FACULTY
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AND PHYSICS
Charles University

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Martin Povišer

Decay of η into three pions

Institute of Particle and Nuclear Physics

Supervisor of the master thesis: Mgr. Martin Zdráhal, Ph.D.
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Author: Martin Povišer

Department: Institute of Particle and Nuclear Physics

Supervisor: Mgr. Martin Zdráhal, Ph.D., Institute of Particle and Nuclear Physics

Abstract: In this work, we summarize a method established in the published literature of dispersive construction of amplitude for the $\eta \to \pi\pi\pi$ decay process. We outline Chiral Perturbation Theory (ChPT) as an effective field theory for the description of low-energy hadron dynamics, and then introduce dispersive methods with the objective of constructing process amplitudes (up to the two-loop order) similar in form to ChPT predictions.
The original contribution of the present work is a software library implementing the "reconstruction procedure" that forms the basis for the dispersive construction of mesonic process amplitudes. This library can be used to construct amplitudes in a computer algebra system (CAS) environment, making those forms of amplitude available to fitting of experimental data and theoretical studies, especially those focusing on the extraction of the up/down quark mass difference.

Keywords: eta-to-pi decay, quantum chromodynamics, chiral perturbation theory, dispersion relations

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Introduction

This work deals with the theoretical construction of abstract forms of amplitude for the $\eta \to \pi\pi\pi$ decay and similar meson processes, the description of which falls within the scope of strong-force physics.

In the Standard Model, strong force is at a fundamental level described by the theory of Quantum Chromodynamics (QCD). The usual perturbative method of enumerating Feynman diagrams can only be readily applied to QCD in a limited regime – in the high-energy regime of quark scattering, where the theory ('perturbative QCD') produces predictions in good agreement with experiment. However, hadron phenomenology, which is to be the low-energy manifestation of QCD, falls outside the scope of simple perturbative treatment.

There are a couple of established approaches for deriving testable low-energy predictions from QCD. One approach is to focus on the consequences of approximate flavor chiral symmetries present in QCD. That approach is systematically implemented in 'Chiral Perturbation Theory' (ChPT) [11][12][22].

ChPT postulates the validity of a 'chiral limit', that is, a limit in which a few of the smallest quark mass parameters of QCD vanish so as to make the resultant theory exactly chiral symmetric. The symmetry breaking effects connected with the physical non-vanishing values of these parameters are then considered a small perturbation on top of the symmetric theory. Predictions of quantities within ChPT are given in the form of expansions, the terms of which are organized by so-called chiral ordering. In practice, the expansion needs to be cut off at some given chiral order. As the considered chiral order cut-off grows, the theory admits an increasing number of free parameters (low-energy coupling constants) – the predictions become more accurate, but the theory gradually loses its predictive power. That is straightforward to understand on the grounds of ChPT being an effective theory with the *approximate* QCD symmetry and a few other assumptions for its only basis, otherwise being unrestricted.

In an approach complementary to evaluating meson process amplitudes from ChPT, one can use dispersive methods to build up general forms of amplitude – taking chiral ordering and unitarity as constructive principles of the method [20][25][14]. These forms of amplitude are then compatible with amplitudes evaluated from ChPT, that is, a ChPT amplitude corresponds to some choice of values for the free parameters of the dispersively-constructed amplitude form. These forms can help organize ChPT results, be a target for the fitting of experimental data, and also be used to investigate models deviating from standard ChPT.

The $\eta \to \pi\pi\pi$ process, in particular, is intimately tied to the explicit breaking of isospin symmetry that is connected to the mass difference between the up and down quarks [10]. The process can only occur in violation of isospin symmetry, and furthermore, there are arguments to show that the electromagnetic contribution to facilitate the violation is small, leaving the explicit isospin-breaking attributes of strong force, that is, the up/down quark mass difference, to be the dominant driver behind the decay. As such, the decay is an interesting testbed to study the quark mass difference.

In Chapter 1, we give an overview of ChPT as an effective theory stemming from the approximate QCD symmetries. In Chapter 2, we introduce the Dalitz plot as the customary presentation of the kinematic variation of $\eta \to \pi\pi\pi$ amplitudes and measured incidences. In Chapter 3, we summarize a dispersive method of mesonic amplitude construction. The method is established in the published literature and is, for example, the basis of Ref. [14]. In Chapter 4, we describe an attached software library with a high-level user interface for carrying out calculations in the dispersive construction of meson process amplitudes. This library is the original contribution of the present work. In Appendix A, we list utility functions that figure in the dispersively-constructed amplitudes, and in Appendix B, we print a transcript of an interactive computer session demonstrating the usage of the library.

Chapter 1

Overview of Chiral Perturbation Theory

1.1. Introduction

Chiral Perturbation Theory ([11], [12], [22]) is an effective field theory for the description of low-energy phenomenology of Quantum Chromodynamics. It builds on considering the symmetry principles manifest in QCD, employing them in the construction of terms in (otherwise most general) Lagrangian density. Predictions are then derived by means of standard perturbation theory except for a special ordering scheme in the perturbative expansion (instead of powers of coupling one considers powers of momenta and quark masses). In what follows some of the technical details are laid out. For a full introduction to the topic, we refer for example to Ref. [19].

The crucial symmetry of QCD in the construction of ChPT is the approximate SU(2) isospin symmetry or, by extension, the SU(3) symmetry among the lightest quark flavors. Famously these symmetries pronounce themselves in a striking way by organizing the hadronic spectrum into SU(2) and SU(3) multiplets of similar particle mass and attributes. The SU(N) ($N=2$ or 3) symmetry of QCD is not exact and is broken by the quark mass term of the Lagrangian. If we neglect the relevant part of the mass term, we decouple[1.1] the left-handed and right-handed quark fields, so the Lagrangian density exhibits an even larger symmetry of flavor multiplets in left- and right-handed quark fields *independently*, the so-called chiral symmetry. We denote by SU$(N)_L$ and SU$(N)_R$ the two symmetry groups acting on left-handed and right-handed fields separately, and we have, overall, an SU$(N)_L \times$ SU$(N)_R$ symmetry (called chiral symmetry) of the QCD Lagrangian, up to the mass term.

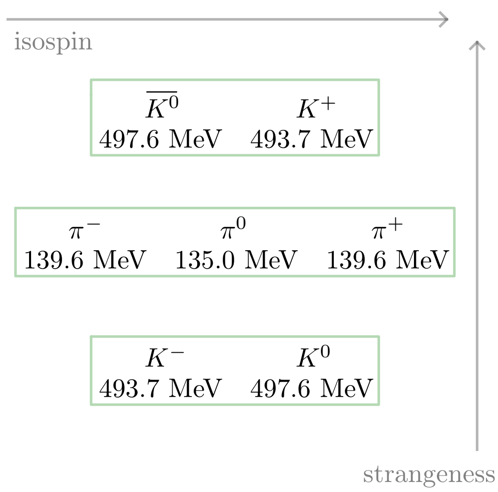

Figure 1.1. The pseudoscalar octet of light mesons arranged in the weight space of the SU(3) Lie group. η meson (mass 547.9 MeV, isospin singlet) is not depicted, but lies at the same position as the neutral pion π^0. Green boxes correspond to isospin doublets/triplets. Mass values were rounded from those available in [27].

1.1. The resulting quantum theory will not exhibit all the symmetries suggested by the formal decouplement of the fields at the classical level. There is a chiral anomaly (discussed at an introductory level e.g. in [8]) that affects the $U(1)$ symmetry of contrariwise change in phase of the left-handed and right-handed fields. However, the SU(N) chiral flavor symmetries that are the subject of discussion here can be preserved through quantization.

The $\mathrm{SU}(N)_L \times \mathrm{SU}_R(N)$ approximate symmetry underpins the construction of ChPT. For further discussion, it is important to recast it, representing each element (L, R) of $\mathrm{SU}_L(N) \times \mathrm{SU}_R(N)$ by two matrices $V, A \in \mathrm{SU}(N)$ such that

$$(L, R) = (VA^\dagger, VA).$$

We use $\mathrm{SU}(N)_V$ and $\mathrm{SU}(N)_A$ to denote the set of all V and A, respectively. The former corresponds to symmetry transformations acting on left-handed and right-handed fields evenhandedly (so the usual flavor symmetry), and the latter corresponds to transformations acting contrariwise on the two fields. We speak of $\mathrm{SU}(N)_V \times \mathrm{SU}(N)_A$ symmetry, but we note that $\mathrm{SU}(N)_A$ is *not* a subgroup of $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ (while $\mathrm{SU}(N)_V$ is). The V and A subscripts stand for *vector* and *axial*, being descriptions of the transformation properties of the associated Noether currents. Under parity, when the left-handed and right-handed quark fields are swapped, the axial current is inverted, while the vector current is unchanged.

For the consideration of the isospin/flavor symmetry effects in QCD we have already invoked what is called the chiral limit – the edge case of taking the mass terms in QCD for the N lightest quarks to zero and thus making the chiral symmetry exact. To account for the low-energy meson phenomenology, the chiral symmetry-breaking effects (non-vanishing quark mass terms in QCD) are treated to be a small perturbation around the chiral limit. Furthermore, in the limit, the full $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ symmetry of the dynamical laws is postulated spontaneously broken down to $\mathrm{SU}(N)_V$. The members of the pseudoscalar meson octet (Fig. 1.1) are then identified with the Goldstone bosons of the Nambu-Goldstone realization of the spontaneously broken $\mathrm{SU}(N)_A$ symmetry.

It can be shown that, in the chiral limit, the interaction among the pseudoscalar mesons (being Goldstone bosons) should vanish in the limit of zero energy [21]. To model the low-energy dynamics of mesons, Chiral Perturbation Theory then systematically implements a perturbation around both the zero-energy and chiral limits.

By the argument given, e.g., in Ref. [15], the low-energy dynamic is presumed dominated by the exchange of light mesons (those being the observed low-energy asymptotic states), and from that it is deduced that an effective low-energy theory can be recast into the form of a perturbative quantum field theory, having an octet of pseudoscalar fields for its dynamical degrees of freedom and an infinite sum of monomials of the fields and their derivatives for its local Lagrangian density. A priori, each term of the effective Lagrangian has an unspecified free parameter, and as such, this doesn't make much for a theory with predictive power, but fortunately, the free parameters in ChPT can be cut down in two steps:

i. By viewing the scattering amplitudes that are the predictions of the effective theory to be an expansion in powers of magnitude of external momenta (and strength of chiral breaking parameters to be introduced later), we can establish an ordering of the terms of Lagrangian. That is, the ordering of the amplitude expansion propagates back into an ordering of the Lagrangian terms. We can then focus on terms with low "chiral order". This issue of chiral ordering is explained in the next section.

ii. Within each chiral order, there's still an a priori infinite number of free parameters of the Lagrangian, but by demanding the symmetries of QCD in the chiral limit to reproduce in the effective theory, we can constraint the free parameters at each chiral order to a finite-dimensional choice. The symmetries to be demanded of the effective theory are not only the global chiral symmetries, but also the *gauge* chiral symmetries once the chiral QCD is made gauge invariant by the addition of appropriate gauge transforming external fields. This is equivalent to demanding the effective theory to satisfy QCD chiral Ward identities. Details on this are to be found in the provided references. We will return to the construction of allowed effective Lagrangian terms in Section 1.3.

The remaining issue is the inclusion of explicit chiral symmetry breaking. The QCD mass term for the N lightest quarks has the form $\mathcal{L}_{\text{mass}} = -\bar{q}\, M\, q$ which can be expanded in terms of the left-handed and right-handed quark fields into $\mathcal{L}_{\text{mass}} = -\bar{q}_R\, M\, q_L + (h.c.)$. If we were to consider M to be an external source field added to a chiral-limit QCD Lagrangian, covariance with an $g(x)$: $\mathbb{R}^4 \to \mathrm{SU}_L(N) \times \mathrm{SU}_R(N)$ gauge transformation would demand of field $M(x)$ the transformation law

$$M(x) \to R\, M(x)\, L^\dagger, \tag{1.1}$$

where $(L, R) = g(x)$.

To include the effects of explicit symmetry breaking that is present in QCD at physical values of quark mass terms, we consider the effective theory not only to be an effective theory of QCD in the chiral limit but also an effective theory of QCD in the chiral limit *with the inclusion of an external field having the transformation law* (1.1). With this external field included, we are still demanding of the effective theory to be gauge invariant like QCD in the sense mentioned within (ii.) above, that is, with the inclusion of the aiding external fields [those are different fields from the one transforming under (1.1)]. Once we wish to apply the effective theory to the physical world, we set the new external field to a constant matrix over the spacetime, just like we do with the field M in QCD.

In ChPT, the external field with transformation law (1.1) is called χ, and by symmetry considerations alone, it is related to M up to a multiplicative constant. This multiplicative constant can have a physical dimension, and as such the dimension of χ is unspecified. The dimension is intimately related to the ordering of the chiral expansion. In standard ChPT, the field χ has dimension mass squared and is normalized for a simple leading order coupling in the effective Lagrangian. (This leading order coupling generates the mass term for the octet of pseudoscalar fields.)

1.2. CHIRAL ORDERING

To restate, in ChPT, we deal with perturbative expansions of scattering amplitudes (and other quantities) around the chiral and zero-energy limits. In the following we will derive what is known as the Weinberg formula [22] for assigning chiral dimension to contributing diagrams. We focus on standard chiral ordering in which the dimension of chiral-breaking field χ is mass squared.

Suppose we have an amplitude $\mathcal{A}(p_i, \chi)$ corresponding to a diagram, which we consider here to be a function of external momenta p_i and the chiral-breaking field χ introduced earlier. To assign a chiral dimension to $\mathcal{A}$, we perform linear scaling of p_i and quadratic scaling of χ. (This choice reflects the mass dimensions of the two quantities: $[p_i] = 1$ and $[\chi] = 2$.) The chiral dimension D of contribution $\mathcal{A}$ is defined such that[1.2]

$$\mathcal{A}(t\, p_i, t^2\, \chi) = t^D\, \mathcal{A}(p_i, \chi),$$

as we vary t. All the while, any constants with a non-zero dimension of mass appearing in $\mathcal{A}$ are held constant. By nature of both p_is and χ being small in application of ChPT, we expect the contribution of diagrams with low D to be dominant.

To assign chiral dimension to a given diagram in practical terms, we can derive a simple formula. It will be expressed in terms of the chiral dimensions of the constituent vertices and the number of loops present in the diagram. We start by considering the energy dimension of $\mathcal{A}$ based on its relation to the S-matrix:

$$[\mathcal{A}] = -N_E + 4, \tag{1.2}$$

where N_E is the number of external lines of the diagram. At the same time, from the definition of the chiral dimension D, using $C(v)$ to denote the coupling constant associated with vertex v, we can state

$$[\mathcal{A}] = D + \sum_v\, [C(v)], \tag{1.3}$$

where the sum is over the vertices of the diagram. Now, we conveniently define the chiral dimension of a term in the effective Lagrangian to be the number of field derivatives plus twice the power of χ appearing in the term's product. We denote by $D(v)$ the chiral dimension of the term generating the vertex v. If we then focus on the mass dimension of the interaction term (here labeled simply $[v]$), it holds

$$4 = [v] = N_F(v) + D(v) + [C(v)], \tag{1.4}$$

1.2. Here, in the definition of the chiral dimension and in the derivation of Weinberg formula, we ignore the issue of renormalization and of regularization of any loop integrals.

where $N_F(v)$ is the number of lines attached to the vertex (number of fields in the interaction term). Solving for $[C(v)]$ in (1.4) and combining it with (1.2) and (1.3), we have

$$-N_E+4=[\mathcal{A}]=D+\sum_v (4-N_F(v)-D(v)).$$

From the above, we can express D. To simplify, we will use the graph identity $\sum_v N_F(v)=2N_I+N_E$, where N_I is the number of internal lines. Therefore, we have

$$D=4-N_E-\sum_v (4-N_F(v)-D(v))=4+2N_I+\sum_v (D(v)-4).$$

Next, we introduce the number of loops N_L and substitute for N_I by $N_I=N_L+N_V-1$, where N_V is the number of vertices:

$$D=2+2N_L+\sum_v (D(v)-2), \tag{1.5}$$

which is the final form of the result. Note that by (1.5), if we are enumerating contributions up to some chiral order D, we only need to consider Lagrangian terms up to the same order. Furthermore diagrams with a high number of loops are suppressed by the $2N_L$ term.

1.3. Terms in the effective Lagrangian

The terms of the Lagrangian in ChPT are constrained by the condition on the effective theory to reproduce the chiral symmetries of QCD. We will show how this is conceptually carried out. We will focus on the SU(3) version of ChPT, pointing out the differences of SU(2) effective theory at the end.

In establishing the chiral symmetry of the effective theory, we need to specify a transformation law for its dynamical degrees of freedom. By the nature of the dynamical fields being Goldstone bosons, and given the symmetry groups involved, we expect the collective value of the fields at a point to correspond to a coset from the quotient group

$$(\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R)/\mathrm{SU}(3)_V.$$

This has relation to the fields' transformation law. A convenient way to achieve the desired chiral symmetry of the effective theory is to make the following choices:

i. Define a field $\phi(x)$ that takes a value from the space of 3×3 Hermitian matrices acting in flavor space. The eight degrees of freedom contained in $\phi(x)$ are the pseudoscalar octet of fields that are to be the dynamical degrees of freedom of the effective theory.

ii. Define an SU(3)-valued field $U(x)$ related to $\phi(x)$ by $U=e^{i\phi/F_0}$, where F_0 is a free constant.[1.3] Prescribe for the field U the gauge transformation law

$$U\to R(x)\,U(x)\,L^\dagger(x). \tag{1.6}$$

Because of this simple and linear transformation law, U is then well-suited to be a building block of chiral invariant terms. Law (1.6) also induces a representation of the chiral group on ϕ – the adjoint representation of the subgroup $\mathrm{SU}(3)_V$ but, in general, a *non-linear* representation of the full group.

iii. Impose the effective theory to have a chiral invariant Lagrangian density.

