Charles University
Faculty of Mathematics and Physics

**HABILITATION THESIS**

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# Topological drawings of graphs

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# Preface

This thesis is based on the following collection of papers, studying topological drawings of graphs in the plane and on surfaces. It is a selection of the results obtained since 2013 in Prague and during my postdoctoral stays at Alfréd Rényi Institute of Mathematics in Budapest and at École Polytechnique Féderale de Lausanne.

[P1] M. Balko, R. Fulek and J. Kynčl, Crossing numbers and combinatorial characterization of monotone drawings of $K_n$, *Discrete and Computational Geometry* **53** (2015), Issue 1, 107–143.
https://dx.doi.org/10.1007/s00454-014-9644-z

[P2] J. Kynčl, Simple realizability of complete abstract topological graphs simplified, *Discrete and Computational Geometry* **64** (2020), Issue 1, 1–27.
https://doi.org/10.1007/s00454-020-00204-0

[P3] J. Kynčl, J. Pach, R. Radoičić and G. Tóth, Saturated simple and $k$-simple topological graphs, *Computational Geometry: Theory and Applications* **48** (2015), Issue 4, 295–310.
https://dx.doi.org/10.1016/j.comgeo.2014.10.008

[P4] R. Fulek, J. Kynčl, I. Malinović and D. Pálvölgyi, Clustered planarity testing revisited, *The Electronic Journal of Combinatorics* **22** (2015), Issue 4, P4.24, 29 pp.
https://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i4p24

[P5] R. Fulek, J. Kynčl and D. Pálvölgyi, Unified Hanani–Tutte theorem, *The Electronic Journal of Combinatorics* **24** (2017), Issue 3, P3.18, 8 pp.
https://www.combinatorics.org/ojs/index.php/eljc/article/view/v24i3p18

[P6] R. Fulek and J. Kynčl, Counterexample to an extension of the Hanani–Tutte theorem on the surface of genus 4, *Combinatorica* **39** (2019), Issue 6, 1267–1279.
https://doi.org/10.1007/s00493-019-3905-7

[P7] R. Fulek and J. Kynčl, The $\mathbb{Z}_2$-genus of Kuratowski minors, submitted.
https://arxiv.org/abs/1803.05085
extended abstract in: *Proceedings of the 34th International Symposium on Computational Geometry (SoCG 2018), Leibniz International Proceedings in Informatics (LIPIcs)* 99, 40:1–40:14, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018.
https://dx.doi.org/10.4230/LIPIcs.SoCG.2018.40

Papers [P1–P7] are attached in Appendices A–G, respectively. In particular, Appendix A contains an extended version of [P1], Appendix G contain a full version of [P7], and Appendices B–F contain the the journal versions of papers [P2–P6].

In Chapter 1 we briefly introduce the topic of the thesis and give basic definitions related to drawings of graphs. In Chapters 2, 3 and 4 we overview the main results obtained in [P1–P7].

# Contents

# 1. Introduction

## 1.1 Graphs

A graph is a finite combinatorial structure representing networks and various relations between objects, people or abstract notions. Real-world examples include transportation networks with cities and roads or airline connections between them, computer networks including the Internet, social networks of friends, relatives or coauthors, phylogenetic trees in biology, hierarchical structures of words within a language or a scientific field, patterns of stars in the sky forming abstract figures, and many others.

A graph may sometimes be defined as a relational structure over a finite set with a single antireflexive symmetric binary relation, or, as a certain 1-dimensional topological space. In combinatorics, the following set-theoretic definition is most common.

A *graph* $G$ is an ordered pair $(V, E)$ where $V$ is a finite set and $E$ is a set of unordered pairs of distinct elements from $V$; equivalently, $E$ is a set of 2-element subsets of $V$. Sometimes we also write $V(G)$ for $V$ and $E(G)$ for $E$, especially if the graph is not clear from the context. The elements of $V(G)$ are called the *vertices* and the elements of $E(G)$ the *edges* of $G$. The motivation for this terminology comes from geometry, from vertices and edges of convex polytopes. For simplicity, an edge $\{u, v\}$ is often written as $uv$. The vertices $u, v$ of an edge $uv$ are often called its *endpoints*. Two vertices $u, v$ are called *adjacent* or *neighbors* in $G$ if $uv$ is an edge of $G$. In this case we also say that $u$ and $v$ are *connected* or *joined* by the edge $uv$. A graph $H$ is a *subgraph* of $G$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $H$ is a *complement* of $G$ if $V(H) = V(G)$ and $E(H) = \{uv; \{u, v\} \subseteq V(G), u \neq v, uv \notin E(G)\}$; in other words, the edges of $H$ are those pair of vertices of $G$ that are not edges of $G$. The *degree* of a vertex $v$ in a graph $G$ is the number of neighbors of $v$ in $G$; equivalently, it is also the number of edges of $G$ containing $v$.

We refer to Diestel's textbook [19] for the basics of graph theory.

## 1.2 Drawings of a graph

Graphs often need to be visualized, not just in applications but also for their theoretical investigation. The most common visualization approach is drawing the graph in the plane, and such drawings come in many flavors. Some real-world graphs, such as road netwoks, already have a natural drawing in the plane, while others, such as social networks, do not, and various techniques are used to find a suitable drawing. We now give basic mathematical definitions related to drawings of graphs. See Figure 1.1 for an illustration of some of the notions.

The plane is defined as the 2-dimensional Euclidean space $\mathbb{R}^2$. A *simple curve*, also called a *Jordan arc*, is a continuous injective map $\gamma : [0, 1] \to \mathbb{R}^2$. We say that $\gamma$ *connects* two points $u, v \in \mathbb{R}^2$ if $\gamma(0) = u$ and $\gamma(1) = v$. The curve $\gamma$ is often identified with its image in the plane.

A *drawing* of a graph $G = (V, E)$ in the plane consists of a set of $|V|$ distinct points, each representing one vertex of $G$, and a set of $|E|$ simple curves, a)

$(\{1,2,3,4\}, \{\{1,2\}, \{2,3\}, \{3,1\}, \{3,4\}\})$

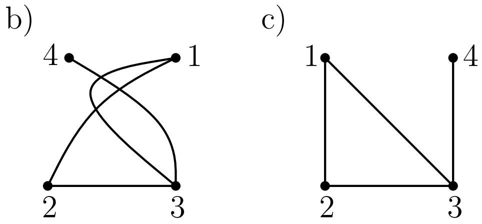


Figure 1.1: Three representations of the same graph: a) an abstract graph $G$, b) a drawing of $G$ in the plane with three crossings, c) a planar embedding of $G$, also a geometric graph.

each representing one edge of $G$, and such that a curve representing the edge $uv$ connects the two points representing the vertices $u$ and $v$. If no confusion is likely to occur, we often talk about the points and curves in the representation as vertices and edges, respectively, and we denote them by the same labels as the vertices and the edges of the graph $G$. We are mostly interested in combinatorial properties of the representation rather than precise positions of the points and the curves. For this reason, we usually require that a drawing of a graph satisfies the following conditions:

(1) the edges pass through no vertices except their endpoints,

(2) every pair of edges has only a finite number of intersection points,

(3) every intersection point of two edges is either a common endpoint or a proper crossing ("touching" of the edges is not allowed), and

(4) no three edges pass through the same crossing.

A *proper crossing*, shortly a *crossing*, of two edges $e, f$ in a drawing is a common intersection point of $e$ and $f$ that is not their common endpoint, and where $e$ passes from one "side" of $f$ to the "other side". A formal definition of a crossing is almost never precisely formulated in the literature on graph drawings. The reason is that general simple curves may be very "wild", which makes defining the two "sides" of a simple curve in a neighborhood of its interior point rather nontrivial, and may require using the Jordan curve theorem or similar topological facts. The Jordan curve theorem states that every simple closed curve (an image of an injective continuous map from the circle) in the plane divides the plane into two conected components, one bounded and the other one unbounded. Although the theorem looks intuitive and almost obvious, all known proofs are quite nontrivial and technically involved. Practically, almost no generality is lost by considering drawings that use only smooth curves or piecewise linear curves, in which case the notion of a crossing may be expressed by requiring the tangent vectors of the two curves at the crossing point be linearly independent. Also the Jordan curve theorem for smooth or piecewise linear curves is significantly easier to prove.

