln a + \dfrac{\epsilon}{a} & \text{if } a > \epsilon, \\ \ln \epsilon + 1 & \text{if } a \le \epsilon. \end{cases}$$

Using the relation $\partial(\ln \det \mathbb{F}_\delta) = \partial \mathbb{F}_\delta : \mathbb{F}_\delta^{-T}$ (whenever $\det \mathbb{F}_\delta > 0$), we see that

$$\partial S_\epsilon(\det \mathbb{F}_\delta) = \partial \mathbb{F}_\delta : \mathbb{A}_\epsilon^\delta.$$

Here, ∂ stands for ∂_t, ∂_{x_1} or ∂_{x_2}.

Finally, we take the scalar product of (4.5.87) with $\mathbb{A}_\epsilon^\delta$ and get

$$\partial_t S_\epsilon(\det \mathbb{F}_\delta) + \text{div}\,(S_\epsilon(\det \mathbb{F}_\delta)\boldsymbol{v}) - (\nabla \boldsymbol{v} \mathbb{F})_\delta : \mathbb{A}_\epsilon^\delta + \frac{1}{2}\left(\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}\right)_\delta : \mathbb{A}_\epsilon^\delta = \mathbb{E}_\delta : \mathbb{A}_\epsilon^\delta.$$

We integrate this identity over $(0,t)\times\mathbb{R}^2$. Since Ω is bounded and $\boldsymbol{v}$ and $(\mathbb{F}-\mathbb{I})$ are extended by zero outside of Ω, we see that $\boldsymbol{v}_\delta$ and also $\mathbb{F}_\delta - \mathbb{I}$ are identically equal to zero outside of any bounded set $O_\delta := \{\boldsymbol{x} \in \mathbb{R}^2;\ B_\delta(\boldsymbol{x}) \cap \Omega \neq \emptyset\}$ (where $B_\delta(\boldsymbol{x})$ is a ball

[3] Note that in order to prove (4.5.89) one needs to use the (in)equality $|\text{adj } \mathbb{F}_\delta| \le |\mathbb{F}_\delta|$, which is (generally) satisfied only in two spatial dimensions. In the three dimensional setting one must use a different approach.

with center $\boldsymbol{x}$ and radius δ), and therefore we deduce after the integration by parts (using the fact that $\operatorname{div}\boldsymbol{v}=0$)

$$\int\limits_{\mathbb{R}^2} S_\epsilon(\det\mathbb{F}_\delta(t)) - S_\epsilon(\det(\mathbb{F}_0)_\delta) - \int\limits_0^t\int\limits_{\mathbb{R}^2}(\nabla\boldsymbol{v}\mathbb{F})_\delta : \mathbb{A}_\epsilon^\delta - \frac{1}{2}\left(\mathbb{F}\mathbb{F}^T\mathbb{F}-\mathbb{F}\right)_\delta : \mathbb{A}_\epsilon^\delta$$
$$= \int\limits_0^t\int\limits_{\mathbb{R}^2}\mathbb{E}_\delta : \mathbb{A}_\epsilon^\delta. \tag{4.5.91}$$

Thanks to (4.5.88) and (4.5.90), the right-hand side of (4.5.91) tends to zero as $\delta\to 0_+$. The limits in all terms on the left-hand side of (4.5.91) are easy to identify (due to the integrability properties of $\mathbb{F}$, $\mathbb{F}_0$ and $\boldsymbol{v}$) and we obtain that

$$\int\limits_{\mathbb{R}^2} S_\epsilon(\det\mathbb{F}(t)) - S_\epsilon(\det(\mathbb{F}_0)) - \int\limits_0^t\int\limits_{\mathbb{R}^2}(\nabla\boldsymbol{v}\mathbb{F}) : \mathbb{A}_\epsilon - \frac{1}{2}\left(\mathbb{F}\mathbb{F}^T\mathbb{F}-\mathbb{F}\right) : \mathbb{A}_\epsilon = 0. \tag{4.5.92}$$

Employing (4.5.92), using the fact that $\operatorname{div}\boldsymbol{v}=0$, the definition of $\mathbb{A}_\epsilon$ and the fact that $\boldsymbol{v}$ is extended by zero and $\mathbb{F}$ is extended by the identity tensor outside of Ω, we get

$$\int\limits_\Omega S_\epsilon(\det\mathbb{F}(t)) - S_\epsilon(\det(\mathbb{F}_0)) + \frac{1}{2}\int\limits_0^t\int\limits_\Omega \frac{(\det\mathbb{F}-\epsilon)^+}{\det\mathbb{F}}\left(|\mathbb{F}\mathbb{F}^T|^2-2\right) = 0. \tag{4.5.93}$$

Consequently, we have for all $t\in(0,T)$

$$\sup_{\epsilon\in(0,1)}\left|\int\limits_\Omega S_\epsilon(\det\mathbb{F}(t))\right| \le \left(1+\int\limits_\Omega |\ln\det\mathbb{F}_0|\right) + \int\limits_{Q_T}(2+|\mathbb{F}\mathbb{F}^T|^2) < +\infty, \tag{4.5.94}$$

which directly gives $\det\mathbb{F}>0$ almost everywhere in Q_T. In addition, we have

$$\sup_{t\in(0,T)}\|\ln\det\mathbb{F}(t)\|_1 \le C(\|\mathbb{F}_0\|_2, \|\ln\det\mathbb{F}_0\|_1).$$

□

4.6 Proof of Theorem 4.2.1 with $G_1=1$, $G_2=0$

To complete the proof of Theorem 4.2.1 for $G_1=1$ and $G_2=0$, it remains to proceed rigorously from (4.4.3) to (4.4.7) (i.e. to derive (4.2.13) from (4.4.9)) and to prove the attainment of the initial condition $\mathbb{B}_0$ (i.e. to complete the proof of (4.2.14)) and the continuity of $\mathbb{B}$ in time. To do so, we multiply (4.5.87) by $\mathbb{F}_\delta^T$ from right, then we take the transpose of (4.5.87) and multiply it by $\mathbb{F}_\delta$ from left. Summing both identities we deduce almost everywhere in Q_T (using the fact that $\operatorname{div}\boldsymbol{v}=0$)

$$\begin{aligned}&\partial_t(\mathbb{F}_\delta\mathbb{F}_\delta^T) + \operatorname{div}((\mathbb{F}_\delta\mathbb{F}_\delta^T)\otimes\boldsymbol{v}) - (\nabla\boldsymbol{v}\mathbb{F})_\delta\mathbb{F}_\delta^T - \mathbb{F}_\delta(\mathbb{F}^T(\nabla\boldsymbol{v})^T)_\delta\\ &+\frac{1}{2}\left((\mathbb{F}\mathbb{F}^T\mathbb{F}-\mathbb{F})_\delta\mathbb{F}_\delta^T + \mathbb{F}_\delta(\mathbb{F}^T\mathbb{F}\mathbb{F}^T-\mathbb{F}^T)_\delta\right) = \mathbb{E}_\delta\mathbb{F}_\delta^T + \mathbb{F}_\delta\mathbb{E}_\delta^T,\end{aligned} \tag{4.6.1}$$

where

$$\mathbb{E}_\delta := \operatorname{div}(\mathbb{F}\otimes\boldsymbol{v})_\delta\mathbb{F}_\delta^T - \operatorname{div}((\mathbb{F}_\delta\mathbb{F}_\delta^T)\otimes\boldsymbol{v}).$$

Consequently, using the integration by parts, we deduce, for all $\mathbb{A} \in (C_c^1((-\infty, T) \times \mathbb{R}^2))^{2\times 2}$, that

$$
\begin{aligned}
&-\int_{Q_T} \Big(\mathbb{F}_\delta \mathbb{F}_\delta^T\Big) : \partial_t \mathbb{A} - \int_{\Omega} \Big((\mathbb{F}_0)_\delta (\mathbb{F}_0^T)_\delta\Big) : \mathbb{A}(0) - \int_{Q_T} \Big((\mathbb{F}_\delta \mathbb{F}_\delta^T) \otimes \boldsymbol{v}\Big) : \nabla \mathbb{A} \\
&\quad - \int_{Q_T} \Big((\nabla \boldsymbol{v} \mathbb{F})_\delta \mathbb{F}_\delta^T + \mathbb{F}_\delta (\mathbb{F}^T (\nabla \boldsymbol{v})^T)_\delta\Big) : \mathbb{A} + \frac{1}{2} \int_{Q_T} \Big((\mathbb{F}\mathbb{F}^T\mathbb{F})_\delta \mathbb{F}_\delta^T + \mathbb{F}_\delta (\mathbb{F}^T\mathbb{F}\mathbb{F}^T)_\delta\Big) : \mathbb{A} \\
&\quad - \int_{Q_T} (\mathbb{F}_\delta \mathbb{F}_\delta^T) : \mathbb{A} = \int_{Q_T} \Big(\mathbb{E}_\delta \mathbb{F}_\delta^T + \mathbb{F}_\delta {\mathbb{E}_\delta}^T\Big) : \mathbb{A}.
\end{aligned} \tag{4.6.2}
$$

Employing $\mathbb{F} \in (L^4((0,T) \times \mathbb{R}^2))^{2\times 2}$, $\boldsymbol{v} \in L^2(0,T; (W^{1,2}(\mathbb{R}^2))^2)$ and Lemma 4.3.1 together with Lebesgue's dominated convergence theorem, we obtain that $\mathbb{E}_\delta \to \mathbb{O}$ strongly in $(L^{\frac{4}{3}}(Q_T))^{2\times 2}$ (as $\delta \to 0_+$). Thus, by means of standard properties of mollifying, we get

$$
\begin{aligned}
&-\int_{Q_T} \mathbb{B} : \partial_t \mathbb{A} - \int_{\Omega} \mathbb{B}_0 : \mathbb{A}(0) - \int_{Q_T} (\mathbb{B} \otimes \boldsymbol{v}) : \nabla \mathbb{A} \\
&\quad - \int_{Q_T} \Big(\nabla \boldsymbol{v} \mathbb{B} + \mathbb{B} (\nabla \boldsymbol{v})^T\Big) : \mathbb{A} + \int_{Q_T} \Big(\mathbb{B}^2 - \mathbb{B}\Big) : \mathbb{A} = 0,
\end{aligned} \tag{4.6.3}
$$

with $\mathbb{B} := \mathbb{F}\mathbb{F}^T$ and $\mathbb{B}_0 := \mathbb{F}_0 \mathbb{F}_0^T$. Hence, due to the properties of $\mathbb{F}$, we have $\mathbb{B} \in (L^2(Q_T))^{2\times 2}$. Since also $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}}) \cap (L^4(Q_T))^2$, $C^1(\overline{\Omega})$ is dense in $W^{1,4}(\Omega)$ and $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, it follows from (4.6.3) that

$$
\partial_t \mathbb{B} \in L^1(0,T; ((W^{1,4}(\Omega))^{2\times 2})^*).
$$

Finally, referring to the Du Bois-Reymond lemma, we conclude (4.2.13) from (4.6.3).

It remains to show

$$
\mathbb{B} \in C([0,T]; (L^1(\Omega))^{2\times 2})
$$

and the attainment of the initial condition $\mathbb{B}_0$. To achieve this goal, we write for all t_0, $t_1 \in [0,T]$

$$
\begin{aligned}
\int_{\Omega} \Big|\mathbb{F}(t_1)\mathbb{F}(t_1)^T - \mathbb{F}(t_0)\mathbb{F}(t_0)^T\Big| &= \int_{\Omega} \Big|\mathbb{F}(t_1)(\mathbb{F}(t_1) - \mathbb{F}(t_0))^T + (\mathbb{F}(t_1) - \mathbb{F}(t_0))\mathbb{F}(t_0)^T\Big|, \\
&\leq \int_{\Omega} |\mathbb{F}(t_1)|\ |\mathbb{F}(t_1) - \mathbb{F}(t_0)| + |\mathbb{F}(t_1) - \mathbb{F}(t_0)|\ |\mathbb{F}(t_0)| \\
&\leq \|\mathbb{F}(t_1)\|_2 \|\mathbb{F}(t_1) - \mathbb{F}(t_0)\|_2 + \|\mathbb{F}(t_1) - \mathbb{F}(t_0)\|_2 \|\mathbb{F}(t_0)\|_2,
\end{aligned}
$$

which converges to zero as $t_1 \to t_0$ if $t_0 \in (0,T)$, as $t_1 \to t_0+$ if $t_0 = 0$, as $t_1 \to t_0-$ if $t_0 = T$, since $\mathbb{F} \in C([0,T]; (L^2(\Omega))^{2\times 2})$. Hence $\mathbb{B} = \mathbb{F}\mathbb{F}^T \in C([0,T]; (L^1(\Omega))^{2\times 2})$ and the property $\mathbb{B}(0,\cdot) = \mathbb{B}_0 = \mathbb{F}_0\mathbb{F}_0^T$ almost everywhere in Ω follows from (4.6.3).

