Report on Jan Kynčl's Habilitation Thesis

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In response to the request of Prof. Zdeněk Doležal dated February 7, 2022, I present this report on Jan Kynčl's Habilitation Thesis. I will begin with some general remarks, then some comments on each of the seven presented papers, and conclude with my summary of the *Turnitn* check.

I have met Jan Kynčl once at a conference, have corresponded with him a few times, and he served as external examiner for one of my recent doctoral students. My desire to have him be the external examiner is a result of the very high regard that I and my colleagues have for him and his work. As a young researcher, he has already been a very influential force in the graph drawing community, covering quite a broad range of topics. As witnessed by the prestigious journals in which his works appear, he is esteemed as one bringing novel ideas into a challenging field.

In the last decade or so there have been several successful efforts to: (a) characterize classes of planar drawings of graphs in combinatorial ways; and (b) comparing different ways of counting "crossings". There have been many people providing examples of such theorems, but there is little doubt that few have been as successful as Kynčl on both topics.

In the context of crossing number problems, a drawing of the graph G (usually in the plane, but other surfaces have been considered) consists of distinct points in the plane, one for each vertex of G, and a simple arc for each edge e of G that has as its ends the two points representing the vertices of G incident with e, but the arc does not contain any other vertex point. Normal requirements include that any two arcs representing edges intersect only finitely often and that there are no points of tangency between such arcs. The arcs may be chosen as smooth or piecewise linear, but this point is rarely explicitly mentioned.

A drawing is *simple* if any two edges intersect in at most one point, either a common incident vertex or a crossing point. The drawing is *x-monotone*, or simply *monotone*, if the x-coordinates of the vertices are all different and every edge-arc α can be described by a function $f : [0, 1] \to \alpha$ that is (strictly) monotonic in the x-coordinate.

The *crossing number* $\mathrm{cr}(D)$ of a drawing D of a graph G is the number of crossings of pairs of edges. (In particular, three edges crossing at the same point counts as three crossings as there are three pairs of edges. Also, if two edges cross multiple times, each crossing is counted separately.) The *crossing number* is

$$\operatorname{cr}(G) := \min\{\operatorname{cr}(D) : D \text{ is a drawing of } G\}\,.$$

An important long-standing conjecture is that of Hill on the crossing number of the complete graph K_n, namely:

$$\operatorname{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor\,.$$

It is usual to set $H(n)$ (the "Hill number") to be the right-hand side of the above equation.

It is an easy theorem to see that any drawing D of G for which $\operatorname{cr}(D) = \operatorname{cr}(G)$ is simple. Thus, the "simple crossing number" is the same as "the crossing number". A more difficult question that has been answered by Pach and Tóth is that "monotone crossing number" can be strictly larger than "crossing number".

On the other hand, if one allows edges to cross multiple times, the "pair crossing number" counts only the number of pairs of edges that cross instead of all the crossings of all the pairs. It is known (and was somewhat surprising at the time) that $\operatorname{pcr}(G)$ can be strictly smaller than $\operatorname{cr}(G)$.

The Hanani–Tutte Theorem asserts that if there is a drawing of G in which every pair of edges crosses an even number of times, then $\operatorname{cr}(G) = 0$. (The converse is obvious.) This result may be expressed differently. The "odd pair crossing number" counts only the number of pairs of edges that cross an odd number of times. Thus, Hanani–Tutte asserts that $\operatorname{opcr}(G) = 0 \Leftrightarrow \operatorname{cr}(G) = 0$.

These represent the main topics of the papers involved in this Habilitation Thesis. I will now turn to remarks on the individual papers.

(1) Crossing numbers and combinatorial characterization of monotone drawings of K_n, with M. Balko and R. Fulek

This work characterizes several variations of monotone drawings of complete graphs and using that characterization to prove that the monotone crossing number of K_n is equal to $H(n)$. This improves an earlier result that the 2-page crossing number of K_n is $H(n)$: every 2-page drawing is homeomorphic to a monotone drawing.

Here, more is proved: the "monotone odd-crossing number" of K_n is also equal to $H(n)$.

The characterization of monotone drawings of the complete graphs is quite interesting. There has long been studies of "ordered point sets", in which every triple $\{x, y, z\}$ is assigned one of its two cyclic orderings (x, y, z) and (x, z, y). For example, in a simple drawing of K_n, every three vertices induce a 3-cycle in K_n, which in turn corresponds in the drawing to a simple closed curve consisting of the vertices and edges of the 3-cycle. Traversing the boundary of the bounded side of this curve in a clockwise orientation provides one of the two cyclic triples, giving the ordered point set.

