FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

**MASTER THESIS**

Matúš Papajčík

# Exact spacetimes in 2+1 gravity

Institute of Theoretical Physics

Supervisor of the master thesis: prof. RNDr. Jiří Podolský, CSc., DSc.
Study programme: Physics
Study branch: Theoretical physics

Prague 2022

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ............. date .............          ....................................

Author's signature

My biggest gratitude goes to my supervisor prof. RNDr. Jiří Podolský, CSc., DSc. for spending countless hours in passionate discussions about the subject. I cannot thank him enough for his guidance not only in the scientific world but also in my academic studies.

I am also grateful for my family and their endless support, especially my mother Jana to whom I am really thankful for enabling me my endeavour in physics.

A big thank you goes also to my friends for making the student life a bit easier.

Title: Exact spacetimes in 2+1 gravity

Author: Matúš Papajčík

Institute: Institute of Theoretical Physics

Supervisor: prof. RNDr. Jiří Podolský, CSc., DSc., Institute of Theoretical Physics

Abstract: We present a study of exact solutions to the coupled system of Einstein–Maxwell equations for Robinson–Trautman and Kundt geometries with a cosmological constant in 2+1 gravity. We also consider an electromagnetic field without any charges or currents. The equations are fully integrated for the nonexpanding Kundt family of spacetimes which only admit an aligned electromagnetic field, and for the aligned Robinson–Trautman class. A special subclass of these solutions is then identified as the charged black hole spacetime in 3D gravity. The nonaligned Robinson–Trautman solution decouples into a separated system of differential equations for the metric and the electromagnetic field. We show that the Robinson–Trautman spacetime admits a nonaligned electromagnetic field by finding a simple particular solution to the equations. Furthermore, we develop a new method of algebraic classification of spacetimes in three dimensions based on the projections of the Cotton tensor onto a suitable null basis. We then show that this classification is equivalent to the Petrov classification of the Cotton–York tensor in 2+1 gravity.

Keywords: 3D gravity, exact spacetimes, Einstein–Maxwell equations, Robinson–Trautman geometry, Kundt geometry, algebraic classification

Abstrakt: V práci studujeme přesná řešení vázaného systému Einsteinových–Maxwellových rovnic pro Robinsonovu–Trautmanovu a Kundtovu geometrii s kosmologickou konstantou v 2+1 gravitaci. Uvažujeme také elektromagnetické pole bez nábojů a proudů. Rovnice jsou plně vyintegrované pro neexpandující Kundtovu rodinu časoprostorů, které připouštějí jen alignované elektromagnetické pole a pro alignovanou Robinsonovu–Trautmanovu třídu. Speciální podtřída těchto řešení je potom identifikována jako nabitá černá díra v 3D gravitaci. Nealignované Robinsonovo–Trautmanovo řešení je přivedeno do separovaného systému diferenciálních rovnic pro metriku a elektromagnetické pole. Dále ukazujeme, že Robinsonův–Trautmanův časoprostor připouští nealignované elektromagnetické pole, a to nalezením jednoduchého partikulárního řešení rovnic. Také prezentujeme novou metodu algebraické klasifikace prostoročasů ve třech dimenzích založenou na projekcích Cottonova tenzoru na vhodnou nulovou bázi. Ukazujeme, že tato klasifikace je ekvivalentní s Petrovovou klasifikací Cotton–Yorkova tenzoru v 2+1 gravitaci.

# Contents

# Introduction

Exact solutions to Einstein's general relativity have always presented a highly interesting field of studies, ever since its formulation in 1915. Not only are exact solutions a good approximation of real physical problems, they also serve as a window to all the intricacies of general relativity which at that time highly differed from a classical theory of gravitation. The first exact solution was published by Schwarzschild [1] only a few months after the publication of general relativity. Nowadays, there exists a large catalogue of exact solutions nicely summarized in books like [2, 3].

The problem of quantum gravity and the need for an alternative to Einstein's theory of relativity lead many researches to more dimensions than four, be it the string theory or just a higher dimensional generalization of relativity summarized in [4]. The interest in gravitational theory in dimensions less than four comes quite late, with the first ever publication on the subject by Staruszkiewicz in 1963 [5], where he studied the motion of point particles in 2+1 dimensional spacetimes. The interest in the theory spiked with the revelation that a quantum theory of gravity is possible in three dimensions. In these spaces the Weyl tensor identically vanishes and the number of independent components of the Riemann and the Ricci tensor is the same. There is no room for dynamical degrees of freedom in three dimensional gravity which is the main reason behind a successful quantization. A nice review of the subject is given in [6].

The lack of dynamical degrees of freedom means that vacuum solutions to Einstein's equations in three-dimensional gravity must necessarily be the maximally symmetric spacetimes. These are the flat Minkowski spacetime or, depending on the sign of the cosmological constant, the de Sitter or anti-de Sitter spacetime for positive and negative cosmological constant, respectively. In this sense there does not exist a Schwarzschild black hole solution. However, it was shown by Bañados, Teitelboim and Zanelli (BTZ) in [7–9] that by nontrivial identifications of a locally anti-de Sitter geometry a black hole like solutions can be constructed for vacuum, pure radiation and electromagnetic field.

The study of exact spacetimes in 2+1 dimensions is not only limited to a toy model for studying quantum gravity. Recently, some solutions found their application in description of 2+1 black holes on a brane [10], while others play important role in the AdS/CFT correspondence [11]. Therefore, the search for more exact solutions can widen the horizons in standard four-dimensional theories of gravity. Plethora of exact solutions have already been found — a nice review of them is [12].

In this study we wish to augment the research recently done by Podolský, Švarc and Maeda in [13], where they systematically investigated the expanding Robinson–Trautman and the nonexpanding Kundt class of spacetimes in three dimensions for vacuum, pure radiation and gyratons. They showed an interesting result, which was also observed by Chow, Pope and Sezgin in [14], that a three-dimensional manifold only allows geometries which are twist-free and shear-free, or in other words, three-dimensional spacetimes which admit a null geodesic field $\mathbf{k}$ must be either Robinson–Trautman or Kundt.

We will focus on generalizing these solutions to also contain an electromagnetic field. The first solution to the Einstein–Maxwell equations in 2+1 gravity belongs to Gott and Alpert, who in 1984 derived an electrostatic solution [15]. More recent study, using the Rainich geometrization of the electromagnetic field, was done in [16]. Another interesting study in this field was done in [17] where the authors found a black hole solution with a real scalar field coupled to a Maxwell field via a duality transformation.

In this work we perform a step-by-step integration of the Einstein–Maxwell equations. In chapter 1 we summarize the results obtained in [13], namely the construction of the metric tensor in canonical coordinates. In section 1.1 we review some basic properties of the electromagnetic field in 2+1 gravity. Section 1.2 presents the coupled Einstein–Maxwell equations, and in section 1.3 we show the correspondence between a massless scalar field and the dual electromagnetic tensor. In chapter 2 we focus on the Kundt class of spacetimes. Sections 2.1-2.3 contain the solution of Einstein–Maxwell equations, and section 2.4 reviews the results. Chapter 3 deals with the Robinson–Trautman spacetimes with an aligned electromagnetic field. Full integration of the equations is performed in sections 3.1-3.3. In section 3.4 we present a summary of obtained results and a identification of a charged black hole solution. We then focus on the nonaligned case for Robinson–Trautman geometry in chapter 4. In sections 4.1-4.3 we try to solve the Einstein–Maxwell equations, but only achieve a separation of variables for the metric functions and the electromagnetic field. We then show a simple nonaligned solution in section 4.4. We already published the results obtained in chapters 1-4 in our paper [18]. It is also contained in appendix B. The last chapter of this thesis, not yet published, is devoted to the problem of algebraic classification of spacetimes in 2+1 gravity. We first review the Cotton and the Cotton–York tensor in sections 5.1 and 5.2. The process of algebraic classification is summarized in section 5.3, and in section 5.4 we show its equivalence to the Penrose-style algebraic classification. In the last section 5.5 we classify the electrovacuum spacetime found in chapter 3.

# 1. Kundt and Robinson–Trautman spacetimes in 2+1 gravity

General spacetimes which admit a null geodesic vector field were previously investigated by Podolský, Švarc and Maeda in [13]. It was proven that under this assumption, the only possible geometries in 2+1 dimensions are the nonexpanding Kundt class and the expanding Robinson–Trautman class of spacetimes; see Theorem 1 in [13]. Another important result was the equivalence between the null vector field being geodesic and hypersurface orthogonal; see Theorem 2 in [13].

At any point we can construct a null triad $\mathbf{e}_I \equiv \{\mathbf{k}, \mathbf{l}, \mathbf{m}\}$ satisfying

$$\mathbf{k} \cdot \mathbf{l} = -1\,, \qquad \mathbf{m} \cdot \mathbf{m} = 1\,, \tag{1.1}$$

with every other scalar product being zero, in which $\mathbf{k}$ is the null vector field along the geodesic congruence. This can be achieved in any spacetime which is at least $C^1$, since we need to guarantee the existence of the covariant derivative. The remaining vector $\mathbf{l}$ is a complementary null vector, and $\mathbf{m}$ is a spacelike vector orthogonal to both $\mathbf{k}$ and $\mathbf{l}$. With respect to this triad, the optical matrix can be defined by

$$\rho \equiv k_{a;b}\, m^a\, m^b\,. \tag{1.2}$$

As a consequence of Theorem 1, the only non-trivial component of $\rho$ is the expansion scalar $\Theta$, while twist and shear vanish identically.

Combining Theorem 1 and Theorem 2, we can establish the canonical coordinates $\{r,\, u,\, x\}$, and it can be shown that the metric in these canonical coordinates takes the form

$$\mathrm{d}s^2 = g_{xx}(r,u,x)\,\mathrm{d}x^2 + 2\,g_{ux}(r,u,x)\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,; \tag{1.3}$$

see Theorem 3 of [13]. The coordinate $r$ is the affine parameter of the geodesic null congruence $\mathbf{k}$, coordinate $u$ labels the hypersurfaces $u = \text{const.}$ to which $\mathbf{k}$ is hypersurface orthogonal, and $x$ span the transverse 1-dimensional subspace with $u$ and $r$ constant. The contravariant metric components are given by the relations

$$g^{xx} = \frac{1}{g_{xx}}\,, \qquad g^{ur} = -1, \qquad g^{rx} = \frac{g_{ux}}{g_{xx}}\,, \qquad g^{rr} = -g_{uu} + \frac{g_{ux}^2}{g_{xx}}\,. \tag{1.4}$$

As suggested in [13], the best choice for a null triad satisfying (1.1) is

$$\mathbf{k} = \partial_r\,, \qquad \mathbf{l} = \frac{1}{2} g_{uu}\,\partial_r + \partial_u\,, \qquad \mathbf{m} = \frac{1}{\sqrt{g_{xx}}}(g_{ux}\,\partial_r + \partial_x)\,. \tag{1.5}$$

Now, the covariant derivative of $\mathbf{k}$, appearing in (1.2), can be easily calculated and it is equal to $k_{a;b} = \frac{1}{2}\, g_{ab,r}$. Using this, the expansion scalar can be evaluated as $\theta = k_{x;x}\, m^x\, m^x$ which gives

$$\theta = \rho = \frac{1}{2}\,\frac{g_{xx,r}}{g_{xx}}\,. \tag{1.6}$$

The spacetimes can now be split into two distinct classes, namely the nonexpanding Kundt class which is invariantly given by vanishing of the expansion $\theta = 0$, and the expanding Robinson–Trautman class which is complementary to the Kundt class and satisfies $\theta \neq 0$.

For later convenience, it is helpful to introduce a shorthand for the square root of the metric component $g_{xx}$. For this purpose we define the function $G(r, u, x)$ as

$$G \equiv \frac{1}{\sqrt{g_{xx}}} \;\Leftrightarrow\; g_{xx} = G^{-2}\,. \tag{1.7}$$

With this notation the equation (1.6), for the scalar expansion, simplifies to

$$\theta = -(\log G)_{,r}\,. \tag{1.8}$$

The Christoffel symbols and the components of the Riemann and Ricci tensors, needed for the gravitational field equations, were already calculated in [13] and for an easy access we present the results in appendix A.

## 1.1 Electromagnetic field in Kundt and Robinson–Trautman spacetimes

Exact solutions for vacuum, gyratons and pure radiation spacetimes in Kundt and Robinson–Trautman classes were already presented in the paper [13]. Therefore, in this work we wish to focus on an important case of nonvacuum spacetimes, namely the ones containing an electromagnetic field.

We will consider a general electromagnetic field given by the Maxwell tensor $F_{ab}$. The antisymmetric Maxwell tensor can also be considered as a 2-form, defined by $\mathbf{F} = \frac{1}{2} F_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b$. In 3D, the electromagnetic tensor has three independent components, and the 2-form explicitly reads

$$\mathbf{F} = F_{ru}\, \mathrm{d}r \wedge \mathrm{d}u + F_{rx}\, \mathrm{d}r \wedge \mathrm{d}x + F_{ux}\, \mathrm{d}u \wedge \mathrm{d}x\,. \tag{1.9}$$

Equivalently, the electromagnetic field can be described by the potential 1-form $\mathbf{A}$ using the relation

$$\mathbf{F} = \mathrm{d}\mathbf{A}\,. \tag{1.10}$$

The contravariant components of the Maxwell tensor are given as $F^{ab} = g^{ac} g^{bd} F_{cd}$, and utilizing the contravariant metric (1.4) we find that

$$F^{ru} = \frac{1}{g_{xx}} (g_{ux}\, F_{rx} - g_{xx}\, F_{ru}), \tag{1.11}$$

$$F^{rx} = \frac{1}{g_{xx}} (g_{ux}\, F_{ru} - g_{uu}\, F_{rx} - F_{ux}), \tag{1.12}$$

$$F^{ux} = -\frac{1}{g_{xx}} F_{rx}\,. \tag{1.13}$$

Interestingly, the Hodge dual of the electromagnetic tensor will play an important role in the field equations later, and it is beneficial to already establish the notion here. We define the Hodge dual ${}^*F^a$ of the electromagnetic field as

$${}^*F^a \equiv \frac{1}{2}\, \varepsilon^{abc}\, F_{bc}\,, \tag{1.14}$$

where $\varepsilon^{abc}$ is the Levi-Civita tensor, and we use the convention

$$\varepsilon^{abc} \equiv \frac{1}{\sqrt{-g}}\,\epsilon^{abc} \;\Rightarrow\; \varepsilon_{abc} = -\sqrt{-g}\,\epsilon_{abc}\,, \tag{1.15}$$

in which $g$ denotes the determinant of the metric, $\epsilon^{abc}$ and $\epsilon_{abc}$ represent the completely antisymmetric Levi-Civita symbol, with the convention that $\epsilon^{rux} = \epsilon_{rux} = 1$ and all even permutations of indices leave the sign unchanged, while odd permutations of indices change the sign.

Straightforward calculation for the determinant of the metric (1.3) yields

$$-g = g_{xx} = G^{-2}\,. \tag{1.16}$$

From the definition (1.14), the corresponding components of the dual electromagnetic tensor are

$${}^*F^r = G\,F_{ux}\,, \qquad {}^*F^u = -G\,F_{rx}\,, \qquad {}^*F^x = G\,F_{ru}\,. \tag{1.17}$$

The dual electromagnetic 1-form given by ${}^*\mathbf{F} = {}^*F_a\,\mathrm{d}x^a$ can be obtained by simply lowering the index of the dual (1.14). Before we continue further, it is useful to introduce a shorthand $F_a$, such that

$$F_a \equiv \frac{{}^*F_a}{G}\,, \tag{1.18}$$

in which we factor out the square root of the metric determinant from the components of the dual 1-form. This definition allows us to simply express the 1-form components as

$${}^*F_a = G\,F_a\,, \tag{1.19}$$

where, after evaluating (1.18), we get

$$F_r = F_{rx}\,, \tag{1.20}$$

$$F_u = g_{ux}\,F_{ru} - F_{ux} - g_{uu}\,F_{rx}\,, \tag{1.21}$$

$$F_x = g_{xx}\,F_{ru} - g_{ux}\,F_{rx}\,. \tag{1.22}$$

It is also true that ${}^*F^a = G\,F^a$, and the explicit expressions for $F^a$ can be immediately seen from (1.17). The inverse relations to the dual electromagnetic tensor are

$$F_{ab} = -\varepsilon_{abc}\,{}^*F^c\,, \qquad \text{or} \qquad F^{ab} = -\varepsilon^{abc}\,{}^*F_c\,. \tag{1.23}$$

The latter equation gives us a convenient way to express the contravariant components of the Maxwell tensor, and in view of this equation we can compute that

$$F^{ru} = -G^2 F_x\,, \qquad F^{rx} = G^2 F_u\,, \qquad F^{ux} = -G^2 F_r\,. \tag{1.24}$$

As the next step we now define the electromagnetic invariants, namely we set

$$F^2 \equiv F_{ab}F^{ab} \qquad \text{and} \qquad {}^*F^2 \equiv {}^*F_a\,{}^*F^a\,. \tag{1.25}$$

The invariant $F_{ab}\,{}^*F^a\,{}^*F^b$ is identically equal to zero because of the symmetry reasons, and the corresponding calculation of (1.25) gives

$$F^2 = -2{}^*F^2 = -2\,G^2\big(g_{uu}\,F_{rx}^2 + 2\,F_{rx}(F_{ux} - g_{ux}\,F_{ru}) + g_{xx}\,F_{ru}^2\big)\,. \tag{1.26}$$

Taking the inspiration from the Newman–Penrose formalism in standard 4D gravity, we define additional set of invariants by projecting the Maxwell tensor onto the frame formed by the null triad (1.5). Due to the antisymmetry of the Maxwell tensor only three such invariants are independent, and they fully describe the three independent components of the electromagnetic field. We choose to define these invariants in the following way

$$\phi_0 \equiv F_{ab}\, k^a\, m^b\,, \tag{1.27}$$
$$\phi_1 \equiv F_{ab}\, k^a\, l^b\,, \tag{1.28}$$
$$\phi_2 \equiv F_{ab}\, m^a\, l^b\,. \tag{1.29}$$

After an easy evaluation we obtain the results

$$\phi_0 = G\, F_{rx}\,, \qquad \phi_1 = F_{ru}\,, \qquad \phi_2 = G(g_{ux}\, F_{ru} - F_{ux} - \tfrac{1}{2}\, g_{uu}\, F_{rx})\,. \tag{1.30}$$

By inverting the relations (1.20)–(1.22), the components of the electromagnetic tensor can be expressed as

$$F_{rx} = F_r\,, \tag{1.31}$$
$$F_{ru} = G^2(F_x + g_{ux}\, F_r)\,, \tag{1.32}$$
$$F_{ux} = g_{ux}\, G^2(F_x + g_{ux}\, F_r) - F_u - g_{uu}\, F_r\,. \tag{1.33}$$

Using this, the scalars (1.30) can be then simplified to

$$\phi_0 = G\, F_r, \qquad \phi_1 = G^2(F_x + g_{ux}\, F_r)\,, \qquad \phi_2 = G(F_u + \tfrac{1}{2} g_{uu}\, F_r)\,. \tag{1.34}$$

Taking a look back at the equation (1.26) and with the help of the results given in (1.30), we can also obtain a compact expression for the electromagnetic scalar, specifically

$$\frac{1}{2} F^2 = 2\,\phi_0\,\phi_2 - \phi_1^2\,. \tag{1.35}$$

In the definition of the Newman–Penrose scalars (1.27)–(1.29), there is a freedom in the choice of the null frame. One can perform Lorentz transformations that leave the normalization (1.1) unchanged. For example one such transformation is a boost

$$\mathbf{k}' = B\,\mathbf{k}\,, \qquad \mathbf{l}' = B^{-1}\,\mathbf{l}\,, \qquad \mathbf{m}' = \mathbf{m}\,, \tag{1.36}$$

which is done by rescaling the null vectors with a constant parameter $B$, and the Newman–Penrose scalars undergo the following change

$$\phi_0' = B\,\phi_0\,, \qquad \phi_1' = \phi_1\,, \qquad \phi_2' = \frac{1}{B}\,\phi_2\,. \tag{1.37}$$

We see that the scalars $\phi_0'$, $\phi_1'$, $\phi_2'$ have respective boost weights of $+1$, $0$, $-1$. In accord with the alignment of general tensors which was established by Milson, Coley, Pravda, and Pravdová in [19], we say that the electromagnetic field is aligned with the null vector field $\mathbf{k}$, if the component with the highest boost weight vanishes. In view of the equations (1.30) and (1.34) this implies the condition

$$\phi_0 = 0 \;\Leftrightarrow\; F_{rx} = 0 \;\Leftrightarrow\; F_r = 0\,. \tag{1.38}$$

By substituting the explicit definition (1.27) for the appropriate Newman–Penrose scalar, we obtain the equation $F_{ab}\, k^a\, m^b = 0$. Thanks to the normalization (1.1), the before mentioned equation is equivalent to the statement

$$F_{ab}\, k^a = \mathcal{N}\, k_b + \mathcal{M}\, l_b\,. \tag{1.39}$$

Since we also have to respect the antisymmetry property of the Maxwell tensor, the projection $F_{ab}\, k^a\, k^b = 0$ must be identically satisfied. This restricts the coefficient $\mathcal{M} = 0$. Thus we have proved an equivalent formulation for the alignment of the electromagnetic tensor, or in other words, the electromagnetic tensor is aligned with the vector field $\mathbf{k}$ if and only if

$$F_{ab}\, k^b = \mathcal{N}\, k_a\,. \tag{1.40}$$

Another type of transformations which do not change the relation (1.1) are null rotations. The most general form of a null rotation preserving the direction $\mathbf{k}$ is

$$\mathbf{k}' = \mathbf{k}\,, \qquad \mathbf{l}' = \mathbf{l} + \sqrt{2}L\,\mathbf{m} + L^2\,\mathbf{k}\,, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}L\,\mathbf{k}\,. \tag{1.41}$$

Such a null rotation transforms the Newmann–Penrose scalars as

$$\phi_0' = \phi_0\,, \tag{1.42}$$
$$\phi_1' = \phi_1 + \sqrt{2}L\,\phi_0\,, \tag{1.43}$$
$$\phi_2' = \phi_2 + \sqrt{2}L\,\phi_1 + L^2\,\phi_0\,. \tag{1.44}$$

It can be noted that whenever we have an aligned electromagnetic field ($\phi_0 = 0$) there exists a frame in which the scalar $\phi_2$ also vanishes, provided $\phi_1 \neq 0$. This is given by the transformation (1.44), where we choose the parameter $L = -\frac{\phi_2}{\sqrt{2}\phi_1}$. Of course, this property is not exclusive to only aligned electromagnetic fields. In the nonaligned case, the vanishing of the scalar $\phi_2'$ is dependent on the existence of a real solution to the quadratic equation given by (1.44).

Similarly, we can perform a null rotation with fixed $\mathbf{l}$, and in parallel with the previous null rotation its general form is

$$\mathbf{k}' = \mathbf{k} + \sqrt{2}E\,\mathbf{m} + E^2\,\mathbf{l}\,, \qquad \mathbf{l}' = \mathbf{l}\,, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}E\,\mathbf{l}\,. \tag{1.45}$$

This invokes the following transformations of the Newmann–Penrose scalars

$$\phi_0' = \phi_0 + \sqrt{2}E\,\phi_1 + E^2\,\phi_2\,, \tag{1.46}$$
$$\phi_1' = \phi_1 + \sqrt{2}E\,\phi_2\,, \tag{1.47}$$
$$\phi_2' = \phi_2\,. \tag{1.48}$$

The existence of a frame in which the electromagnetic field is aligned is determined by the quadratic equation $\phi_0' = 0$, that is

$$\phi_0 + \sqrt{2}E\phi_1 + E^2\phi_2 = 0\,. \tag{1.49}$$

However, it is not clear that a real solution to this quadratic equation exists in general for all instances of the electromagnetic field.

Of course, there also exists a rotation which leaves $\mathbf{m}$ invariant. One example of such transformation is the boost (1.36). Excluding this boost, the general transformation with $\mathbf{m}' = \mathbf{m}$ is

$$\mathbf{k}' = R\,\mathbf{l}\,, \qquad \mathbf{l}' = \frac{1}{R}\,\mathbf{k}\,, \qquad \mathbf{m}' = \mathbf{m}\,, \tag{1.50}$$

which is just a trivial swapping of the vectors $\mathbf{k}$ and $\mathbf{l}$.

Based on the formalism of Newmann–Penrose scalars we can develop an algebraic classification of the electromagnetic tensor. Namely, we distinguish two distinct cases based on the value of the scalar (1.35). For $F^2 = 0$ we say that the field in null and for $F^2 \neq 0$ the field in non-null. In the language of the Newman–Penrose scalars the corresponding classification is

- the electromagnetic field is aligned $\Leftrightarrow \phi_0 = 0$,
- the aligned electromagnetic field is null $\Leftrightarrow \phi_1 = 0$,
- the aligned electromagnetic field is non-null $\Leftrightarrow \phi_2 = 0$.

The aligned electromagnetic field is given by (1.38), and is described by only two components $F_{ru}$ and $F_{ux}$ encoded in $\phi_1 = F_{ru}$ and $\phi_2 = G(g_{ux}\,F_{ru} - F_{ux})$. Furthermore, the electromagnetic field is null if $F_{ru} = 0$. This describes purely radiative electromagnetic field with the single component $\phi_2 = -G\,F_{ux}$, and for such a field the electromagnetic invariant vanishes, $F^2 = 0$. Alternatively, non-radiative and non-null field can be achieved by setting $F_{ux} = g_{ux}F_{ru}$. Here the electromagnetic invariant is characterized by $F^2 = -2\,\phi_1^2$.

Lastly, let us take a look at the energy-momentum tensor of the electromagnetic field which makes an important connection between the curvature and matter in Einstein's general relativity. The energy-momentum tensor in any dimension is defined as

$$T_{ab} = \frac{\kappa_0}{8\pi}\left(F_{ac}\,F_b{}^c - \frac{1}{4}g_{ab}\,F^2\right), \tag{1.51}$$

where $\kappa_0$ is a scaling constant defined for simpler transitions between different unit conventions. The explicit evaluation of the energy-momentum tensor gives the following components

$$\frac{8\pi}{\kappa_0}T_{rr} = G^2F_{rx}^2\,, \tag{1.52}$$

$$\frac{8\pi}{\kappa_0}T_{rx} = G^2F_{rx}(g_{xx}F_{ru} - g_{ux}F_{rx})\,, \tag{1.53}$$

$$\frac{8\pi}{\kappa_0}T_{ru} = \frac{1}{2}G^2(g_{xx}F_{ru}^2 - g_{uu}F_{rx}^2)\,, \tag{1.54}$$

$$\frac{8\pi}{\kappa_0}T_{xx} = -F_{rx}(g_{ux}F_{ru} + F_{ux}) + \frac{1}{2}G^2(2g_{ux}^2 - g_{xx}g_{uu})F_{rx}^2 + \frac{1}{2}g_{xx}F_{ru}^2\,, \tag{1.55}$$

$$\frac{8\pi}{\kappa_0}T_{ux} = \frac{1}{2}G^2\Big(g_{uu}F_{rx}(g_{ux}F_{rx} - 2g_{xx}F_{ru}) + g_{xx}F_{ru}(g_{ux}F_{ru} - 2F_{ux})\Big)\,, \tag{1.56}$$

$$\begin{aligned}\frac{8\pi}{\kappa_0}T_{uu} = \frac{1}{2}G^2\Big(&g_{uu}F_{rx}(g_{uu}F_{rx} - 2g_{ux}F_{ru}) + (2g_{ux}^2 - g_{xx}g_{uu})F_{ru}^2\\ &+ 2F_{ux}(F_{ux} + g_{uu}F_{rx} - 2g_{ux}F_{ru})\Big)\,.\end{aligned} \tag{1.57}$$

In any dimension other than four, the energy-momentum tensor is no longer trace-free. Calculating the trace $T \equiv g^{ab}\, T_{ab}$ gives the result

$$\frac{8\pi}{\kappa_0} T = -\frac{1}{2} G^2 \Big( g_{uu} F_{rx}^2 + 2F_{rx}(F_{ux} - g_{ux} F_{ru}) + g_{xx} F_{ru}^2 \Big) \,. \tag{1.58}$$

It is not surprising that this is exactly equal to $\frac{8\pi}{\kappa_0} T = \frac{1}{4} F^2 = -\frac{1}{2} {}^*F^2$, as this is evident from the definition (1.51). Interestingly, rewriting the energy-momentum tensor in terms of the dual electromagnetic field leads to a huge simplification. The key equation is the inverse relation introduced in (1.23), but first it is good to recall some properties of the Levi-Civita tensor. For the product of two Levi-Civita tensors the following identity holds

$$\varepsilon_{abc}\, \varepsilon^{klm} = -6\, \delta^k_{[a}\, \delta^l_b\, \delta^m_{c]} \,. \tag{1.59}$$

Subsequently, for the contractions of the Levi-Civita tensor we obtain

$$\varepsilon_{abm}\, \varepsilon^{klm} = \delta^k_b\, \delta^l_a - \delta^k_a\, \delta^l_b \,, \tag{1.60}$$

$$\varepsilon_{alm}\, \varepsilon^{klm} = -2\, \delta^k_a \,. \tag{1.61}$$

Now substituting (1.23) into (1.51), and using the property (1.60), we find that the energy-momentum tensor can be simply written in the form

$$T_{ab} = \frac{\kappa_0}{8\pi} \left( {}^*F_a\, {}^*F_b - \frac{1}{2} g_{ab}\, {}^*F^2 \right) , \tag{1.62}$$

where the relation between the electromagnetic invariant and the dual electromagnetic invariant (1.26) was used. This can be independently checked by making use of the second identity of the Levi-Civita tensor (1.61) in the defining equation (1.25), which then leads to the relation $F^2 = -2\, {}^*F^2$.