In the literature, a drawing of a graph is also called a *topological graph*. A drawing is called *rectilinear*, *straight-line*, or a *geometric graph*, if every edge is drawn as a straight-line segment. A drawing or a topological graph is called *simple* if every pair of edges have at most one common point, which is either

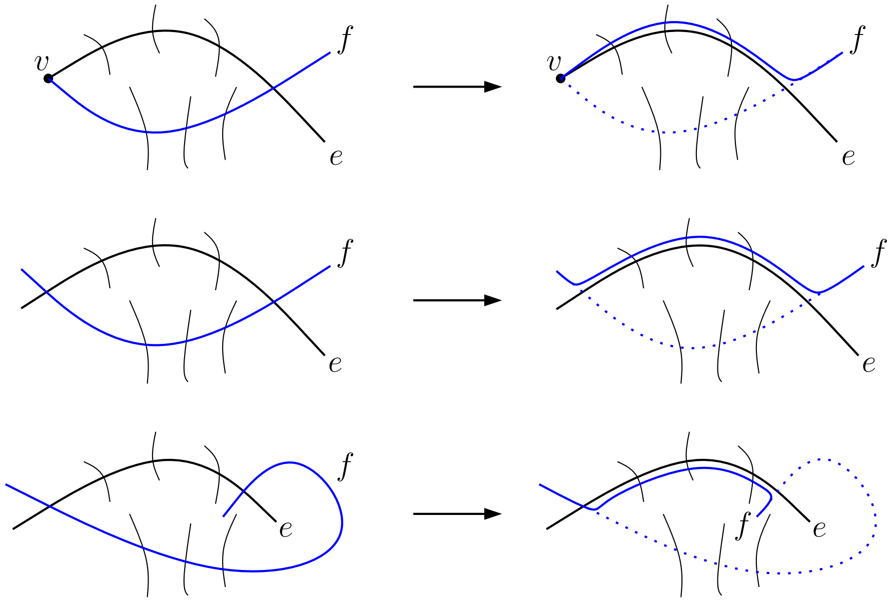


Figure 1.2: Reducing the number of crossings in a drawing where two edges $e$ and $f$ have more than one common point. A portion of $e$ or $f$ between two of their common intersections is redrawn along the other curve. If self-crossings are created, they can be removed easily by another local redrawing; see Figure 1.3

their common endpoint or a proper crossing. Clearly, every straight-line drawing is simple. Moreover, every drawing of a graph with minimum possible number of crossings is simple: if two edges have more than one point in common, one of the edges can be redrawn to decrease the number of crossings; see Figures 1.2 and 1.3.

A large part of the research on graph drawings is motivated by the goal of producing visually pleasing drawings, satisfying various aesthetic criteria. The number of crossings in a drawing is perhaps the most important parameter affecting the readability of a drawing, and thus minimizing the number of crossings in a drawing is one of the central problems in the area. In this sense, the best outcome one can hope for is a drawing with no crossings, which is called a *(planar) embedding*, a *plane drawing*, or a *plane graph*. A graph that has a planar embedding is called a *planar graph*.

In Chapters 2 and 3 we study drawings with crossings. In Chapter 4 we investigate how drawings with crossings relate to embeddings in the plane, and also on other surfaces.

## 1.3 Basic graphs and graph classes

Given a $k$-element set $V = \{v_1, v_2, \ldots, v_k\}$, the graphs $P_k$ and $C_k$ defined by

$$V(P_k) = V,\ E(P_k) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\},$$
$$V(C_k) = V,\ E(C_k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$$

are called a *path* of length $k-1$ and a *cycle* of length $k$, respectively; see Figure 1.4 a) and b). Obviously $P_k$ is a subgraph of $C_k$.

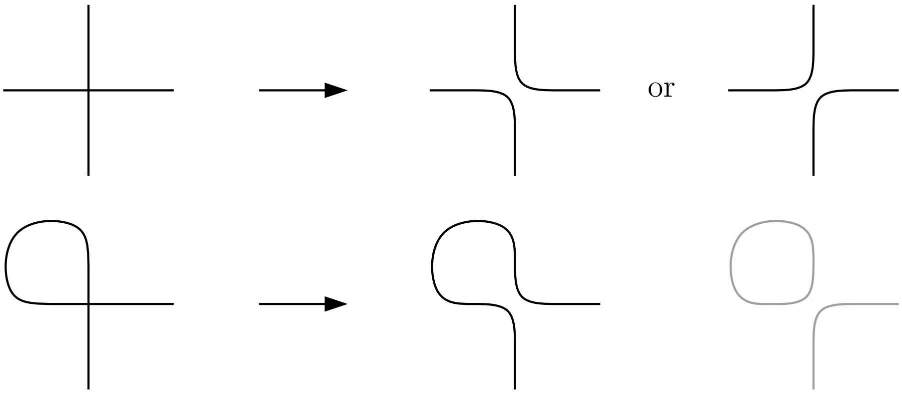


Figure 1.3: Removing a self-crossing of a curve. Exactly one of the two possibilities produces a curve with fewer crossings; the other possibility would produce a curve and a closed curve.

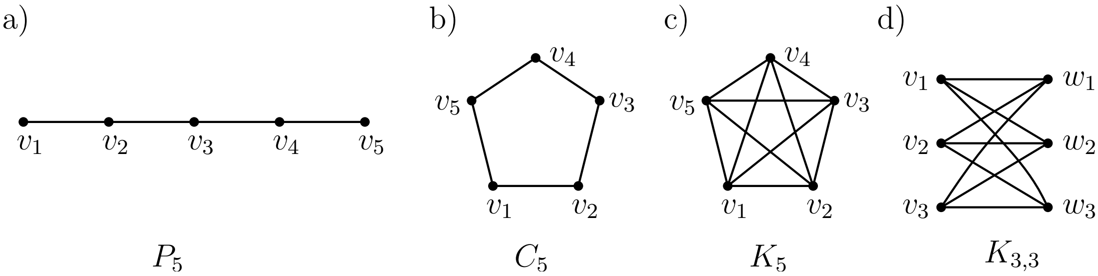


Figure 1.4: Drawings of a few basic graphs: a) the path $P_5$, b) the cycle $C_5$, c) the complete graph $K_5$, d) the complete bipartite graph $K_{3,3}$.

The graph $K_k$ defined by

$$V(K_k) = V,\ E(K_k) = \binom{V}{2} = \{v_iv_j; 1 \le i < j \le k\}$$

is called a *complete graph* with $k$ vertices; it contains all $\binom{k}{2}$ possible edges connecting pairs of vertices in $V$. See Figure 1.4 c).

Let $W = \{w_1, w_2, \dots, w_l\}$ be an $l$-element set disjoint with $V$. The graph $K_{k,l}$ defined by

$$V(K_{k,l}) = V \cup W,\ E(K_{k,l}) = \{v_iw_j; 1 \le i \le k, 1 \le j \le l\}$$

is called a *complete bipartite graph* with parts of size $k$ and $l$; it contains all $k \cdot l$ possible edges with one endpoint in $V$ and the other endpoint in $W$. See Figure 1.4 d).

A graph $G$ is *connected* if every pair of its vertices is contained in a subgraph isomorphic to a path; in other words, for every pair $u, w \in V(G)$, there is a positive integer $k$ and a sequence of distinct vertices $v_1, v_2, \dots, v_k \in V(G)$ such that $v_1 = u$, $v_k = w$, and for every $i \in \{1, 2 \dots, k-1\}$, the vertices $v_i$ and $v_{i+1}$ are joined by an edge in $G$. Maximal connected subgraphs of a graph are called its *components*.

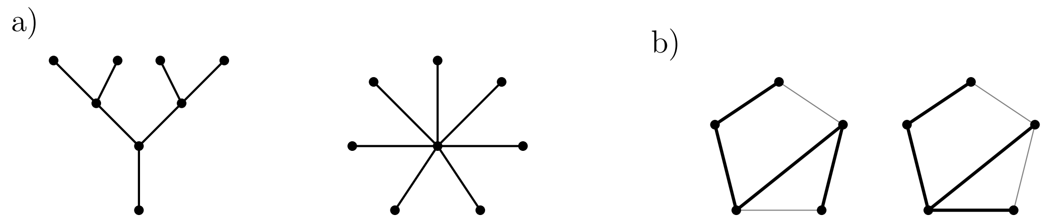


Figure 1.5: a) Examples of trees. b) A connected graph with two of its spanning trees (in bold).