While it is obvious that point (iii) leads to chiral symmetry of the effective theory, and point (ii) leads to a transformation law on the dynamical degrees of freedom that is consistent with physical interpretation of the fields, it is not obvious that the choices made in (ii) and (iii) do not arbitrarily restrict the effective theory beyond what can be inferred from QCD symmetries.

1.3. This constant has a physical interpretation of being the pion-decay constant in the chiral limit.

Ref. [15] shows that this is, in fact, not so, that we are free to make those choices and an effective theory, as long as it is to reproduce also the *local* chiral symmetries of QCD, can be without loss of generality put into form satisfying (ii) and (iii).

So, to build up the effective Lagrangian, at each chiral order, we construct all the chiral invariants that are local products of fields U and χ (taking into account the transformation laws (1.6) and (1.1)), then we select an independent subset of those. We take a freely parametrized linear combination of the independent invariants to make up the part of the effective Lagrangian of the given order.

The Lagrangian is not allowed to have any terms of order $O(p^0)$, since those would imply, in chiral limit, mass terms or interaction terms at zero momenta, both of which we are ruling out. Terms of chiral order $O(p^1)$ are disallowed on symmetry grounds. In fact all terms of odd order are since they would require contracting an odd number of derivatives to a Lorentz scalar. The leading terms are therefore of chiral order $O(p^2)$, the subleading of order $O(p^4)$, and so forth. We use subscripts of $\mathcal{L}$ to group terms of given chiral order, so that the full effective Lagrangian is decomposed by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots.$$

The customary leading order terms are

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F_0^2}{4}\,\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger).$$

The terms $\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$ and $\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)$ are a full set of independent invariants of chiral order $O(p^2)$ up to a total derivative. In place of introducing a free parameter for the coupling of $\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$, we opt to generate a normalized kinetic term[1.4] for ϕ, noting that any freedom afforded by the hypothesized free parameter is equivalent to rescaling of the field and adjusting all other coupling constants. Similarly, we don't need to introduce a free parameter in conjunction with $\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)$ since there is a free parameter in relating the scale of χ to the quark mass field. So in effect $\mathcal{L}_2$ is without free parameters. The free parameter relating χ to quark mass fields is B_0[1.5], defined such that it holds $\chi = 2\,B_0\,M$.

Moving on to the next nonvanishing order, the subleading terms of the effective Lagrangian are

$$\begin{aligned}\mathcal{L}_4 \;=\;& L_1\,\{\mathrm{Tr}[\partial_\mu U\,(\partial^\mu U)^\dagger]\}^2 \\ &+L_2\,\mathrm{Tr}[\partial_\mu U\,(\partial_\nu U)^\dagger]\,\mathrm{Tr}[\partial^\mu U\,(\partial^\nu U)^\dagger] \\ &+L_3\,\mathrm{Tr}[\partial_\mu U\,(\partial^\mu U)^\dagger\,\partial_\nu U\,(\partial^\nu U)^\dagger] \\ &+L_4\,\mathrm{Tr}[\partial_\mu U\,(\partial^\mu U)^\dagger]\,\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\ &+L_5\,\mathrm{Tr}[\partial_\mu U\,(\partial^\mu U)^\dagger\,(\chi U^\dagger + U\chi^\dagger)] \\ &+L_6\,[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 \\ &+L_7\,[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2 \\ &+L_8\,\mathrm{Tr}(U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger) \\ &+H_2\,\mathrm{Tr}(\chi\chi^\dagger),\end{aligned} \tag{1.7}$$

where L_1 to L_8 are free parameters, so-called low-energy coupling constants, and H_2 is a constant controlling a "contact term" (involving an external field only but allowed by symmetry and possibly required for renormalization). One sometimes includes in $\mathcal{L}_4$ further terms with constants L_9, L_{10} and H_1, but those couple to external fields we will not be introducing. The density $\mathcal{L}_4$ quoted above needs to be supplemented by the Wess-Zumino-Witten action ([23],[24]) to account for the effects of an axial anomaly in its leading order (the effect is of order $O(p^4)$), an issue which we are otherwise ignoring.

1.4. Once we decompose ϕ to components by $\phi = \lambda_a\,\phi_a$ such that $\mathrm{Tr}(\lambda_a\,\lambda_b) = \delta_{ab}$, we have $\frac{F_0}{4}\,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) = \frac{1}{2}\,\partial_\mu \phi_a\,\partial^\mu \phi_a + \ldots$

1.5. This parameter is called the "scalar quark condensate in the chiral limit" due to its relation to the vacuum expectation value of quark bilinears.

Finally, to give a concrete relation of the abstract field ϕ to the physical meson states, let us use π^0, $\pi^{\pm}$, K^0, $\bar{K}^0$, $K^{\pm}$, η labels for the fields whose excitations correspond to the light pseudoscalar meson octet. Then, based on a correspondence by the particles' quantum numbers to the generators of adjoint representation of $SU(3)_V$, we can collect the fields like so:

$$\phi = \begin{pmatrix} \pi^0 + \frac{1}{3}\eta & -\sqrt{2}\,\pi^+ & -\sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & -\sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & -\sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

Note that the phases of the meson fields in ϕ are our convention, adopted for agreement with references in the later chapter on dispersive methods. Since ϕ is a Hermitian field, it follows $(\pi^+)^\dagger = -\pi^-$, which we need to keep in mind when relating the ingoing and outgoing asymptotic charged pion states.

1.3.1. SU(2) variety of ChPT

The SU(2) version of ChPT is conceptually similar to the SU(3) version but differs in some specifics. The $O(p^2)$ part of the Lagrangian is the same but the $O(p^4)$ part is composed of fewer independent chiral invariants due to a simpler group structure. This means the set of free parameters $L_1 \ldots L_8$ from Eq. (1.7) is replaced by a smaller set $l_1 \ldots l_6$. In the SU(2) version of ChPT, the field ϕ is related to the pionic fields only, and that can be:

$$\phi = \begin{pmatrix} \pi^0 & -\sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}, \tag{1.8}$$

1.4. Pion-pion scattering in ChPT to one-loop order

We will now discuss the amplitude of $\pi\pi$ scattering in SU(2) ChPT at one-loop order in the isospin limit of $m_u = m_d =: m$. It will be a result to which we will later, for illustration, compare forms of amplitude constructed on dispersion grounds. Up to $O(p^4)$, we have by (1.5) the following contributing diagrams: a single $\mathcal{L}_2$ or $\mathcal{L}_4$ vertex in a tree diagram or two $\mathcal{L}_2$ vertices in loop diagrams (Fig. 1.2).

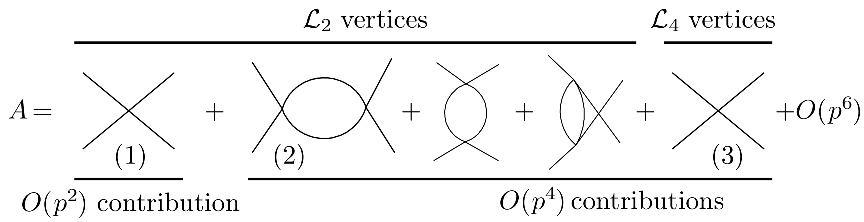

Figure 1.2. Schematic depiction of the leading contributions to pion-pion scattering amplitude in ChPT.

To establish notation, let us write down that we are interested in the process

$$\pi^a(p_a)\,\pi^b(p_b) \to \pi^c(p_c)\,\pi^d(p_d), \tag{1.9}$$

where a, b, c, d are isospin triplet indices of the corresponding incoming and outgoing particles and p_a to p_d are their four-momenta (introduced with a slight abuse of notation).

To obtain the leading tree-level contribution of the graph labeled (1), we expand $\mathcal{L}_2$ with respect to powers of ϕ:

$$\mathcal{L}_2 = \frac{1}{2}\,\partial_\mu\phi_a\,\partial^\mu\phi_a - \frac{M_\pi^2}{2}\,\phi_b\,\phi_b + \frac{1}{6F_0^2}\,[\partial_\mu\phi_i\,\partial^\mu\phi_j\,\phi_i\,\phi_j - \partial_\mu\phi_i\,\partial^\mu\phi_i\,\phi_j\,\phi_j] + \frac{M_\pi^2}{24F_0^2}[(\phi_k\,\phi_k)^2] + O(\phi^6),$$

where we labeled, by $M_\pi^2 = 2\,B_0\,m$, the mass of the pion corresponding to the mass term generated in $\mathcal{L}_2$. It follows that an $O(p^2)$ vertex, as a function of momenta and isospin indices of the lines meeting at the vertex (all momenta ingoing), has the following form:

$$V^{ijkl}(p_1,p_2,p_2,p_3) = \frac{1}{6\,F_0^2}\,\{\delta_{ij}\,\delta_{kl}\,[4\,(p_1+p_2)^2-(p_1-p_2)^2-(p_3-p_4)^2+2\,M_\pi^2]+\cdots\},$$

where the omitted part consists of two symmetric terms, one obtained by exchange of p_1 and p_2 and the other by exchange of p_1 and p_3, accompanied by the corresponding exchanges of isospin indices.

The $O(p^2)$ contribution to the amplitude then quickly follows to be

$$\mathcal{A}_{O(p^2)} = V^{abcd}(p_a,p_b,p_c,p_d) = \frac{1}{F_0^2}\,[\delta_{ab}\,\delta_{cd}\,(s-M_\pi^2)+\delta_{ac}\,\delta_{bd}\,(t-M_\pi^2)+\delta_{ad}\,\delta_{bc}\,(u-M_\pi^2)],$$

where we used the usual Mandelstam variables s, t, u.

For the topic of this thesis which revolves around dispersion relations, we are interested in the non-analytic contributions (non-analytic in kinematic variables) which start appearing at the order of $O(p^4)$. As one can easily convince themself, the tree level $O(p^4)$ contribution is a polynomial and does not contribute a non-analytic part, but the loop diagrams do.

By considering the isospin and crossing symmetries of process (1.9), the amplitude can be expressed in terms of a single a form factor (see Subsection 3.3.1):

$$\mathcal{A}(s,t,u) = \delta^{ab}\,\delta^{cd}A(s,t,u)+\delta^{ac}\,\delta^{bd}\,A(t,s,u)+\delta^{ad}\,\delta^{bc}\,A(u,t,s).$$

We can extract the form factor from the amplitude, e.g., by

$$A\,(s,t,u) = \mathcal{A}(s,t,u)|_{a,b,c,d=1,1,2,2},$$

and that we can also do for the $O(p^2)$, $O(p^4)$ parts of amplitude separately.

So, to account for the $A_{O(p^4)}^{\text{loop}}$ contribution to the form factor from the loop diagrams at $O(p^4)$, we get

$$A_{O(p^4)}^{\text{loop}} = \lim_{\epsilon\to 0+}\frac{(i)^3}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{(\star)}{(k^2-M_\pi^2+i\,\epsilon)[(P-k)^2-M_\pi^2+i\,\epsilon]}$$

where $(\star)$ in the integrand stands for

$$\begin{aligned}(\star) &= V^{11kl}(p_a,p_b,k,-P-k)\,V^{22kl}(p_c,p_d,-k,P+k)\\ &+V^{12kl}(p_a,p_c,k,-P-k)\,V^{12kl}(p_b,p_d,-k,P+k)\\ &+V^{12kl}(p_a,p_c,k,-P-k)\,V^{21kl}(p_d,p_b,-k,P+k).\end{aligned}$$

The integral over loop momenta $A_{O(p^4)}^{\text{loop}}$ evaluated in dimensional regularization and treated by the modified minimal subtraction scheme amounts to

$$\begin{aligned}A_{O(p^4)}^{\text{loop}} &= \frac{s^2-M^4}{2\,F^4}\,\bar{J}(s)+\frac{1}{6F^4}\{(t^2+4u\,M^2-2t\,M^2-2M^4)\,\bar{J}(t)+(u^2+4t\,M^2-2u\,M^2-2\,M^4)\,\bar{J}(u)\}\\ &+\frac{1}{96\pi^2F^4}\left(-3\,s^2-(t-u)^2+\frac{10}{3}\,s\,M^2-\frac{10}{3}\,(t+u)\,M^2+7M^4\right)\log\frac{m^2}{\mu^2}\\ &+\frac{1}{\pi^2F^4}\left(\frac{5}{36}\,M^4-\frac{35}{432}\,s\,M^2-\frac{1}{36}\,(t+u)\,M^2+\frac{1}{8640}\,s^2+\frac{13}{3456}\,s(t+u)+\frac{17}{5760}\,(t^2+u^2)+\right.\\ &\left.\frac{7}{1728}\,t\,u\right),\end{aligned}$$

where $\bar{J}(s)$ is

$$\bar{J}(s) = \lim_{\epsilon\to 0+}\frac{s}{16\,\pi^2}\int_{4m^2}^{\infty}\mathrm{d}x\,\frac{\sqrt{1-4M^2/x}}{x(x-s-i\epsilon)}$$

The $\bar{J}(s)$ terms are the non-analytic contribution to the $O(p^4)$ amplitude. As will be stressed in Chapter 3 discussing dispersion method of amplitude construction, the form of these terms (up to a polynomial difference) is fixed by the condition of the unitarity of the S-matrix.

Chapter 2
Eta-to-pi decay

Experiments involving η decays are important tests of the Standard Model. Recently published data for the decay of η into three neutral pions can be found in Refs. [2], [18], or, for the decay into a pair of charged pions and one neutral pion, in Refs. [1]. Such decays can only occur in violation of isospin symmetry, and as such, they are an interesting link to isospin-breaking parameters. In particular they may provide information [10] on the quark mass ratio

$$R = \frac{m_s - (m_d + m_u)/2}{m_d - m_u}.$$

Literature readily contains results for the ChPT amplitude at orders up to $O(p^4)$ (see Ref. [12]), and the authors of Ref. [5] have worked out the ChPT amplitude up to $O(p^6)$.

2.1. Decay kinematics (Dalitz plot)

In $\eta \to \pi\pi\pi$ and similar processes, the kinematic dependence of process amplitude, or the measured incidences, is customarily presented in the form of a so-called Dalitz plot. In the Dalitz plot, two axes are linearly related to Mandelstam-type variables in such a way that they represent the full variation in decay product kinematics (up to a rotation in the center-of-mass frame of the decaying η).

To elaborate, let us start by defining

$$s_j := (k - p_j)^2 \quad \text{for } j = 1, 2, 3,$$

where k is the four-momentum of the decaying eta meson, and p_j are the four-momenta of the produced pions (each with mass m_j). Over the Mandelstam-type variables a kinematic identity $s_1 + s_2 + s_3 = m_\eta^2 + \sum_{i=1}^{3} m_i^2$ holds, leaving merely two of the variables independent.

As will be stressed in the next chapter, the $\eta \to \pi\pi\pi$ process can be considered related, through crossing, to the scattering processes $\eta\pi \to \pi\pi$. After such crossing, the s_j variables of the decay process are identified with the usual Mandelstam s, t, u variables of a scattering process (the exact pairing depending on the particular identification of pions of the two processes).

We can picture a kinematic plane spanned by any two s_j variables (which up to a scaling of axes and shearing will be the Dalitz plot). The decay process, and the crossed scattering processes, are all confined to their own non-overlapping regions of the kinematic plane. Let us now work out the extent of the decay region. Considering s_1 alone, evaluating it in the rest frame of $p_2 + p_3$, we obtain the lower bound $s_1 = (p_2 + p_3)^2 = (E_2 + E_3)^2 > (m_2 + m_3)^2$ (E_j being the energy of four-momentum p_j). At the same time the center-of-mass energy of $p_2 + p_3$ can be at most $m_\eta - m_1$, and so $s_1 < (m_\eta - m_1)^2$. Similar lower and upper bounds exist on the other s_j by analogy. At this point, we can attest to the separation of kinematic regions by noting that for a scattering process we have $s > (m_\eta + m_\pi)^2$, which is in conflict with the upper bound on s_j in the decay region.

Now, to further confine the decay region, we derive bounds on $(s_2 - s_3)$ given a fixed value of s_1. In the center-of-mass frame of $p_2 + p_3$, it follows from the zero sum of three-momenta $(\vec{p_2} + \vec{p_3}) = \vec{0}$ that $E_2^2 - m_2^2 = E_3^2 - m_3^2$. Now, using this relation to solve for E_2 and E_3 from $s_1 = (E_2 + E_3)^2$, we have

$$E_2 = \frac{s_1 + \Delta_{23}}{2\sqrt{s_1}}, \qquad E_3 = \frac{s_1 - \Delta_{23}}{2\sqrt{s_1}},$$

where we have introduced the notation $\Delta_{AB} = m_A^2 - m_B^2$. Analogously, we have

$$E_\eta = \frac{s_1 + \Delta_{\eta 1}}{2\sqrt{s_1}}, \qquad E_1 = \frac{-s_1 + \Delta_{\eta 1}}{2\sqrt{s_1}}.$$

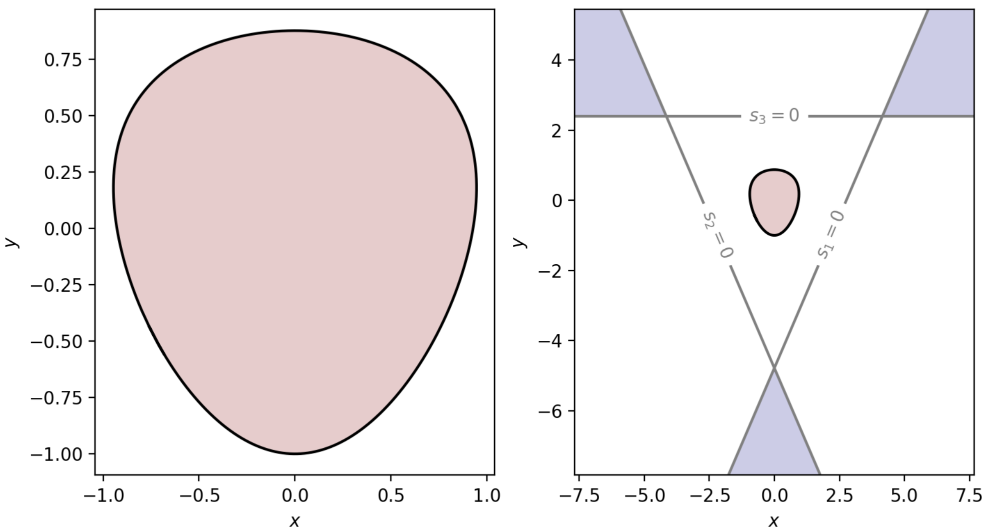

Figure 2.1. Dalitz plot for $\eta \to \pi\pi\pi$ using the usual x, y kinematic variables. The decay region is shaded pink and the scattering regions are shaded purple. Left is detail on the shape of the decay region. Masses of pions are taken equal, i.e. corresponding to $\eta \to \pi^0\pi^0\pi^0$.