## 1.2 Einstein–Maxwell equations with a cosmological constant

After a general exposition of the electromagnetic field, in which we need not specify any field equations, we now turn our focus to establish the equations that govern the dynamics of our spacetimes. It is quite natural to assume that the form of the Einstein equations is not changed in the 3D theory of general relativity. Therefore, we assume that the following field equations hold

$$R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 8\pi\, T_{ab} \,, \tag{1.63}$$

where we also allow a nonvanishing cosmological constant $\Lambda$. Taking the trace of this equation we can express the Ricci scalar as $R = 2\,(3\Lambda - 8\pi\, T)$, and by substituting it back into the equation, we obtain an equivalent form of the field equations, namely

$$R_{ab} = 2\Lambda\, g_{ab} + 8\pi (T_{ab} - T\, g_{ab}) \,. \tag{1.64}$$

Utilizing the form of the energy-momentum tensor (1.62) and its trace $\frac{8\pi}{\kappa_0} T = -\frac{1}{2} {}^*F^2$ we find that the right hand side factors out nicely in terms of the dual electromagnetic tensor as

$$R_{ab} = 2\Lambda\, g_{ab} + \kappa_0\, {}^*F_a\, {}^*F_b \,. \tag{1.65}$$

For computational purposes it is somewhat advantageous to return to the functions $F_a$ given by the relation (1.19), for which we obtain similar field equations

$$R_{ab} = 2\Lambda\, g_{ab} + \kappa_0\, G^2 F_a F_b\,, \tag{1.66}$$

where we recall that the functions $F_a$ are given by (1.20)-(1.22).

For the electromagnetic field to be of any physical importance, it must also satisfy the Maxwell equations. Nice geometrical formulation of these equations is

$$\mathrm{d}^*\mathbf{F} = 4\pi\, {}^*\mathbf{J}\,, \qquad \mathrm{d}\mathbf{F} = 0\,. \tag{1.67}$$

Another equivalent formulation can be achieved using the covariant derivative $\nabla$ instead of the exterior derivative as

$$\nabla\cdot\mathbf{F} = 4\pi\,\mathbf{J}\,, \qquad \nabla\cdot{}^*\mathbf{F} = 0\,. \tag{1.68}$$

In components these Maxwell equations read $F^{ab}{}_{;b} = J^a$ and $F_{[ab;c]} = 0$. We will only consider electromagnetic field in a vacuum, that is, we will not include any electric charges or currents. This simplifies the right hand side of the equations, where we can take the four-current $\mathbf{J} = 0$. Using the identity for the covariant divergence of an antisymmetric second rank tensor

$$F^{ab}{}_{;b} = \frac{1}{\sqrt{-g}}\left(\frac{1}{\sqrt{-g}}F^{ab}\right)_{,b}\,, \tag{1.69}$$

the system of equations can be then brought into an equivalent form

$$(\sqrt{-g}\,F^{ab})_{,b} = 0\,, \qquad F_{[ab,c]} = 0\,. \tag{1.70}$$

Or if we replace the first set by their equivalent counterpart from (1.67), and evaluate the exterior derivative of the Maxwell dual, $\mathrm{d}^*\mathbf{F} = {}^*F_{a,b}\,\mathrm{d}x^a\wedge\mathrm{d}x^b$, using the relation (1.19) we come to the following alternative formulation of the first equation

$$(G\,F_a)_{,b} = (G\,F_b)_{,a}\,. \tag{1.71}$$

In canonical coordinates this gives the following three equations

$$(G\,F_x)_{,r} = (G\,F_r)_{,x}\,, \qquad (G\,F_u)_{,r} = (G\,F_r)_{,u}\,, \qquad (G\,F_u)_{,x} = (G\,F_x)_{,u}\,. \tag{1.72}$$

The second equation of (1.70) in three dimensions adds only one more Maxwell equation

$$F_{ru,x} + F_{ux,r} - F_{rx,u} = 0\,. \tag{1.73}$$

Together with the Einstein field equations (1.63), this represents a set of ten equations (six Einstein field equations and four Maxwell equations) for the three independent components of the metric tensor (1.3) and the three independent components of the Maxwell 2-form (1.9). The forthcoming chapters will be devoted to finding a general solution to this system of equations.

## 1.3 Minimally coupled scalar field in 2+1 gravity

Before we continue with the integration of the Einsten–Maxwell equations, it is worth mentioning a remarkable correspondence between a minimally coupled scalar field and the electromagnetic field, which occurs in 3D gravity. We will consider the following action

$$S = \int \mathrm{d}^3x \,\sqrt{-g}\,\Big(R - 2\Lambda - 8\pi\, g^{ab}\,\nabla_a\Phi\,\nabla_b\Phi\Big), \tag{1.74}$$

where $R$ is the Ricci scalar and $\Phi$ is the minimally coupled massless scalar field. For convenience we will denote the gravitational part of the action as $S_G$ and the Klein–Gordon part, corresponding to the scalar field, as $S_{KG}$. The action can be then written in the form $S = S_G + S_{KG}$, where we define the respective parts by

$$S_G = \int \mathrm{d}^3x \,\sqrt{-g}\,\Big(R - 2\Lambda\Big)\,, \tag{1.75}$$

$$S_{KG} = -8\pi \int \mathrm{d}^3x \,\sqrt{-g}\,\Big(g^{ab}\,\nabla_a\Phi\,\nabla_b\Phi\Big)\,. \tag{1.76}$$

The next step is to vary the action (1.74), obtaining the dynamical equations for our system. The variation of the gravitational part $S_G$ leads to the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda\, g_{ab}$, for proof see appendix E of [20]. By varying the part $S_{KG}$ we get the energy-momentum tensor

$$\delta S_{KG} = -8\pi \int \mathrm{d}^3x \,\sqrt{-g}\,T_{ab}\,\delta g^{ab}\,; \tag{1.77}$$

see equation (E.1.26) of [20]. Explicitly, the variation of the Klein–Gordon part of the action reads

$$\delta S_{KG} = -8\pi \int \mathrm{d}^3x \Big(\delta(\sqrt{-g})\, g^{cd}\,\nabla_c\Phi\,\nabla_d\Phi + \sqrt{-g}\,\nabla_a\Phi\,\nabla_b\Phi\,\delta g^{ab}\Big). \tag{1.78}$$

Employing the equation (E.1.17) in [20] which reads

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\,g_{ab}\,\delta g^{ab}\,, \tag{1.79}$$

the expression (1.78) then amounts to

$$\delta S_{KG} = -8\pi \int \mathrm{d}^3x \,\sqrt{-g}\,\Big(\nabla_a\Phi\,\nabla_b\Phi - \frac{1}{2}\,g_{ab}\,g^{cd}\,\nabla_c\Phi\,\nabla_d\Phi\Big)\,\delta g^{ab}\,. \tag{1.80}$$

By comparing this result with the equation (1.77), we see that the corresponding energy-momentum tensor of the scalar field is

$$T_{ab} = \nabla_a\Phi\,\nabla_b\Phi - \frac{1}{2}\,g_{ab}\,\nabla_c\Phi\,\nabla^c\Phi\,. \tag{1.81}$$

The equation of motion for the field $\Phi$ would arise from the variation of the action (1.74) with respect to the scalar field. Since a similar action was considered in [12], see equation (13.1), where the coupling constant is $b = 0$, $\Lambda \to -\Lambda$ and the scalar fields differ by a trivial gauge transformation, we can use the result given in equation (13.2), which is

$$\nabla^a \nabla_a \Phi = 0\,. \tag{1.82}$$

If we now compare the expression (1.81) with the energy-momentum tensor of the electromagnetic field written in terms of the dual Maxwell 1-form (1.62), we get a correspondence between the scalar field and the electromagnetic field, namely

$$\nabla_a \Phi = \sqrt{\frac{\kappa_0}{8\pi}}\,{}^*F_a\,. \tag{1.83}$$

Indeed, if we were to calculate $T_{ab} - T\,g_{ab}$ for the scalar field, using the result for the trace $T = -\frac{1}{2}\nabla_c\Phi\,\nabla^c\Phi$, which comes directly from (1.81), we get that $T_{ab} - T\,g_{ab} = \nabla_a\Phi\,\nabla_b\Phi$, and the Einstein equations for the scalar field then become

$$R_{ab} = 2\Lambda\,g_{ab} + 8\pi\,\nabla_a\Phi\,\nabla_b\Phi\,, \tag{1.84}$$

This is the same system of equations as for the electromagnetic field given by (1.65), and we see that due to the identification (1.83) the systems are equivalent. The dual Maxwell 1-form can thus be written as

$${}^*\mathbf{F} = \sqrt{\frac{8\pi}{\kappa_0}}\,\mathbf{d}\Phi\,, \tag{1.85}$$

since for the scalar functions it holds that $\nabla\Phi = \mathbf{d}\Phi$. It remains to be proven that the Maxwell equations (1.67) or (1.68) imply the Klein–Gordon equation (1.82), and vice-versa. The vacuum Maxwell equations in terms of the dual Maxwell 1-form read

$$\mathrm{d}{}^*\mathbf{F} = 0\,, \qquad \nabla\cdot{}^*\mathbf{F} = 0\,. \tag{1.86}$$

Substituting (1.85) into the first equation we obtain $\mathrm{d}^2\Phi = 0$, but that is trivially satisfied. The second equation, after the same substitution, becomes $\nabla\cdot(\mathbf{d}\Phi) = 0$. Using the identity $\nabla\Phi = \mathbf{d}\Phi$, we come to the Klein–Gordon equation

$$\nabla\cdot\nabla\Phi = 0\,. \tag{1.87}$$

We have thus proven that from the Maxwell equations, the Klein–Gordon equation follows. The opposite implication is even more trivial. By taking the exterior derivative of (1.85) we obtain the first Maxwell equation. And using the identity for the exterior derivative of scalar function in the Klein–Gordon equation (1.87) we get the second Maxwell equation.

Therefore, we can conclude that the minimally coupled massless scalar field is fully equivalent to the electromagnetic field in 3D gravity.

# 2. All Kundt spacetimes with electromagnetic field in 2+1 gravity

In this chapter we systematically derive and present a general solution to the coupled Einstein–Maxwell equations (1.66) for the Kundt spacetimes. Kundt spacetimes represent a family of geometries satisfying the condition that they are nonexpanding, nontwisting and shear-free. As we mentioned in chapter 1, all spacetimes in three dimensions are nontwisting and shear-free. This implies that the only nontrivial condition for the Kundt spacetime in 2+1 gravity is the vanishing of the expansion scalar, or explicitly

$$\theta = 0\,. \tag{2.1}$$

In view of the equation (1.8) this condition means

$$(\log G)_{,r} = 0\,. \tag{2.2}$$

Immediately it follows from this equation that the function $G(r,u,x)$ does not depend on the coordinate $r$ and we can write a general solution in the form

$$G(r,u,x) = P(u,x)\,, \tag{2.3}$$

where $P(u,x)$ is an arbitrary $r$-independent function. Due to the definition (1.7) we actually obtain a solution for the metric function $g_{xx}$, and we see that

$$g_{xx} = P^{-2}(u,x)\,. \tag{2.4}$$

The gravitational field equations (1.66) can be then written as

$$R_{ab} = 2\Lambda\, g_{ab} + \kappa_0\, P^2 F_a\, F_b\,, \tag{2.5}$$

where we recall that the functions $F_a$ are given by (1.20)-(1.22).

## 2.1 Integration of the Einstein field equations, step one

We are now in the position to start solving the system of equations (2.5), and in this section we will show the solution to the first three of six field equations.

(i) Integration of $R_{rr} = \kappa_0\, P^2 F_r^2$:

Using the condition (2.1) to evaluate the component (A.24) of the Ricci tensor, we get that $R_{rr} = 0$. The corresponding field equation then reads

$$\kappa_0\, P^2 F_r^2 = 0\,, \tag{2.6}$$

and since $P(u,x)$ is arbitrary this implies

$$F_r = 0\,. \tag{2.7}$$

Moreover, as a consequence of the relation (1.20), the component $F_{rx}$ of the electromagnetic tensor is identically equal to

$$F_{rx} = 0\,. \tag{2.8}$$

Evaluating now the Newmann–Penrose scalars (1.34), we come to the result

$$\phi_0 = 0\,, \qquad \phi_1 = P^2 F_x\,, \qquad \phi_2 = P\,F_u\,. \tag{2.9}$$

This shows that in the Kundt geometry of 2+1 gravity the electromagnetic field must be aligned, and that it is in fact aligned with the key vector field $\mathbf{k} = \partial_r$. It parallels the same result that arises in the standard four-dimensional gravity, where it can be also proven that the Kundt spacetime allows only an aligned electromagnetic field, see chapter 18 of [3], or see chapter 31.1 of [2].

(ii) Integration of $R_{rx} = \kappa_0\,P^2 F_r\,F_x$:

The Ricci component (A.25), after substituting $\theta = 0$ amounts to $R_{rx} = -\frac{1}{2}\,g_{ux,rr}$. Including also the previous result (2.7), the field equation simply states

$$g_{ux,rr} = 0\,. \tag{2.10}$$

It can be easily integrated, and the solution can be written as

$$g_{ux} = e(u,x) + f(u,x)\,r\,, \tag{2.11}$$

where $e(u,x)$ and $f(u,x)$ are arbitrary functions independent of $r$.

(iii) Integration of $R_{ru} = -2\Lambda + \kappa_0\,P^2 F_r\,F_u$:

Employing the solution (2.11) alongside with (2.1), the component (A.26) can be expressed as $R_{ru} = -\frac{1}{2}\,g_{uu,rr} + \frac{1}{2}P^2(f_{||x} + f^2)$, and subsequently putting $F_r = 0$, the field equation then reads

$$g_{uu,rr} = 4\Lambda + P^2(f_{||x} + f^2)\,, \tag{2.12}$$

where the symbol $_{||}$ denotes the covariant derivative with respect to the spatial one-dimensional subspace, and it is defined as

$$f_{||x} \equiv f_{,x} - {}^S\Gamma^x_{xx}\,f = f_{,x} + f\,\frac{P_{,x}}{P} = \frac{(fP)_{,x}}{P}\,, \tag{2.13}$$

in which the Christoffel symbol (A.17) evaluates to ${}^S\Gamma^x_{xx} = -\frac{P_{,x}}{P}$.

The right hand side of the equation (2.12) is independent of $r$, and a solution to such differential equation is

$$g_{uu} = a(u,x) + b(u,x)\,r + c(u,x)\,r^2\,, \tag{2.14}$$

where the functions $a(u,x)$ and $b(u,x)$ are arbitrary, and by $c(u,x)$ we denoted the following

$$c = 2\Lambda + \frac{1}{2}P^2(f_{||x} + f^2)\,. \tag{2.15}$$

We have now determined the $r$-dependence of all of the metric functions and instead of continuing further with the Einstein field equations, it is actually easier to solve the Maxwell equations first.

## 2.2 Integration of the Maxwell equations

The electromagnetic 2-form in 3D is given by three independent components. However, in Kundt spacetimes we showed that in canonical coordinates one of the components must necessarily vanish, recall (2.8). The key functions $F_a$, given by (1.20)-(1.22) and using also (2.3), now have the form

$$F_r = 0\,, \qquad F_u = g_{ux}\, F_{ru} - F_{ux}\,, \qquad F_x = P^{-2} F_{ru}\,. \tag{2.16}$$

They encode the remaining two independent components of the electromagnetic tensor, but are yet to be subjected to the Maxwell equations (1.70). The first set of Maxwell equations given in (1.72), after substituting from (2.16), gives

$$(P^{-1} F_{ru})_{,r} = 0\,, \tag{2.17}$$

$$[P(g_{ux}\, F_{ru} - F_{ux})]_{,r} = 0\,, \tag{2.18}$$

$$[P(g_{ux}\, F_{ru} - F_{ux})]_{,x} = (P^{-1} F_{ru})_{,u}\,. \tag{2.19}$$

The last Maxwell equation comes from $F_{[ab,c]} = 0$ and taking $F_{rx} = 0$ in (1.73), it reduces to the equation

$$F_{ux,r} + F_{ru,x} = 0\,. \tag{2.20}$$

Having assembled the system of equations, we can start by solving (2.17) first. Due to the function $P(u, x)$ being $r$-independent, the equation then simplifies to $F_{ru,r} = 0$ and the general solution is

$$F_{ru} = Q(u, x)\,, \tag{2.21}$$

where $Q(u, x)$ is an arbitrary function. Using this in the second equation (2.18), together with (2.11), we obtain $F_{ux,r} = f\, Q$. Since the right hand side is yet again independent of $r$, after an easy integration we get

$$F_{ux} = f Q\, r - \xi(u, x)\,, \tag{2.22}$$

where $\xi(u, x)$ is arbitrary. Instead of continuing to the third Maxwell equation, it is advantageous to integrate the equation (2.20). Utilizing the found electromagnetic component (2.21), the corresponding equation reads $F_{ux,r} = -Q_{,x}$, and its solution is

$$F_{ux} = -Q_{,x}\, r - \tilde{\xi}(u, x)\,. \tag{2.23}$$

The expressions (2.22) and (2.23) should be equal, and by comparing the coefficients we obtain the constraints

$$Q_{,x} = -f Q\,, \tag{2.24}$$

$$\xi = \tilde{\xi}\,. \tag{2.25}$$

Therefore, we can omit the tilde in the expression for $\xi$, and simply write

$$F_{ux} = -Q_{,x}\, r - \xi\,. \tag{2.26}$$

Coming back to the third Maxwell equation (2.19), while also applying (2.11) and (2.22), we get the constraint

$$[P(e\, Q + \xi)]_{,x} = \left(\frac{Q}{P}\right)_{,u}\,. \tag{2.27}$$

Having thus fully integrated the Maxwell equations and obtaining all of the components of the electromagnetic tensor, we can now express the functions (1.20)-(1.22) as

$$F_r = 0\,, \qquad F_x = P^{-2}Q\,, \qquad F_u = e\,Q + \xi\,. \tag{2.28}$$

The Newmann–Penrose scalars for the electromagnetic field (1.34) then amount to

$$\phi_0 = 0\,, \qquad \phi_1 = Q\,, \qquad \phi_2 = P(e\,Q + \xi)\,. \tag{2.29}$$

The field in null ($\phi_1 = 0$) if and only if $Q = 0$. So the function $Q$ is closely related to the structure of the electromagnetic field, specifically it uniquely determines if the field is null. Such a null field is then determined by $\phi_2 = P\xi$. For the field to be non-null, $\phi_2 = 0$, while also $Q \neq 0$, and so to satisfy the condition $\phi_2 = 0$, we see that $e\,Q = -\xi$, and all the information about the field is encoded in the remaining scalar $\phi_1 = Q$.

## 2.3 Integration of the Einstein field equations, step two

We now return to the remaining three Einstein equations. Having found the $r$-dependence of all the metric functions and all the components of the electromagnetic field, these equations will bring constraints on the integration constants established before.

(iv) Integration of $R_{xx} = 2\Lambda\, g_{xx} + \kappa_0\, P^2 F_x^2$:

The condition (2.1) significantly simplifies the component of the Ricci tensor (A.27), and after the substitution of (2.11) the component equates to $R_{xx} = -f_{xx} = -(f_{||x} + \frac{1}{2}f^2)$. Using $g_{xx} = P^{-2}$ and the form of the function $F_x$ given in (2.28), the field equation reads

$$-\left(f_{||x} + \frac{1}{2}f^2\right) = 2\Lambda\, P^{-2} + \kappa_0\, P^{-2}Q^2\,. \tag{2.30}$$

After a trivial manipulation, the equation can be brought into the form

$$\kappa_0\, Q^2 = -2\Lambda - P^2\left(f_{||x} + \frac{1}{2}f^2\right)\,, \tag{2.31}$$

and provided the right hand side is positive, the function $Q$ is entirely determined by the cosmological constant $\Lambda$ and the metric functions $P$ and $f$. Furthermore, we can use the equation (2.30) to express the covariant derivative of $f$ as

$$P^2 f_{||x} = -2\Lambda - \frac{1}{2}F - \kappa_0\, Q^2\,, \tag{2.32}$$

where we defined the new variable $F$ as follows

$$F \equiv P^2 f^2\,, \tag{2.33}$$

and using these expressions in (2.15) we obtain a form of $c$ purely dependent on the cosmological constant $\Lambda$ and the functions $F$ and $Q$, namely

$$c = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\,. \tag{2.34}$$

The equation (2.32) can be rewritten as a first order differential equation by substituting the definition of the covariant derivative (2.13), which then yields the constraint

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0\, Q^2\right) . \tag{2.35}$$

Another formulation of this constraint can be achieved by applying $Pf = \sqrt{F}$, and the resulting equation is

$$F_{,x} + (4\Lambda + F + 2\kappa_0\, Q^2)f = 0 . \tag{2.36}$$

(v) Integration of $R_{ux} = 2\Lambda\, g_{ux} + \kappa_0\, P^2 F_u\, F_x$:

The Ricci component (A.28), after putting $\theta = 0$ and using the found metric functions (2.4), (2.11) and (2.14), can be expressed as a series in the affine parameter $r$, specifically

$$R_{ux} = A_{ux} + B_{ux}\, r , \tag{2.37}$$

where $A_{ux}$ and $B_{ux}$ denotes

$$A_{ux} = \frac{1}{2}\left[f_{,u} - b_{,x} - eP^2(f_{||x} + f^2) - f\frac{P_{,u}}{P}\right] , \tag{2.38}$$

$$B_{ux} = -\frac{1}{4}[F_{,x} - 4\kappa_0\, Q\, Q_{,x} + 2P^2 f(f_{||x} + f^2)] , \tag{2.39}$$

where the new form of the metric function $g_{uu}$ was used, specifically the new form of the coefficient $c$ from (2.34). The right hand side of the equation explicitly reads

$$R_{ux} = 2\Lambda\, e + \kappa_0\, Q(eQ + \xi) + 2\Lambda\, f\, r , \tag{2.40}$$

in which we used (2.28) and $g_{ux} = e + f\, r$. By comparing the coefficients in front of the respective powers of $r$, we obtain the following two equations

$$A_{ux} = 2\Lambda\, e + \kappa_0\, Q(eQ + \xi) , \tag{2.41}$$

$$B_{ux} = 2\Lambda\, f . \tag{2.42}$$

The second equation is not a new independent constraint, but by substituting in from the equation (2.24) and also by substituting in the expression for the covariant derivative of $f$ (2.32) we get after some trivial algebraical manipulation that this equation reduces to (2.36). This leaves us with only one constraint coming from the equation (2.41). It can be simplified by using the expression (2.32) to

$$b_{,x} = f_{,u} - f\frac{P_{,u}}{P} - \frac{1}{2}(4\Lambda + F + 2\kappa_0\, Q^2)\, e - 2\kappa_0\, Q\, \xi . \tag{2.43}$$

Alternatively, we can use the equation (2.35) to rewrite the constraint into the form

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + eP(Pf)_{,x} - 2\kappa_0\, Q\, \xi . \tag{2.44}$$

(vi) Integration of $R_{uu} = 2\Lambda\, g_{uu} + \kappa_0\, P^2 F_u^2$:

The last component of the Ricci tensor (A.29) can also be written as a power series in the $r$ coordinate. Using (2.1) and the metric functions (2.4), (2.11), and (2.14), we thus get

$$R_{uu} = A_{uu} + B_{uu}\, r + C_{uu}\, r^2\,, \tag{2.45}$$

where

$$\begin{aligned} A_{uu} &= a\left(c - \frac{1}{2}F\right) \\ &\quad + P^2\Big[-\frac{1}{2}a_{,xx} + \frac{1}{2}a_{,x}\left(f - \frac{P_{,x}}{P}\right) - \frac{1}{2}b\left(e_{,x} + e\frac{P_{,x}}{P} + \frac{P_{,u}}{P^3}\right) \\ &\qquad\quad + \left(e_{,ux} + e_{,u}\frac{P_{,x}}{P}\right) + (f_{,u} - b_{,x} - ce)e + \frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\Big]\,, \end{aligned} \tag{2.46}$$

$$\begin{aligned} B_{uu} &= b\left(c - \frac{1}{2}F - \frac{1}{2}P(Pf_{,x}\right) \\ &\quad + P^2\Bigg[\left(f_{,u} - \frac{1}{2}b_{,x}\right)_{,x} + \left(f_{,u} - \frac{1}{2}b_{,x}\right)\left(f + \frac{P_{,x}}{P}\right) \\ &\qquad\quad - c\left(e_{,x} + e\frac{P_{,x}}{P} + \frac{P_{,u}}{P^3}\right) - 2e(c_{,x} + fc)\Bigg]\,, \end{aligned} \tag{2.47}$$

$$\begin{aligned} C_{uu} &= c\,(c - F) - P^2\left[\frac{1}{2}c_{,xx} + \frac{1}{2}c_{,x}\left(3f + \frac{P_{,x}}{P}\right)\right. \\ &\qquad\qquad \left. + c\left(f_{,x} + f\frac{P_{,x}}{P} + \frac{1}{2}f^2\right)\right]\,. \end{aligned} \tag{2.48}$$

The corresponding field equation, due to (2.28), then amounts to

$$R_{uu} = 2\Lambda(a + b\,r + c\,r^2) + \kappa_0 P^2(e\,Q + \xi)^2\,. \tag{2.49}$$

Again, by comparing the coefficients in front of the respective powers of $r$, we get the ensuing three equations

$$A_{uu} = 2\Lambda\, a + \kappa_0\, P^2(e\,Q + \xi)^2\,, \tag{2.50}$$
$$B_{uu} = 2\Lambda\, b\,, \tag{2.51}$$
$$C_{uu} = 2\Lambda\, c\,. \tag{2.52}$$

We will now derive a very useful relation for the derivative of the function $c$, given by (2.34). Using (2.24) and (2.36) the first derivative, and subsequently the second derivative of $c$, can be evaluated as

$$c_{,x} = -f\,c\,, \qquad c_{,xx} = (f^2 - f_{,x})\,c\,. \tag{2.53}$$

Taking advantage of these identities, it can be easily shown that the equation (2.52), applying also (2.35), is identically satisfied. It is a little bit more involved to show that the equation (2.51) is satisfied identically as well. The field equation (2.43) can be used to express the first and the second derivatives of function $b$ with respect to $x$. Substituting this into (2.51), while also using the derivative with respect to $u$ of the constraint (2.35), the equation reduces to the already found constraint (2.27), and therefore does not contain any additional information.

Consequently, the only equation left is (2.50) which constrains the last metric function $a$. It can be simplified using (2.35) and (2.43) to

$$\begin{aligned} a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + f\frac{P_{,x}}{P}\right) \\ = -b\left(e_{,x} + e\frac{P_{,x}}{P} + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + e_{,u}\frac{P_{,x}}{P}\right) \\ - Pe^2(Pf)_{,x} + 2e\,f\frac{P_{,u}}{P} + \left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\,\xi^2 \,. \end{aligned} \tag{2.54}$$

It can be brought into a more compact form by utilizing the definition of the covariant derivative $\zeta_{||x} \equiv \zeta_{,x} + \zeta\frac{P_{,x}}{P}$, where $\zeta$ represents $a$, $f$, $e$ or $e_{,u}$. The field equation can be then written as

$$\begin{aligned} a_{||xx} - (fa)_{||x} = -b\left(e_{||x} + \frac{P_{,u}}{P^3}\right) + 2(e_{,u})_{||x} - P^2e^2f_{||x} \\ + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\,\xi^2 \,, \end{aligned} \tag{2.55}$$

where we also denoted $a_{||xx} \equiv a_{,xx} + a_{,x}\frac{P_{,x}}{P}$.

This concludes the integration of all the Einstein–Maxwell equations for the Kundt spacetimes, and we will now discuss some of its special cases.