A graph that has no cycle as a subgraph is called *acyclic*, or also a *forest*. A connected acyclic graph is called a *tree*. Clearly, the components of each forest are trees. The paths $P_k$ and complete bipartite graphs $K_{1,l}$ are examples of trees. Every connected graph $G$ has a tree $T$ that contains all the vertices of $G$; such a tree $T$ is called a *spanning tree* of $G$. See Figure 1.5. A vertex of $T$ of degree 1 is called a *leaf*. Often a tree $T$ is considered *rooted*, which means that a single vertex of $T$ is distinguished as the *root* of $T$.

## 1.4 Planar graphs

Examples of planar graphs include all forests, cycles, complete bipartite graphs $K_{2,l}$, or the complete graph $K_4$; it is an easy observation that all of them have an embedding in the plane. Moreover, the graphs of all convex 3-dimensional polytopes are planar as well: this can be seen by projecting the vertices and edges of the polytope on a sphere and then using a stereographic projection to the plane. On the other hand, not all graphs are planar: $K_5$ and $K_{3,3}$ are two such examples, but it is not so straightforward to prove this. Again, a precise argument needs a variant of the Jordan curve theorem.

Besides their aesthetic appeal, planar graphs form an important class in structural graph theory, and have several interesting combinatorial, algebraic or geometric characterizations and properties. We overview some of them here.

### 1.4.1 Faces, Euler's formula and duals

The edges of a planar embedding $\mathcal{E}$ of a graph $G$ divide the plane into several regions, called *faces*. Exactly one of the regions is unbounded, and it is called the *outer face* of $\mathcal{E}$. For example, every planar embedding of a tree has exactly one face—the outer face, every planar embedding of a cycle has exactly two faces, and the two embeddings in Figure 1.5 b) have three faces. In general, a given planar graph may have many different planar embeddings, but the number of their faces is always the same; this follows from Euler's formula for planar graphs.

**Theorem 1.1** (Euler's formula)**.** *If $G$ is a connected planar graph with $v$ vertices and $e$ edges, and a planar embedding of $G$ has $f$ faces, then*

$$v - e + f = 2.$$

*If $G$ is not connected and has $k$ components, then the formula changes to*

$$v - e + f = k + 1.$$

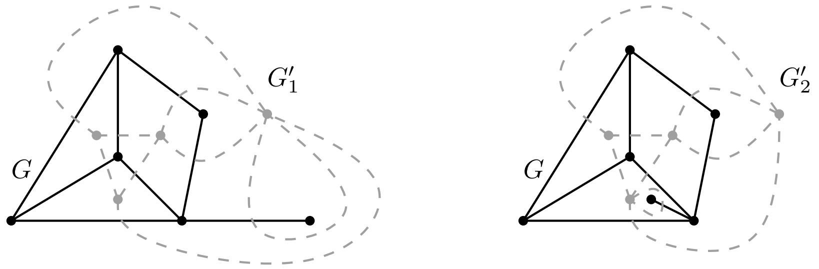


Figure 1.6: Two planar embeddings of a planar graph $G$, and its dual multigraphs $G_1'$ and $G_2'$.

A graph $G' = (V', E')$ is a *dual* of $G = (V, E)$ if there is a one-to-one correspondence between $E$ and $E'$ such that a subgraph $H$ of $G$ is a spanning tree of $G$ if and only if the complement of the subgraph $H' \subseteq G'$ corresponding to $H$ is a spanning tree of $G'$. More generally, the notion of a dual must be extended to *multigraphs*, which allow more than one edge with the same pair of endpoints and also *loops*, which may be regarded as generalized edges whose both endpoints are equal. We do not give a formal definition of multigraphs and refer instead to Figure 1.6, where a graph and its dual multigraph are drawn. A planar graph $G$ has a planar dual $G'$ whose vertices correspond to the faces of a planar embedding of $G$, and $G'$ has a planar embedding where every edge of $G'$ crosses the corresponding edge of $G$. In general, a planar graph may have several different duals, corresponding to different planar embeddings. Whithey showed that planar graphs are the only graphs with a dual.

**Theorem 1.2** (Whitney, 1932 [65]). *A graph is planar if and only if it has a dual multigraph.*

### 1.4.2 Structural characterization

Let $G$ be a graph and let $e = uv$ be an edge of $G$. The operation of *subdividing* the edge $e$ is defined as replacing $e$ with a path $uwv$ where $w$ is a new vertex. This can also be visualized by placing the new vertex $w$ on an interior point of the edge $e$ in a drawing of $G$; see Figure 1.7. A graph obtained from $G$ by several operations of subdividing an edge is called a *subdivision* of $G$. It is an easy observation that for every subdivision $H$ of a graph $G$, the graph $G$ is planar if and only $H$ is planar. In particular, all subdivisions of $K_5$ and $K_{3,3}$ are nonplanar. Kuratowski's theorem shows that these graphs are sufficient to characterize all planar graphs.

**Theorem 1.3** (Kuratowski, 1930 [35]). *A graph $G$ is planar if and only if $G$ contains no subdivision of $K_5$ or $K_{3,3}$ as a subgraph.*

Let $G$ be a graph and let $e = uv$ be an edge of $G$. The operation of *contracting* the edge $e$ is defined as removing the edge $e$ and replacing the vertices $u$ and $v$ with a single vertex $w$ joined by an edge to each neighbor of $u$ and $v$ in $G$ (except $u$ and $v$); see Figure 1.8. A graph $G$ contains $H$ as a *minor* if $H$ can be obtained from $G$ by several operations of deleting vertices, deleting edges, and contracting

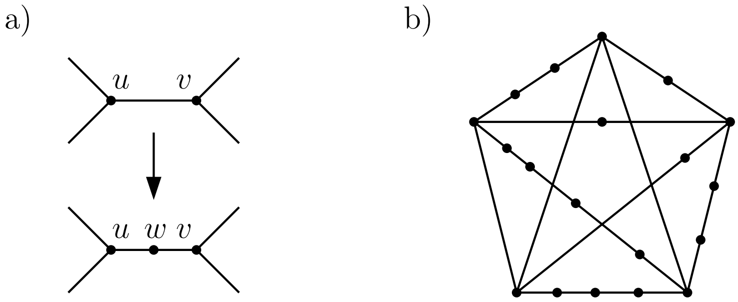


Figure 1.7: a) Subdividing an edge $uv$. b) A subdivision of $K_5$.

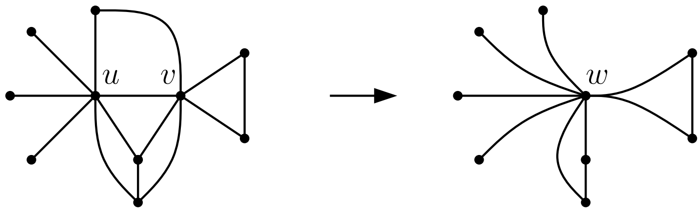


Figure 1.8: Contracting an edge $uv$ to a vertex $w$.

edges. We also say that $H$ is a *minor* of $G$. The following result related to Kuratowski's theorem characterizes planar graphs in terms of forbidden minors.

**Theorem 1.4** (Wagner, 1937 [64]). *A graph $G$ is planar if and only if $G$ contains none of $K_5$ or $K_{3,3}$ as a minor.*

### 1.4.3 Geometric characterization

In general, drawings or embeddings of graphs in the plane may use arbitrarily complicated curves for their edges. For example, it is known that a drawing of $K_8$ minimizing the number of crossings cannot have all its edges drawn as straight-line segments. The following theorem might be a bit surprising if seen for the first time.

**Theorem 1.5** (Fáry, 1948 [20]). *Every planar graph has a straight-line embedding in the plane.*

Fáry's theorem is also a consequence of an even stronger result, known as Koebe–Andreev–Thurston theorem or the circle packing theorem.