Wishing to express the magnitude of three-momenta in terms of s_1, we have

$$|\vec{p}_1| = \sqrt{E_1^2 - m_1^2} = \frac{\sqrt{(s_1 - \Delta_{\eta 1})^2 - 4\, s_1\, m_1^2}}{2\sqrt{s_1}} = \frac{\lambda_{1\eta}^{1/2}(s)}{2\sqrt{s_1}}$$

$$|\vec{p}_3| = \sqrt{E_3^2 - m_3^2} = \frac{\sqrt{(s_1 - \Delta_{23})^2 - 4\, s_1\, m_3^2}}{2\sqrt{s_1}} = \frac{\lambda_{23}^{1/2}(s)}{2\sqrt{s_1}}$$

where we have introduced the square root of the Källen triangle function by $\lambda_{AB}^{1/2}(s) = \sqrt{[s - (m_A + m_B)^2][s + (m_A - m_B)^2]}$. Now using the relations above to express s_2, s_3, we first have

$$s_2 = (p_1 + p_3)^2 = m_1^2 + 2E_1E_3 - 2\vec{p}_1 \cdot \vec{p}_3 + m_3^2 = \frac{3s_0 - s}{2} - \frac{\Delta_{23}\,\Delta_{\eta 1}}{2\,s} - 2\vec{p}_1 \cdot \vec{p}_3$$

$$s_3 = (p_1 + p_2)^2 = m_1^2 + 2E_1E_2 + 2\vec{p}_1 \cdot \vec{p}_3 + m_2^2 = \frac{3s_0 - s}{2} + \frac{\Delta_{23}\,\Delta_{\eta 1}}{2\,s} + 2\vec{p}_1 \cdot \vec{p}_3,$$

where if we subtract the two equations and employ $|\vec{p}_1 \cdot \vec{p}_3| \leqslant |\vec{p}_1|\,|\vec{p}_3| = \frac{\lambda_{1\eta}^{1/2}(s)\,\lambda_{23}^{1/2}(s)}{4s_1}$, we arrive at bounds on $(s_2 - s_3)$ given a fixed value of s_1:

$$\frac{-\Delta_{\eta 1}\,\Delta_{23} - \lambda_{\eta 1}(s_1)\,\lambda_{23}(s_1)}{s_1} \leqslant s_2 - s_3 \leqslant \frac{-\Delta_{\eta 1}\,\Delta_{23} + \lambda_{\eta 1}(s_1)\,\lambda_{23}(s_1)}{s_1}.$$

From this bound and its analogical realization with $s_1 - s_2$ and $s_3 - s_1$, we can draw the shape of the decay region.

For the process $\eta \to \pi^+\pi^-\pi^0$ the standard kinematic variables x, y on the axes of the Dalitz plot are given in terms of s_1, s_2 and s_3 by

$$x = \frac{\sqrt{3}(s_2 - s_1)}{2m_\eta Q} \tag{2.1}$$

$$y = \frac{3[(m_\eta - m_{\pi^0})^2 - s_3] - 1}{2m_\eta Q},$$

where $Q = m_\eta - 2m_{\pi^+} - m_{\pi^0}$ is the kinetic energy of decay products in the center-of-mass frame. For the decay of η into neutral pions, i.e., the process $\eta \to \pi^0\pi^0\pi^0$, the kinematic variables are given by (2.1) also after the replacement of m_{π^+} by m_{π^0}. Using the kinematic variables x, y, we show on Fig. 2.1 the shape of the decay region (of $\eta \to \pi^0\pi^0\pi^0$).

2.2. Dalitz parameters

The customary way of summarizing the results of $\eta \to \pi\pi\pi$ decay experiments is by so-called Dalitz parametrization in which the measured incidences are fitted to a free low-order polynomial in kinematic variables. This is foremost a phenomenological parametrization exploiting the fact that the measured incidences are to a good approximation a smooth function of the kinematic variables.

The Dalitz parametrization does not take into consideration non-analytic features of the amplitude, e.g., the “cusp” [6].

2.2.1. Dalitz parameters of $\eta \to \pi^+\pi^-\pi^0$

In the case of charged decay, the standard parameters are the coefficients of a polynomial in the x, y variables that correspond to the axes of Dalitz plot. Calling the measured incidence by I, its kinematic dependence is fitted to a polynomial with free parameters a, b, c, d, e, f, g like so

$$\frac{I(x,y)}{I(0,0)} = 1 + a\,y + b\,y^2 + c\,x + d\,x^2 + e\,x\,y + f\,y^3 + g\,x^2\,y + \cdots.$$

By charge conjugation, we expect $c = 0$, $e = 0$ since an exchange of the charged pions in the decay product corresponds to flipping the sign of x from where it follows that charge symmetry prohibits terms linear in x. See Table 2.1 for values of the Dalitz parameters from recent experiments.

	a	b	d	f	g
KLOE [7]	−1.090(5)(19)	0.124(6)(10)	0.057(6)(16)	0.14(1)(2)	
KLOE-2 [3]	−1.104(3)(2)	0.142(3)(5)	0.073(3)(4)	0.154(6)(5)	
	−1.095(3)(3)	0.145(3)(5)	0.081(3)(6)	0.141(7)(8)	−0.044(9)(13)

Table 2.1. Estimates of the Dalitz parameters a, b, d, f, g taken from published accounts of few $\eta \to \pi^+\pi^-\pi^0$ experiments.

2.2.2. Dalitz parameters of $\eta \to \pi^0\pi^0\pi^0$

The decay to three neutral pions has additional symmetries, which is reflected in the standard Dalitz parametrization. Defining $z = x^2 + y^2$, the parametrization is

$$\frac{I(x,y)}{I(0,0)} = 1 + \alpha\,z + 2\,\beta\,y(3z - 4y^2) + \gamma\,z^2 + \cdots,$$

where the free parameters are α, β, γ. See Table 2.2 for values of these Dalitz parameters from recent experiments.

	α	β	γ
KLOE 2010 [2]	−0.0301(35)(35)		
A2 Collaboration at MAMI [18]	−0.0302(8)		
	−0.0280(9)	−0.0058(8)	
	−0.0231(33)	−0.0053(8)	−0.0057(37)

Table 2.2. Estimates of the Dalitz parameters α, β, γ taken from published accounts of few $\eta \to \pi^0\pi^0\pi^0$ experiments.

Chapter 3

Dispersive method of amplitude construction

Dispersive methods provide means of amplitude construction (S-matrix construction) based on analyticity and unitarity properties to be satisfied by the elements of the S-matrix. These properties can be justified on the basis of general physical arguments, or may be shown to follow from quantum field theory postulates.

Historically, within the S-matrix programme originally suggested by Heisenberg [13], dispersive methods were used to construct S-matrices accounting for observed scattering phenomena but otherwise detached from any underlying theory, especially one involving local fields.

Within the scope of quantum field theories, dispersive methods can be, for example, of service to aid in modeling of scattering of particles for which it is challenging to define asymptotic states, as is the case of hadrons within QCD, being bound states beyond the reach of direct perturbative treatment. Despite the difficulties in the formulation of hadronic states, we can rely on general properties of the S-matrix elements involving hadronic states.

In the following section a “reconstruction procedure” for meson scattering is introduced. It encapsulates the basic dispersive method of construction of higher-order amplitude (in the sense of chiral ordering) from a lower-order form of it. The present introduction to the procedure is a compilation of an account of the method given in Ref. [25] and [14] which themselves cite Ref. [20] for the establishment of the method in the context of $\pi\pi$ scattering, and cite other references for its subsequent evolution. The procedure is set up in such a way that it can be iterated up to order $O(p^6)$ (that is two-loop order). The application of the reconstruction procedure to processes $\pi\pi\to\pi\pi$ (in isospin limit) and $\eta\to\pi\pi\pi$ (to the first order of isospin breaking) is then commented on.

3.1. Reconstruction procedure

3.1.1. Preliminaries

The reconstruction procedure, as we will establish it, will concern scattering amplitudes $\mathcal{A}(s,t,u)$ of processes with two ingoing and two outgoing particles. The functions $\mathcal{A}(s,t,u)$ will be the usual invariant matrix elements parametrized by Mandelstam variables, that is

$$\langle p_3, p_4|T|p_1, p_2\rangle = (2\pi)^4\,\delta^4(P)\,\mathcal{A}((p_1+p_2)^2, (p_1-p_3)^2, (p_1-p_4)^2),$$

where T is related to the S-matrix by $S=1+iT$, the p_i represent particles by their four-momenta and $P=p_1+p_2-p_3-p_4$ is the momentum balance of the process. Throughout this work, the particles are assumed spinless (as is the case for π, η).

For the use of dispersive methods, we need to extend the domain of the definition of $\mathcal{A}(s,t,u)$ into a complex plane of s, t and u. Nonetheless, the definition domain is restricted to the values satisfying the kinematic identity $s+t+u=\sum m^2$, where the sum is over masses squared of the ingoing and outgoing particles. We will be using $3\,s_0$ to denote the sum. Using analytic continuation, there is still a level of arbitrariness in the definition of $\mathcal{A}$ on the extended domain, but there is a natural choice, and on that point, we simply refer to Ref. [9].

In viewing the function $\mathcal{A}$ as a function of a single complex variable s, t or u (with one of the other variables fixed, and the remaining variable determined from the kinematic identity), it typically has singularities on the real axis. The real axis is, of course, where we are reading off the physical amplitude. To that end, it is crucial that we can, in a simple way, fix the definition of $\mathcal{A}(s,t,u)$ on the real axis in relation to the value it takes on the surrounding complex plane. We can impose the following:

$$\mathcal{A}(s,t,u) = \lim_{\epsilon\to 0+}\mathcal{A}(s+i\epsilon, t, u-i\,\epsilon). \quad (s,t,u\in\text{physical}). \tag{3.1}$$

The label "physical" for the domain of s, t, u represents the set of values that are kinematically feasible. This restriction must be in place for consistency with crossing relations (to be mentioned shortly). The prescription reads, that the functional value of $\mathcal{A}$, when s is on the real axis and in the "physical" set, is the limiting value of approaching the real axis from above. This prescription for the physical amplitude is in natural correspondence to the usual $i\epsilon$ prescriptions in propagators that would appear in perturbative contributions to the S-matrix. This issue is discussed for example in Ref. [9], which also contains a discussion of other properties of $\mathcal{A}$ we are mentioning here.

We assume the validity of crossing relations, which for our purposes state that the amplitude (as a function of complex s, t, u) of process $AB \to CD$ can be made equal to the amplitude of processes $\bar{C}\,\bar{D} \to \bar{A}\,\bar{B}$, $A\,\bar{C} \to \bar{B}\,D$ and $A\,\bar{D} \to \bar{B}\,C$ (bar for antiparticle) under appropriate identification of the s, t, u variables of the original and crossed-into process. Explicitly that means

$$\mathcal{A}_{AB\to CD}(s,t,u) = \mathcal{A}_{\bar{C}\bar{D}\to\bar{A}\bar{B}}(s,t,u) = \epsilon_T \mathcal{A}_{A\bar{C}\to\bar{B}D}(t,s,u) = \epsilon_U \mathcal{A}_{A\bar{D}\to\bar{B}C}(u,s,t),$$

where ϵ_T and ϵ_U are phase factors picked up due to the signs of mesonic fields established in (1.7) and (1.8). The sign convention has the consequence of $\langle\pi^{\pm}|^{\dagger} = -|\pi^{\mp}\rangle$ and analogously for charged kaons. Therefore the phase factor in crossing relations is minus one each time we are crossing an odd number of charged mesons and plus one in all other cases.

For use in dispersion relations, we need to define the Disc symbol for evaluating the discontinuity across a cut on the real axis of s. We define

$$\operatorname{Disc}\mathcal{A}(s,t,u) = \lim_{\epsilon\to 0+} [\mathcal{A}(s+i\,\epsilon, t, u-i\,\epsilon) - \mathcal{A}(s-i\,\epsilon, t, u+i\,\epsilon)].$$

Furthermore, in the case of function $F(s)$ of a single complex variable, we define

$$\operatorname{Disc}F(s) = \lim_{\epsilon\to 0+} [F(s+i\,\epsilon) - F(s-i\,\epsilon)],$$

As a final point, for theories with symmetry such that

$$\langle p_3, p_4|T|p_1, p_2\rangle = \langle p_1, p_2|T|p_3, p_4\rangle, \tag{3.2}$$

it then holds ([9])

$$\operatorname{Disc}\mathcal{A}(s,t,u) = 2\,i\operatorname{Im}\mathcal{A}(s,t,u). \quad (s,t,u\in\mathbb{R}) \tag{3.3}$$

That is, the discontinuity is purely in the imaginary component of the value, and that component flips its sign across the cut.

We can take (3.3) to be true of the $\pi\pi$ scattering amplitude, but in the case of the $\eta\pi\to\pi\pi$ process, the matter is spoiled by the instability of η within the mesonic sector. That can be worked around by a device of analytic continuation in η mass [14], a point to which we will return once we apply the reconstruction procedure to this process.

3.1.2. Dispersion relations

The reconstruction procedure rests on the use of dispersion relations, which we will now derive. Suppose we have a function $F(s)$ of a complex variable s, regular everywhere except for cuts on the real line, constrained to some $s > s_{\text{thr}} > 0$ and (another cut) $s < s'_{\text{thr}} < 0$. The situation is schematically depicted on Fig. 3.1.

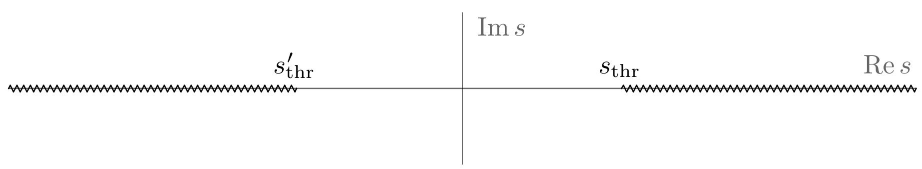

Figure 3.1. The complex plane of s with the indicated location of the two cuts in the definition of $F(s)$.

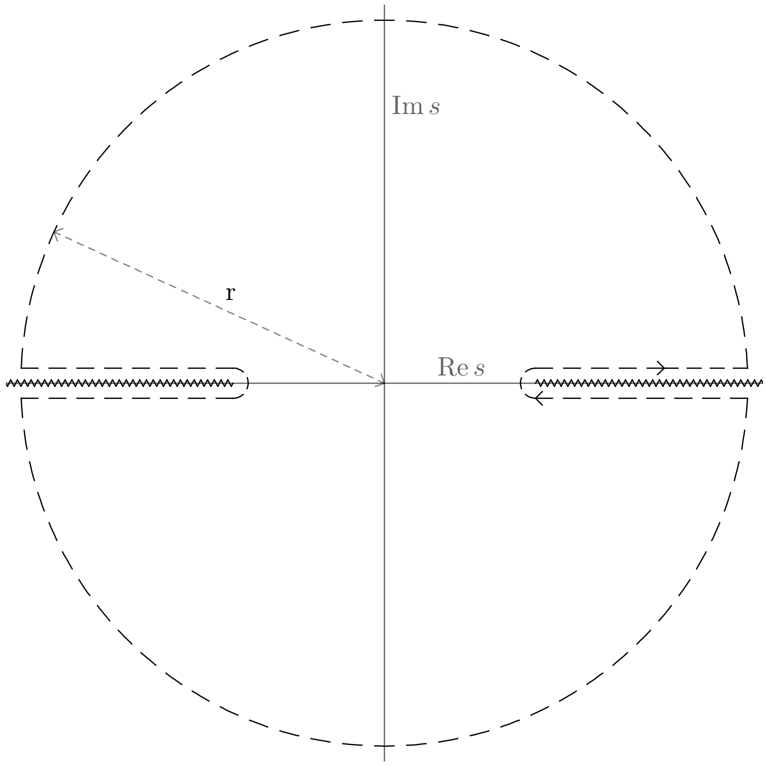

Figure 3.2. Integration contour on the complex plane of s used in derivation of dispersion relations. The contour approaches the cuts of $F(s)$ from below and above in an implicit limit. Inside the contour, $F(s)$ is regular. The outer arches of the contour have radius r.

