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MASTER THESIS

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Cooperative games with partial information

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Author's signature

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Název práce: Kooperativní hry s částečnou informací
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Abstrakt: Částečně definované kooperativní hry jsou zobecněným modelem klasických kooperativních her, ve kterém je známa hodnota pouze některých koalic. Proto se dá na tento model nahlížet jako na jeden z možných přístupů k modelování neurčitosti.

Hlavním cílem této práce je shrnout a rozšířit existující výsledky této teorie. Práce obsahuje výsledky týkající se superaditivity, konvexity, pozitivity a 1-konvexity neúplných her. Pro všechny tyto vlastnosti studujeme popis množiny všech extenzí (úplných her rozšiřujících danou neúplnou hru). V rámci toho se soustředíme na několik tříd neúplných her. Mezi dalšími se jedná o neúplné hry s minimální informací, neúplné hry s definovaným horním vektorem nebo symetrické neúplné hry. Uvádíme i několik výsledků k obecným neúplným hrám.

V rámci studia superaditivity a 1-konvexity definujeme a studujeme koncepty řešení (definované pouze na částečné informaci). Konkrétně pro 1-konvexitu nabízíme rozsáhlou analýzu zadefinovaných konceptů, zahrnující několik ekvivalentních charakterizací.

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Abstract: Partially defined cooperative games are a generalisation of classical cooperative games in which the worth of some of the coalitions is not known. Therefore, they are one of the possible approaches to uncertainty in cooperative game theory.

The main focus of this thesis is to collect and extend the existing results in this theory. We present results on superadditivity, convexity, positivity and 1-convexity of incomplete games. For all the aforementioned properties, a description of the set of all possible extensions (complete games extending the incomplete game) is studied. Different subclasses of incomplete games are considered, among others incomplete games with minimal information, incomplete games with defined upper vector or symmetric incomplete games. Some of the results also apply to fully generalised games.

For superadditivity and 1-convexity, solution concepts (considering only partial information) are introduced and studied. Especially for 1-convexity, a thorough investigation of the defined solution concepts consisting of different characterisations is provided.

Contents

Introduction

The history of humankind taught us that there are many situations and scenarios where cooperation is more profitable than competition. Despite potential advantages, new challenges and questions arise with cooperation. Some of the most urgent questions are the following. How to distribute the payoff of a group between its members? Is there a distribution which is profitable to every player? And if there is more than one way to distribute the payoff, which one is the most fair, stable or efficiently computable?

As a possible answer to these questions, cooperative games with transferable utility were introduced. For every group of players (a so called coalition), a value is assigned to represent the worth of the cooperation. With this simple definition of cooperative games, many tools for distributing payoffs between players were introduced, each focusing on a different goal. These goals are e.g. the fairness of the distribution or a stability in the sense that it is not profitable for any player to divert from this distribution. Also, different ways for modelling markets, solving banktruptcy problems, sharing costs of constructions and many other OR or optimisation problems were developed throughout the years. There is still an extensive research in the theory of cooperative games even nowadays [13, 21].

The main disadvantage of cooperative games is the amount of information necessary when we want to apply the model to the real-world. This gave rise to the theory of partially defined cooperative games and the model of incomplete games. The simplest (and informal) way to describe incomplete game is that it consists of partial information of a classical (complete) cooperative game. This model can be also applied to situations where uncertainty is involved, namely circumstances under which a portion of data was lost or corrupted.

There are two major problems in the model of partially defined cooperative games. The first is to describe possible extensions of an incomplete game to complete games such that the extensions satisfy further properties. The second problem is similar to the one existing in classical cooperative games – to define possible distributions of payoff based on information acquired from the description of extensions.

The theory of partially defined cooperative games is still at its beginnings. Introduced in 1993 by Willson and revisited more than two decades later by Masuya and Inuiguchi with new ideas, the field is nearly unexplored. This thesis tries to fill this gap.

Chapter 1 is devoted to preliminaries and Chapter 2 to the introduction of incomplete cooperative games. Chapter 3 is devoted to superadditivity, mostly covering the research of Masuya and Inuiguchi. Chapter 4 is devoted to convexity and Chapter 5 to positivity. Finally, in Chapter 6 we study 1-convexity.

We want to pinpoint that the results of Chapters 4 and 5 are based on J. Bok, M. Černý, D. Hartman and M. Hladík. *Convexity and positivity in partially defined cooperative games.* arXiv preprint arXiv:2010.08578, 2020 [3]. The author of this thesis made a significant contribution to the results therein.

1. Preliminaries

In Section 1.1, we summarize basics of the theory of convex sets. These results are useful because many of our results are connected to descriptions of specific convex sets. Section 1.2 introduces basic definitions of the theory of cooperative games, among them classes of cooperative games and different solution concepts studied in further chapters of this text.

We denote a *real closed interval* from a to b, $a < b$, by $[a, b]$. For an inequality $L(x) \leq R(x)$, where $L(x)$ is the left-hand side in variable $x \in \mathbb{R}^n$ and $R(x)$ is the right-hand side in variable $x \in \mathbb{R}^n$, we distinguish two cases. For $x^* \in \mathbb{R}^n$, the inequality is *strict* (at x^*) if $L(x^*) < R(x^*)$ and it is *tight* or *binding at* x^* if $L(x^*) = R(x^*)$. For the sake of brevity, we write $\pm$ (or $\mp$) in one inequality instead of two inequalities with $+$ and $-$, e.g. $x \pm y \leq \mp z$ instead $x + y \leq -z$ and $x - y \leq z$. Notice the difference between $\pm$ and $\mp$.

1.1 Convex sets

In the model of partially defined cooperative games, we study subsets of complete games, so called $\mathcal{C}$*-extensions*. All of the classes of $\mathcal{C}$-extensions studied to date form convex sets. In this section, we revise the theory of convex sets and introduce tools for elegant and compact description of $\mathcal{C}$-extensions. We state all the results as facts and refer the reader to the book by Soltan [28] with exhaustive analysis of convex sets.

Definition 1.1. *A set $K \subseteq \mathbb{R}^n$ is called* convex *provided $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $\lambda \in [0, 1]$.*

See Figure 1.1 for an example of a convex and non-convex set.

We can reformulate Definition 1.1, saying that a nonempty set $K \subseteq \mathbb{R}^n$ is convex if and only if it contains all segments with endpoints $x, y \in K$. We can also define a *convex combination* as a linear combination $\sum_{i=1}^{n} \lambda_i x_i$ where $x_i \in K$ and $\sum_{i=1}^{n} \lambda_i = 1$ for $\lambda_i \geq 0$. The convex sets can be characterised using convex combinations.

Theorem 1.1. *A nonempty set $K \subseteq \mathbb{R}^n$ is convex if and only if it contains all convex combinations of points from K.*

The convex sets we study are of a special form as they are intersections of closed halfspaces. A *closed halfspace* is the set $H \coloneqq \{x \in \mathbb{R}^n \mid ax \leq b\}$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The *hyperplane* S is the set $S \coloneqq \{x \in \mathbb{R}^n \mid ax = b\}$.

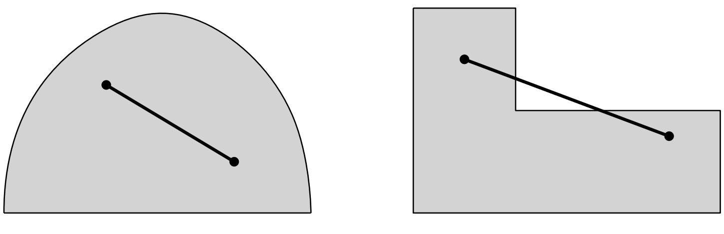

Figure 1.1: A convex set on the left and a non-convex set on the right. The figure is adapted from [28].

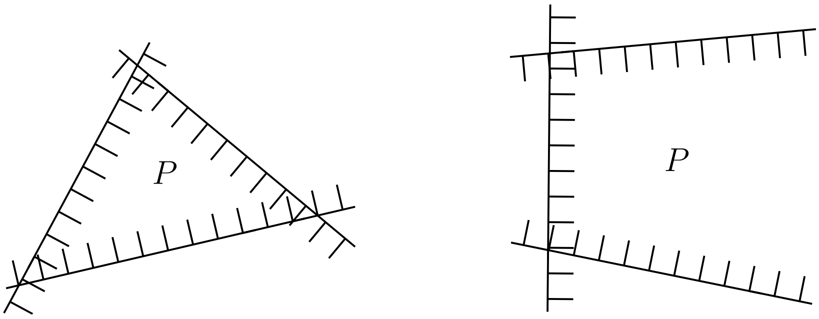

Figure 1.2: An example of two polyhedrons. The one on the left is polytope, the one on the right is not. The figure is adapted from [28].

Definition 1.2. *A set $P \subseteq \mathbb{R}^n$ is called* polyhedron *if it is an intersection of finitely many closed halfspaces, say $H_1, \dots, H_r$:*

$$P = H_1 \cap \dots \cap H_r.$$

The sets $\emptyset$ and $\mathbb{R}^n$ are polyhedrons. A bounded convex polyhedron is called polytope.

See Figure 1.2 for an example of polyhedrons and of polytope.

We say that a hyperplane S_i corresponding to H_i is *supporting* the set P. An important example of polyhedrons are sets $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.

In the study of convex sets, so called *faces* are important. These are the intersections of the convex set and its supporting hyperplanes. *Extreme faces F* satisfy that whenever $\lambda x + (1 - \lambda)y \in F$ for $\lambda \in [0, 1]$, then $x, y \in F$. *Extreme points* and *extreme rays* are extreme faces of a special importance as they fully characterise the polyhedrons.

Definition 1.3. *Let K be a convex set. A point $x \in K$ is an* extreme point *(or* vertex*) of K if there is no way to express x as a convex combination $\lambda y + (1 - \lambda)z$ such that $y, z \in K$ and $0 \leq \lambda \leq 1$, except for taking $y = z = x$.*

Whenever necessary, different characterisations of extreme points are introduced and used throughout the text. If they are used only once, we dare to omit them in this section for the sake of brevity. The following characterisation, however, is used several times in the text.

Theorem 1.2. *Let $P \subseteq \mathbb{R}^n$ be a convex polyhedron. A point $e \in P$ is an extreme point (or vertex) if and only if for every $x \in \mathbb{R}^n$:*

$$(e + x) \in P \wedge (e - x) \in P \implies x = 0.$$

Yet another characterisation of extreme points is in terms of binding of linearly independent constraints.

Theorem 1.3. *A nonzero element x of a polyhedron $P \subseteq \mathbb{R}^n$ is an extreme point if and only if there are n linearly independent constraints binding at x.*

To define extreme rays, we use halflines. A *closed halfline* (from x to y) is the set $\{(1-\lambda)x + \lambda y | \, \lambda \geq 0\}$ and an *open halfline* (from x to y) is the set $\{(1-\lambda)x + \lambda y | \, \lambda > 0\}$. Closed and open halflines form together *halflines.* We call x to be the *endpoint* of the halfline. Notice that halflines form a special case of halfspaces.

Definition 1.4. *An* extreme ray *of a convex set K is a halfline $e \subseteq K$ which is an extreme face of K.*

In this text, we make use of an alternative definition of extreme rays arising from the connection of extreme rays of a polyhedron and extreme rays of a specific *polyhedral cone* (further defined as *recession cone*).

Definition 1.5. *A nonempty set $C \subseteq \mathbb{R}^n$ is a* cone *with apex $s \in \mathbb{R}^n$ provided $s + \lambda(x - s) \in C$ whenever $x \in C$ and $\lambda \geq 0$. A* convex cone *is a cone which is a convex set. Further, a* polyhedral cone *C is a convex cone which can be expressed as $C = \{x \in \mathbb{R}^n \mid Ax \geq 0\}$ for some $A \in \mathbb{R}^{n\times n}$.*

Setting $\lambda = 0$ in the definition of the cone, we can observe the apex s is always a part of the cone. One can reformulate this definition, stating that a nonempty set $C \subseteq \mathbb{R}^n$ is a cone with apex s if and only if every halfline from s to x lies in C whenever $x \in C \setminus \{s\}$. Hence a cone with apex s is either the singleton $\{s\}$ or a union of closed halflines with the common endpoint s.

In the following definition, a polyhedral cone with a connection to extreme rays is introduced.

Definition 1.6. *Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $y \in P$. The* recession cone *of P (at y) is the set*

$$R := \{d \in \mathbb{R}^n \mid y + \lambda d \in P \textit{ for all } \lambda \geq 0\}.$$

From its definition, the recession cone consists of all directions along which we can move indefinitely from y and still remain in P. Notice that $y + \lambda d \in P$ for all $\lambda \geq 0$ if and only if $A(y + \lambda d) \leq b$ for all $\lambda \geq 0$ and this holds if and only if $Ad \leq 0$. Thus R does not depend on a specific vector y.

We call the extreme rays of R associated with P the *extreme rays* of P. Notice, the recession cone allows us to associate extreme rays of polyhedrons with extreme rays of convex cones. Any characterisation of extreme rays of polyhedral cones (including the following one) can be therefore also applied to extreme rays of polyhedrons.

Theorem 1.4. *A nonzero element x of a polyhedral cone $C \subseteq \mathbb{R}^n$ is an extreme ray if and only if there are $n - 1$ linearly independent constraints binding at x.*

The following theorem gives a full description of pointed polyhedron (i.e. unbounded convex set with at least one extreme point) based only on extreme points and extreme rays.

Theorem 1.5. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a nonempty polyhedron with at least one extreme point. Let $x_1, \ldots, x_r$ be its extreme points and $y_1, \ldots, y_\ell$ be its extreme rays. Then*

$$P = \left\{ \sum_{i=1}^{r} \alpha_i x_i + \sum_{j=1}^{\ell} \beta_i y_i \mid \forall i, j : \alpha_i \geq 0, \beta_j \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\}.$$

1.2 Cooperative games

Cooperation of players is an important concept in game theory. This section presents only the basics of cooperative game theory, necessary in our research. For more information on cooperative games, see aforementioned publications [13, 21].

In Subsection 1.2.1, we introduce cooperative games and fix useful abbreviations and notations when working with them. Subsection 1.2.2 presents different classes of cooperative games and in Subsection 1.2.3, we describe how the problem of distribution of individual payoffs between players is tackled, employing so called *payoff vectors* and *solution concepts*.

1.2.1 Main definitions and notation

Definition 1.7. *A* cooperative game *is an ordered pair* (N, v)*, where* N *is a finite set of players (in this text* $\{1, 2, \dots, n\}$*) and* $v\colon 2^N \to \mathbb{R}$ *is the characteristic function of the cooperative game. We further assume that* $v(\emptyset) = 0$*.*

We denote the set of n-person cooperative games by Γ^n. Subsets of N are called *coalitions* and N itself is called the *grand coalition*. We often write v instead of (N, v) whenever there is no confusion over what the player set is. We shall often associate the characteristic functions $v\colon 2^N \to \mathbb{R}$ with vectors $v \in \mathbb{R}^{2^{|N|}}$. This will be more convenient for viewing sets of cooperative games as (possibly convex) sets of points.

We note that the presented definition assumes transferable utility (shortly TU). Therefore, by a cooperative game or a game we mean in fact a cooperative TU game.

Generally, functions $f\colon 2^N \to \mathbb{R}$ (without the restriction $v(\emptyset) = 0$) are called *set functions* and there is a vast research concerning this field of mathematics. We refer the reader to an excellent book by Grabisch [13] unifying the theory of cooperative games and set functions. This connection between cooperative games and set functions will be useful in Chapter 4.

To avoid cumbersome notation, we use the following abbreviations. Instead of $S \cup \{i\}$, we use simply $S \cup i$ and analogously, instead of $S \setminus \{i\}$, we use $S \setminus i$. Also, we often replace singleton set $\{i\}$ with i. We use $\subseteq$ for the relation of "being a subset of" and $\subsetneq$ for the relation "being a proper subset of". By $\emptyset \neq S \subseteq N$, we mean $S \subseteq N$ and $S \neq \emptyset$. To denote the sizes of coalitions e.g. N, S, T, we often use n, s, t, respectively.

1.2.2 Classes of cooperative games

The definition of cooperative game is rather general, therefore in the following chapters, we restrict ourselves to the classes of games introduced in this subsection: *superadditive*, *convex*, *positive* and *1-convex games* (we devote separate chapters to these further in the text). We also introduce *monotonic games*, *symmetric games* and *zero-normalised games* together with the concept of *zero-normalisation*. They are also important for the study of the first four mentioned classes.

Monotonic cooperative games

Definition 1.8. *A cooperative game* (N, v) *is* monotonic *if for every* $S \subseteq T \subseteq N$*, it holds*

$$v(S) \leq v(T).$$

Monotonicity follows the idea that a bigger coalition is able to obtain a higher profit as a group. In cooperative game theory, monotonicity often does not yield strong results on itself. This might change when we consider monotonicity together with some additional properties of cooperative games, e.g. *superadditivity* as can be seen in Chapter 3.

Superadditive cooperative games

Definition 1.9. *A cooperative game* (N, v) *is* superadditive *if for every* $S, T \subseteq N$ *such that* $S \cap T = \emptyset$*, it holds*

$$v(S) + v(T) \leq v(S \cup T). \tag{1.1}$$

We denote the set of all superadditive cooperative n*-person games by* S^n*.*

Intuitively, the condition of superadditivity states that the players from S and T are worth at least as much together as is the sum of worths of the two separate coalitions. The idea behind this follows an argument that the players can always act as if they are working in two separate coalitions, even though, technically, they form a joint coalition.

Of course, in many real world scenarios the superadditivity is violated. Usually, if the number of players of the two coalitions is big, the expenses connected to managing and organising the joint coalition might be too high to be worth it.

There is an alternative characterisation of superadditivity, employing not two, but arbitrarily many disjoint coalition. This characterisation is implicitly used in Chapter 3, which contains results concerning superadditivity.

Theorem 1.6. *[20] A cooperative game* (N, v) *is superadditive if and only if for all* $S_1, \ldots, S_k \subseteq N$ *such that* $\bigcup S_i = N$ *and* $S_j \cap S_\ell = \emptyset$ *for all* $j \neq \ell$*,*

$$\sum_{i=1}^{k} v(S_i) \leq v(N).$$

The relation between monotonic and superadditive cooperative games is sometimes incorrectly understood as that superadditive games are a subset of monotonic games. This is not true and the misunderstanding is probably due to the fact that in many cooperative game theory textbooks, the classes are defined sequentially without any mention of their relation. A 2-person game (N, v) with $v(1) = v(2) = 2$ and $v(12) = 3$ is an example of a monotonic game which is not superadditive. Also, a 2-person game (N, v) with $v(1) = 3$, $v(2) = -1$ and $v(12) = 2$ is superadditive but not monotonic. The relation of monotonic and superadditive games is studied thoroughly in [7].

Convex cooperative games

Definition 1.10. *A cooperative game* (N, v) *is* convex *if for all coalitions* $S, T \subseteq N$*, it holds that*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{1.2}$$

We denote the set of all convex cooperative n*-person games by* C^n*.*

The class of convex cooperative games is a subset of superadditive games. Notice that for $S, T \subseteq N$ such that $S \cap T = \emptyset$, the convexity condition (1.2) is equal to the superadditivity condition (1.1), since $v(S \cap T) = v(\emptyset) = 0$. The difference between the classes is in the conditions for coalitions with nonempty intersection. Intuitively, for these coalitions, the worth of $S \cap T$ is considered twice on the left-hand side of (1.2), therefore $v(S \cap T)$ is added to the right-hand side to balance the inequality.

An alternative characterisation of convexity views the game from the point of view of marginal contributions of players. For player i, the *marginal contribution to* S is $v(S \cup i) - v(S)$. The characterisation states that the games for which marginal contribution grows with the size of the coalition are convex.

Theorem 1.7. *[27] A cooperative game* (N, v) *is convex if and only if for every* $i \in N$ *and every* $S \subseteq T \subseteq N \setminus i$*, it holds that*

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$

Condition (1.2) is sometimes referred to as *supermodularity.* Supermodularity and *submodularity* (a condition where we switch the inequality) is vastly studied property in the theory of set functions [1, 2, 24]. This connection between the theory of cooperative games and the theory of set function helps us in Chapter 4, which concerns convexity.

Positive cooperative games

Positive cooperative games are a subclass of convex cooperative games, however, the relation between them is not as straightforward as the relation between superadditive and convex games. The reason for this is the difference in the definition of these classes. Positive cooperative games employ *unanimity games* and *Harsanyi dividends* in their definition.

Definition 1.11. *For* $\emptyset \neq T \subseteq N$*, the* unanimity game (N, u_T) *is defined as*

$$u_T(S) \coloneqq \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all n-person cooperative games Γ^n forms a vector space and as is shown by Shapley [25], the unanimity games form one of its bases, i.e. every game $v \in \Gamma^n$ can be expressed as $v = \sum_{\emptyset \neq T \subseteq N} d_v(T) u_T$. The coefficients of this linear combination, $d_v(T)$, are called *Harsanyi dividends.* In this text, they are defined in their explicit form.

Definition 1.12. *For* $\emptyset \neq T \subseteq N$*, the* Harsanyi dividend $d_v(T)$ *of a game* (N, v) *is defined as*

$$d_v(T) \coloneqq \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

Positive cooperative games are those games for which all the dividends are non-negative.

Definition 1.13. *A cooperative game* (N, v) *is* positive*, if it holds for all coalitions* $\emptyset \neq T \subseteq N$ *that*

$$d_v(T) \geq 0.$$

We denote the set of all positive cooperative n-person games by P^n*.*

Positive games are sometimes referred to as *totally monotonic games*. This terminology comes from *k-monotonicity*, a property of cooperative games discussed for example in [13]. An interesting fact is that convex games are actually *2-monotonic games*.

Unanimity games are themselves positive games. Also, it is easy to see that every cooperative game $z \in \Gamma^n$ can be expressed as a difference of two positive games $v, w \in P^n$, i.e. $z = v - w$.

1-convex cooperative games

Before we properly introduce 1-convex games, we need the notion of *utopia vector* $b^v \in \mathbb{R}^n$ (sometimes referred to as *upper vector*). It captures each player's marginal contribution to the grand coalition, i.e. $b_i^v := v(N) - v(N \setminus i)$. When there is no ambiguity, we use b instead of b^v. The value b_i^v is considered to be the maximal rightful value that player i can claim when $v(N)$ is distributed among players. If he claims more, it is more advantageous for the rest of the players to form a coalition without player i.

Definition 1.14. *A cooperative game* (N, v) *is called* 1-convex game*, if for all coalitions* $\emptyset \neq S \subseteq N$*, the inequality*

$$v(S) \leq v(N) - b(N \setminus S) \tag{1.3}$$

holds and also

$$b(N) \geq v(N). \tag{1.4}$$

The set of 1-convex n-person games is denoted by C_1^n*.*

From (1.3), (N, v) is 1-convex if even after every player outside the coalition S gets paid his utopia value, there is still more left of the value of the grand coalition for players from S than if they decided to cooperate on their own. This condition challenges the players to remain in the grand coalition and try to find a compromise in the payoff distribution. Also, in (1.4), the utopia vector sums to a value at least as large as the value of the grand coalition N. This was motivated by the idea that the study of possible distributions is not interesting if every player can obtain his maximal rightful (utopia) value.

An equivalent formulation of 1-convexity is in terms of *gap function*, defined as $g^v(S) := b(S) - v(S)$. It captures the gap between the utopia distribution for coalition S and a possible distribution of the profit of S.

Theorem 1.8. *[9] A game* (N, v) *is 1-convex if and only if* $0 \leq g(N) \leq g(S)$ *for all coalitions* $S \subseteq N$*.*

Intuitively, the grand coalition is closest to the utopia distribution among possible coalitions. We can also rewrite conditions (1.3) and (1.4) in terms of the characteristic function as follows. For $\emptyset \neq S \subseteq N$,

$$v(S) + (n - s - 1)v(N) \leq \sum_{i \in N \setminus S} v(N \setminus i), \tag{1.5}$$

and

$$(n - 1)v(N) \geq \sum_{i \in N} v(N \setminus i). \tag{1.6}$$

Other classes of cooperative games

Definition 1.15. *A cooperative game* (N, v) *is* symmetric *if for all* $S, T \subseteq N$ *such that* $|S| = |T|$*, it holds* $v(S) = v(T)$*.*

The characteristic function of a symmetric cooperative game has a simple structure, therefore, these games serve as a good initial object to study new concepts on. This is the reason symmetric cooperative games are usually studied in a connection with further properties.

Definition 1.16. *A cooperative game* (N, v) *is* zero-normalised *if for all singleton coalitions* $\{i\} \subseteq N$*, it holds* $v(\{i\}) = 0$*.*

Zero-normalised games can be considered as those games that capture only the worth of cooperation between players, since the worth of individual players is zero. If a game (N, v) is not zero-normalised, we can define a process of *zero-normalisation* resulting in a zero-normalised game (N, v_0) defined as

$$v_0(S) \coloneqq v(S) - \sum_{i \in S} v(i).$$

All of the classes of cooperative games mentioned above are closed under zero-normalisation, i.e. a class $\mathcal{C} \subseteq \Gamma^n$ is *closed under zero-normalisation* if for $v \in \mathcal{C}$, also $v_0 \in \mathcal{C}$. This fact is used in the proof of Theorem 3.7 and is often used to simplify proofs in the cooperative game theory. We also base a characterisation of one of *solution concepts* for incomplete games on zero-normalisation (see Theorem 6.20).

1.2.3 Solution concepts

The main task of cooperative game theory is to distribute the payoff of the grand coalition $v(N)$ between the players. To be able to work with individual payoffs more easily, *payoff vectors* are introduced. Those are vectors $x \in \mathbb{R}^n$ where x_i represents the individual payoff of player i.

The definition of payoff vector is quite general, therefore, a suitable subset of payoff vectors, so called *imputations* are defined. Those are payoff vectors $x \in \mathbb{R}^n$ such that x is *efficient*, i.e. $\sum_{i \in N} x_i = v(N)$ and *individually rational*, i.e. $x_i \geq v(i)$ for all $i \in N$. This means an imputation distributes the worth of the grand coalition N between the players and only those payoffs where each player is at least as better off as he would be on his own are considered.

To choose between payoff vectors, different *solution concepts* are defined.

Definition 1.17. *Let $\mathcal{C} \subseteq \Gamma^n$ be a class of n-person cooperative games. Then a function $f\colon \mathcal{C} \to 2^{\mathbb{R}^n}$ is a* solution concept *(on class $\mathcal{C}$).*

If the image $f(v)$ of every cooperative game $v \in \mathcal{C}$ is exactly one vector, we write $f\colon \mathcal{C} \to \mathbb{R}^n$ and we say f is a *one-point* solution concept. Otherwise, we say f is a *multi-point* solution concept.

Each solution concept follows a different goal, e.g. *core* (see [21] for definition) is a multi-point solution concept focused on *stability* of the solution. Another example might be the *Shapley value*, a one-point solution concept (defined further in this subsection), which strives to distribute the payoff as *fairly* as possible.

Notice that solution concepts might be equivalently defined as subsets of payoff vectors. There are situations where both approaches are advantageous, usually depending on the specific definition of a solution concept.

In this text, we consider three one-point solution concepts, namely the *τ-value*, the *nucleolus* and the aforementioned *Shapley value*. In the rest of this section, we introduce the three solution concepts, stating their properties and different characterisations. They will be useful further in the analysis of the respective generalisations of solution concepts for incomplete games.

The τ-value $\tau(v)$

The τ-value is a known solution concept for 1-convex games (actually introduced for more general class of quasibalanced games). The concept was defined originally by Tijs in 1981 [29]. We will follow his definition where he describes the τ-value as a compromise between the utopia vector b^v and the *minimal right vector* a^v that is defined as $a^v := b^v - \lambda^v$, where $\lambda_i^v := \min\limits_{S \subseteq N, i \in S} g^v(S)$. The vector λ is called *concession vector*.

The class of quasibalanced games Q^n is defined as

$$Q^n := \{(N, v) \mid \forall i \in N : a_i^v \leq b_i^v \text{ and } a^v(N) \leq v(N) \leq b^v(N)\}.$$

It holds that $C_1^n \subseteq Q^n$.

Definition 1.18. *The* τ-value $\tau\colon Q^n \to \mathbb{R}^n$ *is defined for every $v \in Q^n$ as the unique convex combination of a^v and b^v such that $\sum\limits_{i \in N} \tau_i(v) = v(N)$.*

There is also a formula characterising the τ-value, when we restrict to the set of 1-convex games. It employs the gap function and has a nice interpretation

Theorem 1.9. *[8] For the τ-value $\tau\colon Q^n \to \mathbb{R}^n$, it holds for every $v \in C_1^n$ that*

$$\forall i \in N : \tau_i(v) = b_i^v - \frac{g^v(N)}{n}.$$

From the definition of gap function, $g^v(N)$ can be expressed as the value missing from $v(N)$ that is necessary for the utopia distribution $b^v(N)$, i.e. $v(N) + g^v(N) = b^v(N)$. Theorem 1.9 states that the τ-value is such a compromise of the utopia vector that it distributes equally the losses (given by $g^v(N)$) between all the players.