□

4.7 Proof of Theorem 4.2.1 with G_1, $G_2 > 0$ arbitrary

4.7.1 System with equations for $\mathbb{F}_1$, $\mathbb{F}_2$

Analogously to the case $G_2 = 0$, we start with the evolutionary equation for $\mathbb{F}_i$ instead of the equation for $\mathbb{B}_i$, $i = 1, 2$. Thus, the starting system of governing equations for $\boldsymbol{v}$, p, $\mathbb{F}_1$, $\mathbb{F}_2$ in Q_T reads

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.7.1}$$

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p - \operatorname{div} \mathbb{D} - \operatorname{div}\Big(G_1(\mathbb{F}_1\mathbb{F}_1^T) + G_2(\mathbb{F}_2\mathbb{F}_2^T)\Big) = \mathbf{0}, \quad G_1, G_2 > 0, \tag{4.7.2}$$

$$\partial_t \mathbb{F}_i + \operatorname{div}(\mathbb{F}_i \otimes \boldsymbol{v}) - (\nabla \boldsymbol{v})\mathbb{F}_i + \frac{1}{2}(\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i - \mathbb{F}_i) = \mathbb{O}, \quad i = 1, 2, \tag{4.7.3}$$

$$\det \mathbb{F}_i > 0, \quad i = 1, 2. \tag{4.7.4}$$

This system is completed with the boundary condition

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Sigma_T \tag{4.7.5}$$

and with the initial conditions

$$\boldsymbol{v}(0, \cdot) = \boldsymbol{v}_0 \quad \text{and} \quad \mathbb{F}_i(0, \cdot) = \mathbb{F}_{i_0} (i = 1, 2) \quad \text{in } \Omega. \tag{4.7.6}$$

We prove the existence of weak solutions to (4.7.1)–(4.7.6), i.e. the existence of $\boldsymbol{v}$, $\mathbb{F}_1$, $\mathbb{F}_2$ fulfilling

$$\begin{aligned}
\boldsymbol{v} &\in C([0, T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0, T; W^{1,2}_{\mathbf{0},\mathrm{div}}),\\
\partial_t \boldsymbol{v} &\in L^2(0, T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*),\\
\mathbb{F}_i &\in C([0, T]; (L^2(\Omega))^{2\times 2}) \cap (L^4(Q_T))^{2\times 2} \quad (i = 1, 2),\\
\partial_t \mathbb{F}_i &\in L^{\frac{4}{3}}\left(0, T; ((W^{1,2}(\Omega))^{2\times 2})^*\right) \quad (i = 1, 2),\\
\det \mathbb{F}_i &> 0 \quad \text{a.e. in } Q_T \quad (i = 1, 2)
\end{aligned}$$

and satisfying, for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and for almost all $t \in (0, T)$,

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i(\mathbb{F}_i\mathbb{F}_i^T) : \nabla \boldsymbol{w} = 0, \tag{4.7.7}$$

$$\langle \partial_t \mathbb{F}_i, \mathbb{A} \rangle - \int_\Omega (\mathbb{F}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v})\mathbb{F}_i) : \mathbb{A} + \frac{1}{2} \int_\Omega (\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i - \mathbb{F}_i) : \mathbb{A} = 0, \quad (i = 1, 2) \tag{4.7.8}$$

with initial conditions $\boldsymbol{v}_0$, $\mathbb{F}_{1_0}$ and $\mathbb{F}_{2_0}$ fulfilled in the sense

$$\lim_{t \to 0+} \left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{F}_1(t) - \mathbb{F}_{1_0}\|_2 + \|\mathbb{F}_2(t) - \mathbb{F}_{2_0}\|_2\right) = 0. \tag{4.7.9}$$

4.7.2 Parabolic approximation

We start with approximations, where the term $\varepsilon \int_\Omega \nabla \mathbb{F}_i : \nabla \mathbb{A}$ ($\varepsilon \in (0,1)$ is arbitrary) representing the stress diffusion is added to the left-hand side of (4.7.8). Similarly as in Section 4.5.1 and with the help of the methods used in the proof of Proposition A.3.1, we have that for each $\varepsilon \in (0,1)$ there exist

$$\boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}), \tag{4.7.10}$$

$$\mathbb{F}_i \in C_{weak}\left([0,T]; (L^2(\Omega))^{2\times 2}\right) \quad (i = 1,2) \tag{4.7.11}$$

fulfilling the uniform bounds

$$\begin{aligned}
&\sup_{t\in(0,T)} \left(\|\boldsymbol{v}(t)\|_2^2 + \sum_{i=1}^{2} \|\mathbb{F}_i(t)\|_2^2 \right) + \|\nabla \boldsymbol{v}\|_{2,Q_T}^2 + \sum_{i=1}^{2} \|\mathbb{F}_i\|_{4,Q_T}^4 \\
&\quad + \|\partial_t \boldsymbol{v}\|_{L^2\left(0,T;(W^{1,2}_{0,\mathrm{div}})^*\right)} + \sum_{i=1}^{2} \|\partial_t \mathbb{F}_i\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} + \varepsilon \sum_{i=1}^{2} \|\nabla \mathbb{F}_i\|_{2,Q_T}^2 \\
&\le C(T, \Omega, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_{1_0}\|_2, \|\mathbb{F}_{2_0}\|_2),
\end{aligned} \tag{4.7.12}$$

and satisfying, for $i = 1,2$, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, for all $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and for all $t \in (0,T)$,

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i(\mathbb{F}_i \mathbb{F}_i^T) : \nabla \boldsymbol{w} = 0, \tag{4.7.13}$$

$$\begin{aligned}
\langle \partial_t \mathbb{F}_i, \mathbb{A} \rangle - \int_\Omega (\mathbb{F}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v}) \mathbb{F}_i) : \mathbb{A} + \frac{1}{2} \int_\Omega (\mathbb{F}_i \mathbb{F}_i^T \mathbb{F}_i - \mathbb{F}_i) : \mathbb{A} \\
+ \varepsilon \int_\Omega \nabla \mathbb{F}_i : \nabla \mathbb{A} = 0
\end{aligned} \tag{4.7.14}$$

and attaining the initial data in the following sense

$$\limsup_{t\to 0+} \left(\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 + \sum_{i=1}^{2} G_i \|\mathbb{F}_i(t) - \mathbb{F}_{i_0}\|_2^2 \right) \le 0. \tag{4.7.15}$$

4.7.3 Weak convergence results as $\varepsilon \to 0_+$

Let $\{(\boldsymbol{v}_\varepsilon, \mathbb{F}_{1,\varepsilon}, \mathbb{F}_{2,\varepsilon})\}$ denote the solutions to (4.7.13)–(4.7.14). Employing (4.7.12), we can find subsequences (that we do not relabel) such that $(i = 1, 2)$

$$
\begin{aligned}
\boldsymbol{v}_\varepsilon &\rightharpoonup^* \boldsymbol{v} && \text{weakly-* in } L^\infty(0,T; L^2_{\boldsymbol{n},\mathrm{div}}), && (4.7.16)\\
\boldsymbol{v}_\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}}) \cap (L^4(Q_T))^2, && (4.7.17)\\
\partial_t \boldsymbol{v}_\varepsilon &\rightharpoonup \partial_t \boldsymbol{v} && \text{weakly in } L^2\left(0,T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*\right), && (4.7.18)\\
\mathbb{F}_{i,\varepsilon} &\rightharpoonup^* \mathbb{F}_i && \text{weakly-* in } L^\infty\left(0,T; (L^2(\Omega))^{2\times 2}\right), && (4.7.19)\\
\mathbb{F}_{i,\varepsilon} &\rightharpoonup \mathbb{F}_i && \text{weakly in } (L^4(Q_T))^{2\times 2}, && (4.7.20)\\
\partial_t \mathbb{F}_{i,\varepsilon} &\rightharpoonup \partial_t \mathbb{F}_i && \text{weakly in } L^{\frac{4}{3}}\left(0,T; ((W^{1,2}(\Omega))^{2\times 2})^*\right), && (4.7.21)\\
\boldsymbol{v}_\varepsilon &\to \boldsymbol{v} && \text{strongly in } (L^q(Q_T))^2 \text{ for all } q \in [1,4), && (4.7.22)\\
\varepsilon \nabla \mathbb{F}_{i,\varepsilon} &\to \mathbb{O} && \text{strongly in } (L^2(Q_T))^{2\times 2\times 2}, && (4.7.23)\\
\mathbb{F}_{i,\varepsilon}\mathbb{F}_{i,\varepsilon}^T &\rightharpoonup \overline{\mathbb{F}_i\mathbb{F}_i^T} && \text{weakly in } (L^2(Q_T))^{2\times 2}, && (4.7.24)\\
\mathbb{F}_{i,\varepsilon}\mathbb{F}_{i,\varepsilon}^T\mathbb{F}_{i,\varepsilon} &\rightharpoonup \overline{\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i} && \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}, && (4.7.25)\\
\nabla \boldsymbol{v}_\varepsilon \mathbb{F}_{i,\varepsilon} &\rightharpoonup \overline{\nabla \boldsymbol{v}\mathbb{F}_i} && \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times 2}. && (4.7.26)
\end{aligned}
$$

Then, using (4.7.16)–(4.7.26), we let $\varepsilon \to 0_+$ in (4.7.13) with $\boldsymbol{v} := \boldsymbol{v}_\varepsilon$ and in (4.7.14) with $\mathbb{F}_i := \mathbb{F}_{i,\varepsilon}$, $i = 1,2$, and conclude, for all $\boldsymbol{w} \in W^{1,2}_{\mathbf{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and almost all $t \in (0,T)$, that

$$
\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \sum_{i=1}^{2} \int_\Omega G_i \overline{\mathbb{F}_i\mathbb{F}_i^T} : \nabla \boldsymbol{w} = 0, \quad (4.7.27)
$$

$$
\langle \partial_t \mathbb{F}_i, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}_i \otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega \overline{(\nabla \boldsymbol{v})\mathbb{F}_i} : \mathbb{A} + \frac{1}{2}\int_\Omega (\overline{\mathbb{F}_i\mathbb{F}_i^T\mathbb{F}_i} - \mathbb{F}_i) : \mathbb{A} = 0. \quad (4.7.28)
$$

As $\boldsymbol{v} \in L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, $\partial_t \boldsymbol{v} \in L^2(0,T; (W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, $\mathbb{F}_i \in L^\infty(0,T; (L^2(\Omega))^{2\times 2})$, $\partial_t \mathbb{F}_i$ belong to $L^{\frac{4}{3}}(0,T; (W^{1,2}(\Omega))^{2\times 2})^*)$, $i = 1,2$, and $(W^{1,2}(\Omega))^{2\times 2}$ is dense in $(L^2(\Omega))^{2\times 2}$, the functions $\boldsymbol{v}$, $\mathbb{F}_i$ after a possible change on a zero-measure subset of $(0,T)$ satisfy

$$
\boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}), \quad (4.7.29)
$$

$$
\mathbb{F}_i \in C_{weak}\left([0,T]; (L^2(\Omega))^{2\times 2}\right) \quad (i = 1,2). \quad (4.7.30)
$$

The attainment of initial conditions is done exactly as in Section 4.5.2 and we do not repeat the proof here. Similarly, following step by step the procedure in Section 4.5.3, we get

$$
\mathbb{F}_i \in C([0,T]; (L^2(\Omega))^{2\times 2}) \quad (i = 1,2). \quad (4.7.31)
$$

As $\mathbb{F}_{i,\varepsilon} \rightharpoonup \mathbb{F}_i$, $i = 1,2$ weakly in $(L^4(Q_T))^{2\times 2}$ and $\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v}$ weakly in $L^2(0,T; W^{1,2}_{\mathbf{0},\mathrm{div}})$, in order to conclude (4.7.7)–(4.7.8) from (4.7.27)–(4.7.28) it suffices to prove the compactness of $\{\mathbb{F}_{i,\varepsilon}\}$ in $(L^2(Q_T))^{2\times 2}$.

4.7.4 Compactness of $\mathbb{F}_{1,\varepsilon}, \mathbb{F}_{2,\varepsilon}$ in $(L^2(Q_T))^{2\times 2}$

As $\mathbb{F}_{i,\varepsilon} \rightharpoonup \mathbb{F}_i$ weakly in $(L^4(Q_T))^{2\times 2}$, $i = 1,2$, the compactness of $\mathbb{F}_{i,\varepsilon}$ in $(L^2(Q_T))^{2\times 2}$ is equivalent to the condition

$$
f_i := \overline{|\mathbb{F}_i|^2} - |\mathbb{F}_i|^2 = 0 \text{ a.e. in } Q_T \ (i = 1,2). \quad (4.7.32)
$$

Note that the inequality $f_i \geq 0$ directly follows from the weak lower semicontinuity, see also Step 3 in Section 4.5.4. Hence, our aim is to show that

$$G_1 f_1 + G_2 f_2 \leq 0 \text{ a.e. in } Q_T, \tag{4.7.33}$$

which thanks to the positivity of each G_i implies (4.7.32). Proceeding exactly in the same way as in Section 4.5.4, Step 1, we prove (4.5.57) and (4.5.58) with $\mathbb{F}_{i,\varepsilon}$ in the role of $\mathbb{F}_\varepsilon$ and $\mathbb{F}_i$ in the role of $\mathbb{F}$ ($i = 1, 2$). Next, following the computations from Section 4.5.4, Step 2, we derive from the difference between (4.5.57) and (4.5.58) (with $\mathbb{F}_{i,\varepsilon}$ in the role of $\mathbb{F}_\varepsilon$ and $\mathbb{F}_i$ in the role of $\mathbb{F}$) multiplied by G_i and summed over $i = 1, 2$, for all $\varphi \in C_c^\infty((-\infty, T) \times \Omega)$, $\varphi \geq 0$, the inequality

$$-\int_{Q_T} \sum_{i=1}^{2} (G_i f_i) \partial_t \varphi - \int_{Q_T} \sum_{i=1}^{2} (G_i f_i) \boldsymbol{v} \cdot \nabla \varphi \leq \int_{Q_T} L \sum_{i=1}^{2} (G_i f_i) \varphi \tag{4.7.34}$$

with

$$L := \Big(1 + 2(|\nabla \boldsymbol{v}| + |\mathbb{F}_1|^2 + |\mathbb{F}_1|^2)\Big).$$

The only difference from the computations in Step 2 in Section 4.5.4 is that instead of $\mathbb{F}_\varepsilon$ and $\mathbb{F}$ we work with the sums $\sum_{i=1}^2 G_i \mathbb{F}_{i,\varepsilon}$ and $\sum_{i=1}^2 G_i \mathbb{F}_i$. Then, following Step 3 in Section 4.5.4 and working with $\sum_{i=1}^2 G_i f_i$ instead of f, we conclude (4.7.33) from (4.7.34). This proves the compactness of $\mathbb{F}_{i,\varepsilon}$ ($i = 1, 2$) and consequently we can identify all weak limits in (4.7.27) and (4.7.28) and get the solution to (4.7.1)–(4.7.3) (and (4.7.5)).