Consider the integration contour from Fig. 3.2. Now suppose $|F(s)| \to 0$ as $|s| \to \infty$. If we use the contour in Cauchy's integral formula and take the limit $r \to \infty$, the contribution of the outer arches vanishes. We are led to

$$\begin{aligned} F(s) &= \frac{1}{2\pi i} \lim_{\epsilon\to 0+} \left[\int_{s_{\text{thr}}}^{\infty} \mathrm{d}x \, \frac{F(x+i\epsilon) - F(x-i\epsilon)}{x-s} + \int_{-\infty}^{s'_{\text{thr}}} \mathrm{d}x \, \frac{F(x+i\epsilon) - F(x-i\epsilon)}{x-s} \right] \\ &= \frac{1}{2\pi i} \left[\int_{s_{\text{thr}}}^{\infty} \mathrm{d}x \, \frac{\operatorname{Disc} F(x)}{x-s} + \int_{-\infty}^{s'_{\text{thr}}} \mathrm{d}x \, \frac{\operatorname{Disc} F(x)}{x-s} \right]. \end{aligned}$$

Now suppose that it cannot be assumed $|F(s)| \to 0$ as $|s| \to \infty$, but at least $|F(s)/s^n| \to 0$ for some $n \in \mathbb{N}_0$. If we then define $F'(s) = F(s)/s^n$ and apply Cauchy's integral formula to $F'(s)$ in an analogous manner, we obtain a similar result for $F'(s)$ as we did for $F(s)$ above, except we pick up residual terms due to the introduced pole at $s = 0$. We obtain

$$\begin{aligned} F'(s) &= \frac{1}{2\pi i} \left[\int_{s_{\text{thr}}}^{\infty} \mathrm{d}x \, \frac{\operatorname{Disc} F'(x)}{x-s} + \int_{-\infty}^{s'_{\text{thr}}} \mathrm{d}x \, \frac{\operatorname{Disc} F'(x)}{x-s} \right] \\ & \underbrace{- \frac{1}{(n-1)!} \left[\frac{\mathrm{d}}{\mathrm{d}x^{n-1}} \frac{F(x)}{x-s} \right]_{x=0}}_{\xi}. \end{aligned} \tag{3.4}$$

If we then proceed with the expansion

$$\frac{1}{x-s} = -\frac{1}{s} - \frac{x}{s^2} - \frac{x^2}{s^3} + \cdots - \frac{x^{n-1}}{s^n} + O(x^n),$$

we can conclude that the residual term ξ is a Laurent polynomial in s with terms the power of which ranges from $-n$ to -1 and such that its coefficients depend on the value and first $n-1$ derivatives of $F(x)$ at $x=0$. If we then substitute $F(s)$ into (3.4), we have

$$F(s) \;=\; \frac{s^n}{2\pi i}\left[\int_{s_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,F'(s)}{(x-s)\,x^n}+\int_{-\infty}^{s'_{\rm thr}}\mathrm{d}x\,\frac{\mathrm{Disc}\,F'(x)}{(x-s)\,x^n}\right]+P_n(s), \tag{3.5}$$

where $P_n(s)$ is a so-called subtraction polynomial. It is equal to $-s^n\,\xi$ and so is a polynomial in s of degree $n-1$. Eq. (3.5) is called an n-times subtracted dispersion relation. It is a consequence of the assumed analytic structure of $F(s)$ in combination with the assumption $|F(s)/s^n|\to 0$ as $|s|\to\infty$. Since the integrals as functions of s are regular around $s=0$ and are multiplied by s^n, we can conclude that $P_n(s)$ is in fact the Taylor polynomial of $F(s)$ around $s=0$, as explicit evaluation of $-s^n\,\xi$ would confirm. When we wish to express arbitrary $F(s)$ in terms of its discontinuities at known branch cuts by means of (3.5), the subtraction polynomial $P_n(s)$ represents a fully analytic component of $F(s)$ that is not reproduced by the dispersion integrals but is constrained by the input assumption $|F(s)/s^n|\to 0$.

In application of (3.5) to an amplitude $\mathcal{A}(s,t,u)$, we can substitute e.g. $F(s)=\mathcal{A}(s,t,3\,s_0-u)$, resulting in

$$\mathcal{A}(s,t,3s_0-t-s) \;=\; \frac{s^n}{2\pi i}\left[\int_{s_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(x,t,3s_0-t-x)}{(x-s)\,x^n}+\int_{-\infty}^{s'_{\rm thr}}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(x,t,3s_0-t-x)}{(x-s)\,x^n}\right]+ P_n(s;t), \tag{3.6}$$

where we signified in arguments of $P_n(s;t)$ that it is a polynomial in s with coefficients that are functions of t.

It is natural to replace $s'_{\rm thr}$ by $u_{\rm thr}$ through $s'_{\rm thr}=3s_0-t-u_{\rm thr}$, since $u_{\rm thr}$ then indicates the threshold of the respective cut when the amplitude is viewed as a function of u (with t held fixed). To make the second integral similar in appearance to the first, we can also change

$$\begin{aligned} s^n\int_{-\infty}^{s'_{\rm thr}}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(x,t,3s_0-t-x)}{(x-s)\,x^n} \;&=\; \underbrace{-s^n\int_{u_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(3s_0-t-x,t,x)}{(x-u)\,(3s_0-t-x)^n}}_{(\star)} \\ &=\; -u^n\int_{u_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(3s_0-t-x,t,x)}{(x-u)\,x^n}+M_n(s), \end{aligned}$$

where $M_n(s;t)$ is some polynomial of degree $n-1$ in s. In the first equality, we merely changed the integration variable under $x\to 3s_0-t-x$ and swapped the limits of integration. The second equality is more subtle, one way to prove it is to note that, if $u_{\rm thr}>0$, the expression $(\star)/2\pi i$, as a function of u, on its own satisfies the assumptions of (3.5), so if we apply (3.5) to $F(u):=(\star)/2\pi i$, we obtain the desired result. The introduced polynomial $M_n(s)$ can be absorbed into a redefinition of the subtraction polynomial $P_n(s;t)$, so we have

$$\begin{aligned} \mathcal{A}(s,t,3s_0-t-s) \;=\; & \frac{s^n}{2\pi i}\int_{s_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(x,t,3s_0-t-x)}{(x-s)\,x^n} \\ & -\frac{u^n}{2\pi i}\int_{u_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(3s_0-t-x,t,x)}{(x-u)\,x^n} \\ & +P_n(s;t). \end{aligned} \tag{3.7}$$

The form (3.6) or (3.7) would be called the fixed-t n-times subtracted dispersion relation since in the definition of the underlying function of a single complex variable we held t fixed. In the derivation of (3.7), we so far assumed $s'_{\rm thr}<0<s_{\rm thr}$, $0<u_{\rm thr}$, but it suffices $s'_{\rm thr}<s_{\rm thr}$, $0<u_{\rm thr}$, $0<s_{\rm thr}$, which we do not show here.

3.1.3. Derivation

Exploited by the reconstruction procedure, the discontinuities of $\mathcal{A}$ (in the sense of quantity $\mathrm{Disc}\,\mathcal{A}$) are constrained by unitarity. Specifically, from $SS^\dagger=1+iT-iT^\dagger+TT^\dagger=1$, it follows

$$i\,\langle p_3,p_4|T-T^\dagger|p_1,p_2\rangle=-\langle p_3,p_4|TT^\dagger|p_1,p_2\rangle.$$

We will manipulate the left-hand and right-hand sides separately. On the left-hand side, we have

$$\begin{aligned} i\langle p_3, p_4|T - T^\dagger|p_1, p_2\rangle &= \langle p_3, p_4|T|p_1, p_2\rangle - \langle p_1, p_2|T|p_3, p_4\rangle^\star \\ &= \langle p_3, p_4|T|p_1, p_2\rangle - \langle p_3, p_4|T|p_1, p_2\rangle^\star \\ &= 2i\,\mathrm{Im}\,\langle p_3, p_4|T|p_1, p_2\rangle \\ &= 2i\,\mathrm{Im}\,(2\pi)^4\,\delta^4(P)\,\mathcal{A}(s,t,u), \end{aligned}$$

while on the right-hand side we have (introducing an abstract sum over intermediate states n)

$$\begin{aligned} -\langle p_3, p_4|TT^\dagger|p_1, p_2\rangle &= i\sum_n \langle p_3, p_4|T|n\rangle\,\langle n|T^\dagger|p_1, p_2\rangle \\ &= i\sum_n \langle p_3, p_4|T|n\rangle\,\langle p_1, p_2|T|n\rangle^\star, \end{aligned}$$

and so it follows

$$2i\,\mathrm{Im}\,(2\pi)^4\,\delta^4(P)\,\mathcal{A}(s,t,u) = i\sum_n \langle p_3, p_4|T|n\rangle\,\langle p_1, p_2|T|n\rangle^\star. \tag{3.8}$$

We will make use of (3.8) after the partial-wave projection of $\mathcal{A}$. At this point let us remark that (3.8) will be the basis for the construction of higher-order amplitude approximations from lower-order ones. It mixes different orders of the amplitude in the sense that the product of lower-order components on the right-hand side is made equal to the imaginary part of the higher-order component of the amplitude on the left-hand side. From the imaginary part along a cut given by (3.8) or its variant in spirit, a dispersion relation reproduces the higher-order component in full, *up to the subtraction polynomial*.

To apply the dispersion relations, we need to assume that (for fixed t) the function $\mathcal{A}(s,t,3s_0-t-s)$ (viewed as a function of a single complex variable s) is analytic except for cuts on the real s-axis, which are restricted to regions $s > s_{\mathrm{thr}}$ and $u = 3s_0 - t - s > u_{\mathrm{thr}}$ for some $s_{\mathrm{thr}}, u_{\mathrm{thr}} \in \mathbb{R}^+$. The s_{thr} threshold can be provided by (3.8) in combination with (3.3), analogously u_{thr} can be obtained after considering a crossed amplitude. The fact that the amplitude is otherwise analytic can be attested to by studying perturbative contributions to it, a point on which we again refer to Ref. [9].

To implement the program sketched in the previous paragraphs, we start by projecting $\mathcal{A}$ on the first two partial waves:

$$\mathcal{A}(s,t,u) = 16\pi[t_0(s) + 3t_1(s)\cos\theta] + \mathcal{A}_{\ell>2}(s,t,u), \tag{3.9}$$

where $\cos\theta$ is the cosine of the scattering angle, $t_0(s)$ and $t_1(s)$ are the S and P partial-wave projections and $\mathcal{A}_{\ell>2}$ is the remainder of the decomposition. The cosine of the scattering angle is expressed in terms of Mandelstam variables by the kinematic identity

$$\cos\theta = \frac{s(t-u) + \Delta_{AB}\,\Delta_{CD}}{\lambda_{AB}^{1/2}(s)\,\lambda_{CD}^{1/2}(s)}, \tag{3.10}$$

where $\lambda_{AB}(s) = \lambda(s, m_A^2, m_B^2) = [s - (m_A + m_B)^2][s + (m_A - m_B)^2]$ is the Källen triangle function on the masses of particles A, B and $\Delta_{AB} = m_A^2 - m_B^2$.

The $\mathcal{A}_{\ell>2}$ remainder of (3.9) can be neglected to the order of reconstruction we are interested in, which will be commented on shortly. The explicit expression of $t_\ell(s)$ is given by

$$t_\ell(s) = \frac{1}{32\pi}\int_{-1}^{+1} \mathrm{d}(\cos\theta)\,(\cos\theta)^\ell\,\mathcal{A}(s,t,u). \tag{3.11}$$

The analogue of (3.8) projected on partial waves, neglecting intermediate states with more than two particles (see [26] for discussion), is

$$\mathrm{Im}\,t_\ell^{i\to f}(s) = \sum_k \frac{1}{S_k}\,\frac{\lambda_k^{1/2}(s)}{s}\,t_\ell^{i\to k}(s)\,[t_\ell^{f\to k}(s)]^\star\,\theta(s - s_{\mathrm{thr}}^k), \tag{3.12}$$

where we denoted by $t_\ell^{i\to f}(s)$ the ℓth partial wave of process $i\to f$. The sum is introduced over quantum numbers (selection of particles) in intermediate states. $s_{\rm thr}^k$ is the minimal center-of-mass energy of an intermediate state of kind k. If k contains, say, particles AB, then $s_{\rm thr}^k$ is of course equal to $(m_A+m_B)^2$. S_k is a symmetry factor:

$$S_k=\begin{cases} 2 & \text{if the particles in } k \text{ are undistinguishable} \\ 1 & \text{otherwise}\end{cases}.$$

By means of (3.12) we can show $\mathrm{Im}\,\mathcal{A}_{\ell>2}(s,t,u)=O(p^8)$. First we note a leading $O(p^2)$ amplitude must be analytic since by (3.12) discontinuities are of order $O(p^4)$. Therefore the $O(p^2)$ amplitude must be a first-degree polynomial in s, t, u, and, as such, it cannot contribute to $\mathcal{A}_{\ell>2}$. If then $\mathcal{A}_{\ell>2}=O(p^4)$ it follows from (3.12) again that $\mathrm{Im}\,\mathcal{A}_{\ell>2}(s,t,u)=O(p^8)$.

At this point, we invoke the dispersion relation (3.7) with two modifications. As the first modification, we introduce a crossed amplitude $\mathcal{A}^U(u,t,s){:=}\mathcal{A}(s,t,u)$ into its second integral[3.1]. This is done in advance of using a different expansion in partial waves (the partial waves of the crossed amplitude will be functions of u). As the second modification, we extend the degree of the subtraction polynomial from $n-1$ to n. That is prompted by the reasoning provided, e.g., in Ref. [25], that the additional free terms of the polynomial will account for integrating into high-energy region with an amplitude that is built on the basis of a low-energy effective theory with limited applicability. With the modifications, we have

$$\begin{aligned}\mathcal{A}(s,t,3s_0-t-s) &= \frac{s^n}{2\pi i}\int_{s_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}(x,t,3s_0-t-x)}{(x-s)\,x^n}\\ &+\frac{u^n}{2\pi i}\int_{u_{\rm thr}}^{\infty}\mathrm{d}x\,\frac{\mathrm{Disc}\,\mathcal{A}^U(x,t,3s_0-t-x)}{(x-u)\,x^n}\\ &+P_{n+1}(s;t).\end{aligned}$$

Now we insert the projection (3.9) applied to $\mathcal{A}(s,t,u)$ and separately to $\mathcal{A}^U(s,t,u)$, distinguishing the partial waves of the latter by a U superscript, and we also substitute $\mathrm{Disc}\,t_\ell(s)=2\,\mathrm{Im}\,t_\ell(s)$, finally we arrive at[3.2]

$$\begin{aligned}\mathcal{A}(s,t,3s_0-t-s) &= 16\,s^n\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-s)\,x^n}\,\mathrm{Im}\,t_0(x)\\ &+48\,s^n\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-s)\,x^n}\,\mathrm{Im}\,t_1(x)\,\frac{x(t-u-s+x)+\Delta_{AB}\,\Delta_{CD}}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)}\\ &+16\,u^n\int_{u_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-u)\,x^n}\,\mathrm{Im}\,t_0^U(x)\\ &+48\,u^n\int_{u_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-u)\,x^n}\,\mathrm{Im}\,t_1^U(x)\,\frac{x(t-u-s+x)+\Delta_{AD}\,\Delta_{CB}}{\lambda_{AD}^{1/2}(x)\,\lambda_{CB}^{1/2}(x)}\\ &+P_{n+1}(s;t)+O(p^8),\end{aligned}$$

where we kept the S and P partial waves only and signified that the remainder is $O(p^8)$.

Manipulating the integrals containing $t_1(x)$ and $t_1^U(x)$, we have, respectively

$$\begin{aligned}s^n\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-s)\,x^n}\,\mathrm{Im}\,t_1(x)\,\frac{x(t-u-s+x)}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)} &= s^n\,(t-u)\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{(x-s)\,x^{n-1}}\,\frac{\mathrm{Im}\,t_1(x)}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)}\\ &\underbrace{-s^n\int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}x}{x^{n-1}}\,\frac{\mathrm{Im}\,t_1(x)}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)}}_{(\star)}\end{aligned}$$

3.1. Note the change of sign due to $\mathrm{Disc}\,\mathcal{A}^U(s,t,u)=-\mathrm{Disc}\,\mathcal{A}(u,t,s)$.

3.2. One must be careful to apply the right permutation on A, B, C, D particle symbols corresponding to the identification of s, t, u variables in the original and crossed process.

and

$$u^n\int_{u_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-u)\,x^n}\,{\rm Im}\,t_1^U(x)\,\frac{x(t-u-s+x)}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)} \;=\; u^n\,(t-s)\int_{s_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-s)\,x^{n-1}}\,\frac{{\rm Im}\,t_1^U(x)}{\lambda_{AD}^{1/2}(x)\,\lambda_{BC}^{1/2}(x)} + \underbrace{u^n\int_{s_{\rm thr}}^{\infty}\frac{{\rm d}x}{x^{n-1}}\,\frac{{\rm Im}\,t_1^U(x)}{\lambda_{AD}^{1/2}(x)\,\lambda_{BC}^{1/2}(x)}}_{(\star\star)}.$$

The terms labeled $(\star)$ and $(\star\star)$ can be absorbed into a redefinition of the $P_{n+1}(s;t)$ subtraction polynomial (they are polynomial in s) and, as such, will be omitted going forward.

