FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Tomáš Garaj

Random Dynamical Systems

Department of Probability and Mathematical Statistics

Supervisor of the bachelor thesis: prof. RNDr. Bohdan Maslowski, DrSc.
Study programme: Mathematics
Study branch: General Mathematics

Prague 2023

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

First and foremost, I would like to thank my supervisor, prof. RNDr. Bohdan Maslowski, DrSc., for introducing me to the field of random dynamical systems and its many intricacies and for his guidance during the creation of this thesis. I would also like to express my gratitude to my family for their unrelenting support and for their patience.

Title: Random Dynamical Systems

Author: Tomáš Garaj

Department: Department of Probability and Mathematical Statistics

Supervisor: prof. RNDr. Bohdan Maslowski, DrSc., Department of Probability and Mathematical Statistics

Abstract: Random differential equations are differential equations whose right-hand side contains a random noise. In most applications that noise is modelled by a stochastic process of certain properties or a metric dynamical system. In this thesis we examine random differential equations and find out under which conditions an equation through its solution generates a random dynamical system. To be able to consider a wider variety of functions on the right-hand side of the equation we employ the method of Lyapunov functions, obtaining less restrictive conditions than the ones normally presented. In the latter portion of the thesis we introduce the field of random attractors, present a theorem from literature regarding the conditions for the existence of a random attractor and formulate and prove our own version that is more closely related to the theory we concerned ourselves with before.

Keywords: Carathéodory conditions, random differential equations, explosion, Lyapunov functions, random dynamical systems, cocycle, random attractors

Contents

Introduction

A dynamical system is a mapping describing the evolution of a point in a state space in time. A random dynamical system is a dynamical system enriched by an element of random noise. The theory of random dynamical systems was developed in great detail by L. Arnold in [1]. This thesis will focus mainly on the generation of a random dynamical system as a solution of a random differential equation.

The conditions stated in [1] for a solution to a random differential equation are very restrictive. We will try to relax them using the method of Lyapunov functions.

The method of Lyapunov functions, named after Russian mathematician Aleksandr Mikhailovich Lyapunov, is a method used in the study of stability of differential equations. We take inspiration in [2], where this problematic is described in the deterministic case, and develop similar theory for random ordinary differential equations.

In the first chapter we give a brief overview of the theory of Carathéodory solutions for deterministic systems of differential equations which we use at a later point to create probabilistic analogies.

The focus of the second chapter is finding sufficient conditions for the existence of a local unique solution to a random differential equation. The latter part of this chapter introduces the concept of explosion and provide the conditions for avoiding it and thus guaranteeing a global existence of a solution using the method of Lyapunov functions.

The third chapter contains a summary of the essential definitions regarding random dynamical systems. We quote a theorem from [1] about the generation of random dynamical systems from random differential equation and provide our own improved version for a special case using the results obtained in Chapter 2.

The fourth and final chapter is dedicated to introducing the reader to essential definitions from the field of random attractors. We state a sufficient condition for the existence of a random attractor for a random dynamical system and formulate and prove our own version which corresponds to the results presented in the first three chapters.

Notation

Here we give a brief overview of some of the symbols used in this thesis.

The function $\|\cdot\|_n$ is the Euclidean norm on $\mathbb{R}^n$.
As the dimension of the space will be mostly clear from context, we will omit the lower index and write only $\|\cdot\|$.

$B_n(x, R)$ denotes the closed ball on $\mathbb{R}^n$ with center x and radius R, i.e. the set $\{y \in \mathbb{R}^n : \|x - y\|_n \leq R\}$.
For similar reasons as above, we will omit the lower index denoting the dimension of the space.

$\mathbf{L}^1_{\text{loc}}(\mathbb{R})$ denotes the space of locally integrable real functions, i.e. functions which satisfy the following property:

$$\int_a^b \|f(t)\| dt < \infty$$

for each $-\infty < a < b < \infty$.

1. Deterministic Results

This chapter contains a variety of definitions and theorems which serve to generalise the notion of a solution to a differential equation. The results presented here are mostly drawn from [3] and [4].

1.1 Local Carathéodory Solutions

Consider the differential equation $\dot{x} = f(x,t)$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $n \in \mathbb{N}$.

Normally, for a solution to exist, the equation is required to have a continuous right-hand side. This condition is often too restrictive and does not cover many important cases. For this purpose we introduce the so-called Carathéodory solution to a differential equation and the aptly named Carathéodory conditions for its local existence.

Definition 1. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $I \subset \mathbb{R}$ be an interval and let $t_0 \in I$. A function $x : I \to \mathbb{R}^n$ is called a solution of equation

$$\dot{x} = f(x,t) \tag{1.1}$$

on I with the initial condition

$$x(t_0) = x_0 \tag{1.2}$$

in the sense of Carathéodory, if

- x is absolutely continuous on each compact interval $J \subset I$,
- $(t, x(t)) \in \mathbb{R}^{n+1}$ for each $t \in I$,
- $\frac{d}{dt}x(t) = f(x(t),t)$ almost everywhere on I.

In this thesis we will exclusively deal with Carathéodory solutions. Thus from now on whenever we use the term solution, we mean solution in the sense of Carathéodory.

Remark. For definition and properties of absolute continuity in the multidimensional case, see e.g. [4].

Remark. If x is the solution of equation (1.1) with the initial condition (1.2) in the sense of Definition 1 then for every $t \in I$ it satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(x(t),t)dt.$$

Definition 2. A function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is said to satisfy the Carathéodory conditions, if

- $f(\cdot, t)$ is continuous for almost all fixed $t \in \mathbb{R}$,
- $f(x, \cdot)$ is measurable for all fixed $x \in \mathbb{R}^n$,

- for each $(t_0, x_0) \in \mathbb{R}^{n+1}$ there exist positive constants a, b and a function $m \in \mathbf{L}_1([t_0 - a, t_0 + a])$ such that $||f(x,t)|| \leq m(t)$ for each $(x,t) \in Q = \{(t,x) : |t - t_0| \leq a, ||x - x_0|| \leq b\}$.

Theorem 1. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfy the Carathéodory conditions. Then for every $(x_0, t_0) \in \mathbb{R}^{n+1}$ the initial value problem* (1.1), (1.2) *has a solution on some interval $I \subset \mathbb{R}^n$ containing t_0.*

Proof. The proof can be found in [4] as the proof of Theorem 18.4.2., pp. 332-335. □

In addition to its existence we are often interested whether the solution obtained from the equation (1.1) with the initial condition (1.2) is unique in the following sense:

Whenever x, y are solutions to (1.1) on intervals $I,\ J \subset \mathbb{R}$ respectively such that $x(t_0) = y(t_0)$, for some $t_0 \in I \cap J$ then $x(t) = y(t)$ for each $t \in I \cap J$.

The Carathéodory conditions are sufficient to ensure existence of a local solution. However, if we want the solution to be unique we need to work with additional requirements on the function f, namely its local Lipschitz continuity in x.