The question is: which ordered point sets correspond to monotone drawings of K_n? This question is very fully answered here for "semisimple" monotone drawings and for simple monotone drawings. The very satisfying answer is in terms of the natures of the orderings of the 3-subsets of each 4-subset of the original n points and the 3-subsets of each 5-subset.

Going to an important even more restricted class of graphs, they similarly characterize when an ordering can be realized by a pseudolinear drawing (each edge arc can be extended to a 2-way infinite curve in the plane so that any two such curves intersect exactly once).

This is a lovely paper proving results that are very interesting to the general graph drawing community.

(2) Simple realizability of complete abstract topological graphs simplified

This paper is arguably the most important paper about simple drawings of the complete graphs that has ever been written.

This work and its "unsimplified" predecessor were the ones that really brought Professor Kynčl to my attention. A *rotation system* on n symbols is a family of n cyclic permutations, the i^{th} one being on the symbols $\{1, 2, \ldots, n\} \setminus \{i\}$. These arise naturally for drawings of the complete graph in the plane: at the i^{th} vertex, the $n-1$ edges coming out from i form a cyclic sequence according to the label at the other end. A rotation system arising from a simple drawing is *realizable*.

The main theorem is: if $\mathcal{R}$ is a rotation system on $n \geq 6$ points such that, for every 6 points from the n the corresponding subrotation is realizable as a simple drawing, then $\mathcal{R}$ is realizable as a simple drawing.

About 10 years ago, I was asked about the possibility of such a theorem and my response was, "It surely is not true." Therefore, I was quite surprised that Kynčl was able to prove this theorem. Moreover, the proof in the included paper introduces a beautiful new homotopy tool to apply to planar drawings of K_n.

(3) Saturated simple and k-simple topological graphs, with J. Pach, R. Radoičić, and G. Tóth

Janós Pach and Geza Tóth have been long-time collaborators in the graph drawing community, proving many interesting theorems and raising important questions. It is natural that they would be interested in collaborating with a very strong young researcher such as Kynčl.

Here the topic is: can we add an edge to a drawing of a graph to get a drawing of a larger graph? If we allow arbitrary drawings, then the answer is trivially yes. For simple drawings, it has recently been shown by Arroyo, Derka, and Parada that it is NP-complete to determine if a particular edge can be added to the drawing.

For Kynčl et al, the question is a little different: are there examples for which none of the missing edges can be added? These are the "saturated" examples and their existence follows from the long-known fact that sometimes some edge cannot be added. That there are saturated graphs on n vertices with as few as $\mathrm{O}(n)$ edges is the principal result.

Moreover, such examples exist for each k, with k being the limit to the number of times two edges can cross. In this context, they prove that, in such a circumstance, any missing edge can be added with at most

$2k$ crossings of any other edge and that $2k$ is the best possible upper bound.

As one might expect, the methods involve finding appropriate clever building blocks and putting them together in clever ways.

This article is not going to be the final word on this topic, but is a very substantial advance over previous work.

(4) Clustered planarity testing revisited, with R. Fulek, I. Malinović, and D. Pálvölgyi

I really don't know anything about clustered planarity. It looks like a very interesting topic, but I am unable to evaluate the significance of this work.

(5) Unified Hanani–Tutte theorem, R. Fulek and D. Pálvölgyi

The Hanani–Tutte Theorems (weak and strong) have been getting a lot of useful attention lately. The earlier works of Pelsmajer, Schaefer, and Štefankovič reopened this topic with some very interesting results. In one version, they proved that if D is a drawing in which every pair of edges crossed an even number of times, then there is a planar embedding in which the rotations at the vertices is the same as in D. Their proof was relatively simple.

Despite the adjectives "weak" and "strong", neither version of Hanani–Tutte implies the other. This elegant paper starts with that observation, and then takes the Pelsmajer et al proof and makes it somewhat easier by turning it into an induction that proves a generalization that contains both the weak and strong versions as special cases.

This is the version that will be taught well into the future.

(6) Counterexample to an extension of the Hanani–Tutte theorem on the surface of genus 4, with R. Fulek

See the next paper for comments.