4.7.5 Concluding the result

To complete the proof of Theorem 4.2.1, it suffices to conclude (4.2.10), (4.2.13) and the initial condition (4.2.14). However, in order to conclude (4.2.10), we repeat the procedure from Section 4.5.5 with $\mathbb{F}_i$ in the role of $\mathbb{F}$ ($i = 1, 2$), and in order to conclude (4.2.13) and (4.2.14), we repeat the procedure from Section 4.6 with $\mathbb{F}_i$ in the role of $\mathbb{F}$ and $\mathbb{B}_i$ in the role of $\mathbb{B}$ ($i = 1, 2$).

Conclusion

We developed a robust mathematical theory for selected models describing the mechanical behavior of materials with complex microstructure. We focused, as a fundamental point of the robust mathematical theory, on the long-time and large-data existence theory (the existence of weak solutions satisfying relevant physical principles) for initial- and boundary-value problems associated with two class of models.

In Chapters 2 and 3, we presented the results from the recently published papers [4] and [3]. The model studied in [4], and its generalization treated in [3], describe basic mechanical properties of flows of granular water saturated geological materials and has some relevance to the problem of static liquefaction and enhanced oil recovery. Our motivation for writing [4] was recent research concerning the implicitly constituted materials on one hand and a study by Chupin and Mathé [1] on the other hand. We extended the results presented in [1] in several directions. First, we studied slightly different system of PDEs, namely the one we were able to derive from the basic governing equations of the theory of mixtures, under the cascade of several justified simplifications. Second, the activated system contains in comparison to [1], a non-trivial right-hand side in the equation for the fluid pressure p_{f}. Consequently, we had to use a different approach to get L^∞-estimates for p_{f}. Third, inspired by [1] we provided characterization of the constitutive equation in Proposition 2.1.1. Using one of these equivalent descriptions, we corrected the proof in [1] and developed a useful tool exploited in the proof of Proposition 2.3.3. Fourth, we considered stick-slip boundary conditions that are physically relevant and, on contrary to no-slip boundary condition, guarantees the integrability of the pressure up to the boundary. Finally, we used L^∞-truncation method to analyze three-dimensional flows (while the result in [1] concerns planar flows).

In Chapter 3 (see also [3]) we strengthened the result from Chapter 2 (see also [4]). Incorporating a more general model, we provided a different existence proof for more general data (particularly for the external forces that are merely L^2-integrable).

In Chapter 4, we present the results from [5], where we developed a robust mathematical theory for a viscoelastic rate-type fluid model with two relaxation mechanisms (the mixture of two Giesekus models, while the single Giesekus model was briefly analyzed in [2]) in two spatial dimensions. More specifically, we proved the long-time and large-data existence of weak solutions to unsteady flows of such fluids subject to no-slip boundary conditions. It is the first long-time and large-data existence result for a viscoelastic model of higher (second) order. We also gave a complete rigorous proof of the long-time and large-data existence of weak solutions to the analogous problem associated with the Giesekus model in two spatial dimensions. Doing so, we completed and corrected the theoretical considerations outlined in [2]. The result established here is currently used by the authors as an auxiliary tool for the development of a complete and rigorous long-time and large-data theory for the same viscoelastic models in three dimensions.

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A. Appendix

A.1 Proof of Proposition 3.4.1

Our goal is to prove Proposition 3.4.1. Let us recall that in this section we fix $n \in \mathbb{N}$, we consider G_n smooth function with the properties stated at the beginning of Section 3.4 and the regularization of the material responses given in (3.4.2) and (3.4.3). In what follows, to simplify the notation we drop the indices n.

The proof is split in the following steps.

Step 1. *Approximations.* For any $m \in \mathbb{N}$, we look for

$$\boldsymbol{v}^m(t,\boldsymbol{x}) := \sum_{r=1}^{m} c_r^m(t)\boldsymbol{w}^r(\boldsymbol{x}), \quad p_{\mathrm{f}}^m(t,\boldsymbol{x}) := \sum_{r=1}^{m} d_r^m(t)z^r(\boldsymbol{x}) \tag{A.1.1}$$

satisfying for $r := 1, ..., m$

$$\begin{aligned}\int_\Omega \partial_t \boldsymbol{v}^m \cdot \boldsymbol{w}^r + \int_\Omega \Big(\mathbb{S}^m : \mathbb{D}\boldsymbol{w}^r + \operatorname{div}(\boldsymbol{v}^m \otimes \boldsymbol{v}^m)G(|\boldsymbol{v}^m|^2)\cdot \boldsymbol{w}^r\Big)& \\ + \int_{\partial\Omega} \boldsymbol{s}^m \cdot \boldsymbol{w}^r = \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}^r,&\end{aligned} \tag{A.1.2}$$

where

$$\begin{aligned}\mathbb{S}^m := \mathcal{S}(p_{\mathrm{f}}^m, \mathbb{D}\boldsymbol{v}^m) = \tau(p_{\mathrm{f}}^m)\frac{\mathbb{D}\boldsymbol{v}^m}{|\mathbb{D}\boldsymbol{v}^m| + \frac{1}{n}} + \mathbb{D}\boldsymbol{v}^m\left(1 - \frac{\delta_*}{|\mathbb{D}\boldsymbol{v}^m|}\right)^+& \\ \text{with } \tau(p_{\mathrm{f}}^m) = (p_s - p_{\mathrm{f}}^m)^+,&\end{aligned} \tag{A.1.3}$$

$$\boldsymbol{s}^m := s(\boldsymbol{v}_\tau^m) = s_*\frac{\boldsymbol{v}_\tau^m}{|\boldsymbol{v}_\tau^m| + \frac{1}{n}} + \boldsymbol{v}_\tau^m\left(1 - \frac{\beta_*}{|\boldsymbol{v}_\tau^m|}\right)^+, \tag{A.1.4}$$

and

$$\int_\Omega \partial_t p_{\mathrm{f}}^m z^r - \int_\Omega (p_{\mathrm{f}}^m \boldsymbol{v}^m \cdot \nabla z^r - \nabla p_{\mathrm{f}}^m \cdot \nabla z^r) = \int_\omega (\boldsymbol{f} - p_s \boldsymbol{v}^m)\cdot \nabla z^r, \tag{A.1.5}$$

where $\{\boldsymbol{w}^i\}_{i\in\mathbb{N}}$ is an orthogonal basis in $W^{1,2}_{\mathbf{n},\mathrm{div}}$ consisting of eigenfunctions of the Stokes operator with boundary conditions $\boldsymbol{w}^i \cdot \boldsymbol{n} = 0$ and $[(\mathbb{D}\boldsymbol{w}^i)\boldsymbol{n}]_\tau = \mathbf{0}$ on $\partial\Omega$, while $\{z_j\}_{j\in\mathbb{N}}$ is an orthogonal basis in $W^{1,2}(\Omega)$ consisting of eigenfunctions of the Laplace operator subject to the Neumann homogeneous boundary conditions. The system is supplemented with the corresponding initial conditions $\boldsymbol{v}_0^m$ and p_0^m, obtained by projection $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$ onto the span of $\{\boldsymbol{w}^1, \dots, \boldsymbol{w}^m\}$ and respectively $p_0 \in L^2(\Omega)$ onto the span of $\{z^1, \dots, z^m\}$. Then the local in time existence of $\boldsymbol{v}^m$ and p_{f}^m follows from the Caratheodory theory for systems of ordinary differential equations, whereas the global in time existence is a consequence of the uniform estimates established below.

Step 2. *Uniform estimates.* Multiplying (A.1.2) by $c_r^m(t)$ and (A.1.5) by $d_r^m(t)$ and taking the sum over $r = 1, \dots, m$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m(t)\|_2^2 + \int\limits_{\{|\mathbb{D}\boldsymbol{v}^m|>\delta_*\}} |\mathbb{D}\boldsymbol{v}^m|^2 + \int\limits_{\Omega} \tau(p_{\mathrm{f}}^m)\frac{|\mathbb{D}\boldsymbol{v}^m|^2}{|\mathbb{D}\boldsymbol{v}^m| + \frac{1}{n}} \\ + \int\limits_{\partial\Omega} \boldsymbol{s}^m \cdot \boldsymbol{v}_\tau^m = \int\limits_{\Omega} \boldsymbol{f}\cdot\boldsymbol{v}^m + \int\limits_{\{|\mathbb{D}\boldsymbol{v}^m|>\delta_*\}} \delta_*|\mathbb{D}\boldsymbol{v}^m|, \end{aligned} \tag{A.1.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_{\mathrm{f}}^m(t)\|_2^2 + \|\nabla p_f^m(t)\|_2^2 = \int\limits_{\Omega} (\boldsymbol{f} - p_s\boldsymbol{v}^m)\cdot\nabla p_{\mathrm{f}}^m, \tag{A.1.7}$$

Adding $\int_{\{|\mathbb{D}\boldsymbol{v}^m|\le\delta_*\}} |\mathbb{D}\boldsymbol{v}^m|^2$ to both sides of (A.1.6)

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m(t)\|_2^2 + \int\limits_{\Omega} |\mathbb{D}\boldsymbol{v}^m|^2 + \int\limits_{\Omega} \tau(p_{\mathrm{f}}^m)\frac{|\mathbb{D}\boldsymbol{v}^m|^2}{|\mathbb{D}\boldsymbol{v}^m| + \frac{1}{n}} + \int\limits_{\partial\Omega} s_n(\boldsymbol{v}_\tau^m)\cdot\boldsymbol{v}_\tau^m \\ \le \int\limits_{\Omega} \boldsymbol{f}\cdot\boldsymbol{v}^m + \int\limits_{\Omega} \delta_*|\mathbb{D}\boldsymbol{v}^m| + \delta_*^2|\Omega| \end{aligned}$$

and then by the Young inequality, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m(t)\|_2^2 + \frac{1}{2}\int\limits_{\Omega} |\mathbb{D}\boldsymbol{v}^m|^2 + \int\limits_{\Omega} \tau(p_{\mathrm{f}}^m)\frac{|\mathbb{D}\boldsymbol{v}^m|^2}{|\mathbb{D}\boldsymbol{v}^m| + \frac{1}{n}} + \int\limits_{\partial\Omega} s_n(\boldsymbol{v}_\tau^m)\cdot\boldsymbol{v}_\tau^m \\ \le \int\limits_{\Omega} \boldsymbol{f}\cdot\boldsymbol{v}^m + \frac{3}{2}\delta_*^2|\Omega|. \end{aligned} \tag{A.1.8}$$

Integrating in time, by the Korn and the Young inequalities, using also the fact that the last two terms on the left-hand side of (A.1.8) are non-negative, one concludes that

$$\sup_{t\in[0,T]} \|\boldsymbol{v}^m(t)\|_2^2 + \int\limits_{Q_T} |\mathbb{D}\boldsymbol{v}^m|^2 \le C(\boldsymbol{f}, \boldsymbol{v}_0, \delta_*, |Q_T|). \tag{A.1.9}$$

By the interpolation inequality

$$\|z\|_{\frac{10}{3}} \le \|z\|_2^{\frac{2}{5}}\|z\|_6^{\frac{3}{5}} \le C\|z\|_2^{\frac{2}{5}}\|z\|_{1,2}^{\frac{3}{5}} \tag{A.1.10}$$

and the trace inequalities (see [7, Lemma 1.11]), we obtain

$$\sup_m \Big(\|\boldsymbol{v}^m\|_{\frac{10}{3},Q_T} + \|\boldsymbol{v}_\tau^m\|_{\frac{8}{3},\Sigma_T}\Big) < +\infty. \tag{A.1.11}$$

As a consequence, integrating (A.1.7) in time, we deduce that

$$\sup_{t\in[0,T]} \|p_{\mathrm{f}}^m(t)\|_2^2 + \int\limits_{Q_T} |\nabla p_{\mathrm{f}}^m|^2 \le C\|\boldsymbol{f}\|_{2,Q_T}^2 + C\|p_s\|_{5,Q_T}^2\|\boldsymbol{v}^m\|_{\frac{10}{3},Q_T}^2 + \|p_0\|_2^2, \tag{A.1.12}$$

and thus

$$\sup_{t\in(0,T)} \|p_{\mathrm{f}}^m(t)\|_2 + \|\nabla p_{\mathrm{f}}^m\|_{L^2(Q_T)} \le C(\boldsymbol{f}, p_s, p_0). \tag{A.1.13}$$

Again (A.1.10) gives

$$\sup_m \|p_\mathrm{f}\|_{\frac{10}{3},Q_T} < +\infty. \tag{A.1.14}$$

Recalling the explicit formulas for $\mathbb{S}^m$ and $\boldsymbol{s}^m$ it then follows

$$\sup_m \left(\|\mathbb{S}^m\|_{2,Q_T} + \|\boldsymbol{s}^m\|_{\frac{8}{3},\Sigma_T}\right) < +\infty. \tag{A.1.15}$$

Employing the inequality

$$\|z\|_4 \le \|z\|_2^{\frac{1}{4}} \|z\|_6^{\frac{3}{4}} \le C\|z\|_2^{\frac{1}{4}} \|z\|_{1,2}^{\frac{3}{4}}$$

we deduce corresponding uniform estimates for $\boldsymbol{v}^m$ and p_f^m respectively in $(L^4(Q_T))^3$ and $L^4(Q_T)$, then by virtue of them it results

$$\sup_m \|\partial_t p_\mathrm{f}^m\|_{L^{\frac{4}{3}}(0,T;(W^{1,2}(\Omega))^*)} < +\infty. \tag{A.1.16}$$

Analogously and by virtue of the truncation in the convective term, we also get

$$\sup_m \|\partial_t \boldsymbol{v}^m\|_{L^2(0,T;(W^{1,2}_{\boldsymbol{n},\mathrm{div}})^*)} < +\infty. \tag{A.1.17}$$

Step 3. *Limit.* By virtue of uniform estimates established above there exist subsequences of $\{\boldsymbol{v}^m\}, \{p_\mathrm{f}^m\}$, $\{\mathbb{S}^m\}$ and $\{\boldsymbol{s}^m\}$, converging respectively weakly (or *-weakly) to $\boldsymbol{v}, p_\mathrm{f}, \mathbb{S}$ and $\boldsymbol{s}$ in the corresponding function spaces. Furthermore, the Aubin-Lions compactness lemma implies the following strong convergences:

$$\boldsymbol{v}^m \to \boldsymbol{v} \text{ a.e. in } Q_T \text{ and strongly in } (L^q(Q_T))^3 \text{ for any } q \in \left[1, \frac{10}{3}\right), \tag{A.1.18}$$

$$p_\mathrm{f}^m \to p_\mathrm{f} \text{ a.e. in } Q_T \text{ and strongly in } L^q(Q_T) \text{ for any } q \in \left[1, \frac{10}{3}\right), \tag{A.1.19}$$

$$\boldsymbol{v}_\tau^m \to \boldsymbol{v}_\tau \text{ a.e. in } \Sigma_T \text{ and strongly in } (L^q(\Sigma_T))^3 \text{ for any } q \in \left[1, \frac{8}{3}\right). \tag{A.1.20}$$

As a consequence $\boldsymbol{v}, p_\mathrm{f}, \mathbb{S}$ and $\boldsymbol{s}$ fulfill the weak formulations stated in Proposition 3.4.1.