Theorem 2. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfy the Carathéodory conditions. Suppose in addition that for each $R > 0$ there exists a locally integrable function L_R such that*

$$||f(x_1,t) - f(x_2,t)|| \leq L_R(t)||x_1 - x_2|| \tag{1.3}$$

for every $t \in \mathbb{R}$ and for each $x_1, x_2 \in B(0,R)$.
Then the solution of (1.1), (1.2) *exists for every $(x_0, t_0) \in R^{n+1}$ on some interval $I \subset \mathbb{R}, t_0 \in I$, and is unique.*

Proof. The proof can be found in [4] as proof of Theorem 18.4.13., p. 337. □

1.2 Extension theorems

In this section we formulate some theorems which will later help us find sufficient conditions for global existence of solution.

Theorem 3. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfy the conditions of Theorem 2 and let $x(t)$ be the solution of* (1.1) *with the initial condition* (1.2) *on some interval I containing t_0. Then x can be uniquely extended to its maximal interval of existence of the form (ω_-, ω_+), with $\omega_- \in [-\infty, \infty)$ and $\omega_+ \in (-\infty, \infty]$. Moreover, for any given compact set $K \subset \mathbb{R}^{n+1}$ there exists a time t_K such that $(t, x(t)) \notin K$ for every $t > t_K$.*

Proof. The proof can be found in [5] as proof of Theorem 6, pp. 70-71. □

Corollary. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfy the conditions of Theorem 2 let $x(t)$ be the solution of (1.1) with the initial condition (1.2) on a right maximal interval of existence $I = [t_0, \omega_+)$.
Then either $\omega_+ = \infty$, or $\omega_+ < \infty$ and

$$\lim_{t \to \omega_+} \|x(t)\| = \infty$$

.

2. Random Ordinary Differential Equations

In the whole chapter we assume $(\Omega, \mathcal{A}, \mathbb{P})$ to be a probability space.

2.1 Local Solutions

The main focus of this section will be examining the results from Chapter 1 in a probabilistic setting. We will introduce the concept of a random differential equation and formulate and prove several theorems pertaining to local existence and uniqueness of its solution.

Definition 3. Let $T \subset \mathbb{R}$. A family $\{\xi(t), t \in T\}$ of random variables mapping $(\Omega, \mathcal{A}, \mathbb{P})$ into a measurable space $(E, \mathcal{E})$ is called a stochastic process.
For a fixed $\omega \in \Omega$ the function $x(\cdot, \omega)$ is called the trajectory or the sample path of the process.

In this thesis we will exclusively assume $T \subset \overline{\mathbb{R}}$ to be an interval and E to be $\mathbb{R}^l$, $l \in \mathbb{N}$, with its Borel σ-algebra.

Consider the following problem: Let $t_0 \in \mathbb{R}$ and let $\{\xi(t), t \geq t_0\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^l$, $l \in \mathbb{N}$. Let $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ be a Borel-measurable function.
We call an equation of the form

$$\dot{x} = g(x, t, \xi(t, \omega)) \tag{2.1}$$

with the initial condition

$$x(t_0, \omega) = x_0(\omega) \tag{2.2}$$

a random ordinary differential equation.

Since in this thesis we will work solely with random ordinary differential equations, we will omit the word ordinary in the sequel.

The definition of a solution to a random differential equation is not as straightforward as in the deterministic case, since the interval of existence of a potential solution is generally random and thus different for each $\omega \in \Omega$. We can however prove a similar property as in Theorem 1 when we examine the random differential equation pathwise, i.e. separately for each fixed $\omega \in \Omega$.

Definition 4. Let $t_0 \in \mathbb{R}$ and let $x_0(\omega) \in \mathbb{R}^k$ for each $\omega \in \Omega$. We say that the function $x(t, \omega)$ is the solution to the random differential equation (2.1) with the initial condition (2.2) if for each $\omega \in \Omega$ there exists an interval $I(\omega)$, $t_0 \in I(\omega)$ such that

$$x(t, \omega) = x_0(\omega) + \int_{t_0}^{t} g(x(s, \omega), s, \xi(s, \omega))ds \tag{2.3}$$

for each $t \in I(\omega)$.

We will now search for sufficient conditions under which the equation (2.1) with the initial condition (2.2) has a solution, i.e. satisfies (2.3), on some interval $[t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$, where $\alpha : \Omega \to \mathbb{R}_+$ is a random mapping. For this endeavor we will present an adaptation of the Carathéodory conditions from Definition 2 in the following theorem. First, we will quote the so-called Schauder Fixed Point Theorem which will be useful in its proof.

Theorem 4. *Let $\mathcal{K}$ be a nonempty, closed, convex and bounded subset of a normed linear space X. Assume that $F : \mathcal{K} \to X$ is a compact operator and $F(\mathcal{K}) \subset \mathcal{K}$. Then there is a fixed point of F in $\mathcal{K}$.*

Proof. The proof can be found in [6] as proof of Theorem 5.2.5, p. 254. □

Theorem 5. *Let $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ satisfy the following conditions:*

- *$g(\cdot, t, \cdot)$ is continuous for each fixed $t \in \mathbb{R}$*
- *$g(x, \cdot, z)$ is measurable for each fixed $(x, z) \in \mathbb{R}^{k+l}$*
- *for each $(t_0, x_0) \in \mathbb{R}^{n+1}$ there exist positive constants a, b and function $m(t, \omega)$, $\int_{t_0-a}^{t_0+a} m(t, \omega) < \infty$ for each $\omega \in \Omega$, such that*

$$||f(x, t, \xi(t, \omega))|| \leq m(t, \omega)$$

  for each $(x, t) \in Q = \{(t, x) : |t - t_0| \leq a, ||x - x_0|| \leq b\}$.

Then for each $\omega \in \Omega, (t_0, x_0(\omega)) \in \mathbb{R}^{k+1}$ there exist a function $x(t, \omega)$ and a random variable $\alpha(\omega)$ such that (2.3) holds true for each $t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$.

Proof. Let us fix $\omega \in \Omega$, let $(t_0, x_0(\omega)) \in \mathbb{R}^{k+1}$ and denote

$$M_\omega = \{u(\omega) \in \mathcal{C}([t_0 - \alpha(\omega), t_0 + \alpha(\omega)]) : ||u(t, \omega) - x_0(\omega)|| \leq \beta(\omega))\}$$

for some $\beta(\omega) > 0$, where $\alpha(\omega)$ is to be determined.
Next we shall define the operator $T = T_\omega$ on M_ω by the formula:

$$(Tu(\omega))(t) = x_0(\omega) + \int_{t_0}^{t} g(u(s, \omega), s, \xi(s, \omega))ds$$

for $t \in [t_0 - \alpha(\omega), t_0 - \alpha(\omega)]$.
Let us compute

$$\begin{aligned}
\sup_{\{|t-t_0| \leq \alpha(\omega)\}} ||(Tu(\omega))(t) - x_0(\omega)|| &\leq \sup_{\{|t-t_0| \leq \alpha(\omega)\}} \left| \int_{t_0}^{t} ||g(u(s, \omega), s, \xi(s, \omega))||ds \right| \\
&\leq \int_{t_0-\alpha(\omega)}^{t_0+\alpha(\omega)} ||g(u(s, \omega), s, \xi(s, \omega))||ds \\
&\leq \int_{t_0-\alpha(\omega)}^{t_0+\alpha(\omega)} m(t, \omega) \\
&\leq \beta(\omega)
\end{aligned}$$

for $\alpha(\omega)$ sufficiently small.
Thus we have shown that $T : M_\omega \to M_\omega$.