## 2.4 Summary and discussion of the Kundt solutions

We have fully integrated the system of Einstein–Maxwell equations with a cosmological constant in 2+1 gravity for the Kundt class of spacetimes. Their complete solution lead to the metric functions

$$g_{xx} = P^{-2}(u,x)\,, \tag{2.56}$$
$$g_{ux} = e(u,x) + f(u,x)\,r\,, \tag{2.57}$$
$$g_{uu} = a(u,x) + b(u,x)\,r + c(u,x)\,r^2\,, \tag{2.58}$$

where

$$c = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\,, \qquad \text{with} \qquad F \equiv P^2f^2\,. \tag{2.59}$$

The general metric for such spacetime can thus be written in the form

$$\mathrm{d}s^2 = \frac{1}{P^2}\,\mathrm{d}x^2 + 2(e+f\,r)\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r + \left[a + b\,r + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right]\mathrm{d}u^2\,. \tag{2.60}$$

This geometry allows only aligned electromagnetic field, and the components of the electromagnetic tensor are

$$F_{rx} = 0\,, \tag{2.61}$$
$$F_{ru} = Q(u,x)\,, \tag{2.62}$$
$$F_{ux} = f(u,x)\,Q(u,x)\,r - \xi(u,x)\,. \tag{2.63}$$

The corresponding electromagnetic 2-form can be written as

$$\mathbf{F} = Q\,\mathrm{d}r \wedge \mathrm{d}u + (fQ\,r - \xi)\,\mathrm{d}u \wedge \mathrm{d}x\,. \tag{2.64}$$

Equivalently, we can find the vector potential given by the relation (1.10) as the following 1-form

$$\mathbf{A} = A_r\,\mathrm{d}r + A_x\,\mathrm{d}x\,, \tag{2.65}$$

where

$$A_r = -\int Q\,\mathrm{d}u\,, \qquad A_x = r\int fQ\,\mathrm{d}u - \int \xi\,\mathrm{d}u\,. \tag{2.66}$$

The relation $\mathbf{F} = \mathrm{d}\mathbf{A}$ is preserved thanks to the field equation (2.24). The dual electromagnetic 1-form can be expressed as

$$^{*}\mathbf{F} = P(eQ - \xi)\,\mathrm{d}u + \frac{Q}{P}\,\mathrm{d}x\,. \tag{2.67}$$

We can also obtain the scalar potential, given by the equation (1.85) which correspond to the Klein–Gordon field, and thanks to the constraint (2.27) it reads

$$\Phi = \sqrt{\frac{8\pi}{\kappa_0}}\int \frac{Q}{P}\,\mathrm{d}x\,. \tag{2.68}$$

The Newmann–Penrose scalars of the electromagnetic field amount to

$$\phi_0 = 0 \qquad \phi_1 = Q\,, \qquad \phi_2 = P(eQ + \xi)\,. \tag{2.69}$$

We can perform a null rotation with fixed $\mathbf{k}$, as in (1.42)-(1.44), and by choosing the parameter $L = -\frac{1}{\sqrt{2}}\,eP$ we get a new set of Newmann–Penrose scalars, namely

$$\phi_0' = 0\,, \qquad \phi_1' = Q\,, \qquad \phi_2' = P\xi\,. \tag{2.70}$$

We see that with respect to this new triad, the condition for the field being null is purely determined by the function $\xi$. Actually, this is related to a gauge freedom in the metric (2.60), where we can gauge away the function $e$, as we will show later. Lastly, the Einstein and Maxwell field equations imply the following five constrains for the family of seven parameters, specifically five metric functions $P$, $e$, $f$, $a$, $b$ and two functions $Q$ and $\xi$ connected to the electromagnetic field. They satisfy the field equations

$$Q_{,x} = -fQ\,, \tag{2.71}$$

$$[P(e\,Q + \xi)]_{,x} = \left(\frac{Q}{P}\right)_{,u}\,, \tag{2.72}$$

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0\,Q^2\right)\,, \tag{2.73}$$

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + eP(Pf)_{,x} - 2\kappa_0\,Q\,\xi\,, \tag{2.74}$$

$$\begin{aligned} a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + f\frac{P_{,x}}{P}\right) \\ = -b\left(e_{,x} + e\frac{P_{,x}}{P} + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + e_{,u}\frac{P_{,x}}{P}\right) \\ - Pe^2(Pf)_{,x} + 2e\,f\frac{P_{,u}}{P} + \left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\,\xi^2\,. \end{aligned} \tag{2.75}$$

The metric (2.60) is invariant with respect to two gauges. We can consider the transformation of the coordinates

$$(u,x) \to (u,x') \qquad \text{with} \qquad x = x(u,x')\,. \tag{2.76}$$

This leads to the following transformation of the differential

$$\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial x'}\,\mathrm{d}x'\,. \tag{2.77}$$

The metric in the new gauge then reads

$$\begin{aligned}\mathrm{d}s^2 = \frac{1}{P^2}\left(\frac{\partial x}{\partial x'}\right)^2 \mathrm{d}x'^2 + 2\,\frac{\partial x}{\partial x'}\left(e + \frac{1}{P^2}\frac{\partial x}{\partial u} + f\,r\right)\mathrm{d}u\,\mathrm{d}x'^2 - 2\,\mathrm{d}u\,\mathrm{d}r \\ + \left[\frac{1}{P^2}\left(\frac{\partial x}{\partial u}\right)^2 + 2e\,\frac{\partial x}{\partial u} + a + b\,r + 2\Lambda\,r^2\right]\mathrm{d}u^2\,.\end{aligned} \tag{2.78}$$

We can use this gauge to set the metric function $P$ to $P = 1$, by choosing specifically $x = \int P\,\mathrm{d}x'$. The other gauge freedom is in shifting the $r$ coordinate. We can define a new coordinate

$$r'(r,u,x) = r - r_0(u,x)\,, \tag{2.79}$$

so that the corresponding transformation of the differential is

$$\mathrm{d}r' = \mathrm{d}r - r_{0,u}\,\mathrm{d}u - r_{0,x}\,\mathrm{d}x\,. \tag{2.80}$$

This gives us the metric

$$\begin{aligned}\mathrm{d}s^2 = \frac{1}{P^2}\,\mathrm{d}x^2 + 2(e + r_0\,f + r_{0,x} + f\,r')\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r' \\ + \left[a + b\,r_0 - 2\,r_{0,u} + cr_0^2 + (b + 2c\,r_0)r' + cr'^2\right]\mathrm{d}u^2\,,\end{aligned} \tag{2.81}$$

where $c$ is given in (2.59). This gauge can be used to eliminate the function $e$ by solving the equation $e + r_0\,f + r_{0,x} = 0$. It can be also noted that this transformation does not change the metric component $g_{xx}$. Therefore, we can first do the gauge transformation (2.78) to put $P = 1$ and then set $e = 0$ by the gauge transformation (2.81). In the upcoming discussion we will extensively use this gauge to simplify the field equations.

The field equation (2.71) can be thought of as a hypersurface equation, in the sense that it does not contain any $u$ derivative. This means that if we prescribe some initial data on a hypersurface $u = \text{const.}$, then this data stays on that hypersurface during the evolution. We can therefore prescribe the function $f(u = \text{const.}, x) = f(x)$ and we will now discuss a special subclass $f = 0$.

### 2.4.1 Special subclass of the Kundt solutions

We now consider the case, where the function $f = 0$. The Definition (2.33) implies that also $F = 0$. Due to this choice the field equations significantly simplify to

$$Q_{,x} = 0\,, \tag{2.82}$$

$$[P(e\,Q + \xi)]_{,x} = \left(\frac{Q}{P}\right)_{,u}\,, \tag{2.83}$$

$$\kappa_0\, Q^2 = -2\Lambda\,, \tag{2.84}$$

$$b_{,x} = -2\kappa_0\, Q\,\xi\,, \tag{2.85}$$

$$(Pa_{,x})_{,x} = -b\left((Pe)_{,x} + \frac{P_{,u}}{P^2}\right) + 2(Pe_{,u})_{,x} + 2\left(\frac{P_{,u}}{P^2}\right)_{,u} - 2\kappa_0 P\,\xi^2\,. \tag{2.86}$$

Immediately from the equation (2.84) we have the solution

$$Q = \sqrt{-\frac{2\Lambda}{\kappa_0}}\,. \tag{2.87}$$

It is only valid for $\Lambda \le 0$, and the metric (2.60) for the solution reduces to

$$\mathrm{d}s^2 = \frac{1}{P^2}\,\mathrm{d}x^2 + 2e\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r + (a + b\,r + 2\Lambda\,r^2)\,\mathrm{d}u^2\,. \tag{2.88}$$

We can discuss four different cases based on the algebraic classification of the electromagnetic field.

(i) The case $Q = 0 = \xi$: This leads to the vanishing of all the Newmann–Penrose scalars, and so this is a case of a vacuum solution. Inevitably from (2.87) follows that $\Lambda = 0$, and in 2+1 dimensions a vacuum solution with vanishing cosmological constant can only be the Minkowski spacetime. The metric (2.88) is now in the form

$$\mathrm{d}s^2 = \frac{1}{P^2}\,\mathrm{d}x^2 + 2e\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r + (a + b\,r)\,\mathrm{d}u^2\,. \tag{2.89}$$

It is in fact the Kundt metric obtained in equation (82) in [13], see the case $\mathcal{J} = 0 = \mathcal{N}$. From equation (2.85) we get that $b$ is independent of $x$, specifically $b = b(u)$. Using the gauge $P = 1$ and $e = 0$ the last field equation (2.86) gives that $a$ is linear in the $x$ coordinate. There exist a gauge that transforms the metric (2.89) into the standard Minkowski metric in null coordinates, namely

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}r\,. \tag{2.90}$$

(ii) The case $Q = 0$ and $\xi \neq 0$: The electromagnetic field is null due to $\phi_1 = Q = 0$. The equation (2.87) again implies $\Lambda = 0$, but now the electromagnetic field is described by a nonzero component

$$\mathbf{F} = -\xi\,\mathrm{d}u \wedge \mathrm{d}x\,. \tag{2.91}$$

In the gauge $P = 1$ and $e = 0$ the field equation (2.83) reads $\xi_{,x} = 0$ and necessarily the radiative function $\xi$ is only function of $u$

$$\xi = \gamma(u)\,. \tag{2.92}$$

The same is true for the metric function $b = b(u)$, because of the field equation (2.85). The last metric function can be found using the equation (2.86) and its solution leads to

$$a = a_0(u) + a_1(u)\, x - \kappa_0\, \gamma^2 x^2\,. \tag{2.93}$$

The metric can be then written in the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}r + [a_0(u) + a_1(u)\, x - \kappa_0 \gamma^2 x^2 + b\, r]\,\mathrm{d}u^2\,. \tag{2.94}$$

(iii) The case $Q \neq 0$ and $\xi = 0$: This case complements the previous one as the electromagnetic field for $\xi = 0$ is non-null, see equation (2.70). The electromagnetic field is constant and is given as

$$\mathbf{F} = Q\,\mathrm{d}r \wedge \mathrm{d}u\,, \tag{2.95}$$

where $Q$ is determined by (2.87) and the only allowed value of the cosmological constant is $\Lambda < 0$. Equation (2.85) then limits the function $b$ to $b = b(u)$, and in the gauge $P = 1$ and $e = 0$ the remaining field equation (2.86) can be solved as

$$a = a_0(u) + a_1(u)\, x\,, \tag{2.96}$$

where the functions $a_0(u)$ and $a_1(u)$ are arbitrary functions of $u$. The metric in this gauge reads

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}r + [a_0(u) + a_1(u)\, x + b\, r + 2\Lambda r^2]\,\mathrm{d}u^2\,. \tag{2.97}$$

For $a_0 = 0 = a_1$ and $b = 0$ we get an analogous solution in 2+1 gravity to the exceptional electro-vacuum type D metric found in standard 4D gravity by Plebański and Hacyan in [21]. The metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2\Lambda r^2\,\mathrm{d}u^2\,, \tag{2.98}$$

can be transformed applying $\mathcal{U} = \frac{1}{2\Lambda u}$ and $\mathcal{V} = 2(u + \frac{1}{\Lambda r})$ to the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \frac{2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V}}{(1 - \Lambda\mathcal{U}\mathcal{V})^2}\,, \tag{2.99}$$

which is the direct product of $E^1 \times \mathrm{AdS}_2$. For the four-dimensional counterpart, see equation (7.20) in [3].

(iv) The case $Q \neq 0$ and $\xi \neq 0$: This is the most general case of the family $f = 0$ of the Kundt solutions. The electromagnetic field has its non-null component $Q = \text{const.}$ which is a consequence of (2.87), and the cosmological constant must be $\Lambda < 0$. There is now also the nontrivial null component of the electromagnetic field $\xi$, and the electromagnetic 2-form is given as

$$\mathbf{F} = Q\,\mathrm{d}r \wedge \mathrm{d}u - \xi\,\mathrm{d}u \wedge \mathrm{d}x\,. \tag{2.100}$$

The gauge $P = 1$ and $e = 0$ is still possible, even in this general case, and the field equations can be fully integrated. Equation (2.83) reduces to $\xi_{,x} = 0$ and its solution is

$$\xi = \gamma(u)\,, \tag{2.101}$$

where $\gamma(u)$ is an arbitrary function. Equation (2.85) then implies

$$b = b_0(u) - 2\kappa_0\, Q\, \gamma\, x\,. \tag{2.102}$$

Lastly, the equation (2.86) can be integrated to obtain $a$ in the form

$$a = a_0(u) + a_1(u)\, x - \kappa_0\, \gamma^2\, x^2\,, \tag{2.103}$$

where $a_0(u)$ and $a_1(u)$ are arbitrary functions of the $u$ coordinate. The metric can be then written as

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\,\mathrm{d}u\,\mathrm{d}r + [a_0(u) + a_1(u)\, x - \kappa_0\gamma^2\, x^2 - 2\kappa_0\, Q\,\gamma\, x\, r + b_0\, r + 2\Lambda r^2]\,\mathrm{d}u^2\,. \tag{2.104}$$

### 2.4.2 General case of the Kundt solutions

We will now consider the case $f \neq 0$ which implies $F \neq 0$. The metric is now in its most general form (2.60). We will again consider four different cases based on the vanishing of the Newman–Penrose scalars for the electromagnetic field (2.70).

(i) The case $Q = 0$ and $\xi = 0$: This condition implies that the electromagnetic field identically vanishes which means this solution belongs to the family of vacuum solution. Unlike the case $f = 0$ there is not any constraint on the cosmological constant, and a vacuum solution in 2+1 gravity can either be flat Minkowski spacetime, or spacetimes with constant curvature, namely the de Sitter or the anti-de Sitter spacetimes. The metric (2.60) has now the form

$$\mathrm{d}s^2 = \frac{1}{P^2}\,\mathrm{d}x^2 + 2(e + f\, r)\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\mathrm{d}r + \left[a + b\, r + \left(\Lambda + \frac{1}{4}F\right) r^2\right]\,\mathrm{d}u^2\,. \tag{2.105}$$

This is exactly the same as in equation (85) in [13] for the case $\mathcal{J} = 0 = \mathcal{N}$. The field equations also reduce to the ones derived in appendix C of [13].

(ii) The case $Q = 0$ and $\xi \neq 0$: Since the non-null component of the electromagnetic field is set to zero, the electromagnetic field is null, with the Newman–Penrose scalars $\phi_1 = 0$ and $\phi_2 = P\xi$. The electromagnetic field has now only one independent component given by

$$\mathbf{F} = -\xi\,\mathrm{d}u \wedge \mathrm{d}x\,. \tag{2.106}$$

In the gauge $P = 1$ and $e = 0$, the field equation (2.71) is trivially satisfied and from (2.72) we obtain

$$\xi = \gamma(u)\,. \tag{2.107}$$

The field equation (2.73) reads $f_{,x} = -(2\Lambda + \frac{1}{2}f^2)$ and a general solution is of the form

$$f = -2\sqrt{\Lambda}\tan(\tilde{x})\,, \qquad \text{for } \Lambda > 0\,, \tag{2.108}$$

where we have denoted

$$\tilde{x} = \sqrt{\Lambda}\big(x + x_0(u)\big)\,. \tag{2.109}$$

We see that for $\Lambda = 0$ the function $f = 0$ and we discussed this case in the previous subsection. For $\Lambda < 0$ we just need to replace the tangent function with tanh. From (2.74) we get the metric function $b$ as

$$b = 2\sqrt{\Lambda}\, x_{0,u} \tan(\tilde{x}) + b_0(u)\,, \tag{2.110}$$

Again for $\Lambda < 0$ we replace tan with tanh. The last field equation (2.75) gives $(a_{,x} - af)_{,x} = -2\kappa_0\, \xi^2$. It can be integrated, and the result is a rather complicated form of the metric function $a$, namely

$$a = \cos^2 \tilde{x} \left( -\frac{2\kappa_0\, \gamma}{\Lambda} \log(\cos\tilde{x}) + \frac{1}{\sqrt{\Lambda}} \gamma\, x \tan\tilde{x} + \frac{1}{\sqrt{\Lambda}} b_0 \tan\tilde{x} + a_0(u) \right)\,, \tag{2.111}$$

where $a_0(u)$ is an arbitrary function. For the $\Lambda < 0$ case, the identification $\cos \to \cosh$ and $\tan \to \tanh$ needs to be done.

(iii) The case $Q \neq 0$ and $\xi = 0$: The electromagnetic field in non-null with the only nonzero Newman–Penrose scalar $\phi_1 = Q$. The field has two components,

$$\mathbf{F} = Q\, \mathrm{d}r \wedge \mathrm{d}u + fQ\, r\, \mathrm{d}u \wedge \mathrm{d}x\,. \tag{2.112}$$

Further simplification comes from utilizing the gauge $P = 1$ and $e = 0$. The field equation (2.72) gives $Q_{,u} = 0$, and we find that

$$Q = Q(x)\,. \tag{2.113}$$

The equation (2.71) implies $(\log Q)_{,x} = f$ and since the function $Q$ is just a function of the variable $x$, this means that

$$f = f(x)\,. \tag{2.114}$$

Due to this, we can now integrate the equation (2.74) which gives

$$b = b(u)\,. \tag{2.115}$$

The last Einstein's field equation takes the form

$$a_{,x} - af = a_1(u)\,, \tag{2.116}$$

where $a_1(u)$ is an arbitrary function of the variable $u$. Even though we can write a solution to this system of equations in a closed form, we will not do it, as it is rather complicated and not clear. The metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + 2f(x)r\, \mathrm{d}u\, \mathrm{d}x - 2\, \mathrm{d}u\, \mathrm{d}r + [a + b(u)r + c(x)r^2]\, \mathrm{d}u^2\,, \tag{2.117}$$

where the metric functions satisfy the field equations

$$Q_{,x} = -fQ\,, \quad f_{,x} = -\left(2\Lambda + \frac{1}{2}f^2 + \kappa_0\, Q^2\right)\,, \quad a_{,x} - af = a_1(u)\,. \tag{2.118}$$

(iv) The case $Q \neq 0$ and $\xi \neq 0$: The general case of the electromagnetic field containing both non-null component and a radiative component is too complicated to be solved explicitly. Nevertheless, the gauge $P = 1$ and $e = 0$ is still applicable, so the general metric can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + 2f\,r\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r + \left[a + b\,r + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right]\mathrm{d}u^2\,. \tag{2.119}$$

The electromagnetic field is now described by

$$\mathbf{F} = Q\,\mathrm{d}r \wedge \mathrm{d}u + (fQ\,r - \xi)\,\mathrm{d}u \wedge \mathrm{d}x\,, \tag{2.120}$$

and the field equations are as follows

$$Q_{,x} = -fQ\,, \tag{2.121}$$

$$\xi_{,x} = Q_{,u}\,, \tag{2.122}$$

$$f_{,x} = -\left(2\Lambda + \frac{1}{2}f^2 + \kappa_0 Q^2\right)\,, \tag{2.123}$$

$$b_{,x} = f_{,u} - 2\kappa_0 Q\xi\,, \tag{2.124}$$

$$a_{,xx} - (af)_{,x} = -2\kappa_0\xi^2\,. \tag{2.125}$$

They are too complicated to obtain a general solution, but if we treat the function $f(u, x)$ as an initial data and prescribe this function on some hypersurface $u =$ const., we are then able to solve for $Q$ from equation (2.121). This solution must also satisfy the equation (2.123). From (2.122) we get the remaining electromagnetic component $\xi$. The last two equations (2.124) and (2.125) give us respectively the functions $b$ and $a$.

# 3. All Robinson–Trautman spacetimes with an aligned electromagnetic field in 2+1 gravity

Having investigated the nonexpanding Kundt family of spacetimes in the previous chapter, we now solve the system of equations (1.66) for the Robinson–Trautman family which is defined as twist-free and shear-free class of spacetimes. This means that we now allow a nonvanishing expansion scalar

$$\theta \neq 0\,. \tag{3.1}$$

Before we investigate the general case, in this chapter we consider an electromagnetic field which is aligned with the privileged geometrical direction $\mathbf{k} = \partial_r$. From the discussion of the Newman–Penrose scalars (1.38) we noted that this implies $\phi_0 = 0$, and from the explicit expression (1.30), or (1.34) for this scalar, we get the condition

$$F_{rx} = 0 \;\Leftrightarrow\; F_r = 0\,. \tag{3.2}$$

This significantly simplifies the field equations. In fact, it will allow us to completely integrate the coupled system of Einstein–Maxwell equations.

## 3.1 Integration of the Einstein field equations, step one

We recall that the field equations (1.66) have the following form

$$R_{ab} = 2\Lambda\, g_{ab} + \kappa_0\, G^2 F_a\, F_b\,. \tag{3.3}$$

Because of the condition (3.2), we are able to immediately integrate the first two field equations, namely the equation $R_{rr} = \kappa_0\, G^2 F_r^2$ and $R_{rx} = \kappa_0\, G^2 F_r\, F_x$.

(i) Integration of $R_{rr} = \kappa_0\, G^2 F_r^2$:

The Ricci component (A.24) simply reads $R_{rr} = -(\theta_{,r} + \theta^2)$, by taking $F_r = 0$ the field equation amounts to

$$\theta_{,r} + \theta^2 = 0\,. \tag{3.4}$$

It is a separable differential equation of the first order and its general solution can be written as

$$\theta = \frac{1}{r + r_0(u,x)}\,, \tag{3.5}$$

where the function $r_0(u,x)$ is arbitrary. We note that the metric (1.3) is actually invariant with respect to the gauge transformation $r \to r - r_0(u,x)$, but also the condition (3.2) stays invariant with respect to this transformation. This can be easily seen by substituting $\mathrm{d}r = \mathrm{d}r - r_{0,u}\,\mathrm{d}u - r_{0,x}\,\mathrm{d}x$

into the metric (1.3), and into the expression for the electromagnetic 2-form (1.9). This means that without loss of generality we can set $r_0(u,x) = 0$ and the expansion scalar is now simply given as

$$\theta = \frac{1}{r}\,. \tag{3.6}$$

From the equation (1.8) we get the relation for the metric function $G(r,u,x)$, and we obtain

$$G(r,u,x) = \frac{P(u,x)}{r}\,, \tag{3.7}$$

where $P(u,x)$ is an arbitrary function independent of $r$. The metric function $g_{xx}$ can be expressed from (1.7) accordingly as

$$g_{xx} = \frac{r^2}{P^2(u,x)}\,. \tag{3.8}$$

The contravariant form of the metric is related by $g^{xx} = \frac{1}{g_{xx}}$, and we get explicitly

$$g^{xx} = \frac{P^2}{r^2}\,. \tag{3.9}$$

(ii) Integration of $R_{rx} = \kappa_0\, G^2 F_r\, F_x$:

Applying (3.6) in the Ricci component (A.25) yields $R_{rx} = -\frac{1}{2}(g_{ux,rr} - g_{ux,r}\, r^{-1})$. The right hand side of the Einstein equation vanishes due to (3.2), and the respective field equation is

$$g_{ux,rr} - \frac{1}{r}\, g_{ux,r} = 0\,. \tag{3.10}$$

This is again a separable differential equation and its solution leads to the expression

$$g_{ux} = e(u,x)\, r^2 + f(u,x)\,, \tag{3.11}$$

where $e(u,x)$ and $f(u,x)$ are arbitrary $r$-independent functions. The contravariant component can be calculated from relation (1.4), and is equal to

$$g^{rx} = P^2\Big(e + f\, r^{-2}\Big)\,. \tag{3.12}$$

We will now pause our investigation of Einstein's equations and instead we switch to the Maxwell equations, as we are in a position to completely integrate them. We will see later that this greatly helps us in solving the remaining field equations.

## 3.2 Integration of the Maxwell equations

Setting $F_{rx} = 0 = F_r$ in the Maxwell equations (1.72), while also using the obtained solution (3.7) and the form of the functions $F_a$ from (1.20)-(1.22), we come to the following three equations for our electromagnetic field

$$(r\,F_{ru})_{,r} = 0\,, \tag{3.13}$$

$$\left(\frac{1}{r}(g_{ux}\,F_{ru} - F_{ux})\right)_{,r} = 0\,, \tag{3.14}$$

$$\Big(P(g_{ux}\,F_{ru} - F_{ux})\Big)_{,x} = r^2\left(\frac{F_{ru}}{P}\right)_{,u}\,. \tag{3.15}$$

The last equation coming from $F_{[ab,c]} = 0$ is exactly the same as in the Kundt case, that is

$$F_{ux,r} + F_{ru,x} = 0\,. \tag{3.16}$$

The first of the equations can be easily integrated, and the general solution to (3.13) can be written as

$$F_{ru} = \frac{Q(u,x)}{r}\,, \tag{3.17}$$

where $Q(u,x)$ is arbitrary. This now allows us to solve for the remaining component $F_{ux}$ from the last Maxwell equation (3.16). The general solution is

$$F_{ux} = -Q_{,x}\log|r| - \xi(u,x)\,, \tag{3.18}$$

where $\xi(u,x)$ is an arbitrary function independent of $r$. We now have all of the independent components of the electromagnetic field and the remaining equations will lead to constrains on the functions $Q$ and $\xi$. Plugging the found solutions (3.17) and (3.18) into (3.14) and applying (3.11) as well, the field equation reads

$$\frac{fQ}{r^2} + Q_{,x}\frac{\log|r|}{r} + (\xi - Q_{,x})\frac{1}{r} = 0\,. \tag{3.19}$$

It can be satisfied if and only if

$$fQ = 0\,, \tag{3.20}$$

$$Q_{,x} = 0\,, \tag{3.21}$$

$$\xi = Q_{,x}\,. \tag{3.22}$$

The last two constrains imply that $\xi = 0$, and it follows that $F_{ux}$ is identically equal to zero, or explicitly

$$F_{ux} = 0\,. \tag{3.23}$$

Further restriction comes from (3.21) and we see that the function $Q$ is no longer dependent on $x$

$$Q = Q(u)\,, \qquad \text{or} \qquad F_{ru} = \frac{Q(u)}{r}\,. \tag{3.24}$$

Putting (3.24) and (3.23) in the equation (3.15), together with (3.11), gives the constraint

$$Q(Pe)_{,x} = \left(\frac{Q}{P}\right)_{,u}\,. \tag{3.25}$$

The functions $F_a$ given by (1.20)-(1.22) can now be expressed as

$$F_r = 0\,, \qquad F_x = \frac{Q}{P^2}\,r\,, \qquad F_u = eQ\,r\,. \tag{3.26}$$

The Newman–Penrose scalars given by (1.34) for this solution are thus

$$\phi_0 = 0\,, \qquad \phi_1 = \frac{Q}{r}\,, \qquad \phi_2 = ePQ\,. \tag{3.27}$$

Up to now we have not satisfied the Maxwell equation (3.20). We have two possibilities how to solve this equation, either we take $Q = 0$, or $f = 0$. The first option ($Q = 0$) leads to the vanishing of the electromagnetic tensor and therefore represents a vacuum case. This types of metrics were investigated in [13] and surprisingly the function $f$ stays without any restrain. However, in our work we are interested in spacetimes containing an electromagnetic field, hence we choose the function $f$ to be identically zero, that is

$$f = 0\,, \qquad \text{and} \qquad g_{ux} = e(u,x)\,r^2\,. \tag{3.28}$$

This considerably simplifies the remaining Einstein's field equations, to which we now return.

## 3.3 Integration of the Einstein field equations, step two

Having integrated the Maxwell equations, determining the electromagnetic field up to a one arbitrary function $Q(u)$, and simplifying the metric component $g_{ux}$, we can continue with the prevailing four Einstein's equations.

(iii) Integration of $R_{ru} = -2\Lambda + \kappa_0\, G^2 F_r\, F_u$:

The component of the Ricci tensor (A.26) reduces to $R_{ru} = -\frac{1}{2r}(r\, g_{uu,r})_{,r} + \frac{1}{2r}c + 2P^2e^2$, after applying (3.6) and (3.28), where we also denoted

$$c \equiv 2P^2\left(e_{||x} + \frac{P_{,u}}{P^3}\right) \Leftrightarrow c = 2\left(P(Pe)_{,x} + \frac{P_{,u}}{P}\right)\,, \tag{3.29}$$

and we used the notation for the covariant derivative $e_{||x} = e_{,x} + e\frac{P_{,x}}{P} = \frac{(Pe)_{,x}}{P}$. The right hand side of the equation is trivial, thanks to (3.26), and the corresponding field equation amounts to

$$\frac{1}{r}(r\, g_{uu,r})_{,r} = 4\Lambda + 4P^2e^2 + \frac{c}{r}\,. \tag{3.30}$$

After a straightforward integration we get the result

$$g_{uu} = -a(u,x) - b(u,x)\log|r| + c\, r + (\Lambda + P^2e^2)\, r^2\,, \tag{3.31}$$

where $a(u,x)$ and $b(u,x)$ are arbitrary functions depending only on the coordinates $u$ and $x$.

(iv) Integration of $R_{xx} = 2\Lambda\, g_{xx} + \kappa_0\, G^2 F_x^2$:

Using the acquired solutions (3.6), (3.28) and (3.31) the Ricci component (A.27) simply reads $R_{xx} = 2\Lambda\, P^{-2}r^2 - P^{-2}b$. Putting the expression (3.8) and (3.26) into the right hand side, and after an easy manipulation we get the equation

$$b = -\kappa_0\, Q^2\,, \tag{3.32}$$

since $Q$ is only a function of $u$, this implies that also $b = b(u)$.