To state the result derived from the dispersion relation so far, we have:

$$\begin{aligned}\mathcal{A}(s,t,3s_0-t-s) \;=\;& 16\pi\underbrace{\frac{s^n}{\pi}\int_{s_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-s)\,x^n}\left\{{\rm Im}\,t_0(x)+\frac{3\,\Delta_{AB}\,\Delta_{CD}}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)}\,{\rm Im}\,t_1(x)\right\}}_{W_0(s)}\\ &+16\pi\,(3t-3u)\underbrace{\frac{s^n}{\pi}\int_{s_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-s)\,x^{n-1}}\,\frac{{\rm Im}\,t_1(x)}{\lambda_{AB}^{1/2}(x)\,\lambda_{CD}^{1/2}(x)}}_{W_1(s)}\\ &+16\pi\underbrace{\frac{u^n}{\pi}\int_{u_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-u)\,x^n}\left\{{\rm Im}\,t_0^U(x)+\frac{3\,\Delta_{AD}\,\Delta_{BC}}{\lambda_{AD}^{1/2}(x)\,\lambda_{BC}^{1/2}(x)}\,{\rm Im}\,t_1^U(x)\right\}}_{W_0^U(u)}\\ &+16\pi\,(3t-3s)\underbrace{\frac{u^n}{\pi}\int_{u_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-u)\,x^{n-1}}\,\frac{{\rm Im}\,t_1^U(x)}{\lambda_{AD}^{1/2}(x)\,\lambda_{BC}^{1/2}(x)}}_{W_1^U(u)}\\ &+P_{n+1}(s;t)+O(p^8),\end{aligned} \tag{3.13}$$

where we have labeled parts of the expression by the newly introduced $W_{0,1}(s)/W_{0,1}^U(u)$ family of functions. We can define $W_{0,1}^T(t)$ in terms of partial waves $t_0^T(t), t_1^T(t)$ of $\mathcal{A}^T(t,u,s):=\mathcal{A}(s,t,u)$ in complete analogy to $W_{0,1}^U(u)$. That is

$$\begin{aligned}W_0^T(t) \;&=\; \frac{t^n}{\pi}\int_{t_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-u)\,x^n}\left\{{\rm Im}\,t_0^U(x)+\frac{3\,\Delta_{AC}\,\Delta_{DB}}{\lambda_{AC}^{1/2}(x)\,\lambda_{DB}^{1/2}(x)}\,{\rm Im}\,t_1^U(x)\right\}\\ W_1^T(t) \;&=\; \frac{t^n}{\pi}\int_{t_{\rm thr}}^{\infty}\frac{{\rm d}x}{(x-u)\,x^{n-1}}\,\frac{{\rm Im}\,t_1^U(x)}{\lambda_{AC}^{1/2}(x)\,\lambda_{DB}^{1/2}(x)}\end{aligned} \tag{3.14}$$

We will use these new functions to rewrite (3.13) as follows:

$$\begin{aligned}\mathcal{A}(s,t,u) \;=\;& 16\pi\,[W_0(s)+3(t-u)\,W_1(s)+W_0^U(u)+3(t-s)\,W_0^U(u)+W_0^T(t)+3(u-s)\,W_0^U(t)]\\ &+P_{n+1}(s;t)+O(p^8),\end{aligned} \tag{3.15}$$

where the addition of $W_0^T(t)+3(u-s)\,W_0^U(t)$ (being a polynomial in s with coefficients that are functions of t) can be canceled by yet another redefinition of $P_{n+1}(s;t)$. The expression on the right-hand side of (3.15) appears symmetrical in s, t, u, except for the $P_{n+1}(s;t)$ polynomial, which singles out t (the coefficients can be arbitrary functions of t). To reform the polynomial, we invoke the following argument: We could have redone the derivation up to this point on the amplitude $\mathcal{A}'(s,u,t)=\mathcal{A}(s,t,u)$, by which we would have arrived at (3.15) for $\mathcal{A}(s,t,u)$, except that the subtraction polynomial would be of the form $P'_{n+1}(s;u)$ (signifying it is polynomial in s with coefficients being functions of u). Since in both cases, the left-hand side of (3.15) is the same, and all the other terms on the right-hand side are also the same (up to $O(p^8)$, but we don't consider $n>3$), it must be that the two forms of the subtraction polynomial are equal:

$$P_{n+1}(s;t)=P'_{n+1}(s;u),$$

from where it follows that $P_{n+1}(s;t)$ in (3.15) is, in fact, a polynomial in s, t, u. Inserting this into (3.16), we arrive at the final expression for $\mathcal{A}(s,t,u)$ that forms the basis of the reconstruction procedure. To quote the final result:

$$\begin{aligned}\mathcal{A}(s,t,u) &= 16\pi\,[W_0(s)+3(t-u)\,W_1(s)+W_0^U(u)+3(t-s)\,W_0^U(u)+W_0^T(t)+3(u-s)\,W_0^U(t)]\\ &\quad +P_{n+1}(s,t,u)+O(p^8).\end{aligned} \tag{3.16}$$

Having an approximate expression for the amplitude up to some chiral order $O(p^n)$, we can use (3.16) to obtain the amplitude of order $O(p^{n+2})$ expressed in terms of a free subtraction polynomial and dispersion integrals over S and P partial waves of the lower-order amplitude, that is, up to the $O(p^8)$ remainder.

The number of subtractions n has up to this point not been determined. According to the reasoning found e.g. in [25] considering the tails of the dispersion integrals into the high-energy region beyond the applicability of effective theory, one should use $n=3$ in the derivation of Eq. (3.16).

3.2. Lowering subtractions

In setting $n=3$, the functions $W_{0,1}^{S/U/T}(s)$ appearing in Eq. (3.16) and defined in Eq. (3.13) and (3.14), are expressed in terms of thrice and twice subtracted dispersion integrals over expressions involving the imaginary part of the input partial waves. We can make use of the following equality

$$\begin{aligned} s^n\int_{s_{\text{thr}}}^{\infty}\frac{f(x)\,\mathrm{d}x}{x^n(x-s)} &= s^n\int_{s_{\text{thr}}}^{\infty}\left[\frac{1}{x^{n-1}(x-s)\,s}-\frac{1}{x^n s}\right]f(x)\,\mathrm{d}x \\ &= s^{n-1}\int_{s_{\text{thr}}}^{\infty}\frac{f(x)\,\mathrm{d}x}{x^{n-1}(x-s)}-s^n\int_{s_{\text{thr}}}^{\text{thr}}\frac{f(x)\,\mathrm{d}x}{x^n s}\end{aligned} \tag{3.17}$$

to express an n-times subtracted dispersion integral in terms of an $(n-1)$-times subtracted integral and a remainder, which is monomial in s, provided the integrals on the right-hand side converge. If the monomial remainder can then be absorbed into the redefinition of the subtraction polynomial, or the monomial can be dropped due to its order, one can freely decrease the number of subtractions in the integrals. For a more thorough discussion of the restrictions that may apply, we refer to Section 2.5 of [25].

3.3. Pion-pion scattering

First we will comment on applying the reconstruction procedure to $\pi\pi$ scattering in the isospin limit.

3.3.1. Form-factors

Before applying the reconstruction procedure, we first make use of an assumed isospin symmetry to extract form-factors of $\pi\pi$ scattering. We attach four isospin indices to a general pion-pion scattering amplitude $\mathcal{A}^{ijkl}(s,t,u)$ as follows:

$$\mathcal{A}^{ijkl}(s,t,u)=\mathcal{A}(\pi^i\,\pi^j\to\pi^k\,\pi^l;s,t,u)$$

In the above, π^k, $k\in\{1,2,3\}$, stand for the canonical basis vectors in vectorial representation of the pion isospin triplet. Based on symmetry considerations, $\mathcal{A}^{ijkl}$ must be expressible in terms of to-be-determined isoscalar form factors A, B and C:

$$\mathcal{A}^{ijkl} = A\,\delta^{ij}\,\delta^{kl}+B\delta^{ik}\,\delta^{jl}+C\delta^{il}\,\delta^{jk}$$

Now, for processes of interest involving the charge states π^+, π^-, π^0, we have (s, t, u arguments omitted):

$$\begin{aligned}
\mathcal{A}(\pi^+\pi^- \to \pi^0\pi^0) &= -\frac{1}{2}\mathcal{A}^{1133} - \frac{1}{2}\mathcal{A}^{2233} - i\mathcal{A}^{2133} + i\mathcal{A}^{1233} \\
&= -A \\
\mathcal{A}(\pi^+\pi^- \to \pi^+\pi^-) &= \frac{1}{4}\Big(\underbrace{\mathcal{A}^{1111} + \mathcal{A}^{2222}}_{2A+2B+2C} + \overbrace{\mathcal{A}^{1122} + \mathcal{A}^{2211}}^{2A} + \underbrace{\mathcal{A}^{1212} + \mathcal{A}^{2121}}_{2B} \overbrace{- \mathcal{A}^{1221} - \mathcal{A}^{2112}}^{-2C}\Big) \\
&= A + B \\
\mathcal{A}(\pi^0\pi^0 \to \pi^0\pi^0) &= \mathcal{A}^{3333} = A + B + C \\
\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0) &= \frac{1}{2}\Big(\underbrace{\mathcal{A}^{1313} + \mathcal{A}^{2323}}_{2B} + \underbrace{i\mathcal{A}^{1323} - i\mathcal{A}^{2313}}_{0}\Big) \\
&= B
\end{aligned}$$

We can relate A, B, C to each other. From Bose symmetry of the outgoing particles we have $\mathcal{A}^{ijkl}(s,t,u) = \mathcal{A}^{ijlk}(s,u,t)$, which in turn implies

$$\begin{aligned}
B(s,t,u) &= C(s,u,t) \\
A(s,t,u) &= A(s,u,t)
\end{aligned}$$

Furthermore from the crossing relation[3.3] $\mathcal{A}(\pi^+\pi^- \to \pi^0\pi^0; s,t,u) = -\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0; t,s,u)$, we have

$$-A(s,t,u) = -B(t,s,u).$$

All the form-factors and considered process amplitudes can thus be expressed in terms of a single function A.

We can summarize the result so far:

$$\begin{aligned}
\mathcal{A}(\pi^+\pi^- \to \pi^0\pi^0; s,t,u) = \mathcal{A}_x(s,t,u) &= -A(s,t,u) \\
\mathcal{A}(\pi^+\pi^- \to \pi^+\pi^-; s,t,u) = \mathcal{A}_{+-}(s,t,u) &= A(s,t,u) + A(t,u,s) \\
\mathcal{A}(\pi^0\pi^0 \to \pi^0\pi^0; s,t,u) = \mathcal{A}_{00}(s,t,u) &= A(s,t,u) + A(t,u,s) + A(u,s,t) \\
\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0; s,t,u) = \mathcal{A}_{+0}(s,t,u) &= A(t,u,s)
\end{aligned} \tag{3.18}$$

Here we introduced the x, $+-$, 00, $+0$ shorthands for processes, which we will be using from now on.

3.3.2. Leading-order amplitude

We start by writing down a general form of the amplitude for $\pi\pi$ scattering at the leading chiral order $O(p^2)$. At this order, the amplitude is analytic: (1.5) admits tree-level contributions only, and (3.8) shows any discontinuity is of order $O(p^4)$.

The amplitude must, therefore, be a first-order polynomial in s, t, u, so let us use free parameters α_0, β_0 and write

$$-\mathcal{A}(\pi^+\pi^- \to \pi^0\pi^0; s,t,u) = A(s,t,u) = \underbrace{16\,\pi\,(\alpha_0 + \beta_0\, s)}_{A_0(s,t,u)} + O(p^4),$$

We labeled by A_0 the $O(p^2)$ contribution proper. Linear terms with t or u need not be included from symmetry[3.4].

To obtain form-factor A at $O(p^4)$, we will apply the reconstruction procedure to $\pi^+\pi^- \to \pi^0\pi^0$. Input to the procedure will be partial waves extracted from the $O(p^2)$ form factor A_0 when it is viewed through the lens of some of the processes in (3.18). The $O(p^2)$ partial waves will be labeled by $\varphi_\ell^{i \to f}(s)$, and with the process shorthands introduced earlier, they read

3.3. As a reminder, in the convention we have adopted, a phase factor -1 is picked up when crossing charged pions.

3.4. Taking t for example, we have $2t = (t+u) + (t-u) = (4m_\pi^2 - s) + (t-u)$, where the term in the second pair of parentheses is ruled out by Bose symmetry.

$$\varphi_0^x(s) = -\alpha_0 - \beta_0 s \qquad \varphi_0^{00}(s) = 4M_\pi^2\beta_0 + 3\alpha_0$$
$$\varphi_1^x(s) = 0 \qquad \varphi_1^{00}(s) = 0$$
$$\varphi_0^{+0}(s) = \alpha_0 - \frac{1}{2}\beta_0(s - 4M_\pi^2)$$
$$\varphi_1^{+0}(s) = \frac{1}{6}\beta_0(s - 4M_\pi),$$

where M_π is the average mass of the pion.

3.3.3. Next-to-leading-order amplitude

Using Eq. (3.16) to obtain the imaginary part of $O(p^4)$ partial waves and inserting such into the dispersion integrals contained in the $W_{0,1}^{S/T/U}(s)$ functions, the integrals take on the forms

$$s^3 \int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{\sigma(x, M_\pi)}{x^3(x-s)} Q(x),$$

and

$$s^2 \int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{\sigma(x, M_\pi^2)}{x^2(x-s)} L(x),$$

where $Q(s)$, $L(s)$ are polynomials in s and $\sigma(x, M_\pi^2)$ is defined according to Appendix A. By the change illustrated in Eq. (3.17), we can decrease the number of subtractions in the integral, dealing with the terms of $Q(s)/L(s)$ individually and eventually arriving at

$$Q(s)\, s \int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{\sigma(x, M_\pi)}{x(x-s)},$$

and

$$L(x) \underbrace{s \int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{\sigma(x, M_\pi^2)}{x(x-s)}}_{16\pi \bar{J}(s, M_\pi)},$$

as effectively equal and simplified versions of the initial integrals. These integrals are, up to a multiplicative constant, proportional to the function $\bar{J}(s)$ appearing in the ChPT amplitude for the same process.[3.5]

Now we quote the resulting expressions for $W_{0,1}^{S/T/U}(s)$ functions:

$$W_0(s) = -8\pi\,[8M_\pi^2(\alpha_0 + \beta_0 s)\beta_0 + 7\alpha_0^2 + 8\alpha_0\beta_0 s + \beta_0^2 s^2]\,\bar{J}(s, M_\pi) + O(p^6)$$
$$W_1(s) = O(p^6)$$
$$W_0^T(t) = W_0^U(t) = 4\pi\,[16M_\pi^2(\alpha_0\beta_0 - M_\pi^2\beta_0^2) + 8M_\pi^2\beta_0^2 t + 4\alpha_0(\beta_0 t - \alpha_0) - \beta_0^2 t^2]\,\bar{J}(t, M_\pi) + O(p^6)$$
$$W_1^T(t) = W_1^U(t) = 4\pi\beta_0^2(t - 4M_\pi^2)\,\bar{J}(t, M_\pi)/9 + O(p^6)$$

In terms of the above functions, the $O(p^4)$ amplitude is then

$$\mathcal{A}(s,t,u) = 16\pi\,[W_0(s) + 3(t-u)W_1(s) + W_0^U(u) + 3(t-s)W_0^U(u) + W_0^T(t) + 3(u-s)W_0^T(t)] + P_3(s,t,u) + O(p^6), \tag{3.19}$$

where $P_3(s,t,u)$ is an $O(p^4)$ subtraction polynomial. Comparing the result to ChPT amplitude of Section 1.4, we see that by matching the $O(p^2)$ amplitude ($\alpha_0 = 1/16\pi F_0^2$, $\beta_0 = -M_\pi^2/16\pi F_0^2$) we arrive at the same expressions for the $O(p^4)$ non-analytic part, which is as expected, taking into account its form constrained by unitarity (up to an analytic remainder).

For later use in the construction of the $\eta\pi \to \pi\pi$ amplitude, we will need expressions for S, P partial waves of the reconstructed $O(p^4)$ $\pi^+\pi^- \to \pi^0\pi^0$ amplitude (and of amplitudes related through the form factor A introduced earlier). The dispersive $O(p^4)$ part of the partial waves can be expressed in terms of polynomials in s times a part $k_j(s)/\sigma(s)$, where $k_j(s)$ is a set of utility functions defined in Appendix A. Without giving an explicit expression of the polynomial $\xi_{i\to f,\ell}^{(j)}(s)$, the $O(p^4)$ dispersive part of S, P partial waves $\Psi_{0/1}^{i\to f}(s)$ can be expressed

$$\Psi_\ell^{i\to f}(s) = \frac{1}{\sigma(s)} \sum_{j=0}^{4} \xi_{i\to f,\ell}^{(j)}(s)\, k_j(s).$$

3.5. Appendix A has a closed form expression of the function.

This expression is chosen for convenience in the second iteration of the reconstruction procedure. We will need to perform dispersive integrals of a polynomial times $k_j(s)$, the result of which will be mechanical replacement of $k_j(s)$ in the integrand by appropriately defined $\bar{K}_j(s)$ (see Appendix A).

3.4. Eta-to-pi decay

To dispersively construct an amplitude for the $\eta \to \pi\pi\pi$ decay processes, we use their relation by crossing to $\eta\pi \to \pi\pi$ scattering processes. The reconstruction procedure rests on the assumption of stability of the involved particle states, which does not hold for η. The remedy is to first reconstruct the amplitude at a fictitious value of the mass of η lowered below three times the rest mass of π, and once the amplitude is reconstructed, continue it analytically in taking M_η to its physical value on a path slightly diverted into the upper complex plane. This prescription should recover the correct physical amplitude [4].

We will now comment on the construction of $\eta \to \pi\pi\pi$ amplitude to the first order of isospin breaking. As such, in Eq. (3.12), we only consider the rescattering of $\pi\pi$ states. See Subsection 4.1.3 for representations (tied to the usage of software introduced in the next chapter) of the $\eta \to \pi\pi\pi$ reconstruction to two-loop order.

3.4.1. Leading-order amplitude

Like in the case of $\pi\pi$ scattering the leading-order amplitude will be a first-order polynomial in Mandelstam variables. By symmetries of the processes, the free polynomial parametrization of the amplitudes at $O(p^2)$ is

$$\begin{aligned}\mathcal{A}(\eta\pi^0 \to \pi^0\pi^0; s,t,u) &= A_{00} + O(p^4)\\ \mathcal{A}(\eta\pi^0 \to \pi^+\pi^-; s,t,u) &= A_x + B_x\, s + O(p^4),\end{aligned}$$

where A_{00}, A_x, B_x are free parameters. Amplitudes of the other $\eta\pi \to \pi\,\pi$ processes are related by crossing.

3.4.2. Next-to-leading-order amplitude

In the reconstruction of the $O(p^4)$ dispersive part of the $\eta\,\pi \to \pi\pi$ amplitude, the $O(p^2)$ S and P partial waves of processes $\eta\pi \to \pi\pi$ and $\pi\pi \to \pi\pi$ enter Eq. (3.12). The $W_{0,1}^{S/U/T}(s)$ functions carrying the dispersive part of amplitude then take on the form familiar from the $\pi\pi$ scattering of polynomials times $\bar{J}(s)$ functions (up to $O(p^4)$).