To examine the continuity of T in u, let us choose $\{u_n(\omega)\} \subset M_\omega, u(\omega) \in M_\omega$ such that

$$u_n \to u \text{ in } \mathcal{C}([t_0 - \alpha(\omega), t_0 + \alpha(\omega)]), n \to \infty.$$

We shall employ the following estimate:

$$\begin{aligned}\|Tu_n(\omega) - Tu(\omega)\| &= \sup_{\{|t-t_0| \le \alpha(\omega)\}} \left| \int_{t_0}^{t} g(u_n(s,\omega), s, \xi(s)) - g(u(s,\omega), s, \xi(s)) ds \right| \\ &\le \int_{t_0-\alpha(\omega)}^{t_0+\alpha(\omega)} \|g(u_n(s,\omega), s, \xi(s)) - g(u(s,\omega), s, \xi(s))\| ds\end{aligned}$$

Using the fact that $g(x,t,z)$ is continuous in x, for fixed $(t,z) \in \mathbb{R}^{l+1}$, and Lebesgue theorem, we obtain that the last integral in the inequality above converges to zero as $n \to \infty$.
Finally, let us choose a sequence $\{u_n(\omega)\} \subset M_\omega$. Then for each $\varepsilon > 0$ there exists a $\delta(\omega)$ that for every $s,\ t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$ (we can assume, without loss of generality, that $s < t$)

$$\begin{aligned}\|(Tu_n(\omega))(t) - (Tu_n(\omega))(s)\| &\le \int_s^t \|g(u_n(\tau,\omega), \tau, \xi(\tau,\omega)\| d\tau \\ &\le \int_s^t m(\tau,\omega) d\tau < \varepsilon,\end{aligned} \tag{2.4}$$

whenever $|t - s| < \delta(\omega)$.

By (2.4) we have shown that $\{Tu_n(\omega)\} \subset M_\omega$ is a uniformly bounded and equicontinuous family in $\mathcal{C}([t_0 - \alpha(\omega), t_0 + \alpha(\omega)])$ and from the theorem of Arzelà-Ascoli follows that it has a uniformly convergent subsequence. $\{Tu_n(\omega)\}$ is hence precompact which makes T a compact operator. We can then use Schauder's Fixed Point Theorem to obtain that T has a fixed point in M_ω.
It is easy to see that the fixed-point obtained this way satisfies (2.3) for each $t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$.
This conclude the proof as ω has been chosen arbitrarily. □

Remark. The function $x(t,\omega)$ obtained in Theorem 5 is a Carathéodory solution in the sense of Definition 1 of the equation (2.1) with the initial condition (2.2) for each fixed $\omega \in \Omega$.

We will now explore the uniqueness of the function obtained in Theorem 5. By the remark above, the function $x(t,\omega)$ is a solution of the deterministic equation (2.1) for each fixed $\omega \in \Omega$ and thus we can define uniqueness for $x(t,\omega)$ the same way as in Chapter 1 for each fixed $\omega \in \Omega$.

In the proof of the theorem guaranteeing uniqueness, we will make use of the following lemma, often called Grönwall's inequality.

Lemma 6. *Let K be an interval, $\eta > 0$, $s \in K$, $\rho, \xi : K \to \mathbb{R}$. Let the functions ρ, ξ be continuous and $\rho(t) \ge 0$, $\xi(t) > 0$ for each $t \in K$. Let for each $t \in K$*

$$\xi(t) \le \eta + \left| \int_s^t \rho(u)\xi(u) du \right|.$$

Then we have

$$\xi(t) \leq \eta exp\left|\int_s^t \rho(u)du\right|$$

for each $t \in K$.

Proof. The proof can be found in [4] as proof of Lemma 4.3.1., p. 82. □

Remark. The function ρ in Lemma 6 does not have to be continuous, we can instead require that it is locally integrable.

Theorem 7. *Let* $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ *satisfy the conditions of Theorem 5. Assume in addition that* g *satisfies a local Lipschitz condition, i.e. there exists a stochastic process* $B_R(t,\omega), R > 0$ *with locally integrable trajectories such that*

$$\|g(x_2, t, \xi(t,\omega) - g(x_1, t, \xi(t,\omega)\| \leq B_R(t,\omega)\|x_2 - x_1\| \tag{2.5}$$

whenever $x_1, x_2 \in B(0,R)$.
Then for each $\omega \in \Omega, (t_0, x_0(\omega)) \in \mathbb{R}^{k+1}$ *there exist a uniquely determined function* $x(t,\omega)$ *and a random variable* $\alpha(\omega)$ *such that* (2.3) *holds true for each* $t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$.

Proof. The existence of such function is guaranteed by Theorem 5. It remains to show that it is given uniquely.
Let us fix $\omega \in \Omega$ and let $(t_0, x_0(\omega)) \in \mathbb{R}^{k+1}$. Assume there exist $u_1(t,\omega), u_2(t,\omega)$ satisfying (2.3) on the interval $[t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$.
Since there exists R_0 such that $[t_0 - \alpha(\omega), t_0 + \alpha(\omega)] \subset B(0, R_0)$ then for each $t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$ we have

$$\begin{aligned}
\|u_1(t,\omega) - u_2(t,\omega)\| &= |x_0(\omega) + \int_{t_0}^t g(u_1(s,\omega), s, \xi(s,\omega))ds \\
&\quad - x_0(\omega) - \int_{t_0}^t g(u_2(s,\omega), s, \xi(s,\omega))ds| \\
&\leq \left|\int_{t_0}^t \|g(u_1(s,\omega), s, \xi(s,\omega)) - g(u_2(s,\omega), s, \xi(s,\omega))\|ds\right| \\
&\leq \left|\int_{t_0}^t B_{R_0}(s,\omega)\|u_1(s,\omega) - u_2(s,\omega)\|ds\right| \\
&\leq \eta + \left|\int_{t_0}^t B_{R_0}(s,\omega)\|u_1(s,\omega) - u_2(s,\omega)\|ds\right|
\end{aligned} \tag{2.6}$$

for each $\eta > 0$.
Applying Grönwall's inequality to (2.6), we obtain

$$\|u_1(t,\omega) - u_2(t,\omega)\| \leq \eta \exp\left|\int_{t_0}^t B_{R_0}(s,\omega)ds\right|. \tag{2.7}$$

From (2.7) follows that

$$u_1(t,\omega) = u_2(t,\omega)$$

for each $t \in [t_0 - \alpha(\omega), t_0 + \alpha(\omega)]$ as η can be chosen to be arbitrarily small.
By extending the method above to all $\omega \in \Omega$ we obtain the proposition of the theorem. □

2.2 Sufficient Conditions for Non-explosion

In this section we introduce the concept of explosion and formulate sufficient conditions for the solution of a random differential equation to avoid it.
We illustrate the so-called explosion in the following example.