(v) Integration of $R_{ux} = 2\Lambda\, g_{ux} + \kappa_0\, G^2 F_u\, F_x$:

Plugging the relations (3.6), (3.28) and (3.31), where we also use the equation (3.32), into (A.28) we get the following component of the Ricci tensor, $R_{ux} = 2\Lambda\, e\, r^2 + \kappa_0 e Q^2 - \frac{1}{2r} a_{,x}$. Together with (3.8) and (3.26), the field equation gives only one constraint, namely

$$a_{,x} = 0\,. \tag{3.33}$$

This means that the function $a$ is independent of the coordinate $x$ as well, and the metric function $g_{uu}$ can now be expressed, using also (3.32) as

$$g_{uu} = -a(u) + \kappa_0 Q^2(u) \log|r| + c\, r + (\Lambda + P^2 e^2)\, r^2\,. \tag{3.34}$$

(vi) Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0 G^2 F_u^2$:

The Ricci component (A.29) can be written, using (3.6), (3.28), (3.34) and (3.8), in the form

$$\begin{aligned} R_{uu} = \mathcal{A}_{uu} + \frac{1}{2}\left[a_{,u} - \left(a - \frac{1}{2}b\right)c - \Delta c\right]\frac{1}{r} + \kappa_0\, Q\left(\frac{1}{2}cQ - Q_{,u}\right)\frac{\log|r|}{r} \\ + \kappa_0\, 2\Lambda Q^2 \log|r| + 2\Lambda c\, r + 2\Lambda(\Lambda + P^2 e^2) r^2\,, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \mathcal{A}_{uu} = -2\Lambda a + \kappa_0 P^2 e^2 Q^2 + \frac{1}{4}c^2 + \frac{1}{2}P^2 e c_{,x} - \frac{1}{2}c_{,u} - \frac{1}{2}\Delta(P^2 e^2) \\ + P(P e_{,u})_{,x} - 2\frac{P_{,u}^2}{P^2} + \frac{P_{,uu}}{P}\,. \end{aligned} \tag{3.36}$$

In these expressions we denoted by $\Delta$ the covariant Laplace operator on the transverse spatial one-dimensional subspace, given by the metric $h_{xx}$, defined as

$$g_{xx} \equiv r^2 h_{xx}\,, \qquad \text{or} \qquad g^{xx} \equiv \frac{h^{xx}}{r^2}\,, \tag{3.37}$$

and the explicit expression for the Laplace operator is

$$\Delta c \equiv h^{xx} c_{||xx} = P(P c_{,x})_{,x}\,. \tag{3.38}$$

In the evaluation we used the fact that $h^{xx} = P^2$. Substituting the explicit expression for $c$ from (3.29) into (3.36), it further simplifies to the form

$$\mathcal{A}_{uu} = -2\Lambda a + \kappa_0 P^2 e^2 Q^2\,. \tag{3.39}$$

Applying (3.34) and (3.26) the Einstein field equation reads

$$R_{uu} = 2\Lambda[-a + \kappa_0 Q^2 \log|r| + c\, r + (\Lambda + P^2 e^2) r^2] + \kappa_0 P^2 e^2 Q^2\,. \tag{3.40}$$

Rearranging this equation we get

$$\frac{1}{2}\left[a_{,u} - \left(a - \frac{1}{2}b\right)c - \Delta c\right]\frac{1}{r} + \kappa_0\, Q\left(\frac{1}{2}cQ - Q_{,u}\right)\frac{\log|r|}{r} = 0\,. \tag{3.41}$$

To satisfy this equation we must impose two constraints, namely

$$Q_{,u} = \frac{1}{2} c\, Q\,, \tag{3.42}$$

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c + \Delta c\,. \tag{3.43}$$

The first one, is nothing but a different form of the Maxwell equation (3.25), which can be seen by substituting (3.29) into (3.42). Moreover, the equation (3.42) implies $c = 2\frac{Q_{,u}}{Q}$, and since the right hand side is a function of only $u$, this means that the function $c$ is independent of $x$, and so

$$c(u) = 2(\log Q)_{,u}\,. \tag{3.44}$$

From this it follows that the covariant Laplace operator on $c$ is identically equal to zero, $\Delta c = 0$. This simplifies the equation (3.43) which now has the form

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c\,. \tag{3.45}$$

Using (3.44) the general solution to this equation can be written as

$$a = Q^2(\kappa_0 \log |Q| - \mu)\,, \tag{3.46}$$

where $\mu$ is an arbitrary constant.

We have thus integrated the full system of Einstein–Maxwell equations, and we will continue with a discussion of some special cases.

## 3.4 Summary and discussion of the aligned Robinson–Trautman solutions

Solving the coupled system of Einstein–Maxwell equations for the Robinson–Trautman spacetimes with an aligned electromagnetic field leads us to the following metric functions

$$g_{xx} = \frac{r^2}{P^2(u,x)}\,, \tag{3.47}$$

$$g_{ux} = e(u,x)\, r^2\,, \tag{3.48}$$

$$\begin{aligned} g_{uu} = \mu\, Q^2(u) - \kappa_0\, Q^2(u) \log\left|\frac{Q(u)}{r}\right| + 2\Big(\log Q(u)\Big)_{,u} r \\ + \Big(\Lambda + P^2(u,x)\, e^2(u,x)\Big) r^2\,. \end{aligned} \tag{3.49}$$

The general metric for such spacetimes can thus be written in the form

$$\begin{aligned} \mathrm{d}s^2 =& \frac{r^2}{P^2}(\mathrm{d}x + P^2 e\, \mathrm{d}u)^2 - 2\, \mathrm{d}u\, \mathrm{d}r \\ &+ \left(\mu\, Q^2(u) - \kappa_0\, Q^2(u) \log\left|\frac{Q(u)}{r}\right| + 2\Big(\log Q(u)\Big)_{,u} r + \Lambda r^2\right) \mathrm{d}u^2\,. \end{aligned} \tag{3.50}$$

The aligned electromagnetic field is given by these components

$$F_{rx} = 0\,, \tag{3.51}$$

$$F_{ru} = \frac{Q(u)}{r}\,, \tag{3.52}$$

$$F_{ux} = 0\,. \tag{3.53}$$

The electromagnetic 2-form can then be written simply as

$$\mathbf{F} = \frac{Q}{r}\,\mathrm{d}r \wedge \mathrm{d}u\,. \tag{3.54}$$

This corresponds to the vector potential

$$\mathbf{A} = Q \log\left|\frac{r}{r_0}\right| \mathrm{d}u\,. \tag{3.55}$$

Consequently, this leads to the ensuing dual electromagnetic 1-form

$$^{*}\mathbf{F} = \frac{Q}{P}\,\mathrm{d}x + Pe\,Q\,\mathrm{d}u\,. \tag{3.56}$$

According to the relation (1.85), the Klein–Gordon field corresponding to this dual is equal to

$$\Phi = \sqrt{\frac{8\pi}{\kappa_0}} \int \frac{Q}{P}\,\mathrm{d}x\,, \tag{3.57}$$

thanks to the field equation (3.25).

The metric (3.50) has four parameters, namely the constant $\mu$, the metric functions $P(u,x)$, $e(u,x)$ and the electromagnetic field parameter $Q(u)$. They must satisfy the field equation

$$Q(Pe)_{,x} = \left(\frac{Q}{P}\right)_{,u}\,. \tag{3.58}$$

The Newman–Penrose scalars corresponding to this solution are

$$\phi_0 = 0\,, \qquad \phi_1 = \frac{Q}{r}\,, \qquad \phi_2 = Pe\,Q\,. \tag{3.59}$$

Surprisingly, the solution does not allow a purely null electromagnetic field. The null field is given by $\phi_1 = 0$. This can be only satisfied by setting $Q = 0$, but this also leads to $\phi_2 = 0$, and we see that this is the case for a vacuum solution. However, there exist a solely non-null solution which can be achieved by putting the metric function $e = 0$. Actually, there is a deeper relation, namely that all aligned electromagnetic fields in the Robinson–Trautman geometry are non-null. This can be seen by performing a null rotation (1.41) with the parameter $L = -\frac{1}{\sqrt{2}}Pe\,r$. The corresponding transformation of the Newman–Penrose scalars (1.42)-(1.44) is $\phi_0' = 0$, $\phi_1' = \frac{Q}{r}$ and $\phi_2' = 0$. In such a null triad, the only nonvanishing scalar is the non-null component $\phi_1'$. This is also connected to a gauge transformation of the metric. During its derivation we used the fact that the metric is invariant with respect to the gauge $r \to r - r_0(u,x)$. As in the Kundt case we can perform the gauge transformation $(u,x)\to(u,x')$, where $x=x(u,x')$ which implies

$$\mathrm{d}x=\frac{\partial x}{\partial u}\,\mathrm{d}u+\frac{\partial x}{\partial x'}\,\mathrm{d}x'\,. \tag{3.60}$$

Consequently, the transformed metric takes the form

$$\begin{aligned}\mathrm{d}s^2=&\frac{r^2}{P^2}\left(\frac{\partial x}{\partial x'}\,\mathrm{d}x'+\left(\frac{\partial x}{\partial u}+P^2e\right)\mathrm{d}u\right)^2-2\,\mathrm{d}u\,\mathrm{d}r\\&+\left(\mu\,Q^2(u)-\kappa_0\,Q^2(u)\log\left|\frac{Q(u)}{r}\right|+2\Big(\log Q(u)\Big)_{,u}r+\Lambda r^2\right)\mathrm{d}u^2\,.\end{aligned} \tag{3.61}$$

By putting $\frac{\partial x}{\partial u}=-P^2e$ we can gauge away the function $e$, which is equivalent to taking $e=0$. In this gauge the field equation (3.58) implies that the metric component $P(u,x)$ is factorized as

$$P(u,x)=\frac{Q(u)}{\gamma(x)}\,. \tag{3.62}$$

Rescaling the spatial coordinate as $x'(x)=\int_0^x\gamma(t)\mathrm{d}t$, the general metric (3.50) for an aligned null electromagnetic field takes the form

$$\begin{aligned}\mathrm{d}s^2=&\frac{r^2}{Q^2}\,\mathrm{d}x'^2-2\,\mathrm{d}u\,\mathrm{d}r\\&+\left(\mu\,Q^2(u)-\kappa_0\,Q^2(u)\log\left|\frac{Q(u)}{r}\right|+2\Big(\log Q(u)\Big)_{,u}r+\Lambda r^2\right)\mathrm{d}u^2\,.\end{aligned} \tag{3.63}$$

Another simple solution can be obtained by putting $P=1$. The field equation (3.58) then implies

$$e_{,x}=\frac{Q_{,u}}{Q}\,. \tag{3.64}$$

In this case $e\neq0$, and the solution to the equation is

$$e=x\,\frac{Q_{,u}}{Q}+e_0(u)\,, \tag{3.65}$$

where $e_0(u)$ is arbitrary. This represents a simple example of a solution containing both the Gauss-like component and the null component of the electromagnetic field. The metric for such a solution can be written as

$$\begin{aligned}\mathrm{d}s^2=&r^2\left[\mathrm{d}x+\left(x\frac{Q_{,u}}{Q}+e_0\right)\mathrm{d}u\right]^2-2\,\mathrm{d}u\,\mathrm{d}r\\&+\left(\mu\,Q^2(u)-\kappa_0\,Q^2(u)\log\left|\frac{Q(u)}{r}\right|+2\Big(\log Q(u)\Big)_{,u}r+\Lambda r^2\right)\mathrm{d}u^2\,.\end{aligned} \tag{3.66}$$

The most interesting solution arises if we impose the conditions

$$\left(\frac{Q}{P}\right)_{,u}=0\qquad\text{and}\qquad(Pe)_{,x}=0\,. \tag{3.67}$$

This choice obviously satisfies the only remaining field equation (3.58), and the solution to these two constrains leads to the factorization of the metric functions, namely

$$P = Q(u)\beta(x)\,, \qquad e = \frac{\alpha(u)}{Q(u)\beta(x)}\,. \tag{3.68}$$

The metric now reads

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{r^2}{Q^2}\left(\frac{\mathrm{d}x}{\beta} + \alpha Q\,\mathrm{d}u\right)^2 - 2\,\mathrm{d}u\,\mathrm{d}r \\ & + \left(\mu\, Q^2(u) - \kappa_0\, Q^2(u) \log\left|\frac{Q(u)}{r}\right| + \Lambda r^2\right)\mathrm{d}u^2\,. \end{aligned} \tag{3.69}$$

We can consider a special case of the metric, for which $Q = \text{const.}$, and we define a compact coordinate $\varphi$ as

$$\varphi = \frac{1}{Q}\int_0^x \frac{\mathrm{d}t}{\beta(t)}\,. \tag{3.70}$$

With this choice the metric reduces to

$$\mathrm{d}s^2 = r^2(\mathrm{d}\varphi + \alpha Q\,\mathrm{d}u)^2 - 2\,\mathrm{d}u\,\mathrm{d}r + \left(m - \kappa_0\, Q^2 \log\left|\frac{Q}{r}\right| + \Lambda r^2\right)\mathrm{d}u^2\,,$$

where we defined the new constant $m \equiv \mu\, Q^2$. For $\alpha = 0$ we can perform the transformation

$$\mathrm{d}u = \mathrm{d}t + \left(m - \kappa_0\, Q^2 \log\left|\frac{Q}{r}\right| + \Lambda r^2\right)^{-1}\mathrm{d}r\,, \tag{3.71}$$

for which we obtain the well known metric of a charged black hole in 2+1 gravity as

$$\begin{aligned} \mathrm{d}s^2 = {} & -\left(-m - \kappa_0\, Q^2 \log\left|\frac{Q}{r}\right| + \Lambda r^2\right)\mathrm{d}t^2 \\ & + \frac{\mathrm{d}r^2}{\left(-m - \kappa_0\, Q^2 \log\left|\frac{Q}{r}\right| + \Lambda r^2\right)} + r^2\mathrm{d}\varphi^2\,. \end{aligned} \tag{3.72}$$

This is the form of an electrostatic metric with a cosmological constant investigated by Peldan in 1992 [22]. For $\Lambda = 0$ this reduces to the solution obtained by Gott, Simon and Alpert in [23], or by Deser and Mazur in [24]. In different coordinates the electrostatic and magnetostatic solutions are also contained in [25]. For review of these electrostatic solutions see section 11.2 in [12].

# 4. All Robinson–Trautman spacetimes with a general electromagnetic field in 2+1 gravity

Having investigated the Robinson–Trautman spacetimes with an aligned electromagnetic field, we would now like to proceed to the most general case, specifically we now consider a completely general electromagnetic field. During this chapter we will assume that $F_{rx} \neq 0 \neq F_r$, since the case $F_{rx} = 0 = F_r$ corresponds to an aligned electromagnetic field, and we fully explored this alternative in the previous chapter. Unfortunately, we will see that the general case of nonaligned electromagnetic field is too complex, and we will only achieve a separation of variables of the metric functions and the electromagnetic components.

## 4.1 Integration of the Einstein field equations, step one

We recall that we are trying to solve the coupled system of Einstein–Maxwell equations. They are given by (1.66), and have the form

$$R_{ab} = 2\Lambda\, g_{ab} + \kappa_0\, G^2 F_a F_b\,. \tag{4.1}$$

The functions $F_a$ are given in (1.20)-(1.22), and we start by solving the first three field equations.

(i) Integration of $R_{rr} = \kappa_0\, G^2 F_r^2$:

Taking the exact form of the component (A.24) of the Ricci tensor, the corresponding field equation reads

$$-(\theta_{,r} + \theta^2) = \kappa_0\, G^2 F_r^2\,. \tag{4.2}$$

Since the function $F_r$ is given by (1.20), this equation determines the component $F_{rx}$ of the electromagnetic tensor. Using also (1.7) we can explicitly express this component as

$$\kappa_0\, F_{rx}^2 = -g_{xx}(\theta_{,r} + \theta^2)\,, \tag{4.3}$$

Due to the relation (1.6), the right hand side of the equation is fully determined by the metric function $g_{xx}$. Necessarily, $g_{xx}$ must be a function dependent on the coordinate $r$, otherwise $\theta = 0$. Also, the case $\theta = \frac{1}{r+r_0(u,x)}$ leads to an aligned electromagnetic field, as was seen in the previous chapter, see equation (3.5). For future simplification we will denote by $\alpha$ the following expression

$$\alpha \equiv -g_{xx}(\theta_{,r} + \theta^2)\,. \tag{4.4}$$

With this definition, the field equation (4.3) reduces to

$$\kappa_0\, F_r^2 = \alpha = \kappa_0\, F_{rx}^2\,. \tag{4.5}$$

(ii) Integration of $R_{rx} = \kappa_0\, G^2 F_r\, F_x$:

Using the field equation (4.2), the Ricci component (A.25) can be expressed as $R_{ux} = \frac{1}{2}(\theta\, g_{ux,r} - g_{ux,rr}) - \kappa_0\, g_{ux} G^2 F_r^2$, and the Einstein equation gives

$$\frac{1}{2}(\theta\, g_{ux,r} - g_{ux,rr}) = \kappa_0\, G^2 F_r (F_x + g_{ux}\, F_r)\,. \tag{4.6}$$

Substituting for $F_r$ and $F_x$ from (1.20) and (1.22), we come to the following equation

$$\kappa_0\, F_{rx}\, F_{ru} = \frac{1}{2}(\theta\, g_{ux,r} - g_{ux,rr})\,. \tag{4.7}$$

The system of equations is now beginning to show a specific hierarchy. If we consider that the electromagnetic component $F_{rx}$ is due to (4.3) fully determined by the metric function $g_{xx}$, then from the field equation (4.7) we can express the component $F_{ru}$ of the electromagnetic field, assuming that the metric function $g_{ux}$ is given. As before, we will establish the shorthand $\beta$, defined by

$$\beta \equiv \frac{1}{2} g_{xx} (\theta\, g_{ux,r} - g_{ux,rr})\,. \tag{4.8}$$

This notation helps us to rewrite the equation (4.6) in a compact form

$$\kappa_0\, F_r F_x = \beta - g_{ux}\, \alpha\,. \tag{4.9}$$

(iii) Integration of $R_{ru} = -2\Lambda + \kappa_0\, G^2 F_r F_u$:

Utilizing the explicit form of the Ricci component (A.26), the corresponding field equation can then be written as

$$\begin{aligned} -\frac{1}{2} g_{uu,rr} + \frac{1}{2} g^{rx} g_{ux,rr} + \frac{1}{2} g^{xx}\Big(g_{ux,r||x} + (g_{ux,r})^2\Big) \\ -\Theta_{,u} - \frac{1}{2}\Theta(g^{xx} g_{xx,u} + g^{rx} g_{ux,r} + g_{uu,r}) = -2\Lambda + \frac{\kappa_0}{g_{xx}} F_r F_u\,. \end{aligned} \tag{4.10}$$

Expressing the functions $F_r$ and $F_u$ from (1.20) and (1.21), we can use the previous Einstein equations (4.3) and (4.7) to find that

$$\begin{aligned} \kappa_0\, G^2 F_r\, F_u &= \kappa_0 G^2 F_{rx} (g_{ux} F_{ru} - F_{ux} - g_{uu} F_{rx}) \\ &= -\kappa_0 G^2 F_{ux} F_{rx} + \frac{1}{2} g^{rx} (\theta g_{ux,r} - g_{ux,rr}) + g_{uu}(\theta_{,r} + \theta^2)\,. \end{aligned} \tag{4.11}$$

Substituting this into the right hand side of the field equation it yields

$$\begin{aligned} \kappa_0\, F_{rx}\, F_{ux} =& \frac{1}{2} g_{xx}\Big(g_{uu,rr} + \Theta\, g_{uu,r} + 2(\Theta_{,r} + \Theta^2) g_{uu} - 4\Lambda\Big) \\ & + g_{ux}(\Theta\, g_{ux,r} - g_{ux,rr}) - \frac{1}{2}\Big(g_{ux,r||x} + (g_{ux,r})^2\Big) \\ & + \frac{1}{2}\Theta\, g_{xx,u} + g_{xx}\Theta_{,u}\,. \end{aligned} \tag{4.12}$$

We see that this equation determines the electromagnetic component $F_{ux}$. Indeed, if we are given the metric function $g_{xx}$, from equation (4.3) we obtain $F_{rx}$. Prescribing also the metric components $g_{ux}$ and $g_{uu}$, we can then solve (4.12) for the component $F_{ux}$. We again define a convenient shorthand $\gamma$ by

$$\gamma \equiv \frac{1}{2}\Big(g_{xx}(4\Lambda - g_{uu,rr}) + g_{ux}g_{ux,rr} + g_{ux,r||x} + (g_{ux,r})^2 \\ - 2g_{xx}\Theta_{,u} - \Theta(g_{xx,u} + g_{ux}g_{ux,r} + g_{xx}g_{uu,r})\Big)\,. \tag{4.13}$$

The definition of this notation is engineered to exactly fit the field equation (4.11), namely if we take the contravariant components of the metric (1.4) and put them into the field equation, we will find that

$$\kappa_0\, F_r F_u = \gamma\,. \tag{4.14}$$

The first three Einstein equations thus gave us the relations between the metric components and the components of the electromagnetic tensor. In fact we were able to completely express the electromagnetic tensor in terms of the metric functions $g_{xx}$, $g_{ux}$ and $g_{uu}$. As the next step we will now look at the Maxwell equations.

## 4.2 Integration of the Maxwell equations

In the previous section we obtained a very important result, namely we were able to separate the electromagnetic field from the gravitational field. We showed the explicit dependence of the electromagnetic field on the independent metric functions. This means that we should be able to use the Maxwell equations to find relations for the metric components. For completeness, or just to appreciate how complicated the set of Maxwell equations really is, we present their explicit form by substituting (1.20)-(1.22) into (1.72) which gives

$$g_{xx}F_{ru,r} - g_{ux}F_{rx,r} - F_{rx,x} + \theta(g_{ux}F_{rx} + g_{xx}F_{ru}) \\ -g_{ux,r}F_{rx} + \frac{1}{2}\frac{g_{xx,x}}{g_{xx}}F_{rx} = 0, \tag{4.15}$$

$$F_{ux,r} + F_{rx,u} + g_{uu}F_{rx,r} - g_{ux}F_{ru,r} + \theta(g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}) \\ +g_{uu,r}F_{rx} - g_{ux,r}F_{ru} + \frac{1}{2}\frac{g_{xx,u}}{g_{xx}}F_{rx} = 0, \tag{4.16}$$

$$g_{ux}F_{ru,x} + g_{ux}F_{rx,u} - g_{xx}F_{ru,u} - F_{ux,x} - g_{uu}F_{rx,x} \\ +g_{ux,x}F_{ru} + g_{ux,u}F_{rx} - g_{uu,x}F_{rx} - \frac{1}{2}\frac{g_{xx,u}}{g_{xx}}\Big(g_{xx}F_{ru} + g_{ux}F_{rx}\Big) \\ -\frac{1}{2}\frac{g_{xx,x}}{g_{xx}}\Big(g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}\Big) = 0. \tag{4.17}$$

We could now use the solutions (4.3), (4.7) and (4.12) to obtain a complicated set of equations. Instead, we will use the notation established before, where we expressed the first three Einstein field equations as

$$\kappa_0\, F_r^2 = \alpha, \tag{4.18}$$

$$\kappa_0\, F_r\, F_x = \beta - g_{ux}\,\alpha, \tag{4.19}$$

$$\kappa_0\, F_r\, F_u = \gamma. \tag{4.20}$$

These equations can be used to derive a nice relation for the derivatives of the functions $F_a$, where the partial derivatives with respect to $r, u, x$ (which we here denote by $a$) can be shown to satisfy

$$F_{r,a} = \frac{1}{2\kappa_0 F_r}\,\alpha_{,a}\,, \tag{4.21}$$

$$F_{x,a} = \frac{1}{\kappa_0\, F_r}\left((\beta - \alpha\, g_{ux})_{,a} - \frac{1}{2}(\beta - \alpha\, g_{ux})\frac{\alpha_{,a}}{\alpha}\right)\,, \tag{4.22}$$

$$F_{u,a} = \frac{1}{\kappa_0\, F_r}\left(\gamma_{,a} - \frac{1}{2}\gamma\frac{\alpha_{,a}}{\alpha}\right)\,, \tag{4.23}$$

using the relation

$$\frac{F_x}{F_r} = \frac{\beta}{\alpha} - g_{ux}\,. \tag{4.24}$$

From (1.7) we can derive a similar relationship for the function $G$, namely

$$G_{,a} = -\frac{1}{2}G^3 g_{xx,a}\,. \tag{4.25}$$

Taking advantage of the derived identities, the Maxwell equations (1.72) after some algebraic manipulation give

$$\left(\alpha_{,x} + 2\alpha\frac{G_{,x}}{G}\right) - 2(\beta - \alpha g_{ux})_{,r} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,r}}{\alpha} + 2\theta\right) = 0\,, \tag{4.26}$$

$$\left(\alpha_{,u} + 2\alpha\frac{G_{,u}}{G}\right) - 2\gamma_{,r} + \gamma\left(\frac{\alpha_{,r}}{\alpha} + 2\theta\right) = 0\,, \tag{4.27}$$

$$\gamma_{,x} - \gamma\left(\frac{\alpha_{,x}}{2\alpha} - \frac{G_{,x}}{G}\right) - (\beta - \alpha g_{ux})_{,u} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,u}}{2\alpha} - \frac{G_{,u}}{G}\right) = 0\,. \tag{4.28}$$

The last Maxwell equation (1.73) can be solved by using the relations (1.31)-(1.33), which reveals the following equation

$$\begin{aligned}\beta_{,x} + \beta_{,r}\, g_{ux} + \beta\left[g_{ux,r} - g_{ux}\left(\frac{\alpha_{,r}}{2\alpha} + 2\theta\right) - \left(\frac{\alpha_{,x}}{2\alpha} - 2\frac{G_{,x}}{G}\right)\right] \\ - \frac{1}{2G^2}\left[2\gamma_{,r} - \alpha_{,r}\left(\frac{\gamma}{\alpha} - g_{uu}\right) + \alpha_{,u} + 2\alpha\, g_{uu,r}\right] = 0\,.\end{aligned} \tag{4.29}$$

We now recall the explicit form of the functions $\alpha$, $\beta$ and $\gamma$ given by, respectively, (4.4), (4.8) and (4.13). We see that the function $\alpha$ is entirely determined by the metric function $g_{xx}$. The function $\beta$ additionally contains $g_{ux}$, and the function $\gamma$ intertwines these metric components together with $g_{uu}$. This means that the equations presented here are purely given in terms of the metric components. However their complexity was not reduced, and they pose a challenging and yet unsolved problem. If we were to substitute the metric explicitly, we would obtain a system of nonlinear partial differential equations of the third order and their general integration is still a distant possibility.

## 4.3 Integration of the Einstein field equations, step two

As shown, we were able to reformulate the Maxwell equations as differential equations for the metric components. We still have the remaining three Einstein equations left, and we now return to them. However, due to the complicated nature of the corresponding Ricci components we will not be able to solve them.

(iv) Integration of $R_{xx} = 2\Lambda g_{xx} + \kappa_0\, G^2 F_x^2$:

Multiplying this field equation by $\kappa_0 F_r^2$, and employing the field equations (4.5) and (4.9), the Einstein equation for the Ricci component (A.27) can be written as

$$R_{xx} = 2\Lambda g_{xx} + \frac{G^2}{\alpha}(\beta - g_{ux}\,\alpha)^2\,. \tag{4.30}$$

This equation again encodes only the metric functions $g_{xx}$, $g_{ux}$ and $g_{uu}$, but it is too difficult to obtain any additional information.

(v) Integration of $R_{ux} = 2\Lambda g_{ux} + \kappa_0\, G^2 F_u F_x$:

We can obtain the relation for the component of the Ricci tensor (A.28), by multiplying the field equation with $\kappa_0 F_r^2$ and using the expressions (4.9) and (4.14). Using this we find that

$$R_{ux} = 2\Lambda g_{ux} + \frac{G^2}{\alpha}\gamma\,(\beta - g_{ux}\,\alpha)\,. \tag{4.31}$$

(vi) Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0\, G^2 F_u^2$:

The same procedure can be applied to the last Einstein equation. The multiplication by $\kappa_0 F_r^2$, with the use of (4.14), shows that the last Ricci component (A.29) must satisfy the following

$$R_{uu} = 2\Lambda g_{uu} + \frac{G^2}{\alpha}\gamma^2\,. \tag{4.32}$$

Even though we were not able to solve the coupled Einstein–Maxwell equations, we achieved a separation of variables for the respective fields. The resulting equations are still complicated, but it may open other possibilities how to study them, for example by numerical approximations. However, in the next section we will concentrate on showing that a nonaligned solution exists.