**Theorem 1.6** (Koebe, 1936 [31]; Andreev, 1970 [8]; Thurston, 1980 [61]). *The vertices $v \in V(G)$ of any planar graph $G$ can be represented by closed disks $D_v$ in the plane such that $D_u$ and $D_v$ are tangent to each other if $uv \in E(G)$, and disjoint otherwise.*

Fáry's theorem also follows from Steinitz's theorem characterizing planar 3-connected graphs. For $k \geq 2$, we say that a graph is *$k$-connected* if it is connected, has at least $k+1$ vertices, and by removing an arbitrary subset of at most $k-1$ vertices the resulting graph is still connected. For example, the graph of an arbitrary convex 3-dimensional polytope is 3-connected and planar. Steinitz's theorem shows that the converse is true as well.

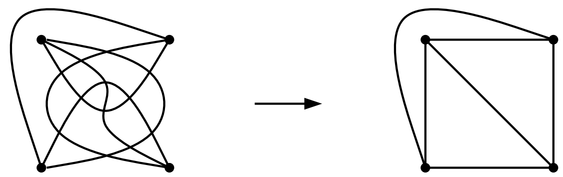

Figure 1.9: An independently even drawing of $K_4$ (left) and its planar embedding guaranteed by the strong Hanani–Tutte theorem (right).

**Theorem 1.7** (Steinitz, 1922 [59]). *Every 3-connected planar graph is a graph of a convex 3-dimensional polytope.*

### 1.4.4 Algebraic characterization

Various algebraic characterizations of planar graphs exist; here we introduce just one of them, which we study in more detail in Chapter 4.

Two edges $ab$ and $cd$ of a graph are called *independent*, also *nonadjacent*, if $\{a, b\} \cap \{c, d\} = \emptyset$; that is, if they do not share any vertex. A drawing of a graph where every pair of independent edges crosses an even number of times is called an *independently even drawing*; see Figure 1.9. Clearly, every planar graph has an independently even drawing in the plane, since every embedding has this property. The strong Hanani–Tutte theorem shows that the converse is true as well.

**Theorem 1.8** (The strong Hanani–Tutte theorem, 1934 [27], 1970 [62]). *A graph $G$ is planar if and only if $G$ has an independently even drawing in the plane.*

*Planarity testing* is the decision problem of determining whether a given graph $G$ is planar. Many algorithms for planarity testing exist; the first linear-time algorithm was published by Hopcroft and Tarjan [30]. However, most of the algorithms are rather complicated. Using the strong Hanani–Tutte theorem, planarity testing can be reduced to solving a system of linear equations over $\mathbb{Z}_2$ [54, Section 1.4.1]. The resulting algorithm is asymptotically slower than the one by Hopcroft and Tarjan, but conceptually very simple.

A drawing of a graph where every pair of edges crosses an even number of times is called an *even drawing*. The *rotation* of a vertex $v$ in a drawing of a graph is the clockwise cyclic order in which the edges incident to $v$ leave the vertex $v$ in the drawing in a small neighborhood of $v$. The collection of the rotations of all vertices in a drawing $D$ is called the *rotation system* of $D$.

**Theorem 1.9** (The weak Hanani–Tutte theorem, 2000+ [15, 45, 48]). *If a graph $G$ has an even drawing $D$ in the plane, then $G$ has an embedding in the plane with the same rotation system as $D$.*

The weak Hanani–Tutte theorem was discovered later than the strong variant, and earned its name because of its stronger assumptions, requiring that all pairs of edges cross evenly rather than just independent pairs. However, the weak variant does not directly follow from the strong variant since its conclusion is stronger.

The strong and weak Hanani–Tutte theorems have many variants; we highly recommend Schaefer's surveys [54, 55] for a comprehensive overview.

# 2. Drawings of complete graphs

## 2.1 Crossing numbers

The *crossing number* of a graph $G$, denoted by $\mathrm{cr}(G)$, is the minimum possible number of crossings in a drawing of $G$ in the plane. The crossing number is an important graph parameter, which has been extensively studied in the literature [63]. Many variants and flavors of the crossing number exist; they are thoroughly explored by Schaefer in his dynamic survey [56]. We mention a few of them in this section.

Despite the popularity and several decades of research on crossing numbers, it may seem surprising that the crossing number of the complete graph $K_n$ is still not known, apart for small values of $n$. According to a famous conjecture by Guy, Harary and Hill [24, 28], which is usually referred to as Hill's conjecture, the crossing number of the complete graph $K_n$ satisfies $\mathrm{cr}(K_n) = Z(n)$ where

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

This conjecture has been verified for $n \leq 10$ by Guy [25], for $n \leq 12$ by Pan and Richter [46], and very recently for $n \leq 14$ by Aichholzer [7]. The conjecture was motivated by two constructions of drawings of $K_n$ with exactly $Z(n)$ crossing: a *cylindrical* and a 2-*page book* drawing [12, 24, 28, 29]; see Figure 2.1. Recently, Ábrego et al. [9] constructed a new family of drawings of $K_n$ with $Z(n)$ crossings, obtained by a small modification of the cylindrical drawings. A large family of drawings of $K_n$ with $Z(n) + O(n^3)$ crossings is obtained by placing the vertices randomly on the sphere and drawing the edges as shortest arcs. Moon [41] showed that the expected value of the number of crossings in such a random spherical drawing is $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$.

For a relatively long time, the best published asymptotic lower bound has been $\mathrm{cr}(K_n) \geq 0.8594Z(n)$, which follows from the lower bound on the crossing number of the complete bipartite graph [32] by an elementary double-counting argument [53]. Very recently, Balogh, Lidický and Salazar [10] improved this to $\mathrm{cr}(K_n) \geq 0.985Z(n)$. All these lower bounds rely on extensive computer calculations.

For rectilinear drawings of $K_n$, Lovász et al. [39] were first to show that their

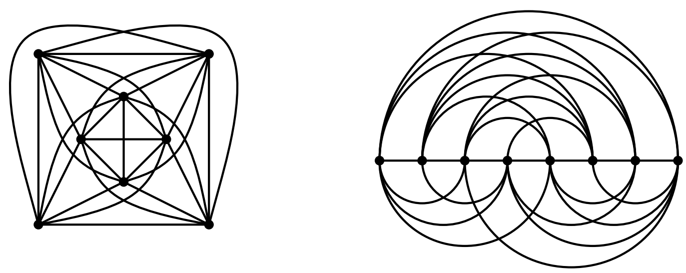

Figure 2.1: A cylindrical and a 2-page book drawing of $K_8$ with $Z(8) = 18$ crossings.

number of crossings is asymptotically strictly larger than $Z(n)$, thus separating the so-called *rectilinear crossing number* of $K_n$ from its crossing number. Determining the rectilinear crossing number of $K_n$ has a rich history, which is explored in a survey by Ábrego, Fernández-Merchant and Salazar [6].

Hill's conjecture has been verified for several special classes of drawings. In a 2-*page book drawing* of a graph, the vertices are placed on a line $l$ and each edge is fully contained in one of the half-planes determined by $l$. Ábrego et al. [2] proved that the number of crossings in every 2-page book drawing of $K_n$ is at least $Z(n)$, matching the upper bound given by the construction of Blažek and Koman [12].

A curve $\alpha$ in the plane is $x$-*monotone* if every vertical line intersects $\alpha$ in at most one point. A drawing of a graph $G$ is called $x$-*monotone* or just *monotone* if every edge is represented by an $x$-monotone curve and no two vertices share the same $x$-coordinate. Clearly, every 2-page book drawing can be transformed into a monotone drawing without creating new crossings, thus the class of monotone drawings is more general. The *monotone crossing number* mon-cr$(G)$ *of a graph* $G$ is the minimum number of crossings in a monotone drawing of $G$.

In our paper **"Crossing numbers and combinatorial characterization of monotone drawings of $K_n$"** (with M. Balko and R. Fulek) [P1, Appendix A], we show that Hill's conjecture is true also for monotone drawings of $K_n$. This was also proved independently by Ábrego et al. [3, 4]. In fact, we show a slightly stronger result, for two other notions of the crossing number.