In the extraction of expressions for S, P waves to be used in the second iteration of reconstruction, we need to evaluate the integral (3.11). It can be evaluated in terms of utility functions $\tilde{k}_j(s)$ in a manner similar to the $k_j(s)$ functions of $\pi\pi$ scattering (see Appendix A for definitions).

3.4.3. Next-to-next-to-leading-order amplitude

To reconstruct the $O(p^6)$ dispersive part of the amplitude, Eq. (3.12) combines $O(p^2)$ $\eta\pi \to \pi\pi$ waves and $O(p^4)$ $\pi\pi \to \pi\pi$ waves, and then separately $O(p^4)$ $\eta\pi \to \pi\pi$ waves and $O(p^2)$ $\pi\pi \to \pi\pi$ waves.

Evaluation of dispersive integrals (3.13) can be expressed in terms of polynomials times functions $\bar{K}_i(s)$, $\bar{\bar{K}}_i(s)$, $\widetilde{K}_i(s)$, $\tilde{K}_i^{(\lambda)}(s)$ (see Appendix A for definitions).

Chapter 4

Attached software

In Chapter 3 we summarized a method of dispersive construction of scattering amplitudes for mesonic processes, producing amplitude forms up to the $O(p^6)$ order in chiral counting. Here we describe an attached[4.1] software library with a high-level user interface available for reproducing the calculations in the dispersive construction of the amplitude forms.

The library is a set of functions and definitions in the Python programming language to complement the SymPy library [16] for symbolic manipulation of mathematical expressions. The library implements the reconstruction procedure generally but is limited in its ability to evaluate the integrals of Eqs. 3.11 and 3.13. As it is, the library is equipped with a ruleset that is tailored to the case of the reconstruction of $\pi\pi$ scattering and $\eta \rightarrow \pi\pi\pi$ decay amplitudes and those in the approximation of equal pion masses. As-is, the ruleset can be sufficient for some related processes, and further processes can be made available to the application of the library by its extension. This includes a generalization to the case of unequal pion masses, which is a plausible direction for future extensions of the library.

The library contains inline documentation on its functions and definitions. In the following section, we describe the key elements of its user interface. Appendix B contains a transcript of a sample usage of the library in an interactive session.

4.1. Future versions of the library are to be found at: `https://github.com/povik/mesonic_displib`

		Python Value Type	Description
Argument	`target`	`ScatterProcess`	Label for the process to be reconstructed. This will be internally used to derive process labels to look up in the `ptable`.
	`ptable`	`dict`	A dictionary of parts of the amplitude at an appropriate chiral order to be the input to reconstruction, indexed by objects of type `ScatterProcess`. A single representative of a class of processes related by crossing and Bose symmetry is sufficient.
	`S`	`list`	A list of kinds of intermediate states to consider in the S channel of the reconstruction (each kind specified by a two-letter string, see Table 4.2).
	`T`	`list`	ditto for T channel
	`U`	`list`	ditto for U channel
Return Value		SymPy expression	The unitarity part of the reconstructed amplitude.

Table 4.1. Inputs and outputs of function `reconstruct`. The function `reconstruct_ext` has a similar interface, but instead of one `ptable` argument, accepts two `ptable1`, `ptable2` arguments which correspond to the two different orders of the lower-order amplitude part to be processed in the desired reconstruction.

4.1. User Interface

In the Python namespace the library is called `displib` and its usage revolves around the functions `reconstruct` and `reconstruct_ext`, the programming inputs and outputs of which are described in Table 4.1. The function performs one iteration of the reconstruction procedure – it takes in the lower-order forms of prerequisite amplitudes and a list of considered kinds of intermediate states and produces the unitarity part of a higher-order amplitude, that is, the terms containing functions $W_{0/1}^{S/T/U}(s)$ on the right-hand side of Eq. 3.16.[4.2] To construct a higher-order amplitude, the user needs to supplement the unitarity part with an appropriate free polynomial. In this way, the labelling of the free polynomial is fully under the user's control.

4.1.1. Process Labels

The library defines a `ScatterProcess` class, which represents a label for a process with defined particle kinds on its input and output. The `ScatterProcess` object also specifies an ordering of the input and output particles and can also carry a phase for the process. That is, it can be a label for a process amplitude with an additional overall phase. This latter feature simplifies the handling of crossing. There is a `scatter` shorthand function for the creation of `ScatterProcess` labels. For

4.2. One can also use the related functions `reconstruct_in_channel` and `reconstruct_in_channel_ext` to obtain the $W_{0/1}^{S/T/U}(s)$ functions in isolation.

the scattering of neutral pions to a pair of charged pions, one creates a label by `scatter("00", "pm")`. See Table 4.2 for the exposition of letter codes in referring to particular particle states.

Particle	Letter Code
π^+	`p`
π^0	`0`
π^-	`m`
η	`E`

Table 4.2. Letter codes used in the library to refer to π and η meson particle states.

4.1.2. Process Table

The `reconstruct` family of functions takes for its input a "process table" (under argument `ptable`). That is, a dictionary from process labels to parts of the amplitude (as SymPy expressions) belonging to those processes. The parts of the amplitude are supplied at an appropriate chiral order with respect to the chiral order that is desired to be reconstructed. That is, one supplies the $O(p^2)$ part to reconstruct $O(p^4)$ by calling of `reconstruct`, or supplies the $O(p^2)$ and $O(p^4)$ parts to reconstruct $O(p^6)$ by calling of `reconstruct_ext`.

A single representative of a class of processes that are mutually related by combinations of crossing and Bose symmetry is sufficient to be present in the process table.

4.1.3. Example

To further illustrate the interface of the library, we show both a graphical (Fig. 4.1) and code (Fig. 4.2) representation of an example $\pi\eta \to \pi\pi$ amplitude reconstruction program.

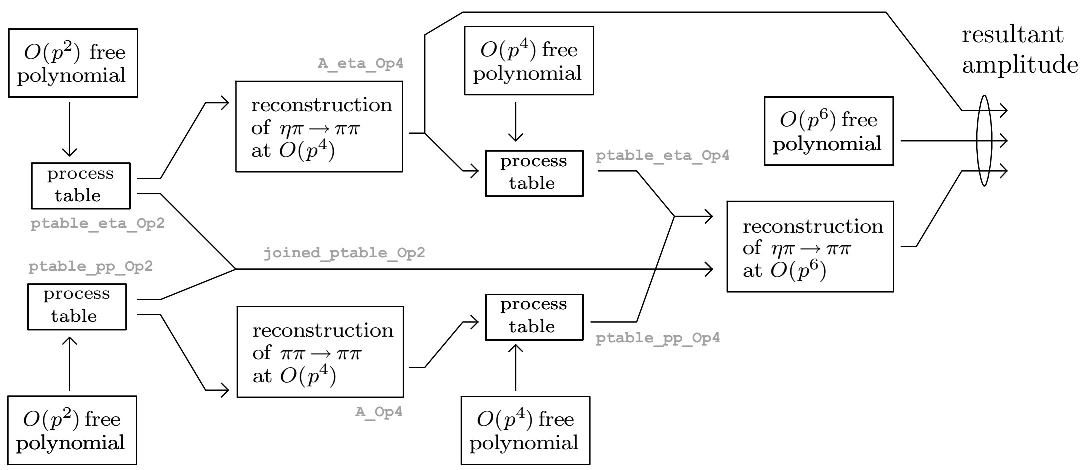

Figure 4.1. Schematic representation of using the interface of the library to reconstruct two-loop level $\eta\pi \to \pi\pi$ scattering amplitude to the first order in isospin breaking (cf. Fig. 1 of Ref. [14]).

```
from displib import *

A = pp_subthreshold_poly(1, subscript=0)
ptable_pp_Op2 = pp_ptable_from_form_factor(A)

A_Op4 = -reconstruct(
        scatter("00", "pm"), ptable_pp_Op2,
        S=["00", "pm"], T=["m0"], U=["p0"],
)
ptable_pp_Op4 = pp_ptable_from_form_factor(-A_Op4)

A_x, B_x = sp.symbols("A_x B_x")
ptable_eta_Op2 = {
    scatter("E0", "pm"): A_x + B_x*(s-s_0)/F_pi**2,
    scatter("E0", "00"): -3*A_x
}

joined_ptable_Op2 = {**ptable_eta_Op2, **ptable_pp_Op2}

A_eta_Op4 = reconstruct(
        scatter("E0", "pm"), joined_ptable_Op2,
        S=["00", "pm"], T=["m0"], U=["p0"],
)

ptable_eta_Op4 = {
    scatter("E0", "pm"): A_eta_Op4,
    scatter("E0", "00"): -A_eta_Op4
        - A_eta_Op4.subs({s: t, t: s}, simultaneous=True)
        - A_eta_Op4.subs({t: u, u: t}, simultaneous=True)
}

joined_ptable_Op4 = {**ptable_eta_Op4, **ptable_pp_Op4}

E0_00_Op6 = reconstruct_ext(
        scatter("E0", "00"),
        joined_ptable_Op2, joined_ptable_Op4,
        S=["00", "pm"], T=["00", "pm"], U=["00", "pm"],
)
```

Figure 4.2. The content of Fig. 4.1 as represented in code by stringing function calls to the library. In addition to the **reconstruct**, **reconstruct_ext** and **scatter** functions introduced in text, the code example makes use of a **pp_ptable_from_form_factor** function to set up the process table from a $\pi\pi$ scattering form factor.

Conclusion

This work follows up on a line of work in published literature dealing with the dispersive construction of mesonic process amplitudes up to the two-loop order. In Ref. [20], the approach was originally applied to construct the two-loop $\pi\pi$ scattering amplitude. Subsequent development has seen the approach generalized, and for this work, our key reference is [14], which contains an account of the two-loop construction of $K \to \pi\pi\pi$ and $\eta \to \pi\pi\pi$ decay amplitudes.

This work supplies a software library implementing the steps of the "reconstruction procedure" that is the basis of the amplitude forms of Ref. [14]. The library can be used to construct the $\eta \to \pi\pi\pi$ decay amplitude to the first order in isospin breaking, up to the two-loop order. The full result of the construction is a long-winded expression, and the library makes the result available for further manipulation in a CAS (computer algebra system) environment.

The forms of amplitude can be fitted to data from high-statistics $\eta \to \pi\pi\pi$ experiments, where in comparison to the usual Dalitz parameters the dispersively-constructed parametrization incorporates the expected non-analytical features of the amplitude surface. Fig. 1 presents the results of an illustratory fit of the one-loop amplitude to the $\eta \to \pi^+\pi^-\pi^0$ data of the KLOE-2 collaboration [3]. Two-loop amplitude results can be fitted as well once suitable numerical representations of the special $\bar{K}_i(s)$ functions appearing in the expressions are supplied.

The dispersively-constructed forms of amplitude can be of use to studies of ChPT generalizations and studies of extraction of the isospin-breaking parameter R.

In future work, the library can be extended to new processes and generalized. One road of extension is to support the construction of amplitude taking into account the mass difference of charged and neutral pions. This would make the resultant constructed amplitude exhibit the feature of the "cusp" in $\eta \to \pi^0\pi^0\pi^0$ decay. By simple modification, the library can be adapted to construct $K \to \pi\pi\pi$ amplitudes (their relation to the $\eta \to \pi\pi\pi$ amplitude is discussed at length in Ref. [14]).

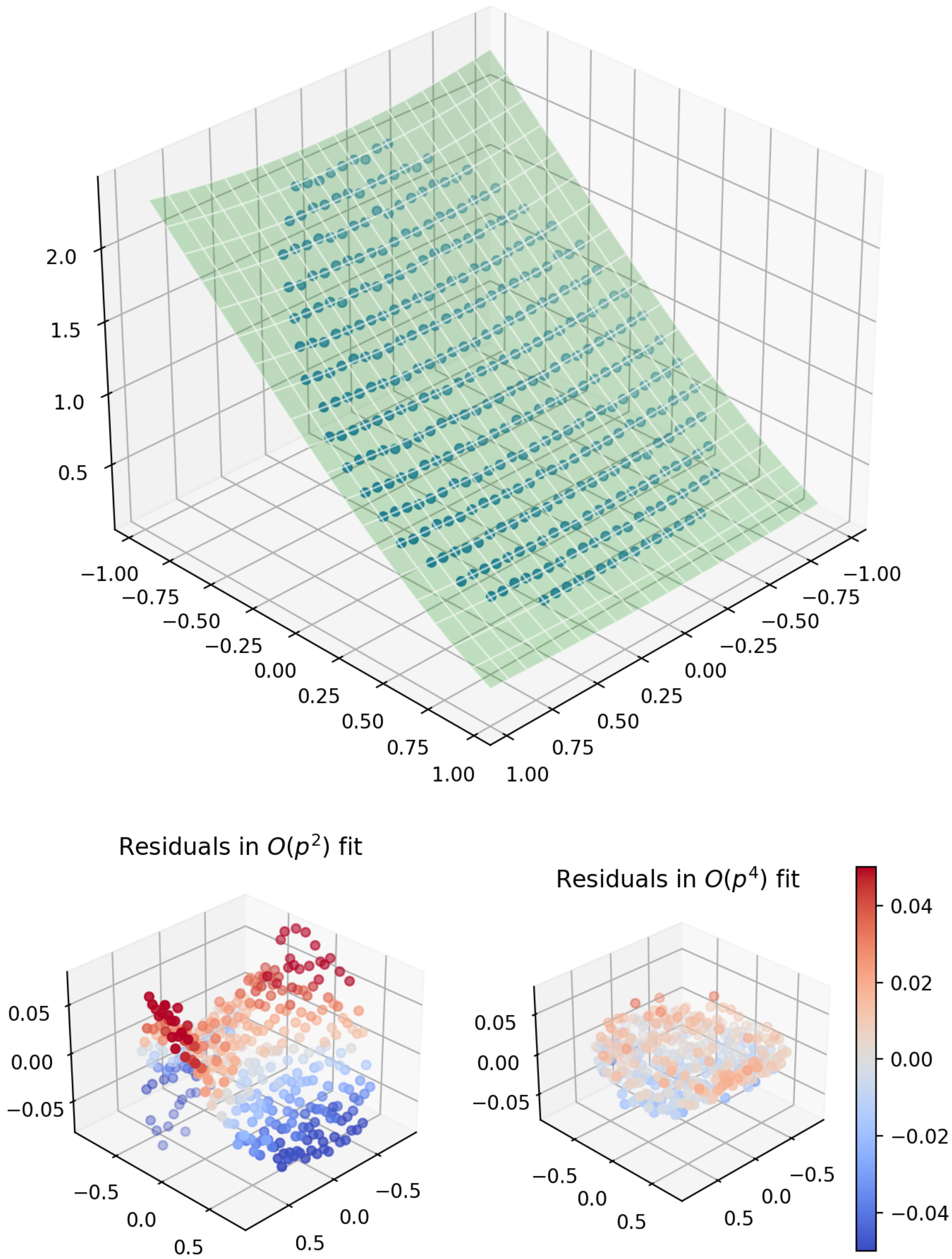

Figure 1. The results of an illustratory fit to the $\eta \to \pi\pi\pi$ decay measurements by the KLOE-2 collaboration [3]. Top figure has the experimental data points in blue and the fitted amplitude surface (from $O(p^4)$ reconstruction) in light green. Bottom left and bottom right are fit residuals to the reconstructed amplitude form at the $O(p^2)$ and $O(p^4)$ order respectively.

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Appendix A
Special Functions

We define a string of special functions to which we refer from the results of dispersive calculations. First, we define a Källen triangle function $\lambda_{AB}(s)$ and the usual $\sigma(s)$ and $\bar{J}(s)$ by:

$$\lambda_{AB}(s) = (s-(M_A^2+M_B^2))(s-(M_A^2-M_B^2)) \qquad \bar{J}(s) = \lim_{\epsilon\to 0+} \frac{s}{16\,\pi^2}\int_{4m^2}^{\infty} \mathrm{d}x \frac{\sigma(s)}{x(x-s-i\epsilon)}$$

$$\sigma(s) = \sigma(s,M_\pi) = \frac{\lambda_{\pi\pi}^{1/2}(s)}{s} = \sqrt{1-\frac{4\,M_\pi^2}{s}} \qquad = \frac{1}{16\pi^2}\left[2+\sigma(s)\log\frac{\sigma(s)-1}{\sigma(s)+1}\right]$$

In the remainder of the Appendix we define functions $k_i(s)$, $\bar{K}_i(s)$ and their decorated variants, matching those definitions made in Ref. [14].