## 4.4 Particular solution for a nonaligned electromagnetic field

To show that the Robinson–Trautman geometry admits also a nonaligned electromagnetic field and, to show the usefulness of the notation established before, we will now look for a simple solution with the condition $F_{rx} \neq 0$, since the case when $F_{rx} = 0$ corresponds to an aligned electromagnetic field. To simplify the field equations we will impose the following assumptions

$$g_{ux} = 0\,, \qquad F_u = 0\,. \tag{4.33}$$

In view of (4.8), the first condition implies

$$\beta = 0 \quad \Rightarrow \quad F_{ru} = 0\,, \tag{4.34}$$

where the second part of the implication is just a consequence of the field equation (4.7), considering $F_{rx} \neq 0$. Using (1.21) the second condition can now be rewritten as

$$F_{ux} = -g_{uu}\, F_{rx}\,. \tag{4.35}$$

Another simplification comes from the Einstein equation (4.14), where we find that

$$\gamma = 0\,. \tag{4.36}$$

This will now allow us to solve the Maxwell equations (4.26) and (4.27), using (4.33), (4.34) and $\gamma = 0$. These equations read

$$\frac{\alpha_{,x}}{\alpha} + 2\frac{G_{,x}}{G} = 0\,, \tag{4.37}$$

$$\frac{\alpha_{,u}}{\alpha} + 2\frac{G_{,u}}{G} = 0\,. \tag{4.38}$$

The simultaneous solution of both of these equations yields

$$\alpha = G^{-2} f(r)\,, \tag{4.39}$$

where $f(r)$ is an arbitrary function dependent only on the coordinate $r$. Comparing this with the first field equation (4.5), we can express the electromagnetic component as

$$F_{rx} = \sqrt{\frac{f(r)}{\kappa_0\, G}}\,. \tag{4.40}$$

Utilizing the explicit definition of $\alpha$ from (4.4), we obtain the following differential equation for the expansion scalar

$$\theta_{,r} + \theta^2 = -f(r). \tag{4.41}$$

This is a Riccati type differential equation. Unfortunately, a solution to this equation cannot be written in a closed form unless we prescribe the function $f(r)$. Keeping it in line with trying to find the simplest solution, we now choose the function $f(r)$ to be

$$f(r) = C^2\,, \qquad \text{where} \qquad C = \text{const.}\,. \tag{4.42}$$

By this choice the equation (4.41) reduces to

$$\theta_{,r} + \theta^2 = -C^2\,, \tag{4.43}$$

and its general solution has the form

$$\theta = -C \tan\Big[C\Big(r + r_0(u,x)\Big)\Big]\,. \tag{4.44}$$

Since the expansion scalar is related to the function $G$ by (1.8), we find that

$$G = \frac{P(u,x)}{\cos\Big(C(r + r_0)\Big)}\,, \tag{4.45}$$

where $P(u,x)$ is an arbitrary function independent of $r$. Consequently, the corresponding metric function can be expressed from (1.7) as

$$g_{xx} = \frac{\cos^2\Big(C(r+r_0)\Big)}{P^2}\,. \tag{4.46}$$

Now the only independent metric function is $g_{uu}$. The third Maxwell equation (4.28) is trivially satisfied, using (4.34) and $\gamma = 0$. Applying the same relations in (4.29) and utilizing (4.45), the Maxwell equation reads

$$g_{uu,r} + \theta g_{uu} = \frac{P_{,u}}{P} + C\, r_{0,u} \tan\Big[C\Big(r+r_0\Big)\Big]\,. \tag{4.47}$$

Substituting the expression for the expansion scalar (4.44), the general solution to this equation is

$$g_{uu} = \frac{1}{C}\frac{P_{,u}}{P}\tan\Big[C\Big(r+r_0\Big)\Big] + \frac{a(u,x)}{\cos\Big[C\Big(r+r_0\Big)\Big]} - r_{0,u}, \tag{4.48}$$

where $a(u,x)$ is an arbitrary $r$-independent function.

We have thus depleted all of the Maxwell equations, and have only four Einstein equations left. Using the Ricci component (A.27) in the equation (4.30) together with (4.33), (4.34), (4.45) and the obtained solution for $g_{uu}$ (4.48), we come to the equation

$$\begin{aligned}\frac{CP_{,u}}{P}\sin^3\Big[C\Big(r+r_0\Big)\Big] + (C^2 r_{0,u} - 2\Lambda)\cos^3\Big[C\Big(r+r_0\Big)\Big]&\\ - 2C^2\, r_{0,u}\cos\Big[C\Big(r+r_0\Big)\Big] + C^2(a - 2r_{0,u}) = 0.&\end{aligned} \tag{4.49}$$

For this to hold for any value of $r$ it is clear that the following must hold

$$P_{,u} = 0\,, \qquad r_{0,u} = 0\,, \qquad a = 0\,, \qquad \Lambda = 0\,. \tag{4.50}$$

Therefore, we see that by satisfying the Einstein equation, we necessarily get

$$g_{uu} = 0\,, \qquad P = P(x)\,, \qquad r = r(x)\,. \tag{4.51}$$

This result, together with (4.33) and (4.34), is enough to trivially satisfy the remaining Einstein equations, namely (4.31), (4.32) and (4.36). So, to summarize our results, we obtained the metric functions

$$g_{xx} = \frac{\cos^2\Big[C\Big(r+r_0(x)\Big)\Big]}{P^2(x)}\,, \tag{4.52}$$

$$g_{ux} = 0\,, \tag{4.53}$$

$$g_{uu} = 0\,. \tag{4.54}$$

The corresponding electromagnetic field has the components

$$F_{rx} = \frac{C}{\kappa_0 P}\cos\Big[C\Big(r+r_0(x)\Big)\Big]\,, \qquad F_{ru} = 0\,, \qquad F_{ux} = 0\,. \tag{4.55}$$

We can simplify the metric further by employing the gauge transformation $\mathrm{d}x \to \frac{\mathrm{d}x}{P}$, or simply set $P = 1$, and $r \to r - r_0$, which allows us to put $r_0 = 0$, but also introduces a nondiagonal term $g_{ux} = r_0(x)$, which we will denote for clarity as $g_{ux} = B(x)$. The metric then reads

$$\mathrm{d}s^2 = \cos^2(C\,r)\,\mathrm{d}x^2 - B(x)\,\mathrm{d}u\,\mathrm{d}x - 2\,\mathrm{d}u\,\mathrm{d}r\,. \tag{4.56}$$

We would like to point out that we could relax some of the conditions under which we derived this metric. Specifically, we could consider the case $g_{ux} = B(x)$, since we saw that such metric component can be gauged away by introducing shift in the $r$ variable. Also the condition $\beta = 0$ and $F_{ru} = 0$ still holds due to (4.8) and (4.7). The electromagnetic field and its dual, in this gauge read

$$\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\cos(C\,r)\,\mathrm{d}r\wedge\mathrm{d}x\,, \qquad {}^*\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\,\mathrm{d}r\,. \tag{4.57}$$

The corresponding vector and scalar potentials are

$$\mathbf{A} = \frac{1}{\sqrt{\kappa_0}}\sin(C\,r)\,\mathrm{d}x\,, \qquad \Phi = \frac{C}{\sqrt{\kappa_0}}\,r\,. \tag{4.58}$$

We conclude this exploration with the Newman–Penrose scalars corresponding to this solution

$$\phi_0 = \frac{C}{\sqrt{\kappa_0}}\,, \qquad \phi_1 = 0\,, \qquad \phi_2 = 0\,. \tag{4.59}$$

It may seem that this solution belongs to the family of nonaligned Robinson–Trautman solutions, but the problem is more subtle. It is true that the scalar $\phi_0 \neq 0$, and from its definition this would mean that the electromagnetic field is nonaligned. However the scalar $\phi_2$ vanishes identically and this means that actually the electromagnetic field is aligned with the geometrical direction given by $\mathbf{l}$ defined in (1.5). Explicitly, this can be seem by applying the spatial rotation (1.50) with the parameter $R = 1$. This transformation just swaps the $\phi_0 \leftrightarrow \phi_2$ scalars, and we would obtain alignment exactly as we defined. So the problem if the Robinson–Trautman geometry allows a genuine nonaligned electromagnetic field is still open.

# 5. Algebraic classification in 2+1 gravity

In this chapter we review the current procedure of algebraic classification of gravitational fields in 2+1 gravity developed by García-Díaz in [12], see section 20.5. We will then establish a new approach to algebraic classification, based on projections of the Cotton tensor onto a suitable null basis. This more convenient classification is an 2+1 dimensional analogue of the Penrose classification in standard four-dimensional theory of relativity. Moreover, we will show that such classification is completely equivalent to the one developed by García-Díaz.

Let us first establish the notation in this chapter, which partly deviates from the one used in the text before. We consider a general three-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with the metric signature $(-,+,+)$. On such a manifold we have the frame triad $\mathbf{e}_a$ with its dual $\boldsymbol{\omega}^a$. In local coordinates they can be expressed as

$$\mathbf{e}_a = e^{\alpha}{}_{a}\,\partial_\alpha \qquad \boldsymbol{\omega}^a = e_{\alpha}{}^{a}\,\mathrm{d}x^\alpha\,. \tag{5.1}$$

We denote by Latin indices $a$ the triad indices, and by Greek letters we denote the coordinate indices. The metric, in terms of the dual basis, reads

$$\mathrm{d}s^2 = g_{ab}\,\boldsymbol{\omega}^a\,\boldsymbol{\omega}^b\,. \tag{5.2}$$

We will now start with the exposition of the two key objects in the algebraic classification in 2+1 gravity, namely the Cotton tensor and the Cotton–York tensor.

## 5.1 Cotton tensor

The role of the key conformally invariant tensor in three-dimensional spaces plays the Cotton tensor, as was investigated already by Cotton in [26], or later by Schouten [27]. It is a suitable replacement for the Weyl tensor which identically vanishes in these spaces. Therefore, taking inspiration from the standard four-dimensional theory of gravity, it is natural to use the Cotton tensor for algebraic classification of spacetimes in 3D gravity. The Cotton tensor in three dimensions is defined as

$$C_{abc} = 2\left(\nabla_{[a}R_{b]c} - \frac{1}{4}\nabla_{[a}R g_{b]c}\right), \tag{5.3}$$

see equation (20.39) in [12], or equation (3.89) in [2], where unfortunately a different convention is used for the position of the antisymetric indices. From the definition (5.3) it can be immediately seen that the Cotton tensor is antisymmetric in the first two indices,

$$C_{abc} = -C_{bac}\,. \tag{5.4}$$

The Cotton tensor also satisfies the properties

$$C_{[abc]} = 0\,, \tag{5.5}$$

$$C_{ab}{}^{a} = 0\,, \tag{5.6}$$

as is shown in [12] in equations (20.52) and (20.55). This restricts the Cotton tensor to have only 5 independent components. Indeed, due to (5.4), the Cotton tensor has nine independent components, constrained by one condition (5.5) and three independent conditions (5.6).

We can now obtain the convenient components by projecting the Cotton tensor onto a suitable basis. We choose a basis triad on the tangent space as $\{\mathbf{e}_a\} = \{\mathbf{k},\, \mathbf{l},\, \mathbf{m}\}$, given by the relations

$$\mathbf{k}\cdot\mathbf{l} = -1\,, \qquad \mathbf{m}\cdot\mathbf{m} = 1\,, \tag{5.7}$$

or explicitly in components

$$k_a\, l^a = -1\,, \qquad m_a\, m^a = 1\,, \tag{5.8}$$

where $\mathbf{k}$, and $\mathbf{l}$ are null, $\mathbf{k}\cdot\mathbf{k} = 0 = \mathbf{l}\cdot\mathbf{l}$ and $\mathbf{k}\cdot\mathbf{m} = 0 = \mathbf{l}\cdot\mathbf{m}$. The dual basis will be denoted by $\{\boldsymbol{\omega}^b\}$ and is given by the relation $e_a^\alpha\, \omega_\alpha^b = \delta_a^b$. In view of the scalar products (5.7), the dual basis can be constructed as $\{\boldsymbol{\omega}^b\} = \{-\mathbf{l}, -\mathbf{k}, \mathbf{m}\}$. By this notation we mean that the dual to the vector $\mathbf{k}$ is $\boldsymbol{\omega}^1 = -l_\alpha\, \mathrm{d}x^\alpha$, and so on for the remaining basis vectors.

The dimension of the space of 2-forms (bivectors) in 3D is three, and we can construct a bivector basis to easily express them. We denote this basis as $\{Z^I_{ab}\} = \{U_{ab}\,, V_{ab},\, W_{ab}\}$. A natural choice for these three bivectors, using the vectors of the null basis (5.7), is

$$U_{ab} = 2\,\mathbf{m}\wedge\mathbf{l}\ \equiv m_a\, l_b - l_a\, m_b\,, \tag{5.9}$$

$$V_{ab} = 2\,\mathbf{k}\wedge\mathbf{m} \equiv k_a\, m_b - m_a\, k_b\,, \tag{5.10}$$

$$W_{ab} = 2\,\mathbf{l}\wedge\mathbf{k}\ \ \equiv l_a\, k_b - k_a\, l_b\,, \tag{5.11}$$

see the analogous definition in 4D given by the equation (3.40) in [2]. A direct calculation reveals that they satisfy the relations

$$U_{ab}V^{ab} = 2\,, \qquad W_{ab}W^{ab} = -2\,, \tag{5.12}$$

while all other contractions are zero.

The Cotton tensor can be expressed in the basis given by all combinations of the tensor product of a basis bivector and a covector, so the Cotton tensor can be written as

$$C_{abc} = \sum_{I,J=1}^{3} C_{IJ}\, Z^I_{ab}\, \omega^J_c\,, \tag{5.13}$$

or explicitly writing its nine terms:

$$\begin{aligned} C_{abc} = &- C_{11}\, U_{ab}\, l_c - C_{12}\, U_{ab}\, k_c + C_{13}\, U_{ab}\, m_c \\ &- C_{21}\, V_{ab}\, l_c - C_{22}\, V_{ab}\, k_c + C_{23}\, V_{ab}\, m_c \\ &- C_{31}\, W_{ab}\, l_c - C_{32}\, W_{ab}\, k_c + C_{33}\, W_{ab}\, m_c\,. \end{aligned} \tag{5.14}$$

Since bivectors are antisymmetric the condition (5.4) is trivially satisfied. Next we employ the condition (5.6) and use the following relations for contractions of the bivectors

$$\begin{aligned} U_{ab}\, l^a &= 0\,, & U_{ab}\, k^a &= m_b\,, & U_{ab}\, m^a &= l_b\,, \\ V_{ab}\, l^a &= -m_b\,, & V_{ab}\, k^a &= 0\,, & V_{ab}\, m^a &= -k_b\,, \\ W_{ab}\, l^a &= l_b\,, & W_{ab}\, k^a &= -k_b\,, & W_{ab}\, m^a &= 0\,. \end{aligned} \tag{5.15}$$

We thus come to the equation

$$(C_{13} - C_{31})\, l_b + (C_{32} - C_{23})\, k_b + (C_{21} - C_{12})\, m_b = 0\,. \tag{5.16}$$

This equation must be satisfied identically, and so we obtain three conditions

$$C_{13} = C_{31}\,, \qquad C_{23} = C_{32}\,, \qquad C_{12} = C_{21}\,. \tag{5.17}$$

The general form of the Cotton tensor can thus be written as

$$\begin{aligned} C_{abc} = &- C_{11}\, U_{ab}\, l_c - C_{12}(U_{ab}\, k_c + V_{ab}\, l_c) + C_{13}(U_{ab}\, m_c - W_{ab}\, l_c) \\ &- C_{22}\, V_{ab}\, k_c + C_{23}(V_{ab}\, m_c - W_{ab}\, k_c) + C_{33}\, W_{ab}\, m_c\,. \end{aligned} \tag{5.18}$$

For the remaining condition (5.5), it is helpful to do the following calculation first

$$3!\,(U_{[ab}\, k_{c]} + V_{[ab}\, l_{c]}) = 4(U_{ab}\, k_c + V_{ab}\, l_c + W_{ab}\, m_c)\,, \tag{5.19}$$

$$3!\, W_{[ab}\, m_{c]} = 2(U_{ab}\, k_c + V_{ab}\, l_c + W_{ab}\, m_c)\,, \tag{5.20}$$

where the factorial was included just to cancel out the factor in the definition of the antisymmetrization. All other terms of the tensor product basis are trivially zero under the complete antisymmetrization, namely

$$\begin{aligned} U_{[ab}\, l_{c]} &= 0\,, & U_{[ab}\, m_{c]} &= 0\,, \\ V_{[ab}\, k_{c]} &= 0\,, & V_{[ab}\, m_{c]} &= 0\,, \\ W_{[ab}\, k_{c]} &= 0\,, & W_{[ab}\, l_{c]} &= 0\,. \end{aligned} \tag{5.21}$$

Thus from the equation (5.5) we get

$$(-2C_{12} + C_{33})(U_{ab}\, k_c + V_{ab}\, l_c + W_{ab}\, m_c) = 0\,, \tag{5.22}$$

from which we obtain the last condition

$$C_{33} = 2\, C_{12}\,. \tag{5.23}$$

In view of (5.17), (5.18) and (5.23), the generic Cotton tensor in the null basis takes the form

$$\begin{aligned} C_{abc} = &- C_{11}\, U_{ab}\, l_c - C_{12}(U_{ab}\, k_c + V_{ab}\, l_c - 2W_{ab}\, m_c) \\ &+ C_{13}(U_{ab}\, m_c - W_{ab}\, l_c) - C_{22}\, V_{ab}\, k_c + C_{23}(V_{ab}\, m_c - W_{ab}\, k_c)\,. \end{aligned} \tag{5.24}$$

Using (5.7) and (5.12), the individual coefficients of the Cotton tensor can then be expressed as

$$C_{11} = \frac{1}{2} C_{abc}\, V^{ab}\, k^c\,, \tag{5.25}$$

$$C_{12} = \frac{1}{2} C_{abc}\, V^{ab}\, l^c = \frac{1}{2} C_{abc}\, U^{ab}\, k^c = -\frac{1}{4} C_{abc}\, W^{ab}\, m^c\,, \tag{5.26}$$

$$C_{13} = \frac{1}{2} C_{abc}\, V^{ab}\, m^c = -\frac{1}{2} C_{abc}\, W^{ab}\, k^c\,, \tag{5.27}$$

$$C_{22} = \frac{1}{2} C_{abc}\, U^{ab}\, l^c\,, \tag{5.28}$$

$$C_{23} = \frac{1}{2} C_{abc}\, U^{ab}\, m^c = -\frac{1}{2} C_{abc}\, W^{ab}\, l^c\,. \tag{5.29}$$

We now propose the following definition of real Newman–Penrose-type curvature scalars in 3D gravity:

$$\Psi_0 \equiv C_{abc}\, k^a\, m^b\, k^c\,, \tag{5.30}$$
$$\Psi_1 \equiv C_{abc}\, k^a\, l^b\, k^c\,, \tag{5.31}$$
$$\Psi_2 \equiv C_{abc}\, k^a\, m^b\, l^c\,, \tag{5.32}$$
$$\Psi_3 \equiv C_{abc}\, l^a\, k^b\, l^a\,, \tag{5.33}$$
$$\Psi_4 \equiv C_{abc}\, l^a\, m^b\, l^c\,, \tag{5.34}$$

which are fully analogous to the definition of the Newman–Penrose scalars constructed from the Weyl curvature tensor in 4D (see [2]) and in higher dimensional gravity (see [28]). For this definition we get a simple correspondence between the five independent coefficients of the Cotton tensor, namely

$$C_{11} = \Psi_0\,,\quad C_{12} = \Psi_2\,,\quad C_{13} = \Psi_1\,,\quad C_{22} = -\Psi_4\,,\quad C_{23} = -\Psi_3\,, \tag{5.35}$$

Indeed, after explicitly putting the bivectors (5.9)-(5.11) to (5.25)-(5.29) and using the antisymmetry of the Cotton tensor (5.4) one arrives at (5.35). Moreover, the four remaining frame components are not independent, because

$$C_{12} = C_{abc}\, m^a\, l^b\, k^c \quad = \frac{1}{2} C_{abc}\, k^a\, l^b\, m^c\,, \tag{5.36}$$
$$C_{13} = C_{abc}\, k^a\, m^b\, m^c\,, \tag{5.37}$$
$$C_{23} = C_{abc}\, m^a\, l^b\, m^c\,, \tag{5.38}$$

and therefore they are related through the relations (5.35) to the Newman–Penrose scalars, as well. Explicitly, the following expressions for $\Psi_1$, $\Psi_2$ and $\Psi_3$ are valid:

$$\Psi_1 = C_{abc}\, k^a\, m^b\, m^c\,, \tag{5.39}$$
$$\Psi_2 = C_{abc}\, m^a\, l^b\, k^c \quad = \frac{1}{2} C_{abc}\, k^a\, l^b\, m^c\,, \tag{5.40}$$
$$\Psi_3 = C_{abc}\, l^a\, m^b\, m^c\,. \tag{5.41}$$

Consequently, the general form (5.24) of the Cotton tensor in the null basis is

$$\begin{aligned} C_{abc} = &- \Psi_0\, U_{ab}\, l_c + \Psi_1 (U_{ab}\, m_c - W_{ab}\, l_c) \\ &- \Psi_2 (U_{ab}\, k_c + V_{ab}\, l_c - 2W_{ab}\, m_c) \\ &- \Psi_3 (V_{ab}\, m_c - W_{ab}\, k_c) + \Psi_4\, V_{ab}\, k_c\,, \end{aligned} \tag{5.42}$$

which is analogous to the expression valid in 4D, see equation (3.58) in [2].

## 5.2 Cotton–York tensor

The number of independent components of the Cotton tensor in three dimensions, being five, is exactly equal to the number of components of a symmetric and traceless second-rank tensor. This is of course exclusive to only three-dimensional spaces, and a mapping can be achieved between the two by a means of the Hodge dual. Here, we follow the conventions given in [12], with only slight alterations.

The Cotton–York tensor is defined by equation (20.111) in [12] as

$$Y_{ab} \equiv \mathbf{e}_a \rfloor {}^*\mathbf{C}_b = {}^*(\mathbf{C}_b \wedge \boldsymbol{\omega}_a)\,, \tag{5.43}$$

where we denote by $\boldsymbol{\omega}_a$ the following linear combination of the basis 1-forms, $\boldsymbol{\omega}_a = g_{ab}\,\boldsymbol{\omega}^b$, and $\mathbf{C}_b$ is the Cotton (vector valued)

$$\text{2-form:} \qquad \mathbf{C}_b = \frac{1}{2} C_{bmn}\,\boldsymbol{\omega}^m \wedge \boldsymbol{\omega}^n\,. \tag{5.44}$$

The symbol $\rfloor$ stands for the interior product defined on a general p-form $\boldsymbol{\sigma}$ as

$$\mathbf{e}_a \rfloor \boldsymbol{\sigma} \equiv \frac{1}{(p-1)!}\,\sigma_{ab_2\ldots b_p}\,\boldsymbol{\omega}^{b_2} \wedge \ldots \wedge \boldsymbol{\omega}^{b_p}\,. \tag{5.45}$$

The exterior calculus notation and definitions used here, are mainly taken from appendix A in [29]. Another quite common notation for this operation is $\iota_{\mathbf{e}}\boldsymbol{\sigma}$. A metric independent Hodge dual operator can be defined by employing the so called $\eta$-basis which is constructed by subsequent interior products of the Levi-Civita tensor $\boldsymbol{\varepsilon}$. Explicitly the $\eta$-basis is then equal to

$$\text{2-form:} \quad \boldsymbol{\eta}_a \ = \mathbf{e}_a \rfloor \boldsymbol{\varepsilon} \ \ = \frac{1}{2}\varepsilon_{abc}\,\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c\,, \tag{5.46}$$

$$\text{1-form:} \quad \boldsymbol{\eta}_{ab} = \mathbf{e}_b \rfloor \boldsymbol{\eta}_a \ = \varepsilon_{abc}\,\boldsymbol{\omega}^c\,, \tag{5.47}$$

$$\text{0-form:} \quad \eta_{abc} = \mathbf{e}_c \rfloor \boldsymbol{\eta}_{ab} = \varepsilon_{abc}\,, \tag{5.48}$$

where the Levi-Civita tensor can be defined using the metric on the manifold as $\boldsymbol{\varepsilon} = -3!\,\sqrt{-g}\,\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2$, where we denoted the determinant of the metric $g_{ab}$ as $g$. In components this tensor explicitly reads

$$\varepsilon_{abc} = -\sqrt{-g}\,\epsilon_{abc}\,, \qquad \text{or} \qquad \varepsilon^{abc} = \frac{1}{\sqrt{-g}}\,\epsilon^{abc}\,, \tag{5.49}$$

where $\epsilon^{abc} = \epsilon_{abc}$ is the Levi-Civita symbol. Without loss of generality we assume that the null basis has the following orientation

$$\varepsilon_{abc}\,l^a\,k^b\,m^c = -1\,. \tag{5.50}$$

It is equivalent to defining the Levi-Civita symbol in the null basis $\{\boldsymbol{\omega}^b\}$ as $\epsilon^{123} = \epsilon_{123} = -1$. This orientation ensures that, in an orthonormal frame, the spatial part will have the right-handed orientation. The correspondence between the Hodge dual operation and the $\eta$-basis is then through the relations

$$\boldsymbol{\eta}_a \equiv {}^*\boldsymbol{\omega}_a\,, \qquad \boldsymbol{\eta}_{ab} \equiv {}^*(\boldsymbol{\omega}_a \wedge \boldsymbol{\omega}_b)\,, \qquad \boldsymbol{\eta}_{abc} \equiv {}^*(\boldsymbol{\omega}_a \wedge \boldsymbol{\omega}_b \wedge \boldsymbol{\omega}_c)\,. \tag{5.51}$$

Applying this to the construction of the Hodge dual of the vector valued Cotton 2-form (5.44), we get ${}^*\mathbf{C}_b = \frac{1}{2}C_{mnb}\,\boldsymbol{\eta}^{mn}$, or explicitly we obtain the following expression for the dual (vector valued)

$$\text{1-form:} \qquad {}^*\mathbf{C}_b = \frac{1}{2}\,g_{na}\,\varepsilon^{kmn}\,C_{kmb}\,\boldsymbol{\omega}^a\,. \tag{5.52}$$

Now putting together the equation (5.52) and the definition (5.45) we get from the equation (5.53) an explicit prescription for the Cotton–York tensor, namely

$$Y_{ab} = \frac{1}{2}\, g_{na}\, \varepsilon^{kmn}\, C_{kmb}\,. \tag{5.53}$$

This alternative form of the Cotton tensor first appeared in [30], but was already discussed before by ADM in [31]. It encodes the same information as the Cotton tensor, but it is a rank lower tensor, and one of its major advantages is that it is symmetric

$$Y_{ab} = Y_{ba}\,, \tag{5.54}$$

and also traceless

$$Y_a{}^a = 0\,. \tag{5.55}$$

The Cotton–York tensor is the key tensor in 3D gravity, whose algebraic classification is done. Since it is a second-rank tensor, an eigenvalue problem can be formulated as a standard eigenvalue problem for matrices. Therefore, our motivation is to also express this tensor in the null basis (5.7) as

$$Y_{ab} = \sum_{I,J=1}^{3} Y_{IJ}\, \omega_a^I\, \omega_b^J\,, \tag{5.56}$$

and in the next sections to investigate the relation between the Newman–Penrose scalars (5.30)-(5.34) and the algebraic classification of spacetimes. Writing the sum (5.56) explicitly, and only using the symmetry property (5.54) of the Cotton–York tensor we get

$$\begin{aligned} Y_{ab} =& Y_{11}\, l_a\, l_b + Y_{12}(l_a\, k_b + k_a\, l_b) - Y_{13}(l_a\, m_b + m_a\, l_b) \\ &+ Y_{22}\, k_a\, k_b - Y_{23}(k_a\, m_b + m_a\, k_b) + Y_{33}\, m_a\, m_b\,. \end{aligned} \tag{5.57}$$

It is also possible not to assume any symmetry at all, but after explicit calculation of the coefficients the symmetry would again be obtained. The coefficients can be expressed from the above expression as

$$Y_{11} = Y_{ab}\, k^a\, k^b\,, \tag{5.58}$$
$$Y_{12} = Y_{ab}\, k^a\, l^b\,, \tag{5.59}$$
$$Y_{13} = Y_{ab}\, k^a\, m^b\,, \tag{5.60}$$
$$Y_{22} = Y_{ab}\, l^a\, l^b\,, \tag{5.61}$$
$$Y_{23} = Y_{ab}\, l^a\, m^b\,, \tag{5.62}$$
$$Y_{33} = Y_{ab}\, m^a\, m^b\,. \tag{5.63}$$

Using (5.43), (5.50) and (5.42) in the evaluation of the coefficients (5.58)-(5.63), one arrives at the following result

$$\begin{aligned} &Y_{11} = -\Psi_0\,, \qquad && Y_{12} = -\Psi_2\,, \qquad && Y_{13} = -\Psi_1\,, \\ &Y_{22} = \Psi_4\,, && Y_{23} = \Psi_3\,, && Y_{33} = -2\Psi_2\,. \end{aligned} \tag{5.64}$$

The general form of the Cotton–York tensor (5.57), thus takes the form

$$\begin{aligned} Y_{ab} =& -\Psi_0\, l_a\, l_b + \Psi_1(l_a\, m_b + m_a\, l_b) - \Psi_2(l_a\, k_b + k_a\, l_b + 2m_a\, m_b) \\ &- \Psi_3(k_a\, m_b + m_a\, k_b) + \Psi_4\, k_a\, k_b\,. \end{aligned} \tag{5.65}$$

This is the key expression that will allow us to relate the Newman–Penrose scalars $\Psi_A$ to the algebraic classification of spacetimes in 3D gravity.

## 5.3 Algebraic classification of spacetimes in 2+1 gravity

We will recall the algebraic classification employed by García-Díaz in section 1.2 of [12]. The algebraic classification is based on the Jordan form of the Cotton–York tensor. It corresponds to the classical eiqenvalue problem, $Y_a{}^b - \lambda\delta_a{}^b = 0$, where $Y_a{}^b$ can be thought of as a matrix, where $a$ counts the number of rows while $b$ denotes the columns. Expressing the corresponding Cotton–York tensor in the Jordan orthonormal basis, the Petrov types of spacetimes are defined by García-Díaz in table 5.1, which is actually the table 1.2.1 in [12].