We call a drawing of a graph *semisimple* if adjacent edges do not cross but independent edges may cross more than once. The *monotone semisimple odd crossing number* of $G$ (called *monotone odd* + by Schaefer [56]), denoted by mon-ocr$_+(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone semisimple drawing of $G$. We call a drawing of a graph *weakly semisimple* if every pair of adjacent edges cross an even number of times, and independent edges may cross arbitrarily. The *monotone weakly semisimple odd crossing number* of $G$, denoted by mon-ocr$_\pm(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone weakly semisimple drawing of $G$. Clearly, mon-ocr$_\pm(G) \leq$ mon-ocr$_+(G) \leq$ mon-cr$(G)$.

Our main result from [P1] is the following.

**Theorem 2.1** ([P1]). *For every $n \in \mathbb{N}$, we have*

$$\text{mon-ocr}_\pm(K_n) = \text{mon-ocr}_+(K_n) = \text{mon-cr}(K_n) = Z(n).$$

Ábrego et al. [4] further extended the class of drawings for which Hill's conjecture is true to so-called *shellable* drawings; we omit their precise definition, but we note that they are better suited to the method used to show the result for monotone drawings. In [P1] we also verify that the results of Theorem 2.1 are still true for shellable drawings and even a slightly more general class of so-called *weakly shellable* drawings.

After the publication of our paper [P1], Hill's conjecture has been verified for a few more classes of drawings further generalizing shellable drawings, including so-called *bishellable* drawings [5], *seq-shellable* drawings [43] and *semi-pair-shellable* drawings [44].

## 2.2 Combinatorial characterization

### 2.2.1 Monotone drawings

In [P1] we also give a combinatorial characterization of several types of monotone drawings of $K_n$ in terms of orientations of their triangles, called *signature functions*, and show that the signature functions of these drawings can be characterized by a finite list of forbidden configurations. In particular, we characterize simple, semisimple and pseudolinear classes of monotone drawings of $K_n$, where *pseudolinear* drawings are those whose edges can be extended to unbounded simple curves that cross each other exactly once, thus forming an arrangement of pseudolines.

**Theorem 2.2** ([P1]). *A collection of triangle orientations is realizable as a signature function of*

1) *a simple monotone drawing of $K_n$ if and only if every subcollection corresponding to a subgraph with 5 vertices is realizable in this way.*

2) *a semisimple monotone drawing of $K_n$ if and only if every subcollection corresponding to a subgraph with 4 vertices is realizable in this way (and also as a simple monotone drawing of $K_4$).*

3) *a pseudolinear monotone drawing of $K_n$ if and only if every subcollection corresponding to a subgraph with 4 vertices is realizable in this way (and also as a rectilinear drawing of $K_4$).*

### 2.2.2 Simple drawings

The results on monotone drawings motivated further research on a similar characterization of general simple drawings of $K_n$. In the paper **"Simple realizability of complete abstract topological graphs simplified"** [P2, Appendix B], we show that such a characterization is indeed possible. However, instead of triangle orientations we use a different combinatorial representation of the drawings.

An *abstract topological graph* (briefly an *AT-graph*), a notion introduced by Kratochvíl, Lubiw and Nešetřil [34], is a pair $(G, \mathcal{X})$ where $G$ is a graph and $\mathcal{X} \subseteq \binom{E(G)}{2}$ is a set of pairs of its edges. We require, in addition, that $\mathcal{X}$ consists only of independent pairs of edges. For a simple topological graph $T$ that is a drawing of $G$, let $\mathcal{X}_T$ be the set of pairs of edges having a common crossing. A simple topological graph $T$ is a *simple realization* of $(G, \mathcal{X})$ if $\mathcal{X}_T = \mathcal{X}$. We say that $(G, \mathcal{X})$ is *simply realizable* if $(G, \mathcal{X})$ has a simple realization.

An AT-graph $(G, \mathcal{X})$ is *complete* if $G$ is a complete graph. An AT-graph $(H, \mathcal{Y})$ is an *AT-subgraph* of an AT-graph $(G, \mathcal{X})$ if $H$ is a subgraph of $G$ and $\mathcal{Y} = \mathcal{X} \cap \binom{E(H)}{2}$. Clearly, a simple realization of $(G, \mathcal{X})$ restricted to the vertices and edges of $H$ is a simple realization of $(H, \mathcal{Y})$.

The main result of [P2] is the following.

**Theorem 2.3** ([P2]). *A complete AT-graph is simply realizable if and only if each of its AT-subgraphs with at most six vertices is simply realizable.*

We also show that AT-subgraphs with five vertices are not sufficient to characterize simple realizability, thus Theorem 2.3 is tight.

Theorem 2.3 implies a straightforward polynomial algorithm for testing the simple realizability of complete AT-graphs, running in time $O(n^6)$ for graphs with $n$ vertices. It is likely that this running time can be improved relatively easily. Compared to the first polynomial algorithm for simple realizability of complete AT-graphs [37], the new algorithm may be more suitable for implementation and for practical applications, such as generating all simply realizable complete AT-graphs of given size or computing the crossing number of the complete graph [16, 42]. On the other hand, the new algorithm does not directly provide the drawing itself, unlike the original algorithm [37]. The explicit list of 102 simply realizable AT-graphs on six vertices was given by Rafla [51], under the assumption that they contain a noncrossing Hamiltonian cycle. Ábrego et al. [1] verified that the assumption is always satisfied and thus the list is complete, and also generated a database of small simple complete topological graphs up to 9 vertices.

### 2.2.3 Parity of crossings in general drawings

In [P2] we also show an analogous characterization for general drawings of complete graphs where only the parity of the number of crossings for each pair of independent edges is specified.

A topological graph $T$ is an *independent $\mathbb{Z}_2$-realization* of an AT-graph $(G, \mathcal{X})$ if $\mathcal{X}$ is the set of pairs of independent edges that cross an odd number of times in $T$. We say that $(G, \mathcal{X})$ is *independently $\mathbb{Z}_2$-realizable* if $(G, \mathcal{X})$ has an independent $\mathbb{Z}_2$-realization.

Clearly, every simple realization of an AT-graph is also its independent $\mathbb{Z}_2$-realization. The converse is not true, since every simple realization of $K_4$ has at most one crossing, but there are independently $\mathbb{Z}_2$-realizable AT-graphs $(K_4, \mathcal{X})$ with $|\mathcal{X}| = 2$ or $|\mathcal{X}| = 3$. Thus, independent $\mathbb{Z}_2$-realizability is only a necessary condition for simple realizability. However, independent $\mathbb{Z}_2$-realizability of arbitrary AT-graphs can be tested in polynomial time since it is equivalent to the solvability of a system of linear equations over $\mathbb{Z}_2$. In contrast, testing simple realizability of arbitrary AT-graphs is an NP-complete problem [33, 37].

Independent $\mathbb{Z}_2$-realizability has been usually considered only in the special case when $\mathcal{X} = \emptyset$; we explore this problem more in Chapter 4. A related concept, the *independent odd crossing number* of a graph $G$, denoted by $\mathrm{iocr}(G)$, measuring the minimum cardinality of $\mathcal{X}$ for which $(G, \mathcal{X})$ has an independent $\mathbb{Z}_2$-realization, has been introduced by Székely [60]. The asymptotic value of $\mathrm{iocr}(K_n)$ is not known, and computing $\mathrm{iocr}(G)$ for a general graph $G$ is NP-complete [50]. See Schaefer's survey [56] for more information.

We call an AT-graph $(G, \mathcal{X})$ *even* (or an *even* $G$) if $|\mathcal{X}|$ is even, and *odd* (or an *odd* $G$) if $|\mathcal{X}|$ is odd. By $2K_3$ we denote the graph that is a disjoint union of two triangles. The following theorem is an analogue of Theorem 2.3 for independent $\mathbb{Z}_2$-realizability.

**Theorem 2.4** ([P2])**.** *A complete AT-graph is independently $\mathbb{Z}_2$-realizable if and only if it contains no even $K_5$ and no odd $2K_3$ as an AT-subgraph.*

Theorem 2.4 again implies a straightforward $O(n^6)$-time algorithm for testing the independent $\mathbb{Z}_2$-realizability of complete AT-graphs with $n$ vertices.