For utility in two-loop calculations of pion rescattering, we define functions $k_i(s)$. First, we define, for $s \geqslant 4M_\pi^2$,

$$L(s) = \log\frac{1-\sigma(s,M_\pi)}{1+\sigma(s,M_\pi)},$$

in terms of which the $k_i(s)$ are (again for $s \geqslant 4M_\pi^2$):

$$\begin{aligned}
k_0(s) &= \frac{1}{16\pi}\sigma(s,M_\pi)\\
k_1(s) &= \frac{1}{8\pi}L(s)\\
k_2(s) &= \frac{1}{8\pi}\left(1-\frac{4M_\pi^2}{s}\right)L(s)\\
k_3(s) &= \frac{3}{16\pi}\frac{M_\pi^2}{s\,\sigma(s,M_\pi)}L^2(s)\\
k_4(s) &= \frac{1}{16\pi}\frac{M_\pi^2}{s\,\sigma(s,M_\pi)}\left\{1+\frac{1}{\sigma(s,M_\pi)}L(s)+\frac{M_\pi^2}{s-4M_\pi^2}L^2(s)\right\}
\end{aligned}$$

We label $\bar{K}_i(s)$ the once-subtracted dispersion integral of each $k_i(s)$, and $\bar{\bar{K}}_i(s)$ the twice-subtracted dispersion integral, that is

$$\bar{K}_i(s) = \frac{s}{\pi}\lim_{\epsilon\to 0^+}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\frac{k_i(x)}{x-s-i\epsilon},$$

$$\bar{\bar{K}}_i(s) = \frac{s^2}{\pi}\lim_{\epsilon\to 0^+}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x^2}\frac{k_i(x)}{x-s-i\epsilon}.$$

Similarly for utility in two-loop calculations of pion decay, we define functions $\tilde{k}_i(s)$ to be

$$\begin{aligned}
\tilde{k}_0(s) &= \frac{1}{16\pi}\sigma(s,M_\pi)\\
\tilde{k}_1(s) &= \frac{1}{16\pi}L(s)\\
\tilde{k}_2(s) &= \frac{1}{16\pi}\sigma(s,M_\pi)\,s\,\frac{M(s)}{\lambda_{\eta\pi}^{1/2}(s)}\\
\tilde{k}_3(s) &= \frac{1}{16\pi}\sigma(s,M_\pi)\,s\,\frac{M(s)}{\lambda_{\eta\pi}^{1/2}(s)}L(s),
\end{aligned}$$

where the function $M(s)$ is defined to be

$$M(s) = -\log\left(1-\frac{M_\eta^2-M_\pi^2}{s}+\frac{\lambda_{\eta\pi}^{1/2}(s)}{s}\right)+\log\left(1-\frac{M_\eta^2-M_\pi^2}{s}-\frac{\lambda_{\eta\pi}^{1/2}(s)}{s}\right),$$

and for each $\tilde{k}_i(s)$ we define its once- and twice-subtracted dispersion integrals

$$\tilde{K}_i(s) = \frac{s}{\pi} \lim_{\epsilon \to 0^+} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}x}{x} \frac{\tilde{k}_i(x)}{x - s - i\epsilon},$$

$$\bar{\tilde{K}}_i(s) = \frac{s}{\pi} \lim_{\epsilon \to 0^+} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}x}{x} \frac{\tilde{k}_i(x)}{x - s - i\epsilon}.$$

In addition, to express the $\eta \to \pi\pi\pi$ two-loop amplitude, we define the function

$$\tilde{K}_i^{(\lambda)}(s) = \frac{s}{\pi} \lim_{\epsilon \to 0^+} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}x}{x} \frac{M_\eta M_\pi^3}{\lambda_{\eta\pi}(x)} \frac{\tilde{k}_i(x)}{x - s - i\epsilon}.$$

Appendix B

Transcript of using the attached library

The following is a transcript of an interactive session with a Python interpreter. After-the-fact comments were written interspersed into the transcript.

```
SymPy 1.9 under Python 3.9.10
```

Import all symbols of the library.

```
>>> from displib import *
```

Assign to a variable a free form of the amplitude of $\pi\pi$ scattering at the $O(p^2)$ order in subthreshold parametrization.

```
>>> A_Op2 = pp_subthreshold_poly(1)
>>> A_Op2
```

$$\frac{M_\pi^2\,\alpha}{3\,F_\pi^2} + \frac{\beta\left(-\frac{4\,M_\pi^2}{3} + s\right)}{F_\pi^2}$$

Derive a 'process table' from the amplitude above. That is, knowing the above is the amplitude of a $\pi^+\pi^- \to \pi^0\pi^0$ process, obtain a table with amplitudes of processes related by isospin symmetry. The function **render_ptable** presents the process table in tabular form for inspection.

```
>>> ptable_Op2 = pp_ptable_from_form_factor(A_Op2)
>>> render_ptable(ptable_Op2)
```

$$\begin{bmatrix} \pi^+\,\pi^- \to \pi^0\,\pi^0 & \dfrac{-M_\pi^2\,\alpha + \beta\,(4\,M_\pi^2 - 3\,s)}{3\,F_\pi^2} \\ \pi^+\,\pi^- \to \pi^+\,\pi^- & \dfrac{\frac{2\,M_\pi^2\,\alpha}{3} - \frac{8\,M_\pi^2\,\beta}{3} + \beta\,s + \beta\,t}{F_\pi^2} \\ \pi^0\,\pi^0 \to \pi^0\,\pi^0 & \dfrac{M_\pi^2\,\alpha - 4\,M_\pi^2\,\beta + \beta\,s + \beta\,t + \beta\,u}{F_\pi^2} \end{bmatrix}$$

Reconstruct the dispersive part of amplitude at the $O(p^4)$ order, taking as input the $O(p^2)$ process table.

```
>>> A_Op4 = -reconstruct(
            scatter("00", "pm"), ptable_Op2,
            S=["00", "pm"], T=["m0"], U=["p0"],
    )
```

As before, derive from $\pi^+\pi^- \to \pi^0\pi^0$ amplitude (this time of the $O(p^4)$ order), the process table with amplitudes of isospin-related processes.

```
>>> ptable_Op4 = pp_ptable_from_form_factor(A_Op4)
```

Look up S and P waves of a couple of processes. Take suitable linear combinations to extract the isospin form factors of $\pi\pi \to \pi\pi$ amplitudes at $O(p^4)$.

```
>>> A_00_S, A_00_P = lookup_SP_wave(scatter("00", "00"), ptable_Op4)
    A_x_S,  A_x_P  = lookup_SP_wave(scatter("00", "pm"), ptable_Op4)
    A_pm_S, A_pm_P = lookup_SP_wave(scatter("pm", "pm"), ptable_Op4)
>>> psi_0 = sp.cancel(A_00_S - 2*A_x_S)
    psi_1 = sp.cancel(A_pm_S*2)
    psi_2 = sp.cancel(A_00_S + A_x_S)
```

Calculate and display the $\xi_I^{(i)}$ polynomials (as defined in (5.7) of [14]).

```
>>> def derive_xi(psi):
        xi_ = sp.cancel(attempt_poly_reduce(psi * sigma(s, M_pi), s))
        return normalize_k4(xi_, s, M_pi)
>>> analyze_table(derive_xi(psi_0), s)
```

$$\begin{bmatrix} k_1(s, M_\pi) & \dfrac{5\,M_\pi^4\,\alpha^2}{96\,\pi^2\,F_\pi^4} + \dfrac{5\,M_\pi^4\,\beta^2}{24\,\pi^2\,F_\pi^4} - \dfrac{5\,M_\pi^2\,\beta^2\,s}{36\,\pi^2\,F_\pi^4} + \dfrac{7\,\beta^2\,s^2}{288\,\pi^2\,F_\pi^4} \\ k_3(s, M_\pi) & -\dfrac{5\,M_\pi^4\,\alpha^2}{144\,\pi^2\,F_\pi^4} - \dfrac{5\,M_\pi^4\,\beta^2}{36\,\pi^2\,F_\pi^4} + \dfrac{M_\pi^2\,\beta^2\,s}{24\,\pi^2\,F_\pi^4} \\ k_0(s, M_\pi) & \dfrac{35\,M_\pi^4\,\alpha^2}{72\,\pi^2\,F_\pi^4} - \dfrac{5\,M_\pi^4\,\alpha\,\beta}{9\,\pi^2\,F_\pi^4} + \dfrac{49\,M_\pi^4\,\beta^2}{27\,\pi^2\,F_\pi^4} + \dfrac{5\,M_\pi^2\,\alpha\,\beta\,s}{12\,\pi^2\,F_\pi^4} - \dfrac{617\,M_\pi^2\,\beta^2\,s}{432\,\pi^2\,F_\pi^4} + \dfrac{311\,\beta^2\,s^2}{864\,\pi^2\,F_\pi^4} \\ k_2(s, M_\pi) & \dfrac{25\,M_\pi^4\,\alpha^2}{576\,\pi^2\,F_\pi^4} - \dfrac{5\,M_\pi^4\,\alpha\,\beta}{36\,\pi^2\,F_\pi^4} + \dfrac{M_\pi^4\,\beta^2}{9\,\pi^2\,F_\pi^4} + \dfrac{5\,M_\pi^2\,\alpha\,\beta\,s}{48\,\pi^2\,F_\pi^4} - \dfrac{M_\pi^2\,\beta^2\,s}{6\,\pi^2\,F_\pi^4} + \dfrac{\beta^2\,s^2}{16\,\pi^2\,F_\pi^4} \end{bmatrix}$$

```
>>> analyze_table(derive_xi(psi_1), s)
```

$$\left[k_1(s,\,M_\pi),\,\frac{13\,M_\pi^4\,\alpha^2}{288\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\alpha\,\beta}{72\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\beta^2}{18\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\alpha\,\beta\,s}{96\,\pi^2\,F_\pi^4}-\frac{29\,M_\pi^2\,\beta^2\,s}{288\,\pi^2\,F_\pi^4}+\frac{13\,\beta^2\,s^2}{576\,\pi^2\,F_\pi^4}+\frac{-M_\pi^4\,\alpha\,\beta+4\,M_\pi^4\,\beta^2}{48\,\pi^2\,F_\pi^4};\,k_3(s,\,M_\pi),\,-\frac{13\,M_\pi^4\,\alpha^2}{432\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\alpha\,\beta}{54\,\pi^2\,F_\pi^4}-\frac{4\,M_\pi^4\,\beta^2}{27\,\pi^2\,F_\pi^4}+\frac{M_\pi^2\,\beta^2\,s}{48\,\pi^2\,F_\pi^4}+\frac{-M_\pi^4\,\alpha\,\beta+4\,M_\pi^4\,\beta^2}{72\,\pi^2\,F_\pi^4};\,k_0(s,\,M_\pi),\,\frac{19\,M_\pi^4\,\alpha^2}{48\,\pi^2\,F_\pi^4}-\frac{7\,M_\pi^4\,\alpha\,\beta}{24\,\pi^2\,F_\pi^4}+\frac{61\,M_\pi^4\,\beta^2}{54\,\pi^2\,F_\pi^4}+\frac{19\,M_\pi^2\,\alpha\,\beta\,s}{96\,\pi^2\,F_\pi^4}-\frac{227\,M_\pi^2\,\beta^2\,s}{216\,\pi^2\,F_\pi^4}+\frac{503\,\beta^2\,s^2}{1728\,\pi^2\,F_\pi^4}+\frac{-M_\pi^4\,\alpha\,\beta+4\,M_\pi^4\,\beta^2}{24\,\pi^2\,F_\pi^4};\,k_2(s,\,M_\pi),\,\frac{M_\pi^4\,\alpha^2}{32\,\pi^2\,F_\pi^4}-\frac{M_\pi^4\,\alpha\,\beta}{12\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\beta^2}{12\,\pi^2\,F_\pi^4}+\frac{M_\pi^2\,\alpha\,\beta\,s}{16\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\beta^2\,s}{8\,\pi^2\,F_\pi^4}+\frac{3\,\beta^2\,s^2}{64\,\pi^2\,F_\pi^4}\right]$$

```
>>> analyze_table(derive_xi(psi_2), s)
```

$$\begin{bmatrix} k_1(s,M_\pi) & \frac{M_\pi^4\,\alpha^2}{32\,\pi^2\,F_\pi^4}-\frac{M_\pi^4\,\alpha\,\beta}{48\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\alpha\,\beta\,s}{32\,\pi^2\,F_\pi^4}-\frac{7\,M_\pi^2\,\beta^2\,s}{288\,\pi^2\,F_\pi^4}+\frac{11\,\beta^2\,s^2}{576\,\pi^2\,F_\pi^4} \\ k_3(s,M_\pi) & -\frac{M_\pi^4\,\alpha^2}{48\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\alpha\,\beta}{72\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\beta^2\,s}{48\,\pi^2\,F_\pi^4} \\ k_0(s,M_\pi) & \frac{31\,M_\pi^4\,\alpha^2}{144\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\alpha\,\beta}{9\,\pi^2\,F_\pi^4}+\frac{7\,M_\pi^4\,\beta^2}{27\,\pi^2\,F_\pi^4}-\frac{23\,M_\pi^2\,\alpha\,\beta\,s}{96\,\pi^2\,F_\pi^4}-\frac{8\,M_\pi^2\,\beta^2\,s}{27\,\pi^2\,F_\pi^4}+\frac{265\,\beta^2\,s^2}{1728\,\pi^2\,F_\pi^4} \\ k_2(s,M_\pi) & \frac{M_\pi^4\,\alpha^2}{144\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\alpha\,\beta}{36\,\pi^2\,F_\pi^4}+\frac{M_\pi^4\,\beta^2}{36\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\alpha\,\beta\,s}{48\,\pi^2\,F_\pi^4}-\frac{M_\pi^2\,\beta^2\,s}{24\,\pi^2\,F_\pi^4}+\frac{\beta^2\,s^2}{64\,\pi^2\,F_\pi^4} \end{bmatrix}$$

Follow on by reconstructing the $\eta \to \pi\pi\pi$ amplitude at up to the $O(p^6)$ order.

```
>>> A_x, B_x = sp.symbols("A_x B_x")
    A_00 = sp.symbols("A_00")

    ptable_eta_Op2 = {
        scatter("E0", "pm"): A_x + B_x*(s-s_0)/F_pi**2,
        scatter("E0", "00"): A_00,
    }

    joined_ptable = {**ptable_eta_Op2, **ptable_Op2}

    E0_pm_Op4 = reconstruct(
            scatter("E0", "pm"), joined_ptable,
            S=["00", "pm"], T=["m0"], U=["p0"],
    )
    E0_00_Op4 = reconstruct(
            scatter("E0", "00"), joined_ptable,
            S=["00", "pm"], T=["00", "pm"], U=["00", "pm"],
    )

    ptable_eta_Op4 = {
        scatter("E0", "pm"): E0_pm_Op4,
        scatter("E0", "00"): E0_00_Op4
    }

    joined_ptable_Op4 = {**ptable_eta_Op4, **ptable_Op4}
    E0_pm_Op6 = reconstruct_ext(
            scatter("E0", "pm"), joined_ptable, joined_ptable_Op4,
            S=["00", "pm"], T=["0m"], U=["0p"],
    )
```

Perform last step of the reconstruction once more, but this time ask for the $W_{0/1}^{S/T/U}(s)$ unitarity components separately.

```
>>> ptable1, ptable2 = joined_ptable, joined_ptable_Op4
    target = scatter("E0", "pm")
    W_0_S, W_1_S = reconstruct_in_channel_ext("S", target, ["00", "pm"], ptable1, ptable2)
    W_0_T, W_1_T = reconstruct_in_channel_ext("T", target, ["0m"], ptable1, ptable2)
    W_0_U, W_1_U = reconstruct_in_channel_ext("U", target, ["0p"], ptable1, ptable2)
```