Table 5.1: Algebraic classification of the Cotton–York tensor $Y_a{}^b$

| Petrov type | Jordan form of $Y_a{}^b$ | eigenvalues relation |
|---|---|---|
| I | $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}$ | $\lambda_1 \neq \lambda_2,\ \lambda_3 = -\lambda_1 - \lambda_2$ |
| II | $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$ | $\lambda_1 = \lambda_2 \neq 0,\ \lambda_3 = -2\lambda_1$ |
| III | $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ | $\lambda_1 = \lambda_2 = \lambda_3 = 0$ |
| D | $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$ | $\lambda_1 = \lambda_2 \neq 0,\ \lambda_3 = -2\lambda_1$ |
| N | $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\lambda_1 = \lambda_2 = \lambda_3 = 0$ |
| O | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | |

The corresponding orthonormal basis ($\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$) can be constructed from the null triad (5.7) via the usual relations

$$\mathbf{E}_0 = \frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{l}) \,, \qquad \mathbf{E}_1 = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{l}) \,, \qquad \mathbf{E}_2 = \mathbf{m} \,. \tag{5.66}$$

The inverse relations are thus

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{E}_0 + \mathbf{E}_1)\,, \qquad \mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{E}_0 - \mathbf{E}_1)\,, \qquad \mathbf{m} = \mathbf{E}_2\,. \tag{5.67}$$

Indeed, the orthonormal frame (5.66) satisfies the conditions

$$\mathbf{E}_0 \cdot \mathbf{E}_0 = -1\,, \qquad \mathbf{E}_1 \cdot \mathbf{E}_1 = 1\,, \qquad \mathbf{E}_2 \cdot \mathbf{E}_2 = 1\,, \tag{5.68}$$

with all other scalar products equal to zero. Using the inverse relation (5.67) in the explicit expression for the Cotton–York tensor (5.65) we obtain the Cotton–York tensor in an orthonormal basis, and we represent it by a symmetric matrix in the form

$$Y_{ab} = \begin{pmatrix} -\frac{1}{2}\Psi_0 - \Psi_2 + \frac{1}{2}\Psi_4 & \frac{1}{2}(\Psi_0 + \Psi_4) & \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) \\ \frac{1}{2}(\Psi_0 + \Psi_4) & -\frac{1}{2}\Psi_0 + \Psi_2 + \frac{1}{2}\Psi_4 & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) \\ \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix}. \tag{5.69}$$

Since the metric in an orthonormal frame is

$$g_{ab} = \mathrm{diag}(-1, 1, 1)\,, \tag{5.70}$$

we can raise the index by $Y_a{}^b = Y_{ac}\, g^{cb}$, and obtain the main traceless matrix for the algebraic classification, namely

$$Y_a{}^b = \begin{pmatrix} \frac{1}{2}\Psi_0 + \Psi_2 - \frac{1}{2}\Psi_4 & \frac{1}{2}(\Psi_0 + \Psi_4) & \frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) \\ -\frac{1}{2}(\Psi_0 + \Psi_4) & -\frac{1}{2}\Psi_0 + \Psi_2 + \frac{1}{2}\Psi_4 & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) \\ -\frac{1}{\sqrt{2}}(\Psi_1 - \Psi_3) & -\frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix}. \tag{5.71}$$

The Jordan forms given in table 5.1, corresponding to the respective Petrov types, give relations for the Newman–Penrose scalars defined by (5.30)-(5.34). These relations are presented in table 5.2, together with the normal forms. By a normal form of the matrix $Y_a{}^b$ we mean that there exists a similarity transformation between the Jordan form $J$ and the respective normal form $N$, such that

$$N = A\, J\, A^{-1}\,, \tag{5.72}$$

where $A$ is the similarity matrix and $A^{-1}$ its inverse. For enthusiasts of matrix multiplication, we can indeed check that for the type II spacetime the similarity transformation reads

$$\begin{pmatrix} \lambda_1 - 1 & 1 & 0 \\ -1 & \lambda_1 + 1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.73}$$

A special subclass of this transformation arises for $\lambda_1 = 0$, which then gives the similarity transformation for type N, while for type III we have

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 \end{pmatrix}. \tag{5.74}$$

Table 5.2: Algebraic classification according to the Newman–Penrose scalars $\Psi_A$

| Petrov type | Normal form of ${Y_a}^b$ | Newman–Penrose scalars relation |
|---|---|---|
| I | $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}$ | $\Psi_1 = 0 = \Psi_3$<br>$\Psi_0 = \frac{\lambda_1-\lambda_2}{2} = -\Psi_4$<br>$\Psi_2 = \frac{\lambda_1+\lambda_2}{2}$ |
| II | $\begin{pmatrix} \lambda_1 - 1 & 1 & 0 \\ -1 & \lambda_1 + 1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$ | $\Psi_0 = \Psi_1 = \Psi_3 = 0$<br>$\Psi_4 = 2$<br>$\Psi_2 = \lambda_1$ |
| III | $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ | $\Psi_0 = 0 = \Psi_1$<br>$\Psi_2 = 0 = \Psi_4$<br>$\Psi_3 = \sqrt{2}$ |
| D | $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$ | $\Psi_1 = 0 = \Psi_3$<br>$\Psi_0 = 0 = \Psi_4$<br>$\Psi_2 = \lambda_1$ |
| N | $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\Psi_0 = \Psi_1$<br>$\Psi_2 = 0 = \Psi_3$<br>$\Psi_4 = 2$ |
| O | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\Psi_0 = 0 = \Psi_1$<br>$\Psi_2 = 0 = \Psi_3$<br>$\Psi_4 = 0$ |

Therefore, we see that the normal forms are analogous to the Jordan forms given in table 5.1. The normal forms of the Cotton–York matrix were chosen here instead of the Jordan forms, because for certain Petrov types the Jordan forms cannot be obtained by some special choice of the Newman–Penrose scalars. The statement we have made is that there exists an orthonormal frame, which we may call the principal Cotton–York frame, in which the Penrose scalars take the values given in the last column of table 5.2. We will now prove that this frame implies the same classification scheme based on the number of multiplicities of the null vector $\mathbf{k}$, as in standard 4D gravity, see section 4.3 of [2].

To find the relation to the multiplicities of the null direction $\mathbf{k}$, we perform a null rotation which leaves the normalizations (5.7) invariant. Such rotation has the general form

$$\mathbf{k}' = \mathbf{k} + \sqrt{2}E\,\mathbf{m} + E^2\,\mathbf{l}\,, \qquad \mathbf{l}' = \mathbf{l}\,, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}E\,\mathbf{l}\,. \tag{5.75}$$

The Penrose scalars (5.30)–(5.34) undergo the following transformation

$$\Psi_0' = \Psi_0 + 2\sqrt{2}E\,\Psi_1 + 6E^2\,\Psi_2 - 2\sqrt{2}E^3\,\Psi_3 - E^4\Psi_4\,, \tag{5.76}$$
$$\Psi_1' = \Psi_1 + 3\sqrt{2}E\,\Psi_2 - 3E^2\,\Psi_3 - \sqrt{2}E^3\Psi_4\,, \tag{5.77}$$
$$\Psi_2' = \Psi_2 - \sqrt{2}E\,\Psi_3 - E^2\,\Psi_4\,, \tag{5.78}$$
$$\Psi_3' = \Psi_3 + \sqrt{2}E\,\Psi_4\,, \tag{5.79}$$
$$\Psi_4' = \Psi_4\,. \tag{5.80}$$

The inverse relations are given as

$$\Psi_0 = \Psi_0' - 2\sqrt{2}E\,\Psi_1' + 6E^2\,\Psi_2' + 2\sqrt{2}E^3\,\Psi_3' - E^4\,\Psi_4'\,, \tag{5.81}$$
$$\Psi_1 = \Psi_1' - 3\sqrt{2}E\,\Psi_2' - 3E^2\,\Psi_3' + \sqrt{2}E^3\,\Psi_4'\,, \tag{5.82}$$
$$\Psi_2 = \Psi_2' + \sqrt{2}E\,\Psi_3' - E^2\,\Psi_4'\,, \tag{5.83}$$
$$\Psi_3 = \Psi_3' - \sqrt{2}E\,\Psi_4'\,, \tag{5.84}$$
$$\Psi_4 = \Psi_4'\,. \tag{5.85}$$

We can now define the principle null frame. Such a frame is defined by the condition that the Cotton tensor $C_{abc}$ is aligned with the direction $\mathbf{k}$, which means that the scalar with the highest boost weight vanishes. Explicitly, the principle null frame is a frame in which $\Psi_0 = 0$, and from (5.81) we obtain the following algebraic equation

$$\Psi_0' - 2\sqrt{2}E\,\Psi_1' + 6E^2\,\Psi_2' + 2\sqrt{2}E^3\,\Psi_3' - E^4\,\Psi_4' = 0\,. \tag{5.86}$$

This algebraic equation can in general have four complex solutions for $E$. We also know that in the principle Cotton–York frame, the $\Psi_A$ must satisfy the conditions for different algebraic types of spacetimes given in the last column of table 5.2. We can now establish a classification based on the number of solutions to this quartic equation, which stays invariant. We denote by $\Psi_A$ the Newman–Penrose scalars in the principle null frame, while $\Psi_A'$ are the Newman–Penrose scalars in the principle Cotton–York frame. The different solutions to the algebraic equation (5.86) of the fourth order are given in table 5.3.

We can also check that for types D and III there exists a second null vector with the multiplicity of two and one, respectively, by making a null rotation with $\mathbf{k}' = \mathbf{k}$, and its general form is

$$\mathbf{k}' = \mathbf{k}\,, \qquad \mathbf{l}' = \mathbf{l} + \sqrt{2}B\,\mathbf{m} + B^2\,\mathbf{l}\,, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}B\,\mathbf{k}\,. \tag{5.87}$$

This rotation results in the change of the Newman–Penrose scalars

$$\Psi_0 = \Psi_0'\,, \tag{5.88}$$
$$\Psi_1 = \Psi_1' - \sqrt{2}B\,\Psi_0'\,, \tag{5.89}$$
$$\Psi_2 = \Psi_2' - \sqrt{2}B\,\Psi_1' + B^2\,\Psi_0'\,, \tag{5.90}$$
$$\Psi_3 = \Psi_3' + 3\sqrt{2}B\,\Psi_2' - 3B^2\,\Psi_1' + \sqrt{2}B^3\,\Psi_0'\,, \tag{5.91}$$
$$\Psi_4 = \Psi_4' - 2\sqrt{2}B\,\Psi_3' - 6B^2\,\Psi_2' + 2\sqrt{2}B^3\,\Psi_1' - B^4\,\Psi_0'\,, \tag{5.92}$$

Table 5.3: Multiplicities of the principal null direction **k**

| Petrov type | Equation (5.86) | Roots of E | Multiplicities |
|---|---|---|---|
| I | $1 + 6bE^2 + E^4 = 0$ | $E = \pm\sqrt{-3b \pm 2\sqrt{D}}$, where $b = \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$ and $\sqrt{D} = \frac{\sqrt{2\lambda_1^2+5\lambda_1\lambda_2+2\lambda_2^2}}{\lambda_1-\lambda_2}$ | (1,1,1,1) |
| II | $E^2(\frac{3}{2}\lambda_1 - E^2) = 0$ | $E = 0, \quad E = \pm\sqrt{\frac{3}{2}\lambda_1}$ | (2,1,1) |
| III | $E^3 = 0$ | $E = 0$ | (3,1) |
| D | $E^2 = 0$ | $E = 0$ | (2,2) |
| N | $E^4 = 0$ | $E = 0$ | (4) |

and we say that **l** is principle null vector if $\Psi_4 = 0$. If we put the values for $\Psi'_A$ from the table 5.2 into the equation (5.92), we obtain for

$$\text{Type D:} \qquad B^2 = 0 \tag{5.93}$$
$$\text{Type III:} \qquad B = 0\,, \tag{5.94}$$

and we see that they indeed have the respective multiplicities.

## 5.4 The Penrose classification in 2+1 gravity

The Penrose classification in standard general relativity is based on performing the general null rotation (5.81) and looking for the principle null frame, given by the condition $\Psi'_0 = 0$, for review see section 4.3 in [2]. Upon finding this frame and expressing the Newman–Penrose scalars in this frame, based on the vanishing of the remaining Newman–Penrose scalars we may classify the types of spacetimes according to table 5.4.

We will now prove that these conditions indeed imply the correct multiplicities, and therefore are equivalent to the Petrov classification done in the previous section.

(i) Type I: We assume that after evaluation of the Newman–Penrose scalars in the principal null frame only $\Psi_0$ is zero, $\Psi_0 = 0$. Putting this into (5.76), we come to the following equation for other principal null directions, namely

$$2\sqrt{2}E\,\Psi_1 + 6E^2\,\Psi_2 - 2\sqrt{2}E^3\,\Psi_3 - E^4\Psi_4 = 0\,. \tag{5.95}$$

We see that $E = 0$ is a simple root of the equation and by factoring it out, the remaining equation is of the third order which means that there exist

Table 5.4: Penrose classification of 2+1 spacetimes

| Petrov type | Penrose conditions |
|---|---|
| Type I | $\Psi_0 = 0$ |
| Type II | $\Psi_0 = \Psi_1 = 0$ |
| Type III | $\Psi_0 = \Psi_1 = \Psi_2 = 0$ |
| Type D | $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ |
| Type N | $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ |

three complex roots different than zero due to $\Psi_1 \neq 0$. But this implies that in general there really exist four different principle null directions and therefore the spacetime is of Type I.

(ii) Type II: The starting assumptions for this case are that $\Psi_0 = 0 = \Psi_1$. We again put this conditions into (5.76) and the resulting equation for the null directions is

$$6E^2\,\Psi_2 - 2\sqrt{2}E^3\,\Psi_3 - E^4\Psi_4 = 0\,. \tag{5.96}$$

This equation has the solution $E = 0$ with the multiplicity two, and the other solutions are given by a quadratic equation which, thanks to $\Psi_2 \neq 0$, has two different complex roots, meaning that the multiplicities of the null direction **k** are (2,1,1) and that is exactly the case for a type II spacetime.

(iii) Type III: Let $\Psi_0 = \Psi_1 = \Psi_2 = 0$, then the equation (5.76) takes the form

$$2\sqrt{2}E^3\,\Psi_3 + E^4\Psi_4 = 0\,. \tag{5.97}$$

Immediately, we get that there exist a root $E = 0$ of multiplicity three, and the other solution satisfies a linear equation with a single root. The null direction is therefore degenerate as (3,1), which means that this spacetime is of type III.

(iv) Type D: We now assume that all of the Newmann–Penrose scalars vanish except for $\Psi_2 \neq 0$. Substituting $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ into (5.76), the equation reads

$$E^2 = 0\,. \tag{5.98}$$

We see, that it has a trivial solution of $E = 0$ which has the multiplicity two. We can use the inverse of the relation (5.92) which is

$$\Psi_4' = \Psi_4 + 2\sqrt{2}B\Psi_3 - 6B^2\Psi^2 - 2\sqrt{2}B^3\Psi_1 - B^4\Psi_0\,, \tag{5.99}$$

and by setting $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ the equation gives

$$B^2 = 0\,, \tag{5.100}$$

which means that the direction **l** is also a principal direction of multiplicity two. This implies that the conditions for the spacetime to be of type D are met, and therefore we proved the statement given in table 5.4.

(v) Type N: Finally, we consider the case when all of the Newman–Penrose scalars vanish except for $\Psi_4 \neq 0$. This leads to a trivial equation of (5.76), namely

$$E^4 = 0\,. \tag{5.101}$$

Since this is exactly the condition we demanded that a spacetime of type N must satisfy, we see that for $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ the spacetime is necessarily of type N.

This proves that the simple Penrose classification summarized in table 5.4 is equivalent to the Petrov García-Díaz classification. Another prove of this fact can be done by putting the respective values of the Newman–Penrose scalars into the matrix (5.71) and showing that it can be decomposed into the corresponding Jordan forms. This alternative is however very tedious, but it leads to the same result.

We will also list the Bel–Debever conditions, which in 3D can be written as

$$k_{[d}C_{a]bc}\,k^b\,k^c = 0 \qquad \Leftrightarrow \qquad \Psi_0 = 0\,, \tag{5.102}$$

$$C_{abc}\,k^b\,k^c = 0 \qquad \Leftrightarrow \qquad \Psi_0 = \Psi_1 = 0 \tag{5.103}$$

$$k_{[d}C_{a]bc}\,k^b = 0 \qquad \Leftrightarrow \qquad \Psi_0 = \Psi_1 = \Psi_2 = 0\,, \tag{5.104}$$

$$C_{abc}\,k^c = 0 \qquad \Leftrightarrow \qquad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,. \tag{5.105}$$

For type D not only the condition (5.103) must hold but also the complementary relation must be true:

$$C_{abc}\,l^b\,l^c = 0 \qquad \Leftrightarrow \qquad \Psi_3 = \Psi_4 = 0\,. \tag{5.106}$$

The prove is quite straightforward. We start with (5.102), using (5.42) we get the following result

$$C_{abc}\,k^b\,k^c = -\Psi_0\,m_a + \Psi_1\,k_a\,. \tag{5.107}$$

The antisymmetrization then yields

$$k_{[d}C_{a]bc}\,k^b\,k^c = \Psi_0\,V_{da}\,. \tag{5.108}$$

From which we obtain the desired condition. The equivalence of (5.103) is obtained immediately from (5.107). As for (5.104), we get from (5.42)

$$C_{abc}\,k^b = \Psi_0\,m_a\,l_c - \Psi_1\,(m_a\,m_c + k_a\,l_c) + \Psi_2\,(m_a\,k_c + 2k_a\,m_c) + \Psi_3\,k_a\,l_c\,, \tag{5.109}$$

and after antisymmetrization we find that

$$k_{[d}C_{a]bc}\,k^b = \Psi_0\,V_{ad}\,l_c + \Psi_1\,V_{ad}\,m_c - \Psi_2\,V_{ad}\,k_c\,, \tag{5.110}$$

from which the equivalence is clear. The last relation immediately follows from (5.109). For type D the remaining condition to be satisfied is

$$C_{abc}\,l^b\,l^c = \Psi_3\,l_a - \Psi_4\,m_a\,, \tag{5.111}$$

which concludes the proof of the Bell–Debever relations in 2+1 gravity.

Finally, we would like to remind that the algebraic classification in 2+1 dimensions is more involved. The complications arise from the fact that the matrix

$Y_a{}^b$ (5.71) is not symmetric and there is no guarantee that the eigenvalues are real. Nevertheless, the common practice is to embrace the complex classification. The consequence of this unwelcome property is that the equation (5.86) can in general have complex roots. This means that some of the null vectors $\mathbf{k}$ may formally be complex. A classification based on only real roots can be done, see for example section 20.5.2 in [12]. By restricting the solutions to only real numbers, a new general type of spacetime must be added, namely the spacetime denoted by García-Díaz as I′. The development of the detailed equivalence of this classification to the Penrose form, is a prospect for our future work.

## 5.5 Application of the Penrose classification on the aligned Robinson–Trautman solutions

We will now show the usefulness of the Penrose classification summarized in table 5.4 by applying it to a specific case, namely the Robinson–Trautman spacetimes with an aligned electromagnetic field derived in chapter 3. The general form of the metric can be written as

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{r^2}{P^2}\left(\mathrm{d}x + e\,P^2\mathrm{d}u\right)^2 - 2\,\mathrm{d}\,u\mathrm{d}r \\ & + \left(\mu\,Q^2 - \kappa_0\,Q^2\log\left|\frac{Q}{r}\right| + 2\,(\ln Q)_{,u}\,r + \Lambda\,r^2\right)\mathrm{d}u^2\,, \end{aligned} \tag{5.112}$$

see equation (3.50). Using the components of the Ricci tensor (A.24)-(A.29), the Cotton tensor corresponding to this solution can be constructed from the definition (5.3), and its nonzero components are

$$C_{rru} = \frac{\kappa_0\,Q^2}{2\,r^3}\,, \tag{5.113}$$

$$C_{uru} = \kappa_0\,L_r\,Q^4\,\frac{1}{2\,r^3} + A_{uru}\,\frac{1}{r^2} + \kappa_0(e^2P^2 - \Lambda)\,Q^2\,\frac{1}{2\,r}\,, \tag{5.114}$$

$$C_{urx} = \frac{\kappa_0\,e\,Q^2}{2\,r}\,, \tag{5.115}$$

$$C_{uux} = A_{uux} - \left(\frac{1}{2}\,\kappa_0 + L_r\right)\left(P(eP)_x + \frac{P_{,u}}{P}\right)_{,x} Q^2\,\frac{1}{r}\,, \tag{5.116}$$

$$C_{xru} = \frac{\kappa_0\,e\,Q^2}{2\,r}\,, \tag{5.117}$$

$$C_{xrx} = \frac{\kappa_0\,Q^2}{2\,P^2\,r}\,, \tag{5.118}$$

$$C_{xux} = \left(\frac{3}{2}\,\kappa_0 + L_r\right)\left((eP)_{,x} + \frac{P_{,u}}{P^2}\right)\frac{Q^2}{P} - (2\,\kappa_0 + L_r)\frac{Q\,Q_{,u}}{P^2}\,, \tag{5.119}$$

where

$$A_{uru} = (\kappa_0 + L_r) Q\, Q_{,u} - \Big(\frac{3}{2}\,\kappa_0 + L_r\Big)\Big(P(eP)_{,x} + \frac{P_{,u}}{P}\Big)Q^2\,, \tag{5.120}$$

$$\begin{aligned} A_{uux} = {} & \Big(\frac{3}{2}\,\kappa_0 + L_r\Big)\Big(P(eP)_{,x} + \frac{P_{,u}}{P}\Big)eQ^2 - (2\kappa_0 + L_r)e\,Q\,Q_{,u} \\ & + \Big[P(eP)_{,x} + \frac{P_{,u}}{P}\Big]_{,x}\frac{Q_{,u}}{Q} + P_{,x}\,(P\,e_{,u})_{,x} + P\,(P_{,x}\,e_{,u})_{,x} + P\,(P\,e_{,ux})_{,x} \\ & - [2e^2P^2P_{,x}^2 + P^4(e_{,x}^2 + e_{,x}e_{,xx}) + e\,P^3(5e_{,x}P_{,x} + e\,P_{,xx})]_{,x}\,, \end{aligned} \tag{5.121}$$

and $L_r$ denotes

$$L_r = \kappa_0\,\log\left|\frac{Q}{r}\right| - \mu\,. \tag{5.122}$$

Using the null basis (1.5), which reads

$$\mathbf{k} = \partial_r\,, \qquad \mathbf{l} = \frac{1}{2}\,g_{uu}\,\partial_r + \partial_u\,, \qquad \mathbf{m} = \frac{1}{\sqrt{g_{xx}}}\Big(g_{ux}\,\partial_r + \partial_x\Big)\,, \tag{5.123}$$

and the field equation (3.58), we obtain the following Newman–Penrose scalars defined in (5.30)-(5.34), namely

$$\Psi_0 = 0\,, \tag{5.124}$$

$$\Psi_1 = \frac{\kappa_0 Q^2}{2r^3}\,, \tag{5.125}$$

$$\Psi_2 = \frac{\kappa_0 ePQ^2}{2r^2}\,, \tag{5.126}$$

$$\Psi_3 = \frac{\kappa_0 Q^4}{4r^3}\Big(\mu - \kappa_0\log\left|\frac{Q}{r}\right|\Big) + \frac{\kappa_0 Q^2}{4r}(\Lambda - 3(eP)^2)\,, \tag{5.127}$$

$$\Psi_4 = \frac{\kappa_0 ePQ^4}{2r^2}\Big(\mu - \kappa_0\log\left|\frac{Q}{r}\right|\Big) + \frac{\kappa_0 ePQ^2}{2}(\Lambda - (eP)^2)\,. \tag{5.128}$$

For $eP = \alpha = 0$ we were able to identify the class of charged black hole solutions, see equations (3.68) and (3.72), and the Newman–Penrose scalars reduce to

$$\Psi_0 = 0\,, \tag{5.129}$$

$$\Psi_1 = \frac{\kappa_0 Q^2}{2r^3}\,, \tag{5.130}$$

$$\Psi_2 = 0\,, \tag{5.131}$$

$$\Psi_3 = \frac{\kappa_0 Q^2}{4r^3}\Big(m - \kappa_0 Q^2\log\left|\frac{Q}{r}\right|\Big) + \frac{\kappa_0 Q^2}{4r}\Lambda\,, \tag{5.132}$$

$$\Psi_4 = 0\,. \tag{5.133}$$

According to the Penrose classification in table 5.4, this spacetime would be of algebraic type I. In fact, the same result was obtained by García-Díaz using the Petrov classification, see section 11.1.5 in [12]. Another check is that both the Newman–Penrose scalars for the special aligned solution and for a general aligned solution vanish when $Q = 0$, which corresponds to the vacuum Robinson–Trautmann solution, and thus necessarily must be a spacetime of constant curvature which are conformally flat.

# Conclusion

We investigated the Robinson–Trautman and Kundt families of spacetimes with an electromagnetic field and a cosmological constant in 2+1 gravity. We started by summarizing the previous results of the work by Podolský, Švarc and Maeda in [13], namely the construction of the metric in canonical coordinates (1.3) and the interesting property of three-dimensional spaces, that virtually all of them must belong to either the nonexpanding Kundt class or the expanding Robinson–Trautman class.

We then showed that, in terms of the components of the dual electromagnetic tensor, the Einstein equations have a nice factorized form (1.65). We also proved another interesting property of 3D gravity, specifically the correspondence between the minimally coupled massless scalar field and the dual electromagnetic 1-form (1.85).

We were able to completely solve the coupled system of Einstein–Maxwell equations for the Kundt and Robinson–Trautman spacetimes with an aligned electromagnetic field.

The general form of the metric for the Kundt solution could be written as (2.60), and the corresponding electromagnetic field is given by (2.64). Along the way we demonstrated that the nonexpanding Kundt class admits only an aligned electromagnetic field with the privileged null vector field **k**.

The aligned Robinson–Trautman case lead to the metric (3.50) with the electromagnetic field (3.54). We then recovered a special solution (3.72) corresponding to a charged black hole solution in 2+1 gravity obtained by Peldan in [22]. This corresponds to an electrostatic solution, however there exist other stationary solutions which are yet to be identified, for references see section 11.5.2 in [12]. This poses an interesting problem as to how to obtain them from the expanding and nontwisting class of Robinson–Trautman spacetimes.

The general Robinson–Trautman case with a nonaligned electromagnetic field was brought into a separated form of the Einstein–Maxwell equations. We were able to fully express the components of the electromagnetic tensor (4.3), (4.7) and (4.12), which meant that the remaining Einstein–Maxwell equations could be rewritten as equations containing only the metric components (4.26)-(4.32). Due to the complicated nature of these equations we could not obtain a general solution. It is still desirable to obtain such fully general solution, but for now this poses an open problem.

To show that at least some solutions with the $F_{rx} \neq 0$ component of the electromagnetic tensor exist we derived under the assumptions (4.33) a particular solution. We presented the metric (4.56) and the electromagnetic field (4.57). However we then argued that even though the solution satisfies the definition of nonalignment along **k**, with a trivial transformation we were able to prove that actually the obtained electromagnetic field is aligned with the null vector field given by **l**. Therefore, it is still not clear whether the Robinson–Trautman geometry allows a general nonaligned electromagnetic field. In principle, we can always do the transformation (1.46) which leads to the quadratic equation (1.49). The question that remains unanswered is, whether this equation has always real roots. For this to be true it would need to hold $\phi_1^2 \geq 2\phi_0\phi_2$. As it stands, there are still open problems left in the nonaligned Robinson–Trautman case.

In the last chapter we dwelt into the problem of algebraic classification in 2+1 gravity. We reviewed the current algorithm of algebraic classification of the Cotton and Cotton–York tensors, and reformulated it to the formalism of null basis. We defined the convenient Newman–Penrose scalars for the gravitational field (5.30)-(5.34). We then showed that these Newman–Penrose scalars must satisfy the relations given in table 5.2 for the respective algebraic types. This allowed us to develop the algebraic classification based on the number of multiplicities of the principal null vector, given in table 5.3. This paved the way to showing that the Penrose classification is fully equivalent to the Petrov one. However, the property of employing complex null vectors remained, and it is demanding to reformulate the classification to only real domain. The problem of establishing a possible classification based on spinors and showing its equivalence is also another open question. Nevertheless, we demonstrated the effectiveness of our new approach to the classification on the aligned Robinson–Trautman solution. We showed that the electrostatic black hole solution is of a correct type I given by (5.129)-(5.133), comparing it to the classification done in [12]. It would be nice to test the algebraic classification on other explicit solutions.

All these unanswered problems remain as suitable topics for our future work.

# Bibliography


[1] K. Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **7** (1916) 189-196.

[2] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003).

[3] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, 2009).

[4] G. T. Horowitz, ed., *Black holes in higher dimensions* (Cambridge University Press, Cambridge, 2012).

[5] A. Staruszkiewicz, Gravitation theory in three-dimensional space, Acta Physica Polonica **6** (1963) 735-740.

[6] S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, Cambridge, 2003).

[7] M. Bañados, C. Teitelboim and J. Zanelli, The Black Hole in Three-Dimensional Space-Time, Phys. Rev. Lett. **69** (1992) 1849.

[8] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the (2+1) black hole, Phys. Rev. D **48** (1993) 1506.

[9] C. Martinez, C. Teitelboim and J. Zanelli, Charged rotating black hole in three space-time dimensions, Phys. Rev. D **61** (2000) 104013.

[10] R. Emparan, G. T. Horowitz and R. C. Myers, Exact description of black holes on branes, JHEP **01** (2000) 007.

[11] G. Bernardi de Freitas and H. S. Reall, Algebraically special solutions in AdS/CFT, JHEP **06** (2014) 148.

[12] A. A. García–Díaz, *Exact Solutions in Three-Dimensional Gravity* (Cambridge University Press, Cambridge, 2017).

[13] J. Podolský, R. Švarc, and H. Maeda, All solutions of Einstein's equations in 2+1 dimensions: Λ-vacuum, pure radiation, or gyratons, Class. Quantum Grav. **36** (2019) 015009 (31pp).

[14] D. D. K. Chow, C. N. Pope and E. Sezgin, Kundt spacetimes as solutions of topological massive gravity, Class. Quantum Grav. **27** (2010) 105002 (19pp).