The τ-value can be also described axiomatically, being uniquely characterised by its properties.

Theorem 1.10. *[30] The τ-value is the only function $f\colon Q^n \to \mathbb{R}^n$ such that the following properties hold for every $v \in Q^n$:*

1. *(efficiency)* $\sum_{i\in N} f_i(v) = v(N)$,
2. *(minimal right property)* $f(v) = a^v + f(v - a^v)$, *where the game* $(v - a^v)$ *is defined as* $(v - a^v)(S) := v(S) - \sum_{i\in S} a_i^v$,
3. *(restricted proportionality property)* $f(v_0) = \alpha b^{v_0}$, *where* $\alpha \in \mathbb{R}$ *and* (N, v_0) *is the zero-normalisation of* (N, v).

The second characterisation can be found in work of Tijs [6]. It consists of five axioms, namely *efficiency*, *translation equivalence*, *bounded aspirations*, *convexity*, and *restricted linearity*. On top of that, there are further results concerning axioms of τ-value, which help with a better comparison with other solution concepts. In the next theorem, we state several of them.

Theorem 1.11. *[29] For the τ-value $\tau\colon Q^n \to \mathbb{R}^n$, the following properties hold for every $v \in Q^n$:*

1. *(individual rationality)* $\forall i \in N : \tau_i(v) \geq v(i)$,
2. *(efficiency)* $\sum_{i\in N} \tau_i(v) = v(N)$,
3. *(dummy player)* $\forall i \in N$ *and* $\forall S \subseteq N \setminus i : v(S \cup i) = v(S) \implies \tau_i(v) = 0$,
4. *(S-equivalence property)* $\forall k \in [0, \infty], \forall c \in \mathbb{R} : \tau(kv + c) = k\tau(v) + c$.

We note the τ-value does not satisfy *additivity*, which is going to be crucial in our generalisation of this concept (see Subsection 6.2.2). Surprisingly, we show that our generalisation of the τ-value for *incomplete games with minimal information* satisfies a certain form of *additivity* on the class of incomplete games with minimal information (see Theorem 6.19).

The nucleolus $\eta(v)$

An essential component of the definition of nucleolus is the *excess* $e(S, x)$, which is a function dependent on a coalition S and an imputation x. It computes the remaining potential of S when the payoff is distributed according to x, i.e. $e(S, x) := v(S) - x(S)$. Further, $\theta(x) \in \mathbb{R}^{2^{|N|}}$ is a vector of excesses with respect to x, which is arranged in a non-increasing order.

Definition 1.19. *The* nucleolus, $\eta\colon \Gamma^n \to \mathbb{R}^n$, *is the solution concept that assigns to a given game the minimal imputation x with respect to the lexicographical ordering $\theta(x)$, defined as:*

$$\theta(x) < \theta(y) \text{ if } \exists k : \forall i < k : \theta_i(x) = \theta_i(y) \text{ and } \theta_k(x) < \theta_k(y).$$

It is a basic result in cooperative game theory that the nucleolus is a one-point solution concept [21]. In general, the nucleolus can be computed by means of linear programming [15]. For 1-convex games, however, the notion of the nucleolus and the τ-value coincide.

Theorem 1.12. *[8] For every $v \in C_1^n$ it holds that $\eta(v) = \tau(v)$.*

The Shapley value ϕ

The *Shapley value* is one of the oldest and one of the most well-known one-point solution concepts, introduced already in 1953 by Shapley [25]. It computes player's payoff by summing up his marginal contributions to each coalition. As these values do not sum up to $v(N)$ when summing across all players, the values are normalised by a factor dependent on its size.

Definition 1.20. *The* Shapley value $\phi\colon \Gamma^n \to \mathbb{R}^n$ *is defined as*

$$\forall i \in N : \phi_i(v) := \sum_{S \subseteq N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S \setminus i)).$$

There are many different alternative formulas for the Shapley value, including the one from the next theorem.

Theorem 1.13. *[22] The Shapley value $\phi\colon \Gamma^N \to \mathbb{R}^n$ can be expressed as follows:*

$$\forall i \in N : \phi_i(v) = \frac{1}{n} \sum_{S \subseteq N \setminus i} \binom{n-1}{s}^{-1} (v(S \cup i) - v(S)).$$

There are many axiomatic characterisations of the Shapley value. Among those there are the following few: [23, 32, 33, 35]. In the next theorem, a characterisation proposed and proved by Shapley in the aforementioned paper from 1953 is given.

Theorem 1.14. *[25] The Shapley value is the only function $f\colon \Gamma^n \to \mathbb{R}^n$ such that the following properties hold for every $v, w \in \Gamma^n$:*

1. *(efficiency)* $\sum_{i \in N} f_i(v) = v(N)$,
2. *(symmetry)* $\forall i,j \in N, \forall S \subseteq N \setminus \{i,j\} : v(S+i) = v(S+j) \implies f_i(v) = f_j(v)$,
3. *(null player)* $\forall \in N$ *and* $\forall S \subseteq N \setminus i : v(S) = v(S+i) \implies f_i(v) = 0$,
4. *(additivity)* $f(v+w) = f(v) + f(w)$.

The Shapley value also satisfies all of the axioms from Theorem 1.11 except for individual rationality.

2. Incomplete cooperative games

In this chapter, we introduce the model of partially defined cooperative games. In Section 2.1, we define incomplete game and present the fundamental objects of the study, $\mathcal{C}$*-extensions*, together with typical questions concerning them. After that we mention special complete games that are used in bounding and describing the set of $\mathcal{C}$-extensions.

In Section 2.2, we present classes of incomplete games that are studied in this text. We discuss their meaning in real-world applications as well as advantages and disadvantages of their study.

The idea of incomplete games, as considered in this text, was introduced in literature by Willson [34] in 1993. Wilson gave the basic notion of the incomplete game and introduced a solution concept generalising the Shapley value for such games. After more than two decades, Masuya and Inuiguchi revived the research. In [19], they focused on the class of S^n-extensions and analysed a few different generalisations of the Shapley value. Subsequently, Masuya pursued the research for more general classes of S^n-extendable incomplete games in [17, 18], however, most of the results were published without proofs. We discuss these results in Chapter 3. Apart from that, Yu [36] introduced a generalisation of incomplete games to games with coalition structures and studied the proportional Owen value (which is a generalisation of the Shapley value for these games). Unfortunately for the general public, the paper of Yu is published only in Chinese.

2.1 Definitions and notation

Definition 2.1. (Incomplete game) *An incomplete game is a tuple* $(N, \mathcal{K}, v)$ *where* N *is a finite set of players (in this text* $\{1, \ldots, n\}$*),* $\mathcal{K} \subseteq 2^N$ *is the set of coalitions with known values and* $v\colon 2^N \to \mathbb{R}$ *is the characteristic function of the incomplete game. We further assume that* $\emptyset \in \mathcal{K}$ *and* $v(\emptyset) = 0$*.*

We denote the set of n-person incomplete games with $\mathcal{K}$ by $\Gamma^n(\mathcal{K})$. An incomplete game can be viewed from several perspectives. In one of the views, there is an underlying complete game (N, v) from a class of n-person games $\mathcal{C} \subseteq \Gamma^n$. The presence of (N, v) in $\mathcal{C}$ implies further properties of the characteristic function, e.g. superadditivity. Unfortunately, only partial information (captured by $(N, \mathcal{K}, v)$) is known and there is no way to acquire more knowledge. This can be caused by an error in storing or transmitting information about the game as well as by a lack of resources or funds while obtaining information about it. The goal is then to reconstruct (N, v) as accurately as possible. This leads to the definition of $\mathcal{C}$-extensions.

Definition 2.2. ($\mathcal{C}$-extension) *Let* $\mathcal{C} \subseteq \Gamma^n$ *be a class of* n*-person games. A cooperative game* $(N, w) \in \mathcal{C}$ *is a* $\mathcal{C}$-extension *of an incomplete game* $(N, \mathcal{K}, v)$ *if* $w(S) = v(S)$ *for every* $S \in \mathcal{K}$*.*

The set of all $\mathcal{C}$-extensions of an incomplete game $(N, \mathcal{K}, v)$ is denoted by $\mathcal{C}(v)$. We write $\mathcal{C}(v)$*-extension* whenever we want to emphasize the game $(N, \mathcal{K}, v)$.

To view the incomplete games from a different point, imagine we are modelling a real-world problem and we acquire only partial information $(N, \mathcal{K}, v)$.

Therefore, no further information about the underlying complete game is known. In this case, it even makes sense to study if there is a $\mathcal{C}(v)$-extension. In case there is at least one $\mathcal{C}(v)$-extension, we say the game $(N, \mathcal{K}, v)$ is *$\mathcal{C}$-extendable.* The set of all $\mathcal{C}$-extendable incomplete games with fixed $\mathcal{K}$ is denoted by $\mathcal{C}(\mathcal{K})$.

The sets of $\mathcal{C}$-extensions studied in this text are always convex. One of the main goals of the model of partially defined cooperative games is to describe these sets using their extreme points and extreme rays whenever the description is possible. We refer to the extreme points as to *extreme games.*

If the structure of $\mathcal{C}(v)$ is too difficult to describe and it is bounded from either above or from below, we introduce the lower and the upper game.

Definition 2.3. (The lower game and the upper game of a set of $\mathcal{C}$-extensions) *Let $(N, \mathcal{K}, v)$ be a $\mathcal{C}$-extendable incomplete game. Then the* lower game $(N, \underline{v})$ *of $\mathcal{C}(v)$ and the* upper game $(N, \overline{v})$ *of $C(v)$ are complete games such that for any $(N, w) \in \mathcal{C}(v)$ and any $S \subseteq N$, it holds*

$$\underline{v}(S) \leq w(S) \leq \overline{v}(S),$$

and for each $S \subseteq N$, there are $(N, w_1), (N, w_2) \in \mathcal{C}(v)$ such that

$$\underline{v}(S) = w_1(S) \text{ and } \overline{v}(S) = w_2(S).$$

These games delimit the area of $\mathbb{R}^{2^{|N|}}$ that contains the set of $\mathcal{C}$-extensions. Even if we know the description of $\mathcal{C}(v)$, the lower and the upper game are still useful as they encapsulate a range of possible profits $[\underline{v}(S), \overline{v}(S)]$ of coalition S across all possible $\mathcal{C}$-extensions.

In many situations in the cooperative game theory, full information of a cooperative game is not necessary for a satisfiable answer. For example, the τ-value of a 1-convex cooperative game (N, v) depends only on values $v(N)$ and $v(N \setminus i)$ for $i \in N$. What if there are other satisfiable ways to distribute the the payoff between players that can be computed only from partial information encoded by an incomplete game? Based on this question, we can generalise the solution concept to incomplete games.

Definition 2.4. *Let $\mathcal{C}(\mathcal{K})$ be a class of $\mathcal{C}$-extendable n-person incomplete games. Then function $f \colon \mathcal{C}(\mathcal{K}) \to 2^{\mathbb{R}^n}$ is a* solution concept *(on class $\mathcal{C}(\mathcal{K})$).*

If the image $f((N, \mathcal{K}, v))$ of every game $(N, \mathcal{K}, v) \in \mathcal{C}(\mathcal{K})$ is exactly one vector, we write $f \colon \mathcal{C}(\mathcal{K}) \to \mathbb{R}^n$ and we say f is a *one-point* solution concept. Otherwise, we say f is a *multi-point* solution concept.

If f always returns at most one vector, we write $f \colon \mathcal{C}(\mathcal{K}) \to \mathbb{R}^n$, instead of writing $2^{\mathbb{R}^n}$ as the domain and we say f is a *one-point* solution concept. Otherwise, we say f is *multi-point* solution concept.

The model of partially defined cooperative games is still at its beginnings. One of the most significant downsides of classical cooperative games is the complexity of information required. For an n-person game, we have to consider 2^n different coalitions with corresponding values of the characteristic function and to be able to apply the model, we often need all this information (the τ-value of 1-convex games is rather an exception). As we have already stated, what if in certain applications, not exponential, but, let us say, polynomial information in n is required to find an acceptable solution?

2.2 Classes of incomplete games

For complete cooperative games, we usually restrict ourselves to certain classes. These classes determine properties of the characteristic function $v\colon 2^N \to \mathbb{R}$. For incomplete games, we do the same thing with one distinction. Not only we define a class of incomplete games based on the properties of $v\colon 2^{\mathcal{K}} \to \mathbb{R}$, we can also impose a specific structure on $\mathcal{K}$ itself. In this text, we have already encountered classes of incomplete games when we discussed $\mathcal{C}$-extendability. Incomplete games that are $\mathcal{C}$-extendable are an example of such classes. We can also define different types of classes that do not presume $\mathcal{C}$-extendability itself, but rather restrictions of both $\mathcal{K}$ and v. These restrictions reflect both the needs of a certain model of partial information or the need for simpler situations that can be more easily studied and analysed.

In this subsection, we define several important classes of incomplete games and discuss their applicability as well as advantages and disadvantages in the analysis of incomplete games.

Incomplete games with minimal information

Definition 2.5. *An* incomplete game with minimal information $(N, \mathcal{K}, v)$ *satisfies* $\mathcal{K} = \{\emptyset, N\} \cup \{\{i\} | i \in N\}$.

We say that an incomplete game $(N, \mathcal{K}, v)$ *contains minimal information* when we want to talk about incomplete games for which $\mathcal{K}$ might contain other coalitions than singletons and the grand coalition, i.e. $\{\emptyset, N\} \cup \{\{i\} | i \in N\} \subseteq \mathcal{K}$. Furthermore, we define $\mathcal{K}_{\min} := \{\emptyset, N\} \cup \{\{i\} \mid i \in N\}$ to be able to distinguish between general incomplete games and incomplete games with minimal information only by employing the set of coalitions with known values. When using $\mathcal{K}_{\min}$, the set N is always clear from the context.

In our definition of minimal information we do not enforce any restriction on values of v as opposed to the definition of minimal information in [19] where $v(i) \geq 0$ for $i \in N$. We refer to their definition as to *non-negative games with minimal information*. Also, the *minimality* might be misleading since there is also a class with $\mathcal{K} = N$, which is studied, but this class does not posses a special name in our text.

The condition $N \in \mathcal{K}$ is not unreasonably restrictive in the study of incomplete games. First, it often implies an upper bound for the sets of different extensions (for example for P^n-extensions, $N \in \mathcal{K}$ characterises the boundedness of $P^n(v)$, see Theorem 5.4). Also, many of the solution concepts for complete cooperative games (and all of those considered in this text) satisfy efficiency of payoff vectors, i.e. the payoff vector $x \in \mathbb{R}^n$ satisfies $\sum_{i \in N} x_i = v(N)$.

Similarly, the singleton coalitions $\{i\}$ for $i \in N$ often imply a lower bound in many situations. However, for example, the set of C_1^n-extensions of incomplete games with minimal information is not bounded from below (see 6.14).

The main advantage of the class of games with minimal information is in its simplicity. Usually, for this class of incomplete games, the structure of the set of $\mathcal{C}$-extensions is simpler than for more general structures of $\mathcal{K}$, thus also the description of this set is more compact and yields a more elegant form. Therefore, it serves as a good starting point when studying an unknown set of $\mathcal{C}$-extensions.

Also, from the point of view of applications, it is relatively easy to get all information needed to study these classes.

Incomplete games with defined upper vector

Definition 2.6. *An* incomplete game with defined upper vector $(N, \mathcal{K}, v)$ *satisfies* $\{N \setminus i | i \in N\} \cup \{N\} \subseteq \mathcal{K}$.

The upper vector b^v of a complete game (N, v) depends precisely on values of coalitions N and $N \setminus i$ for $i \in N$. This is key information for the study of C_1^n-extensions as well as generalisations of the τ-value (see Section 6.3 for C_1^n-extendable incomplete games with defined upper vector). Also, it might be interesting to study this class of incomplete games on further generalisations of C_1^n-extensions like extensions of *balanced games*, *quasibalanced games* and *semibalanced games* (see [31] for analysis and relation of corresponding classes of complete games).

When we want to stress that $\mathcal{K} = \{N, \setminus i | i \in N\} \cup \{N\}$, we call the game an *incomplete game with exactly the defined upper vector*.

As we already know, the upper vector represents marginal contributions of each player to the grand coalition N. In many real world situations, this is still relatively accessible and meaningful information.

Incomplete games with non-negative singletons

Definition 2.7. *An* incomplete game with non-negative singletons $(N, \mathcal{K}, v)$ *satisfies* $\{\emptyset, N\} \cup \{\{i\} | i \in N\} \subseteq \mathcal{K}$ *and* $v(i) \geq 0$ *for all* $i \in N$.

This class was introduced in Masuya [19] as the most *general* case for the study of S^n-extensions. The non-negativity of values of singletons, together with minimal information, enforce boundedness of the set $S^n(v)$ as well as a relatively simple structure.

We believe this is still a reasonable class as there are many real-world problems representing the values of characteristic functions as profits or any sorts of payment for which the non-negativity is natural.

The main disadvantage is, in our opinion, the loss of a complex structure of the set of $\mathcal{C}$-extensions. In the case of S^n-extensions, non-negativity together with superadditivity enforce monotonicity of the characteristic function, a strong property that does not hold in general for S^n-extensions of an incomplete game.

Symmetric incomplete games

Definition 2.8. *An incomplete game* $(N, \mathcal{K}, v)$ *is* symmetric *if for all* $S, T \in \mathcal{K}$ *such that* $|S| = |T|$*, it holds* $v(S) = v(T)$.

For this class of incomplete games, we are interested in $\mathcal{C}$-extensions that are also symmetric complete games. These classes (denoted as $\mathcal{C}_\sigma$-extensions) are studied for their nice theoretical aspects. The symmetry usually causes a simpler structure of $\mathcal{C}_\sigma$-extensions, however, it might still preserve key properties of the set and might help to build-up intuition for the study of $\mathcal{C}$-extensions. Although lacking any empirical evidence, this approach helped us in the study of C_1^n-extensions (Chapter 6) and also gave us a better understanding of C^n-extensions in Chapter 4.

Many one-point solution concepts distribute the payoff equally between all players when applied to symmetric cooperative games. Therefore, if the set of $\mathcal{C}_\sigma$-extensions is nonempty, equal distribution is a good candidate for a payoff.

In this text, we also consider other classes of games but as they are always used on a single occasion, we omit them in this subsection.

3. Superadditivity

In this chapter, we focus on superadditivity and the set of superadditive extensions, S^n-extensions. We summarize, review, and sometimes extend the results given by Masuya and Inuiguchi [17, 18, 19]. Some of the proofs missing in the mentioned publications are also provided. In Section 3.1, we discuss results for incomplete games with non-negative singletons. We extend the known results with a characterisation of $S^n(v)$-extendability (since it was not considered in mentioned publications). We also review the lower game and the upper game of the set of S^n-extensions as well as a subset of extreme games. For the subset of extreme games, we provide missing proofs. The full description of the set of S^n-extensions, however, remains unsolved. Section 3.2 is a survey of results concerning non-negative games with minimal information from [19].

3.1 Games with non-negative singletons

In their first publication [19], Masuya and Inuiguchi considered what they called *general* case of incomplete games. However, the set of coalitions of known values $\mathcal{K}$ as well as the characteristic function v were restricted. We decided to refer to these games as *incomplete games with non-negative singletons*. The class is reasonable from the point of view of boundedness of $S^n(v)$. There would be no upper bound on the profit of N as well as there would be no lower bound on the profits of singletons if these coalitions were not part of the set $\mathcal{K}$. The non-negativity of profit of all singletons together with superadditivity imply non-negativity of values of all coalitions $S \subseteq N$, since for any S^n-extension (N, w),

$$0 \leq \sum_{i \in N} v(i) = \sum_{i \in N} w(i) \leq w(S).$$

We want to point out that by maintaining the condition of $N \in \mathcal{K}$ and non-negativity of the S^n-extensions, one could arrive at a similar, yet more general, result without forcing the singletons to be a part of $\mathcal{K}$.

Non-negative superadditive complete games are also monotonic games. The monotonicity of the characteristic function enforces a simpler structure of $S^n(v)$. We therefore believe it is interesting to study S^n-extensions on more general classes of incomplete games.

For the rest of this section, we consider $(N, \mathcal{K}, v)$ to be an incomplete game with non-negative singletons, i.e. $\{\{i\} \mid i \in N\} \cup \{\emptyset, N\} \subseteq \mathcal{K}$ and $v(i) \geq 0$.

S^n-extendability

The S^n-extendability of $(N, \mathcal{K}, v)$ was never considered in the work of Masuya and Inuiguchi because of the different approach they took in their research. Instead of defining an incomplete game and only then studying the set of possible S^n-extensions, they supposed they already had a superadditive complete game but only partial information was known. This slight difference simply expels the question of S^n-extendability. In the following theorem, we present two characterisations of S^n-extendability, one following immediately from the other.

Theorem 3.1. *Let $(N, \mathcal{K}, v)$ be an incomplete game with non-negative singletons. It is S^n-extendable if and only if one of the following characterisations holds.*

1. *For all $S, S_1, \dots, S_s \in \mathcal{K}$ such that $\bigcup_{i=1}^{s} S_i = S$ and $S_i \cap S_j = \emptyset$, it holds*

$$\sum_{i=1}^{s} v(S_i) \leq v(S). \tag{3.1}$$

2. *There is $S^n(v)$-extension $(N, \underline{v})$ described as*

$$\underline{v}(S) \coloneqq \max_{\mathcal{S}} \sum_{i=1}^{s} v(S_i), \tag{3.2}$$

where $\mathcal{S} \coloneqq \{S_1, \dots, S_s \subseteq N \mid \bigcup_{i=1}^{s} S_i = S$ and $S_i \cap S_j = \emptyset\}$.

Proof. If $(N, \underline{v})$ is S^n-extension, the incomplete game is clearly S^n-extendable. Also, if there are $S, S_1, \dots, S_s \in \mathcal{K}$ such that the inequality (3.1) does not hold, there is no S^n-extension (N, w), since $w(S) = v(S)$ and $w(S_i) = v(S_i)$ for $i \in \{1, \dots, s\}$, therefore (N, w) is not superadditive. To conclude the proof, we show the characterisations are equivalent.

First, suppose that conditions (3.1) hold. For $S \in \mathcal{K}$, $\underline{v}(S) = v(S)$ as $\underline{v}(S) \geq v(S)$ follows from (3.1) and $\underline{v}(S) \leq v(S)$ follows from (3.2) (for $\{S\} \in \mathcal{S}$).

To prove the superadditivity of $(N, \underline{v})$, let $A, B \subseteq N$ such that $A \cap B = \emptyset$. Define

$$\mathcal{A} \coloneqq \{A_1, \dots, A_a \subseteq N \mid \bigcup_{i=1}^{a} A_i = A \text{ and } A_i \cap A_j = \emptyset\},$$

$$\mathcal{B} \coloneqq \{B_1, \dots, B_b \subseteq N \mid \bigcup_{i=1}^{b} B_i = B \text{ and } B_i \cap B_j = \emptyset\}, \text{ and}$$

$$\mathcal{C} \coloneqq \{C_1, \dots, C_c \subseteq N \mid \bigcup_{i=1}^{c} C_i = A \cup B \text{ and } C_i \cap C_j = \emptyset\}.$$

For collections $\{A_1, \dots, A_a\} \in \mathcal{A}$ and $\{B_1, \dots, B_b\} \in \mathcal{B}$, it holds from $A \cap B = \emptyset$ that together, $\{A_1, \dots, A_a, B_1, \dots, B_b\} \in \mathcal{C}$, and

$$\underline{v}(A) + \underline{v}(B) = \max_{\mathcal{A}} \left(\sum_{i=1}^{a} v(A_i) \right) + \max_{\mathcal{B}} \left(\sum_{i=1}^{b} v(B_i) \right) \leq \max_{\mathcal{C}} \left(\sum_{i=1}^{c} v(C_i) \right) = \underline{v}(A \cup B).$$

This implies $(N, \underline{v})$ is superadditive and concludes $(N, \underline{v}) \in S^n(v)$.

Second, suppose $(N, \underline{v})$, defined by (3.2), is an $S^n(v)$-extension. As $(N, \underline{v})$ is superadditive and for $S \in \mathcal{K}$, $\underline{v}(S) = v(S)$, the inequalities in (3.1) must hold. □

The upper and the lower game

It is not a coincidence that the game $(N, \underline{v})$ resembles the lower game from Definition 2.3. The next theorem introduces its counterpart, the upper game, and states that together they bound the set of $S^n(v)$.

Theorem 3.2. *[19] Let $(N, \mathcal{K}, v)$ be S^n-extendable incomplete game with non-negative singletons. Then complete game $(N, \underline{v})$ from Theorem 3.1 and complete game $(N, \overline{v})$ defined as*

$$\overline{v}(S) \coloneqq \min_{T \in \mathcal{K} : S \subseteq T} v(T) - \underline{v}(T \setminus S).$$

are the lower game and the upper game, respectively.

For the lower game $(N, \underline{v})$, we have already seen that it is a part of $S^n(v)$. For the upper game, however, this is not always truth, since it might not be superadditive in general. Let $(N, \mathcal{K}, v)$ be a 4-player incomplete game with minimal information as defined in Table 3.1.

$v(1)$	$v(2)$	$v(3)$	$v(4)$	$v(1234)$
8	7	3	1	30

Table 3.1: An example of non-superadditive upper game $(N, \overline{v})$.

For $S = \{1,2\}$ and $T = \{3,4\}$, $v(S) + v(T) = 26 + 15 \not\leq 30 = \overline{v}(S \cup T)$, therefore the game is not superadditive.

As we have already discussed, the games of $S^n(v)$ are monotonic and so is the upper game.

Theorem 3.3. *[19] For an S^n-extendable incomplete game $(N, \mathcal{K}, v)$, the upper game $(N, \overline{v})$ is monotonic.*

A question under what circumstances $(N, \overline{v})$ is superadditive remains unanswered and we leave it as an open problem.

Extreme games of the set $S^n(v)$

Extreme games of S^n-extendable incomplete games with non-negative singletons were considered in [18]. All of the results in the publication are stated without proofs. We provide proofs for a part of the results and leave out the main result (Theorem 2 in the publication) as well as all results concerning solution concepts. We do so after a recommendation by the author as these are only preliminary results.

Let $\mathcal{T} = \{T_1, \dots, T_k\}$ be a set of coalitions such that $T_i \notin \mathcal{K}$ for every i and $T_i \not\subseteq T_j$ for $i \neq j$. For an S^n-extendable game $(N, \mathcal{K}, v)$, define games $(N, v^{\mathcal{T}})$ as

$$v^{\mathcal{T}}(S), := \begin{cases} \overline{v}(S), & \text{if } S \notin \mathcal{K} \text{ and there is } T \in \mathcal{T},\, T \subseteq S, \\ \underline{v}(S), & \text{if } S \notin \mathcal{K} \text{ and for all } T \in \mathcal{T},\, T \not\subseteq S, \\ v(S), & \text{if } S \in \mathcal{K}. \end{cases} \tag{3.3}$$

These games are not superadditive in general, but if they are, they are extreme games of $S^n(v)$.

Theorem 3.4. *Let $(N, \mathcal{K}, v)$ be an S^n-extendable incomplete game with non-negative singletons. A complete game $(N, v^{\mathcal{T}})$ is an extreme game of $S^n(v)$ if and only if $(N, v^{\mathcal{T}}) \in S^n(v)$.*

Proof. If $(N, v^{\mathcal{T}}) \notin S^n(v)$, it cannot be considered as an extreme game. Let $(N, v^{\mathcal{T}}) \in S^n(v)$ and suppose, for a contradiction, $(N, v^{\mathcal{T}})$ is not an extreme game of $S^n(v)$. By Definition 1.3, there are S^n-extensions (N, x) and (N, y) such that $v^{\mathcal{T}} = \lambda x + (1-\lambda) y$ for $0 < \lambda < 1$. In other words, there is a coalition S such that $x(S) < v^{\mathcal{T}}(S) < y(S)$. If $v^{\mathcal{T}}(S) = \underline{v}(S)$, the first inequality leads to a contradiction as $\underline{v}(S)$ is the lower bound on the profit of S among all S^n-extensions. Similarly, if $v^{\mathcal{T}}(S) = \overline{v}(S)$, the second inequality leads to a contradiction as $\overline{v}(S)$ is, in turn, the upper bound on the profit of S among all S^n-extensions. Therefore, $(N, v^{\mathcal{T}})$ is an extreme game of $S^n(v)$. ☐

Unfortunately, in general, not all games $(N, v^{\mathcal{T}})$ are $S^n(v)$-extensions. However, there is a subset of these games with a simple sufficient condition ensuring superadditivity of $(N, v^{\mathcal{T}})$.