Example. Consider the one-dimensional case

$$\dot{x} = \phi(t,\omega)x^2$$

with the initial condition

$$x(t_0,\omega) = x_0(\omega) > 0$$

where

$$\int_{t_1}^{t_2} \phi(t,\omega)dt < \infty$$

for each $\omega \in \Omega$ and $t_0 \leq t_1 < t_2 < \infty$ and $\phi(t,\omega) > 0$ for each $\omega \in \Omega$.
By separating $x(t,\omega)$ and integrating both sides of the equation with respect to t we obtain

$$-\frac{1}{x(s,\omega)} = \Phi(s,\omega) + c,$$

where $\Phi(s,\omega) = \int_{t_0}^{s} \phi(t,\omega)dt$.

Using the initial condition we can easily calculate that

$$c = -\frac{1}{x_0(\omega)}.$$

Then we can express

$$x(s,\omega) = \frac{1}{\frac{1}{x_0(\omega)} - \Phi(s,\omega)}.$$

From this expression we can clearly see that the function $x_\omega(s)$ for a fixed ω generally escapes to infinity in finite time τ which satisfies $\Phi(\tau,\omega) = \frac{1}{x_0(\omega)}$.

Our goal in this section is to find such requirements on the function g to avoid cases such as the one described in the example above. One possible approach can be seen in the following theorem from [2].

Theorem 8. *Let $\xi(t,\omega)$ be a separable stochastic process with values in $\mathbb{R}^l$, $t_0 \in \mathbb{R}$ and let $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ be a Borel-measurable function satisfying the following conditions:*

- *There exists a stochastic process $B(t,\omega)$ with locally integrable trajectories such that for all $x_1,\ x_2 \in \mathbb{R}^k$*

$$\|g(x_2,t,\xi(t,\omega)) - g(x_1,t,\xi(t,\omega))\| \leq B(t,\omega)\|x_2 - x_1\|.$$

- $\mathbb{P}\left\{\int_0^T \|g(0,t,\xi(t,\omega))\| < \infty\right\} = 1$ *for every $T > 0$.*

Then the equation (2.1) *with the initial condition $x(t_0,\omega) = x_0(\omega)$ has a unique solution $x(t,\omega)$ for each $t \in [t_0,\infty)$.*

Proof. The proof can be found in [2] as proof of Theorem 1.5, p. 9. □

This theorem gives us fairly concise conditions for determining whether a global solution exists. However, the global Lipschitz continuity requirement turns out to be too strict and fails to cover many cases of practical importance, e.g. functions polynomial in x of a higher degree than 1.

In this section we will introduce an alternative approach using the method of Lyapunov functions. This method for exploring stability of differential equations in the deterministic case is fairly well established and its results can be found for example in [2].

The following part will be dedicated to adapting those results for the random case. First we start with the definition of the Lyapunov operator:

Definition 5. Let $V \in \mathcal{C}^1(\mathbb{R}^k \times (t_0, \infty))$ for some $t_0 \in \mathbb{R}$. Then we define the action of a Lyapunov operator associated with the random differential equation (2.1) by the following formula:

$$\begin{aligned}(\mathcal{L}^z V)(x,t) &= \frac{\partial V}{\partial t}(x,t) + \sum_{i=1}^{k} g_i(x,t,z)\frac{\partial V}{\partial x_i}(x,t) \\ &= \frac{\partial V}{\partial t}(x,t) + \langle g(x,t,z), (\nabla V)(x,t)\rangle,\end{aligned}$$

where $x \in \mathbb{R}^k, t \in (t_0, \infty)$ and $z \in \mathbb{R}^l$.

Another way to express the Lyapunov operator when we plug in $x(t,\omega)$, the local solution to (2.1) for a fixed $\omega \in \Omega$, is given in the following lemma which will be useful in the proof of the main theorem of this section.

Lemma 9. *Let V be a function such that $V \in \mathcal{C}^1(\mathbb{R}^k \times (t_0, \infty))$ for some $t_0 \in \mathbb{R}$. Let $x(t,\omega)$ satisfy* (2.3) *on the interval $[t_0, t_1], -\infty < t_0 < t_1 < \infty$. Then*

$$(\mathcal{L}^{\xi(t,\omega)} V)(x(t,\omega),t) = \frac{d}{dt}V(x(t,\omega),t)$$

for each $\omega \in \Omega$ and for a.e. $t \in [t_0, t_1]$.

Proof. Using the chain rule for derivatives we obtain

$$\begin{aligned}\frac{d}{dt}V(x(t,\omega),t) &= \frac{\partial V}{\partial t}(x(t,\omega),t) + \sum_{i=1}^{k}\frac{\partial V}{\partial x_i}(x(t,\omega),t)\dot{x}_i(t,\omega) \\ &= \frac{\partial V}{\partial t}(x(t,\omega),t) + \sum_{i=1}^{k}\frac{\partial V}{\partial x_i}(x(t,\omega),t)g_i(x(t,\omega),t,\xi(t,\omega)) \\ &= (\mathcal{L}^{\xi(t,\omega)} V)(x(t,\omega),t).\end{aligned} \tag{2.8}$$

□

Another lemma which will be useful later is this special version of Grönwall's inequality:

Lemma 10. *Let the function $y : [t_0, t_1) \to \mathbb{R}$ be differentiable almost everywhere on (t_0, t_1) and let the derivative $\frac{dy}{dt}$ satisfy the following inequality:*

$$\frac{dy}{dt} \leq A(t)y(t) + B(t) \tag{2.9}$$

for almost all $t \in (t_0, t_1)$, where A, B are locally integrable functions. Then for a.e. $t \in [t_0, t_1)$ we have

$$y(t) \leq y(t_0) exp\left(\int_{t_0}^{t} A(s)ds\right) + \int_{t_0}^{t} exp\left(\int_{s}^{t} A(u)du\right) B(s)ds.$$

Proof. By dividing both sides of the inequality (2.9) by the term $\exp\left(\int_{t_0}^{t} A(s)ds\right)$ we obtain

$$\frac{d}{dt} y(t)\exp\left(-\int_{t_0}^{t} A(s)ds\right) \leq y(t)A(t)\exp\left(-\int_{t_0}^{t} A(s)ds\right) + B(t)\exp\left(-\int_{t_0}^{t} A(s)ds\right)$$

or equivalently

$$\frac{d}{dt}\left(y(t)\exp\left(-\int_{t_0}^{t} A(s)ds\right)\right) \leq B(t)\exp\left(-\int_{t_0}^{t} A(s)ds\right)$$

for almost all $t \in (t_0, t_1)$.

Integration of this inequality with respect to t yields the result of the lemma. □

Theorem 11. *Let $t_0 \in \mathbb{R}$ and let $\{\xi(t), t \geq t_0\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^l, l \in \mathbb{N}$.*
Let $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ be a function satisfying the conditions of Theorem 7.
Suppose there exist a function $V \in \mathcal{C}^1(\mathbb{R}^{l+1})$, functions $c_1, c_2 \in \mathbf{L}^1_{loc}(\mathbb{R})$ and function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(\|\xi(t,\omega)\|)$ is locally integrable for each $\omega \in \Omega$.
Let the inequality

$$(\mathcal{L}^z V)(x, t) \leq c_1(t)h(||z||)V(x, t) + c_2(t),$$

be satisfied for almost all $t \in (t_0, \infty), x \in \mathbb{R}^k$ and $z \in \mathbb{R}^l$.