[15] J. R. Gott and M. Alpert, General relativity in a (2+1)-dimensional space-time, Gen. Rel. Grav. **16** (1984) 243.

[16] D. S. Krongos and C. G. Torre, Rainich conditions in (2 + 1)-dimensional gravity, J. Math. Phys. (N.Y.) **58** (2017) 012501.

[17] P. Bueno, P. A. Cano, J. Moreno and G. van der Velde, Regular black holes in three dimensions, Phys. Rev. D **104** (2021) L021501.

[18] J. Podolský and M. Papajčík, All solutions of Einstein–Maxwell equations with a cosmological constant in 2+1 dimensions, Phys. Rev. D **105** (2022) 064004.

[19] R. Milson, A. Coley, V. Pravda, and A. Pravdová, Alignment and algebraically special tensors in Lorentzian geometry, Int. J. Geom. Meth. Mod. Phys. **2** (2005) (41pp)

[20] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[21] J. F. Plebański and S. Hacyan, Some exceptional electrovac type D metrics with cosmological constant, J. Math. Phys. **20** (1979) 1004.

[22] P. Peldan, Unification of gravity and Yang-Mills theory in (2 + 1)-dimensions, Nucl. Phys. **B395** 239 (1993).

[23] J. R. Gott, J. Z. Simon and M. Alpert, General relativity in a (2 + 1)-dimensional spacetime: An electrically charged solution, Gen. Relativ. Gravit. **18** (1986) 1019.

[24] S. Deser and P. O. Mazur, Static solutions in $D = 3$ Einstein–Maxwell theory, Class. Quantum Grav. **2** (1985) L51.

[25] M. A. Melvin, Exterior solutions for electric and magnetic stars in 2 + 1 dimensions, Class. Quantum Grav. **3** (1986) 117.

[26] É. Cotton, Sur les variétés à trois dimensions, Annales de la faculté des sciences de Toulouse (II) **1** (1899) 385-438.

[27] J. A. Schouten, *Ricci-Calculus* (Springer–Verlag, Berlin, 1954).

[28] P. Krtouš and J. Podolský, Asymptotic structure of radiation in higher dimensions, Class. Quantum Grav. **23** (2006) 1603 (14pp).

[29] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, Metric-Affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Repts. **258** (1995) 1 (171pp).

[30] J. W. York Jr., Gravitational Degrees of Freedom and the Initial-Value Problem, Phys. Rev. Lett. **26** (1971) 1656 (3pp)

[31] R. Arnowitt, S. Deser and C. W. Misner *The dynamics of General Relativity.* In: *Gravitation: An Introduction to Current Research,* L. Witten, ed. (Wiley, New York, 1962)

# A. Connection and curvature components in canonical coordinates

The Christoffel symbols for a general Robinson–Trautman and Kundt metric (1.3), using also the relation for the expansion scalar (1.6), are

$$\Gamma^r_{rr} = 0\,, \tag{A.1}$$

$$\Gamma^r_{ru} = -\frac{1}{2}g_{uu,r} + \frac{1}{2}g^{rx}g_{ux,r}\,, \tag{A.2}$$

$$\Gamma^r_{rx} = -\frac{1}{2}g_{ux,r} + \Theta g_{ux}\,, \tag{A.3}$$

$$\Gamma^r_{uu} = \frac{1}{2}\Big[-g^{rr}g_{uu,r} - g_{uu,u} + g^{rx}(2g_{ux,u} - g_{uu,x})\Big]\,, \tag{A.4}$$

$$\Gamma^r_{ux} = \frac{1}{2}\Big[-g^{rr}g_{ux,r} - g_{uu,x} + g^{rx}g_{xx,u}\Big]\,, \tag{A.5}$$

$$\Gamma^r_{xx} = -\Theta g^{rr}g_{xx} - g_{ux||x} + \frac{1}{2}g_{xx,u}\,, \tag{A.6}$$

$$\Gamma^u_{rr} = \Gamma^u_{ru} = \Gamma^u_{rx} = 0\,, \tag{A.7}$$

$$\Gamma^u_{uu} = \frac{1}{2}g_{uu,r}\,, \tag{A.8}$$

$$\Gamma^u_{ux} = \frac{1}{2}g_{ux,r}\,, \tag{A.9}$$

$$\Gamma^u_{xx} = \Theta g_{xx}\,, \tag{A.10}$$

$$\Gamma^x_{rr} = 0\,, \tag{A.11}$$

$$\Gamma^x_{ru} = \frac{1}{2}g^{xx}g_{ux,r}\,, \tag{A.12}$$

$$\Gamma^x_{rx} = \Theta\,, \tag{A.13}$$

$$\Gamma^x_{uu} = \frac{1}{2}\Big[-g^{rx}g_{uu,r} + g^{xx}(2g_{ux,u} - g_{uu,x})\Big]\,, \tag{A.14}$$

$$\Gamma^x_{ux} = \frac{1}{2}\Big[-g^{rx}g_{ux,r} + g^{xx}g_{xx,u}\Big]\,, \tag{A.15}$$

$$\Gamma^x_{xx} = -\Theta g^{rx}g_{xx} + {}^S\Gamma^x_{xx}\,, \tag{A.16}$$

where

$${}^S\Gamma^x_{xx} \equiv \frac{1}{2}g^{xx}g_{xx,x} = -\frac{G_{,x}}{G} \tag{A.17}$$

is the connection on the transverse one-dimensional space spanned by the spatial coordinate $x$. The nonzero components of the Riemann tensor are

$$R_{rxrx} = -\Big(\Theta_{,r} + \Theta^2\Big) g_{xx}\,, \tag{A.18}$$

$$R_{rxru} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r}\,, \tag{A.19}$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} g^{xx} (g_{ux,r})^2\,, \tag{A.20}$$

$$R_{rxux} = \frac{1}{2} g_{ux,r||x} + \frac{1}{4}(g_{ux,r})^2 - g_{xx}\Theta_{,u} - \frac{1}{2}\Theta\Big(g_{xx,u} + g_{xx} g_{uu,r}\Big)\,, \tag{A.21}$$

$$\begin{aligned} R_{ruux} = {} & g_{u[u,x],r} + \frac{1}{4} g^{rx}(g_{ux,r})^2 - \frac{1}{4} g^{xx} g_{xx,u} g_{ux,r} \\ & + \Theta\Big(g_{ux,u} - \frac{1}{2} g_{uu,x} - \frac{1}{2} g_{ux} g_{uu,r}\Big)\,, \end{aligned} \tag{A.22}$$

$$\begin{aligned} R_{uxux} = {} & -\frac{1}{2}(g_{uu})_{||xx} + g_{ux,u||x} - \frac{1}{2} g_{xx,uu} + \frac{1}{4} g^{rr}(g_{ux,r})^2 \\ & - \frac{1}{2} g_{uu,r} e_{xx} + \frac{1}{2} g_{uu,x} g_{ux,r} - \frac{1}{2} g^{rx} g_{xx,u} g_{ux,r} + \frac{1}{4} g^{xx}(g_{xx,u})^2 \\ & - \frac{1}{2}\Theta g_{xx}\Big[g^{rr} g_{uu,r} + g_{uu,u} - g^{rx}(2g_{ux,u} - g_{uu,x})\Big]\,. \end{aligned} \tag{A.23}$$

The components of the Ricci tensor then become

$$R_{rr} = -\Big(\Theta_{,r} + \Theta^2\Big)\,, \tag{A.24}$$

$$R_{rx} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r} + \Big(\Theta_{,r} + \Theta^2\Big) g_{ux}\,, \tag{A.25}$$

$$\begin{aligned} R_{ru} = {} & -\frac{1}{2} g_{uu,rr} + \frac{1}{2} g^{rx} g_{ux,rr} + \frac{1}{2} g^{xx}\Big(g_{ux,r||x} + (g_{ux,r})^2\Big) \\ & - \Theta_{,u} - \frac{1}{2}\Theta\Big(g^{xx} g_{xx,u} + g^{rx} g_{ux,r} + g_{uu,r}\Big)\,, \end{aligned} \tag{A.26}$$

$$\begin{aligned} R_{xx} = {} & -g_{xx}\, g^{rr}\Big(\Theta_{,r} + \Theta^2\Big) + 2g_{xx}\Big(\Theta_{,u} - g^{rx}\Theta_{,x}\Big) + 2g_{ux}\Theta_{,x} - f_{xx} \\ & + \Theta\Big[2g_{ux||x} + 2g_{ux,r} g_{ux} + g_{xx}\Big(g_{uu,r} - 2g^{rx} g_{ux,r}\Big) - 2e_{xx}\Big]\,, \end{aligned} \tag{A.27}$$

$$\begin{aligned} R_{ux} = {} & -\frac{1}{2} g^{rr} g_{ux,rr} - \frac{1}{2} g_{uu,rx} + \frac{1}{2} g_{ux,ru} - \frac{1}{2} g^{rx}\Big[g_{ux,r||x} + (g_{ux,r})^2\Big] \\ & + g^{xx}\Big(\frac{1}{2} g_{ux,r} g_{ux||x} - \frac{1}{2} e_{xx} g_{ux,r}\Big) + g_{ux}\Theta_{,u} \\ & + \Theta\Big[g_{ux} g_{uu,r} - \frac{1}{2}(g_{uu} g_{ux,r} - g_{uu,x}) - g_{ux,u} + \frac{1}{2} g^{rx} g_{ux,r} g_{ux} + \frac{1}{2} g^{rx} g_{xx,u}\Big]\,, \end{aligned} \tag{A.28}$$

$$\begin{aligned} R_{uu} = {} & -\frac{1}{2} g^{rr} g_{uu,rr} - g^{rx} g_{uu,rx} - \frac{1}{2} g^{xx} e_{xx} g_{uu,r} + g^{rx} g_{ux,ru} - \frac{1}{2} g^{xx} g_{xx,uu} \\ & + g^{xx}(g_{ux,u||x} - \frac{1}{2} g_{uu||xx}) + \frac{1}{2}(g^{rr} g^{xx} - g^{rx} g^{rx})(g_{ux,r})^2 + \frac{1}{2} g^{xx} g_{ux,r} g_{uu,x} \\ & + \frac{1}{4}(g^{xx} g_{xx,u})^2 + \frac{1}{2}\Theta\Big[-g^{rx}(2g_{ux,u} - g_{uu,x} - g_{ux} g_{uu,r}) + g_{uu} g_{uu,r} - g_{uu,u}\Big]\,, \end{aligned} \tag{A.29}$$

with the Ricci scalar equal to

$$R = g_{uu,rr} - 2g^{rx}g_{ux,rr} - 2g^{xx}g_{ux,r||x} - \frac{3}{2}g^{xx}(g_{ux,r})^2$$
$$+ 2\Theta_{,r}\, g_{uu} + 4\Theta_{,u} + 2\Theta^2 g_{uu} + \Theta(2g_{uu,r} + 2g^{rx}g_{ux,r} + 2g^{xx}g_{xx,u})\,. \qquad \text{(A.30)}$$

The symbol $_{||}$ denotes the covariant derivative with respect to the one-dimensional transverse space, that is

$$g_{ux||x} = g_{ux,x} - g_{ux}\,{}^S\Gamma^x_{xx}\,, \qquad \text{(A.31)}$$
$$g_{ux,r||x} = g_{ux,rx} - g_{ux,r}\,{}^S\Gamma^x_{xx}\,, \qquad \text{(A.32)}$$
$$g_{ux,u||x} = g_{ux,ux} - g_{ux,u}\,{}^S\Gamma^x_{xx}\,, \qquad \text{(A.33)}$$
$$(g_{uu})_{||xx} = g_{uu,xx} - g_{uu,x}\,{}^S\Gamma^x_{xx}\,, \qquad \text{(A.34)}$$

where $e_{xx}$ and $f_{xx}$ are convenient shorthand defined as

$$e_{xx} \equiv g_{ux||x} - \frac{1}{2}g_{xx,u}\,, \qquad \text{(A.35)}$$
$$f_{xx} \equiv g_{ux,r||x} + \frac{1}{2}(g_{ux,r})^2\,. \qquad \text{(A.36)}$$

# B. Published article in Physical Review D

PHYSICAL REVIEW D **105,** 064004 (2022)

## All solutions of Einstein-Maxwell equations with a cosmological constant in 2+1 dimensions

Jiří Podolský* and Matúš Papajčík†
*Charles University, Faculty of Mathematics and Physics, Institute of Theoretical Physics, V Holešovičkách 2, 18000 Prague 8, Czech Republic*

(Received 27 August 2021; accepted 16 December 2021; published 2 March 2022)

We present a *general* solution of the coupled Einstein-Maxwell field equations (without the source charges and currents) in three spacetime dimensions. We also admit any value of the cosmological constant. The whole family of such Λ-electrovacuum local solutions splits into two distinct subclasses, namely the nonexpanding Kundt class and the expanding Robinson-Trautman class. While the Kundt class only admits electromagnetic fields which are aligned along the geometrically privileged null congruence, the Robinson-Trautman class admits both aligned and also more complex nonaligned Maxwell fields. We derive all the metric and Maxwell field components, together with explicit constraints imposed by the field equations. We also identify the most important special spacetimes of this type, namely the coupled gravitational-electromagnetic waves and charged black holes.

DOI: 10.1103/PhysRevD.105.064004

### I. INTRODUCTION

Recently, in paper [1] we derived the most general solution of the Einstein equations with a cosmological constant Λ and also an aligned pure radiation matter field (possibly gyrating null dust/particles) in three spacetime dimensions. Here we extend this study to another important nonvacuum case, which is the presence of an electromagnetic field. In fact, we explicitly derive *all* solutions of the Einstein-Maxwell field equations with any value of Λ.

For many decades, the $2+1$-dimensional Einstein gravity has attracted a great deal of attention. The main reason is that such gravity theory is mathematically simpler than standard general relativity because the number of independent components of the curvature tensor is much lower. In fact, the Weyl tensor identically vanishes, and the Riemann and Ricci tensors have the same number of components. Consequently, there is *no classic dynamical degree of freedom in* $2+1$ *spacetimes*. The Ricci tensor—directly given by the Einstein field equations—fully determines the *local curvature* of the spacetime. This implies that a general *vacuum* solution of Einstein's equations is just the maximally symmetric Minkowski, de Sitter (dS), or anti–de Sitter (AdS) spacetime for $\Lambda = 0$, $\Lambda > 0$, or $\Lambda < 0$, respectively.

Despite such local simplicity/triviality of the $2+1$ gravity theory, it can serve as a very useful playground for various investigations, ranging from the black hole properties and cosmology to high-energy physics and quantum gravity. While the Einstein equations determine the spacetime locally, there can be *global topological degrees of freedom* reflected in the appropriate domains of the coordinates employed: It is possible to construct globally different geometries from locally identical spacetimes by various identifications. In the context of black holes, this has been successfully used for construction of famous Bañados-Teitelboim-Zanelli (BTZ)-type solutions with horizons when $\Lambda < 0$ by performing nontrivial identifications of the local AdS vacuum spacetime, pure radiation solutions, or spacetimes with electromagnetic field [2–4]. The corresponding topological degrees of freedom play a crucial role in quantum gravity models [5]. However, it is still not clear if they represent all possible nonvacuum spacetimes. It is thus desirable to obtain and investigate more general exact solutions in the presence of matter.

Many exact spacetimes in $2+1$-dimensional Einstein gravity have already been found. They are nicely summarized, classified, and described in a helpful comprehensive catalog [6]. Such solutions were found in a great number of works by making various specific assumptions on their symmetry, algebraic structure, or other geometrical or physical constraints. A general study of solutions of $2+1$-dimensional Einstein-Maxwell theory using the Rainich geometrization of the electromagnetic field was presented in [7]. Using a different approach, in this paper we solve the Einstein-Maxwell field equations generically, *without making any assumption*. In fact, we systematically derive all possible such spacetimes, extending and generalizing previously known exact electrovacuum solutions.

Specifically, in Sec. II we recall the key result of [1] that (virtually) all $2+1$ geometries belong either to the family

*podolsky@mbox.troja.mff.cuni.cz
†MatusPapajcik@centrum.sk

 

of (nonexpanding) Kundt spacetimes or to the family of (expanding) Robinson-Trautman spacetimes. We also present the canonical metric form and the natural null triad. The related Appendix contains the corresponding Christoffel symbols and all components of the Riemann and Ricci tensors. In Sec. III we present the most general electromagnetic 2-form field in $2+1$ gravity, together with its dual 1-form, the equivalent Newman-Penrose scalars, and the energy-momentum tensor. In Sec. IV we formulate the (source-free) Einstein-Maxwell field equations with $\Lambda$, expressed in a simple form. Section V contains an explicit step-by-step integration of these field equations in the Kundt case, while Sec. VI contains an analogous procedure for the complementary Robinson-Trautman case. In both cases, the electromagnetic field is aligned with the privileged null direction of the gravitational field. The resulting complete families of such spacetimes are summarized in Secs. V H and VI H, respectively. The distinct family of Robinson-Trautman geometries with nonaligned electromagnetic fields is presented in Sec. VII, with a specific particular solution obtained in Sec. VII F. Final summary and further remarks can be found in concluding Sec. VIII.

## II. ALL GEOMETRIES AND THEIR CANONICAL FORM IN 2+1 GRAVITY

In Sec. II of our previous work [1], we investigated general three-dimensional Lorentzian spacetimes $(\mathcal{M}, g_{ab})$ with the metric signature $(++-)$. We proved the uniqueness theorem, namely that the only possible such spacetimes are either *expanding geometries* of the *Robinson-Trautman type* (with $\Theta \neq 0$) or *nonexpanding geometries* of the *Kundt type* (with $\Theta = 0$). They are necessarily twist-free and shear-free; see Theorem 1 in [1] (this observation was already made in [8]).

In a $C^1$ spacetime there exists a *geodesic null* vector field $\mathbf{k}$ (defined as a tangent vector of null geodesics at any point), which in $D=3$ is equivalent to *hypersurface-orthogonality*; see Theorem 2 in [1]. Recall that the *expansion* $\Theta$ is the only nontrivial optical scalar,

$$\Theta = \rho \equiv k_{a;b} m^a m^b, \tag{1}$$

which characterizes the properties of a null congruence generated by $\mathbf{k}$, in a triad $\mathbf{e}_I \equiv \{\mathbf{k}, \mathbf{l}, \mathbf{m}\}$ of two null vectors $\mathbf{k}$, $\mathbf{l}$ and one spatial vector $\mathbf{m}$, normalized as

$$\mathbf{k}\cdot\mathbf{l} = -1, \qquad \mathbf{m}\cdot\mathbf{m} = 1. \tag{2}$$

In [1], we also introduced *canonical coordinates* $\{r, u, x\}$ for all Robinson-Trautman and Kundt metrics; see Theorem 3,

$$\begin{aligned} \mathrm{d}s^2 = {} & g_{xx}(r,u,x)\mathrm{d}x^2 + 2g_{ux}(r,u,x)\mathrm{d}u\mathrm{d}x \\ & - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2. \end{aligned} \tag{3}$$

These coordinates are adapted to their unique geometry, namely $r$ is an affine parameter along the null congruence generated by $\mathbf{k}$, the coordinate $u$ labels null hypersurfaces (such that $k_a \propto u_{,a}$) which naturally foliate the spacetimes, and the spatial coordinate $x$ spans the one-dimensional "transverse" subspace with constant $u$ and $r$.

It is also convenient to recall that the nonvanishing contravariant metric components $g^{ab}$ are

$$\begin{aligned} & g^{xx} = 1/g_{xx}, \qquad g^{ur} = -1, \qquad g^{rx} = g_{ux}/g_{xx}, \\ & g^{rr} = -g_{uu} + g_{ux}^2/g_{xx}, \end{aligned} \tag{4}$$

equivalent to the inverse relations

$$\begin{aligned} & g_{xx} = 1/g^{xx}, \qquad g_{ur} = -1, \qquad g_{ux} = g_{xx}g^{rx}, \\ & g_{uu} = -g^{rr} + g_{xx}(g^{rx})^2. \end{aligned} \tag{5}$$

The most natural choice of the null triad frame $\{\mathbf{k}, \mathbf{l}, \mathbf{m}\}$ satisfying (2) is

$$\mathbf{k} = \partial_r, \qquad \mathbf{l} = \frac{1}{2}g_{uu}\partial_r + \partial_u, \qquad \mathbf{m} = \frac{1}{\sqrt{g_{xx}}}(g_{ux}\partial_r + \partial_x). \tag{6}$$

A direct calculation for the metric (3) reveals that $k_{a;b} = \frac{1}{2}g_{ab,r}$. An explicit form of the expansion scalar (1) thus becomes $\Theta = k_{x;x}m^x m^x$, implying an important relation:

$$g_{xx,r} = 2\Theta g_{xx}, \tag{7}$$

For our next investigation it seems convenient to introduce a new function $G(r,u,x)$, which *fully* encodes the spatial metric function $g_{xx} > 0$ via the simple relation

$$G \equiv \frac{1}{\sqrt{g_{xx}}} \Leftrightarrow g_{xx} = G^{-2}. \tag{8}$$

The key relation (7) then takes the form

$$\Theta = -(\ln G)_{,r}. \tag{9}$$

Now it immediately follows that for vanishing expansion, $\Theta = 0$, the function $G$ and thus also the spatial metric $g_{xx}(u,x)$ must be *independent of the coordinate r*. It yields the *Kundt class* of nonexpanding, twist-free, and shear-free geometries [9–13]. The complementary case $\Theta \neq 0$ gives the expanding *Robinson-Trautman class* of geometries [10,11,13–18], as summarized in Theorem 4 of our work [1].

The Christoffel symbols and all coordinate components of the Riemann and Ricci curvature tensors for the general metric (3), calculated using the relation (7), are listed in the Appendix.

### III. GENERIC ELECTROMAGNETIC FIELD IN 2+1 GRAVITY

The aim of this work is to systematically investigate all possible gravitational and electromagnetic fields in $2+1$ dimensions, solving the coupled Einstein-Maxwell field equations.

Based on the results summarized in previous Sec. II, all such spacetimes can be conveniently written in the canonical coordinates $\{r, u, x\}$ for the general metric (3). Consequently, *generic electromagnetic field* takes the form of an antisymmetric $3\times 3$ Maxwell tensor

$$F_{ab} = \begin{pmatrix} 0 & F_{ru} & F_{rx} \\ -F_{ru} & 0 & F_{ux} \\ -F_{rx} & -F_{ux} & 0 \end{pmatrix}, \tag{10}$$

which is equivalent to considering the 2-form $\mathbf{F} = \frac{1}{2}F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$, that is explicitly

$$\mathbf{F} = F_{ru}\mathrm{d}r\wedge\mathrm{d}u + F_{rx}\mathrm{d}r\wedge\mathrm{d}x + F_{ux}\mathrm{d}u\wedge\mathrm{d}x. \tag{11}$$

The field has *only three independent components*. These can be obtained from the electromagnetic potential 1-form $\mathbf{A} = A_a\mathrm{d}x^a$ by the standard relation

$$\mathbf{F} = \mathrm{d}\mathbf{A}. \tag{12}$$

Using (4), the corresponding contravariant components $F^{ab} \equiv g^{ac}g^{bd}F_{cd}$ read

$$F^{ru} = -\frac{F_x}{g_{xx}}, \qquad F^{rx} = \frac{F_u}{g_{xx}}, \qquad F^{ux} = -\frac{F_r}{g_{xx}}, \tag{13}$$

where the useful functions are

$$F_r \equiv F_{rx}, \tag{14}$$

$$F_x \equiv g_{xx}F_{ru} - g_{ux}F_{rx}, \tag{15}$$

$$F_u \equiv g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}. \tag{16}$$

In fact, these three functions are directly related to the components of the *dual Maxwell field 1-form* ${}^*\mathbf{F} = {}^*F_a\mathrm{d}x^a$ defined using the Hodge star operator,

$${}^*F^a \equiv \frac{1}{2}\omega^{abc}F_{bc}, \quad \text{where } \omega^{abc} = \frac{1}{\sqrt{-g}}\epsilon^{abc}. \tag{17}$$

Here $g$ denotes the *determinant of the metric* $g_{ab}$, while $\epsilon^{abc}$ is the completely antisymmetric Levi-Civita symbol, for which we employ the convention that $\epsilon^{abc} = \epsilon_{abc} \equiv +1$ if $abc$ is an even permutation of $rux$, it is $-1$ for odd permutation of $rux$, and 0 otherwise. For the metric (3) we immediately get

$$-g = g_{xx} \equiv G^{-2}, \tag{18}$$

and in view of (10) we obtain

$${}^*F^r = GF_{ux}, \qquad {}^*F^u = -GF_{rx}, \qquad {}^*F^x = GF_{ru}. \tag{19}$$

Using (14)–(16), the corresponding covariant components ${}^*F_a = g_{ab}{}^*F^b$ are

$${}^*F_a = GF_a, \tag{20}$$

so that the dual 1-form Maxwell field reads

$${}^*\mathbf{F} = G(F_r\mathrm{d}r + F_u\mathrm{d}u + F_x\mathrm{d}x). \tag{21}$$

For completeness let us also recall the inverse relation to (17),

$$F_{ab} = -\omega_{abc}{}^*F^c \quad \text{where } -\omega_{abc} = \sqrt{-g}\epsilon_{abc} = G^{-1}\epsilon_{abc}. \tag{22}$$

Next, it is necessary to evaluate the *electromagnetic invariants*

$$F^2 \equiv F_{ab}F^{ab}, \qquad {}^*F^2 \equiv {}^*F_a{}^*F^a. \tag{23}$$

A direct evaluation yields

$$F^2 = -2{}^*F^2 = -2G^2(g_{uu}F_{rx}^2 + 2F_{rx}(F_{ux} - g_{ux}F_{ru}) + g_{xx}F_{ru}^2). \tag{24}$$

Moreover, $F_{ab}{}^*F^a{}^*F^b = 0$ due to the symmetry reasons.

Similarly as for general relativity in $D=4$, it is convenient to define *Newman-Penrose scalars of the Maxwell field* by its three independent projections onto the frame (6),

$$\phi_0 \equiv F_{ab}k^a m^b,$$
$$\phi_1 \equiv F_{ab}k^a l^b,$$
$$\phi_2 \equiv F_{ab}m^a l^b. \tag{25}$$

Explicit calculation reveals that

$$\phi_0 = GF_{rx} = GF_r, \tag{26}$$

$$\phi_1 = F_{ru} = G^2(F_x + g_{ux}F_r), \tag{27}$$

$$\phi_2 = G\left(g_{ux}F_{ru} - F_{ux} - \frac{1}{2}g_{uu}F_{rx}\right) = G\left(F_u + \frac{1}{2}g_{uu}F_r\right), \tag{28}$$

so that the invariant can be expressed as

$$\frac{1}{2}F^2 = 2\phi_0\phi_2 - \phi_1^2. \tag{29}$$

These scalars have *distinct boost weights* $+1$, $0$, $-1$, respectively, and can be used for invariant *algebraic classification* of the electromagnetic field [13], based on its (non-)alignment with the geometrically privileged null vector field $\mathbf{k} = \partial_r$ of the metric. By definition the field is *aligned* if its component with the highest boost weight vanishes. From (26) we immediately observe that

$$\text{electromagnetic field is aligned with } \mathbf{k} \Leftrightarrow \phi_0 = 0 \Leftrightarrow F_{rx} = 0 \Leftrightarrow F_r = 0. \tag{30}$$

It can also be shown that this is equivalent to the special property of the field, namely

$$F_{ab}k^b = \mathcal{N}k_a. \tag{31}$$

Such an aligned field has just two components, namely $\phi_1 = F_{ru}$ and $\phi_2 = G(g_{ux}F_{ru} - F_{ux})$, and $F^2 = -2\phi_1^2$. When $\phi_1 = 0 \Leftrightarrow F_x = 0$, the field is *null*. When $\phi_2 = 0 \Leftrightarrow F_u = 0$, it is *non-null*.

In the case when the electromagnetic field is *both aligned and null*, the invariant vanishes, $F^2 = 0$. This describes *purely radiative field*, i.e., a propagating electromagnetic wave characterized by the only nonvanishing component $F_{ux}$.

There is a freedom in the choice of the frame normalized as (2), given by the local *Lorentz transformations*. It consists of a boost $\mathbf{k}' = B\mathbf{k}$, $\mathbf{l}' = B^{-1}\mathbf{l}$ which determines the distinct boost weights $+1$, $0$, $-1$ of (25), respectively. The second Lorentz transformation is a null rotation with fixed $\mathbf{k}$ of the form

$$\mathbf{k}' = \mathbf{k}, \qquad \mathbf{l}' = \mathbf{l} + \sqrt{2}L\mathbf{m} + L^2\mathbf{k}, \qquad \mathbf{m}' = \mathbf{m} + \sqrt{2}L\mathbf{k}. \tag{32}$$

There is also an analogous null rotation with fixed $\mathbf{l}$ which changes $\mathbf{k}$. However, in our case the direction of $\mathbf{k}$ is geometrically privileged (being twist-free and shear-free). Only (32) thus needs to be considered. It is easy to prove that the Maxwell scalars (25) transform as

$$\begin{aligned}
\phi_0' &= \phi_0,\\
\phi_1' &= \phi_1 + \sqrt{2}L\phi_0,\\
\phi_2' &= \phi_2 + \sqrt{2}L\phi_1 + L^2\phi_0.
\end{aligned} \tag{33}$$

Of course, the expression (29) is invariant since $2\phi_0'\phi_2' - \phi_1'^2 = 2\phi_0\phi_2 - \phi_1^2$.