Theorem 3.5. *Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a set of coalitions such that $T_i \notin \mathcal{K}$ for every i and $T_i \nsubseteq T_j$ for $i \neq j$ and let for every $T_i \in \mathcal{T}$, $|T_i| > \lceil \frac{n}{2} \rceil$. Then $(N, v^{\mathcal{T}})$ in an $S^n(v)$-extension.*

Proof. For coalitions $S, T \subseteq N$ such that $S \cap T = \emptyset$, there are only three possible forms of the $v^{\mathcal{T}}(S) + v^{\mathcal{T}}(T) \leq v^{\mathcal{T}}(S \cup T)$ depending on the values of $v^{\mathcal{T}}$. The first form

$$\underline{v}(S) + \underline{v}(T) \leq \underline{v}(S \cup T)$$

holds, since $(N, \underline{v})S^n(v)$ is superadditive. The second form

$$\underline{v}(S) + \underline{v}(T) \leq \overline{v}(S \cup T)$$

holds since $\underline{v}(S) + \underline{v}(T) \leq \underline{v}(S \cup T) \leq \overline{v}(S \cup T)$. Finally, the third form

$$\underline{v}(S) + \overline{v}(T) \leq \overline{v}(S \cup T).$$

holds, because for $(N, w) \in S^n(v)$ such that $w(T) = \overline{v}(T)$,

$$\underline{v}(S) + \overline{v}(T) \leq w(S) + w(T) \leq w(S \cup T) \leq \overline{v}(S \cup T).$$

We note such game (N, w) exists from the definition of the upper game (see Definition 2.3). □

There are two questions concerning extreme games of $S^n(v)$ that remain unanswered. First, there is the characterisation of all collections $\mathcal{T}$ for which $(N, v^{\mathcal{T}})$ is superadditive. Second, it is not clear if there are other extreme games different from $(N, v^{\mathcal{T}})$.

3.2 Non-negative incomplete games with minimal information

In this section, non-negative incomplete games with minimal information are considered. Those are incomplete games with minimal information such that $v(i) \geq 0$ for all $i \in N$. Following results by Masuya and Inuiguchi [19], in Subsection 3.2.1, a description of the set of $S^n(v)$-extensions is proposed. An interesting aspect of the description is that the set is not described by its extreme games but by extreme games of positive extensions (P^n-extensions). Solution concepts for non-negative incomplete games with minimal information are considered in Subsection 3.2.2. Those are defined as solution concepts for special complete S^n-extensions. Namely the Shapley value of the lower game, the upper game and any *almost symmetric* game (N, s) (properly defined further in the subsection) are studied. Surprisingly, all three solution concepts coincide for the class of non-negative incomplete games with minimal information. A similar research was done for the nucleolus, however, all six solution concepts proved to be equal.

3.2.1 Description of the set of S^n-extensions

Following from Theorems 3.1 and 3.2, the lower and the upper games can be expressed as

$$\underline{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ \sum_{i\in S} v(i), & \text{if } S \notin \mathcal{K}, \end{cases} \text{ and } \overline{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N), & \text{if } S \notin \mathcal{K}. \end{cases}$$

Equivalently, when we employ the *total excess* $\Delta \coloneqq v(N) - \sum_{i\in N} v(i)$, the games can be rewritten as

$$\underline{v}(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \sum_{i\in S} v(i) + \Delta, & \text{if } S = N, \\ \sum_{i\in S} v(i), & \text{otherwise,} \end{cases} \text{ and } \overline{v}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ v(i), & \text{if } S = i, \\ \sum_{i\in S} v(i) + \Delta, & \text{otherwise.} \end{cases}$$

For the lower game, the total excess Δ is assigned only to the grand coalition N. However, for the upper game, the total excess is assigned to every non-singleton coalition S.

It was proved in [19] that the lower game $(N, \underline{v})$ of S^n-extensions defined in Theorem 3.1 is actually a positive game. This will help us to state a simplified characterisations of S^n-extendability.

Theorem 3.6. *Let $(N, \mathcal{K}, v)$ be a non-negative incomplete game with minimal information. It is S^n-extendable if and only if $\Delta \geq 0$.*

Proof. For the lower game $(N, \underline{v})$, it holds $d_w(i) = v(i)$ for any $i \in N$. Furthermore, it holds

$$v(N) = w(N) = \sum_{\emptyset \neq S \subseteq N} d_w(S) \geq \sum_{i\in N} d_w(i) = \sum_{i\in N} v(i),$$

where the inequality holds from positivity of $(N, \underline{v})$. It follows $\Delta \geq 0$ is a necessary condition for $P^n(v)$-extension $(N, \underline{v})$.

If $\Delta \geq 0$, a game (N, w) defined by dividends

- $d_w(i) \coloneqq v(i)$ for $i \in N$,
- $d_w(N) \coloneqq \Delta$, and
- $d_w(S) \coloneqq 0$ otherwise

is clearly a $P^n(v)$-extension, thus an $S^n(v)$-extension. □

The description of $S^n(v)$ is based on games (N, v^T), parameterised by coalitions $\emptyset \neq T \subseteq N$, defined as

$$v^T(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ \overline{v}(S), & \text{if } S \notin \mathcal{K} \text{ and } T \subseteq S, \\ \underline{v}(S), & \text{if } S \notin \mathcal{K} \text{ and } T \subsetneq S. \end{cases} \tag{3.4}$$

These games actually form the extreme games of P^n-extensions (we will address this in Section 5.2.2). When zero-normalised, these games are described as

$$v_0^T(S) = \begin{cases} \Delta, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

Any such game (N, v_0^T) can be rewritten as $v_0^T = \Delta u_T$ where (N, u_T) is a unanimity game. The unanimity games form a basis of cooperative games, therefore, any zero-normalised game (N, w_0) can be expressed as a linear combination of games (N, v_0^T) for $|T| > 1$ with coefficients $\alpha_S = \frac{d_{w_0}(S)}{\Delta}$. To remain in the set of S^n-extensions, we have to enforce $\sum_{S \subseteq N, |S|>1} \alpha_S = 1$. To enforce superadditivity of (N, w_0), i.e.

$$w_0(S_1) + w_0(S_2) = \sum_{S' \subseteq S_1} d_{w_0}(S') + \sum_{S' \subseteq S_2} d_{w_0}(S') \leq \sum_{S' \subseteq S_1 \cup S_2} d_{w_0}(S') = v(S_1 \cup S_2),$$

the middle inequality must hold for every $S_1, S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$. By subtracting the left-hand side of the inequality from the right-hand side of the inequality, we get

$$0 \leq \sum_{S' \subseteq S_1 \cup S_2, S' \nsubseteq S_1, S' \nsubseteq S_2} d_{w_0}(S').$$

We denote the set of such coalitions S' by $E(S_1, S_2)$. It holds,

$$0 \leq \sum_{S' \in E(S_1, S_2)} d_{w_0}(S') \iff 0 \leq \sum_{S' \in E(S_1, S_2)} \alpha_{S'}.$$

All these conditions hold for zero-normalised S^n-extensions if and only if they holds for $S^n(v)$-extensions. In the theorem, we employ $N_1 \coloneqq \{T \in 2^N \mid |T| > 1\}$ and $N_2 \coloneqq \{(S_1, S_2) \in 2^N \times 2^N \mid S_1 \cap S_2 = \emptyset\}$.

Theorem 3.7. *[19] Let $(N, \mathcal{K}, v)$ be an S^n-extendable non-negative incomplete game with minimal information and let (N, v^T) for $T \in N_1$ be games from (3.4). The set of S^n-extension can be expressed as*

$$S^n(v) = \left\{ \sum_{T \in N_1} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, (S_1, S_2) \in N_2 : \sum_{T \in E(S_1, S_2)} \alpha_T \geq 0 \right\}, \quad (3.5)$$

where $E(S_1, S_2) \coloneqq \{T \subseteq S_1 \cup S_2 \mid T \nsubseteq S_1$ and $T \nsubseteq S_2\}$.

3.2.2 Solution concepts

There are three different solution concepts for non-negative incomplete games with minimal information that are considered in [19]. For all of them, a special S^n-extension is considered, namely $(N, \underline{v})$, $(N, \overline{v})$ and (N, s) which is an arbitrary S^n-extension such that for all $S, T \subseteq N$, $|S| = |T| > 1$, it holds $s(S) = s(T)$. Notice, game (N, s) is *almost symmetric* (except for values of $i \in N$) and we will refer to such games by this name. Authors of [19] believe such games are of special importance. It is because the only information about $S, T \notin \mathcal{K}$ within $(N, \mathcal{K}, v)$ is these coalitions are of the same size and without a further assumption, it would seem unreasonable to assign higher profit to either of them. For all of these three complete games, the Shapley value is computed, and surprisingly, for all three games, the Shapley value coincide.

Theorem 3.8. *[19] Let $(N, \mathcal{K}, v)$ be an S^n-extendable non-negative incomplete game with minimal information and let $(N, \underline{v}), (N, \overline{v})$ and (N, s) be the lower* *game, the upper game and any almost symmetric S^n-extension. Then it holds that*

$$\phi(\underline{v}) = \phi(\overline{v}) = \phi(s),$$

explicitly,

$$\phi_i(\underline{v}) = \phi_i(\overline{v}) = \phi_i(s) = v(i) + \frac{\Delta}{n}.$$

A similar result is derived for the nucleolus. This is only thanks to the result by Driessen [10], stating that for symmetric complete games (N, v), the Shapley value and the nucleolus coincide. This is important, because under zero-normalisation, games $(N, \underline{v}_0), (N, \overline{v}_0)$ and (N, s_0) are symmetric. The proof of the following theorem is based on these two facts.

Theorem 3.9. *[19] Let $(N, \mathcal{K}, v)$ be an S^n-extendable non-negative incomplete game with minimal information and let $(N, \underline{v}), (N, \overline{v})$ and (N, s) be the lower game, the upper game and any almost symmetric S^n-extension. It holds that*

$$\eta(\underline{v}) = \eta(\overline{v}) = \eta(s),$$

explicitly,

$$\eta_i(\underline{v}) = \eta_i(\overline{v}) = \eta_i(s) = v(i) + \frac{\Delta}{n}.$$

We see that the value $v(i) + \frac{\Delta}{n}$ is a good candidate for the payoff of player i under S^n-extendable non-negative incomplete games with minimal information. We will prove ((in Section 6.2)) the same value is a good candidate for the payoff of player i under C_1^n-extendable incomplete game with minimal information. However, we shall use a different approach.

4. Convexity

In this chapter, we investigate C^n-extensions of $(N, \mathcal{K}, v)$. The condition of convexity of cooperative games can be viewed from the point of view of set functions as supermodularity. If we change the relation in the convexity condition, we arrive to submodularity. Both super- and submodularity are properties of set functions that are widely studied in discrete optimization as well as in other areas of mathematics [1, 2, 11, 16]. In Section 4.1, we utilize the connection between these two worlds, arriving to characterisation of C^n-extendability, followed by an example of an application of this characterisation. After that, in Section 4.2, we study a set of convex extensions that is further restricted by symmetry of the players. Even though these extensions are not interesting from the point of view of solution concepts (as every player receives the same payoff in these games), they serve as a good illustration and, as we believe, a good starting point for the study of more general settings. We give a full characterisation of the lower and the upper game of this set and a description of the set employing extreme points. We conclude the section by yet another geometrical interpretation of the set. Finally, in Section 4.3, we described the set of C^n-extensions of non-negative incomplete games with minimal information. This description is inspired by a description of S^n-extensions from Subsection 3.2.1.

4.1 C^n-extendability

As we already mentioned at the beginning of this chapter, *submodularity* of a set function $v\colon 2^N \to \mathbb{R}$ is closely connected to convexity of the corresponding cooperative game. This is because a characteristic function is *submodular* if for every $S, T \subseteq N$,

$$v(S) + v(T) \geq v(S \cup T) + v(S \cap T).$$

For any submodular function v, we can construct corresponding convex (or sometimes called supermodular) function $-v$. Therefore, results arising from the study of one concept can be considered for the other and vice versa.

The study of extendability of submodular functions initiated by Seshadhri and Vondrák in 2014 [24]. They introduced *path certificate*, a combinatorial structure whose existence certifies that a submodular function is not extendable. They also showed an example of a partial function defined on almost all coalitions that is not extendable, but by removing a value for any coalition, the game becomes extendable. Later in 2018, Bhaskar and Kumar [1] studied extendability of several classes of set functions, including submodular functions. Inspired by the results of Seshadhri and Vondrák, they introduced a more natural combinatorial certificate of non-extendability — *square certificate*. Using this concept, they were able to show that a submodular function is extendable on the entire domain if and only if it is extendable on the lattice closure of the sets with defined values. In 2019, the same authors showed that the problem of extendability for a subclass of submodular functions, so called *coverage functions*, is NP-complete. For more information on coverage functions, see [2].

We present here a characterisation that employs lattice closure. The *lattice closure* $LC(\mathcal{K})$ of a set of points $\mathcal{K} \subseteq 2^N$ in a partially ordered set $(2^N, \subseteq)$ is the inclusion-minimal subset of 2^N that contains $\mathcal{K}$ and that is closed under the operation of union and intersection of sets.

Theorem 4.1. *[1] Let $f\colon \mathcal{K} \to \mathbb{R}$ be a partial function on the power set 2^N. Let $\mathcal{F} := LC(\mathcal{K}) \cap \{S : \exists T_i, T_j \in \mathcal{K} \text{ s.t. } T_j \subseteq S \subseteq T_i\}$ be the sets obtained by an intersection of the lattice closure of $\mathcal{K}$ and the sets that are also both contained in and contained by sets in $\mathcal{K}$. If the partial function f can be extended to a submodular function on $\mathcal{F}$, then it can be extended to a submodular function on 2^N.*

For incomplete games with special structure of $\mathcal{K}$, Theorem 4.1 yields even stronger results. We take as an example an incomplete game where the set of known coalitions forms a chain, i.e. for every $S, T \in \mathcal{K}$, it holds that either $S \subseteq T$ or $T \subseteq S$. This result indicates that there are structures of $\mathcal{K}$ for which the C^n-extendability is independent on the values of v.

Theorem 4.2. *Let $(N, \mathcal{K}, v)$ be an incomplete game with $\mathcal{K}$ forming a chain. Then $(N, \mathcal{K}, v)$ is C^n-extendable.*

Proof. For $\mathcal{K}$ forming a chain, suppose $A, B \in \mathcal{K}$ such that $A \subseteq B$. It holds $A \cap B = A \in \mathcal{K}$ and $A \cup B = B \in \mathcal{K}$, thus the lattice closure of $\mathcal{K}$ as well as $\mathcal{F}$ is the set $\mathcal{K}$ itself. Since the function v is defined on $\mathcal{F}$, we only need to show that condition $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ holds for any $A, B \in \mathcal{K}$. But we already know (for $A \subseteq B$) that $v(A \cup B) = v(B)$ and $v(A \cap B) = v(A)$. Hence the conditions hold and by a result analogous to Theorem 4.1, $(N, \mathcal{K}, v)$ is C^n-extendable. □

4.2 Symmetric incomplete games

In this section, we restrict ourselves to the set of symmetric convex extensions denoted by C^n_σ or, if we want to stress the connection to a game $(N, \mathcal{K}, v)$, we denote it by $C^n_\sigma(v)$. We exploit properties of the characteristic function of symmetric convex games to describe the lower and the upper game of symmetric convex extensions. The additional property of symmetry yields compact (and by our opinion elegant) descriptions.

The fundamental idea behind our results is based on the following characterisation of symmetric convex games.

Proposition 4.3. *Let (N, v) be a symmetric cooperative game. Then for every $S \subsetneq N \setminus j$ and $i \in S$, it holds that*

$$v(S) \leq \frac{v(S \setminus i) + v(S \cup j)}{2} \tag{4.1}$$

if and only if the game is convex.

Proof. If the game is symmetric convex, we consider the characterisation from Theorem 1.7 for coalitions $S, S \cup j$ and $i \in S$, obtaining

$$v(S) - v(S \setminus i) \leq v(S \cup j) - v(S \cup j \setminus i). \tag{4.2}$$

Because $|(S \cup j) \setminus i| = |S|$, we have $v((S \cup j) \setminus i) = v(S)$ by symmetry. By adding $v(S)$ to (4.2) and rearranging the inequality, we get (4.1)

$$v(S) \leq \frac{v(S \setminus i) + v(S \cup j)}{2}.$$

For the opposite implication, suppose that conditions (4.1) hold and (N, v) is not convex. Then there is a player $k \in N$ and coalitions $T_1 \subsetneq T_2 \subseteq N \setminus k$ for which the condition from Theorem 1.7 is violated, i.e.

$$v(T_1 \cup k) - v(T_1) > v(T_2 \cup k) - v(T_2). \tag{4.3}$$

We choose player k and coalitions T_1, T_2 such that the difference $|T_2| - |T_1|$ is minimal. We distinguish two possible cases.

1. If $|T_2| - |T_1| = 1$, then by symmetry of v, we have that $v(T_2) = v(T_1 \cup k)$. In that case, we get

$$v(T_2) > \frac{v(T_1) + v(T_2 \cup k)}{2}.$$

Furthermore, there exists a unique $\ell \in T_2 \setminus T_1$ such that $T_1 \cup \ell = T_2$. Thus we can write

$$v(T_2) > \frac{v(T_2 \setminus \ell) + v(T_2 \cup k)}{2},$$

which leads to a contradiction with (4.1).

2. If $|T_2| - |T_1| > 1$, then there is a coalition T_3 such that $T_1 \subsetneq T_3 \subsetneq T_2 \subseteq N \setminus k$. By minimality of $|T_2| - |T_1|$, we know that

$$v(T_1 \cup k) - v(T_1) \leq v(T_3 \cup k) - v(T_3) \tag{4.4}$$

and

$$v(T_3 \cup k) - v(T_3) \leq v(T_2 \cup k) - v(T_2). \tag{4.5}$$

By adding (4.4) and (4.5) together, we get

$$v(T_1 \cup k) - v(T_1) \leq v(T_2 \cup k) - v(T_2),$$

which is a contradiction with (4.3). □

We note that the characterisation from Proposition 4.3 does not hold for general convex games. This can be seen in the following example.

Example. (A convex game not satisfying conditions from Proposition 4.3)
The game (N, v) given in Table 4.2 is convex, as can be easily checked. However, the inequality

$$v(\{1,3\}) \leq \frac{v(\{1\}) + v(\{1,2,3\})}{2}$$

is not satisfied, as $6 \not\leq \frac{1+9}{2}$.

For symmetric games, we can denote by $s(k)$ the value of $v(S)$ of any $S \subseteq N$ such that $|S| = k$. This allows us to formulate the following characterisation of symmetric convex games.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	1	1	4	6	4	9

Table 4.1: The game (N, v) from Example 4.2 with its characteristic function given in the table.

Theorem 4.4. *A game (N, v) is symmetric convex if and only if for all $k \in \{1, \dots, n-1\}$,*

$$s(k) \leq \frac{s(k-1) + s(k+1)}{2}. \tag{4.6}$$

Hence we can associate every symmetric convex game (N, v) with a function $s\colon \{0, \dots, n\} \to \mathbb{R}$ having the above property. Similarly, we can apply this to $(N, \mathcal{K}, v)$ with a function $\sigma\colon \mathcal{X} \to \mathbb{R}$ where $\mathcal{X} \subseteq \{0, \dots, n\}$ is constructed from $\mathcal{K}$. To formalise these constructions, we define reduced forms of games (N, v) and $(N, \mathcal{K}, v)$.

Definition 4.1. *Let (N, v) be a symmetric game and $(N, \mathcal{K}, v)$ a symmetric incomplete game.*

- *The* reduced form of a game (N, v) *is an ordered pair (N, s), where the function $s\colon \{0, \dots, n\} \to \mathbb{R}$ is a reduced characteristic function such that $s(k) \coloneqq v(S)$ for any $S \subseteq N$ with $|S| = k$.*
- *The* reduced form of an incomplete game $(N, \mathcal{K}, v)$ *is a tuple $(N, \mathcal{X}, \sigma)$ where $\mathcal{X} = \{i | \, i \in \{0, \dots, n\}, \exists S \in \mathcal{K} : |S| = i\}$ and the function $\sigma\colon \mathcal{X} \to \mathbb{R}$ is defined as $\sigma(k) \coloneqq v(S)$ for any $S \in \mathcal{K}$ such that $|S| = k$.*

We also call (N, s) and $(N, \mathcal{X}, \sigma)$ *the reduced game* and *the reduced incomplete game*, respectively.

Since $\emptyset$ always belongs to $\mathcal{K}$, for every reduced incomplete game $(N, \mathcal{X}, \sigma)$, it also holds that $0 \in \mathcal{X}$ and $\sigma(0) = 0$. When we consider a reduced game (N, s) of a $C^n_\sigma(v)$-extension, we often denote this, for brevity, by $(N, s) \in C^n_\sigma(v)$. By $\overline{X}$, we denote the complement of $\mathcal{X}$ in $\{0, \dots, n\}$, i.e. $\overline{\mathcal{X}} \coloneqq \{0, \dots, n\} \setminus \mathcal{X}$.

Notice that a game (N, v) is symmetric convex if and only if the function s of its reduced form (N, s) satisfies property (4.6) from Theorem 4.4.

We can visualize the reduced form (N, s) of a symmetric convex game (N, v) by a graph in $\mathbb{R}^2$. On the x-axis we put the coalition sizes and on the y-axis the values of s. The point $(0, 0)$ is fixed for all reduced games. Now by Theorem 4.4, the conditions for $k \in \{1, \dots, n-1\}$ enforce that for $i \in \{0, \dots, n\}$, points $(i, s(i))$ lie in a *convex* position. More precisely, if we connect the neighbouring pairs $(i, s(i)), (i+1, s(i+1))$ (where $i \in \{0, \dots, n-1\}$) by line segments, we obtain a graph of a convex function. The graph is illustrated in an example in Figure 4.1. Further in this text, we refer to this function as the *line chart* of (N, s). Similarly, for $(N, \mathcal{X}, \sigma)$, the line chart is obtained by connecting consecutive elements from $\mathcal{X}$ by line segments. If $n \in \overline{\mathcal{X}}$, the rightmost line segment is extended to end at x-coordinate n. The values of s are then set to lie on the union of these line segments.

For an incomplete game in reduced form, i.e. $(N, \mathcal{X}, \sigma)$, the first question that arises is that of C^n_σ-extendability. For $\mathcal{X} = \{0, i\}$ with $i \in \{1, \dots, n\}$, the game is

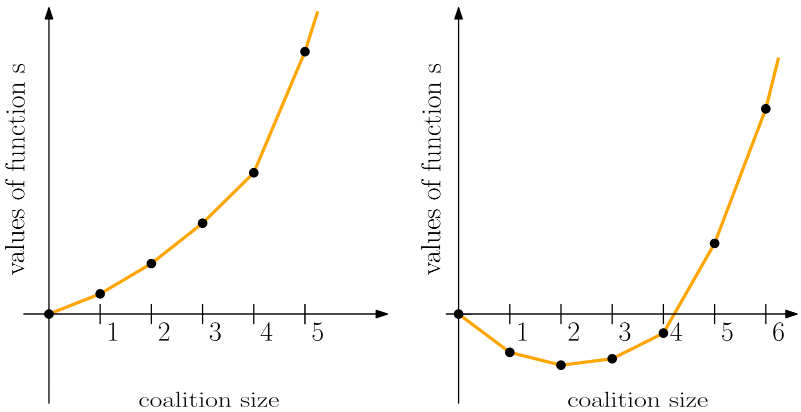

Figure 4.1: Examples of line charts of symmetric convex games in their reduced forms. The figure on the left depicts a game (N, s) where $s(1) > 0$, the graph on the right a situation where $s(1) < 0$. The slopes of the line segments are bounded by convexity of the function.

always C^n_σ-extendable (a possible C^n_σ-extension is the one where the values of each coalition size lie on the line coming through $(0, \sigma(0))$ and $(i, \sigma(i))$). Therefore, in the following theorem, we consider $|\mathcal{X}| > 2$.

C^n_σ-extendability

Theorem 4.5. *Let $(N, \mathcal{X}, \sigma)$ be a reduced form of a symmetric incomplete game $(N, \mathcal{K}, v)$ where $|\mathcal{X}| > 2$. The game is C^n_σ-extendable if and only if*

$$\sigma(k_2) \leq \sigma(k_1) + (k_2 - k_1)\frac{\sigma(k_3) - \sigma(k_1)}{k_3 - k_1},$$

for all consecutive elements $k_1 < k_2 < k_3$ from $\mathcal{X}$.

Proof. If the game is C^n_σ-extendable, let (N, s) be the reduced form of any of its C^n_σ-extension. By Theorem 4.4, the line chart of (N, s) is a convex function that coincides with σ on the values of $\mathcal{X}$. Therefore, for any consecutive elements k_1, k_2, k_3 from $\mathcal{X}$, the inequality must hold.

For the opposite implication, we construct a $C^n_\sigma(v)$-extension by setting the values of s to lie on the line chart of $(N, \mathcal{X}, \sigma)$. The construction is illustrated in Figure 4.2.

Notice that $s(k) = \sigma(k)$ for $k \in \mathcal{X}$ and also, because the inequalities for consecutive elements k_1, k_2, k_3 from $\mathcal{X}$ hold, the line chart represents a convex function. Thus for all $k \in \{1, \ldots, n-1\}$, it holds

$$s(k) \leq \frac{s(k-1) + s(k+1)}{2}$$

and by Theorem 4.4, the game $(N, s) \in C^n_\sigma(v)$. □

As a direct consequence of the previous theorem, the problem of C^n_σ-extendability of symmetric incomplete games can be decided in linear time with respect to the size of the original game (i.e. the size of the characteristic function).

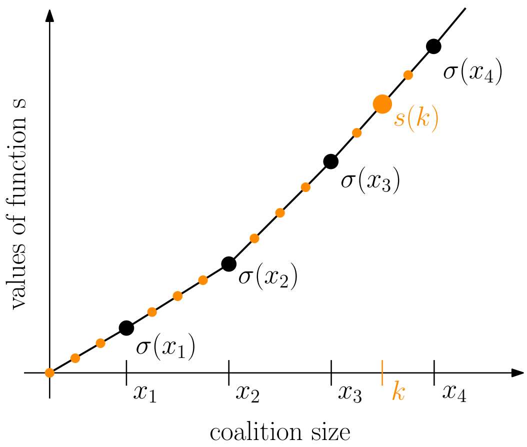

Figure 4.2: The construction of a C_σ^n-extension of $(N, \mathcal{X}, \sigma)$ where $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, using the line chart of $(N, \mathcal{X}, \sigma)$. The value $s(k)$ lies on the line segment connecting $(x_3, \sigma(x_3))$ and $(x_4, \sigma(x_4))$.

The lower game and the upper game

The following proposition addresses the boundedness of the set of C_σ^n-extensions. The restriction to $|N| \geq 3$ is without loss of generality, because for $|N| \leq 2$, when the game $(N, \mathcal{X}, \sigma)$ is not complete and is C_σ^n-extendable, the set of C_σ^n-extensions is always unbounded.

Proposition 4.6. *Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a C_σ^n-extendable symmetric incomplete game $(N, \mathcal{K}, v)$ with $|N| \geq 3$. The $C_\sigma^n(v)$ is bounded if and only if $|\mathcal{X}| \geq 3$ and $n \in \mathcal{X}$.*

Proof. Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a C_σ^n-extendable incomplete game. If $n \in \overline{\mathcal{X}}$, clearly, from Theorem 4.4 there is no upper bound on the profit of n. Let $n \in \mathcal{X}$ and suppose for a contradiction that there is $k \in N$ such that there is no upper bound on its profit. Choose a $C_\sigma^n(v)$-extension (N, s) such that $s(k) > k\frac{\sigma(n)}{n}$. The line chart of (N, s) is not a convex function (the property is violated for $(0, s(0)), (k, s(k)), (n, s(n)))$, therefore $(N, s) \notin C_\sigma^n(v)$.

If $|\mathcal{X}| \leq 2$, then $\mathcal{X} = \{0, n\}$ (otherwise the set of C_σ^n-extensions is not bounded from above). Let ℓ be a negative value smaller than or equal to $\sigma(n)$. Any game (N, s_ℓ) with $s_\ell(k) = \ell$ for $k \in \{1, \ldots, n-1\}$ and $s_\ell(0) = \sigma(0), s_\ell(n) = \sigma(n)$ is a $C_\sigma^n(v)$-extension of $(N, \mathcal{X}, \sigma)$. Thus, there is no lower bound on values of $1, \ldots, n-1$.