Step 4. *Attainment of the constitutive equations.* The convergence

$$\boldsymbol{s}^m \rightharpoonup \boldsymbol{s} \text{ weakly in } (L^{\frac{8}{3}}(\Sigma_T))^3$$

together with (A.1.20) ensures that

$$\lim_{m\to+\infty} \int_{\Sigma_T} \boldsymbol{s}^m \cdot \boldsymbol{v}_\tau^m = \int_{\Sigma_T} \boldsymbol{s} \cdot \boldsymbol{v}_\tau. \tag{A.1.21}$$

Then, thanks to the monotonicity it is standard to prove that

$$\boldsymbol{s} = s(\boldsymbol{v}_\tau).$$

Next, it follows from the monotonicity that

$$0 \le \int_{Q_T} (\mathbb{S}^m - \boldsymbol{\mathcal{S}}(p_\mathrm{f}^m, \mathbb{A})) : (\mathbb{D}\boldsymbol{v}^m - \mathbb{A}) \quad \text{for all } \mathbb{A} \in (L^2(Q_T))^{3\times 3}. \tag{A.1.22}$$

Now, note that by (A.1.19),

$$\begin{aligned} &\mathcal{S}(p_{\mathrm{f}}^m,\mathbb{A}) := (p_s-p_{\mathrm{f}}^m)^+\tfrac{\mathbb{A}}{|\mathbb{A}|+\frac{1}{n}}+\mathbb{A}\left(1-\tfrac{\delta_*}{|\mathbb{A}|}\right)^+ \\ &\to (p_s-p_{\mathrm{f}})^+\tfrac{\mathbb{A}}{|\mathbb{A}|+\frac{1}{n}}+\mathbb{A}\left(1-\tfrac{\delta_*}{|\mathbb{A}|}\right)^+ =: \mathcal{S}(p_{\mathrm{f}},\mathbb{A}) \text{ strongly in } (L^2(Q_T))^{3\times 3} \end{aligned} \tag{A.1.23}$$

while, as $\boldsymbol{v}$ can play the role of a test function in the established weak formulation, it is standard to obtain

$$\limsup_{m\to+\infty}\int_{Q_T}\mathbb{S}^m:\mathbb{D}\boldsymbol{v}^m \le \int_{Q_T}\mathbb{S}:\mathbb{D}\boldsymbol{v}. \tag{A.1.24}$$

Finally, thanks to the convergences

$$\begin{aligned} &\mathbb{D}\boldsymbol{v}^m \rightharpoonup \mathbb{D}\boldsymbol{v} \text{ weakly in } (L^2(Q_T))^{3\times 3},\\ &\mathbb{S}^m \rightharpoonup \mathbb{S} \text{ weakly in } (L^2(Q_T))^{3\times 3}, \end{aligned}$$

and (A.1.23), the limit as $m\to+\infty$ in (A.1.22) gives

$$0\le \int_{Q_T}(\mathbb{S}-\mathcal{S}(p_{\mathrm{f}},\mathbb{A})):(\mathbb{D}\boldsymbol{v}-\mathbb{A}) \quad \text{for all } \mathbb{A}\in (L^2(Q_T))^{3\times 3}. \tag{A.1.25}$$

At this point, it is standard to choose $\mathbb{A}=\mathbb{D}\boldsymbol{v}\pm\varepsilon\mathbb{B}$ for arbitrary $\mathbb{B}\in L^2(Q_T)$ and $\varepsilon>0$ and arrive at

$$0=\int_{Q_T}\mathbb{B}:(\mathbb{S}-\mathcal{S}(p_{\mathrm{f}},\mathbb{D}\boldsymbol{v})) \quad \text{for all } \mathbb{B}\in (L^2(Q_T))^{3\times 3},$$

which implies $\mathbb{S}=\mathcal{S}(p_{\mathrm{f}},\mathbb{D}\boldsymbol{v})$ almost everywhere in Q_T. The proof of Proposition 3.4.1 is complete. □

A.2 Proof of Proposition 4.5.1

Before making the proof of Proposition 4.5.1 let us quote from [81] the following lemma concerning stationary Stokes problems.

Lemma A.2.1. Let $d\ge 2$, $m\ge -1$, $q\in(1,\infty)$, $\tilde\Omega\subset\mathbb{R}^d$, $\tilde\Omega\in C^{\max\{m+2,2\}}$, let $\boldsymbol{g}\in (W^{m,q}(\tilde\Omega))^d$, $\boldsymbol{w}^*\in (W^{m+2-\frac{1}{q},q}(\partial\tilde\Omega))^d$ and $\int_{\partial\tilde\Omega}\boldsymbol{w}^*\cdot\boldsymbol{n}=0$. Then there exists unique weak solution $[\boldsymbol{w},\tilde p]$, $\int_{\tilde\Omega}\tilde p=0$, to the Stokes problem

$$\begin{aligned} -\Delta\boldsymbol{w}+\nabla\tilde p&=\boldsymbol{g} && \text{in } \tilde\Omega,\\ \operatorname{div}\boldsymbol{w}&=0 && \text{in } \tilde\Omega,\\ \boldsymbol{w}&=\boldsymbol{w}^* && \text{on } \partial\tilde\Omega, \end{aligned}$$

more specifically, there exists unique couple $[\boldsymbol{w},\tilde p]$ fulfilling

$$\boldsymbol{w}\in (W^{m+2,q}(\tilde\Omega))^d,\quad \boldsymbol{w}-\boldsymbol{w}^*\in (W_0^{1,q}(\tilde\Omega))^d,\quad \tilde p\in W^{m+1,q}(\tilde\Omega),\quad \int_{\tilde\Omega}\tilde p=0$$

and

$$\begin{aligned} \int_{\tilde\Omega}\nabla\boldsymbol{w}:\nabla\boldsymbol{\Phi}-\int_{\tilde\Omega}\tilde p\operatorname{div}\boldsymbol{\Phi}&=\langle\boldsymbol{g},\boldsymbol{\Phi}\rangle && \text{for all } \boldsymbol{\Phi}\in (W_0^{1,q'}(\tilde\Omega))^d,\\ \int_{\tilde\Omega}(\operatorname{div}\boldsymbol{w})\phi&=0 && \text{for all } \phi\in L^{q'}(\tilde\Omega), \end{aligned}$$

which is equivalent to the existence of unique

$$\boldsymbol{w} \in (W^{m+2,q}(\tilde\Omega))^d, \quad \boldsymbol{w} - \boldsymbol{w}^* \in (W_0^{1,q}(\tilde\Omega))^d, \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \tilde\Omega$$

fulfilling

$$\int_{\tilde\Omega} \nabla \boldsymbol{w} : \nabla \boldsymbol{\Phi} = \langle \boldsymbol{g}, \boldsymbol{\Phi} \rangle \qquad \text{for all } \boldsymbol{\Phi} \in \widetilde{W}^{1,q'}_{\boldsymbol{0},\mathrm{div}}.$$

Moreover, the solution satisfies the estimate

$$\|\boldsymbol{w}\|_{(W^{m+2,q}(\tilde\Omega))^d} + \|\tilde p\|_{W^{m+1,q}(\tilde\Omega)} \le \|\boldsymbol{f}\|_{(W^{m,q}(\tilde\Omega))^d} + \|\boldsymbol{w}^*\|_{(W^{m+2-\frac{1}{q},q}(\partial\tilde\Omega))^d}$$

with the convention $W^{-1,q}(\tilde\Omega) = (W_0^{1,q}(\tilde\Omega))^*$, $W^{0,q}(\tilde\Omega) = L^q(\tilde\Omega)$.

Proof. See [81]. □

For lucidity let us recall the formulation of Proposition 4.5.1.

Proposition A.2.2. Let $\tilde\Omega \subset \overline{\tilde\Omega} \subset \Omega$, $\tilde\Omega \in C^\infty$. Then for every $\varepsilon \in (0,1)$ there exists p_ε of the form $p_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon}$, where

$$p_{1,\varepsilon} \in L^2(0,T; W^{2,2}(\tilde\Omega)), \tag{A.2.1}$$

$$p_{2,\varepsilon} \in L^2((0,T) \times \tilde\Omega), \tag{A.2.2}$$

$$\partial_t \left(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}\right) \in L^2\left(0,T; ((W_0^{1,2}(\tilde\Omega))^2)^*\right) \tag{A.2.3}$$

and, for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde\Omega))^2$ and almost all $t \in (0,T)$, it holds

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}), \boldsymbol{w} \rangle = \int_{\tilde\Omega} (\mathbb{G}_\varepsilon : \nabla \boldsymbol{w}) + \int_{\tilde\Omega} p_{2,\varepsilon} \operatorname{div} \boldsymbol{w}, \tag{A.2.4}$$

where $\mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T$. Next, there exists p of the form $p = p_1 + p_2$, where

$$p_1 \in L^2(0,T; W^{2,2}(\tilde\Omega)), \tag{A.2.5}$$

$$p_2 \in L^2((0,T) \times \tilde\Omega), \tag{A.2.6}$$

$$\partial_t \left(\boldsymbol{v} + \nabla p_1\right) \in L^2\left(0,T; ((W_0^{1,2}(\tilde\Omega))^2)^*\right) \tag{A.2.7}$$

and, for all $\boldsymbol{w} \in (W_0^{1,2}(\tilde\Omega))^2$ and almost all $t \in (0,T)$, it holds

$$\langle \partial_t(\boldsymbol{v} + \nabla p_1), \boldsymbol{w} \rangle = \int_{\Omega} (\mathbb{G} : \nabla \boldsymbol{w}) + \int_{\Omega} p_2 \operatorname{div} \boldsymbol{w}, \tag{A.2.8}$$

where $\mathbb{G} := (\boldsymbol{v} \otimes \boldsymbol{v}) - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}$. Moreover, as $\varepsilon \to 0_+$, we have

$$p_{1,\varepsilon} \to p_1 \qquad \text{strongly in } L^2(0,T; W^{2,2}_{loc}(\tilde\Omega)), \tag{A.2.9}$$

$$p_{2,\varepsilon} \rightharpoonup p_2 \qquad \text{weakly in } L^2((0,T) \times \tilde\Omega). \tag{A.2.10}$$

The functions $\nabla p_{1,\varepsilon}$ and ∇p_1 belong to $C([0,T]; (L^2(\tilde\Omega))^2)$ and

$$\nabla p_{1,\varepsilon}(0,\cdot) = \nabla p_1(0,\cdot) \quad \text{a.e. in } \tilde\Omega. \tag{A.2.11}$$

Proof. Let $\tilde{\Omega}$ be an arbitrary smooth domain fulfilling $\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \Omega$. For every $t \in [0,T]$ let us introduce the Stokes problems

$$-\Delta \boldsymbol{w}_{1,\varepsilon} + \nabla p_{1,\varepsilon} = \boldsymbol{v}_\varepsilon \qquad \text{in } \tilde{\Omega}, \tag{A.2.12}$$

$$\operatorname{div} \boldsymbol{w}_{1,\varepsilon} = 0 \qquad \text{in } \tilde{\Omega}, \tag{A.2.13}$$

$$\boldsymbol{w}_{1,\varepsilon} = \boldsymbol{0} \qquad \text{on } \partial\tilde{\Omega}, \tag{A.2.14}$$

$$-\Delta \boldsymbol{w}_{2,\varepsilon} + \nabla p_{2,\varepsilon} = \operatorname{div} \mathbb{G}_\varepsilon \qquad \text{in } \tilde{\Omega}, \tag{A.2.15}$$

$$\operatorname{div} \boldsymbol{w}_{2,\varepsilon} = 0 \qquad \text{in } \tilde{\Omega}, \tag{A.2.16}$$

$$\boldsymbol{w}_{2,\varepsilon} = \boldsymbol{0} \qquad \text{on } \partial\tilde{\Omega}. \tag{A.2.17}$$