Suppose further that the following holds true:

$$V_R = \inf_{(t_0,\infty)\times\{||x||>R\}} V(x, t) \to +\infty \text{ as } R \to +\infty.$$

Then for all $\omega \in \Omega$, each $\alpha(\omega) > 0$ and $x(t, \omega)$ such that $x(t, \omega)$ satisfies (2.3) on $[t_0, t_0 + \alpha(\omega))$, we have

$$\hat{S}(\omega) = \sup_{[t_0,t_0+\alpha(\omega))} ||x(t, \omega)|| < \infty.$$

Proof. Let us fix $\omega \in \Omega$ and choose $\alpha(\omega) > 0$. Denote $x_\omega(t) = x(t, \omega)$, $\xi(t) = \xi(t, \omega)$, where $x(t, \omega)$ is a function that satisfies (2.3) for all $t \in [t_0, t_0 + \alpha(\omega))$.

The conditions of Lemma 9 are satisfied and thus we have

$$(\mathcal{L}^{\xi(t)} V)(x_\omega(t), t) = \frac{d}{dt} V(x_\omega(t), t) \leq c_1(t)h(||\xi(t)||)V(x_\omega(t), t) + c_2(t)$$

for each $t \in [t_0, t_0 + \alpha(\omega)]$.
Using Lemma 10 on the inequality above, we get

$$\begin{aligned} V(x_\omega(t), t) \leq & V(x_\omega(t_0), t_0)\exp\left(\int_{t_0}^{t} c_1(s)h(||\xi(s)||)ds\right) \\ & + \int_{t_0}^{t} \exp\left(\int_{s}^{t} c_1(u)h(||\xi(u)||)du\right) c_2(s)ds \end{aligned} \tag{2.10}$$

for all $t \in [t_0, t_0 + \alpha(\omega)]$.

By the assumptions of the theorem, the functions c_1, c_2 and $h(||\xi(t)||)$ are all integrable and thus we can simplify the inequality above to yield

$$V(x_\omega(t), t) \leq C_1(\omega)V(x_\omega(t_0), t_0) + C_2(\omega),$$

for all $t \in [t_0, t_0 + \alpha(\omega)]$, where $C_1(\omega), C_2(\omega) \geq 0$ are constant.

Using the fact that $V(x_\omega(t_0), t_0)$ is also constant for a fixed ω, we obtain

$$\sup_{(t_0, t_0+\alpha(\omega))} V(x_\omega(t), t) \leq K(\omega),$$

where $K(\omega) \geq 0$.

The inequality above implies that there exists an $R_0 > 0$ such that $V_{R_0} > K(\omega)$ and thus it follows, that

$$\sup_{[t_0, t_0+\alpha(\omega))} ||x_\omega(t)|| \leq R_0.$$

Since $\omega \in \Omega$ is arbitrary, this concludes the proof. □

Remark. Left-sided and both-sided variants of this theorem, i.e. for $t < t_0$ or for any $t \in \mathbb{R}$, respectively, can be proven.

Corollary. Let $t_0 \in \mathbb{R}$ and assume the function $g : \mathbb{R}^{k+l+1} \to \mathbb{R}^k$ satisfies the conditions of Theorem 11.
Then there exists a function $x(t, \omega)$ with values in $\mathbb{R}^k$ that for every $\omega \in \Omega$ satisfies (2.3) for each $t \in [t_0, \infty)$.

Proof. Let us fix $\omega \in \Omega$ and examine $x(t, \omega)$ as the solution of

$$\dot{x} = g(x, t, \xi(t, \omega))$$

with the initial condition

$$x(t_0, \omega) = x_0(\omega).$$

Choose $a(\omega) > 0$. By Theorem 11, the expression

$$||x(t, \omega)||$$

is bounded on $[t_0, t_0 + a(\omega))$ and together with the corollary after Theorem 3 it implies that its maximal interval of existence is $[t_0, \infty)$.

□

Example. Let $g : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x,z) = -z_1 x^3 + z_2 x$, where $z = (z_1, z_2),\ z_1 \in \mathbb{R}^+,\ z_2 \in \mathbb{R}$ and let $t_0 \in \mathbb{R}$. Let $\xi(t,\omega) = (\xi_1(t,\omega), \xi_2(t,\omega))$ be a stochastic process defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^2$ such that

$$\int_{t_0}^{t_1} ||\xi(t,\omega)||dt < \infty$$

for all $\omega \in \Omega$ and all $-\infty \leq t_0 < t_1 < +\infty$ and $\xi_1(t,\omega) > 0$ for each $t \geq t_0,\ \omega \in \Omega$. We shall prove that under these conditions each solution of the equation $\dot{x} = g(x, \xi(t,\omega))$ with the initial condition $x(t_0,\omega) = x_0(\omega)$ satisfies

$$\sup_{[t_0,t_1)} ||x(t,\omega)|| < +\infty$$

for each $t_1 < +\infty$ and for all $\omega \in \Omega$.

Proof. Firstly, the function g clearly is not globally Lipschitz continuous in x and thus we cannot use Theorem 8. We shall verify that the function g satisfies the conditions of Theorem 11:
The function $g(x,z)$ is clearly locally Lipschitz continuous in (x,z). Moreover, for $(x,t) \in Q = \{|t - t_0| \leq a,\ |x - x_0| \leq b\}$ (without loss of generality we can assume $b \geq 1$) we have

$$\begin{aligned}|g(x,\xi(t,\omega))| &= |\xi_1(t,\omega)x^3 + \xi_2(t,\omega)x| \leq |\xi_1(t,\omega)||x|^3 + |\xi_2(t,\omega)||x| \\ &\leq |\xi_1(t,\omega)|b^3 + |\xi_2(t,\omega)|b \leq \|\xi(t,\omega)\|b^3 = m(t,\omega)\end{aligned}$$

which we assumed to be locally integrable for each $\omega \in \Omega$.

Next let us take a Lyapunov function $V(x) = x^2 + 1, x \in \mathbb{R}$. Then $V \in \mathcal{C}^1(\mathbb{R})$ and $\frac{d}{dx}V(x) = 2x,\ x \in \mathbb{R}$. Furthermore, we have

$$V_R = \inf_{\{|x|>R\}} V(x) = R^2 + 1$$

which clearly exceeds all boundaries as R approaches infinity.

The Lyapunov operator associated with the differential equation $\dot{x} = g(x, \xi(t,\omega))$ is of the form:

$$\begin{aligned}(\mathcal{L}^z V)(x) &= g(x,z)\frac{d}{dx}V(x) = (-z_1 x^3 + z_2 x)2x \\ &= -2z_1 x^4 + 2z_2 x^2.\end{aligned}$$

Using that $x < x^2 + 1$ for each $x \in \mathbb{R}$ we get the following inequality:

$$\begin{aligned}(\mathcal{L}^z V)(x) &= -2z_1 x^4 + 2z_2 x^2 = 2(-z_1 x^4 + z_2 x^2) \\ &= 2(-z_1(x^4 + 1) + z_1 + z_2 x^2) \leq 2(-|z_1|x^2 + |z_1| + |z_2|x^2) \\ &\leq 2(|z_1|x^2 + |z_1| + |z_2|x^{\leq}6||z||x^2 + 2||z|| \leq 6||z||(x^2 + 1) \\ &= 6||z||V(x)\end{aligned}$$

for each $x \in \mathbb{R}, z \in \mathbb{R}^3$.
Let us set $c_1(t) \equiv 6,\ c_2(t) \equiv 0$. Both of these functions are clearly integrable over every finite interval and thus we have verified the conditions of Theorem 11 which concludes the proof. $\square$

3. Random Dynamical Systems

3.1 Basic Definitions

The definitions presented here are formulated with a general time $\mathbb{T}$ which can be any additive group or semigroup. Most often it is one of the following: $\mathbb{R}$, $\mathbb{R}^+$, in which case we speak about continuous time, or $\mathbb{Z}$, $\mathbb{Z}^+$ which we call discrete time. In this thesis we exclusively assume continuous time.