# 3. Simple and k-simple drawings

A considerable amount of knowledge has been obtained about the structure of simple drawings of complete graphs, including the results discussed in the previous chapter. A natural approach for studying simple drawings of general graphs would be extending the simple drawing of a graph to a simple drawing of a complete graph with the same vertex set and applying the results for complete graphs. Unfortunately, such an extension is not possible in general, as shown by the drawing of in Figure 3.1 [36, Figure 1.3], [38, Figure 9].

In the paper **"Saturated simple and $k$-simple topological graphs"** (with J. Pach, R. Radoičić and G. Tóth) [P3, Appendix C], we study the extension problem more broadly and deeply.

We call a simple drawing of a noncomplete graph *saturated* if no edge can be added to the drawing while keeping it simple. The drawing in Figure 3.1 is not saturated, because it can be extended to a simple drawing by adding all the missing edges except $uv$. We are interested in the following question: how few edges can a saturated simple drawing with $n$ vertices have?

We also generalize the concept of a simple drawing as follows. We call a drawing of a graph *$k$-simple* if every pair of edges has at most $k$ points in common; one of them can be their common endpoint, the remaining common points are proper crossings of the two edges. In particular, a 1-simple drawing is a simple drawing. Similarly, we call a $k$-simple drawing of a noncomplete graph *saturated* if no edge can be added to the drawing while keeping it $k$-simple.

As our first main result we show that there exist saturated simple and $k$-simple drawings with only a linear number of edges. We also show that a linear number of edges is necessary.

**Theorem 3.1** ([P3]). *For any positive integers $k$ and $n \geq 4$, let $s_k(n)$ be the minimum number of edges that a saturated $k$-simple topological graph on $n$ vertices can have. Then*

1) *we have $1.5n \leq s_1(n) \leq 17.5n$, and*

2) *for $k \geq 2$ we have $n \leq s_k(n) \leq 16n$.*

We have slightly better upper bounds for $k \geq 3$; for example, we show that $s_k(n) \leq 7n$ for $k \geq 11$. The upper bounds for $k = 1$ and $k = 2$ have been improved by Hajnal et al. [26] to $s_1(n) \leq 7n$ and $s_2(n) \leq 14.5n$, respectively.

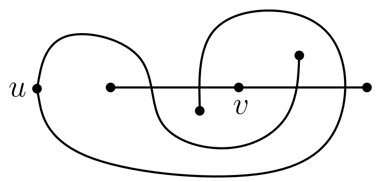


Figure 3.1: A simple drawing of $2P_3$ that cannot be extended by an edge $uv$ while keeping the drawing simple: no face contains both $u$ and $v$ on its boundary, so the edge $uv$ would have to cross some other edge incident with $u$ or $v$, which is forbidden in a simple drawing.

Since there are saturated $k$-simple drawings of noncomplete graphs, it is natural to ask whether a particular missing edge can be added to a given $k$-simple drawing so that it crosses every other edge only $f(k)$ times where $f$ is some function. We show that this is indeed the case, with $f(k) = 2k$, and also that this cannot be improved to $2k - 1$.

**Theorem 3.2** ([P3]). *Let $k$ be a positive integer.*

1) *If $D$ is a $k$-simple drawing of a graph and $u, v$ are vertices of $D$ not connected by an edge, then a curve joining $u$ and $v$ can be added to the drawing such that the resulting drawing is $2k$-simple.*

2) *There is a $k$-simple drawing $D$ of a graph with vertices $u, v$ not connected by an edge such that every curve connecting $u$ and $v$ crosses some edge of $D$ at least $2k$ times.*

We also have a saturation result for the case when we can choose which missing edge to add to a given $k$-simple drawing. Let $k$ and $l$ be positive integers such that $k < l$. A drawing $D$ of a noncomplete graph is $(k, l)$*-saturated* if $D$ is $k$-simple and any curve joining any pair of nonadjacent vertices has at least $l$ points in common with at least one edge of $D$. Obviously, every saturated $k$-simple drawing is $(k, k+1)$-saturated. Thus, Theorem 3.1 implies the existence of $(k, k+1)$-saturated drawings, while Theorem 3.2 1) shows that $(k, 2k+1)$-saturated drawings do not exist. We can show the following.

**Theorem 3.3** ([P3]). *For every positive integer $k$, there exists a $(k, \lceil 3k/2 \rceil)$-saturated drawing of a noncomplete graph.*

# 4. Hanani–Tutte theorems and Z2-embeddings

The strong and weak Hanani–Tutte theorems, stated in the Introduction as Theorem 1.8 and Theorem 1.9, respectively, have many variants and extensions, for drawings in the plane and also on other surfaces. We explore some of the variants in this chapter.

## 4.1 Clustered planarity

Clustered planarity is a variant of the planarity problem for which no polynomial algorithm is known, and it is not known to be NP-hard either. Roughly speaking, an instance of clustered planarity is a graph whose vertices are partitioned into a nested hierarchy of clusters, and the question is whether the graph can be drawn in the plane so that the vertices in the same cluster belong to the same topological disc and no edge crosses the boundary of a particular disc more than once.

More precisely, a *clustered graph* is a pair $(G, T)$ where $G = (V, E)$ is a graph and $T$ is a rooted tree whose set of leaves is $V$. The non-leaf vertices of $T$ represent the clusters, in the following way. Let $C(T)$ be the set of non-leaf vertices of $T$. For each $\nu \in C(T)$, the *cluster* $V(\nu)$ is the set of leaves of the subtree of $T$ rooted at $\nu$. Clearly, every pair of clusters are either disjoint or one contains the other. If $\rho$ is the root of $T$, the *root cluster* $V(\rho)$ contains all the vertices of $G$. A clustered graph $(G, T)$ is *flat* if all non-root clusters are children of the root cluster; that is, if every root-leaf path in $T$ has at most three vertices.

A clustered graph $(G, T)$ is *clustered planar* (or briefly *c-planar*) if $G$ has an embedding in the plane such that

(i) for every $\nu \in C(T)$, there is a topological disc $\Delta(\nu)$ containing all the vertices in $V_\nu$ and no other vertices of $G$,

(ii) if $V(\mu) \subseteq V(\nu)$ then $\Delta(\mu) \subseteq \Delta(\nu)$,

(iii) if $V(\mu_1)$ and $V(\mu_2)$ are disjoint, then $\Delta(\mu_1)$ and $\Delta(\mu_2)$ are internally disjoint, and

(iv) for every $\nu \in C(T)$, every edge of $G$ intersects the boundary of the disc $\Delta(\nu)$ at most once.

A *clustered drawing (or embedding)* of a clustered graph $(G, T)$ is a drawing (or embedding, respectively) of $G$ satisfying (i)–(iv). The word "cluster" is often used for both the topological disc $\Delta(\nu)$ and the subset of vertices $V(\nu)$.

In the paper **"Clustered planarity testing revisited"** (with R. Fulek, I. Malinović and D. Pálvölgyi) [P4, Appendix D], we extend the Hanani–Tutte theorem to several cases of the clustered planarity problem.

We call a clustered graph $(G, T)$ *two-clustered* if the root of $T$ has exactly two children, $A$ and $B$, and every vertex of $G$ is a child of either $A$ or $B$ in $T$. In other words, $A$ and $B$ are the only non-root clusters and they form a partition of the vertex set of $G$. Obviously, two-clustered graphs form a subclass of flat clustered graphs.

First we extend both the weak and the strong variant of the Hanani–Tutte theorem to two-clustered graphs.

**Theorem 4.1** ([P4]). *If a two-clustered graph $(G, T)$ has an even clustered drawing $\mathcal{D}$ in the plane then $(G, T)$ is c-planar. Moreover, $(G, T)$ has a clustered embedding with the same rotation system as $\mathcal{D}$.*

**Theorem 4.2** ([P4]). *If a two-clustered graph $(G, T)$ has an independently even clustered drawing in the plane then $(G, T)$ is c-planar.*

Theorem 4.1 has been recently generalized by Fulek to the case of strip planarity [22].

A clustered graph $(G, T)$ is *c-connected* if every cluster of $(G, T)$ induces a connected subgraph of $G$. We prove a strong Hanani–Tutte theorem for c-connected clustered graphs.