If $|\mathcal{X}| \geq 3$, then let $i \in \mathcal{X} \setminus \{0, n\}$. For $k \in \{1, \ldots, i-1\}$, the point $(k, s(k))$ must lie on or above the line coming through points $(i, \sigma(i)), (n, \sigma(n))$, otherwise the convexity of line chart of (N, s) is violated, leading to a contradiction. Similarly, for any $k \in \{i+1, \ldots, n-1\}$ the value $s(k)$ must lie on or above the line coming through points $(0, \sigma(0)), (i, \sigma(i))$, otherwise the convexity is violated, again. The profit of every k is therefore bounded from below. □

Theorem 4.7. *Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a C_σ^n-extendable symmetric incomplete game. Suppose that $C_\sigma^n(v)$ is bounded. Furthermore, for every $k \in \overline{\mathcal{X}}$, denote by i_1, i_2, j_1, j_2 the closest distinct elements from $\mathcal{X}$ such that it holds*

$i_1 < i_2 < k < j_1 < j_2$, *if they exist. Then the lower game has the following form:*

$$\underline{s}(k) := \begin{cases} \sigma(k), & \text{if } k \in \mathcal{K}, \\ \sigma(i_1) + (k - i_1)\frac{\sigma(i_2)-\sigma(i_1)}{i_2-i_1}, & \text{if } k \notin \mathcal{K} \text{ and } j_2 \text{ does not exist,} \\ \sigma(j_1) + (k - j_1)\frac{\sigma(j_2)-\sigma(j_1)}{j_2-j_1}, & \text{if } k \notin \mathcal{K} \text{ and } i_1 \text{ does not exist,} \\ \max\begin{Bmatrix} \sigma(i_1) + (k - i_1)\frac{\sigma(i_2)-\sigma(i_1)}{i_2-i_1}, \\ \sigma(j_1) + (k - j_1)\frac{\sigma(j_2)-\sigma(j_1)}{j_2-j_1} \end{Bmatrix}, & \text{if } k \notin \mathcal{K} \text{ and } i_1, i_2, j_1, j_2 \text{ exist.} \end{cases}$$

The upper game has the following form:

$$\overline{s}(k) := \begin{cases} \sigma(k), & \text{if } k \in \mathcal{X}, \\ \sigma(i_2) + (k - i_2)\frac{\sigma(j_1)-\sigma(i_2)}{j_1-i_2}, & \text{otherwise.} \end{cases}$$

Proof. To prove that $(N, \underline{s})$ is the lower game, we start by showing that for every C^n_σ-extension (N, w) and every coalition size $k \in N$, it holds that $\underline{s}(k) \le w(k)$. If $k \in \mathcal{X}$, trivially $\underline{s}(k) = \sigma(k) = w(k)$. If $k \notin \mathcal{X}$, then since any C^n_σ-extension must have a convex line chart, the value $w(k)$ must lie on or above the lines coming through pairs of points $(i_1, \sigma(i_1)), (i_2, \sigma(i_2))$ and $(j_1, \sigma(j_1)), (j_2, \sigma(j_2))$. The three cases in the definition of the lower game capture this fact by setting the value of $\underline{s}(k)$ so that it lies on either one of the lines (if the other one does not exist) or on the maximum of both of them.

Now it remains to show that for every $k \in N$, the value $\underline{s}(k)$ is attained for at least one C^n_σ-extension. We introduce a C^n_σ-extension $(N, s^{\{a,b\}})$ for consecutive $a, b \in \mathcal{X}$ such that $a < b$, described as

$$s^{\{a,b\}}(\ell) := \begin{cases} \sigma(\ell), & \text{if } \ell \in \mathcal{X}, \\ \underline{s}(\ell), & \text{if } \ell \notin \mathcal{X} \text{ and } a < \ell < b, \\ \overline{s}(\ell), & \text{if } \ell \notin \mathcal{X} \text{ and either } \ell < a, \text{ or } b < \ell. \end{cases}$$

Clearly the game is an extension of $(N, \mathcal{X}, \sigma)$. For $i \in \{2, \dots, n-1\}$ such that all three values $s^{\{a,b\}}(i-1), s^{\{a,b\}}(i), s^{\{a,b\}}(i+1)$ coincide with the respective values of the upper game $\overline{s}$, it holds $s^{\{a,b\}}(i) \le \frac{s^{\{a,b\}}(i-1)+s^{\{a,b\}}(i+1)}{2}$, because $(N, \overline{s})$ is a symmetric convex game (as we show further in this proof) so by Theorem 4.4, the same inequality holds for values of $\overline{s}$. In the rest of the cases, either all the three points $(i-1, s^{\{a,b\}}(i-1)), (i, s^{\{a,b\}}(i)), (i+1, s^{\{a,b\}}(i+1))$ lie on the same line and the inequality holds with the equal sign, or the three points lie on the maximum of two lines coming through pairs of points $(a_2, \sigma(a_2)), (a, \sigma(a))$ and $(b, \sigma(b)), (b_2, \sigma(b_2))$ where $a_2 < a$ and $b < b_2$ are consecutive pairs from $\mathcal{X}$. If $s^{\{a,b\}}(i) > \frac{s^{\{a,b\}}(i-1)+s^{\{a,b\}}(i+1)}{2}$, then either $\sigma(a) > \sigma(a_2) + (a - a_2)\frac{\sigma(b)-\sigma(a_2)}{b-a_2}$ or $\sigma(b) > \sigma(a) + (b-a)\frac{\sigma(b_2)-\sigma(a)}{b_2-a}$, both resulting, by Theorem 4.5, in a contradiction with the C^n_σ-extendability of $(N, \mathcal{X}, \sigma)$. Now for $k \in \mathcal{X}$, we choose $(N, s^{\{a,b\}})$ such that $a = k$ and for $k \notin \mathcal{X}$, we choose $(N, s^{\{a,b\}})$ such that $a < k < b$ are the closest coalition sizes with defined value.

For the upper game $(N, \overline{s})$, suppose for a contradiction that there is the reduced form (N, s) of a $C^n_\sigma(v)$-extension such that for $k \in N$, $\overline{s}(k) < s(k)$. As for $k \in \mathcal{X}$, $\overline{s}(k) = \sigma(k) = s(k)$, it must be that $k \notin \mathcal{X}$. But if $k \notin \mathcal{X}$ and $\overline{s}(k) = \sigma(i_2) + (k - i_2)\frac{\sigma(j_1)-\sigma(i_2)}{j_1-i_2} < s(k)$, the convexity of the line chart is violated,

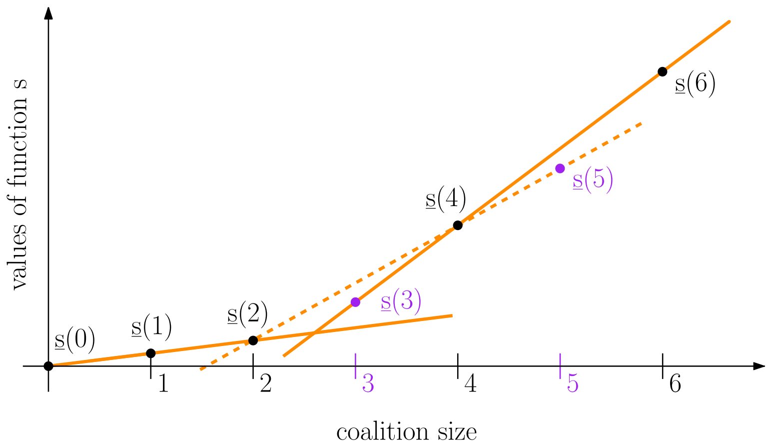

Figure 4.3: An example of a reduced game $(N, \mathcal{X}, \sigma)$ with $\mathcal{X} = \{0, 1, 2, 4, 6\}$ where the condition $\frac{\underline{s}(3)+\underline{s}(5)}{2} \not\geq \underline{s}(4)$ from Theorem 4.4 is not satisfied. This implies that $(N, \underline{s})$ is not a C^n_σ-extension of $(N, \mathcal{X}, \sigma)$.

because $(k, s(k))$ lies above the line segment between points $(i_2, \sigma(i_2))$,$(j_1, \sigma(j_1))$. This is a contradiction.

Now we prove that $(N, \overline{s})$ is a C^n_σ-extension of $(N, \mathcal{X}, \sigma)$. First, it is clearly an extension. Furthermore, notice that the values of $(N, \overline{s})$ lie on the line chart of $(N, \mathcal{X}, \sigma)$. Since the game is C^n_σ-extendable, the line chart is a convex function, therefore inequalities (4.6) from Theorem 4.4 hold, meaning $(N, \overline{s}) \in C^n_\sigma(v)$. □

The game $(N, \overline{s})$ is always a C^n_σ-extension, however, this is not true for $(N, \underline{s})$ in general, as can be seen in the example in Figure 4.3.

Extreme games

The games $(N, s^{\{a,b\}})$ are actually even more important because they are extreme games of $C^n_\sigma(v)$.

Proposition 4.8. *Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a C^n_σ-extendable symmetric incomplete game $(N, \mathcal{K}, v)$. The games $(N, s^{\{a,b\}})$ for consecutive $a, b \in \mathcal{X}$, where $a < b$, and $(N, \overline{s})$, are extreme games of $C^n_\sigma(v)$.*

Proof. For a contradiction, suppose that for some a, b, $(N, s^{\{a,b\}})$ is not an extreme game of $C^n_\sigma(v)$. By Definition 1.3, there are two $C^n_\sigma(v)$-extensions (N, s_1) and (N, s_2) such that $(N, s^{\{a,b\}})$ is their nontrivial convex combination and without loss of generality, there is $i \in \{0, \ldots, n\}$ such that $s_1(i) < s^{\{a,b\}}(i) < s_2(i)$. For $i \in \mathcal{X}$, this is not possible as $s_1(i) = s_2(i) = s^{\{a,b\}}(i)$. Furthermore, for $i \notin \mathcal{X}$ and $a < i < b$, this is a contradiction with $s_1(i) < s^{\{a,b\}}(i) = \underline{s}(i)$ and finally for $i \notin \mathcal{X}$ and either $i < a$ or $b < i$, we get again a contradiction because $\overline{s}(i) = s^{\{a,b\}}(i) < s_2(i)$. Following a similar argument, we conclude that the upper game $(N, \overline{s})$ is also an extreme game. □

In general, $(N, \overline{s})$ and $(N, s^{\{a,b\}})$ are not the only extreme games. In the following theorem, we describe all the extreme games of $C^n_\sigma(v)$.

Theorem 4.9. *Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a C^n_σ-extendable symmetric incomplete game such that $C^n_\sigma(v)$ is bounded. For $k \in \{0, \ldots, n\} \setminus \mathcal{X}$ and $i, j \in \mathcal{X}$*

closest to k such that $i < k < j$, the games (N, s^k) defined as

$$s^k(m) \coloneqq \begin{cases} \sigma(m), & \text{if } m \in \mathcal{X}, \\ \overline{s}(m), & \text{if } m \notin \mathcal{X} \text{ and either } m < i \text{ or } j < m, \\ \underline{s}(m), & \text{if } m = k, \\ \sigma(j) + (m - j)\frac{\sigma(j) - \underline{s}(k)}{j-k}, & \text{if } m \notin \mathcal{X} \text{ and } k < m < j, \\ \sigma(i) + (m - i)\frac{\underline{s}(k) - \sigma(i)}{k-i}, & \text{if } m \notin \mathcal{X} \text{ and } i < m < k \end{cases}$$

together with $(N, \overline{s})$ form all the extreme games of $C^n_\sigma(v)$.

Proof. We divide the proof into two parts. In the first part, we show that any $C^n_\sigma(v)$-extension (N, s) is a convex combination of games $(N, \overline{s})$ and (N, s^k) for $k \in \overline{\mathcal{X}}$. In the second part, we show that every game (N, s^k) is an extreme game, thus (together with the upper game $(N, \overline{s})$) they form all the extreme games.

Before we begin, let us define a *gap* as an inclusion-wise maximal nonempty sequence of consecutive coalition sizes with undefined profit. In other words, we can say that there is a gap between i and j if $i, j \in \mathcal{X}$, $i < j$, $j - i > 1$, and for every i' such that $i < i' < j$, it holds $i' \in \overline{\mathcal{X}}$. The *size* of the gap between i and j is defined as $j - i - 1$, that is the number of coalition sizes with unknown values in the given gap. It is immediate that the size of every gap is at least one.

We shall now prove the first part of the theorem. First, let us suppose that there is only one gap in $(N, \mathcal{X}, \sigma)$. We shall prove this case by induction on the size of the gap.

If the size of the gap is 1, there is only one game (N, s^k) that is equal to $(N, s^{\{k-1,k+1\}})$. Any C^n_σ-extension (N, s) can be expressed as a convex combination of this game and the upper game $(N, \overline{s})$ as $s = \alpha s^k + (1 - \alpha)\overline{s}$ with

$$\alpha = \frac{s(k) - \overline{s}(k)}{s^k(k) - \overline{s}(k)} \in [0, 1].$$

For the induction step, suppose that the size of the gap between i and j is ℓ, $\ell > 1$. Hence there are ℓ games

$$(N, s^{i+1}), (N, s^{i+2}), \ldots, (N, s^{j-1}) \text{ together with } (N, \overline{s}).$$

We construct a new system of $\ell - 1$ games

$$(N, (s^{i+2})'), (N, (s^{i+3})'), \ldots, (N, (s^{j-1})') \text{ together with } (N, (s^{i+1})'),$$

$$\text{where } (s^m)' \coloneqq \alpha s^m + (1 - \alpha)\overline{s} \text{ and } \alpha = \frac{s(i + 1) - \overline{s}(i + 1)}{s^{i+1}(i + 1) - \overline{s}(i + 1)}.$$

These games correspond to the extreme games of an incomplete game $(N, \mathcal{X}', \sigma')$ where $\mathcal{X}' \coloneqq \mathcal{X} \cup \{i + 1\}$, and the function σ' is defined as $\sigma'(m) \coloneqq \sigma(m)$ for $m \in \mathcal{X}$ and $\sigma'(i + 1) \coloneqq s(i + 1)$. The game $(N, (s^{i+1})')$ represents the upper game of $C^n_\sigma(v)$. Since the new system of ℓ games forms the extreme games of $C^n_{\sigma'}$-extensions of $(N, \mathcal{X}', \sigma')$, the game (N, s) (which is also a $C^n_{\sigma'}$-extension of $(N, \mathcal{X}', \sigma')$) is, by induction hypothesis, their convex combination. And as each game $(N, (s^m)')$ is a convex combination of $(N, \overline{s})$ and (N, s^m), the game (N, s) is also a convex combination of the former system

$$(N, s^{i+1}), (N, s^{i+2}), \ldots, (N, s^{j-1}) \text{ together with } (N, \overline{s}).$$

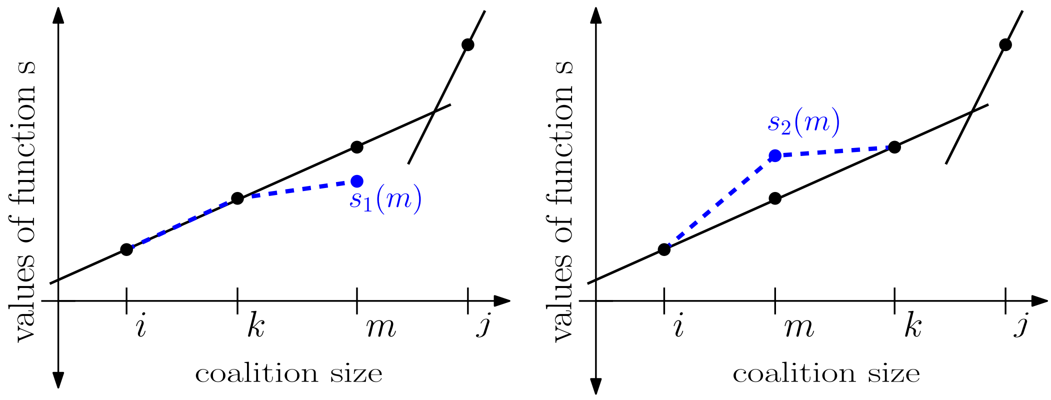

Figure 4.4: Examples of a violation of convexity of the line chart of both (N, s_1) and (N, s_2). The full lines depict the line chart of $(N, \underline{s})$ and the dotted lines depict the line charts of (N, s_1) and (N, s_2). On the left, the situation where $k < m$ is shown. We have values $s^k(i) = s_1(k)$ and $s^k(k) = s_1(k)$, yet $s_1(m)$ is too small. Similarly, on the right, the situation where $m < k$ is shown, with $s^k(i) = s_2(k)$, $s^k(k) = s_2(k)$. However, in this case, the value $s_2(m)$ is too big.

Notice that if there is more than one gap between the coalition sizes in $\mathcal{X}$, then we can follow a similar construction as in the situation with precisely one gap. This is because any two extreme games parametrised by two coalition sizes from one gap assign the same profit to any coalition size from a different gap. Thus, we start our construction by filling in the first gap, after that, taking the extreme games of the extended incomplete game and so on, until there is no gap left.

As for the second part of the proof, suppose for a contradiction that (N, s^k) for $k \in \overline{\mathcal{X}}$ is not an extreme game of $C_\sigma^n(v)$. By Definition 1.3, there are C_σ^n-extensions $(N, s_1), (N, s_2)$ and $m \in N$ such that $s_1(m) < s^k(m) < s_2(m)$. Clearly, $m \notin \mathcal{X}$ (since $s_1(m) = s^k(m) = s_2(m) = \sigma(m)$) and if m is such that $s^k(m) = \overline{s}(k)$ or $s^k(m) = \underline{s}(m)$, we arrive at a contradiction. Therefore, the only case that remains is $m \notin \mathcal{X}$ together with $i < m < j$ and $m \neq k$. For any such m, the convexity of the line chart is violated either for $(i, s_1(i)), (k, s_1(k)), (m, s_1(m))$ (if $k < m$), or for $(i, s_2(i)), (m, s_2(m)), (k, s_2(k))$ (if $m < k$). Both cases are depicted in Figure 4.4. □

For a C_σ^n-extendable symmetric incomplete game in a reduced form $(N, \mathcal{X}, \sigma)$ with $C_\sigma^n(v)$ bounded and $|C_\sigma^n(v)| > 1$, the number of extreme games is always $|\overline{\mathcal{X}}| + 1 = n - |\mathcal{X}| + 2$, no matter what the values of σ are.

Algebraically, we can describe the set $C_\sigma^n(v)$ as

$$C_\sigma^n(v) = \left\{ \left(N, \overline{\alpha}\ \overline{s} + \sum_{k \in \overline{\mathcal{X}}} \alpha_k s^k \right) \middle| \ \overline{\alpha} + \sum_{k \in \overline{\mathcal{X}}} \alpha_k = 1, \overline{\alpha}, \alpha_k \geq 0, k \in \overline{\mathcal{X}} \right\}, \tag{4.7}$$

namely as the set of convex combinations of extreme games $\overline{s}$ and s^k for $k \in \overline{\mathcal{X}}$.

Geometrically, we can describe the set $C_\sigma^n(v)$ when we restrict the game $(N, \mathcal{X}, \sigma)$ a little. First, suppose $\mathcal{X} = \{0, n\}$ and $\sigma(0) = \sigma(n) = 0$. According to Theorem 4.4, we can describe $C_\sigma^n(v)$ by a system of $n - 1$ inequalities with

$n-1$ unknowns, $Ay \leq 0$, where

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}.$$

The matrix A is an *M-matrix* [14], therefore it is nonsingular and $A^{-1} \geq 0$. Nonsingularity of A implies that $C^n_\sigma(v)$ is a pointed polyhedral cone, which is translated such that its vertex is not necessarily in the origin of the coordinate system. Furthermore, because $A^{-1} \geq 0$, the *normal cone* $C^n_\sigma(v)^*$ of $C^n_\sigma(v)$ (see [5]) contains the whole nonnegative orthant. Thus, the vertex of polyhedral cone $C^n_\sigma(v)$ is the biggest element of $C^n_\sigma(v)$ when restricted to each coordinate (this corresponds with the statement that the upper game is a C^n_σ-extension). Therefore, geometrically, the set $C^n_\sigma(v)$ looks like *squeezed* negative orthant. For an incomplete game $(N, \mathcal{X}', \sigma')$ where $\{0, n\} \subseteq \mathcal{X}'$ and $\sigma'(0) = \sigma(n) = 0$, the set of C^n_σ-extensions is $C^n_\sigma(v)$ with some of the coordinates fixed, i.e.

$$C^n_\sigma(v) \cap_{k \in \mathcal{X}'} \{s(k) = \sigma(k)\}.$$

4.3 Non-negative incomplete games with minimal information

For non-negative incomplete games with minimal information, the sets of S^n-extensions and P^n-extensions are described in [19]. For the sake of completeness, in this section we derive similar results for the set of C^n-extensions.

Theorem 4.10. *Let $(N, \mathcal{K}, v)$ be a non-negative incomplete game with minimal information. It is C^n-extendable if and only if $\Delta \geq 0$.*

Proof. The proof immediately follows from the proof of Theorem 3.6 and from inclusions $P^n(v) \subseteq C^n(v) \subseteq S^n(v)$. □

In [19], they showed the lower and the upper games of P^n-extensions coincide with those of S^n-extensions, thus they must coincide with the lower and the upper games of C^n-extensions as well. Therefore $(N, \underline{v})$ and $(N, \overline{v})$ are defined as

$$\underline{v}(S) \coloneqq \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ \sum_{i \in S} v(i), & \text{if } S \notin \mathcal{K}, \end{cases} \text{ and } \overline{v}(S) \coloneqq \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N), & \text{if } S \notin \mathcal{K}, \end{cases}.$$

Further, they showed the lower game $(N, \underline{v})$ is positive (thus also convex) and the upper game is not, however, it is monotonic. Finally, we derive a description of the set of C^n-extensions. We employ $N_1 \coloneqq \{T \subseteq N \mid |T| > 1\}$.

Theorem 4.11. *Let $(N, \mathcal{K}, v)$ be a non-negative incomplete game with minimal information, and let (N, v^T) for $T \in N_1$ be games from (3.4). The set of C^n-extension can be expressed as*

$$C^n(v) = \left\{ \sum_{T \in N_1} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, \forall S_1, S_2 \subseteq N : \sum_{T \in E(S_1, S_2)} \alpha_T \geq 0 \right\}, \quad (4.8)$$

where $E(S_1, S_2) \coloneqq \{T \subseteq S_1 \cup S_2 \,|\, T \not\subseteq S_1 \text{ and } T \not\subseteq S_2\}$.

Proof. The proof follows from the proof of Theorem 3.7, which can be found in [19]. The only difference is in the condition for coefficients α_T. For the description of the set of S^n-extensions, a condition $\sum_{T\in E(S_1,S_2)} \alpha_T \geq 0$ for every pair of conditions $S_1 \cap S_2 = \emptyset$ is enforced (see (3.5)). This condition corresponds to the fact that for $S_1, S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$, it holds $v(S_1)+v(S_2) \leq v(S_1 \cup S_2)$. In terms of Harsanyi dividends, it is equivalent to $\sum_{T\in E(S_1,S_2)} \delta_v(T) \geq 0$. For convex games and $S_1, S_2 \subseteq N$ (not necessarily disjoint coalitions), the conditions $v(S_1) + v(S_2) \leq v(S_1 \cap S_2) + v(S_1 \cup S_2)$ can be equivalently expressed in terms of Harsanyi dividends as

$$\sum_{T\subseteq S_1\cup S_2, T\not\subseteq S_1, T\not\subseteq S_2} \alpha_T \geq 0.$$

Notice that coalitions T are exactly those from the set $E(S_1, S_2)$. $\square$

5. Positivity

In this chapter, we investigate P^n-extensions of $(N, \mathcal{K}, v)$. In Section 5.1, we study incomplete games in general. We tackle questions considering P^n-extendability, boundedness of the set of P^n-extensions and show a characterisation of extreme games. In Section 5.2, we restrict ourselves to special cases – different classes of incomplete games. The mentioned characterisation of extreme games from the previous section is applied in an analysis of several classes of incomplete games. Also, P^n_σ-extensions of incomplete games and P^n-extensions of incomplete games with minimal information are considered in this section.

5.1 Description of $P^n(v)$ for general case

In Subsection 5.1.1, we focus on P^n-extendability. We provide a characterisation based on duality of linear programming and give an example of its application in a time-complexity analysis of the question of P^n-extendability. After that, in Subsection 5.1.2, we characterise the boundedness of $P^n(v)$, and in Subsection 5.1.3, we investigate extreme games of the set of P^n-extensions. For the characterisation of extreme games, we follow and modify the proof of the sharp form of Bondareva-Shapley theorem (the theorem was introduced independently by Bondareva in 1963 [4] and Shapley in 1967 [26]).

5.1.1 P^n-extendability

To provide a certificate for non-P^n-extendability of an incomplete game, we employ duality of linear systems. This approach was motivated by the aforementioned work by Seshadhri and Vondrák [24] and the so called *path certificate* for non-extendability of submodular functions. Although its size is exponential in the number of players in general, for special cases, the solvability of the dual system is polynomial in n and therefore, the P^n-extendability is polynomially decidable in n for such cases. In the proof of the characterisation, we use the seminal result of Farkas [12].

Lemma 5.1. (Farkas' lemma, [12]) *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$. Then exactly one of the following two statements is true.*

1. *There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.*
2. *There exists $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y \leq -1$.*

Theorem 5.2. *Let $(N, \mathcal{K}, v)$ be an incomplete game. The game is P^n-extendable if and only if the following system of linear equations is not solvable:*

1. *$\forall T \subseteq N, T \neq \emptyset : \sum_{S \in \mathcal{K}, T \subseteq S} y(S) \geq 0,$*
2. *$\sum_{S \in \mathcal{K}} v(S)y(S) \leq -1.$*

Proof. Let $M \coloneqq 2^n - 1$ and $U' \in \mathbb{R}^{M \times M}$ be a matrix with characteristic vectors of unanimity games u_T as its columns. Then it holds that $U'd = w$ for every game (N, w) and its vector of Harsanyi dividends d.

For a partially defined cooperative game $(N, \mathcal{K}, v)$, we reduce matrix U' by deleting the rows corresponding to coalitions with non-defined utility, reaching a system $Ud = v$. This adjustment eliminates unknowns on the right hand side of the equation, yet no information about the complete game is lost since the vector of Harsanyi dividends carries full information.

The game $(N, \mathcal{K}, v)$ is P^n-extendable if and only if the system of linear equations above is solvable for $d \geq 0$. By Farkas' lemma (Lemma 5.1) this happens if and only if the following system has no solution,

$$U^T y \geq 0 \text{ and } v^T y \leq -1. \tag{5.1}$$

The conditions given by (5.1) correspond to those from the statement of the theorem. □

Notice that even though the number of inequalities $\sum_{S\in K, T\subseteq S} y(S) \geq 0$ is $2^n - 1$ (since we have one inequality for every $\emptyset \neq T \subseteq N$), the actual number of *distinct* inequalities is not larger than $2^{|\mathcal{K}|} - 1$ because each inequality sums over a subset of $\mathcal{K}$. Depending on the structure of $\mathcal{K}$, the actual number might be even smaller as is shown in the following result.

Theorem 5.3. *Let $(N, \mathcal{K}, v)$ be an incomplete game such that sizes of all $S \in \mathcal{K}$ are bounded by a fixed constant c. Then the problem of P^n-extendability is polynomially-time solvable in n.*

Proof. Let $(N, \mathcal{K}, v)$ be the incomplete game from the claim. The number of coalitions with a defined value is at most $\sum_{i=1}^{c} \binom{n}{i}$, which is a polynomial in n. Also, if we consider the linear system from Theorem 5.2, every $T \subseteq N$ such that $|T| > c$ yields an empty sum in its corresponding inequality. Therefore, the number of unique conditions in the problem is bounded by the number of coalitions with defined value, that is by the sum $\sum_{i=1}^{c} \binom{n}{i}$. We conclude that the linear system can be solved in polynomial time by means of linear programming. □

5.1.2 Boundedness

In this subsection, we state a simple condition for the boundedness of the set of P^n-extensions. Notice, the set of P^n-extensions is always bounded from below, as for every P^n-extension (N, w), $w(S) = \sum_{\emptyset \neq T \subseteq S} d_w(T)$ and $d_w(T) \geq 0$ for every $\emptyset \neq T \subseteq N$. Therefore, 0 serves as a lower bound on the profit of any coalition $S \subseteq N$. To find the lower bound that is binding the profit of every coalition as well as the binding upper bound (hence the lower and the upper games) remains an open problem.