Definition 6. Let $(\Omega, \mathcal{A})$ be a measurable space. A family $\{\theta(t), t \in \mathbb{T}\}$ of mappings of $(\Omega, \mathcal{A})$ into itself is called a measurable dynamical system with time $\mathbb{T}$ if it satisfies the following conditions:

- $(\omega, t) \mapsto \theta(t)\omega$ is measurable
- $\theta(0) = \mathrm{id}_\Omega$, i.e. $\theta(0)\omega = \omega$ for each $\omega \in \Omega$
- $\theta(s+t) = \theta(s) \circ \theta(t)$ for all $s, t \in \mathbb{T}$.

Definition 7. Let θ be a measurable mapping of $(\Omega, \mathcal{A}, \mathbb{P})$ to $(\Omega, \mathcal{A})$. The measure $\theta\mathbb{P}$ defined by

$$\theta\mathbb{P}(A) = \mathbb{P}\left\{\theta^{-1}(A)\right\},$$

where $A \in \mathcal{A}$, is called the image of $\mathbb{P}$ with respect to θ. We say that θ is an endomorphism if $\theta\mathbb{P} = \mathbb{P}$.

Definition 8. A measurable dynamical system $\{\theta(t), t \in \mathbb{T}\}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\theta(t)$ is an endomorphism for each $t \in \mathbb{T}$ is called a metric dynamical system (sometimes a measure preserving dynamical system) and is denoted by $\Sigma = (\Omega, \mathcal{A}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$.

Definition 9. A measurable random dynamical system on the measurable space $(X, \mathcal{F})$ over a metric dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ with time $\mathbb{T}$ is a mapping

$$\varphi : \mathbb{T} \times \Omega \times X \to X,\ (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

with the following properties:

- φ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{A} \otimes \mathcal{F}, \mathcal{F}$-measurable
- the mappings $\varphi(t, \omega) = \varphi(t, \omega, \cdot)$ form a cocycle over X, i.e. they satisfy

$$\varphi(0, \omega) = \mathrm{id}_X \text{ for all } \omega \in \Omega$$

$$\varphi(t+s, \omega) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega) \text{ for all } s, t \in \mathbb{T}, \omega \in \Omega.$$

Similarly as in the case of time, we define a random dynamical system for a general state space X but we mainly work with $\mathbb{R}^k$, $k \in \mathbb{N}$.

Definition 10. A continuous random dynamical system on the metric space X over the metric dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}}$ is a measurable random dynamical system which satisfies in addition the following property: The mapping

$$\varphi(\cdot, \omega, \cdot) : \mathbb{T} \times X \to X,\ (t, x) \mapsto \varphi(t, \omega, x)$$

is continuous for each $\omega \in \Omega$.

3.2 Generation

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}^+$, $X = \mathbb{R}^k$, $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(\Omega, \mathcal{A}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ be a metric dynamical system.
In this section we use the results of Chapter 2 but we choose a different approach. Instead of focusing on a single solution to a random differential equation, determined uniquely by the initial condition, we will examine $x(t, \omega, t_0, x_))$, i.e. the solution to a random differential equation dependent on the chosen initial value at time t_0. The focus of this and the next section is to show that this approach yields a random dynamical system.

Definition 11. Let $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$. We say that the equation

$$\dot{x} = f(x, \theta(t)\omega) \tag{3.1}$$

generates φ if for each $t \in \mathbb{R}$, $\omega \in \Omega$ the following holds true:

$$\varphi(t, \omega)x = x + \int_0^t f(\varphi(s, \omega)x, \theta(t)\omega)ds. \tag{3.2}$$

Theorem 12. *Consider the equation* (3.1) *and denote* $f_\omega(x, t) = f(x, \theta(t)\omega)$. *Suppose that for each* $\omega \in \Omega$ *the function* f_ω *has the following properties:*

- $f_\omega(\cdot, t)$ *is locally Lipschitz continuous for almost all* $t \in \mathbb{R}$,
- $\sup_{x\in\mathbb{R}^k} \dfrac{\|f_\omega(x, t)\|}{1 + \|x\|} \leq \alpha(t, \omega)$ *for each* $t \in \mathbb{R}$, *where* $\alpha(t, \omega)$ *is locally integrable,*
- $\displaystyle\int_a^b \sup_{x\in\mathbb{R}^k} \frac{\|f_\omega(x, t)\|}{1 + \|x\|}dt < \infty$ *for each* $-\infty < a < b < \infty$.

Under these conditions the equation (3.1) *uniquely generates continuous a random dynamical system for all* $t \in \mathbb{R}$

Proof. The proof can be found in [1] as proof of Theorem 2.2.1, pp. 58-60. □

Remark. A more general condition can be considered instead of the second condition in Theorem 12: $\|f_\omega(x, t)\| \leq \alpha(t, \omega)\|x\| + \beta(t, \omega)$ for all $t \in \mathbb{R}$ where $\alpha(t, \omega)$, $\beta(t, \omega)$ are locally integrable.

Both the conditions in Theorem 12 and in the remark that follows are very strict and exclude many important practical cases. In the following section we will show how we can use the results of Chapter 2 to obtain less restrictive conditions for generating a random dynamical system in a special case called The Memoryless Case.

3.3 The Memoryless Case

Very often we encounter the case

$$\dot{x} = f(x, \theta(t)\omega) = g(x, \xi(t, \omega)) \tag{3.3}$$

where $\xi(t,\omega)$ is a stochastic process. We will show that this type of equation also generates a random dynamical system.

As per section 3.2, a measure preserving dynamical system is required rather than a stochastic process as an argument of the function on the right-hand side of the equation. However, with additional requirements on the process $\xi(t,\omega)$ this problem can be circumvented and both cases are equivalent. First, let us now present some definitions necessary for the construction to work.

Definition 12. Let $\{\xi(t),\ t \in \mathbb{T}\}$ be a stochastic process with values in $(E, \mathcal{E})$ and let $\{t_1, ..., t_r\}$ be a finite subset of $\mathbb{T}$. The probability measure $\mathbb{P}_{t_1,...,t_r}$ defined by

$$\mathbb{P}_{t_1,...,t_r}(A) = \mathbb{P}\left(\xi(t_1) \in A_1, ..., \xi(t_r) \in A_r\right),$$

where $A = A_1 \times ... \times A_r$, $A_1, ..., A_r \in \mathcal{E}$, is called a finite-dimensional distribution of $\xi(t,\omega)$.

