Referee's comments on the Doctoral Thesis

# Mathematical Analysis of Selected Problems for Complex Fluids

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Complex fluids play an important role in a wide range of applications, and the analysis of mathematical models of complex fluids has been a subject of active research worldwide. Models of complex fluids are systems of nonlinear partial differential equations and there are numerous difficult mathematical problems associated with their study. Given a particular mathematical model, one of the most fundamental analytical questions one can ask is whether the system of nonlinear partial equations involved possesses a solution in a suitable sense. The Ph.D. thesis of Tomáš Los is concerned with the proof of existence of global-in-time large-data weak solutions to three classes of models of complex fluids:

1. Unsteady flows of pore-pressure-activated Bingham fluids in three space dimensions;
2. Unsteady flows of pore-pressure-activated granular materials in three space dimensions;
3. Planar flows of viscoelastic fluids of Burgers type.

The results in the three central chapters (Chapters 2–4) of the thesis are based on three journal papers co-authored by the Candidate, the first two of which have already been published, while the third paper has been submitted for publication:

- A. Abbatiello, T. Los, J. Málek, O. Souček: On unsteady flows of pore pressure activated Bingham fluids. *Mathematical Models and Methods in Applied Sciences*, 29(11): 2089–2125, 2019.
- A. Abbatiello, M. Bulíček, T. Los, J. Málek, O. Souček: On unsteady flows of pore pressure activated granular materials. *Zeitschrift für angewandte in Mathematik und Physik*, 72(1):1–18, 2021.
- M. Bulíček, T. Los, Y. Lu and J. Málek: On planar flows of viscoelastic fluids of the Giesekus type. Preprint submitted to *Nonlinearity*, 2022.

## Summary of the main contributions of the thesis

The thesis spans 109 pages and is structured into an introductory chapter, the three central chapters (Chapters 2, 3 and 4), Conclusions, a list of references consisting of 81 entries, a list of the author's own publications, and a three-part Appendix covering just over 16 pages.

In Chapters 2 and 3, the Candidate presents the results from his two recently published papers (papers 1. and 2. listed above). The mathematical models studied in these papers describe basic mechanical properties of flows of granular water-saturated geological materials, and are of relevance to problems of static liquefaction and enhanced oil recovery. The Candidate's motivation for exploring these models has been recent research concerning implicitly constituted materials and a paper by Chupin & Mathé published in the European Journal of Mechanics/Fluids in 2011 concerning the existence of solutions to a model of homogeneous incompressible Navier–Stokes equations with variable rheology. In his Ph.D. thesis the Candidate has successfully extended the results of Chupin & Mathé in several directions: he studies a slightly different system of PDEs, namely one that is rigorously derived from the basic governing equations of the theory of mixtures; secondly, the activated system studied by the Candidate in his thesis contains, in comparison with the paper of Chupin & Mathé a nontrivial right-hand side in the equation for the fluid pressure $p_f$. Consequently, he had to use a different approach than Chupin & Mathé in order to obtain an $L^\infty$-bound on $p_f$. Thirdly, the candidate has provided a characterization of the constitutive equations, featuring in the bulk and in the boundary conditions, in terms of equivalence relations, which play a helpful role in the subsequent analysis (cf. Proposition 2.1.1 on p.7). A particularly important contribution by the Candidate here is that, using one of the equivalent descriptions of the constitutive equation(s) appearing in Proposition 2.1.1, he has successfully corrected an error in the proof of the key theorem in Chupin & Mathé. Fourthly, he considers a stick-slip boundary condition that is physically relevant and, in contrast with a no-slip boundary condition, guarantees the integrability of the pressure up to the boundary of the flow domain. Fifthly, using an $L^\infty$-truncation method, he has managed to analyze three-dimensional flows, whereas the analysis in the paper by Chupin & Mathé is restricted to the case of two space dimensions.

In Chapter 3 of the thesis the Candidate then strengthens the results developed in Chapter 2 by working with a more general class of models, still in three space dimensions. He also provides a different proof of existence of global-in-time large-data weak solutions for more general data, and in particular for external forces which are merely square-integrable.

Finally, in Chapter 4, the Candidate shows the existence of large-data global weak solutions to a viscoelastic rate-type fluid model with two relaxation mechanisms (the mixture of two Giesekus models). This represents an important new contribution compared with the 2011 paper of Nader Masmoudi published in *Journal des Mathèmatiques Pures et Appliquées*, which only deals with a single Giesekus model in two space dimensions (and there are both gaps and unclarities in Masmoudi's paper). The Candidate manages to prove in Chapter 4 the long-time and large-data existence of weak solutions to unsteady flows of such fluids subject to a no-slip boundary condition. The results in Chapter 4 are therefore the fist complete long-time and large-data existence results for a viscoelastic model of higher (second) order. He has also given a rigorous and complete proof of the long-time and large-data existence of weak solutions to the analogous problem associated with the Giesekus model in two spatial dimensions; thereby he has corrected the theoretical considerations in Masmoudi's paper.