Theorem 5.4. *Let $(N, \mathcal{K}, v)$ be a P^n-extendable incomplete game. The set of positive extensions $P^n(v)$ is bounded if and only if $N \in \mathcal{K}$.*

Proof. If $N \in \mathcal{K}$, then for any $P^n(v)$-extension (N, w), $\sum_{T\subseteq N} d_w(T) = v(N)$, and since for all $\emptyset \neq T \subseteq N$, $d_w(T) \geq 0$, we can deduce that $d_w(T) \in [0, v(N)]$. This yields a bound (possibly an overestimation) for all possible values of $d_w(T)$. Since the dividends are bounded, the set $P^n(v)$ is also bounded.

If $N \notin \mathcal{K}$, then the value of coalition N can be arbitrarily large, since there is no upper bound on $d_w(N)$ for a $P^n(v)$-extension (N, w). Thus, $P^n(v)$ is not bounded. □

5.1.3 Extreme games

Following the proof of the sharp form of Bondareva-Shapley theorem from [21], we give an insight into the description of extreme games of $P^n(v)$. We show that for these games, the set of coalitions with values of dividends equal to zero is maximal with respect to the inclusion.

We know that the set of P^n-extensions of $(N, \mathcal{K}, v)$ can be described as

$$P^n(v) = \left\{(N,w) \middle| \forall S \in \mathcal{K} : w(S) = v(S) \text{ and } \forall T \subseteq N : d_w(T) \geq 0\right\},$$

or equivalently in terms of dividends and $M := 2^n - 1$ we can write

$$P^n_d(v) := \left\{d_w \in \mathbb{R}^M \middle| \forall S \in \mathcal{K} : \sum_{T \subseteq S} d_w(T) = v(S), \forall T \subseteq N : d_w(T) \geq 0\right\}.$$

Notice that $P^n(v) \neq P^n_d(v)$ since the former is a set of cooperative games and the latter is a set of vectors of dividends.

We see that both sets are closed convex polytopes since they are formed by intersections of closed half-spaces. If we suppose that $(N, \mathcal{K}, v)$ is P^n-extendable, then both sets are nonempty. Furthermore, the sets are bounded if and only if $N \in \mathcal{K}$. Bounded, closed and convex polytopes are convex hulls of their extreme points.

To be able to freely neglect the distinction between the extreme points of both sets, we introduce a basic result from linear algebra. An image of a convex set under a linear mapping is again a convex set. Moreover, the extreme points of the preimage set correspond to those of the image set.

Lemma 5.5. *Let P be a convex subset of $\mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ a nonsingular matrix, and $x \in P$ an extreme point of P. Then Ax is an extreme point of the convex set $A(P) := \{Au | u \in P\}$.*

Proof. Suppose that $x \in P$ is an extreme point of P and the image Ax is not an extreme point of $A(P)$. Therefore, there are $Au, Av \in A(P)$ and $\alpha \in (0, 1)$ such that $\alpha Au + (1 - \alpha)Av = Ax$. But then $\alpha Au + (1 - \alpha)Av = A(\alpha u + (1 - \alpha)v) = Ax$, and therefore, x is not an extreme point of P, as it is a nontrivial convex combination of $u, v \in P$. This is a contradiction. □

Let $U \in \mathbb{R}^{M \times M}$ be a matrix with vectors of unanimity games $u_T \in \mathbb{R}^M$ as columns. It holds that $Ud_w = w$ where $w \in \mathbb{R}^M$ is a characteristic vector of game (N, w) and $d_w \in \mathbb{R}^M$ represents a vector of Harsanyi dividends of the game. Since unanimity games form a basis of $\mathbb{R}^M$, the matrix U is nonsingular and thus, by Lemma 5.5, the extreme points of $P^n(v)$ correspond to those of $P^n_d(v)$, allowing us to further consider those instead of the former ones.

Our result is based on the following lemma stating a characterisation of extreme points of a convex polyhedral set.

Lemma 5.6. *[21] Let P be a convex polyhedral set in $\mathbb{R}^k$ given by*

$$P := \left\{x \in \mathbb{R}^k \middle| \sum_{j=1}^{k} a_{tj} x_j \geq b_t, t = 1, \dots, m\right\}.$$

For $x \in P$, let $S(x) \coloneqq \Big\{t \in \{1, \dots, m\} \,|\, \sum_{j=1}^{k} a_{tj} x_j = b_t\Big\}$. The point $x \in P$ is an extreme point of P if and only if the system of linear equations

$$\sum_{j=1}^{k} a_{ij} y_j = b_t \text{ for all } t \in S(x)$$

has x as its unique solution.

When we apply Lemma 5.6 to our situation, d_e is an extreme game of $P_d^n(v)$ if and only if there is no $d_w \neq d_e$ such that $d_w(T) = 0 \iff d_e(T) = 0$. For any $P^n(v)$-extension (N, w), we denote by $E(w)$ the *set of negligible coalitions* defined as $E(w) \coloneqq \{T \subseteq N \mid d_w(T) = 0\}$. This set proves itself useful in the following lemma. The lemma states that inclusion-maximality of $E(e)$ across $E(x)$ for $d_x \in P_d^n(v)$ is equivalent with uniqueness of $E(e)$ across $E(x)$ for $d_x \in P_d^n(v)$. Together with Lemma 5.6, this connects the extremality of games with the inclusion-maximality of sets $E(e)$.

Lemma 5.7. *Let $(N, \mathcal{K}, v)$ be a P^n-extendable incomplete game and $d_e \in P_d^n(v)$. Then the following are equivalent:*

1. *there is no $d_x \in P_d^n(v)$ such that $E(e) \subsetneq E(x)$,*
2. *there is no $d_y \in P_d^n(v)$ different from d_e, such that $E(e) = E(y)$.*

Proof. First, suppose that there is $d_x \in P_d^n(v)$ such that $E(e) \subsetneq E(x)$. We show that there is not only one, but infinitely many vectors $d_{y^\alpha} \in P_d(v)$ different from d_e such that $E(e) = E(y^\alpha)$. The idea is to take any non-trivial convex combination $d_{y^\alpha} \coloneqq \alpha d_e + (1 - \alpha) d_x$ for $0 < \alpha < 1$. Such game is clearly positive (a convex combination of non-negative dividends remains non-negative) as it is also an extension of $(N, \mathcal{K}, v)$, because for every $S \in \mathcal{K}$,

$$\sum_{T \subseteq S} d_{y^\alpha}(T) = \alpha \sum_{T \subseteq S} d_e(T) + (1 - \alpha) \sum_{T \subseteq S} d_x(T) = \alpha v(S) + (1 - \alpha) v(S) = v(S).$$

And since $d_x \neq d_e$, there is $S \notin \mathcal{K}$ such that $x(S) \neq e(S)$ for which

$$y^\alpha(S) = \sum_{T \subseteq S} d_{y^\alpha}(T) = \alpha \sum_{T \subseteq S} d_x(T) + (1 - \alpha) \sum_{T \subseteq S} d_e(T) = \alpha x(S) + (1 - \alpha) e(S).$$

Therefore, any two parameters α_1, α_2 such that $0 < \alpha_1 < \alpha_2 < 1$ yield different values $y^{\alpha_1}(S) \neq y^{\alpha_2}(S)$, thus $d_{y^{\alpha_1}} \neq d_{y^{\alpha_2}}$.

Now suppose that there is $d_y \in P_d^n(v)$ different from d_e such that $E(e) = E(y)$. We take a combination $d_z = d_e - \beta(d_y - d_e)$ with β such that for at least one $S \notin E(e)$, $d_z(S) = 0$. Thus $E(e) \subseteq E(z)$ and still, $d_z \in P_d^n(v)$. For such S, it must hold

$$d_z(S) = d_e(S) - \beta(d_y(S) - d_e(S)) = 0,$$

therefore $\beta = \frac{d_e(S)}{d_y(S) - d_e(S)}$. We have to choose S such that $d_y(S) \neq d_e(S)$. Furthermore, we have to secure that for every $T \notin E(e)$, $d_z(T) \geq 0$, or equivalently

$$d_z(T) = d_e(T) - \beta(d_y(T) - d_e(T)) = d_e(T) - \frac{d_e(S)}{d_y(S) - d_e(S)}(d_y(T) - d_e(T)) \geq 0.$$

This can be done by taking minimum for S over all such coalitions T, i.e.

$$\beta := \min_{T \notin E(e): d_e(T) \neq d_y(T)} \frac{d_e(T)}{d_y(T) - d_e(T)}.$$

Then for $T \notin E(e)$, $d_z(T) \geq 0$, since it is equal to

$$d_e(T) - \frac{d_e(S)}{d_y(S) - d_e(S)}(d_y(T) - d_e(T)) \geq d_e(T) - \frac{d_e(T)}{d_y(T) - d_e(T)}(d_y(T) - d_e(T)).$$

Clearly, the last expression is equal to zero. Finally, for $K \in \mathcal{K}$,

$$z(K) = \sum_{C \subseteq K} d_z(K) = \sum_{C \subseteq K} d_e(K) - \beta \left(\sum_{C \subseteq K} d_y(K) - \sum_{C \subseteq K} d_e(K) \right),$$

and since all three sums in the last expression are equal to $v(K)$, we conclude $z(K) = v(K)$, thus $d_z \in P^n_d(v)$. $\square$

From a direct application of Lemma 5.6 and 5.7 follows a characterisation of the extreme points.

Theorem 5.8. *For a P^n-extendable incomplete game $(N, \mathcal{K}, v)$, a P^n-extension (N, e) is an extreme game of $P^n(v)$ if and only if its set of negligible coalitions $E(e)$ is inclusion-maximal, i.e. there is no $(N, w) \in P^n(v)$ such that $E(e) \subsetneq E(w)$.*

5.2 Description of $P^n(v)$ for special cases

This section contains an analysis of P^n-extensions of several classes of incomplete games. In subsection 5.2.1, we employ the characterisation of extreme games from Theorem 5.8 in an analysis of three classes. Subsection 5.2.2 is focused on the class of incomplete games with minimal information and results from [19] connected to the class are presented.

5.2.1 Classes employing the characterisation of extreme games

In this subsection, we show a direct application of Theorem 5.8 in the description of the set of P^n-extensions for two classes of incomplete games. We do not show only a derivation of extreme games but also a derivation of the lower and the upper games as well as the P^n-extendability. For both cases, $P^n(v)$ is bounded,i.e. $N \in \mathcal{K}$. From the second case, we derive a result for P^n_σ-extensions of symmetric incomplete games.

Pairwise disjoint coalitions of known values

For the first class of incomplete games it holds that the coalitions with known values (excluding N) are pairwise-disjoint.

Theorem 5.9. *Let $(N, \mathcal{K}, v)$ be a P^n-extendable incomplete game, where $\mathcal{K} = \{S_1, \ldots, S_{k-1}, N\}$ and for all $i, j \in \{1, \ldots, k-1\}$, it holds that $S_i \cap S_j = \emptyset$. Then the extreme games $v^{\mathcal{T}}$, the lower game $\underline{v}$, and the upper game $\overline{v}$ can be described as follows:*

$$v^{\mathcal{T}}(S) \coloneqq \begin{cases} 0, & \text{if } \nexists T \in \mathcal{K} : T \subseteq S, \\ \sum_{i:T_i \subseteq S} v(S_i), & \text{if } \exists T \in \mathcal{K} : T \subseteq S \text{ and } T_N \nsubseteq S \\ v(N) - \sum_{i:T_i \nsubseteq S} v(S_i), & \text{if } \exists T \in \mathcal{K} : T \subseteq S \text{ and } T_N \subseteq S, \end{cases}$$

$$\underline{v}(S) \coloneqq v^{\mathcal{K}}(S) = \begin{cases} 0, & \text{if } \nexists T \in \mathcal{K} : T \subseteq S, \\ \sum_{i:S_i \subseteq S} v(S_i), & \text{if } \exists T \in \mathcal{K} : T \subseteq S \text{ and } N \neq S, \\ v(N), & \text{if } \exists T \in \mathcal{K} : T \subseteq S \text{ and } N = S, \end{cases}$$

$$\overline{v}(S) \coloneqq \begin{cases} v(S_i), & \text{if } S \subseteq S_i, \\ v(N) - \sum_{i:S_i \nsubseteq S} v(S_i), & \text{otherwise,} \end{cases}$$

where $\mathcal{T} \coloneqq \{T_1, \ldots, T_{k-1}, T_N\}$ such that $T_i \subseteq S_i$, $T_N \subseteq N$ and $T_N \nsubseteq S_\ell$ for any $\ell \in \{1, \ldots, k-1\}$.
Furthermore, the P^n-extendability of $(N, \mathcal{K}, v)$ is characterised by a condition

$$v(N) \geq \sum_{i=1}^{k-1} v(S_i).$$

Proof. Let $(N, \mathcal{K}, v)$ be an incomplete game with the properties above. For any $P^n(v)$-extension (N, w), from the fact that the coalitions in $\mathcal{K} \setminus \{N\}$ are disjoint, at least one subcoalition T_i of each coalition $S_i \in \mathcal{K} \setminus \{N\}$ must have a nonzero dividend $d_w(T_i)$, otherwise $v(S_i) = 0$. By Theorem 5.8, there is at most one such subcoalition if we consider an extreme game. If there were two nonzero dividends $d_w(T_i^1)$, $d_w(T_i^2)$ for one S_i, then the corresponding set of negligible coalitions would not be maximal. Setting the dividend of T_i^1 to $d_w(T_i^1) + d_w(T_i^2)$ yields a set E, such that $E(w) \subsetneq E$. By this, for the extreme game, it holds $d_w(T_i) = v(S_i)$. We further see, since $v(N) = \sum_{T \subseteq N} d_w(T)$, that $v(N) \geq \sum_{S_i \in \mathcal{K} \setminus \{N\}} v(S_i)$ holds. If the inequality does not hold, then that there is no extreme game of $P^n(v)$ and hence, since the set is bounded ($N \in \mathcal{K}$), it is not P^n-extendable. Now, if the inequality is strict, there has to be another nonzero dividend of a coalition $T_N \subseteq N$ such that $T_N \nsubseteq S_i$ for $S_i \in \mathcal{K} \setminus \{N\}$, otherwise T_i, T_N are two distinct subsets of S_i and $E(w)$ is not maximal. Again, since we are interested in extreme games, by Theorem 5.8, there is only one such coalition T_N, resulting in $d_w(T_N) = v(N) - \sum_{S_i \in \mathcal{K} \setminus \{N\}} v(S_i)$. Any game parameterised by a collection $\mathcal{T} \coloneqq \{T_1, \ldots, T_{k-1}, T_N\}$ and expressed as $v^{\mathcal{T}}$ from the statement of the theorem is thus an extreme game of $P^n(v)$.

Now let us show that the game $v^{\mathcal{K}}$ is the lower game. For a coalition S with no subcoalition contained in $\mathcal{K}$, $v^{\mathcal{K}}(S) = 0 = \underline{v}(S)$. For a coalition S such that there is $T \in K, T \subseteq S$, the value of $w(S)$ of any $P^n(v)$-extension cannot be smaller than the sum $\sum_{T:T \in \mathcal{K}, T \subseteq S} v(T) = v^{\mathcal{K}}(S)$. And since $N \in \mathcal{K}$, $v^{\mathcal{T}}(N) = v(N) = \underline{v}(N)$.

Finally, we show that each value of the upper game is achieved by a different extreme game. If S is a proper subcoalition of S_i, the value $v(S_i)$ is, thanks to the non-negativity of dividends, an upper bound for the value of S. For any extreme game $v^{\mathcal{T}}$ such that $S \in \mathcal{T}$, this bound is tight. If S is not a subcoalition of any S_i, its value cannot exceed $v(N) - \sum_{T_i \in \mathcal{T} \setminus T_N} d_w(T_i)$, otherwise the characterisation of P^n-extendability is not satisfied for the grand coalition N. By taking an extreme game with $T_N = S$, we see that this bound is tight. $\square$

Set of known values $\mathcal{K}$ closed on subsets

The second class of incomplete games satisfies that the set $\mathcal{K} \setminus \{N\}$ is closed on subsets, i.e. $S \in K, T \subseteq S \implies T \in \mathcal{K}$. The analysis of this case will help us in the study of symmetric positive extensions (P^n_σ-extensions).

Theorem 5.10. *Let $(N, \mathcal{K}, v)$ be a P^n-extendable incomplete game such that $N \in \mathcal{K}$ and for every $S \in \mathcal{K} \setminus \{N\}, T \subseteq S \implies T \in \mathcal{K}$. Furthermore, for $S \in \mathcal{K}$, let δ_S be defined as $\delta_{\{i\}} = v(\{i\})$ and $\delta_S = v(S) - \sum_{T \subsetneq S} \delta_T$. Then the extreme games v^C, the lower game $\underline{v}$, and the upper game $\overline{v}$ can be described as follows:*

$$v^C(S) \coloneqq \begin{cases} \delta_N + \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T, & \text{if } C \subseteq S, \\ \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T, & \text{otherwise,} \end{cases}$$

for $C \notin \mathcal{K} \setminus \{N\}$, and

$$\underline{v}(S) \coloneqq v^N(S) = \begin{cases} \delta_N + \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T, & \text{if } S = N, \\ \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T, & \text{otherwise,} \end{cases}$$

$$\overline{v}(S) \coloneqq \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v^S(S), & \text{otherwise.} \end{cases}$$

Furthermore, $(N, \mathcal{K}, v)$ is P^n-extendable if and only if $\delta_S \geq 0$ for all $S \in \mathcal{K}$.

Proof. Let $(N, w) \in P^n(v)$. Thanks to the structure of $\mathcal{K}$, the dividends $d_w(S)$ for $S \in \mathcal{K} \setminus \{N\}$ are the same for any $(N, w) P^n(v)$and they are equal to δ_S. As a consequence, for any S such that $\delta_S = 0$ it holds $S \in E(w)$ and this holds for any $P^n(v)$-extension. Now if the uniquely defined value $\delta_N = v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S > 0$, there has to be at least one $C \notin \mathcal{K} \setminus \{N\}$ such that its dividend $d_w(C) \neq 0$. By Theorem 5.8, following a similar argument as in the proof of the previous theorem, $E(w)$ is maximal if and only if there is only one such C, otherwise if there are $C_1 \neq C_2$ such that $d_w(C_1) \neq 0$ and $d_w(C_2) \neq 0$, by taking $(N, x) \in P^n(v)$ such that $d_x(C_1) = 0$, $d_x(C_2) = d_w(C_1) + d_w(C_2)$ we arrive into contradiction with maximality, since $E(w) \subsetneq E(x)$. Thus choosing $(N, w) \in P^n(v)$, such that $d_w(C) = \delta_n$ yields an extreme game v^C of $P^n(v)$ for any $C \notin \mathcal{K} \setminus \{N\}$.

For any coalition S, its value in any $P^n(v)$-extension has to be larger or equal to $\sum_{T \in \mathcal{K}, T \subseteq S} \delta_S$. Notice that $v^N(S)$ is equal to this number for any S, thus being the lower game.

For any coalition S, its maximal value is either $v(S)$ if $S \in \mathcal{K}$, or at most $v(N) - \sum_{T \in \mathcal{K} \setminus \{N\}: T \not\subseteq S} \delta_T = \delta_N + \sum_{T \in \mathcal{K}, T \subseteq S} \delta_T$, which is equal to $v^S(S)$ and thus it is the upper game. $\square$

For both studied classes of incomplete games, it holds $\underline{v}(S) \in P^n(v)$. Also notice that the number of extreme games $v^{\mathcal{C}}$ equals the number of coalitions C such that $C \notin K \setminus \{N\}$, that is $2^n - |K| + 1$ if $v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S > 0$, otherwise $P^n(v)$ contains precisely one game (in case $v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S = 0$) or no game at all (if $v(N) - \sum_{S \in \mathcal{K} \setminus \{N\}} \delta_S < 0$).

Symmetric positive extensions

We denote the set of symmetric positive extensions of $(N, \mathcal{K}, v)$ by $P^n_\sigma(v)$. Analogously to study of $C^n_\sigma(v)$, we shall make use of the *reduced forms* (N, s) and $(N, \mathcal{X}, \sigma)$ of games (N, v) and $(N, \mathcal{K}, v)$, respectively, which are defined in Definition 4.1. We can easily obtain the following result as a corollary of Theorem 5.10.

Theorem 5.11. *Let $(N, \mathcal{X}, s)$ be the reduced form of a symmetric incomplete game such that $n \in \mathcal{X}$ and $i \in N, i \leq k \implies i \in \mathcal{X}$. Then the lower game and the upper game of $P^n_\sigma(v)$ can be described as*

$$\underline{s}(i) \coloneqq \begin{cases} s(i), & \textit{for } i \in \mathcal{X}, \\ s(k), & \textit{otherwise,} \end{cases} \quad \textit{and} \quad \overline{s}(i) \coloneqq \begin{cases} s(i), & \textit{for } i \in \mathcal{K}, \\ s(n), & \textit{otherwise.} \end{cases}$$

The following game illustrates that even in the symmetric scenario, there is $(N, \mathcal{X}, \sigma)$ such that $(N, \underline{s}) \notin P^n_\sigma(v)$.

Example. (The lower game is not necessarily a P^4_σ-extension) Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a symmetric 4-person incomplete game such that $\mathcal{X} = \{2, 4\}$. From the properties of symmetric positive games we know that any $(N, s) \in P^4_\sigma(v)$ is given by 4 non-negative dividends with corresponding values d_1, d_2, d_3, d_4 such that

- $s(1) = d_1$,
- $s(2) = 2d_1 + d_2$,
- $s(3) = d_3 + 3d_2 + 3d_1$,
- $s(4) = d_4 + 4d_3 + 6d_2 + 4d_1$.

By setting $d_1 \coloneqq 0, d_2 \coloneqq \sigma(2), d_3 \coloneqq 0$, and $d_4 \coloneqq \sigma(4) - 6d_2$ we get a P^4_σ-extension where $s(1) = 0$ (clearly the minimum) and it is achieved if and only if $d_1 = 0$. Setting $d_1 = 0$ yields $s(3) = 3\sigma(2)$. However, to minimize $s(3)$, we can choose $d_1 \coloneqq \frac{\sigma(2)}{2}$, $d_2 \coloneqq 0$, $d_3 \coloneqq 0$, and $d_4 \coloneqq \sigma(4) - 4d_1$, obtaining $s(3) = 3d_1 = \frac{3}{2}\sigma(2)$. We cannot minimize both values simultaneously and thus $(N, \underline{s}) \notin P^n_\sigma(v)$.

It is not difficult to generalise this example for symmetric n-person games. For similar reasons, even the lower game of (non-symmetric) P^n-extensions of non-symmetric incomplete games is not contained in $P^n(v)$. This is contrary to what we showed for the classes of incomplete games in Theorem 5.9 and 5.10.

5.2.2 Incomplete games with minimal information

Most of the results concerning P^n-extensions for incomplete games with minimal information are from Masuya and Inuiguchi [19]. In their work, they considered non-negative incomplete games with minimal information. However, since $v(i) = d_v(i) \geq 0$ for every $(N, v) \in P^n$, these classes coincide. A characterisation of P^n-extendability, employing the total excess $\Delta \coloneqq v(N) - \sum_{i \in N} v(i)$, is equivalent to the characterisation of S^n-extendability and C^n-extendability. Its proof follows immediately from the proof of Theorem 3.6.

Theorem 5.12. *Let $(N, \mathcal{K}, v)$ be a non-negative incomplete game with minimal information. It is P^n-extendable if and only if $\Delta \geq 0$.*

In [19], they showed the lower and the upper games coincide with the lower and the upper games of S^n-extensions, i.e.

$$\underline{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ \sum_{i \in S} v(i), & \text{if } S \notin \mathcal{K}, \end{cases} \text{ and } \overline{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N), & \text{if } S \notin \mathcal{K}. \end{cases}$$

Further, they showed that the lower game $(N, \underline{v})$ is positive and the upper game is not, however, it is monotonic. Finally, we state a description of the set of P^n-extensions. We employ $N_1 \coloneqq \{T \subseteq N \mid |T| > 1\}$.

Theorem 5.13. *[19] Let $(N, \mathcal{K}, v)$ be a non-negative incomplete game with minimal information and let (N, v^T) for $T \in N_1$ be games from (3.4). The set of P^n-extension can be expressed as*

$$P^n(v) = \left\{ \sum_{T \in N_1} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, \alpha_T \geq 0 \right\}. \tag{5.2}$$

6. 1-convexity

In this chapter, we focus on C_1^n-extensions of incomplete games. The difficulty in the analysis of the set of C_1^n-extensions lies in the values of almost-grand coalitions $N \setminus i$ for $i \in N$. In Section 6.1, we investigate the set of symmetric C_1^n-extensions to build up our intuition.

After that, in Section 6.2, we restrict ourselves to incomplete games with minimal information. We derive a compact description of the set of C_1^n-extensions (Subsection 6.2.1), and in Subsection 6.2.2, we investigate generalisations of three solution concepts for complete games, namely the τ-value, the Shapley value and the nucleolus. We consider two variants and we show that all the variants coincide. We call the solution concept the average value $\tilde{\zeta}$. The analysis of $\tilde{\zeta}$ (Subsection 6.2.3) concludes the section.

In Section 6.3, we derive similar results for incomplete games with defined upper vector. Subsection 6.3.1 contains a description of C_1^n-extensions and in Subsection 6.3.2, we show the average Shapley value does not in general coincide with the conic Shapley value.

6.1 Symmetric incomplete games

We denote the set of symmetric 1-convex extensions of $(N, \mathcal{K}, v)$ by $C_{1,\sigma}^n(v)$. Analogously to study of $C_\sigma^n(v)$ and $P_\sigma^n(v)$, we shall make use of the *reduced forms* (N, s) and $(N, \mathcal{X}, \sigma)$ of games (N, v) and $(N, \mathcal{K}, v)$, respectively, which are defined in Definition 4.1.

Geometrical interpretation of 1-convex symmetric games

Let (N, s) be the reduced form of a 1-convex symmetric game. In terms of the reduced game, 1-convexity is given by

$$ns(n-1) \leq (n-1)s(n) \tag{6.1}$$

and for $i \in \{1, n-2\}$,

$$s(i) + (n-i-1)s(n) \leq (n-i)s(n-1). \tag{6.2}$$

We reformulate the condition (6.1) as $s(n) \geq \frac{n}{n-1}s(n-1)$ and distinguish two cases depending on whether the condition is binding or holds with a strict inequality.

First, suppose that $s(n) = \frac{n}{n-1}s(n-1)$. Substituting $s(n)$ into the conditions (6.2) for $i \in \{1, \ldots, n-2\}$ yields

$$s(i) + (n-i-1)\frac{n}{n-1}s(n-1) \leq (n-i)s(n-1). \tag{6.3}$$

Rearranging and simplifying (6.3) leads to

$$s(i) \leq \frac{i}{n-1}s(n-1). \tag{6.4}$$

Equivalently, the points $(i, s(i))$ for $i \in \{1, \ldots, n-2\}$ lie on or below the line intersecting the points $(0, 0)$ and $(n-1, s(n-1))$ (see Figure 6.1).

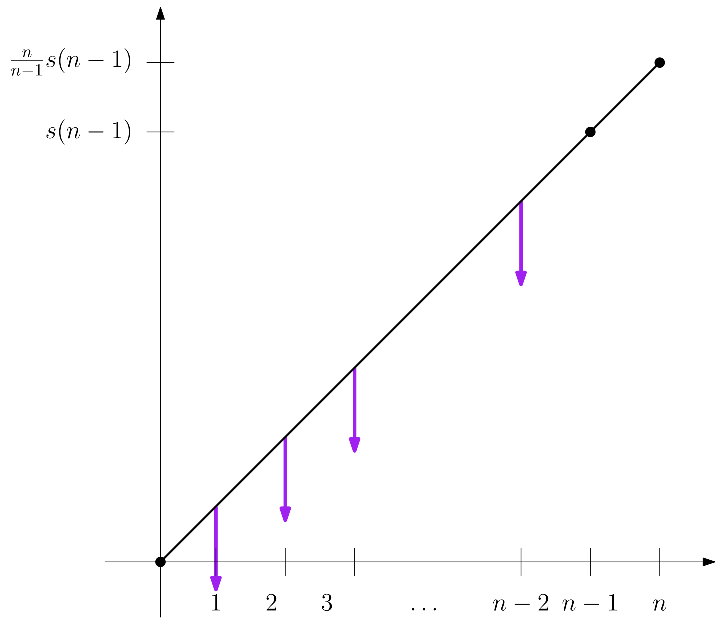

Figure 6.1: The situation in which $s(n) = \frac{n}{n-1}s(n-1)$. The restriction here is that the values of 1 to $n-2$ lie below the depicted line.