Since $\boldsymbol{v}_\varepsilon \in L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$, Lemma A.2.1 implies for all $t \in [0,T]$ the existence of unique weak solution $[\boldsymbol{w}_{1,\varepsilon}, p_{1,\varepsilon}]$, $\int_{\tilde{\Omega}} p_{1,\varepsilon} = 0$, to the system (A.2.12)–(A.2.14), more precisely $[\boldsymbol{w}_{1,\varepsilon}, p_{1,\varepsilon}]$ satisfy for all $\boldsymbol{\Phi} \in (W^{1,2}_0(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$ and all $t \in [0,T]$

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_{1,\varepsilon} : \nabla \boldsymbol{\Phi} - \int_{\tilde{\Omega}} p_{1,\varepsilon} \operatorname{div} \boldsymbol{\Phi} = \int_{\tilde{\Omega}} \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\Phi}, \tag{A.2.18}$$

$$\int_{\tilde{\Omega}} (\operatorname{div} \boldsymbol{w}_{1,\varepsilon}) \phi = 0 \tag{A.2.19}$$

and the estimate

$$\|\boldsymbol{w}_{1,\varepsilon}\|_{(W^{m+2,2}(\tilde{\Omega}))^2} + \|p_{1,\varepsilon}\|_{W^{m+1,2}(\tilde{\Omega})} \le \|\boldsymbol{v}_\varepsilon\|_{(W^{m,2}(\Omega))^2}, \quad m \in \{-1,0,1\}. \tag{A.2.20}$$

Let us note that (A.2.19) and (A.2.20) imply the condition

$$\operatorname{div} \boldsymbol{w}_{1,\varepsilon} = 0 \quad \text{a.e. in } \tilde{\Omega}. \tag{A.2.21}$$

Next, $\mathbb{G}_\varepsilon := (\boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon - \mathbb{D}_\varepsilon - \mathbb{F}_\varepsilon \mathbb{F}_\varepsilon^T)$ belongs to $(L^2(Q_T))^{2\times 2}$, hence $\operatorname{div} \mathbb{G}_\varepsilon$ belongs to $L^2(0,T; ((W^{1,2}_0(\tilde{\Omega}))^2)^*)$, and by Lemma A.2.1 there exists for almost all $t \in (0,T)$ unique weak solution $[\boldsymbol{w}_{2,\varepsilon}, p_{2,\varepsilon}]$, $\int_{\tilde{\Omega}} p_{2,\varepsilon} = 0$, to the Stokes problem (A.2.15)–(A.2.17), more precisely, $[\boldsymbol{w}_{2,\varepsilon}, p_{2,\varepsilon}]$ satisfy for all $\boldsymbol{\Phi} \in (W^{1,2}_0(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$ and almost all $t \in (0,T)$

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_{2,\varepsilon} : \nabla \boldsymbol{\Phi} - \int_{\tilde{\Omega}} p_{2,\varepsilon} \operatorname{div} \boldsymbol{\Phi} = - \int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla \boldsymbol{\Phi}, \tag{A.2.22}$$

$$\int_{\tilde{\Omega}} \operatorname{div} \boldsymbol{w}_{2,\varepsilon} \phi = 0 \tag{A.2.23}$$

and the estimate

$$\|\boldsymbol{w}_{2,\varepsilon}\|_{(W^{1,2}(\tilde{\Omega}))^2} + \|p_{2,\varepsilon}\|_{L^2(\tilde{\Omega})} \le \|\operatorname{div} \mathbb{G}_\varepsilon\|_{((W^{1,2}_0(\tilde{\Omega}))^2)^*}. \tag{A.2.24}$$

Let us note that (A.2.23) and (A.2.24) imply the condition

$$\operatorname{div} \boldsymbol{w}_{2,\varepsilon} = 0 \quad \text{a.e. in } \tilde{\Omega}. \tag{A.2.25}$$

Let $\theta \in C_c^\infty((0,T))$, $\boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ be arbitrary. In (A.2.18) set $\boldsymbol{\Phi} := \boldsymbol{\Phi}_0$, multiply the result by $\partial_t\theta$ and integrate over $(0,T)$ to obtain (use also the estimate (A.2.20))

$$\int_0^T \int_{\tilde{\Omega}} (\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\Phi}_0)\partial_t\theta = -\int_0^T \int_{\tilde{\Omega}} (\Delta \boldsymbol{w}_{1,\varepsilon} \cdot \boldsymbol{\Phi}_0)\partial_t\theta. \tag{A.2.26}$$

Since $\partial_t \boldsymbol{v}_\varepsilon \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, from (A.2.26) it follows $\partial_t\Delta\boldsymbol{w}_{1,\varepsilon} \in L^2(0,T;(\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})^*)$ and

$$\langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{\Phi}_0\rangle = -\langle \partial_t \Delta \boldsymbol{w}_{1,\varepsilon}, \boldsymbol{\Phi}_0\rangle = \int_{\Omega} (\partial_t \nabla \boldsymbol{w}_{1,\varepsilon} : \nabla\boldsymbol{\Phi}_0) \ \text{ a.e. in } (0,T). \tag{A.2.27}$$

Next, setting in (A.2.22) $\boldsymbol{\Phi} := \boldsymbol{\Phi}_0$ yields

$$\int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla\boldsymbol{\Phi}_0 = -\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_{2,\varepsilon} : \nabla\boldsymbol{\Phi}_0 \ \text{ a.e. in } (0,T). \tag{A.2.28}$$

Summing (A.2.27) with (A.2.28) leads to

$$0 = \langle \partial_t \boldsymbol{v}_\varepsilon, \boldsymbol{\Phi}_0\rangle - \int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla\boldsymbol{\Phi}_0 = \int_{\tilde{\Omega}} \nabla(\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon}) : \nabla\boldsymbol{\Phi}_0 \quad \text{a.e. in } (0,T), \tag{A.2.29}$$

where the first equality follows from (4.5.11). Since $\boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ is arbitrary, the relations (A.2.19), (A.2.23) and (A.2.29) implies that $\boldsymbol{w}_\varepsilon := \partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ solves for almost all $t \in (0,T)$ the Stokes problem

$$\int_{\tilde{\Omega}} \nabla \boldsymbol{w}_\varepsilon : \nabla\boldsymbol{\Phi}_0 = 0 \qquad \text{for all } \boldsymbol{\Phi}_0 \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}. \tag{A.2.30}$$

Lemma A.2.1 guarantees the existence of unique solution $\boldsymbol{w}_\varepsilon \in \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}$ to the Stokes problem (A.2.30), hence (the quadratic integrability of $\|\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon}\|_{W^{1,2}_{0,\mathrm{div}}}$ over $(0,T)$ follows from (A.2.29) and the properties $\partial_t \boldsymbol{v} \in L^2(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*)$, $\mathbb{G}_\varepsilon \in (L^2(Q_T))^{2\times 2}$)

$$\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}) \tag{A.2.31}$$

and

$$\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon} = \mathbf{0} \quad \text{a.e. in } (0,T)\times\tilde{\Omega}. \tag{A.2.32}$$

Now we are able to write for all $\boldsymbol{\Phi} \in (C_c^\infty(\tilde{\Omega}))^2$ and $\theta \in C_c^\infty((0,T))$ (in the first equality we use (A.2.18), (A.2.22) and the integration by parts, the second and the third equality is just the integration by parts, the last equality follows from (A.2.32))

$$\begin{aligned}
\int_0^T \int_{\tilde{\Omega}} -(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \cdot (\partial_t\theta)\boldsymbol{\Phi} - \int_0^T \int_{\Omega} (\mathbb{G}_\varepsilon : \nabla\boldsymbol{\Phi})\theta + \int_0^T \int_{\tilde{\Omega}} (p_{2,\varepsilon} \operatorname{div} \boldsymbol{\Phi})\theta \\
= \int_0^T \int_{\tilde{\Omega}} \Delta \boldsymbol{w}_{1,\varepsilon} \cdot (\partial_t\theta)\boldsymbol{\Phi} + \int_0^T \int_{\tilde{\Omega}} (\nabla \boldsymbol{w}_{2,\varepsilon} : \nabla\boldsymbol{\Phi})\theta \\
= \int_0^T \int_{\tilde{\Omega}} \boldsymbol{w}_{1,\varepsilon} \cdot (\partial_t\theta)\Delta\boldsymbol{\Phi} - \int_0^T \int_{\tilde{\Omega}} (\boldsymbol{w}_{2,\varepsilon} \cdot \Delta\boldsymbol{\Phi})\theta \\
= \int_0^T \int_{\tilde{\Omega}} -(\partial_t \boldsymbol{w}_{1,\varepsilon} + \boldsymbol{w}_{2,\varepsilon}) \cdot (\theta\Delta\boldsymbol{\Phi}) = 0.
\end{aligned}$$

Since $\mathbb{G}_\varepsilon \in (L^2(Q_T))^{2\times 2}$ and $p_{2,\varepsilon} \in L^2((0,T)\times\tilde{\Omega})$ (which follows from the estimate (A.2.24) and the fact that $\|\mathbb{G}_\varepsilon\|_2$ is quadratically integrable over $(0,T)$), the last chain yields

$$\partial_t(\boldsymbol{v}_\varepsilon + \nabla p_{1,\varepsilon}) \in L^2(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*),$$

which is (A.2.3), and (recall that $C_c^\infty(\tilde{\Omega})$ is dense in $W_0^{1,2}(\tilde{\Omega})$) for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and almost all $t \in (0,T)$ it holds

$$\langle \partial_t(\boldsymbol{v}_\varepsilon + p_{1,\varepsilon}), \boldsymbol{\Phi}\rangle = \int_{\tilde{\Omega}} \mathbb{G}_\varepsilon : \nabla\boldsymbol{\Phi} + \int_{\tilde{\Omega}} p_{2,\varepsilon} \operatorname{div}\boldsymbol{\Phi}, \tag{A.2.33}$$

which is (A.2.4).

Next we prove (A.2.7) and (A.2.8). As $\boldsymbol{v}_\varepsilon$ is uniformly bounded in $L^2(0,T;W^{1,2}_{\mathbf{0},\mathrm{div}})$, the estimate (A.2.20) implies that $\boldsymbol{w}_{1,\varepsilon}$ is uniformly bounded in $L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2)$, $p_{1,\varepsilon}$ is uniformly bounded in $L^2(0,T;W^{2,2}(\tilde{\Omega}))$, and since $\boldsymbol{w}_{1,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\fint_{\tilde{\Omega}} p_{1,\varepsilon} = 0$ for every $\varepsilon \in (0,1)$, there exists $\boldsymbol{w}_1 \in L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2 \cap \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $p_1 \in L^2(0,T;W^{2,2}(\tilde{\Omega}))$, $\fint_{\tilde{\Omega}} p_1 = 0$, such that (for suitable subsequences of $\{\boldsymbol{w}_{1,\varepsilon}\}, \{p_{1,\varepsilon}\}$, which we do not relabel)

$$\boldsymbol{w}_{1,\varepsilon} \rightharpoonup \boldsymbol{w}_1 \qquad \text{weakly in } L^2(0,T;(W^{3,2}(\tilde{\Omega}))^2 \cap \widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}), \tag{A.2.34}$$

$$p_{1,\varepsilon} \rightharpoonup p_1 \qquad \text{weakly in } L^2(0,T;W^{2,2}(\tilde{\Omega})). \tag{A.2.35}$$

Let $t \in (0,T)$ be fixed. As $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$ strongly in $(L^2(Q_T))^2$ (see (4.5.25)), taking the limit $\varepsilon \to 0+$ in (A.2.18) and (A.2.19), employing the convergences (A.2.34) and (A.2.35), we observe that $[\boldsymbol{w}_1, p_1]$ satisfy for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \nabla\boldsymbol{w}_1 : \nabla\boldsymbol{\Phi} - \int_{\tilde{\Omega}} p_1 \operatorname{div}\boldsymbol{\Phi} = \int_{\tilde{\Omega}} \boldsymbol{v}\cdot\boldsymbol{\Phi}, \tag{A.2.36}$$

$$\int_{\tilde{\Omega}} (\operatorname{div}\boldsymbol{w}_1)\phi = 0. \tag{A.2.37}$$

Next, due the estimate (A.2.24) and due to the facts that $\mathbb{G}_\varepsilon$ is uniformly bounded in $(L^2(Q_T))^{2\times 2}$, $\boldsymbol{w}_{2,\varepsilon} \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$ and $\fint_{\tilde{\Omega}} p_{2,\varepsilon} = 0$ for every $\varepsilon \in (0,1)$, there exist $\boldsymbol{w}_2 \in L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}})$, $p_2 \in L^2((0,T)\times\tilde{\Omega})$, $\fint_{\tilde{\Omega}} p_2 = 0$, such that (for suitable subsequences)

$$\boldsymbol{w}_{2,\varepsilon} \rightharpoonup \boldsymbol{w}_2 \qquad \text{weakly in } L^2(0,T;\widetilde{W}^{1,2}_{\mathbf{0},\mathrm{div}}), \tag{A.2.38}$$

$$p_{2,\varepsilon} \rightharpoonup p_2 \qquad \text{weakly in } L^2((0,T)\times\tilde{\Omega}), \tag{A.2.39}$$

$$\mathbb{G}_\varepsilon \rightharpoonup \mathbb{G} \qquad \text{weakly in } (L^2(Q_T))^{2\times 2}, \tag{A.2.40}$$

where $\mathbb{G} := \boldsymbol{v}\otimes\boldsymbol{v} - \mathbb{D} - \overline{\mathbb{F}\mathbb{F}^T}$. Taking the limit $\varepsilon \to 0+$ in (A.2.22) and (A.2.23), we observe that $\boldsymbol{w}_2$, p_2 satisfy for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$, $\phi \in L^2(\tilde{\Omega})$

$$\int_{\tilde{\Omega}} \nabla\boldsymbol{w}_2 : \nabla\boldsymbol{\Phi} - \int_{\tilde{\Omega}} p_2 \operatorname{div}\boldsymbol{\Phi} = -\int_{\tilde{\Omega}} \mathbb{G} : \nabla\boldsymbol{\Phi}, \tag{A.2.41}$$

$$\int_{\tilde{\Omega}} (\operatorname{div}\boldsymbol{w}_2)\phi = 0. \tag{A.2.42}$$

Lemma A.2.1 guarantees the uniqueness of the solution to problems (A.2.36)–(A.2.37) and (A.2.41)–(A.2.42) and the estimates

$$\|\boldsymbol{w}_1\|_{(W^{m+2,2}(\tilde{\Omega}))^2} + \|p_1\|_{W^{m+1,2}(\tilde{\Omega})} \le \|\boldsymbol{v}\|_{(W^{m,2}(\Omega))^2},\ m \in \{-1,0,1\},$$
$$\|\boldsymbol{w}_2\|_{(W^{1,2}(\tilde{\Omega}))^2} + \|p_2\|_{L^2(\tilde{\Omega})} \le \|\operatorname{div}\mathbb{G}\|_{((W_0^{1,2}(\Omega))^2)^*}.$$

Now proceeding in the same way as on the approximate level, we conclude

$$\partial_t(\boldsymbol{v} + \nabla p_1) \in L^2(0,T;((W_0^{1,2}(\tilde{\Omega}))^2)^*),$$

which is (A.2.7), and for all $\boldsymbol{\Phi} \in (W_0^{1,2}(\tilde{\Omega}))^2$ and almost all $t \in (0,T)$

$$\langle \partial_t(\boldsymbol{v} + p_1), \boldsymbol{\Phi}\rangle = \int_{\tilde{\Omega}} \mathbb{G} : \nabla\boldsymbol{\Phi} + \int_{\tilde{\Omega}} p_2 \operatorname{div}\boldsymbol{\Phi}, \tag{A.2.43}$$

which is (A.2.8).

