Opponent report on doctoral thesis **Mathematical Analysis of Selected Problems for Complex Fluids** by Tomáš Los

In the thesis, the author studies several models of incompressible fluids with complex behaviour which are described by rather complicated generalizations of classical incompressible Navier-Stokes system. Altogether, three systems of partial differential equations are studied and in each case, the author proves global existence of weak solutions.

The thesis is well structured. In the first chapter, the author introduces the models he studies in further chapters, in particular he describes the steps and assumptions leading from the general model to the studied pore pressure activated Bingham fluid studied in Chapter 2 and similarly the activated granular material studied in Chapter 3. He also introduces the rate-type fluid model studied later in Chapter 4.

In Chapter 2, the author proves a global existence theorem for weak solutions for the activated Bingham fluid model. The main problem in the proof and the key contribution of the author is the proof that the constitutive relation between the symmetric velocity gradient and the stress tensor is satisfied in the limit from the approximated system.

In Chapter 3, a similar model is studied, however with more complicated constitutive relation between the symmetric velocity gradient and the stress tensor. Moreover, a similar type of condition is prescribed also at the boundary of the physical domain, where a rather complicated relation is given between the tangential part of the velocity and tangential part of normal stresses. The proof follows the ideas from the previous chapter, namely the approximation is very similar and again, the key problem is to prove that the constitutive relations are satisfied in the limit.

Finally, in Chapter 4, a fundamentally different system of PDEs is studied, the stress tensor here is assumed to be split into a standard viscous part and two elastic parts, for each of which one prescribes another partial differential equation. This system is studied in two space dimensions, which allows the author to use in the proof some properties not directly available in 3D. Using a kind of stress diffusion approximation and a limiting process, the author proves existence of weak solutions to the studied problem.

Some additional supplementary material is presented in the Appendix of the thesis. The decision, what to include in the Appendix and what to present in the Chapters of the thesis seems a bit unclear to me, for example Appendix A.1 would in my opinion be more suitable to include directly in Chapter 3, but that is more a matter of taste than an objection from my side. There are naturally some misprints in the thesis but their number is quite low and do not have an impact on readability of the thesis.

It needs to be emphasized that global existence results for weak solutions without any smallness assumptions for such complex problems are valuable and highly non-trivial. The presented proofs required the author to master rather robust and difficult mathematical apparatus. The results presented in Chapter 2 and Chapter 3 were already published in very good journals, the result of Chapter 4 is submitted for publication. By achieving these results, the author clearly demonstrated that he is capable of independent research and without a doubt, **I recommend the thesis to be defended and the author to be awarded a Ph.D. title**.

There are however some small issues which I would like the author to address during the defence of this thesis.

- Unlabeled formula on page 5: Is there really a single total energy equation and a single entropy equation for the mixture and if so, what are the quantities ρ and $\boldsymbol{v}$ in these equations and how are they related to ρ_α and $\boldsymbol{v}_\alpha$?
- Chapter 2: In the introduction to the chapter, there is a sentence about commenting on possible results for no-slip boundary conditions and further extensions in the concluding section. There is no such concluding section in the Thesis. Therefore the question of other boundary conditions should be addressed during the defence. This also applies to the Navier boundary conditions achieved by setting $s_* = 0$. This is not allowed by the main theorem of Chapter 2, so does this proof work also for $s_* = 0$?

- Definition 1 of Chapter 2: The test functions in the equations (2.2.4) and (2.2.5) are assumed to be time-independent and the equations to hold almost everywhere in time. However in the proof, the weak formulation is developed for time-dependent test functions, see (2.4.34). The author should comment this inconsistency between the definition of the weak solution and proof of its existence.
- The calculation behind (4.1.4) does not seem to be completely straightforward, I would appreciate to see more details during the defence.

In Prague, August 31, 2022

Mgr. Ondřej Kreml, Ph.D.