Second, suppose that $s(n) = \frac{n}{n-1}s(n-1) + c$ for $c > 0$. It means $(n, s(n))$ lies on the line coming through $(0, c)$ and $(n-1, s(n-1)+c)$. Inequality (6.2) for $s(i)$ differs from (6.4) (where $c = 0$) as

$$s(i) \leq \frac{i}{n-1}s(n-1) - c(n-i-1) = \frac{i}{n-1}s(n-1) - c(n-1) + ci. \qquad (6.5)$$

In (6.5), the values of $s(i)$ are still bounded from above. This time, there is a different bounding line for every value. All the lines have the same slope as the line coming through $(0, 0)$ and $(n-1, s(n-1))$. Furthermore, smaller the i is, further the line is moved along the vertical axis. The vertical distance between all consecutive parallel lines is c (see Figure 6.2).

We base the results concerning $C^n_{1,\sigma}$-extendability and the description of the set $C^n_{1,\sigma}$ on this geometrical interpretation.

Analysis of the set of $C^n_{1,\sigma}$-extensions

In our analysis, we restricted ourselves to the case where $n \in \mathcal{X}$. Still, we have to distinguish between two cases depending on whether $n-1 \in \mathcal{X}$ or $n-1 \notin \mathcal{X}$.

Theorem 6.1. *Let $(N, \mathcal{X}, \sigma)$ be the reduced form of a symmetric incomplete game such that $\{n-1, n\} \subseteq \mathcal{X}$. The game is $C^n_{1,\sigma}$-extendable if and only if*

$$\frac{\sigma(n-1)}{n-1} \leq \frac{\sigma(n)}{n} \qquad (6.6)$$

and for every $i \in \mathcal{X} \setminus \{0\}$,

$$\sigma(i) \leq \sigma(n) - (n-i)\left(\sigma(n) - \sigma(n-1)\right). \qquad (6.7)$$

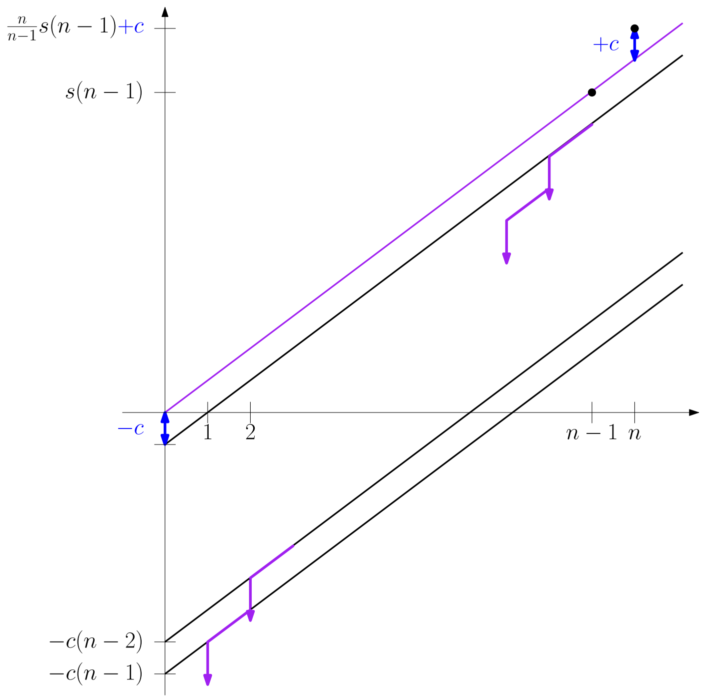

Figure 6.2: The situation in which $s(n) = \frac{n}{n-1}s(n-1) + c$ for $c > 0$.

Proof. Any $C^n_{1,\sigma}$-extension (N, e) in the reduced form must satisfy conditions (6.1) and (6.2), namely $ne(n-1) \leq (n-1)e(n)$ and for every $i \in \{1, \ldots, n-2\}$, condition $e(i) + (n-i-1)e(n) \leq (n-i)e(n-1)$. The first condition can be rewritten as $\frac{e(n-1)}{n-1} \leq \frac{e(n)}{n}$, which is, since $\{n-1, n\} \subseteq X$, equivalent to

$$\frac{\sigma(n-1)}{n-1} \leq \frac{\sigma(n)}{n}. \tag{6.8}$$

Hence, the condition (6.6) must hold. From the second-type conditions, we can subtract $(n-i-1)e(n)$, arriving at

$$e(i) \leq e(n) - (n-i)\left(e(n) - e(n-1)\right). \tag{6.9}$$

For $i \in \mathcal{X}$, it follows from (6.9) that conditions (6.7) must hold.

Now if conditions (6.8) and (6.9) holds, we define a $C^n_{1,\sigma}$-extension $(N, \overline{s}_1)$ as

$$\overline{s}_1(i) := \begin{cases} \sigma(n) - (n-i)\left(\sigma(n) - \sigma(n-1)\right), & \text{if } i \notin \mathcal{X}, \\ \sigma(i), & \text{if } i \in \mathcal{X}. \end{cases}$$

The values of $\overline{s}_1$ are chosen such that conditions (6.8) and (6.9) hold therefore $(N, \overline{s}_1) \in C^n_{1,\sigma}$. □

Notice that the reduced game $(N, \overline{s}_1)$ is actually the upper game of $C^n_{1,\sigma}$. This is because the inequalities in conditions (6.8) and (6.9) are binding, meaning any reduced game (N, x) for which there is $i \notin \mathcal{X}$ such that $x(i) > \overline{s}_1(i)$ violates inequality from condition (6.9) corresponding to i. Therefore, $(N, \overline{s}_1 + x)$ is not 1-convex. Also, whenever the upper game of a set of $\mathcal{C}$-extensions is part of the set, it is also its extreme game. This follows immediately from Theorem 1.2, because no reduced game $(N, \overline{s}_1 + x)$ where $x(S) > 0$ for any coalition $S \subseteq N$ can be an extension.

Theorem 6.2. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that it holds $\{n-1, n\} \subseteq \mathcal{X}$, the game $(N, \overline{s}_1)$ is both the upper game and the only extreme game of $C^n_{1,\sigma}$-extensions.*

Proof. For a contradiction, let (N, e) be a reduced extreme game of $C^n_{1,\sigma}$ different from $(N, \overline{s}_1)$. It means there is (N, x) such that $x := \overline{s}_1 - e \neq 0$. Furthermore, it holds that $e \pm x = \pm\overline{s}_1$. $(N, \overline{s}_1)$ is clearly a $C^n_{1,\sigma}$-extension and since all inequalities (6.1) and (6.2) are binding for $\overline{s}_1$, also $(N, -\overline{s}_1) \in C^n_{1,\sigma}$. By Theorem 1.2, (N, e) is not an extreme game. □

Theorem 6.3. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that it holds $\{n-1, n\} \subseteq \mathcal{X}$, the set $C^n_{1,\sigma}$ can be described as*

$$C^n_{1,\sigma} = \left\{ \overline{s}_1 + \sum_{i \notin \mathcal{X}} \alpha_i e_i \mid \textit{for } \alpha_i \geq 0 \right\},$$

where (N, e_i) is defined as

$$e_i(j) := \begin{cases} -1, & \textit{if } j = i, \\ 0, & \textit{if } j \neq i. \end{cases}$$

Proof. Any game $(N, \overline{s}_1 + \alpha e_i)$ for any $\alpha > 0$ is $C^n_{1,\sigma}$-extension. This is because for $i \notin \mathcal{X}$, the 1-convexity condition (6.2) involving i holds as

$$\overline{s}_1(i) - \alpha + (n-i-1)\overline{s}_1(n) < \overline{s}_1(i) + (n-i-1)\overline{s}_1(n) = (n-i)\overline{s}_1(n-1)$$

The rest of the 1-convexity conditions (6.1) and (6.2) of $(N, \overline{s}_1 + \alpha e_i)$ is the same as for $(N, \overline{s}_1)$, thus they also hold. Since any $(N, \overline{s}_1 + \alpha e_i)$ is 1-convex, (N, e_i) is a part of the recession cone of $C^n_{1,\sigma}$. All but one conditions are binding (for $j \in \mathcal{X}$, we consider condition $(\overline{s}_1 + e_i(j)) = \sigma(j))$. By Theorem 1.4, (N, e_i) is an extreme ray of $C^n_{1,\sigma}$. Clearly, any $C^n_{1,\sigma}$-extension is of the form $\overline{s}_1 - x$ where $x(i) = 0$ if $i \in \mathcal{X}$ and $x(i) \geq 0$ if $i \notin \mathcal{X}$. Notice, $\overline{s}_1 - x = \overline{s}_1 - \sum_{i \notin \mathcal{X}} x(i)e_i$, therefore there are no more extreme rays. By Theorem 1.5, the proof is concluded. □

For the case where $n - 1 \notin \mathcal{X}$, the analysis is similar. The condition for $C^n_{1,\sigma}$-extendability is even simpler. However, when $\mathcal{X} \neq \{n\}$, there might be two extreme games instead of just one. Also, if $\mathcal{X} = \{n\}$, there is another extreme ray of $C^n_{1,\sigma}$.

Theorem 6.4. *Let $(N, \mathcal{X}, \sigma)$ be a reduced form of a symmetric incomplete game such that $n \in \mathcal{X}$ and $n - 1 \notin \mathcal{X}$. The game is $C^n_{1,\sigma}$-extendable if and only if for every $k \in \mathcal{X} \setminus \{0\}$,*

$$\frac{\sigma(k)}{k} \leq \frac{\sigma(n)}{n}. \tag{6.10}$$

Proof. Any reduced game (N, e) which is $C^n_{1,\sigma}$-extension must satisfy conditions (6.1) and (6.2), namely $ne(n-1) \leq (n-1)e(n)$ and for every $i \in \{1, \ldots, n-2\}$, $e(i) + (n-i-1)e(n) \leq (n-i)e(n-1)$. The first condition can be rewritten as

$$e(n-1) \leq (n-1)\frac{e(n)}{n}. \tag{6.11}$$

For the second type conditions, we can use (6.11) to bound $e(n-1)$ from above, arriving at $e(i)+(n-i+1)e(n) \leq (n-i)(n-1)\frac{e(n)}{n}$, which is equivalent to

$$e(i) \leq i\frac{e(n)}{n}. \tag{6.12}$$

We see conditions (6.11) and (6.12) for $i \in \mathcal{X}$ are equivalent with (6.10) and must hold if the game is $C^n_{1,\sigma}$-extendable.

Now if conditions (6.10) hold, we define a $C^n_{1,\sigma}$-extension $(N, \overline{s}_2)$ as

$$\overline{s}_2(i) := \begin{cases} i\frac{\sigma(n)}{n}, & \text{if } i \notin \mathcal{X}, \\ \sigma(i), & \text{if } i \in \mathcal{X}. \end{cases}$$

The values of $\overline{s}_2$ are chosen such that conditions (6.11) and (6.12) hold, therefore $(N, \overline{s}_2) \in C^n_{1,\sigma}$. □

By an analogous argument as for $\overline{s}_1$, the reduced game $(N, \overline{s}_2)$ is the upper game and also an extreme game of the set of $C^n_{1,\sigma}$-extensions. However, if $\mathcal{X} \neq \{n\}$ and there is $i \in \mathcal{X}$ for which the condition is not binding, there are actually two extreme games instead of one.

Theorem 6.5. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that $n-1 \notin \mathcal{X}$ and $\{n\} \subsetneq \mathcal{X}$, the game (N, s^*) defined as*

$$s^*(i) := \begin{cases} \overline{s}_2(i) - \alpha, & \textit{if } i = n-1, \\ \overline{s}_2(i) - (n-i)\alpha, & \textit{if } i \notin \mathcal{X}, i \neq n-1, \\ \sigma(i), & \textit{if } i \in \mathcal{X}, \end{cases}$$

where $\alpha := \min\limits_{i \notin \mathcal{X} \cup \{n-1\}} \frac{1}{n-i}\left(\frac{i}{n}\sigma(n) - \sigma(i)\right)$ is an extreme game of $C^n_{1,\sigma}$.

Proof. For $n-1 \in \mathcal{X}$, we want to find the lowest possible value. If (N, s^*) is a $C^n_{1,\sigma}$-extension, then for every $i \in \mathcal{X}$, it must hold $s^*(i) + (n-i-1)s^*(n) \leq (n-i)s^*(n-1)$. That is equivalent to $\sigma(i) + (n-i-1)\sigma(n) \leq (n-i)s^*(n-1)$, or

$$\frac{\sigma(i) + (n-i-1)\sigma(n)}{n-1} \leq \sigma(n-1).$$

We set $s^*(n-1) := \max\limits_{i \in \mathcal{X} \setminus \{0\}} \frac{\sigma(i)+(n-i-1)\sigma(n)}{n-1}$, which is a lower bound for the profit of $n-1$. Further, for the rest of $i \notin \mathcal{X}$ different from $n-1$, we want the condition $s^*(i) + (n-i-1)s^*(n) \leq (n-i)s^*(n-1)$ to be binding and we know that for the upper game $(N, \overline{s}_2)$, the conditions are binding, i.e. $\overline{s}_2(i) + (n-i-1)\overline{s}_2(n) = (n-i)\overline{s}_2(n-1)$. The difference $\overline{s}_2(n-1) - s^*(n-1)$ is equal to α, therefore, we can rewrite the condition as

$$s^*(i) + (n-i-1)s^*(n) \leq (n-i)\left(\overline{s}_2(n-1) - \alpha\right). \tag{6.13}$$

We know $s^*(n)$ is fixed and the right-hand side of (6.13) differs from the right-hand side of the corresponding condition for $(N, \overline{s}_2)$ by $(n-i)\alpha$. Thus, setting $s^*(i) := \overline{s}_2(i) - (n-i)\alpha$ ensures the condition is binding for (N, s^*). Since all the conditions are binding, (N, s^*) is an extreme game. □

We see the set $C^n_{1,\sigma}(v)$ has two different extreme games if and only if $\alpha \neq 0$. Also, these games are the only extreme games of the set.

Theorem 6.6. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that $n-1 \notin \mathcal{X}$ and $\{n\} \subsetneq \mathcal{X}$, the games $(N, \overline{s}_2)$ and (N, s^*) are the only extreme games.*

Proof. Let $(N, e) \in C^n_{1,\sigma}$ be an extreme game different from $(N, \overline{s}_2)$ and (N, s^*). It means, there is $i \notin \mathcal{X}$ such that $e(i) \neq \overline{s}_2(i)$ and $e(i) \neq s^*(i)$. If $i = n-1$, then neither condition $ne(n-1) \leq (n-1)e(n)$, nor condition

$$\max_{i \in \mathcal{X} \setminus 0} e(i) + (n-i-1)e(n) \leq (n-i)e(n-1)$$

is binding, therefore (N, e) is not an extreme game. If $i \neq n-1$, and $e(n-1) = \overline{s}_2(n-1)$, then $e(i) < (n-i)e(n-1) - (n-i-1)e(n)$, because

$$e(i) < \overline{s}_2(i) = (n-i)\overline{s}_2(n-1) - (n-i-1)\overline{s}_2(n) = (n-i)e(n-1) - (n-i-1)e(n).$$

Similarly for the case where $e(n-1) = s^*(n-1)$. As one of the conditions is not binding, again, the game (N, e) is not an extreme game. □

Theorem 6.7. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that $\mathcal{X} = \{n\}$, the set $C^n_{1,\sigma}$ can be described as*

$$C^n_{1,\sigma} = \left\{ \overline{s}_2 + \sum_{i \notin \mathcal{X}} \alpha_i e_i \mid \alpha_i \geq 0 \right\},$$

where for $i \notin \mathcal{X}$ such that $i \neq n-1$,

$$e_i(j) := \begin{cases} -1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad \text{and} \quad e_{n-1}(j) := \begin{cases} -1, & \text{if } j = n-1, \\ -(n-i), & \text{if } j \neq i. \end{cases}$$

Proof. For $\mathcal{X} = \{n\}$, we can derive for all $i \notin \mathcal{X}$, $i \neq n-1$, the extreme rays (N, e_i) similarly as in Theorem 6.3. For (N, e_{n-1}), the condition $ne_{n-1}(n-1) \leq (n-1)e_{n-1}e(n)$ is not binding. To keep the conditions for the rest of coalition sizes binding, we have to set them such that for $(N, \overline{s}_2 + \alpha e_{n-1})$, it holds

$$(\overline{s}_2 + \alpha e_{n-1})(i) = (n-i)(\overline{s}_2 + \alpha e_{n-1})(n-1) - (n-i-1)(\overline{s}_2 + \alpha e_{n-1})(n).$$

As the right-hand side differs from $(n-i)\overline{s}_2(n-1) - (n-i-1)\overline{s}_2(n)$ in $(n-i)\alpha$, we have to set $e(i)$ such that $(\overline{s}_2 + \alpha e_{n-1})(i)$ differs from $\overline{s}_2(i)$ in this value. In other words, we set it to $e(i) := -(n-i)$. This yields the only possible extreme ray for unbinding condition for the coalition size $n-1$. Therefore, there are no more extreme rays and by Theorem 1.5, the proof concludes. □

Theorem 6.8. *For a $C^n_{1,\sigma}$-extendable reduced game $(N, \mathcal{X}, \sigma)$ such that $n-1 \notin \mathcal{X}$ and $\{n\} \subsetneq \mathcal{X}$, the set $C^n_{1,\sigma}$ can be described as*

$$C^n_{1,\sigma} = \left\{ \alpha\overline{s}_2 + (1-\alpha)s^* + \sum_{i \notin \mathcal{X}, i \neq n-1} \alpha_i e_i \mid \alpha_i \geq 0 \right\},$$

where

$$e_i(j) := \begin{cases} -1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Proof. By Theorem 6.6, we know $(N, \overline{s}_2)$ and (N, s^*) are the only extreme games, with possibly $\overline{s}_2 = s^*$. It is a similar argument to the one in Theorem 6.3, showing (N, e_i) are rays of $C^n_{1,\sigma}$, binding all but one condition of $C^n_{1,\sigma}$. Note that (N, e_{n-1}) defined in Theorem 6.7 is not an extreme ray of $C^n_{1,\sigma}$. The proof is once again concluded by Theorem 1.5. □

6.2 Games with minimal information

In this section, we restrict ourselves to incomplete games with minimal information. We derive a compact description of the set of C^n_1-extensions employing its extreme games and its extreme rays (Subsection 6.2.1). Then, in Subsection 6.2.2, we investigate generalisations of three solution concepts for complete games, namely the τ-value, the Shapley value and the nucleolus. We consider two variants where we compute the centre of gravity of either extreme games or of a combination of extreme games and extreme rays. We show that all of the generalised values coincide for games with minimal information and we call the solution concept the average value $\tilde{\zeta}$. Finally, in Subsection 6.2.3, we provide three different axiomatisations of the average value and outline a method to generalise several of the axiomatisations of the τ-value and the Shapley value into an axiomatisation of the average value.

6.2.1 Description of the set of C^n_1-extensions

For the class of incomplete games with minimal information, we define the *total excess* as $\Delta \coloneqq v(N) - \sum_{i \in N} v(i)$, which we will widely use in this section.

The first step towards understanding the set of $C^n_1(v)$-extensions is to describe when it is empty.

Theorem 6.9. *An incomplete game with minimal information $(N, \mathcal{K}, v)$ is C^n_1-extendable if and only if $\Delta \geq 0$.*

Proof. Let $(N, w) \in C^n_1(v)$. Since it is 1-convex, it must hold for each $i \in N$,

$$w(i) \leq w(N) - b(N \setminus i).$$

We sum the inequalities over all n players to get

$$\sum_{i \in N} w(i) \leq nw(N) - \sum_{i \in N} b^w(N \setminus i).$$

We now expand expressions $b^w(N \setminus i)$ and rearrange the inequality into

$$\sum_{i \in N} w(i) + n(n-2)w(N) \leq (n-1) \sum_{i \in N} w(N \setminus j). \tag{6.14}$$

Since $b^w(N) \geq w(N)$ is equivalent to $\sum_{i \in N} w(N \setminus i) \leq (n-1)w(N)$, we bound the right-hand side of (6.14) by $(n-1)^2 w(N)$ and by rearranging, we conclude that $\Delta \geq 0$.

For the opposite direction, let us consider C_1^n-extensions (N, v^i) for $i \in N$ defined as

$$v^i(S) := \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N) - \sum_{j \notin S} v(j), & \text{if } S \notin \mathcal{K} \wedge i \in S, \\ v(N) - \sum_{j \notin S} v(j) - \Delta, & \text{if } S \notin \mathcal{K} \wedge i \notin S. \end{cases} \tag{6.15}$$

Notice that such games coincide on values of $S \in \mathcal{K}$. We claim that for any i, the game $v^i \in C_1^n(v)$. The condition $b^{v^i}(N) \geq v^i(N)$ holds since

$$b^{v^i}(N) = nv^i(N) - \sum_{j \in N} v^i(N \setminus j) = nv(N) - nv(N) + \sum_{j \in N} v(j) + \Delta = \sum_{j \in N} v(j) + \Delta.$$

Furthermore, $v^i(N) = v(N) = \sum_{j \in N} v(j) + \Delta$ and hence the condition is clearly satisfied.

Now to verify the condition $v^i(S) \leq v^i(N) - b^{v^i}(N \setminus S)$ for each $S \subseteq N$, we distinguish two cases based on if $i \in S$ or $i \notin S$.

For $i \in S$, $v^i(S) = v(N) - \sum\limits_{j \notin S} v(j)$, which is equal to $v^i(N) - b^{v^i}(N \setminus S)$. Therefore, the condition is satisfied and in fact, its upper bound is attained.

For $i \notin S$, $v^i(S) = v(N) - \sum\limits_{j \notin S} v(j) - \Delta$ and $v^i(N) - b^{v^i}(N \setminus S) = v(N) - \sum\limits_{j \notin S} v(j) - \Delta$. Again, the condition holds and the upper bound is attained. □

We note that if $\Delta = 0$, the set of C_1^n-extensions is rather simple and consists only of $(N, \overline{v})$ (the upper game defined in Theorem 6.10). Therefore, we are naturally more interested in situations when $\Delta > 0$.

The set of C_1^n-extensions is not bounded from below. For a C_1^n-extendable incomplete game $(N, \mathcal{K}, v)$ and its $C_1^n(v)$-extension (N, w), we can construct yet another $C_1^n(v)$-extension (N, w_S) dependent on a coalition $S \subseteq N$ such that $1 < |S| < n - 1$. We set the characteristic functions of the two games to differ only in values of S, so that $w_S(S) < w(S)$. The 1-convexity of (N, w_S) is easy to check from 1-convexity of (N, w) and it can be immediately seen that any arbitrarily small number ε satisfying $\varepsilon < v(S)$ could be chosen for the worth of coalition S in (N, w_S). Even though not bounded from below, the set of C_1^n-extensions is bounded from above.

Theorem 6.10. *Let $(N, \mathcal{K}, v)$ be a C_1^n-extendable game with minimal information. Then the upper game $(N, \overline{v})$ has the following form:*

$$\overline{v}(S) := \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N) - \sum_{i \notin S} v(i), & \text{if } S \notin \mathcal{K}. \end{cases}$$

Proof. To show that this is an upper bound for the value of each coalition $T \subseteq N$, we formulate the following optimization problem:

$$\begin{aligned} \max_{(N,w) \in C_1^n(v)} \quad & w(T) \\ \text{s.t.} \quad & w(N) \leq b^w(N), \\ & w(S) \leq w(N) - b^w(N \setminus S) \text{ for } S \subseteq N, S \neq \emptyset. \end{aligned} \tag{6.16}$$

Clearly, the optimal value of the optimization problem (if it exists) is the value $\overline{v}(T)$. Also notice that from the condition for T, i.e. $w(T) \leq w(N) - b^w(N \setminus T)$, that the upper bound of $w(T)$ is dependent only on value $w(N)$ (which is a constant since $N \in \mathcal{K}$) and n values $w(N \setminus i)$ for $i \in N$ (which are variables). The sum of these variables is bounded from above by $(n-1)v(N)$ (since $w(N) \leq b^w(N) \iff \sum_{i \in N} v(N \setminus i) \leq (n-1)v(N)$). From below, we have to consider only conditions $w(i) \leq w(N) - b^w(N \setminus i)$, because for $S \notin \mathcal{K}$, we can always choose a C_1^n-extension such that the value $w(S)$ is small enough to satisfy $w(S) \leq w(N) - b^w(N \setminus S)$.

Therefore, we can simplify the optimization problem by:

1. removing conditions for $S \notin \mathcal{K}$,
2. removing variables $w(S)$ for $S \notin \mathcal{K}$, and
3. substituting objective function $w(T)$ for $w(N) - b^w(N \setminus T)$.

By these simplifications, we get an optimization problem

$$\begin{aligned} \max \quad & w(N) - b^w(N \setminus T) \\ \text{s. t.} \quad & w(i) \leq w(N) - b^w(N \setminus i) \\ & i = 1, \dots, n. \end{aligned} \tag{6.17}$$

The set of feasible solutions is now $w \in \mathbb{R}^n$ where $w_i = w(N \setminus i)$ and $w(N)$ together with $w(i)$ (for $i \in N$) are constants. A feasible solution $w \in \mathbb{R}^n$ of problem (6.17) is equivalent to a feasible solution of problem (6.16) by setting $w(S) := -(n-s-1)w(N) + \sum_{k \in N \setminus S} w_k = w(N) - b^w(N \setminus S)$. Notice that the optimal values for both problems with corresponding feasible solutions equal.

We restate the problem in terms of the characteristic function w and we substitute $w(N \setminus i)$ for w_i, arriving at

$$\begin{aligned} \max_{w \in \mathbb{R}^n} \quad & \sum_{i \in N \setminus S} w_i - (n-s-1)w(N) \\ \text{s.t.} \quad & \sum_{j \in N} w_j \leq (n-1)w(N) \\ & w(k) \leq \sum_{j \neq i} w_j - (n-2)w(N) \\ & k = 1, \dots, n. \end{aligned} \tag{6.18}$$

Problem (6.18) is an instance of linear programming. Therefore, we can construct its dual program:

$$\begin{aligned} \min_{y \in \mathbb{R}^{n+1}} \quad & \sum_{i \in N} [(-(n-2)w(N) - w(i))y_i] + (n-1)w(N)y_{n+1} - (n-s-1)w(N) \\ \text{s.t.} \quad & -\sum_{j \neq i} y_j + y_{n+1} = 1 \text{ for } i \notin T \\ & -\sum_{j \neq i} y_j + y_{n+1} = 0 \text{ for } i \in T \\ & y_i \geq 0 \ i = 1, \dots, n+1. \end{aligned} \tag{6.19}$$

Let us define the vector $y^* \in \mathbb{R}^{n+1}$ as

$$y_j^* = \begin{cases} 0, & \text{if } j \in T, \\ 1, & \text{if } j \notin T, \\ n-t, & \text{if } j = n+1. \end{cases}$$

We deduce that

- $y_j^* \geq 0$ for all $j \in N$,
- $-\sum_{j \neq i} y_j^* + y_{n+1}^* = -(n-t-1)1 + (n-t) = 1$ for $i \notin T$,
- $-\sum_{j \neq i} y_j^* + y_{n+1}^* = -(n-t)1 + (n-t) = 0$ for $i \in T$.

Hence y^* is a feasible solution of (6.19). Further, the value of the objective function for y^* equals $w(N) - \sum_{i \notin T} w(i) = v(N) - \sum_{i \notin T} v(i)$. This means (from the duality of linear programming) that the primal program is feasible and the value of its objective function is bounded from above by this value.

To see that this upper bound is attained, take a game (N, v^i) (from the proof of Theorem 6.9) such that $i \notin T$. □

It is important (and by our opinion interesting) that the upper game of the set of S^n-extensions of non-negative incomplete games with minimal information coincide with the upper game of C_1^n-extensions from Theorem 6.10.

The upper game $\overline{v}$ is not 1-convex in general. For example, a 3-person incomplete game $(N, \mathcal{K}, v)$ with minimal information such that, $v(N) = 1$ and $v(i) = 0$ for all $i \in N$, is C_1^n-extendable because $\Delta = 1$, but $1 = \overline{v}(N) \nleq b^{\overline{v}}(N) = 0$. From the condition $\overline{v}(N) \leq b^{\overline{v}}(N)$ we can derive that $v(N) \leq \sum_{i \in N} v(i)$, therefore $\Delta = 0$. For $\emptyset \neq S \subsetneq N$ and conditions $\overline{v}(S) \leq \overline{v}(N) - b^{\overline{v}}(N \setminus S)$ we can easily derive from the definition of the upper game $(N, \overline{v})$ that

$$v(N) \leq \min_{\emptyset \neq S \subsetneq N} \left\{ \frac{2}{n-s} \sum_{i \in N \setminus S} v(i) \right\}.$$

Theorem 6.11. *Let $(N, \mathcal{K}, v)$ be an incomplete game with minimal information. Then it holds that the upper game $(N, \overline{v}) \in C_1^n(v)$ if and only if*

$$\Delta = 0 \text{ and } v(N) \leq \min_{\emptyset \neq S \subsetneq N} \left\{ \frac{2}{n-s} \sum_{i \in N \setminus S} v(i) \right\}.$$

So far, we showed that the set $C_1^n(v)$ is a convex polyhedron, since it can be described by a set of inequalities. It is bounded from above by $(N, \overline{v})$ and unbounded from below. Such polyhedrons (if having at least one vertex) can be characterised by the set of extreme points and the cone of extreme rays (see Theorem 1.5).