Display the content of the $W_1^T(s)$ component.

```
>>> sp.simplify(W_1_T)
```

$$\Big(12\,M_\pi^2\,\beta\,\big(-3\,B_x\,M_\pi^2\,\beta\,(M_\eta^6-3\,M_\eta^4\,M_\pi^2+3\,M_\eta^2\,M_\pi^4-M_\pi^6)\,\tilde{K}_1^{(\lambda)}(t,\,M_\pi,\,M_\eta)+(3\,A_{00}\,F_\pi^2\,M_\eta^6\,\beta-9\,A_{00}\,F_\pi^2\,M_\eta^4\,M_\pi^2\,\beta+9\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\beta-3\,A_{00}\,F_\pi^2\,M_\pi^6\,\beta-6\,A_x\,F_\pi^2\,M_\eta^6\,\beta+18\,A_x\,F_\pi^2\,M_\eta^4\,M_\pi^2\,\beta-18\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\beta+6\,A_x\,F_\pi^2\,M_\pi^6\,\beta-3\,B_x\,M_\eta^8\,\beta-5\,B_x\,M_\eta^6\,M_\pi^2\,\alpha+8\,B_x\,M_\eta^6\,M_\pi^2\,\beta+6\,B_x\,M_\eta^6\,\beta\,s_0+15\,B_x\,M_\eta^4\,M_\pi^4\,\alpha-6\,B_x\,M_\eta^4\,M_\pi^4\,\beta-18\,B_x\,M_\eta^4\,M_\pi^2\,\beta\,s_0-15\,B_x\,M_\eta^2\,M_\pi^6\,\alpha+18\,B_x\,M_\eta^2\,M_\pi^4\,\beta\,s_0+5\,B_x\,M_\pi^8\,\alpha+B_x\,M_\pi^8\,\beta-6\,B_x\,M_\pi^6\,\beta\,s_0)\,\tilde{K}_2^{(\lambda)}(t,\,M_\pi,\,M_\eta)\big)+t\,\big(120\,B_x\,M_\pi^4\,\beta^2\,\bar{J}(t,\,M_\pi)+72\,B_x\,M_\pi^4\,\beta^2\,\bar{K}_1(t,\,M_\pi)-96\,B_x\,M_\pi^4\,\beta^2\,\bar{K}_2(t,\,M_\pi)+96\,B_x\,M_\pi^4\,\beta^2\,\bar{K}_3(t,\,M_\pi)+288\,B_x\,M_\pi^4\,\beta^2\,\bar{K}_4(t,\,M_\pi)+24\,B_x\,M_\pi^4\,\beta\,(5\,\alpha-2\,\beta)\,\bar{K}_1(t,\,M_\pi)-288\,B_x\,M_\pi^4\,\beta\,(5\,\alpha-2\,\beta)\,\bar{K}_4(t,\,M_\pi)-12\,B_x\,M_\pi^4\,\beta\,(10\,\alpha-\beta)\,\bar{K}_1(t,\,M_\pi)+40\,B_x\,M_\pi^4\,\beta\,(11\,\alpha-14\,\beta)\,\bar{J}(t,\,M_\pi)+4\,B_x\,M_\pi^4\,(5\,\alpha^2-40\,\alpha\,\beta+8\,\beta^2)\,\bar{K}_3(t,\,M_\pi)-6\,B_x\,M_\pi^4\,(5\,\alpha^2-20\,\alpha\,\beta+6\,\beta^2)\,\bar{J}(t,\,M_\pi)+24\,B_x\,M_\pi^4\,(5\,\alpha^2+20\,\alpha\,\beta+2\,\beta^2)\,\bar{K}_4(t,\,M_\pi)+108\,B_x\,M_\pi^2\,\beta^2\,t^3\,\tilde{K}_3^{(\lambda)}(t,\,M_\pi,\,M_\eta)-30\,B_x\,M_\pi^2\,\beta^2\,t\,\bar{J}(t,\,M_\pi)-42\,B_x\,M_\pi^2\,\beta^2\,t\,\bar{K}_1(t,\,M_\pi)+48\,B_x\,M_\pi^2\,\beta^2\,t\,\bar{K}_2(t,\,M_\pi)-48\,B_x\,M_\pi^2\,\beta^2\,t\,\bar{K}_3(t,\,M_\pi)-216\,B_x\,M_\pi^2\,\beta^2\,t\,\bar{K}_4(t,\,M_\pi)-6\,B_x\,M_\pi^2\,\beta\,t\,(5\,\alpha-3\,\beta)\,\bar{K}_1(t,\,M_\pi)+72\,B_x\,M_\pi^2\,\beta\,t\,(5\,\alpha-3\,\beta)\,\bar{K}_4(t,\,M_\pi)-10\,B_x\,M_\pi^2\,\beta\,t\,(11\,\alpha-21\,\beta)\,\bar{J}(t,\,M_\pi)+14\,B_x\,\beta^2\,t^4\,\tilde{K}_0^{(\lambda)}(t,\,M_\pi,\,M_\eta)+3\,B_x\,\beta^2\,t^4\,\tilde{K}_1^{(\lambda)}(t,\,M_\pi,\,M_\eta)+3\,B_x\,\beta^2\,t^4\,\tilde{K}_2^{(\lambda)}(t,\,M_\pi,\,M_\eta)-10\,B_x\,\beta^2\,t^2\,\bar{J}(t,\,M_\pi)+3\,B_x\,\beta^2\,t^2\,\bar{K}_1(t,\,M_\pi)-6\,B_x\,\beta^2\,t^2\,\bar{K}_2(t,\,M_\pi)+36\,B_x\,\beta^2\,t^2\,\bar{K}_4(t,\,M_\pi)-36\,M_\pi^2\,\beta\,t^2\,(-A_{00}\,F_\pi^2\,\alpha+A_{00}\,F_\pi^2\,\beta+2\,A_x\,F_\pi^2\,\alpha-2\,A_x\,F_\pi^2\,\beta-B_x\,M_\eta^2\,\alpha+4\,B_x\,M_\eta^2\,\beta+2\,B_x\,M_\pi^2\,\alpha+8\,B_x\,M_\pi^2\,\beta-2\,B_x\,\alpha\,s_0+2\,B_x\,\beta\,s_0)\,\tilde{K}_3^{(\lambda)}(t,\,M_\pi,\,M_\eta)-36\,M_\pi^2\,\beta\,t\,(A_{00}\,F_\pi^2\,M_\eta^2\,\alpha-A_{00}\,F_\pi^2\,M_\eta^2\,\beta+A_{00}\,F_\pi^2\,M_\pi^2\,\alpha-7\,A_{00}\,F_\pi^2\,M_\pi^2\,\beta-2\,A_x\,F_\pi^2\,M_\eta^2\,\alpha+2\,A_x\,F_\pi^2\,M_\eta^2\,\beta-2\,A_x\,F_\pi^2\,M_\pi^2\,\alpha+$$

$$14\,A_x\,F_\pi^2\,M_\pi^2\,\beta + B_x\,M_\eta^4\,\alpha - B_x\,M_\eta^4\,\beta - B_x\,M_\eta^2\,M_\pi^2\,\alpha - 2\,B_x\,M_\eta^2\,M_\pi^2\,\beta + 2\,B_x\,M_\eta^2\,\alpha\,s_0 - 2\,B_x\,M_\eta^2\,\beta\,s_0 + 8\,B_x\,M_\pi^4\,\alpha +$$
$$7\,B_x\,M_\pi^4\,\beta + 2\,B_x\,M_\pi^2\,\alpha\,s_0 - 14\,B_x\,M_\pi^2\,\beta\,s_0)\,\tilde{K}_3^{(\lambda)}(t,\,M_\pi,\,M_\eta) - 2\,M_\pi^2\,\beta\,(84\,A_{00}\,F_\pi^2\,M_\eta^4\,\beta - 168\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta +$$
$$84\,A_{00}\,F_\pi^2\,M_\pi^4\,\beta - 168\,A_x\,F_\pi^2\,M_\eta^4\,\beta + 336\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta - 168\,A_x\,F_\pi^2\,M_\pi^4\,\beta - 81\,B_x\,M_\eta^6\,\beta - 140\,B_x\,M_\eta^4\,M_\pi^2\,\alpha +$$
$$35\,B_x\,M_\eta^4\,M_\pi^2\,\beta + 168\,B_x\,M_\eta^4\,\beta\,s_0 + 280\,B_x\,M_\eta^2\,M_\pi^4\,\alpha + 173\,B_x\,M_\eta^2\,M_\pi^4\,\beta - 336\,B_x\,M_\eta^2\,M_\pi^2\,\beta\,s_0 - 140\,B_x\,M_\pi^6\,\alpha -$$
$$127\,B_x\,M_\pi^6\,\beta + 168\,B_x\,M_\pi^4\,\beta\,s_0)\,\tilde{K}_0^{(\lambda)}(t,M_\pi,M_\eta) + 18\,M_\pi^2\,\beta\,(-3\,A_{00}\,F_\pi^2\,M_\eta^4\,\beta - 2\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\alpha + 14\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta +$$
$$2\,A_{00}\,F_\pi^2\,M_\pi^4\,\alpha - 11\,A_{00}\,F_\pi^2\,M_\pi^4\,\beta + 6\,A_x\,F_\pi^2\,M_\eta^4\,\beta + 4\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\alpha - 28\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta - 4\,A_x\,F_\pi^2\,M_\pi^4\,\alpha +$$
$$22\,A_x\,F_\pi^2\,M_\pi^4\,\beta + 3\,B_x\,M_\eta^6\,\beta + 3\,B_x\,M_\eta^4\,M_\pi^2\,\alpha + B_x\,M_\eta^4\,M_\pi^2\,\beta - 6\,B_x\,M_\eta^4\,\beta\,s_0 - 14\,B_x\,M_\eta^2\,M_\pi^4\,\alpha - 7\,B_x\,M_\eta^2\,M_\pi^4\,\beta -$$
$$4\,B_x\,M_\eta^2\,M_\pi^2\,\alpha\,s_0 + 28\,B_x\,M_\eta^2\,M_\pi^2\,\beta\,s_0 + 11\,B_x\,M_\pi^6\,\alpha + 3\,B_x\,M_\pi^6\,\beta + 4\,B_x\,M_\pi^4\,\alpha\,s_0 - 22\,B_x\,M_\pi^4\,\beta\,s_0)\,\tilde{K}_1^{(\lambda)}(t,\,M_\pi,\,M_\eta) -$$
$$\beta\,t^3\,(-33\,A_{00}\,F_\pi^2\,\beta + 66\,A_x\,F_\pi^2\,\beta + 64\,B_x\,M_\eta^2\,\beta + 55\,B_x\,M_\pi^2\,\alpha + 56\,B_x\,M_\pi^2\,\beta - 66\,B_x\,\beta\,s_0)\,\tilde{K}_0^{(\lambda)}(t,\,M_\pi,\,M_\eta) -$$
$$3\,\beta\,t^3\,(-3\,A_{00}\,F_\pi^2\,\beta + 6\,A_x\,F_\pi^2\,\beta + 5\,B_x\,M_\eta^2\,\beta + 5\,B_x\,M_\pi^2\,\alpha + 9\,B_x\,M_\pi^2\,\beta - 6\,B_x\,\beta\,s_0)\,\tilde{K}_1^{(\lambda)}(t,M_\pi,M_\eta) -$$
$$3\,\beta\,t^3\,(-3\,A_{00}\,F_\pi^2\,\beta + 6\,A_x\,F_\pi^2\,\beta + 6\,B_x\,M_\eta^2\,\beta + 5\,B_x\,M_\pi^2\,\alpha + 8\,B_x\,M_\pi^2\,\beta - 6\,B_x\,\beta\,s_0)\,\tilde{K}_2^{(\lambda)}(t,M_\pi,M_\eta) +$$
$$3\,\beta\,t^2\,(-9\,A_{00}\,F_\pi^2\,M_\eta^2\,\beta - 21\,A_{00}\,F_\pi^2\,M_\pi^2\,\beta + 18\,A_x\,F_\pi^2\,M_\eta^2\,\beta + 42\,A_x\,F_\pi^2\,M_\pi^2\,\beta + 12\,B_x\,M_\eta^4\,\beta + 15\,B_x\,M_\eta^2\,M_\pi^2\,\alpha +$$
$$36\,B_x\,M_\eta^2\,M_\pi^2\,\beta - 18\,B_x\,M_\eta^2\,\beta\,s_0 + 35\,B_x\,M_\pi^4\,\alpha + 4\,B_x\,M_\pi^4\,\beta - 42\,B_x\,M_\pi^2\,\beta\,s_0)\,\tilde{K}_2^{(\lambda)}(t,M_\pi,M_\eta) + 3\,\beta\,t^2\,(-6\,A_{00}\,F_\pi^2\,M_\eta^2\,\beta -$$
$$24\,A_{00}\,F_\pi^2\,M_\pi^2\,\beta + 12\,A_x\,F_\pi^2\,M_\eta^2\,\beta + 48\,A_x\,F_\pi^2\,M_\pi^2\,\beta + 7\,B_x\,M_\eta^4\,\beta + 10\,B_x\,M_\eta^2\,M_\pi^2\,\alpha + 36\,B_x\,M_\eta^2\,M_\pi^2\,\beta - 12\,B_x\,M_\eta^2\,\beta\,s_0 +$$
$$40\,B_x\,M_\pi^4\,\alpha + 9\,B_x\,M_\pi^4\,\beta - 48\,B_x\,M_\pi^2\,\beta\,s_0)\,\tilde{K}_1^{(\lambda)}(t,\,M_\pi,\,M_\eta) + 2\,\beta\,t^2\,(-33\,A_{00}\,F_\pi^2\,M_\eta^2\,\beta + 9\,A_{00}\,F_\pi^2\,M_\pi^2\,\alpha -$$
$$171\,A_{00}\,F_\pi^2\,M_\pi^2\,\beta + 66\,A_x\,F_\pi^2\,M_\eta^2\,\beta - 18\,A_x\,F_\pi^2\,M_\pi^2\,\alpha + 342\,A_x\,F_\pi^2\,M_\pi^2\,\beta + 43\,B_x\,M_\eta^4\,\beta + 64\,B_x\,M_\eta^2\,M_\pi^2\,\alpha +$$
$$182\,B_x\,M_\eta^2\,M_\pi^2\,\beta - 66\,B_x\,M_\eta^2\,\beta\,s_0 + 252\,B_x\,M_\pi^4\,\alpha + 81\,B_x\,M_\pi^4\,\beta + 18\,B_x\,M_\pi^2\,\alpha\,s_0 - 342\,B_x\,M_\pi^2\,\beta\,s_0)\,\tilde{K}_0^{(\lambda)}(t,M_\pi,M_\eta) -$$
$$3\,\beta\,t\,(-9\,A_{00}\,F_\pi^2\,M_\eta^4\,\beta - 36\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta + 12\,A_{00}\,F_\pi^2\,M_\pi^4\,\alpha - 3\,A_{00}\,F_\pi^2\,M_\pi^4\,\beta + 18\,A_x\,F_\pi^2\,M_\eta^4\,\beta + 72\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta -$$
$$24\,A_x\,F_\pi^2\,M_\pi^4\,\alpha + 6\,A_x\,F_\pi^2\,M_\pi^4\,\beta + 10\,B_x\,M_\eta^6\,\beta + 15\,B_x\,M_\eta^4\,M_\pi^2\,\alpha + 48\,B_x\,M_\eta^4\,M_\pi^2\,\beta - 18\,B_x\,M_\eta^4\,\beta\,s_0 + 72\,B_x\,M_\eta^2\,M_\pi^4\,\alpha -$$
$$54\,B_x\,M_\eta^2\,M_\pi^4\,\beta - 72\,B_x\,M_\eta^2\,M_\pi^2\,\beta\,s_0 - 39\,B_x\,M_\pi^6\,\alpha + 20\,B_x\,M_\pi^6\,\beta + 24\,B_x\,M_\pi^4\,\alpha\,s_0 - 6\,B_x\,M_\pi^4\,\beta\,s_0)\,\tilde{K}_2^{(\lambda)}(t,M_\pi,M_\eta) -$$
$$3\,\beta\,t\,(-3\,A_{00}\,F_\pi^2\,M_\eta^4\,\beta - 30\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta + 12\,A_{00}\,F_\pi^2\,M_\pi^4\,\alpha - 15\,A_{00}\,F_\pi^2\,M_\pi^4\,\beta + 6\,A_x\,F_\pi^2\,M_\eta^4\,\beta + 60\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta -$$
$$24\,A_x\,F_\pi^2\,M_\pi^4\,\alpha + 30\,A_x\,F_\pi^2\,M_\pi^4\,\beta + 3\,B_x\,M_\eta^6\,\beta + 5\,B_x\,M_\eta^4\,M_\pi^2\,\alpha + 37\,B_x\,M_\eta^4\,M_\pi^2\,\beta - 6\,B_x\,M_\eta^4\,\beta\,s_0 + 62\,B_x\,M_\eta^2\,M_\pi^4\,\alpha +$$
$$97\,B_x\,M_\eta^2\,M_\pi^4\,\beta - 60\,B_x\,M_\eta^2\,M_\pi^2\,\beta\,s_0 - 19\,B_x\,M_\pi^6\,\alpha - 113\,B_x\,M_\pi^6\,\beta + 24\,B_x\,M_\pi^4\,\alpha\,s_0 - 30\,B_x\,M_\pi^4\,\beta\,s_0)\,\tilde{K}_1^{(\lambda)}(t,M_\pi,M_\eta) -$$
$$\beta\,t\,(-33\,A_{00}\,F_\pi^2\,M_\eta^4\,\beta + 18\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\alpha - 342\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta + 126\,A_{00}\,F_\pi^2\,M_\pi^4\,\alpha - 585\,A_{00}\,F_\pi^2\,M_\pi^4\,\beta +$$
$$66\,A_x\,F_\pi^2\,M_\eta^4\,\beta - 36\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\alpha + 684\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^2\,\beta - 252\,A_x\,F_\pi^2\,M_\pi^4\,\alpha + 1170\,A_x\,F_\pi^2\,M_\pi^4\,\beta + 36\,B_x\,M_\eta^6\,\beta +$$
$$73\,B_x\,M_\eta^4\,M_\pi^2\,\alpha + 358\,B_x\,M_\eta^4\,M_\pi^2\,\beta - 66\,B_x\,M_\eta^4\,\beta\,s_0 + 630\,B_x\,M_\eta^2\,M_\pi^4\,\alpha + 620\,B_x\,M_\eta^2\,M_\pi^4\,\beta + 36\,B_x\,M_\eta^2\,M_\pi^2\,\alpha\,s_0 -$$
$$684\,B_x\,M_\eta^2\,M_\pi^2\,\beta\,s_0 + 513\,B_x\,M_\pi^6\,\alpha + 266\,B_x\,M_\pi^6\,\beta + 252\,B_x\,M_\pi^4\,\alpha\,s_0 - 1170\,B_x\,M_\pi^4\,\beta\,s_0)\,\tilde{K}_0^{(\lambda)}(t,\,M_\pi,\,M_\eta) -$$
$$3\,\beta\,(3\,A_{00}\,F_\pi^2\,M_\eta^6\,\beta + 27\,A_{00}\,F_\pi^2\,M_\eta^4\,M_\pi^2\,\beta - 24\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\alpha + 33\,A_{00}\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\beta + 24\,A_{00}\,F_\pi^2\,M_\pi^6\,\alpha -$$
$$63\,A_{00}\,F_\pi^2\,M_\pi^6\,\beta - 6\,A_x\,F_\pi^2\,M_\eta^6\,\beta - 54\,A_x\,F_\pi^2\,M_\eta^4\,M_\pi^2\,\beta + 48\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\alpha - 66\,A_x\,F_\pi^2\,M_\eta^2\,M_\pi^4\,\beta - 48\,A_x\,F_\pi^2\,M_\pi^6\,\alpha +$$
$$126\,A_x\,F_\pi^2\,M_\pi^6\,\beta - 3\,B_x\,M_\eta^8\,\beta - 5\,B_x\,M_\eta^6\,M_\pi^2\,\alpha - 32\,B_x\,M_\eta^6\,M_\pi^2\,\beta + 6\,B_x\,M_\eta^6\,\beta\,s_0 - 69\,B_x\,M_\eta^4\,M_\pi^4\,\alpha + 18\,B_x\,M_\eta^4\,M_\pi^4\,\beta +$$
$$54\,B_x\,M_\eta^4\,M_\pi^2\,\beta\,s_0 + 57\,B_x\,M_\eta^2\,M_\pi^6\,\alpha + 72\,B_x\,M_\eta^2\,M_\pi^6\,\beta - 48\,B_x\,M_\eta^2\,M_\pi^4\,\alpha\,s_0 + 66\,B_x\,M_\eta^2\,M_\pi^4\,\beta\,s_0 + 17\,B_x\,M_\pi^8\,\alpha -$$
$$55\,B_x\,M_\pi^8\,\beta + 48\,B_x\,M_\pi^6\,\alpha\,s_0 - 126\,B_x\,M_\pi^6\,\beta\,s_0)\,\tilde{K}_2^{(\lambda)}(t,M_\pi,M_\eta)))/(331776\,\pi^3\,F_\pi^6\,t)$$