**Theorem 4.3** ([P4]). *If a c-connected clustered graph $(G, T)$ has an independently even clustered drawing in the plane then $(G, T)$ is c-planar.*

Similarly to other variants of the Hanani–Tutte theorem, as a byproduct of Theorem 4.2 and Theorem 4.3 we immediately obtain a polynomial-time algorithm for testing c-planarity in these special cases. Faster but more complicated algorithms were known before.

On the other hand, we show examples of clustered graphs with more than two disjoint clusters that are not c-planar, but admit an even clustered drawing. This shows that a straightforward extension of Theorem 4.1 and Theorem 4.2 to flat clustered graphs with more than two clusters is not possible.

**Theorem 4.4** ([P4]). *For every $k \geq 3$ there exists a flat clustered cycle with $k$ clusters that is not c-planar but has an even clustered drawing in the plane.*

After the publication of our paper [P4] we have learned that the same example as the one in Theorem 4.4 was found earlier by Repovš and Skopenkov [52] in the related context of approximations of maps by embeddings.

## 4.2 Unified Hanani–Tutte theorem

In the paper **"Unified Hanani–Tutte theorem"** (with R. Fulek and D. Pálvölgyi) [P5, Appendix E], we introduce a common generalization of the strong Hanani–Tutte theorem and the weak Hanani–Tutte theorem, which seems to have been overlooked in the literature.

**Theorem 4.5** (Unified Hanani–Tutte theorem [P5]). *Let $G$ be a graph and let $W \subseteq V(G)$. Let $\mathcal{D}$ be a drawing of $G$ where every pair of edges that are independent or have a common endpoint in $W$ cross an even number of times. Then $G$ has a planar embedding where the rotations of vertices from $W$ are the same as in $\mathcal{D}$.*

By setting $W = \emptyset$ in Theorem 4.5 we obtain the strong Hanani–Tutte theorem, while $W = V(G)$ gives the weak variant.

Theorem 4.5 directly follows from the proof of the strong Hanani–Tutte theorem by Pelsmajer, Schaefer and Štefankovič [48]. We give a new, slightly simpler proof by induction, based on case distinction of the connectivity of $G$ and using the weak Hanani–Tutte theorem as a base case. Our proof of Theorem 4.5 also gives an alternative proof of the strong Hanani–Tutte theorem, by reducing it to the weak variant for 3-connected graphs. We also show an extension of Theorem 4.5 to multigraphs.

The unified Hanani–Tutte theorem became an important tool in our further research. For example, it may be used to simplify the proof of Theorem 4.2. It also found an unexpected application to drawings of graphs on surfaces, which we discuss in the next section.

## 4.3 Drawings on surfaces

We recommend the monograph by Mohar and Thomassen [40] for a detailed introduction into surfaces and graph embeddings. By a *surface* we mean a connected compact 2-dimensional topological manifold. Every surface is either *orientable* (has two sides) or *nonorientable* (has only one side). Every orientable surface $S$ is obtained from the sphere by attaching $g \geq 0$ *handles*, and this number $g$ is called the *genus* of $S$. Similarly, every nonorientable surface $S$ is obtained from the sphere by attaching $g \geq 0$ *crosscaps*, and this number $g$ is called the *(nonorientable) genus* of $S$. The simplest orientable surfaces are the sphere (with genus 0) and the torus (with genus 1). The simplest nonorientable surfaces are the projective plane (with genus 1) and the Klein bottle (with genus 2). We denote the orientable surface of genus $g$ by $M_g$ and the nonorientable surface of genus $g$ as $N_g$.

The *Euler characteristic* of a surface $S$ of genus $g$, denoted by $\chi(S)$, is defined as $\chi(S) = 2 - 2g$ if $S$ is orientable, and $\chi(S) = 2 - g$ if $S$ is nonorientable. The *Euler genus* $\mathrm{eg}(S)$ of $S$ is defined as $2 - \chi(S)$. In other words, the Euler genus of $S$ is equal to the genus of $S$ if $S$ is nonorientable, and to twice the genus of $S$ if $S$ is orientable.

A *drawing* and an *embedding* of a graph $G$ on a surface $S$ are defined analogously as a drawing and an embedding in the plane. The *embedding scheme* of a drawing $\mathcal{D}$ on a surface $S$ consists of a rotation at each vertex and a signature $+1$ or $-1$ assigned to every edge, representing the parity of the number of crosscaps the edge is passing through. If $S$ is orientable, the signature of every edge is $+1$, thus the embedding scheme is determined by the rotation system in this case.

The *genus* $g(G)$ of a graph $G$ is the minimum $g$ such that $G$ has an embedding on $M_g$. The $\mathbb{Z}_2$-*genus* of a graph $G$ is the minimum $g$ such that $G$ has an independently even drawing on $M_g$. The *Euler genus* $\mathrm{eg}(G)$ of $G$ is the minimum $g$ such that $G$ has an embedding on a surface of Euler genus $g$. The *Euler* $\mathbb{Z}_2$-*genus* $\mathrm{eg}_0(G)$ of $G$ is the minimum $g$ such that $G$ has an independently even drawing on a surface of Euler genus $g$.

Cairns and Nikolayevsky [15] extended the weak Hanani–Tutte theorem to every orientable surface. Pelsmajer, Schaefer and Štefankovič [49] extended it further to every nonorientable surface.

**Theorem 4.6** (The weak Hanani–Tutte theorem on surfaces [15, Lemma 3], [49, Theorem 3.2])**.** *If a graph $G$ has an even drawing $\mathcal{D}$ on a surface $S$, then $G$ has an embedding on $S$ that preserves the embedding scheme of $\mathcal{D}$.*

Pelsmajer, Schaefer and Stasi [47] extended the strong Hanani–Tutte theorem to the projective plane, using the list of forbidden minors. Colin de Verdière et al. [18] recently provided an alternative proof, which does not rely on the list of forbidden minors.

**Theorem 4.7** (The (strong) Hanani–Tutte theorem on the projective plane [18, 47])**.** *If a graph $G$ has an independently even drawing on the projective plane, then $G$ has an embedding on the projective plane.*

Whether the strong Hanani–Tutte theorem can be extended to some other surface than the plane or the projective plane has been an open problem. Schaefer and Štefankovič [57] showed that a minimal counterexample to the strong Hanani–Tutte theorem on any surface must be 2-connected.

In the paper **"Counterexample to an extension of the Hanani–Tutte theorem on the surface of genus 4"** (with R. Fulek) [P6, Appendix F], we give a negative answer to the problem for the orientable surface of genus 4.

**Theorem 4.8** ([P6])**.** *There is a graph of genus 5 that has an independently even drawing on $M_4$.*

Theorem 4.8 disproves a conjecture of Schaefer and Štefankovič [57, Conjecture 1] that the $\mathbb{Z}_2$-genus of a graph is equal to its genus; but the question whether the Euler $\mathbb{Z}_2$-genus of a graph is equal to its Euler genus remains open.

As a base step in the construction, we use a counterexample to the extension of the unified Hanani–Tutte theorem on the torus.

**Theorem 4.9** ([P6])**.** *There is a graph $G$ with the following two properties.*

1) *The graph $G$ has an independently even drawing $\mathcal{D}$ on the torus, with a set $W$ of four vertices such that every pair of edges with a common endpoint in $W$ crosses an even number of times.*

2) *There is no embedding of $G$ on the torus with the same rotations of the vertices in $W$ as in $\mathcal{D}$.*

In our proof of Theorem 4.9 the graph $G$ is isomorphic to $K_{3,4}$. The graph in Theorem 4.8 is obtained by attaching three stars $K_{1,4}$ to a sufficiently large grid.

Using the additivity of the genus [11] and the $\mathbb{Z}_2$-genus [57] of a graph over its components, by taking the disjoint union of the graph from Theorem 4.8 with $k$ copies of $K_5$ we obtain a counterexample to an extension of the strong Hanani–Tutte theorem on an arbitrary orientable surface of genus larger than 4. Moreover, by taking $k$ disjoint copies of the graph from Theorem 4.8, we obtain a separation of the genus and the $\mathbb{Z}_2$-genus by a multiplicative factor of $5/4$.