We initiate the derivation of the full description of the set of C_1^n-extensions by proving that games (N, v^i) (defined as (6.15)) are actually extreme points of the set. To prove this, we use a characterisation of extreme points from Theorem 1.2.

Theorem 6.12. *For C_1^n-extendable game $(N, \mathcal{K}, v)$ with minimal information, the games (N, v^i) are extreme games of $C_1^n(v)$.*

Proof. Let $x \in \mathbb{R}^{2^{|N|}}$ be a vector such that both $(N, v^i \pm x) \in C_1^n(v)$. We will show that in such case, inevitably $x(S) = 0$ for all $S \subseteq N$, thus by Theorem 1.2 (N, v^i) is an extreme game.

Define $f^+ := v^i + x$ and $f^- := v^i - x$. For $S \in \mathcal{K}$, clearly $x(S) = 0$. It remains to show that for $S \notin \mathcal{K}$, $x(S) = 0$.

For $S \notin \mathcal{K}$ and $i \notin S$, for the sake of contradiction, suppose w.l.o.g. that $x(S) > 0$. Then $f^+(S) = v^i(S) + x(S) = \overline{v}(S) + x(S) > \overline{v}(S)$, therefore $(N, f^+) \notin C_1^n(v)$, a contradiction.

For $S \notin \mathcal{K}$ and $i \in S$, again, suppose $x(S) = \delta > 0$. Because (N, f^+) and (N, f^-) are both 1-convex, conditions

$$f^+(S) + (n - s - 1)f^+(N) \leq \sum_{j \notin S} f^+(N \setminus j)$$

and

$$f^-(S) + (n - s - 1)f^-(N) \leq \sum_{j \notin S} f^-(N \setminus j)$$

must hold. We can rewrite both of the inequalities (and aggregate them by $\pm$) as

$$v^i(S) + (n - s - 1)v(N) \pm x(S) \pm (n - s - 1)x(N) \leq \sum_{j \notin S} v^i(N \setminus j) \pm \sum_{j \notin S} x(N \setminus j),$$

which is equivalent to

$$v(N) - \sum_{j \notin S} v(j) - c + (n - s - 1)v(N) \pm x(S) \leq (n - s)v(N) - \sum_{j \notin S} v(j) - c \pm \sum_{j \notin S} x(N \setminus j)$$

or

$$\pm x(S) \leq \pm \sum_{j \notin S} x(N \setminus j).$$

Since both inequalities hold, we conclude $x(S) = \sum_{j \notin S} x(N \setminus j)$. We already showed that $x(N \setminus j) = 0$ if $i \in N \setminus j$ if and only if $j \neq i$. But since $i \notin S$ we conclude $0 < \delta = x(S) = \sum_{j \notin S} x(N \setminus j) = 0$, which is a contradiction.

We proved that x is necessarily a vector of zeroes and thus we conclude the proof by taking Theorem 1.2 into account. □

Not only are games (N, v^i) for $i \in N$ the extreme games of $C_1^n(v)$, they are also the only extreme games.

Theorem 6.13. *For a C_1^n-extendable game $(N, \mathcal{K}, v)$ with minimal information, the games (N, v^i) are the only extreme games of $C_1^n(v)$.*

Proof. We will prove this theorem by showing that any extreme game (N, e) has the form of one of the (N, v^i) games. Since there are n different games, we have to enforce that the game coincides with (N, v^i) for a specific i.

To do so, realise there is i such that $e(N \setminus i) < \overline{v}(N \setminus i)$. If there was no such i, then $\forall k : e(N \setminus k) \geq \overline{v}(N \setminus k)$. The sum of these conditions leads to

$$\sum_{k \in N} e(N \setminus k) > \sum_{k \in N} \overline{v}(N \setminus k) = n\overline{v}(N) - \sum_{k \in N} v(k) = (n-1)\overline{v}(N) + \Delta \geq (n-1)\overline{v}(N).$$

But this is a contradiction, because the opposite inequality holds. Now we proceed to prove that $e = v^i$.

First, we show i is the unique coalition of size $n-1$ with its coalition value $e(N\setminus i)$ different from $\overline{v}(N\setminus i)$, i.e. there is no $j \neq i$ such that $e(N\setminus j) < \overline{v}(N\setminus j)$. For a contradiction, if there is such j, denote $\varepsilon_i = \overline{v}(N\setminus i) - e(N\setminus i), \varepsilon_j = \overline{v}(N\setminus j) - e(N\setminus j)$ and $\varepsilon = \min\{\varepsilon_i, \varepsilon_j\}$. We define a non-trivial game (N,x) such that both $(N, e+x) \in C_1^n(v)$ and $(N, e-x) \in C_1^n(v)$, contradicting (by Theorem 1.2) that (N,e) is an extreme game. The game (N,x) can be described as

$$x(S) = \begin{cases} \varepsilon, & \text{if } S = N\setminus i \text{ or } S \notin \mathcal{K} \wedge i \notin S \wedge j \in S, \\ -\varepsilon, & \text{if } S = N\setminus j \text{ or } S \notin \mathcal{K} \wedge i \in S \wedge j \notin S, \\ 0, & \text{otherwise.} \end{cases}$$

The condition (1.3) from Definition 1.14 for both $(N, e+x)$ and $(N, e-x)$ now reads as

$$\sum_{k\in N} e(N\setminus k) \pm x(N\setminus i) \pm x(N\setminus i) \leq (n-1)e(N)$$

or equivalently

$$\sum_{k\in N} e(N\setminus k) \pm \varepsilon \mp \varepsilon \leq (n-1)e(N)$$

is equivalent to the respective condition of (N,e). Furthermore, for any nonempty coalition S such that $i \notin S$ and $j \in S$, the condition (1.4) from Definition 1.14 for both games is

$$e(S) \pm x(S) - (n-s-1)e(N) \leq \sum_{k\in N\setminus S} v(N\setminus k) \pm x(N\setminus i).$$

Since $x(S) = x(N\setminus i)$, it is equivalent to the respective condition of (N,e). The rest of the cases for non-empty S can be dealt with in a similar manner, therefore both games $(N, e\pm x) \in C_1^n$. But this leads to a contradiction, because (N,e) is an extreme game.

Second, we show that $e(N\setminus i) = v(N) - v(i) - \Delta$. Clearly, for $\alpha > 0$,

$$(n-1)e(N) = \sum_{k\in N} e(N\setminus k) = nv(N) - \sum_{k\in N} v(k) - \alpha.$$

We note the first equality holds, otherwise there is a game (N,x) where $x(N\setminus j) := \beta$ and for $S, i \notin S : x(S) := -\beta$ that leads to a contradiction with (N,e) being an extreme game. Therefore $\alpha = v(N) - \sum_{k\in N} v(k) = \Delta$.

Finally, it is elementary to prove that $e(S) = e(N) - b(N\setminus S) = v^i(S)$. If this was not true, yet another game (N,x) with $x(S) = e(N) - b(N\setminus S)$ and $x(T) = 0$ would lead to a contradiction with extremality of (N,e). □

Now let us proceed with the investigation of the extreme rays. For a game $(N, v^i + \lambda e)$ to be 1-convex (thus being in the recession cone of $C_1^n(v)$), the following conditions must hold for every nonempty $S \subseteq N$:

- $b^{v^i}(N) + b^{\lambda e}(N) \geq v^i(N) + \lambda e(N)$,
- $v^i(S) + \lambda e(S) \leq v^i(N) + \lambda e(N) - b^{v^i}(N\setminus S) - b^{\lambda e}(N\setminus S)$.

Notice from the proof of Theorem 6.9 that

- $b^{v^i}(N) = v^i(N)$,

- $v^i(S) = v^i(N) - b^{v^i}(N \setminus S)$.

We can therefore simplify the conditions, arriving at

- $b^{\lambda e}(N) \geq \lambda e(N)$,
- $\lambda e(S) \leq \lambda e(N) - b^{\lambda e}(N \setminus S)$.

Furthermore, we can factor out λ since it is non-negative. Notice that for each $j \in N$, $e(j) = e(N) = 0$, otherwise $(N, v^i + \lambda e) \notin C_1^n(v)$. Taking all this into consideration, we obtain the following conditions for (N, e), representing an unbounded direction in C_1^n:

1. $b^e(N) \geq e(N) \iff \sum_{j\in N} e(N \setminus j) \leq 0$,
2. $\forall S \subseteq N, S \neq \emptyset : e(S) \leq e(N) - b^e(N \setminus S) \iff e(S) \leq \sum_{j\in N\setminus S} e(N \setminus j)$,
3. $\forall k \in N : 0 \leq \sum_{j\in N\setminus k} e(N \setminus k)$,
4. $\forall j \in N : e(j) = 0$,
5. $e(N) = 0$.

Conditions 1 and 2 show that (N, e) has to be 1-convex game, itself. Moreover, if it is 1-convex, for any $\lambda \geq 0$, the game $(N, \lambda e)$ is also 1-convex. Therefore, the game (N, e) is (not necessarily an extreme) ray of the recession cone of the set of C_1^n-extensions. It is a zero-normalised game with $e(N) = 0$ (thanks to conditions 4 and 5). Observe that condition 3 is a special case of condition 2 (take $S = \{k\}$ for $k \in N$). We state it separately, since it will come in handy to refer just to this special case in further text. Notice an interesting fact: values of the game (N, e) do not depend on the value of $(N, \mathcal{K}, v)$. Therefore, the recession cone is the same for every incomplete game with minimal information.

Further, to simplify conditions 1 to 5, suppose that there is a C_1^n-extension $(N, v^i + e)$ such that $\sum_{j\in N} e(N \setminus j) < 0$. Then there is $k \in N$ such that $\sum_{j\in N\setminus k} e(N\setminus j) < 0$. But this is a contradiction with $0 \leq \sum_{j\in N\setminus k} e(N\setminus j)$. Further, suppose that $\sum_{j\in N} e(N \setminus j) = 0$ and there is k such that $e(N \setminus k) \neq 0$. We distinguish two cases. If $e(N \setminus k) > 0$, then $e(N \setminus k) = -\sum_{j\in N\setminus k} e(N \setminus j) > 0$, which is a contradiction because both $0 \leq \sum_{j\in N\setminus k} e(N \setminus j)$ and $\sum_{j\in N\setminus k} e(N \setminus j) < 0$. If $e(N \setminus k) < 0$, there is $\ell \in N$ such that $e(N \setminus \ell) > 0$ and we arrive into a similar contradiction. Hence it must hold for every $i \in N$, that $e(N \setminus i) = 0$. We can now rewrite the conditions as

1. $\forall S \subseteq N, S \neq \emptyset : e(S) \leq 0$,
2. $\forall i \in N : e(i) = e(N) = e(N \setminus i) = 0$.

Let us now select the extreme rays. From Definition 1.4, all but one of conditions 1 or 2 have to be satisfied with equality for game $(N, v^i + e)$ to be an extreme ray. We see that the extreme rays are given by 1-convex games (N, e_T) for coalitions $T \in E = 2^N \setminus (\{0, N\} \cup \{N \setminus i \mid i \in N\} \cup \{\{i\} \mid i \in N\})$, where

$$e_T(S) := \begin{cases} -1, & \text{if } S = T, \\ 0, & \text{if } S \neq T. \end{cases} \tag{6.20}$$

With such knowledge, we are ready to fully describe the set of C_1^n-extensions of games with minimal information.

Theorem 6.14. *For a C_1^n-extendable game $(N, \mathcal{K}, v)$ with minimal information, the set of C_1^n-extensions can be described as*

$$C_1^n(v) = \left\{ \sum_{i \in N} \alpha_i v^i + \sum_{T \in E} \beta_T e_T \mid \sum_{i \in N} \alpha_i = 1 \text{ and } \alpha_i, \beta_T \geq 0 \right\}.$$

Proof. We have already proved that games (N, v^i) for $i \in N$ from (6.15) are the extreme games of $C_1^n(v)$ and games (N, e_T) for $T \in E$ from (6.20) are the extreme rays of $C_1^n(v)$. The rest of the proof follows from Theorem 1.5. □

6.2.2 Solution concepts

In this subsection, we present generalisations of the τ-value and the Shapley value based on two ideas. The first idea is to consider solely the vertices of the set of C_1^n-extensions, compute their centre of gravity and for the resulting game, compute its τ-value (see Definition 6.1) or its Shapley value (Definition 6.4). The second idea considers also the recession cone, which is completely neglected in the first approach (Definitions 6.3, 6.5). We show that from the symmetry of recession cone, both approaches for both generalisations of the τ-value and the Shapley value lead to the same solution concept for games with minimal information. We call it the *average value*.

The average τ-value

The first solution concept considers the centre of gravity of the extreme games, that is

$$\tilde{v} = \sum_{i \in N} \frac{v^i}{N}.$$

Note that $(N, \tilde{v})$ is 1-convex if (N, v^i) are 1-convex for all $i \in N$. Since the additivity does not hold for the τ-value in general, $\tau(\tilde{v}) \neq \sum_{i \in N} \frac{\tau(v^i)}{N}$ in general. We consider both variants in the next definition.

Definition 6.1.

1. *The* average τ-value $\tilde{\tau} \colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ *is defined as*

$$\tilde{\tau}(v) \coloneqq \tau(\tilde{v}), \text{ and}$$

2. *the* solidarity τ-value $\tau^s \colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ *is defined as*

$$\tau^s(v) \coloneqq \sum_{i \in N} \frac{\tau(v^i)}{n},$$

where $(N, \tilde{v})$ and (N, v^i) for $i \in N$ are the centre of gravity and the extreme games of $C_1^n(v)$, respectively.

The justification for the name of the solidarity τ-value is given in the following theorem.

Theorem 6.15. *The average τ-value $\tilde{\tau}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ and the solidarity τ-value $\tau^s\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ can be expressed as follows:*

1. *$\forall i \in N : \tilde{\tau}_j(v) = v(j) + \frac{\Delta}{n}$,*
2. *$\forall i \in N : \tau_j^s(v) = \frac{v(N)}{n}$.*

Proof. Both expressions can be easily derived from the definition of $\tilde{\tau}$, τ^s and (N, v^i). First of all, the game $(N, \tilde{v})$ can be expressed as

$$\tilde{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N) - \left(\sum\limits_{j \in N \setminus S} v(j)\right) - \frac{n-s}{n}\Delta, & \text{if } S \notin \mathcal{K}. \end{cases}$$

The values of its utopia vector are $b_j^{\tilde{v}} = v(j) + \frac{n-1}{n}\Delta$, and by summing them, we arrive at $b^{\tilde{v}}(N) = \sum_{j \in N} v(j) + (n-1)\Delta = v(N) + (n-2)\Delta$. The gap function for N is $g^{\tilde{v}}(N) = (n-2)\Delta$ and finally (by Theorem 1.9), from the definition of $\tilde{\tau}(v)$, using $\tilde{\tau}(v) = \tau(\tilde{v})$, we get

$$\tau_j(\tilde{v}) = b_j^{\tilde{v}} - \frac{g^{\tilde{v}}(N)}{n} = v(j) + \frac{n-1}{n}\Delta - \frac{n-2}{n}\Delta = v(j) + \frac{\Delta}{n}.$$

The main reason behind the formula for the solidarity τ-value is the fact that $\tau(v^i) = b^{v^i}$, which immediately follows from $g^{v^i}(N) = 0$, and from the form of b^{v^i} which is

$$b_j^{v^i} = \begin{cases} v(j), & \text{if } j = i, \\ v(j) + \Delta, & \text{if } j \neq i. \end{cases}$$

By summing the values of vector b^{v^i}, we get that $b^{v^i}(N) = \sum_{j \in N} v(j) + \Delta = v(N)$. Therefore, $g^{v^i}(N) = v(N) - b^{v^i}(N) = 0$. Finally, if we take

$$\tau_j^s = \sum_{i \in N} \frac{\tau_j(v^i)}{n} = \sum_{i \in N} \frac{b_j^{v^i}}{n} = \frac{\sum_{i \in N} b_j^{v^i}}{n},$$

we arrive at the formula stated above, since $\sum_{i \in N} b_j^{v^i} = \sum_{i \in N} v(i) + \Delta = v(N)$. □

We immediately see that the solidarity τ-value is not a very reasonable solution concept if we consider that under such value, every player should get an equal share of $\frac{v(N)}{n}$ no matter his contribution.

The conic τ-value

It might seem that the main downside of the previous two solution concepts is that we do not consider the recession cone of the set of C_1^n-extensions. Here we provide an argument showing that with no further assumptions, this is not the case for games with minimal information. We define a solution concept dependent on the recession cone, which will serve as a foundation for the study of incomplete games with more general sets $\mathcal{K}$.

Let (N, v^i) for $i \in N$ be the extreme games of $C_1^n(v)$ and (N, e_T) for $T \in E$ be the extreme rays of $C_1^n(v)$. $(N, \tilde{v})$ denotes again the centre of gravity of extreme games and $(N, \tilde{e})$ the centre of gravity of extreme rays, i.e.

$$\tilde{e} = \sum_{T \in E} \frac{e_T}{|E|} = \sum_{T \in E} \frac{e_T}{|2^n - 2n - 2|}.$$

The *conic τ-value* $\tau^{<}$, introduced in the following definition, is computed on the sum of these games. It considers the extreme games as well as extreme rays, thus the information from the *shape* of the conic cone is also included.

Definition 6.2. *The* conic τ-value $\tau^{<}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ *is defined as*

$$\tau^{<}(v) \coloneqq \tau(\tilde{v} + \tilde{e}),$$

where the games $(N, \tilde{v})$ *and* $(N, \tilde{e})$ *are the centres of gravity of extreme points and of extreme rays of* $C_1^n(v)$*.*

Surprisingly, the average τ-value and conic τ-value are the same function for incomplete games with minimal information. The reason is hidden in the symmetry of $(N, \tilde{e})$.

Theorem 6.16. *The conic τ-value* $\tau^{<}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ *can be expressed as follows:*

$$\forall i \in N : \tau_i^{<}(v) = v(i) + \frac{\Delta}{n}.$$

Proof. The proof is a straightforward derivation from the definitions. First, we already know the description of $(N, \tilde{v})$ from the proof of Theorem 6.15. The description of $(N, \tilde{e})$ is

$$\tilde{e}(S) = \begin{cases} -\frac{1}{\varepsilon}, & \text{if } S \in \mathcal{K} \vee S = N \setminus j \text{ for } j \in N, \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon = 2^n - 2n - 2$. From this description we derive that $b_i^{\tilde{v}+\tilde{e}} = v(i) + \frac{n-1}{n}\Delta - \frac{1}{\varepsilon}$ as $b_i^{\tilde{v}+\tilde{e}}$ is equal to

$$(\tilde{v} + \tilde{e})(N) - (\tilde{v} + \tilde{e})(N \setminus j) = v(N) - \tilde{v}(N \setminus i) - \tilde{e}(N \setminus i)$$

and the right-hand side can be rewritten as $v(N) - \left(v(N) - v(i) - \frac{n-1}{n}\Delta\right) + \frac{1}{\varepsilon}$. Further, $b^{(\tilde{v}+\tilde{e})}(N) = \sum\limits_{i \in N} v(i) + (n-1)\Delta + \frac{n}{\varepsilon} = v(N) + (n-2)\Delta + \frac{n}{\varepsilon}$. The last equality follows by using the fact that $\sum\limits_{i \in N} v(i) + \Delta = v(N)$. The gap function $g^{(\tilde{v}+\tilde{e})}(N) = b^{(\tilde{v}+\tilde{e})}(N) - v(N) = (n-2)\Delta + \frac{n}{\varepsilon}$.

Finally, by Theorem 1.9, $\tau_i^{<}(v) = b_i^{(\tilde{v}+\tilde{e})} - \frac{g^{(\tilde{v}+\tilde{e})}}{n} = v(i) + \frac{n-1}{n}\Delta + \frac{1}{\varepsilon} - \frac{(n-2)\Delta + \frac{n}{\varepsilon}}{n}$. Since $\frac{(n-2)\Delta + \frac{n}{\varepsilon}}{n} = \frac{n-2}{n}\Delta + \frac{1}{\varepsilon}$, we have

$$\tau_i^{<}(v) = v(i) + \frac{n-1}{n}\Delta + \frac{1}{\varepsilon} - \frac{n-2}{n}\Delta - \frac{1}{\varepsilon} = v(i) + \frac{\Delta}{n}. \qquad \square$$

As mentioned before, the reason for this (one might say) surprising result is the symmetry of $(N, \tilde{e})$. Actually, consider a more general setting, where we take the expression

$$\frac{1}{\gamma}\left(\beta \sum_{i \in N} v^i + \alpha \sum_{T \in E} e_T\right). \tag{6.21}$$

This is a generalization of $\tilde{v} + \tilde{e}$ (for $\beta = \frac{1}{n}, \alpha = \frac{1}{\varepsilon}$, and $\gamma = 1$, we get $\tilde{v} + \tilde{e}$). Also, if $\beta \neq \gamma$, it can be shown that the game from (6.21) does not lie in $C_1^n(v)$. Fixing $\beta = \gamma$, the τ-value of this expression equals $\tilde{\tau}$ for any $\alpha \in \mathbb{R}$.

On the other hand, if $(N, \tilde{e})$ would not be symmetric or would depend on values of $v \in C_1^n(\mathcal{K}_{\min})$, the information about the cone might matter and $\tau^<(v) \neq \tilde{\tau}(v)$. This motivates the definition for games $(N, \mathcal{K}, v)$ with a more general structure of $\mathcal{K}$.

Definition 6.3. *Let $\mathcal{K} \subseteq 2^N$ and suppose that $\forall v \in C_1^n(\mathcal{K})$, the set $C_1^n(v)$ is a polyhedron described by its extreme points and extreme rays. Then the* α-conic τ-value *$\tau^\alpha\colon C_1^n(\mathcal{K}) \to \mathbb{R}^n$ is defined as*

$$\tau^\alpha(v) \coloneqq \tau(\tilde{v} + \alpha\tilde{e}),$$

where $(N, \tilde{v})$, $(N, \tilde{e})$ are the centres of gravity of extreme points and of extreme rays of $C_1^n(\mathcal{K})$, respectively.

As mentioned before, for games with minimal information, $\tilde{\tau}(v) = \tau^\alpha(v)$. However, for more general sets $\mathcal{K}$, this definition might yield a different solution. This is supported by the investigation of incomplete games with defined upper vector in Section 6.3. In there, we show that a similar solution concept, *the α-conic Shapley value*, does not coincide with *the average Shapley value*. The idea behind these solution concepts is the same as behind the average and the conic τ-value.

Once more, the fact that additivity does not hold for the τ-value in general leads to a question whether $\tau(\tilde{v}+\tilde{e})$ and $\tau(\tilde{v})+\tau(\tilde{e})$ yield a different function. For games with minimal information, this is not the case since $\tau(\tilde{e}) = 0$. Therefore, $\tau(\tilde{v}) + \tau(\tilde{e}) = \tau(\tilde{v}) = \tilde{\tau}(v)$.

The average Shapley value

The average Shapley value $\tilde{\phi}$ was already studied by Masuya and Inuiguchi in [19] for the set of S^n-extensions of non-negative incomplete games with minimal information. We show that in the context of 1-convexity, the average Shapley value coincides with their definition, which is also equal to the average τ-value. Yet again, the consideration of the recession cone (thanks to its symmetry) does not come to fruition.

Definition 6.4. *The* average Shapley value *$\tilde{\phi}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$, is defined as*

$$\tilde{\phi}(v) \coloneqq \phi(\tilde{v}),$$

where $(N, \tilde{v})$ is the centre of gravity of extreme games of $C_1^n(v)$.

Theorem 6.17. *The average Shapley value $\tilde{\phi}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ can be expressed as follows:*

$$\forall i \in N : \tilde{\phi}_i(v) = v(i) + \frac{\Delta}{n}.$$

Proof. The proof is based on the characterisation of the Shapley value from Theorem 1.13 and the fact that for every $S \subseteq N \setminus i$, $\tilde{v}(S \cup i) - \tilde{v}(S) = v(i) + \frac{\Delta}{n}$. This holds as $\tilde{v}(S \cup i) - \tilde{v}(S)$ is from the definition of $(N, \tilde{v})$ equal to

$$v(N) - \sum_{j \in N \setminus (S \cup i)} v(j) - \frac{(n-(s+1))}{n}\Delta - \left(v(N) - \sum_{j \in N \setminus S} v(j) - \frac{(n-s)}{n}\Delta \right),$$

which can be rewritten to $v(i) + \frac{\Delta}{n}$. Observe that $v(i) + \frac{\Delta}{n}$ is independent of coalition S. We know that $\tilde{\phi}_i(v) = \phi_i(\tilde{v})$ and substituting into the expression from Theorem 1.13, we get

$$\phi_i(\tilde{v}) = \frac{1}{n} \sum_{S \subseteq N \setminus i} \binom{n-1}{s}^{-1} \left(v(i) + \frac{\Delta}{n}\right) = \left(v(i) + \frac{\Delta}{n}\right) \frac{1}{n} \sum_{S \subseteq N \setminus i} \binom{n-1}{s}^{-1}.$$

Modifying the sum is a mere exercise, using the following identity:

$$\sum_{S \subseteq N \setminus i} \binom{n-1}{s}^{-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n-1}{j}^{-1} = n.$$

Combining together, we arrive at the desired formula. □

Similarly to the investigation of the conic τ-value, we arrive to a conclusion that any sensible integration of the recession cone in the definition of the generalised Shapley value does not yield a different result. This is since $\phi(\tilde{v} + \alpha\tilde{e}) = \phi(\tilde{v}) + \phi(\alpha\tilde{e}) = \phi(\tilde{v})$ as for symmetric game $(N, \alpha\tilde{e})$, the Shapley value for any player i equals $\phi_i(\alpha\tilde{e}) = 0$. Nonetheless, similar argument for the definition of ϕ^α for games with general $\mathcal{K}$ holds. For the conic Shapley value of incomplete games with defined upper vector, we show in Section 6.3 that the two concepts do not coincide in general.

Definition 6.5. *Let $\mathcal{K} \subseteq 2^N$ and suppose that $\forall v \in C_1^n(\mathcal{K})$, the set $C_1^n(v)$ is a polyhedron described by its extreme points and extreme rays. Then the α-*conic Shapley-value *$\phi^\alpha \colon C_1^n(\mathcal{K}) \to \mathbb{R}^n$ is defined as*

$$\phi^\alpha(v) \coloneqq \phi(\tilde{v} + \alpha\tilde{e}),$$

where $(N, \tilde{v}), (N, \tilde{e})$ are the centres of gravity of extreme points and of extreme rays, respectively.

To sum it up, we considered generalisation of three values τ, n, ϕ of complete cooperative games in two variants (including/excluding the information from the recession of C_1^n-extensions) and showed that actually all of them coincide thanks to 1-convexity and symmetry of the recession cone of $C_1^n(v)$. From now on, we will refer to this solution concept of incomplete games with minimal information as the *average value* $\tilde{\zeta}$.

6.2.3 Axiomatisation of the average value $\tilde{\zeta}$

In this subsection, we focus on axiomatisation of the average value. In the first part, we consider known characterisations of the τ-value and the Shapley value of complete games. We show how to generalise these characterisation for the average value. This is done through the fact that the average value is both the τ-value and Shapley value of a specific complete game $(N, \tilde{v})$. In the second part, we offer three axiomatisations where the axioms are defined in the context of values of $v \in C_1^n(\mathcal{K}_{\min})$.

Generalisations of known axiomatisations

The idea behind generalisations of known axiomatisations is based on the fact that the average value is defined as either the τ-value or the Shapley value of the centre of gravity $(N, \tilde{v})$. Since these solution concepts satisfy certain axioms, also the average value satisfies these axioms when restricted to $\tilde{v}$. The uniqueness of the average value is then given by the uniqueness of $\tilde{\zeta}(v)$ for each $v \in C_1^n(\mathcal{K}_{\min})$. If we had a function f satisfying the restricted axioms different from $\tilde{\zeta}$, we would have a game $v \in C_1^n(\mathcal{K}_{\min})$ such that $\tilde{\zeta}(v) = \tau(\tilde{v}) \neq f(v)$. But this means that for $(N, \tilde{v})$, we have two solution concepts for complete games satisfying the axioms of the τ-value (or the Shapley value) that differ in the imputation assigned to $(N, \tilde{v})$. This is a contradiction with the uniqueness of these values.