### Minor stylistic points/suggestions

p.2, line –2 "apriori" $\longrightarrow$ "a priori";

p.10, line –11: "response" $\longrightarrow$ "respond"'; on the same page, line –13: "criteria" $\longrightarrow$ "criterion";

p.13, 3 lines above eq. (1.2.10): 'well sounded" $\longrightarrow$ "sound" (or "well-founded");

p.14, lines –9 and –10: "multidimensional" $\longrightarrow$ "multicomponent";

p.52, 5 lines below eq. (4.1.2): "sounding" $\longrightarrow$ "sound";

p.63, line –5: "and limited as" $\longrightarrow$ "and passing to the limit as"; same page, line –9: "in virtue of" $\longrightarrow$ "by";

p.69, line –5: “Schwartz” $\longrightarrow$ “Schwarz”;
p.92, line 1 of Appendix A2: “making” $\longrightarrow$ “stating” (or “presenting”);
p.102, 2 lines below eq. (A.3.20): “Schwartz” $\longrightarrow$ “Schwarz”.

There are also a few places where the definite article “the” is missing (or is redundant and should be deleted); or “the” should be “a”. I have not listed these here.

**Recommendation**

The thesis submitted by the Candidate is an excellent piece of work: and this is true of the results of all three of the central chapters of the thesis (Chapters 2, 3 and 4). The contents of Chapter 4 are particularly impressive, in light of the number of technical obstacles that the Candidate had to overcome to complete the proofs of the large-data global-in-time existence results stated therein.

The presentation in the thesis is clear and scholarly throughout. The mathematics is correct and I have only spotted a small number of very minor stylistic slips in the English (which, by the way, is also of a high standard throughout).

The results of the thesis are important new mathematical contributions that are of relevance to the broader area of complex fluids, clearly demonstrating the Candidate’s ability for creative scientific work.

There is therefore no doubt in my mind that based on the excellent Ph.D. thesis submitted by the Candidate, he unquestionably merits the award of the academic degree of Ph.D.

**Questions for the oral examination**

1. p.5: looking at the system of PDEs at the top of the page, 2nd equation from the top, it seems to me that div $\mathbb{T}_\alpha$ is missing from the right-hand side. Could you clarify?

2. p.17, 4 lines below eq. (2.1.6): Could you elaborate further where the error in the cited paper by Chupin & Mathé is?

3. p.33, eq. (2.4.22): missing “$(\Omega)$” after “$W^{1,2}$”.

4. p.40, last 4 lines: Could you elaborate on the work of Abatiello & Feireisl (reference [70]) that allows those authors to cover the range of $q \in [1, 6/5]$, which you have not covered here?

5. p.65, 4 lines above eq. (4.5.59): insert “the limit as $\varepsilon \to 0$ of” after ‘the term containing”; in the same line delete “there is no more” and insert “is absent” after “$\varepsilon\Delta\mathbb{F}_\varepsilon : \mathbb{F}_\varepsilon$”.

6. p.66, line 6: I think that $C_c((-\infty, T) \times \Omega_{\delta_0})$ should be either $C^2_c((-\infty, T) \times \Omega_{\delta_0})$ or $C^\infty_c((-\infty, T) \times \Omega_{\delta_0})$, because in the middle line of (4.5.60) the Laplacian of $\varphi$ and $\partial_t\varphi$ appear, so suitable differentiability of $\varphi$ is needed so as to ensure that (4.5.60) makes sense.

7. On p.88, the Candidate's Master's thesis is listed among the "author's publications". If this Master's thesis is publicly available, it would be helpful to provide the URL of the PDF file of the Master's thesis.

8. Pages 82–85: it is not clear to me why in the Bibliography the various bibliographical items have been listed in their order of appearance rather than in alphabetical order according to the surname of the first author, as is standard in books and in Ph.D. theses.

9. Pages 100–102: Shouldn't the discussion here be confined to $t \in [0, \tilde{t})$ instead of $t \in [0, T]$? In the previous section the existence of approximating solutions has only been shown for $t \in [0, \tilde{t})$, so it is only after the energy estimates on pages 100–102 have been proved for $t \in [0, \tilde{t})$ that one can deduce that $\tilde{t}$ is in fact equal to $T$.