Concerning the convergence results, as the weak convergence (A.2.10) follows immediately from (A.2.39), it suffices to prove only the strong convergence (A.2.9). By subtracting (A.2.36) from (A.2.18) and (A.2.37) from (A.2.19) it is obvious that $\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1$, $p_{1,\varepsilon} - p_1$ solve the Stokes problem with the right hand side $\boldsymbol{v}_\varepsilon - \boldsymbol{v}$. By Lemma A.2.1 the solution is unique and satisfies the estimate

$$\|\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1\|_{(W^{m+2,2}(\tilde{\Omega}))^2} + \|p_{1,\varepsilon} - p_1\|_{W^{m+1,2}(\tilde{\Omega})} \le \|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{(W^{m,2}(\Omega))^2}, \quad m \in \{-1,0,1\}. \tag{A.2.44}$$

From (A.2.44) with $m = 0$ it follows

$$\lim_{\varepsilon\to 0+} \int_0^T \|p_{1,\varepsilon} - p_1\|^2_{W^{1,2}(\tilde{\Omega})} \le \lim_{\varepsilon\to 0+} \int_0^T \|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|^2_{(L^2(\Omega))^2} = 0, \tag{A.2.45}$$

where the second equality is achieved by the strong convergence (4.5.25). The relation (A.2.45) yields

$$p_{1,\varepsilon} \to p_1 \qquad \text{strongly in } L^2(0,T;W^{1,2}(\tilde{\Omega})). \tag{A.2.46}$$

We need to show that also

$$\nabla^2 p_{1,\varepsilon} \to \nabla^2 p_1 \qquad \text{strongly in } L^2(0,T;(L^2_{loc}(\tilde{\Omega}))^{2\times 2}). \tag{A.2.47}$$

Let us denote

$$\tilde{p}_\varepsilon := p_{1,\varepsilon} - p_1.$$

To prove (A.2.47) we will use the property

$$\Delta\tilde{p}_\varepsilon = 0 \quad \text{a.e. in } (0,T)\times\tilde{\Omega}, \tag{A.2.48}$$

which follows from the equality

$$-\Delta(\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1) + \nabla\tilde{p}_\varepsilon = \boldsymbol{v}_\varepsilon - \boldsymbol{v} \quad \text{a.e. in } (0,T)\times\tilde{\Omega}$$

($\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1$, $\tilde{p}_\varepsilon$ is the solution to the Stokes problem with the right hand side $\boldsymbol{v}_\varepsilon - \boldsymbol{v}$ and $\Delta(\boldsymbol{w}_{1,\varepsilon} - \boldsymbol{w}_1)$, $\nabla\tilde{p}_\varepsilon$ are integrable over Q_T) and from the fact $\operatorname{div}\boldsymbol{v}_\varepsilon = \operatorname{div}\boldsymbol{v} = 0 = \operatorname{div}\boldsymbol{w}_{1,\varepsilon} = \operatorname{div}\boldsymbol{w}_1$ almost everywhere in $(0,T)\times\tilde{\Omega}$. The equation (A.2.48) implies

$$\int_{\tilde{\Omega}} \nabla\tilde{p}_\varepsilon \cdot \nabla\varphi = 0 \quad \text{for all } \varphi \in W_0^{1,2}(\tilde{\Omega}). \tag{A.2.49}$$

Taking $\varphi := \phi\xi^2$, where $\phi \in W^{1,2}(\tilde{\Omega})$, $\xi \in C_c^\infty(\tilde{\Omega})$, (A.2.49) can be rewritten as

$$\int_{\tilde{\Omega}} \nabla(\tilde{p}_\varepsilon \xi) \cdot \nabla(\phi\xi) = -\int_{\tilde{\Omega}} \operatorname{div}(\tilde{p}_\varepsilon \nabla\xi)\phi\xi - \int_{\tilde{\Omega}} (\nabla\tilde{p}_\varepsilon \cdot \nabla\xi)\phi\xi. \tag{A.2.50}$$

Denote $\psi := \phi\xi$, $g_\varepsilon := -\operatorname{div}(\tilde{p}_\varepsilon\nabla\xi) - (\nabla\tilde{p}_\varepsilon \cdot \nabla\xi)$. Since every $\psi \in W_0^{1,2}(\tilde{\Omega})$ can be written in the form $\phi\xi$, where the functions ϕ, ξ have the properties described above, (A.2.50) gives

$$\int_{\tilde{\Omega}} \nabla(\tilde{p}_\varepsilon\xi) \cdot \nabla\psi = \int_{\tilde{\Omega}} g_\varepsilon\psi \quad \text{for all } \psi \in W_0^{1,2}(\tilde{\Omega}). \tag{A.2.51}$$

Employing the local regularity of weak solutions to elliptic problems, we have

$$\|\tilde{p}_\varepsilon\xi\|_{W^{2,2}(\tilde{\Omega})} \leq \|g_\varepsilon\|_{L^2(\tilde{\Omega})}. \tag{A.2.52}$$

Since $\xi \in C_c^\infty(\tilde{\Omega})$ is arbitrary, the inequality (A.2.52) together with the definitions of $\tilde{p}_\varepsilon$ and g_ε implies for every $\tilde{\tilde{\Omega}} \subset \overline{\tilde{\tilde{\Omega}}} \subset \tilde{\Omega}$

$$\int_0^T \|\nabla^2 p_{1,\varepsilon} - \nabla^2 p_1\|_{(L^2(\tilde{\tilde{\Omega}}))^{2\times 2}} \leq C \int_0^T \|p_{1,\varepsilon} - p_1\|_{W^{1,2}(\tilde{\Omega})}, \tag{A.2.53}$$

and since the right hand side of (A.2.53) converges to zero by (A.2.46), we obtain (A.2.47), which together with (A.2.46) concludes the strong convergence (A.2.9).

To finish the proof of the lemma, it remains to show the continuity of $p_{1,\varepsilon}$, p_1 with respect to time and the convergence of the initial conditions $p_{1,\varepsilon}(0)$, $p_1(0)$. Let t_1 and t_2 from $[0,T]$ be arbitrary. The functions $\boldsymbol{w}_{1,\varepsilon}(t_1) - \boldsymbol{w}_{1,\varepsilon}(t_2)$, $p_{1,\varepsilon}(t_1) - p_{1,\varepsilon}(t_2)$ solve the Stokes problem with the right hand side $\boldsymbol{v}_\varepsilon(t_1) - \boldsymbol{v}_\varepsilon(t_2)$. By Lemma A.2.1 this solution is unique and it satisfies the estimate

$$\|\boldsymbol{w}_{1,\varepsilon}(t_1) - \boldsymbol{w}_{1,\varepsilon}(t_2)\|_{(W^{2,2}(\tilde{\Omega}))^2} + \|p_{1,\varepsilon}(t_1) - p_{1,\varepsilon}(t_2)\|_{W^{1,2}(\tilde{\Omega})} \leq \|\boldsymbol{v}_\varepsilon(t_1) - \boldsymbol{v}_\varepsilon(t_2)\|_{(L^2(\Omega))^2}.$$

Since the right hand side converges to zero whenever $t_2 \to t_1$ (as $\boldsymbol{v}_\varepsilon$ belongs to $C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$), we conclude that $\nabla p_{1,\varepsilon}$ belongs to $C([0,T]; (L^2(\tilde{\Omega}))^2)$. The fact $\nabla p_1 \in C([0,T]; (L^2(\tilde{\Omega}))^2)$ is proved in the same way.

Finally, from the relation (A.2.44) we have for all $t \in [0,T]$ (recall that $\boldsymbol{v}_\varepsilon, \boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}})$ and $\nabla p_{1,\varepsilon}, \nabla p_1 \in C([0,T]; (L^2(\tilde{\Omega}))^2)$)

$$\|\nabla p_{1,\varepsilon}(t) - \nabla p_1(t)\|_{(L^2(\tilde{\Omega}))^2} \leq \|\boldsymbol{v}_\varepsilon(t) - \boldsymbol{v}(t)\|_{(L^2(\Omega))^2}. \tag{A.2.54}$$

And since $\boldsymbol{v}_\varepsilon(0) = \boldsymbol{v}(0) = \boldsymbol{v}_0$ almost everywhere in Ω, we conclude $\nabla p_{1,\varepsilon}(0) = \nabla p_1(0)$ almost everywhere in $\tilde{\Omega}$, which is the relation (A.2.11) completing the proof of the proposition. □

A.3 Existence of solution to parabolic approximation to (4.4.1)–(4.4.6)

In this section, for arbitrary $\varepsilon \in (0,1)$, we establish the existence of a weak solution to parabolic approximation to (4.4.1)–(4.4.6) defined in Section 4.5. More precisely, we study the problem (4.5.1)–(4.5.5).

Proposition A.3.1. Let $\varepsilon \in (0,1)$, $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}$, $\mathbb{F}_0 \in (L^2(\Omega))^{2\times 2}$. Then there exists a weak solution to the system (4.5.1)–(4.5.5), i.e. there exists a couple $(\boldsymbol{v}, \mathbb{F})$ fulfilling

$$\begin{aligned}&\boldsymbol{v} \in C([0,T]; L^2_{\boldsymbol{n},\mathrm{div}}) \cap L^2(0,T; W^{1,2}_{\boldsymbol{0},\mathrm{div}}), \quad \partial_t \boldsymbol{v} \in L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*),\\ &\mathbb{F} \in C_{weak}([0,T];(L^2(\Omega))^{2\times 2}) \cap L^2(0,T;(W^{1,2}(\Omega))^{2\times 2}), \partial_t \mathbb{F} \in L^{\frac{4}{3}}\left(0,T;((W^{1,2}(\Omega))^{2\times 2})^*\right),\end{aligned}$$

and satisfying, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and for almost all $t \in (0,T)$,

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle - \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v}) : \nabla \boldsymbol{w} + \int_\Omega \mathbb{D} : \nabla \boldsymbol{w} + \int_\Omega (\mathbb{F}\mathbb{F}^T) : \nabla \boldsymbol{w} = 0, \tag{A.3.1}$$

$$\langle \partial_t \mathbb{F}, \mathbb{A}\rangle - \int_\Omega (\mathbb{F}\otimes \boldsymbol{v}) : \nabla \mathbb{A} - \int_\Omega ((\nabla \boldsymbol{v})\mathbb{F}) : \mathbb{A} + \frac{1}{2}\int_\Omega (\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}) : \mathbb{A} + \varepsilon \int_\Omega \nabla \mathbb{F} : \nabla \mathbb{A} = 0, \tag{A.3.2}$$

and attaining the initial conditions in the following sense

$$\lim_{t\to 0+} (\|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 + \|\mathbb{F}(t) - \mathbb{F}_0\|_2) = 0. \tag{A.3.3}$$

Moreover, the following uniform estimates holds true

$$\sup_{t\in(0,T)} \left(\|\boldsymbol{v}(t)\|_2^2 + \|\mathbb{F}(t)\|_2^2\right) + \int_0^T \|\mathbb{D}\|_2^2 + \|\mathbb{F}\|_4^4 + \varepsilon\|\nabla \mathbb{F}\|_2^2 \le C(\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}_0\|_2^2) \tag{A.3.4}$$

and

$$\|\partial_t \boldsymbol{v}\|_{L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)} + \|\partial_t \mathbb{F}\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} \le \overline{C}(T,\Omega,\|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2), \tag{A.3.5}$$

where $\overline{C}(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2)$ denotes a generic constant depending only on T, Ω, $\|\boldsymbol{v}_0\|_2$ and $\|\mathbb{F}_0\|_2$.

The rest of this section is devoted to the proof of Proposition A.3.1.