We demonstrate this proof method on two examples, generalising both an axiomatisation of the τ-value and the Shapley value.

Theorem 6.18. *The average value $\tilde{\zeta}$ is the only function $f\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v \in C_1^n(\mathcal{K}_{\min})$:*

1. *(efficiency) $\sum_{i \in N} f_i(v) = v(N)$,*
2. *(restricted proportionality property of $\tilde{v}$) $f(v_0) = \alpha b^{\tilde{v}_0}$,*
3. *(minimal right property of $\tilde{v}$) $f(v) = a^{\tilde{v}} + f(v - a^{\tilde{v}})$,*

where $\alpha \in \mathbb{R}$ and $(v - a^{\tilde{v}})(S) \coloneqq v(S) - \sum_{i \in S} a_i^{\tilde{v}}$ for every $S \subseteq N$.

Proof. To prove that the average value satisfies the mentioned properties, remember the definition $\tilde{\zeta}(v) = \tau(\tilde{v})$ and Theorem 1.10. Since $\tilde{v}(S) = v(S)$ for $S \in \mathcal{K}$, all three properties hold.

As for the uniqueness, suppose there is a function $g\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ such that the properties hold and there is a game $v \in C_1^n(\mathcal{K}_{\min})$, $\tilde{\zeta}(v) \neq g(v)$. We can construct a function $\gamma\colon C_1^n \to \mathbb{R}^n$ such that $\gamma(w) \coloneqq \tau(w)$ for every $w \in C_1^n$, $w \neq \tilde{v}$ and $\gamma(\tilde{v}) \coloneqq g(v)$. Clearly, γ satisfies all axioms from Theorem 1.10, which leads (together with $\gamma(\tilde{v}) = g(v) \neq \tau(\tilde{v})$) to a contradiction with the uniqueness of the τ-value. □

It can be showed that in the context of incomplete games the second axiom is equivalent to restricted proportionality property of $\tilde{v}$ where $\alpha = 1$.

The alternative characterisation of the τ-value was proposed in [6] and it can be generalised in a similar manner. Let us proceed with yet another example, generalising axiomatisation of the Shapley value.

Theorem 6.19. *The average value $\tilde{\zeta}$ is the only function $f\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v, w \in C_1^n(\mathcal{K}_{\min})$:*

1. *(efficiency) $\sum_{i \in N} f_i(v) = v(N)$,*
2. *(symmetry of $\tilde{v}$) $\forall i, j \in N$ and $\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j) \implies f_i(v) = f_j(v)$,*
3. *(null player of $\tilde{v}$) $\forall i \in N$ and $\forall S \subseteq N \setminus i : \tilde{v}(S) = \tilde{v}(S + i) \implies f_i(v) = 0$,*
4. *(additivity of $\tilde{v}$) if $v + w \in C_1^n(\mathcal{K}_{\min}) : f(\tilde{v} + \tilde{w}) = f(\tilde{v}) + f(\tilde{w})$.*

Proof. Since the average value of $v \in C_1^n(\mathcal{K}_{\min})$ acts as the Shapley value of $\tilde{v} \in C_1^n$, the axioms are satisfied. We note that considering efficiency, $\sum_{i \in N} f_i(v) = v(N) = \tilde{v}(N)$, therefore it is equivalent with $\phi(\tilde{v}) = \tilde{v}(N)$ and for additivity, $(\tilde{v} + \tilde{w}) = \tilde{v} + \tilde{w}$, where $(\tilde{v} + \tilde{w})$ is the centre of gravity of vertices of $v + w$ and $\tilde{v} + \tilde{w}$ is the sum of centres of gravity of v and w. The uniqueness of $\tilde{\zeta}$ is given by the uniqueness of the Shapley value. □

From the alternative characterisations of the Shapley value, we generalised [32, 35]. To do the same for the one by Roth in [23] is more challenging and we leave it as an open problem.

Axiomatisations in the context of values of $(N, \mathcal{K}, v)$

The previously mentioned characterisations do not tell us anything new about the average value that we do not already know from its definition $\tilde{\zeta}(v) := \tau(\tilde{v}) = \phi(\tilde{v})$. In this subsection, we derive three axiomatisations in the context of values of $(N, \mathcal{K}, v)$.

Theorem 6.20. *The average value $\tilde{\zeta}$ is the only function $f \colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v \in C_1^n(\mathcal{K}_{\min})$:*

1. *(efficiency) $\sum_{i \in N} f_i(v) = v(N)$,*
2. *(elementary symmetry) $\forall i, j \in N : i \neq j \wedge v(i) = v(j) \implies f_i(v) = f_j(v)$,*
3. *(zero-normalisation invariance) $\forall i \in N : f_i(v) = v(i) + f_i(v_0)$.*

Proof. Let us prove that $\tilde{\zeta}$ satisfies all three properties. First,

$$\sum_{i \in N} \tilde{\zeta}_i(v) = \sum_{i \in N} v(i) + n\frac{\Delta}{n} = v(N).$$

Furthermore, for $v(i) = v(j)$, it holds $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta}{n} = v(j) + \frac{\Delta}{n} = \tilde{\zeta}_j(v)$. For the third property, as $v(i) + \frac{\Delta}{n} = \tilde{\zeta}_i(v) = v(i) + \tilde{\zeta}_i(v_0)$ holds for every player i, it suffices to show that $\tilde{\zeta}_i(v_0) = \frac{\Delta}{n}$. For any $v \in C_1^n(\mathcal{K}_{\min})$, the zero-normalisation v_0 can be described as

$$v_0(S) = \begin{cases} v(N) - \sum_{i \in N} v(i), & \text{if } S = N, \\ 0, & \text{if } S \neq N. \end{cases}$$

The total excess Δ_0 of $(N, \mathcal{K}, v_0)$ is equal to $v(N) - \sum_{i \in N} v(i) = \Delta$. Therefore $\tilde{\zeta}_i(v_0) = v_0(i) + \frac{\Delta_0}{n} = \frac{\Delta}{n}$, thus the third property also holds.

Now, let us prove that $f \colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ satisfying the three properties must be $\tilde{\zeta}$. First, from zero-normalisation invariance, it holds $\forall i : f_i(v) = v(i) + f_i(v_0)$. Because $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta}{n}$ for all $i \in N$, it suffices to prove that $f_i(v_0) = \frac{\Delta}{n}$ for any zero-normalisation v_0 of $v \in C_1^n(\mathcal{K}_{\min})$. From the first property, we have $\sum_{i \in N} f_i(v_0) = \Delta = v_0(N)$. Also, $v_0(i) = v_0(j)$ for all pairs of players i, j implies $f_i(v_0) = f_j(v_0)$. Combining both together, we get $f_i(v_0) = \frac{\sum_{j \in N} f_i(v_0)}{n} = \frac{\Delta}{n}$. □

Another characterisation can be obtained by substituting the axiom of zero-normalisation invariance for additivity axiom. Such replacement has to be compensated by adding yet another axiom, because without it, they do not characterise the function uniquely (for example, the solidarity τ-value τ^s also satisfies these three axioms). We deal with this by providing two different axioms: *zero-excess* axiom (employing the total excess Δ) and a more familiar axiom of *individual rationality.*

Theorem 6.21. *The average value $\tilde{\zeta}$ is the only function $f\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$ such that the following properties hold for every $v, w \in C_1^n(\mathcal{K}_{\min})$:*

1. *(efficiency)* $\sum_{i\in N} f_i(v) = v(N)$,
2. *(elementary symmetry)* $\forall i, j \in N : i \neq j \wedge v(i) = v(j) \implies f_i(v) = f_j(v)$,
3. *(elementary additivity) if* $v + w \in C_1^n(\mathcal{K}_{\min}) : f(v + w) = f(v) + f(w)$,
4. *(zero-excess axiom) if* $\Delta_v = 0 \implies \forall i : f_i(v) = v(i)$,
5. *(individual rationality)* $\forall i \in N : f_i(v) \geq v(i)$.

Proof. We have already proved in Theorem 6.20 that the first two axioms are satisfied by $\tilde{\zeta}$. To prove additivity, we have $\tilde{\zeta}_i(v + w) = v(i) + w(i) + \frac{\Delta_{v+w}}{n}$ and

$$\tilde{\zeta}_i(v) + \tilde{\zeta}_i(w) = v(i) + \frac{\Delta_v}{n} + w(i) + \frac{\Delta_w}{n} = v(i) + w(i) + \frac{\Delta_v}{n} + \frac{\Delta_w}{n}.$$

for any player i. Clearly, if $\Delta_{v+w} = \Delta_v + \Delta_w$, elementary additivity is satisfied. However,

$$\Delta_{v+w} = v(N) + w(N) - \sum_{i\in N}(v(i) + w(i)) = v(N) - \sum_{i\in N} v(i) + w(N) - \sum_{i\in N} w(i)$$

and

$$\Delta_v + \Delta_w = v(N) - \sum_{i\in N} v(i) + w(N) - \sum_{i\in N} w(i).$$

Zero-excess axiom is satisfied because for $\Delta_v = 0$ and any player i, $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta_v}{n} = v(i)$. Individual rationality is also satisfied as for any player i: $\tilde{\zeta}_i(v) = v(i) + \frac{\Delta_v}{n} \geq v(i)$.

To substitute elementary additivity for zero-normalisation in our proof of uniqueness, we define a game with minimal information $(N, \mathcal{K}, \Sigma)$ such that $v = v_0 + \Sigma$. We do so by setting $\Sigma(i) \coloneqq v(i)$ and $\Sigma(N) \coloneqq \sum_{i\in N} v(i)$. Notice that $\Delta_\Sigma = 0$ and thus, $\Sigma \in C_1^n(\mathcal{K}_{\min})$. Now, from elementary additivity $f_i(v) = f_i(v_0) + f_i(\Sigma)$. We already proved, that $f_i(v_0) = \frac{\Delta}{n}$, thus all that remains is to prove that for every player i, $f_i(\Sigma) = v(i)$.

From zero-excess axiom, this already holds as $\Delta_\Sigma = 0$. Without zero-excess axiom, by efficiency $\sum_{i\in N} f_i(\Sigma) = \Sigma(N) = 0$ and individual rationality, each summand $f_i(\Sigma) \geq 0$, which leads together to the desired $f_i(\Sigma) = v(i)$. □

We conclude this section with further axioms, which are connected with different definitions of the Shapley value [32, 35].

Both of the following properties can be easily derived from the definition of the average value $\tilde{\zeta}$.

Theorem 6.22. *For the average value* $\tilde{\zeta}\colon C_1^n(\mathcal{K}_{\min}) \to \mathbb{R}^n$*, the following properties hold for every* $v, w \in C_1^n(\mathcal{K}_{\min})$*:*

(elementary triviality) $\tilde{\zeta}(v_\emptyset) = 0$*, where* $v_\emptyset(S) \coloneqq 0$ *for* $S \in \mathcal{K}$*,*

(elementary fairness) $\tilde{\zeta}_i(v + w) - \tilde{\zeta}_i(v) = \tilde{\zeta}_j(v + w) - \tilde{\zeta}_j(v)$ *if* $w(i) = w(j)$*.* □

Notice that a property similar to *null player* cannot hold when $\Delta > 0$. that is because if $v(i) = 0$ for a player i, then $\tilde{\zeta}_i = \frac{\Delta}{n} > 0$. This might seem a surprising result as in the characterisation of the Shapley value, the axiom of null player is satisfied. This corresponds with the idea that even though it might seem from the known information given by $\mathcal{K}$ that the player does not have any worth in the game, since we cannot be sure, we act as if he has some.

6.3 Games with defined upper vector

In this section, we study the class of incomplete games with defined upper vector, i.e. $(N, \mathcal{K}, v)$ such that $\{\emptyset, N\} \cup \{N \setminus i \mid i \in N\} \subseteq \mathcal{K}$. We derive the description of the set $C_1^n(v)$ in Subsection 6.3.1 and in Subsection 6.3.2, we prove that the average τ-value and the average Shapley value do not coincide for games with at least four players.

6.3.1 Description of the set of C_1^n-extensions

Theorem 6.23. *Let* $(N, \mathcal{K}, v)$ *be an incomplete game with defined upper vector. It is* C_1^n*-extendable if and only if*

$$\forall S \in K : v(S) \le v(N) - b(N \setminus S) \tag{6.22}$$

and

$$b(N) \ge v(N). \tag{6.23}$$

Proof. If the conditions hold, we can define a complete game $(N, \overline{v})$ such that

$$\overline{v}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{K}, \\ v(N) - b(N \setminus S), & \text{if } S \notin \mathcal{K}. \end{cases}$$

The game is 1-convex, because $v(S) \le v(N) - b(N \setminus S)$ for $S \notin \mathcal{K}$ holds since the left-hand side is actually equal to the right-hand side. For $S \in \mathcal{K}$ the conditions hold from the assumption as well as the condition $b(N) \ge v(N)$. Therefore, it is a C_1^n-extension of $(N, \mathcal{K}, v)$

If one of the conditions (6.22) or (6.23) fails, the condition does not hold for any extension, therefore the extension is not 1-convex. □

We denoted the C_1^n-extension using the line over v. This is not a coincidence, as the game is really the upper game of the set of $C_1^n(v)$-extensions. On the top of that, it is also the only extreme game of the set.

Theorem 6.24. *Let* (N, v) *be a* C_1^n*-extendable incomplete game with defined upper vector. Then the game* $(N, \overline{v})$ *is the only extreme game of* $C_1^n(v)$*.*

Proof. First, let us prove it is an extreme game. Following Theorem 1.2, let us consider (N, x) such that both $\overline{v} + x$ and $\overline{v} - x$ are $C_1^n(v)$-extensions. If for any S, the value $x(S) > 0$, then either $\overline{v} + x$ is not a C_1^n-extension (if $S \in \mathcal{K}$, then $v(S) \neq (\overline{v} + x)(S)$) or the complete game is not 1-convex (if $S \notin \mathcal{K}$, then $(\overline{v} + x)(S) = \overline{v}(S) + x(S) = v(N) - b(N \setminus S) + x(S) > v(N) - b(N \setminus S)$.

Further, suppose there is an extreme game (N, y) different from $(N, \overline{v})$. It means there is a coalition $S \notin \mathcal{K}$ such that $y(S) < \overline{v}(S)$. If we take (N, x) such that $x(S) = \overline{v}(S) - y(S)$ and $x(T) = 0$ otherwise, we immediately conclude that both $y + x$ and $y - x$ are in $C_1^n(v)$, and since $x \neq 0$, we conclude by Theorem 1.2 that (N, y) is not an extreme game. □

We further define games (N, e_T) for $T \notin \mathcal{K}$ as

$$e_T(S) := \begin{cases} -1, & \text{if } S = T, \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to see that games $(N, \overline{v} + \alpha e_T)$ are $C_1^n(v)$-extensions for any $\alpha \geq 0$, therefore (N, e_T) are rays of $C_1^n(v)$. Moreover, all but one conditions are satisfied for $(N, \overline{v} + e_T)$ with equality, therefore they are even the extreme rays. The following theorem shows they are the only extreme rays.

Theorem 6.25. *Let (N, v) be a C_1^n-extendable incomplete game with defined upper vector. Then the set of $C_1^n(v)$-extensions can be described as*

$$C_1^n(v) = \left\{ \overline{v} + \sum_{T \notin \mathcal{K}} \alpha_T e_T \mid \alpha_T \geq 0 \right\}.$$

Proof. For a C_1^n-extension (N, w), we show it can we expressed as a combination of the upper game and games (N, e_T) for $T \notin \mathcal{K}$. Since $(N, w) \in C_1^n(v)$, it holds for every $T \notin \mathcal{K}$ that $w(T) \leq \overline{v}(T)$. Therefore, we define $\alpha_T := w(T) - \overline{v}(T)$. Immediately, it follows that

$$(w + \alpha_T e_T)(T) = w(T) - w(T) + \overline{v}(T) = \overline{v}(T).$$

Setting α_T for every $T \notin \mathcal{K}$ in this manner concludes the proof. □

6.3.2 Solution concepts

From the point of view of the τ-value, a game with defined upper vector is equivalent with any of its C_1^n-extensions. This is because, for a complete game v, $\tau(v) = b^v - \frac{g^v(N)}{n}$ and both b^v and $g^v(N)$ depend only on values $v(N)$ and $v(N \setminus i)$ for $i \in N$.

Incomplete games with defined upper vector are a good example for showing that in general, $\tilde{\phi}(v) \neq \phi^\alpha(v)$. From the definition of the conic Shapley value, additivity and S-equivalence axiom, we have

$$\phi^\alpha(v) = \phi(\tilde{v} + \alpha \tilde{e}) = \phi(\tilde{v}) + \alpha \phi(\tilde{e}).$$

Therefore, $\phi^\alpha(v) = \tilde{\phi}(v)$ for $\alpha > 0$ if and only if $\alpha \phi(\tilde{e}) = 0$, which is equivalent to $\phi(\tilde{e}) = 0$. In order to compute the Shapley value, we need to obtain the marginal contributions of player i to all coalitions S, i.e. $\tilde{e}(S \cup i) - \tilde{e}(S)$ for $i \in N$ and $S \subseteq N \setminus i$.

Lemma 6.26. *Let $(N, \mathcal{K}, v)$ be a C_1^n-extendable incomplete game with defined upper vector and for $C_1^n(v)$, let the game $(N, \tilde{e})$ be the centre of gravity of its extreme rays. Then we have*

$$\tilde{e}(S \cup i) - \tilde{e}(S) = \begin{cases} 0, & \text{if } S \in \mathcal{K} \text{ and } S \cup i \in \mathcal{K}, \\ 0, & \text{if } S \notin \mathcal{K} \text{ and } S \cup i \notin \mathcal{K}, \\ \frac{1}{|E|}, & \text{if } S \notin \mathcal{K} \text{ and } S \cup i \in \mathcal{K}, \\ -\frac{1}{|E|}, & \text{if } S \in \mathcal{K} \text{ and } S \cup i \notin \mathcal{K}, \end{cases}$$

where $E = \{T \subseteq N \mid T \in 2^N \setminus \mathcal{K}\}$ and thus $|E| = 2^{|N|} - |\mathcal{K}|$.

Proof. We denote by $\overline{\mathcal{K}}$ the coalitions with unknown values, that is $\overline{\mathcal{K}} := 2^N \setminus \mathcal{K}$. From the definition of $(N, \tilde{e})$, we have

$$\tilde{e}(S \cup i) - \tilde{e}(S) = \frac{1}{|E|} \left(\sum_{T \in \overline{\mathcal{K}}} e_T(S \cup i) - \sum_{T \in \overline{\mathcal{K}}} v(S) \right).$$

Remember, that $e_T(S) = -1$ if and only if $T = S$, otherwise $e_T(S) = 0$. It means that if $S \in \overline{\mathcal{K}}$, the sum $\sum_{T \in \overline{\mathcal{K}}} e_T(S)$ is equal to zero and similarly for $S \cup i$. Let us now distinguish the following cases.

- If $S \in \mathcal{K}$ and $S \cup i \in \mathcal{K}$, we have

$$\left(\sum_{T \in \overline{\mathcal{K}}} e_T(S \cup i) - \sum_{T \in \overline{\mathcal{K}}} v(S) \right) = 0 - 0 = 0.$$

- If $S \in \overline{\mathcal{K}}$ and $S \cup i \in \overline{\mathcal{K}}$, we have

$$\left(\sum_{T \in \overline{\mathcal{K}}} e_T(S \cup i) - \sum_{T \in \overline{\mathcal{K}}} v(S) \right) = -1 - (-1) = 0.$$

- If $S \in \mathcal{K}$ and $S \cup i \in \overline{\mathcal{K}}$, we have

$$\left(\sum_{T \in \overline{\mathcal{K}}} e_T(S \cup i) - \sum_{T \in \overline{\mathcal{K}}} v(S) \right) = 0 - (-1) = 1.$$

- If $S \in \overline{\mathcal{K}}$ and $S \cup i \in \mathcal{K}$, then

$$\left(\sum_{T \in \overline{\mathcal{K}}} e_T(S \cup i) - \sum_{T \in \overline{\mathcal{K}}} v(S) \right) = -1 - 0 = -1.$$

This case analysis concludes the proof. ☐

Lemma 6.27. *Let $(N, \mathcal{K}, v)$ be C_1^n-extendable incomplete game with defined upper vector and for $C_1^n(v)$, game $(N, \tilde{e})$ the centre of gravity of its extreme rays. Then it holds*

$$\phi_i(\tilde{e}) = \frac{1}{|E|n!} \left(\sum_{\substack{S \subseteq N \setminus i \\ S \in \mathcal{K} \\ S \cup i \in \overline{\mathcal{K}}}} s!(n-s-1)! - \sum_{\substack{S \subseteq N \setminus i \\ S \in \overline{\mathcal{K}} \\ S \cup i \in \mathcal{K}}} s!(n-s-1)! \right).$$

Proof. The result immediately follows from the definition of the Shapley value (Definition 1.20) and Lemma 6.26, since $\phi_i(\tilde{e}) = \frac{1}{|E|n!}\sum_{S\subseteq N\setminus i}(\dots)$ and substituting corresponding marginal contributions and dividing the sum into 4 sums according to presence of S and $S\cup i$ in $\mathcal{K}$ yields the formula above. $\square$

From Lemma 6.27, we can conclude that for cooperative games with at most 3 players, $\tilde{\phi}$ and ϕ^α always coincide. However, if we consider games with more players, the two solution concepts differ.

Theorem 6.28. *Let $(N,\mathcal{K},v)$ be a C_1^n-extendable incomplete game with defined upper vector.*

1. *If $|N| \leq 3$, then $\tilde{\phi}(v) = \phi^\alpha(v)$,*
2. *If $|N| \geq 4$ and $\mathcal{K} = 2^N \setminus \{\{i\} \mid i \in N\}$, then $\tilde{\phi}(v) \neq \phi^\alpha(v)$.*

Proof. If $|N| \leq 3$, then if there is $S \notin \mathcal{K}$, it is a singleton coalition $S = \{j\}$. This means, that in the case $S \notin \mathcal{K}$ and $S\cup i \in \mathcal{K}$, the element of the sum is $s!(n-s-1)! = 0$. Also, if we consider the other sum where $T \in \mathcal{K}$ and $T\cup i \notin \mathcal{K}$, the only possibility is $T = \emptyset$, therefore, again $t!(n-t-1)! = 0$. Thus $\phi_i(\tilde{e}) = 0$ for any such game and any player i, leading to coincidence of $\tilde{\phi}$ and ϕ^α.

For $|N| \geq 4$ and $\mathcal{K} = 2^N \setminus \{i \mid i \in N\}$, the coalition S satisfying $S \in \mathcal{K}$ and $S\cup i \notin \mathcal{K}$ is only $S = \emptyset$. for $j \neq k$. Since there are $\binom{N}{2}$ coalitions $\{j,k\}$ for every j and there are n players, we get that the second sum is equal to $n(n-1)1!(n-2)! = n!$. Further, the coalitions S satisfying $S \notin \mathcal{K}$ and $S\cup i \in \mathcal{K}$ are only those satisfying $|S| = n-2$. For every such coalition $s!(n-s-1)! = (n-2)!(n-(n-2)-1)! = (n-2)!$ and there are $2\binom{N}{2} = 2\frac{n!}{(n-2)!}2! = n!$. Therefore, $\tilde{\phi}_i(v) \neq \phi^\alpha(v)$. $\square$

Conclusion

This thesis considers the theory of partially defined cooperative games. In Chapters 1 and 2, we revised necessary preliminaries and introduced basic definitions connected to incomplete games, mostly inspired by the definitions introduced by Masuya and Inuiguchi in [19].

In Chapter 3, we summarized results of the two authors concerning superadditivity. We extended them with two characterisations of S^n-extendability (Theorem 3.1) and provided not yet published proofs for results concerning extreme games of the set of S^n-extensions (Theorem 3.4, Lemma 3.5). These results were stated without proofs in [18].

In Chapter 4, we considered convexity, which has not been investigated before in the theory of partially defined cooperative games. In Section 4.1, we presented a connection to the research of submodular set functions and concluded a characterisation of C^n-extendability (Theorem 4.1) together with an application of the characterisation for a specific class of incomplete games (Theorem 4.2). Further, in Section 4.2, we investigated the set of C^n_σ-extensions, namely the characterisation of C^n_σ-extendability (Theorem 4.5), a description of the lower and the upper game (Theorem 4.7), and a description of the set $C^n_\sigma(v)$, given by (4.7). In Subsection 4.3, we provided a description of the set of C^n-extensions for non-negative incomplete games with minimal information.

Chapter 5 investigates positivity. In Section 5.1, we gave a characterisation of P^n-extendability for general incomplete games (Theorem 5.2) based on duality of linear programming. Also, in Theorem 5.3 we gave an example of polynomial time complexity (in n) of P^n-extendability for a specific class of incomplete games. A simple characterisation of boundedness of $P^n(v)$ (Theorem 5.4) was also proposed. The main result of this chapter is the characterisation of extreme games of $P^n(v)$ (Theorem 5.8) based on an idea similar to the proof of Bondareva-Shapley theorem. This characterisation can be used as a tool to derive extreme games for specific classes of incomplete games (we provide two examples of such classes in Subsection 5.2.1). In Subsection 5.2.2, we considered incomplete games with minimal information. Apart from Subsection 5.2.2, all the results in this chapter are new.

Finally, in Chapter 6, we focused on 1-convexity, a not yet studied property of incomplete games. In Section 6.1, we investigated symmetric extensions – a characterisation for $C^n_{1,\sigma}$-extendability, the upper game and the description of the set $C^n_{1,\sigma}(v)$. Then, a similar analysis was done for two classes of incomplete games – games with minimal information (Section 6.2) and games with defined upper vector (Section 6.3). In these sections, we also study solution concepts. We introduced the average τ-value (Definition 6.1), the conic τ-value (Definition 6.3), the average Shapley value (Definition 6.4), and the conic Shapley value (Definition 6.5). For incomplete games with minimal information, we proved that all these generalisations coincide (Theorems 6.15 and 6.17). Therefore, we called them by a unified name – the average value. Different axiomatisations of the average value were considered (Theorems 6.18, 6.19, 6.20, 6.21). For incomplete games with defined upper vector with at least four players, the average Shapley value and the conic Shapley value do not coincide (Theorem 6.28).

Future research

In the nearest future, we want to focus on other classes of $\mathcal{C}$-extensions, namely the hierarchy of *balanced*, *quasibalanced* and *semibalanced* extensions. We also want to introduce multi-point solution concepts for incomplete games based on an idea that we can take the intersection of the solution concepts of all $\mathcal{C}$-extensions. As the resulting payoff vectors are part of the solution concept of every $\mathcal{C}$-extension, they are good candidates for a distribution of payoff in incomplete games. The author of this thesis will focus on this research during his doctoral studies.

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List of Abbreviations

Cooperative games

(N, v)	cooperative game
Γ^n	class of n-player cooperative games
$\Gamma^n(\mathcal{K})$	class of n-player partially defined cooperative games with $\mathcal{K}$
S^n	superadditive games of n players
C^n	convex games of n players
P^n	positive games of n players
C_1^n	1-convex games
Q^n	quasibalanced games

Incomplete games

$(N, \mathcal{K}, v)$	incomplete game
$C^n(\mathcal{K})$	convex extendable incomplete games with $\mathcal{K}$
$C^n(\mathcal{X})$	symmetric-convex extendable incomplete games with $\mathcal{X}$
$C_1^n(\mathcal{K})$	1-convex extendable incomplete games with $\mathcal{K}$
$Q^n(\mathcal{K})$	quasibalanced extendable incomplete games with $\mathcal{K}$

Extensions

$S^n(v)$	superadditive extensions of $(N, \mathcal{K}, v)$
$C^n(v)$	convex extensions of $(N, \mathcal{K}, v)$
$C_\sigma^n(v)$	symmetric convex extensions of $(N, \mathcal{K}, v)$
$P^n(v)$	positive extensions of $(N, \mathcal{K}, v)$
$P_\sigma^n(v)$	symmetric positive extensions of $(N, \mathcal{K}, v)$
$C_1^n(v)$	1-convex extensions of $(N, \mathcal{K}, v)$
$C_{1,\sigma}^n(v)$	symmetric 1-convex extensions of $(N, \mathcal{K}, v)$

Solution concepts

Payoff vectors

a^v	minimal right vector of game (N, v) (lower vector)
b^v	utopia vector of game (N, v) (upper vector)
λ^v	concession vector

Complete games

$\tau(v)$	the τ-value
$\eta(v)$	the nucleolus
$\phi(v)$	the Shapley value

Incomplete games

$\tilde{\tau}(v)$	the average τ-value
$\tilde{\phi}(v)$	the average Shapley value
$\tilde{\zeta}(v)$	the average value