Definition 13. A stochastic process $\{\xi(t),\ t \in \mathbb{T}\}$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in state space $(E, \mathcal{E})$ is called stationary if for all $t, t_1, ..., t_r \in \mathbb{T}$ we have

$$\mathbb{P}_{t_1+t,...,t_r+t} = \mathbb{P}_{t_1,...,t_r}.$$

Definition 14. We say that a function $\lambda : \mathbb{R} \to \mathbb{R}^m$, $m \in \mathbb{N}$, is càdlàg if for every $t \in \mathbb{R}$

$$\lim_{s \to t-} \lambda(s) \in \mathbb{R}^m \text{ exists,}$$
$$\lim_{s \to t+} \lambda(s) = f(t).$$

If $\xi(t,\omega)$ is a stationary stochastic process with càdlàg trajectories for each $\omega \in \Omega$ then the equation (3.3) holds true and thus we can examine whether the equation $\dot{x} = g(x, \xi(t,\omega)$ generates a random dynamical system. The detailed explanation of this equivalence can be found in [1] on pp. 64 and 542-545.

Theorem 13. *Let the function* $g : \mathbb{R}^{k+l} \to \mathbb{R}^k$, $\xi(t,\omega)$ *satisfy the conditions of Theorem 11. Assume, in addition, that the function* $x(t,\omega)$ *obtained as a solution to* (2.1) *is measurable in* ω. *Then the equation* $\dot{x} = g(x, \xi(t,\omega))$ *uniquely generates a random dynamical system defined on* $t \in \mathbb{R}^+$.

Proof. The global existence and uniqueness of a solution to the equation $\dot{x} = g(x, \xi(t,\omega)$ follow directly from Theorem 11 with a special choice $t_0 = 0$.
It remains to show that the mapping φ defined by (3.2) satisfies the cocycle property of a random dynamical system. For this, we use the equivalence in (3.3).
First, it follows immediately from the definition of φ that $\varphi(0,\omega)x = x$ for each $\omega \in \Omega,\ x \in \mathbb{R}^k$.
Let $s,\ t \in \mathbb{R},\ s,\ t > 0$. Then, using the equation (3.2) and the properties of a metric dynamical system, we have

$$\begin{aligned}\varphi(s+t,\omega)x &= x + \int_0^{s+t} f(\varphi(r,\omega)x, \theta(r)\omega)dr \\ &= x + \int_0^{s} f(\varphi(r,\omega)x, \theta(r)\omega)dr + \int_s^{s+t} f(\varphi(r,\omega)x, \theta(r)\omega)dr \\ &= \varphi(s,\omega)x + \int_0^{t} f(\varphi(r+s,\omega)x, \theta(r)\theta(s)\omega)dr\end{aligned}$$

where all the integrals are finite.
Now denote $\bar{\omega} = \theta(s)\omega$ and $\bar{x} = \varphi(s,\omega)$ and put $\psi(r,\bar{\omega}) = \varphi(r+s,\omega)$. Then we will find that $\psi(s,\bar{\omega})\bar{x}$ also satisfies

$$\psi(t,\bar{\omega})\bar{x} = \bar{x} + \int_0^t f(\psi(r,\bar{\omega}), \theta(r)\omega)dr.$$

Using the fact that the solution to (3.1) is unique we obtain

$$\varphi(t,\bar{\omega})\bar{x} = \varphi(t,\theta(s)\omega)\varphi(s,\omega) = \psi(t,\bar{\omega})\bar{x} = \varphi(t+s,\omega)x.$$

Finally, when at least one of s, t is equal to zero then we have (without loss of generality assume $t = 0$)

$$\begin{aligned}\varphi(s,\omega)x = \varphi(s+t,\omega)x = x + \int_0^{s+t} f(\varphi(r,\omega)x, \theta(r)\omega)dr \\ = \int_0^s f(\varphi(r,\omega)x, \theta(r)\omega)dr\end{aligned}$$

which is true from definition of φ. Thus we have verified that φ satisfies the cocycle property and this concludes the proof. □

Remark. The random dynamical system obtained in Theorem 13 is also continuous in the sense of Definition 10. This can be proven by applying a straightforward modification to the method described in the proof of Theorem 9 in [5], pp. 72-73.

4. Random Attractors

In this final chapter our main object of interest will be random attractors for random dynamical systems. In the study of deterministic dynamical systems, an attractor is a set toward which the system tends to evolve. For us to be able to create an analogy we will first need to start with some definitions cited from [7]. Let (X, d) be a metric space, $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. Let, in addition, $\theta = (\Omega, \mathcal{A}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ be a metric dynamical system and φ a continuous random dynamical system over θ.

The most important definition of this section will be that of a random set, which is a set-valued extension to a random variable.

Definition 15. A set-valued map $K : \Omega \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X, taking values in the closed subsets of X is said to be measurable if for every $x \in X$ the map $\omega \mapsto d(x, K(\omega))$ is measurable, where we define

$$d(A, B) = \sup \{\inf \{d(x, y),\ y \in B\},\ x \in A\},$$

for A, B non-void subsets of X, and $d(x, B) = d(\{x\}, B)$.
A closed set valued measurable map $K : \Omega \to \mathcal{P}(X)$ will be called a random closed set.

Definition 16. A random set K is said to be φ-forward invariant if

$$\varphi(t, \omega)K(\omega) \subset K(\theta(t)\omega)$$

for each $t > 0$.
K is said to be strictly φ-forward invariant if

$$\varphi(t, \omega)K(\omega) = K(\theta(t)\omega)$$

for each $t > 0$.

Definition 17. Let K be a random set.

$$\Omega_K(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta(-t)\omega)K(\theta(-t)\omega)}$$

is said to be the Ω-limit set of K.

Remark. The Ω-limit set is, by definition, closed.

An important definition for this chapter will be that of an attracting and an absorbing random set.

Definition 18. A random set A is said to attract another random set B if

$$d(\varphi(t, \theta(-t)\omega)B(\theta(-t)\omega), A(\omega)) \to 0$$

as $t \to \infty$ for almost every $\omega \in \Omega$.

Definition 19. Let K and B be random sets such that for almost all $\omega \in \Omega$ there exists $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta(-t)\omega)B(\theta(-t)\omega) \subset K(\omega)$$

then K is said to absorb B and $t_B(\omega)$ is called the time of absorption.

Definition 20. Let φ be a random dynamical system such that there exists a random compact set A satisfying the following conditions:

- A is strictly φ-forward invariant,
- A attracts every bounded deterministic set $B \subset X$.

Then we say that A is a globally attracting set or a global attractor for φ.

Theorem 14. *Suppose φ is a random dynamical system on a Polish space X, and suppose that there exists a compact set $\omega \mapsto K(\omega)$ absorbing every bounded non-random set $B \subset X$. Then the set*

$$A(\omega) = \overline{\bigcup_{B \subset X} \Omega_B(\omega)}$$

is a global attractor for φ.

Proof. Proof can be found in [7] as proof of Theorem 3.11, p. 371. □

Using the results of the theorem above we can now provide sufficient conditions for the existence of an attracting set for a random dynamical system generated from a random differential equation.