A.3.1 Galerkin approximations

Let $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ be a basis of $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ composed of eigenfunctions of the Stokes operator subject to the boundary condition $\boldsymbol{w} = \boldsymbol{0}$ on $\partial\Omega$, orthogonal in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, orthonormal in $L^2_{\boldsymbol{n},\mathrm{div}}$. Let $\{\mathbb{A}_j\}_{j\in\mathbb{N}}$ be a basis of $(W^{1,2}(\Omega))^{2\times 2}$ composed of eigenfunctions of the Laplace operator subject to the boundary condition $(\nabla\mathbb{A})\boldsymbol{n} := \{\nabla A_{kl}\cdot \boldsymbol{n}\}_{k,l=1}^2 = \mathbb{O}$ on $\partial\Omega$, orthogonal in $(W^{1,2}(\Omega))^{2\times 2}$, orthonormal in $(L^2(\Omega))^{2\times 2}$. Let us denote $W_n := \mathrm{span}\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$, $X_n := \mathrm{span}\{\mathbb{A}_1,\dots,\mathbb{A}_n\}$. The orthogonal projection from $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ to W_n is denoted by P_n and the orthogonal projection from $(W^{1,2}(\Omega))^{2\times 2}$ to X_n by Q_n. The projection P_n is continuous in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ and the projection Q_n is continuous in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$. In addition, the norm of this projection is independent on n. From the Carathéodory theory for ordinary differential equations (when the external forces $\boldsymbol{f}$ are identically equal to zero, it suffices to use the Peano theory) it follows that there exist time dependent coefficients $\alpha_1^n(t),\dots,\alpha_n^n(t)$, $\beta_1^n(t),\dots,\beta_n^n(t)$ such that if we define $\boldsymbol{v}_n$ and $\mathbb{F}_n$ as

$$\boldsymbol{v}_n(t,\boldsymbol{x}) := \sum_{j=1}^n \alpha_j^n(t)\boldsymbol{w}_j(\boldsymbol{x}) \quad \text{and} \quad \mathbb{F}_n(t,\boldsymbol{x}) := \sum_{j=1}^n \beta_j^n(t)\mathbb{A}_j(\boldsymbol{x}), \tag{A.3.6}$$

then for all $j \in \{1, \dots, n\}$ and for all $t \in (0, \tilde{t})$, where $\tilde{t}$ is certain positive number, the following system of equations (we denote $\mathbb{D}_n := \frac{1}{2}\left(\nabla \boldsymbol{v}_n + (\nabla \boldsymbol{v}_n)^T\right)$) holds true:

$$\int_\Omega \partial_t \boldsymbol{v}_n \cdot \boldsymbol{w}_j - \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla \boldsymbol{w}_j + \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{w}_j + \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{w}_j = 0, \tag{A.3.7}$$

$$\begin{aligned} \int_\Omega \partial_t \mathbb{F}_n : \mathbb{A}_j - \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla \mathbb{A}_j - \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{A}_j + \frac{1}{2} \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n) : \mathbb{A}_j \\ - \frac{1}{2} \int_\Omega \mathbb{F}_n : \mathbb{A}_j + \varepsilon \int_\Omega \nabla \mathbb{F}_n : \nabla \mathbb{A}_j = 0. \end{aligned} \tag{A.3.8}$$

The functions $\boldsymbol{v}_n$ are absolutely continuous in $[0, \tilde{t})$ with values in W_n, the functions $\mathbb{F}_n$ are absolutely continuous in $[0, \tilde{t})$ with values in X_n and they satisfy the initial conditions

$$\boldsymbol{v}_n(0, \cdot) = P_n(\boldsymbol{v}_0), \quad \mathbb{F}_n(0, \cdot) = Q_n(\mathbb{F}_0) \text{ in } \Omega. \tag{A.3.9}$$

The fact that $\tilde{t} = T$ is an easy consequence of the uniform estimates that follow.

A.3.2 Uniform n-independent estimates

Multiplying (A.3.7) by α_j^n, (A.3.8) by β_j^n and taking the sum over $j = 1, \dots, n$, we obtain (using also the symmetry of $\mathbb{F}_n \mathbb{F}_n^T$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\|\boldsymbol{v}_n\|_2^2}{2} - \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla \boldsymbol{v}_n + \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{v}_n + \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{v}_n = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\|\mathbb{F}_n\|_2^2}{2} - \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla \mathbb{F}_n - \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n + \frac{\|\mathbb{F}_n \mathbb{F}_n^T\|_2^2 - \|\mathbb{F}_n\|_2^2}{2} + \varepsilon \|\nabla \mathbb{F}_n\|_2^2 = 0.$$

Integrating both equations over $(0, t)$, where $t \in (0, T)$ is arbitrary, and employing the integration by parts and the properties $\operatorname{div} \boldsymbol{v}_n = 0$ in Q_T, $\boldsymbol{v}_n = \mathbf{0}$ on Σ_T, we arrive at

$$\frac{\|\boldsymbol{v}_n(t)\|_2^2}{2} + \int_0^t \int_\Omega \mathbb{D}_n : \nabla \boldsymbol{v}_n + \int_0^t \int_\Omega (\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{v}_n = \frac{\|\boldsymbol{v}_n(0)\|_2^2}{2},$$

$$\frac{\|\mathbb{F}_n(t)\|_2^2}{2} - \int_0^t \int_\Omega (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n + \int_0^t \frac{\|\mathbb{F}_n \mathbb{F}_n^T\|_2^2 - \|\mathbb{F}_n\|_2^2}{2} + \varepsilon \int_0^t \|\nabla \mathbb{F}_n\|_2^2 = \frac{\|\mathbb{F}_n(0)\|_2^2}{2}.$$

By the symmetry of $\mathbb{D}_n$ it holds $\mathbb{D}_n : \nabla \boldsymbol{v}_n = |\mathbb{D}_n|^2$ and by the symmetry of $\mathbb{F}_n \mathbb{F}_n^T$ it holds $(\mathbb{F}_n \mathbb{F}_n^T) : \nabla \boldsymbol{v}_n = (\nabla \boldsymbol{v}_n \mathbb{F}_n) : \mathbb{F}_n$, thus by summing the last two equations (both multiplied by 2), we get for all $t \in (0, T)$

$$\begin{aligned} \|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2 + \int_0^t \left(2\|\mathbb{D}_n\|_2^2 + \|\mathbb{F}_n \mathbb{F}_n^T\|_2^2 + 2\varepsilon \|\nabla \mathbb{F}_n\|_2^2\right) \\ = \|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2 + \int_0^t \|\mathbb{F}_n\|_2^2 \le \|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2 + \int_0^t \left(\|\boldsymbol{v}_n\|_2^2 + \|\mathbb{F}_n\|_2^2\right). \end{aligned} \tag{A.3.10}$$

Since $\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2$ is estimated by the right-hand side of (A.3.10), the Gronwall lemma applied on (A.3.10) (the functions $\|\boldsymbol{v}_n(\cdot)\|_2$ and $\|\mathbb{F}_n(\cdot)\|_2$ are continuous in $[0,T)$) together with the condition (A.3.9) and the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ and of Q_n in $(L^2(\Omega))^{2\times 2}$ implies

$$\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2 \le e^t\left(\|\boldsymbol{v}_n(0)\|_2^2 + \|\mathbb{F}_n(0)\|_2^2\right) \le e^t\left(\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}_0\|_2^2\right). \tag{A.3.11}$$

Let us note that the inequality (A.3.11) will be useful in the proof of attainment of the initial conditions (4.5.5). In addition, the combination of (A.3.10) with (A.3.11) yields

$$\sup_{t\in(0,T)}\left(\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2\right) + \int_0^T \left(\|\mathbb{D}_n\|_2^2 + \|\mathbb{F}_n\mathbb{F}_n^T\|_2^2 + \varepsilon\|\nabla\mathbb{F}_n\|_2^2\right) \le C(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2). \tag{A.3.12}$$

The matrix $\mathbb{F}_n\mathbb{F}_n^T$ acting in (A.3.12) is symmetric, hence it is a diagonalizable matrix, let us denote the corresponding diagonal matrix as $\mathbb{J}_n$. The inequality $\pm ab \le \frac{1}{2}(a^2+b^2)$ for all $a, b \in \mathbb{R}$ then gives in Q_T

$$|\mathbb{F}_n|^4 = \left(\mathrm{tr}(\mathbb{F}_n\mathbb{F}_n^T)\right)^2 = (\mathrm{tr}\ \mathbb{J}_n)^2 \le 2\ \mathrm{tr}(\mathbb{J}_n^2) = 2\ \mathrm{tr}((\mathbb{F}_n\mathbb{F}_n^T)^2) = 2\ |\mathbb{F}_n\mathbb{F}_n^T|^2, \tag{A.3.13}$$

hence for all $t \in (0,T)$ it holds

$$\|\mathbb{F}_n\|_4^4 \le 2\|\mathbb{F}_n\mathbb{F}_n^T\|_2^2. \tag{A.3.14}$$

Taking the supremum over $t \in (0,T)$ at each term of (A.3.12) and using the Korn inequality and (A.3.14) leads finally to

$$\begin{aligned}\sup_{t\in(0,T)}(\|\boldsymbol{v}_n(t)\|_2^2 + \|\mathbb{F}_n(t)\|_2^2) + \|\nabla\boldsymbol{v}_n\|_{2,Q_T}^2 + \|\mathbb{F}_n\|_{4,Q_T}^4 + \varepsilon\|\nabla\mathbb{F}_n\|_{2,Q_T}^2 \\ \le \tilde{C}(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2).\end{aligned} \tag{A.3.15}$$

It remains to estimate the time derivatives of $\boldsymbol{v}_n$ and $\mathbb{F}_n$. Obviously, we can replace in (A.3.7) the base functions $\boldsymbol{w}_j$ by any function belonging to W_n and in (A.3.8) the base functions $\mathbb{A}_j$ by any function belonging to X_n, and consequently, for arbitrary $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ and for all $t \in (0,T)$, it holds

$$\int_\Omega (\partial_t\boldsymbol{v}_n \cdot P_n(\boldsymbol{w})) = \int_\Omega \left((\boldsymbol{v}_n \otimes \boldsymbol{v}_n) - \mathbb{D}_n - \mathbb{F}_n\mathbb{F}_n^T\right) : \nabla P_n(\boldsymbol{w}). \tag{A.3.16}$$

Thanks to the orthogonality and the continuity of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, employing the Hölder inequality, we derive from (A.3.16) that, for all $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$ and for all $t \in (0,T)$,

$$\begin{aligned}|\langle\partial_t\boldsymbol{v}_n, \boldsymbol{w}\rangle| &= \left|\int_\Omega(\partial_t\boldsymbol{v}_n \cdot \boldsymbol{w})\right| = \left|\int_\Omega(\partial_t\boldsymbol{v}_n \cdot P_n(\boldsymbol{w}))\right| \\ &\le \int_\Omega \left|(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) - \mathbb{D}_n - \mathbb{F}_n\mathbb{F}_n^T\right| |\nabla P_n(\boldsymbol{w})| \\ &\le \left(\|\boldsymbol{v}_n\|_4^2 + \|\nabla\boldsymbol{v}_n\|_2 + \|\mathbb{F}_n\|_4^2\right)\|\nabla\boldsymbol{w}\|_2.\end{aligned} \tag{A.3.17}$$

To estimate the right hand side, we use the Ladyzenskaya inequality and thanks to (A.3.15) it holds

$$\|\boldsymbol{v}_n\|_{4,Q_T}^4 = \int_0^T \|\boldsymbol{v}_n\|_4^4 \le \int_0^T \|\boldsymbol{v}_n\|_2^2 \|\nabla \boldsymbol{v}_n\|_2^2 \le \hat{C}\left(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2\right). \tag{A.3.18}$$

Integrating the second power of (A.3.17) over $(0,T)$, using (A.3.15), (A.3.18) and the Minkowski inequality, we can write

$$\|\partial_t \boldsymbol{v}_n\|^2_{L^2\left(0,T;(W^{1,2}_{\mathbf{0},\mathrm{div}})^*\right)} \le 2\int_0^T \left(\|\boldsymbol{v}_n\|_4^4 + \|\nabla \boldsymbol{v}_n\|_2^2 + \|\mathbb{F}_n\|_4^4\right) \le \overline{C}\left(T, \|\boldsymbol{v}_0\|_2, \|\mathbb{F}_0\|_2\right). \tag{A.3.19}$$

Analogously we estimate $\|\partial_t \mathbb{F}_n\|_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)}$. Let $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$, then by (A.3.8) it holds for all $t \in (0,T)$

$$\int_\Omega (\partial_t \mathbb{F}_n : Q_n(\mathbb{A})) = \int_\Omega (\mathbb{F}_n \otimes \boldsymbol{v}_n) : \nabla Q_n(\mathbb{A}) + \int_\Omega \left(\nabla \boldsymbol{v}_n \mathbb{F}_n - \mathbb{F}_n \mathbb{F}_n^T \mathbb{F}_n + \mathbb{F}_n\right) : Q_n(\mathbb{A}) - \varepsilon \int_\Omega \nabla \mathbb{F}_n : \nabla Q_n(\mathbb{A}). \tag{A.3.20}$$

Employing the orthogonality and the continuity of Q_n in $(L^2(\Omega))^{2\times 2}$ and also in $(W^{1,2}(\Omega))^{2\times 2}$, the Cauchy-Schwartz and the Hölder inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ (if $a \in W^{1,2}(\Omega)$, then $\|a\|_4 \le \hat{C}\|a\|_{1,2}$, where $\hat{C} = \hat{C}(\Omega)$), we obtain from (A.3.20) for all $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$ and $t \in (0,T)$

$$\begin{aligned}
|\langle \partial_t \mathbb{F}_n, \mathbb{A}\rangle| &= \left|\int_\Omega (\partial_t \mathbb{F}_n : \mathbb{A})\right| = \left|\int_\Omega (\partial_t \mathbb{F}_n : Q_n(\mathbb{A}))\right| \\
&\le \left(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2\right) \|\nabla \mathbb{A}\|_2 + \|\mathbb{F}_n\|_2 \|\mathbb{A}\|_2 \\
&\quad + \left(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3\right) \|Q_n(\mathbb{A})\|_4 \\
&\le \left(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2 + \|\mathbb{F}_n\|_2\right) \|\mathbb{A}\|_{1,2} \\
&\quad + \hat{C}\left(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3\right) \|Q_n(\mathbb{A})\|_{1,2} \\
&\le \left(\|\mathbb{F}_n\|_4 \|\boldsymbol{v}_n\|_4 + \varepsilon \|\nabla \mathbb{F}_n\|_2 + \|\mathbb{F}_n\|_2 + \hat{C}(\|\nabla \boldsymbol{v}_n\|_2 \|\mathbb{F}_n\|_4 + \|\mathbb{F}_n\|_4^3)\right) \|\mathbb{A}\|_{1,2}.
\end{aligned}$$