(7) The $\mathbb{Z}_2$-genus of Kuratowski minors, with R. Fulek

I will take these last two papers together, as they make a very nice pair that provide a wonderful contribution to the Hanani–Tutte problems. Up to this point, we have been discussing only planar drawings. In these two works, the ambient space becomes a surface (such as the torus or Klein bottle).

Each graph G has three different different values that represent the simplest surface upon which G has an embedding: the orientable genus $\gamma(G)$; the non-orientable genus $\widetilde{\gamma}(G)$; and the Euler genus $\overline{\gamma}(G)$. Not surprisingly:

- $\gamma(G)$ is the smallest number of handles needed to add to the sphere in order for G to have an embedding in the resulting *orientable* surface;
- $\widetilde{\gamma}(G)$ is the smallest number of crosscaps needed to add to the sphere in order for G to have an embedding in the resulting *non-orientable* surface; and
- $\overline{\gamma}(G) = \min\{2\gamma(G), \widetilde{\gamma}(G)\}$.

Likewise, we can define the $\mathbb{Z}_2$-*genera* $\gamma_2(G)$, $\widetilde{\gamma}_2(G)$, and $\overline{\gamma}_2(G)$ as the smallest genus surfaces in with G has a drawing in which two independent edges cross an even number of times.

The fundamental question is: for each $\delta \in \{\gamma, \widetilde{\gamma}, \overline{\gamma}\}$ and for each graph G, is $\delta_2(G) = \delta(G)$? Evidently, $\delta_2(G) \leq \delta(G)$. The sphere (= the plane) has all genera equal to 0 and Pelsmajer, Schaefer, and Stasi proved the strong version (independent edges cross an even number of times) for the projective plane (non-orientable $\widetilde{\gamma}(G) = 1$).

Pelsmajer, Schaefer, and Šefankovič have proved the weak version of these questions: if EVERY pair of edges cross an even number of times, then the corresponding genera are equal and the embedding preserves the "combinatorial type" of the drawing (rotations and 1- or 2-sidedness of each cycle).

Fulek and Kynčl's two papers combine nicely to provide very interesting bookends towards the general situation. In "Counterexample" they give an example of a graph G for which $\gamma_2(G) = 4 < 5 = \gamma(G)$, and, with an easy extension, that, for every integer $k \geq 1$, there is a graph G_k such that $\gamma_2(G_k) = 4k < 5k = \gamma(G_k)$.

Like most examples, this involves a clever construction that, once explained, is quite easy to understand. The proof involves standard ideas for the area, but one needs the very bright idea to get going.

A principal take-away from this work is that, if there is a function $f(n)$ such that, for every graph G, $\gamma(G) \leq f(\gamma_2(G))$, then $f(n) \geq \frac{5}{4}n$.

In "The $\mathbb{Z}_2$-genus" article, Fulek and Kynčl provide (modulo an old unpublished result) a function f as in the preceding paragraph. One can easily obtain graphs of growing (in t) genus by piecing together t copies of either $K_{3,3}$ or K_5 by identifying 0, 1 or 2 vertices in each copy. Additionally, the complete bipartite graph $K_{3,t}$ has genus that grows with t.

Robertson and Seymour long ago announced that they proved that if $\{G_n\}$ is any set of graphs such that $\overline{\gamma}(G_n) \to_n \infty$, then the largest t_n such that G_n contains one of the above graphs for $t = t_n$ goes to infinity as well. There have been related results published, but this specific result does not appear anywhere.

Essentially, in this last paper, Fulek and Kynčlshow that these special graphs have equal genera and Euler genera. Together with the Robertson and Seymour "result", one sees that there is a function f such that $\gamma(G) \leq f(\gamma_2(G))$ and $\overline{\gamma}(G) \leq f(\gamma_2(G))$.

The arguments for the equality of the genera are homological and quite non-trivial. Although they make use of earlier similar discussions, there is a wealth of useful techniques developed here for future use. From my own experience, I know that these arguments are quite subtle and difficult to express accurately.

I am very impressed with this work, which superficially seems very specialized. Anyone interested in the interplay between homology theory and graph drawings will benefit from carefully reading this work.

Comments on TURNITIN report

The “TURNITIN” report has found many sources that contain similar or the same content. This is hardly surprising, as many of the highlighted paragraphs are occurring in the papers that are included with the thesis. The rest are simply the natural use of common symbols, words, and phrases in the particular area of mathematics covered by these works. There is no question in my mind that the seven presented works are completely original and carefully make appropriate reference to previous works whose results are mentioned and/or used.