Theorem 15. *Let $g : \mathbb{R}^{k+l} \to \mathbb{R}^k$, $\xi(t, \omega)$ satisfy the conditions of Theorem 13. Let the following hold true:*

$$2\langle x, g(x, z)\rangle \leq -k_1\|x\|^2 + k_2\|z\|^2 + k_3 \tag{4.1}$$

for some $k_1 > 0$, k_2, $k_3 \in \mathbb{R}$ for each $x \in \mathbb{R}^k$, $z \in \mathbb{R}^l$.
Assume that $\xi(t, \omega)$ has locally integrable trajectories and it satisfies

$$\int_t^0 e^s \xi(s, \omega) ds < \infty, \; t < 0$$

for each $\omega \in \Omega$. Then there exists $r(\omega) > 0$ such that for each $\rho > 0$ there exists a $\bar{t} \leq 0$ such that for each $t_0 \leq \bar{t}$, $x_0 = x(t_0, \omega)$, $\|x_0\| < \rho$ we have

$$\|x(0, \omega, t_0, x_0)\|^2 \leq r^2(\omega).$$

Proof. Let us fix an $\omega \in \Omega$.
First, it is easy to see that $\frac{d}{dt}\|x(t, \omega)\|^2 = 2\langle x(t, \omega), g(x(t, \omega), \xi(t, \omega)\rangle$, for each $t \geq 0$, and thus from (4.1) it follows that

$$\frac{d}{dt}\|x(t, \omega)\|^2 \leq k_1\|x(t, \omega)\|^2 + k_2\|\xi(t, \omega)\|^2 + k_3 \tag{4.2}$$

for each $t \geq 0$. By integrating both sides of the inequality (4.2) we obtain:

$$\|x(t,\omega)\|^2 \leq e^{-k_1(t-t_0)}\|x_0\|^2 + \int_{t_0}^{t} e^{-k_1(t-s)}(k_2\|\xi(t,\omega)\|^2 + k_3 ds).$$

By plugging in $t = 0$ and using the fact that $\|x_0\| < \rho$ we obtain:

$$\begin{aligned}\|x(0,\omega)\|^2 &\leq e^{k_1 t_0}\rho^2 + \int_{t_0}^{0} e^{k_1 s}(k_2\|\xi(s,\omega)\|^2 + k_3)ds \\ &\leq e^{k_1 t_0}\rho^2 + \int_{-\infty}^{0} e^{k_1 s}(k_2\|\xi(s,\omega)\|^2 + k_3)ds \\ &\leq e^{k_1 t_0}\rho^2 + C(\omega).\end{aligned}$$

By choosing $\bar{t}$ sufficiently small that $e^{k_1\bar{t}}\rho^2 < 1$ and setting $r(\omega) = \sqrt{1 + C(\omega)}$ we obtain the result of the theorem since ω has been chosen arbitrarily. □

Corollary. A straightforward application of Theorem 15 is the existence of a global attracting set for each random dynamical system obtained as a solution to a random differential equation which has the required properties.

Example. Let $g : \mathbb{R}^2 \to \mathbb{R},\ g(x,z) = f(x) + \sigma(x)z$ where $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$xf(x) \leq -k_1 x^2 + k_2,$$

where $k_1 > 0,\ k_2 \in \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies

$$\sup_{x\in\mathbb{R}} |\sigma(x)| < \infty.$$

Assume further that f, σ are locally Lipschitz continuous. Let $\xi(t,\omega)$ be a stationary stochastic process with locally integrable, càdlàg trajectories. Then the random dynamical system obtained as a solution to (3.3) has a global attractor.

Proof. Denote $\hat{\sigma} = \sup_{x\in\mathbb{R}} |\sigma(x)|$. Consider the function $V(x) = x^2 + 1$ and let us examine the Lyapunov operator:

$$\begin{aligned}(\mathcal{L}^z V)(x) &= 2xf(x) + 2x\sigma(x)z \\ &\leq -2k_1x^2 + 2k_2 + 2|x\sigma(x)z| \\ &\leq -2k_1x^2 + 2k_2 + 2\hat{\sigma}(x^2+1)|z| \\ &\leq K(x^2+1) + 2\hat{\sigma}|z|(x^2+1) \\ &= (K + 2\hat{\sigma}|z|)(x^2+1) = h(|z|)V(x)\end{aligned}$$

where $K = \max\{2|k_1|, 2|k_2|\}$.
The local integrability of $h(|\xi(t,\omega)|)$ follows from the local integrability of $\xi(t,\omega)$. Thus we have verified the conditions of Theorem 13 and so the equation (3.3) generates a random dynamical system.
Next, by means of Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}x^2(t,\omega) &= 2x(t)g(x(t,\omega),\xi(t,\omega)) \\ &\leq 2x(t,\omega)f(x(t,\omega)) + 2x(t)\sigma(x(t,\omega))\xi(t,\omega) \\ &\leq -2k_1x^2(t,\omega) + 2k_2 + 2x(t)\sigma(x(t,\omega))\xi(t,\omega) \\ &\leq -2k_1x^2(t,\omega) + 2k_2 + \varepsilon x^2(t,\omega)\sigma^2(x(t,\omega)) + \frac{1}{\varepsilon}\xi^2(t,\omega) \\ &\leq (-2k_1 + \hat{\sigma}^2\varepsilon)x^2(t,\omega) + \frac{1}{\varepsilon}\xi^2(t,\omega) + k_2\end{aligned}$$

Taking $\varepsilon < \frac{2k_1}{\hat{\sigma}^2}$ we see that the conditions of Theorem 15 are verified and that concludes the proof. $\square$

Conclusion

We have examined the deterministic results which were our starting point and through combining theory from various resources we managed to create probabilistic analogies.
We familiarised ourselves with the phenomenon of explosion and through theory of Lyapunov functions we discovered requirements for a function to avoid it.
We explored the field of random dynamical systems from the angle of random differential equations and found sufficient conditions for a random dynamical system to be generated by a random differential equation.
We managed to improve on the current results pertaining to the existence of a global solution to random differential equations by making the conditions less restrictive and thus allowing for a wider variety of functions to be considered.
Finally, we presented some important results from the theory of random attractors and by making use of previous chapters we came up with sufficient conditions for their existence.

Bibliography

[1] L. Arnold. *Random Dynamical Systems.* Corrected 2nd printing 2003. Springer-Verlag, Berlin Heidelberg, 2003.

[2] R. Khasminskii. *Stochastic Stability of Differential Equations.* Completely Revised and Enlarged 2nd Edition. Springer-Verlag, Berlin Heidelberg, 2012.

[3] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations.* McGraw-Hill Book Company, Inc., New York Toronto London, 1955.

[4] J. Kurzweil. *Ordinary Differential Equations.* Elsevier, Amsterdam Oxford New York Tokyo, 1986.

[5] K. Schmitt and R. C. Thompson. *Nonlinear Analysis and Differential Equations, An Introduction.* Lecture Notes. University of Utah, 2004.

[6] P. Drábek and J. Milota. *Methods of Nonlinear Analysis.* Second Editon. Springer Basel, Heidelberg New York Dordrecht London, 2013.

[7] H. Crauel and F. Flandoli. Attractors for random dynamical systems. *Probability Theory and Related Fields*, 100:365–393, 1994.