Integrating the $\frac{4}{3}$-power of the last chain over $(0,T)$, using (A.3.15), (A.3.18) and the Hölder and the Minkowski inequalities, we conclude (note that $\varepsilon \in (0,1)$)

$$\begin{aligned}
\|\partial_t \mathbb{F}_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times 2})^*)} &\le (1 + C^*(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2)) \int_0^T (\varepsilon \|\nabla \mathbb{F}_n\|_2)^{\frac{4}{3}} \\
&\quad + \tilde{C}\left(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2\right) \\
&\le \left(\varepsilon^{\frac{2}{3}} T^{\frac{1}{3}} + C^*(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2)\right) \left(\int_0^T \varepsilon \|\nabla \mathbb{F}_n\|_2^2\right)^{\frac{2}{3}} \\
&\quad + \tilde{C}\left(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2\right) \\
&\le \overline{C}\left(T,\Omega,\|\boldsymbol{v}_0\|_2,\|\mathbb{F}_0\|_2\right).
\end{aligned} \tag{A.3.21}$$

A.3.3 Limit $n \to +\infty$

The uniform estimates (A.3.15), (A.3.19) and (A.3.21) imply the existence of $\boldsymbol{v}$, $\mathbb{F}$ satisfying the following convergence relations (the relations hold true for suitable subsequences of $\{\boldsymbol{v}_n\}$, $\{\mathbb{F}_n\}$, which we do not relabel):

$$\boldsymbol{v}_n \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly-* in } L^\infty(0,T;L^2_{\boldsymbol{n},\mathrm{div}}), \tag{A.3.22}$$

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}}) \cap (L^4(Q_T))^2, \tag{A.3.23}$$

$$\partial_t\boldsymbol{v}_n \rightharpoonup \partial_t\boldsymbol{v} \qquad \text{weakly in } L^2\left(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*\right), \tag{A.3.24}$$

$$\mathbb{F}_n \rightharpoonup^* \mathbb{F} \qquad \text{weakly-* in } L^\infty\left(0,T;(L^2(\Omega))^{2\times2}\right), \tag{A.3.25}$$

$$\mathbb{F}_n \rightharpoonup \mathbb{F} \qquad \text{weakly in } L^2\left(0,T;(W^{1,2}(\Omega))^{2\times2}\right) \cap (L^4(Q_T))^{2\times2}, \tag{A.3.26}$$

$$\partial_t\mathbb{F}_n \rightharpoonup \partial_t\mathbb{F} \qquad \text{weakly in } L^{\frac{4}{3}}\left(0,T;((W^{1,2}(\Omega))^{2\times2})^*\right). \tag{A.3.27}$$

Let us note that thanks to the properties $\boldsymbol{v} \in L^2(0,T;W^{1,2}_{\boldsymbol{0},\mathrm{div}})$, $\partial_t\boldsymbol{v} \in L^2(0,T;(W^{1,2}_{\boldsymbol{0},\mathrm{div}})^*)$, $\mathbb{F} \in L^\infty(0,T;(L^2(\Omega))^{2\times2})$ and $\partial_t\mathbb{F} \in L^{\frac{4}{3}}(0,T;((W^{1,2}(\Omega))^{2\times2})^*)$ together with the density of $(W^{1,2}(\Omega))^{2\times2}$ in $(L^2(\Omega))^{2\times2}$, the functions $\boldsymbol{v}$, $\mathbb{F}$ after a possible change in a zero-measure subset of $(0,T)$ enjoy

$$\boldsymbol{v} \in C([0,T];L^2_{\boldsymbol{n},\mathrm{div}}), \tag{A.3.28}$$

$$\mathbb{F} \in C_{weak}\left([0,T];(L^2(\Omega))^{2\times2}\right). \tag{A.3.29}$$

Employing (A.3.23), (A.3.24), (A.3.26), (A.3.27) and the Aubin–Lions compactness lemma, we get

$$\boldsymbol{v}_n \to \boldsymbol{v} \qquad \text{strongly in } (L^q(Q_T))^2 \text{ for all } q \in [1,4), \tag{A.3.30}$$

$$\mathbb{F}_n \to \mathbb{F} \qquad \text{strongly in } (L^q(Q_T))^{2\times2} \text{ for all } q \in [1,4). \tag{A.3.31}$$

From (A.3.23), (A.3.26), (A.3.30) and (A.3.31) we also obtain the following relations:

$$\boldsymbol{v}_n \otimes \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \otimes \boldsymbol{v} \qquad \text{weakly in } (L^2(Q_T))^{2\times2}, \tag{A.3.32}$$

$$\mathbb{F}_n \otimes \boldsymbol{v}_n \rightharpoonup \mathbb{F} \otimes \boldsymbol{v} \qquad \text{weakly in } (L^2(Q_T))^{2\times2\times2}, \tag{A.3.33}$$

$$\nabla\boldsymbol{v}_n\mathbb{F}_n \rightharpoonup \nabla\boldsymbol{v}\mathbb{F} \qquad \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times2}, \tag{A.3.34}$$

$$\mathbb{F}_n\mathbb{F}_n^T \rightharpoonup \mathbb{F}\mathbb{F}^T \qquad \text{weakly in } (L^2(Q_T))^{2\times2}, \tag{A.3.35}$$

$$\mathbb{F}_n\mathbb{F}_n^T\mathbb{F}_n \rightharpoonup \mathbb{F}\mathbb{F}^T\mathbb{F} \qquad \text{weakly in } (L^{\frac{4}{3}}(Q_T))^{2\times2}. \tag{A.3.36}$$

By the convergence results stated above and the facts that $\bigcup_{n\in\mathbb{N}} W_n$ is dense in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, $\bigcup_{n\in\mathbb{N}} X_n$ is dense in $(W^{1,2}(\Omega))^{2\times2}$, we conclude from (A.3.7) and (A.3.8) the validity of the equations (A.3.1) and (A.3.2). Due to (A.3.15), (A.3.22), (A.3.25), (A.3.28), (A.3.29), due to (A.3.23), (A.3.26) and the weak lower semicontinuity of the relevant norms, we obtain the uniform bound (A.3.4). The uniform bound (A.3.5) follows from the uniform estimates (A.3.19) and (A.3.21) (using also (A.3.24), (A.3.27) and weak lower-semicontinuity of the relevant norms).

A.3.4 Attainment of the initial data

Multiplying (A.3.7) by any $\phi \in C_c^\infty(-\infty,T)$, $\phi(0) \neq 0$, integrating the result over $(0,T)$ and employing the orthogonality of P_n in $L^2_{\boldsymbol{n},\mathrm{div}}$ (together with the condition

(A.3.9)), we have, for every $j \leq n$, $\boldsymbol{w}_j \in W_j$, that

$$-\int_\Omega \boldsymbol{v}_0 \cdot \phi(0)\boldsymbol{w}_j - \int_{Q_T} \boldsymbol{v}_n \cdot (\partial_t \phi)\boldsymbol{w}_j + \int_{Q_T} \left(-(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) + \mathbb{D}_n + \mathbb{F}_n\mathbb{F}_n^T\right) : (\phi \nabla \boldsymbol{w}_j) = 0. \tag{A.3.37}$$

Multiplying (A.3.1) by the same ϕ and integrating over $(0, T)$, we have, for every $\boldsymbol{w} \in W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, that

$$-\int_\Omega \boldsymbol{v}(0) \cdot \phi(0)\boldsymbol{w} - \int_{Q_T} \boldsymbol{v} \cdot (\partial_t \phi)\boldsymbol{w} + \int_{Q_T} \left(-(\boldsymbol{v} \otimes \boldsymbol{v}) + \mathbb{D} + \mathbb{F}\mathbb{F}^T\right) : (\phi \nabla \boldsymbol{w}) = 0. \tag{A.3.38}$$

Subtracting (A.3.37) from (A.3.38), applying (A.3.23), (A.3.32), (A.3.35), the density of $\bigcup_{n\in\mathbb{N}} W_n$ in $L^2_{\boldsymbol{n},\mathrm{div}}$ and in $W^{1,2}_{\boldsymbol{0},\mathrm{div}}$, letting $n \to \infty$, $j \to \infty$ and dividing the result by $\phi(0)$, we obtain

$$\int_\Omega \boldsymbol{v}(0) \cdot \boldsymbol{w} = \int_\Omega \boldsymbol{v}_0 \cdot \boldsymbol{w} \qquad \text{for all } \boldsymbol{w} \in L^2_{\boldsymbol{n},\mathrm{div}}. \tag{A.3.39}$$

Similarly, multiplying (A.3.8) by $\phi \in C_c^\infty(-\infty, T)$, $\phi(0) \neq 0$, integrating the result over $(0, T)$ and employing the orthogonality of Q_n in $(L^2(\Omega))^{2\times 2}$ (together with the condition (A.3.9)), we have, for every $j \leq n$, $\mathbb{A}_j \in X_j$, that

$$\begin{aligned} -\int_\Omega \mathbb{F}_0 : \phi(0)\mathbb{A}_j - \int_{Q_T} \mathbb{F}_n : (\partial_t \phi)\mathbb{A}_j - \int_{Q_T} (\mathbb{F}_n \otimes \boldsymbol{v}_n) : (\phi \nabla \mathbb{A}_j) \\ + \int_{Q_T} \left(-\nabla \boldsymbol{v}_n \mathbb{F}_n + \frac{\mathbb{F}_n\mathbb{F}_n^T\mathbb{F}_n - \mathbb{F}_n}{2}\right) : (\phi \mathbb{A}_j) + \varepsilon \int_{Q_T} \nabla \mathbb{F}_n : (\phi \nabla \mathbb{A}_j) = 0. \end{aligned} \tag{A.3.40}$$

Multiplying (A.3.2) by the same ϕ and integrating over $(0, T)$, we have, for every $\mathbb{A} \in (W^{1,2}(\Omega))^{2\times 2}$, that

$$\begin{aligned} -\int_\Omega \mathbb{F}(0) : \phi(0)\mathbb{A} - \int_{Q_T} \mathbb{F} : (\partial_t \phi)\mathbb{A} - \int_{Q_T} (\mathbb{F} \otimes \boldsymbol{v}) : (\phi \nabla \mathbb{A}) - \int_{Q_T} (\nabla \boldsymbol{v}\mathbb{F}) : (\varphi \mathbb{A}) \\ + \int_{Q_T} \frac{\mathbb{F}\mathbb{F}^T\mathbb{F} - \mathbb{F}}{2} : (\phi \mathbb{A}) + \varepsilon \int_{Q_T} \nabla \mathbb{F} : (\phi \nabla \mathbb{A}) = 0. \end{aligned}$$

Subtracting (A.3.40) from the last equation, applying (A.3.26), (A.3.33), (A.3.34) and (A.3.36) and the density of $\bigcup_{n\in\mathbb{N}} X_n$ in $(L^2(\Omega))^{2\times 2}$ and in $(W^{1,2}(\Omega))^{2\times 2}$, letting $n \to \infty$, $j \to \infty$ and dividing the result by $\phi(0)$, we obtain

$$\int_\Omega \mathbb{F}(0) : \mathbb{A} = \int_\Omega \mathbb{F}_0 : \mathbb{A} \qquad \text{for all } \mathbb{A} \in (L^2(\Omega))^{2\times 2}. \tag{A.3.41}$$

In order to prove the attainment of the initial conditions in the sense of (A.3.3) we take the limit $n \to \infty$ in (A.3.11). From (A.3.22) and (A.3.25) we deduce that $\boldsymbol{v}_n(t) \rightharpoonup \boldsymbol{v}(t)$ weakly in $L^2_{\boldsymbol{n},\mathrm{div}}$ and $\mathbb{F}_n(t) \rightharpoonup \mathbb{F}(t)$ weakly in $(L^2(\Omega))^{2\times 2}$ for almost all $t \in (0, T)$, hence by the weak lower semicontinuity of $L^2(\Omega)$ norm and the (weak) continuity with respect to time (A.3.28) and (A.3.29), we have

$$\|\boldsymbol{v}(t)\|_2^2 + \|\mathbb{F}(t)\|_2^2 \leq e^t \left(\|\boldsymbol{v}_0\|_2^2 + \|\mathbb{F}_0\|_2^2\right) \qquad \text{for all } t \in (0, T). \tag{A.3.42}$$

Consequently,

$$\limsup_{t\to 0+}\Big(\|\boldsymbol{v}(t)\|_2^2+\|\mathbb{F}(t)\|_2^2\Big)\le\|\boldsymbol{v}_0\|_2^2+\|\mathbb{F}_0\|_2^2 \tag{A.3.43}$$

and employing (A.3.39) with $\boldsymbol{w}:=\boldsymbol{v}_0$ and (A.3.41) with $\mathbb{A}:=\mathbb{F}_0$ (and using again (A.3.28) and (A.3.29)), we conclude that

$$\limsup_{t\to 0+}\Big(\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2+\|\mathbb{F}(t)-\mathbb{F}_0\|_2^2\Big)\le 0, \tag{A.3.44}$$

which gives (A.3.3).