**Corollary 4.10** ([P6])**.** *For every positive integer $k$ there is a graph of genus $5k$ and $\mathbb{Z}_2$-genus at most $4k$.*

Very recently, Fulek, Pelsmajer and Schaefer [23] proved that the strong Hanani–Tutte theorem extends to the torus.

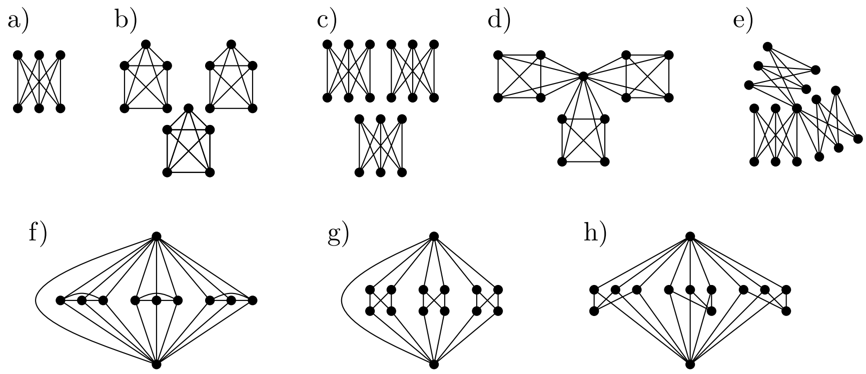


Figure 4.1: The eight 3-Kuratowski graphs.

### 4.3.1 Genus and Z2-genus of graphs

Schaefer and Štefankovič [57] asked whether the genus of a graph can be bounded by a function of its $\mathbb{Z}_2$-genus, which may be considered as an "approximate" version of the strong Hanani–Tutte theorem for orientable surfaces. They also posed an analogous question for the Euler genus.

**Problem 1** ([57])**.** *Is there a function $f$ such that $\mathrm{g}(G) \leq f(\mathrm{g}_0(G))$ for every graph $G$? Is there a function $f$ such that $\mathrm{eg}(G) \leq f(\mathrm{eg}_0(G))$ for every graph $G$?*

In the paper **"The $\mathbb{Z}_2$-genus of Kuratowski minors"** (with R. Fulek) [P7, Appendix G], we give a conditional positive answer, following from an unpublished Ramsey-type result.

A graph is called a *$t$-Kuratowski graph* if it is one of the following: $K_{3,t}$, or $t$ copies of $K_5$ or $K_{3,3}$ sharing at most 2 common vertices. See Figure 4.1 for an illustration.

The following Ramsey-type statement for graphs of large Euler genus is a folklore unpublished result.

**Claim 4.11** (Robertson–Seymour [13, 58], unpublished)**.** *There is a function $g$ such that for every $t \geq 3$, every graph of Euler genus $g(t)$ contains a $t$-Kuratowski graph as a minor.*

For 7-connected graphs, Claim 4.11 follows from the result of Böhme, Kawarabayashi, Maharry and Mohar [13], stating that for every positive integer $t$, every sufficiently large 7-connected graph contains $K_{3,t}$ as a minor. Böhme et al. [14] later generalized this to graphs of larger connectivity and $K_{a,t}$ minors for every fixed $a > 3$. Fröhlich and Müller [21] gave an alternative proof of this generalized result.

Christian, Richter and Salazar [17] proved a similar statement for graph-like continua.

For a positive integer $n$ we denote the set $\{1, \ldots, n\}$ by $[n]$. Let $r, s \geq 3$. The *projective $r \times s$ grid* is the graph with vertex set $[r] \times [s]$ and edge set

$$\{\{(i,j),(i',j')\}; |i-i'|+|j-j'|=1\} \ \cup \ \{\{(i,1),(r+1-i,s)\}; i \in [r]\}.$$

In other words, the projective $r \times s$ grid is obtained from the planar $r \times (s+1)$ grid by identifying pairs of opposite vertices and edges in its leftmost and rightmost column. See Figure 4.2, left.

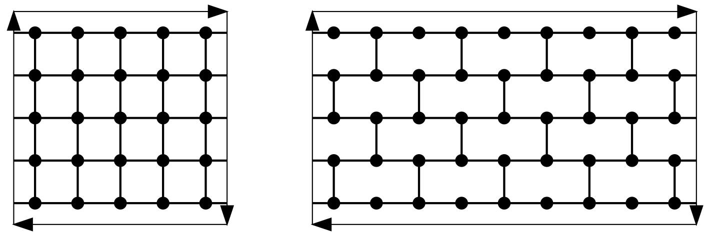

Figure 4.2: Left: a projective $5 \times 5$ grid. Right: a projective 5-wall.

For an odd $t \geq 3$, a *projective t-wall* is obtained from the projective $t \times (2t-1)$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for $i$ odd and $1 \leq j \leq t-1$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for $i$ even and $1 \leq j \leq t$. Similarly, for an even $t \geq 4$, a *projective t-wall* is obtained from the projective $t \times 2t$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for $i$ odd and $1 \leq j \leq t$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for $i$ even and $1 \leq j \leq t$. The projective $t$-wall has maximum degree 3 and can be embedded on the projective plane as a "twisted wall" with inner faces bounded by 6-cycles forming the "bricks", and with the "outer" face bounded by a $(4t-2)$-cycle for $t$ odd and a $4t$-cycle for $t$ even. See Figure 4.2, right.

As an almost direct consequence of Claim 4.11, we obtain an analogous Ramsey-type statement for graphs of large genus.

**Theorem 4.12** ([P7]). *Claim 4.11 implies that there is a function $h$ such that for every $t \geq 3$, every graph of genus $h(t)$ contains, as a minor, a $t$-Kuratowski graph or the projective $t$-wall.*

As our main result in [7] we complete a proof that the $\mathbb{Z}_2$-genus of each $t$-Kuratowski graph and the projective $t$-wall grows to infinity with $t$; in fact, the $\mathbb{Z}_2$-genus of each of these graphs is equal to their genus. Analogously, we also show that the Euler $\mathbb{Z}_2$-genus of each $t$-Kuratowski graph is equal to its Euler genus. Schaefer and Štefankovič [57] proved this for those $t$-Kuratowski graphs that consist of $t$ copies of $K_5$ or $K_{3,3}$ sharing at most one vertex. For the projective $t$-wall, the result follows directly from the weak Hanani–Tutte theorem on orientable surfaces [15, Lemma 3] (Theorem 4.6): indeed, all vertices of the projective $t$-wall have degree at most 3, therefore pairs of adjacent edges crossing oddly in an independently even drawing can be redrawn in a small neighborhood of their common vertex so that they cross evenly, and the weak Hanani–Tutte theorem can be applied. Thus, the remaining cases are $K_{3,t}$ and $t$ copies of $K_5$ or $K_{3,3}$ sharing 2 vertices; we refer to them as $t$-Kuratowski graphs of type a), f), g) and h) as in Figure 4.1.

**Theorem 4.13** ([P7]). *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each $t$-Kuratowski graph of type a), f), g) and h) is equal to its genus, and also its Euler $\mathbb{Z}_2$-genus is equal to its Euler genus. In particular,*

a) $\mathrm{g}_0(K_{3,t}) \geq \lceil (t-2)/4 \rceil$, $\mathrm{eg}_0(K_{3,t}) \geq \lceil (t-2)/2 \rceil$, *and*

b) *if $G$ consists of $t$ copies of $K_5$ or $K_{3,3}$ sharing a pair of adjacent or nonadjacent vertices, then $\mathrm{g}_0(G) \geq \lceil t/2 \rceil$ and $\mathrm{eg}_0(G) \geq t$.*

Combining Theorem 4.13 with the result of Schaefer and Štefankovič [57] and the simple argument for the projective $t$-wall we obtain the following result.

**Corollary 4.14** ([P7]). *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each $t$-Kuratowski graph and the projective $t$-wall is equal to its genus, and the Euler $\mathbb{Z}_2$-genus of each $t$-Kuratowski graph is equal to its Euler genus.*

Combining Corollary 4.14 with Theorem 4.12 we get the following implication.

**Corollary 4.15** ([P7]). *Claim 4.11 implies a positive answer to both parts of Problem 1.*

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