Finally, we need to evaluate the *energy-momentum tensor* for a generic electromagnetic field which (in any dimension, including $D = 3$) is defined as

$$T_{ab} \equiv \frac{\kappa_0}{8\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2\right), \tag{34}$$

where $\kappa_0 > 0$ is a constant depending on the choice of the physical units. Interestingly, in arbitrary dimension $D \geq 3$ the Maxwell field *satisfies all the standard energy conditions*; see Proposition 21 in [19].

A straightforward (but somewhat lengthy) calculation reveals that

$$\begin{aligned}
\frac{8\pi}{\kappa_0}T_{rr} &= G^2F_{rx}^2,\\
\frac{8\pi}{\kappa_0}T_{rx} &= G^2F_{rx}(g_{xx}F_{ru} - g_{ux}F_{rx}),\\
\frac{8\pi}{\kappa_0}T_{ru} &= \frac{1}{2}G^2(g_{xx}F_{ru}^2 - g_{uu}F_{rx}^2),\\
\frac{8\pi}{\kappa_0}T_{xx} &= -F_{rx}(g_{ux}F_{ru} + F_{ux}) + \frac{1}{2}G^2(2g_{ux}^2 - g_{xx}g_{uu})F_{rx}^2\\
&\quad + \frac{1}{2}g_{xx}F_{ru}^2,\\
\frac{8\pi}{\kappa_0}T_{ux} &= \frac{1}{2}G^2[g_{ux}g_{uu}F_{rx}^2 - 2g_{xx}g_{uu}F_{ru}F_{rx}\\
&\quad + g_{xx}F_{ru}(g_{ux}F_{ru} - 2F_{ux})],\\
\frac{8\pi}{\kappa_0}T_{uu} &= \frac{1}{2}G^2[2F_{ux}^2 + 2g_{uu}F_{rx}F_{ux} + g_{uu}^2F_{rx}^2 - 4g_{ux}F_{ru}F_{ux}\\
&\quad - 2g_{ux}g_{uu}F_{rx}F_{ru} + (2g_{ux}^2 - g_{xx}g_{uu})F_{ru}^2],
\end{aligned} \tag{35}$$

and the corresponding trace $T \equiv g^{ab}T_{ab}$ is

$$\frac{8\pi}{\kappa_0}T = G^2F_{rx}(g_{ux}F_{ru} - F_{ux}) - \frac{1}{2}G^2(g_{xx}F_{ru}^2 + g_{uu}F_{rx}^2). \tag{36}$$

Now, it is a nice fact that, by combining (35) with (36) as $T_{ab} - Tg_{ab}$, the result for *all components* can be written in a *simple factorized form* as

$$\frac{8\pi}{\kappa_0}(T_{ab} - Tg_{ab}) = G^2F_aF_b, \tag{37}$$

in terms of the functions $F_a$ encoding the electromagnetic field, which we have introduced in (14)–(16).

## IV. EINSTEIN-MAXWELL FIELD EQUATIONS WITH Λ

Having identified all three-dimensional Lorentzian geometries—which can be written in the canonical form (3)—and also the generic form of the electromagnetic field (10) with the energy-momentum tensor of the form (35) implying (37), we can now apply the field equations.

Einstein's equations are $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, in which we also admit a nonvanishing *cosmological constant* $\Lambda$. Their trace is $R = 2(3\Lambda - 8\pi T)$, so that the equations can be put into the form $R_{ab} = 2\Lambda g_{ab} + 8\pi(T_{ab} - T g_{ab})$. For the generic electromagnetic field $F_{ab}$ we have derived the nice relation (37), and thus the *Einstein field equations* in $2+1$ gravity with $\Lambda$, coupled to an electromagnetic field, are simply

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{38}$$

where the functions $F_a$ are defined by (14)–(16). Expressed in terms of the dual Maxwell field ${}^*\mathbf{F}$ 1-form components [see (21) and (20)] these are even simpler, namely

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 {}^*F_a {}^*F_b. \tag{39}$$

In addition to these equations for the gravitational field represented by the metric $g_{ab}$, we must also satisfy the *Maxwell equations* $\mathrm{d}{}^*\mathbf{F} = 4\pi {}^*\mathbf{J}$ and $\mathrm{d}\mathbf{F} = 0$ for the electromagnetic field $F_{ab}$. In the *absence of electric charges and currents*, in components these read $F^{ab}{}_{;b} = 0$, $F_{[ab;c]} = 0$. They are equivalent to

$$(\sqrt{-g} F^{ab})_{,b} = 0, \tag{40}$$

$$F_{[ab,c]} = 0, \tag{41}$$

where, using (18),

$$\sqrt{-g} = \sqrt{g_{xx}} = G^{-1}. \tag{42}$$

Recall also that the source-free Maxwell equation $\mathrm{d}{}^*\mathbf{F} = 0$, which is equivalent to (40), in components reads ${}^*F_{[a,b]} = 0$. In view of (20), it can be directly written as

$$(G F_a)_{,b} = (G F_b)_{,a}. \tag{43}$$

Our task is to completely integrate the coupled system of the field equations (38) and (40), (41) [or, equivalently, (43) instead of (40)] in $2+1$ dimensions for (3) and (10), both for the nonexpanding Kundt spacetimes (Sec. V) and the expanding Robinson-Trautman spacetimes (Sec. VI and Sec. VII). Explicit components of the Ricci tensor $R_{ab}$, which enter (38), for these twist-free and shear-free geometries are given by Eqs. (A24)–(A29) in the Appendix.

### A. Einstein field equations with a massless scalar field

Let us also remark that in three dimensions there is a relation between the Einstein-Maxwell system (39) and the Einstein gravity equations (minimally) coupled to a *massless scalar field* $\Phi$ such that

$$g^{ab} \Phi_{;ab} = 0. \tag{44}$$

Indeed, the corresponding energy-momentum tensor reads

$$T_{ab} \equiv \Phi_{,a} \Phi_{,b} - \frac{1}{2} g_{ab} \Phi_{,c} \Phi^{,c}, \tag{45}$$

implying the trace $T = -\frac{1}{2} \Phi_{,c} \Phi^{,c}$, so that the Einstein equations $R_{ab} = 2\Lambda g_{ab} + 8\pi(T_{ab} - T g_{ab})$ become

$$R_{ab} = 2\Lambda\, g_{ab} + 8\pi\, \Phi_{,a} \Phi_{,b}. \tag{46}$$

With the identification

$$\Phi_{,a} \equiv \sqrt{\frac{\kappa_0}{8\pi}}\, {}^*F_a, \tag{47}$$

this system of equations is clearly equivalent to (39). The dual Maxwell field 1-form is thus

$${}^*\mathbf{F} = \sqrt{\frac{8\pi}{\kappa_0}}\, \mathrm{d}\Phi. \tag{48}$$

## V. ALL KUNDT SOLUTIONS

In this section, we explicitly perform a step-by-step integration of the field equations in the nonexpanding case $\Theta = 0$, which defines the Kundt family of spacetimes. Recall a consequence of (8) and (9), namely that the function $G$ is now $r$ independent. It can be renamed as $G(u,x) \equiv P(u,x)$. Also, the one-dimensional spatial metric $g_{xx} = G^{-2}$ must be $r$ independent, that is

$$g_{xx} \equiv P^{-2}(u,x). \tag{49}$$

Of course, $g^{xx} = P^2$. Now, we will employ the Einstein field equations (38), which for the Kundt spacetimes take the form

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 P^2 F_a F_b. \tag{50}$$

### A. Integration of $R_{rr} = \kappa_0 P^2 F_r^2$

In view of Eq. (A24), $R_{rr} = 0$ for $\Theta = 0$. Therefore, this Einstein equation immediately requires $F_r = 0$, that is

$$F_{rx} = 0. \tag{51}$$

It means that, inevitably, *any electromagnetic field in the* $2+1$ *Kundt spacetimes must be aligned with* $\mathbf{k} = \partial_r$. Such fields are fully described by the functions

$$F_r = 0, \qquad F_x = P^{-2} F_{ru}, \qquad F_u = g_{ux} F_{ru} - F_{ux}. \tag{52}$$

There are only *two* possible components of the electromagnetic field, namely $F_{ru}$ and $F_{ux}$.

In fact, this result is analogous to the situation in standard $3+1$ general relativity, for which it is well known that (due to the Mariot-Robinson theorem) any Einstein-Maxwell field (including a cosmological constant $\Lambda$) in the Kundt class of geometries must be aligned; see the introductions to Chapter 31 of [10] and Chapter 18 of [11].

### B. Integration of $R_{rx} = \kappa_0 P^2 F_r F_x$

The Ricci tensor component (A25) for $\Theta = 0$ reduces to $R_{rx} = -\frac{1}{2} g_{ux,rr}$. Since $F_r = 0$, we obtain a general solution of this Einstein equation:

$$g_{ux} = e(u,x) + f(u,x)r, \qquad (53)$$

where $e$ and $f$ are arbitrary functions of $u$ and $x$. In view of Eqs. (4) and (49), the corresponding contravariant component of the Kundt metric is

$$g^{rx} = P^2[e(u,x) + f(u,x)r]. \qquad (54)$$

### C. Integration of $R_{ru} = -2\Lambda + \kappa_0 P^2 F_r F_u$

Using Eqs. (49) and (53), the Ricci tensor component (A26) is $R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} P^2 (f_{||x} + f^2)$, where

$$f_{||x} \equiv f_{,x} + \frac{P_{,x}}{P} f \Leftrightarrow P f_{||x} \equiv (Pf)_{,x}. \qquad (55)$$

Actually, the symbol $_{||}$ denotes the covariant derivative (of a 1-form $f$) related to the spatial metric $g_{xx}$ on the one-dimensional "transverse" subspace with constant $u$ and $r$, namely $f_{||x} = f_{,x} - {}^S\Gamma^x_{xx} f$, where ${}^S\Gamma^x_{xx} \equiv \frac{1}{2} g^{xx} g_{xx,x}$ is the corresponding Christoffel symbol (see the Appendix). Although this notation seems to be superficial here, we employ it in order to see the relation to our previous studies [20–22] of Kundt and Robinson-Trautman spacetimes in any higher dimension $D \geq 4$ where this geometric notation plays a key role.

Because $F_r = 0$, the corresponding Einstein equation thus simplifies, and can be integrated to

$$g_{uu} = a(u,x) + b(u,x)r + c(u,x)r^2, \qquad (56)$$

where $a(u,x)$ and $b(u,x)$ are *arbitrary* functions, while

$$c(u,x) \equiv 2\Lambda + \frac{1}{2} P^2 (f_{||x} + f^2). \qquad (57)$$

### D. Integration of the Maxwell equations

The crucial $r$ dependence of all metric functions for the $2+1$ Kundt spacetimes is thus determined. In general, $g_{uu}$ is quadratic, $g_{ux}$ is linear, and $g_{xx} \equiv P^{-2}(u,x)$ is independent of $r$. Now, applying the Maxwell equations (40), (41) with $\sqrt{-g} = P^{-1}$, we will determine the $r$ dependence of the electromagnetic field.

In the present setting, there are only four independent Maxwell equations, namely three components of $(\sqrt{-g} F^{ab})_{,b} = 0$ and just one component of $F_{[ab,c]} = 0$. Because (13) with (52) implies

$$F^{ru} = -F_{ru}, \qquad F^{rx} = P^2 (g_{ux} F_{ru} - F_{ux}), \qquad F^{ux} = 0, \qquad (58)$$

these four equations for the electromagnetic field have the form

$$F_{ru,r} = 0, \qquad (59)$$

$$(g_{ux} F_{ru} - F_{ux})_{,r} = 0, \qquad (60)$$

$$(P(g_{ux} F_{ru} - F_{ux}))_{,x} = \left(\frac{F_{ru}}{P}\right)_{,u}, \qquad (61)$$

$$F_{ux,r} + F_{ru,x} = 0. \qquad (62)$$

These equations can be completely solved for the two nontrivial components $F_{ru}$ and $F_{ux}$. Starting with (59), we immediately obtain that

$$F_{ru} = Q(u,x), \qquad (63)$$

where $Q(u,x)$ is an arbitrary function independent of $r$. By employing (62), we thus get

$$F_{ux} = -Q_{,x} r - \xi(u,x), \qquad (64)$$

where $\xi(u,x)$ is another arbitrary function. Equation (60) gives the constraint

$$Q_{,x} = -fQ, \qquad (65)$$

and (61) reduces to the equation

$$(P(eQ + \xi))_{,x} = \left(\frac{Q}{P}\right)_{,u}. \qquad (66)$$

To summarize, by integrating all the Maxwell equations we obtained explicit components of the (necessarily aligned) *electromagnetic field in any* $2+1$ *Kundt spacetime*,

$$F_{rx} = 0, \qquad F_{ru} = Q, \qquad F_{ux} = fQr - \xi, \qquad (67)$$

where the functions $Q(u,x)$ and $\xi(u,x)$ are constrained by Eqs. (65), (66). Consequently,

$$F_r = 0, \qquad F_x = P^{-2} Q, \qquad F_u = eQ + \xi, \qquad (68)$$

and, due to (26)–(28),

$$\phi_0 = 0, \qquad \phi_1 = Q, \qquad \phi_2 = P(eQ + \xi). \quad (69)$$

When $\phi_1 = 0 \Leftrightarrow Q = 0$, the field is *null*, and then $\phi_2 = P\xi$. When $\phi_2 = 0 \Leftrightarrow eQ = -\xi$, it is *non-null*, and then $\phi_1 = Q$.

Now, we can integrate the remaining three Einstein equations, which impose the unique relation between the gravitational and electromagnetic field components.

### E. Integration of $R_{xx} = 2\Lambda g_{xx} + \kappa_0 P^2 F_x^2$

For $\Theta = 0$, using Eqs. (A36) and (53), the Ricci tensor component (A27) reduces to $R_{xx} = -f_{,xx} \equiv -(f_{\|x} + \frac{1}{2}f^2)$. The field equation $R_{xx} = 2\Lambda g_{xx} + \kappa_0 P^2(P^{-2}Q)^2 = (2\Lambda + \kappa_0 Q^2)P^{-2}$ implies

$$\kappa_0 Q^2 = -\left[2\Lambda + P^2\left(f_{\|x} + \frac{1}{2}f^2\right)\right]. \quad (70)$$

The electromagnetic field component $F_{ru} \equiv \phi_1 = Q(u,x)$ is thus *explicitly determined by the cosmological constant* $\Lambda$ *and by the metric functions* $P$, $f$ [provided the right-hand side of (70) is non-negative]. It is now convenient to introduce a rescaled form of $f$ entering the metric function $g_{ux} = e + fr$ [see (53)], namely

$$F \equiv P^2 f^2. \quad (71)$$

Then the field equation (70) can be rewritten as

$$P^2(f_{\|x} + f^2) = \frac{1}{2}F - 2\Lambda - \kappa_0 Q^2. \quad (72)$$

We can thus simplify the metric function $g_{uu}$, namely its coefficient $c$ in (56) given by (57), to

$$c(u,x) = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2. \quad (73)$$

At this stage, the most general Kundt solution in $D = 3$ takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2(e + fr)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + \left[a + br + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right]\mathrm{d}u^2, \quad (74)$$

and the Einstein-Maxwell field equation (72) using (55) reads

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0 Q^2\right). \quad (75)$$

### F. Integration of $R_{ux} = 2\Lambda g_{ux} + \kappa_0 P^2 F_u F_x$

Equation (A28) with $\Theta = 0$ for the metric (74) gives $R_{ux} = \frac{1}{2}[f_{,u} - b_{,x} - eP^2(f_{\|x} + f^2) - f(\ln P)_{,u}] - \frac{1}{4}[(F - 2\kappa_0 Q^2)_{,x} + 2fP^2(f_{\|x} + f^2)]r$. Applying (72) and (68), (53), the corresponding field equation $R_{ux} = 2\Lambda g_{ux} + \kappa_0 Q(eQ + \xi) = 2\Lambda e + \kappa_0(eQ^2 + Q\xi) + 2\Lambda fr$ splits into two conditions, resulting from the coefficients for the powers $r^1$ and $r^0$, namely

$$F_{,x} - 2\kappa_0(Q^2)_{,x} + (F - 4\Lambda - 2\kappa_0 Q^2)f = -8\Lambda f, \quad (76)$$

$$f_{,u} - b_{,x} - \left(\frac{1}{2}F - 2\Lambda - \kappa_0 Q^2\right)e - f(\ln P)_{,u} = 4\Lambda e + 2\kappa_0(eQ^2 + Q\xi). \quad (77)$$

Using the field equation (75), Eq. (76) simplifies to $(Q^2)_{,x} = -2Q^2 f$ which is *identically satisfied* due to (65). Only the constraint (77) thus remains, which can be put into the form

$$b_{,x} = f_{,u} - f(\ln P)_{,u} - \frac{1}{2}(F + 4\Lambda + 2\kappa_0 Q^2)e - 2\kappa_0 Q\xi, \quad (78)$$

that is

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + Pe(Pf)_{,x} - 2\kappa_0 Q\xi. \quad (79)$$

This is an explicit expression *determining the metric function* $b(u,x)$.

### G. Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0 P^2 F_u^2$

For $\Theta = 0$ and the Kundt metric (74), using the relation $e_{\|x} \equiv e_{,x} + eP_{,x}/P$ and similar for $f_{\|x}$, $e_{,u\|x}$, $f_{,u\|x}$, $a_{\|xx}$, $b_{\|xx}$ and $c_{\|xx}$ (see the Appendix), the last Ricci tensor component (A29) reads

$$R_{uu} = A + Br + Cr^2, \quad (80)$$

where

$$A = a\left(c - \frac{1}{2}F\right) + P^2\left[-\frac{1}{2}a_{,xx} + \frac{1}{2}a_{,x}\left(f - \frac{P_{,x}}{P}\right) - \frac{1}{2}b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) + (f_{,u} - b_{,x} - ce)e + \left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right) + \frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right], \quad (81)$$

$$B = b\left(c - \frac{1}{2}F - \frac{1}{2}P(Pf)_{,x}\right) + P^2\left[\left(f_{,u} - \frac{1}{2}b_{,x}\right)_{,x} + \left(f_{,u} - \frac{1}{2}b_{,x}\right)\left(f + \frac{P_{,x}}{P}\right) - c\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) - 2e(c_{,x} + fc)\right], \tag{82}$$

$$C = c(c - F) - P^2\left[\frac{1}{2}c_{,xx} + \frac{1}{2}c_{,x}\left(3f + \frac{P_{,x}}{P}\right) + c\left(f_{,x} + \frac{P_{,x}}{P}f + \frac{1}{2}f^2\right)\right]. \tag{83}$$

Due to (56), (68), the corresponding field equation is $R_{uu} = 2\Lambda(a + br + cr^2) + \kappa_0 P^2(eQ + \xi)^2$, which splits into the following three constraints:

$$A = 2\Lambda a + \kappa_0 P^2(eQ + \xi)^2, \tag{84}$$

$$B = 2\Lambda b, \tag{85}$$

$$C = 2\Lambda c. \tag{86}$$

From (73), (75), (65) we easily derive interesting identities for spatial derivatives of $c$,

$$c_{,x} = -fc, \qquad c_{,xx} = (f^2 - f_{,x})c. \tag{87}$$

By using (87), the expression (83) reduces to $C = c[c - \frac{1}{2}F - \frac{1}{2}P(Pf)_{,x}]$, and substituting from (73), (75) we obtain $C = 2\Lambda c$. Equation (86) *is thus identically satisfied*.

Surprisingly, Eq. (85) *is also identically satisfied*. Applying (75), the first term in (82) yields $2\Lambda b$, while the complicated combination of various terms in the square brackets vanishes by using the relations (87), (78), (73) and the field equations (65), (66). Therefore, $B = 2\Lambda b$, which is Eq. (85).

We are thus left with *only one equation*, namely (84). Using (70), (73), (75), and (78), it can be simplified to

$$a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + \frac{P_{,x}}{P}f\right) = -b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right) - Pe^2(Pf)_{,x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\xi^2. \tag{88}$$

This equation *determines the last metric function* $a(u, x)$.

Alternatively, it can be understood as an *explicit expression for the* $\xi(u, x)$ component of the Maxwell field, in terms of the metric functions $P, e, f, a, b$. Such an equation can be expressed in a covariant form as

$$2\kappa_0\xi^2 = -a_{\|xx} + (fa)_{\|x} - b\left(e_{\|x} + \frac{P_{,u}}{P^3}\right) + 2(e_{,u})_{\|x} - P^2e^2f_{\|x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right), \tag{89}$$

where $a_{\|xx} \equiv a_{,xx} + \frac{P_{,x}}{P}a_{,x}$ and $\psi_{\|x} \equiv \psi_{,x} + \psi P_{,x}/P$, for $\psi$ representing $a_{,x}$, $f$, $e$, and $e_{,u}$.

### H. Summary of the Kundt solutions

We have thus solved all the Einstein-Maxwell equations with a cosmological constant $\Lambda$ in $2 + 1$ gravity for the complete Kundt family of nonexpanding spacetimes. The generic gravitational field of this type is

$$g_{xx} = P^{-2}(u, x),$$
$$g_{ux} = e(u, x) + f(u, x)r$$
$$g_{uu} = a(u, x) + b(u, x)r + c(u, x)r^2, \tag{90}$$

where

$$c = \Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2, \tag{91}$$

with

$$F \equiv P^2 f^2, \tag{92}$$

cf. (73), (71),while the electromagnetic field (67) reads

$$F_{rx} = 0,$$
$$F_{ru} = Q(u, x)$$
$$F_{ux} = f(u, x)Q(u, x)r - \xi(u, x). \tag{93}$$

Written explicitly in a compact form,

$$ds^2 = \frac{dx^2}{P^2} + 2(e + fr)du dx - 2du dr + \left(a + br + \left(\Lambda + \frac{1}{4}F - \frac{\kappa_0}{2}Q^2\right)r^2\right)du^2, \tag{94}$$

and

$$\mathbf{F} = Q dr \wedge du + (fQr - \xi)du \wedge dx, \tag{95}$$

corresponding to the potential

$$\mathbf{A} = A_r dr + A_x dx, \tag{96}$$

where, considering (65),

$$A_r \equiv -\int Q\mathrm{d}u, \qquad A_x \equiv r\int fQ\mathrm{d}u - \int \xi\mathrm{d}u. \tag{97}$$

It is now important to recall the *Maxwell scalars* given by (69),

$$\begin{aligned}\phi_0 &= 0,\\ \phi_1 &= Q,\\ \phi_2 &= P(eQ+\xi).\end{aligned} \tag{98}$$

We have thus proved that *all electromagnetic fields in the Kundt spacetimes* in $2+1$ gravity *are necessarily aligned* ($\phi_0 = 0$). Moreover, they split into *two distinct subclasses*:

(i) The case $\phi_1 = 0 \Leftrightarrow Q = 0$: The field is *null*, in which case $\phi_2 = P\xi$ and $F_{ux} = -\xi$, so that

$$\mathbf{F} = -\xi\mathrm{d}u\wedge\mathrm{d}x. \tag{99}$$

(ii) The case $\phi_2 = 0 \Leftrightarrow \xi = -eQ$: The field is *non-null* with only $\phi_1 = Q$, corresponding to

$$\mathbf{F} = Q\mathrm{d}r\wedge\mathrm{d}u + Q(e+fr)\mathrm{d}u\wedge\mathrm{d}x. \tag{100}$$

Notice also that, applying the Lorentz null rotation (32) with fixed $\mathbf{k}$ and the uniquely chosen parameter $L = -\frac{1}{\sqrt{2}}eP$ in (33), the scalars (98) transform to

$$\begin{aligned}\phi'_0 &= 0,\\ \phi'_1 &= Q,\\ \phi'_2 &= P\xi.\end{aligned} \tag{101}$$

Therefore, with respect to the triad with $\mathbf{m}' = \mathbf{m} + \sqrt{2}L\mathbf{k} = P(\partial_x + fr\partial_r)$, the condition for the Maxwell field being non-null is $\phi'_2 = 0 \Leftrightarrow \xi = 0$.

The *two electromagnetic components* $Q$, $\xi$ and the *five metric functions* $P$, $e$, $f$, $a$, $b$ describing the gravitational field are mutually constrained by the following Einstein-Maxwell field equations:

$$Q_{,x} = -fQ, \tag{102}$$

$$(QPe + P\xi)_{,x} = \left(\frac{Q}{P}\right)_{,u}, \tag{103}$$

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}F + \kappa_0 Q^2\right), \tag{104}$$

$$b_{,x} = P\left(\frac{f}{P}\right)_{,u} + Pe(Pf)_{,x} - 2\kappa_0 Q\xi, \tag{105}$$

$$\begin{aligned}a_{,xx} - a_{,x}\left(f - \frac{P_{,x}}{P}\right) - a\left(f_{,x} + \frac{P_{,x}}{P}f\right) &= -b\left(e_{,x} + \frac{P_{,x}}{P}e + \frac{P_{,u}}{P^3}\right) + 2\left(e_{,ux} + \frac{P_{,x}}{P}e_{,u}\right)\\ &\quad - Pe^2(Pf)_{,x} + 2ef\frac{P_{,u}}{P} + 2\left(\frac{P_{,uu}}{P^3} - 2\frac{P_{,u}^2}{P^4}\right) - 2\kappa_0\xi^2;\end{aligned} \tag{106}$$

see Eqs. (65), (66), (75), (79), and (88).

Interestingly, the form of the electromagnetic field (95) and also the same field equations (102)–(106) can *formally* be obtained by setting $D = 3$ in the corresponding equations for *higher-dimensional* Kundt spacetimes with an aligned Maxwell field [12].

Let us now separately discuss *two geometrically distinct subclasses*, namely $f = 0$ and $f \neq 0$.

### 1. The subclass $f = 0$

From (92) it follows that $f = 0 \Leftrightarrow F = 0$, so that Eqs. (102)–(106) considerably simplify to

$$Q_{,x} = 0, \tag{107}$$

$$(QPe + P\xi)_{,x} = \left(\frac{Q}{P}\right)_{,u}, \tag{108}$$

$$\kappa_0 Q^2 = -2\Lambda, \tag{109}$$

$$b_{,x} = -2\kappa_0 Q\xi, \tag{110}$$

$$\begin{aligned}(Pa_{,x})_{,x} &= -b\left((Pe)_{,x} + \frac{P_{,u}}{P^2}\right) + 2(Pe_{,u})_{,x}\\ &\quad + 2\left(\frac{P_{,u}}{P^2}\right)_{,u} - 2\kappa_0 P\xi^2.\end{aligned} \tag{111}$$

In this case, $Q$ *is necessarily a constant*, and $\Lambda \leq 0$ because

$$2\Lambda = -\kappa_0 Q^2. \tag{112}$$

Therefore, the electromagnetic component $\phi_1$ is also independent of $u$ and $x$,

$$F_{ru} = \phi_1 = Q = \sqrt{-\frac{2}{\kappa_0}\Lambda}. \tag{113}$$

Keeping both the functions $P(u,x)$ and $\xi(u,x)$ *arbitrary*, Eq. (108) determines the metric function $e(u,x)$. Moreover, the function $b(u,x)$ is directly determined by the spatial integral of $\xi$ via (110). Finally, integrating (111) we obtain $a(u,x)$.

Thus, we have obtained a complete and explicit family of such electrovacuum Kundt spacetimes in $2+1$ gravity, namely

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2e\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + (a + br + 2\Lambda r^2)\mathrm{d}u^2, \tag{114}$$

and

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u - \xi\mathrm{d}u \wedge \mathrm{d}x. \tag{115}$$

It admits *four physically distinct subcases*:

(i) The case $Q = 0 = \xi$: The electromagnetic field $\mathbf{F}$ *vanishes*, and necessarily $\Lambda = 0$. The metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2e\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + (a + br)\mathrm{d}u^2, \tag{116}$$

where $b(u)$ is independent of $x$. It is a vacuum solution without a cosmological constant, and thus in $2+1$ gravity it must be *flat Minkowski space*. We derived this metric in our previous work [1]; see Eq. (82) with $\mathcal{J} = 0 = \mathcal{N}$ therein.

(ii) The case $Q = 0$: Again, $\Lambda = 0$ and $b = b(u)$, so that the metric has the form (116), but there is now a *radiative (null) electromagnetic field*

$$\mathbf{F} = -\xi\mathrm{d}u \wedge \mathrm{d}x. \tag{117}$$

The amplitude $\xi(u,x)$ must satisfy the field equation (108), which is $(P\xi)_{,x} = 0$. Therefore,

$$\xi(u,x) = \frac{\gamma(u)}{P(u,x)}, \tag{118}$$

where $\gamma(u)$ is *an arbitrary profile function* of the retarded time $u$. Finally, $a(u,x)$ is then obtained by integrating the remaining field equation (111).

(iii) The case $\xi = 0$: The *electromagnetic field is non-null*, and has the form

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u, \tag{119}$$

where $Q$ is a *constant uniquely determined by negative cosmological constant* $\Lambda$ via (113). The electromagnetic field is thus uniform, and positive (or zero) $\Lambda$ is not allowed.

The metric is of the form (114). The field equation (110) implies that $b = b(u)$, while the remaining (108) and (111) reduce to

$$(Pe)_{,x} = -\frac{P_{,u}}{P^2}, \tag{120}$$

$$(Pa_{,x})_{,x} = 2(Pe_{,u})_{,x} - 2(Pe)_{,ux}. \tag{121}$$

The latter can be immediately integrated to

$$a_{,x} = 2e_{,u} - \frac{2}{P}(Pe)_{,u} + \frac{\delta(u)}{P}, \tag{122}$$

where $\delta(u)$ is any function of $u$. After prescribing an arbitrary metric function $P(u,x)$, we obtain $e(u,x)$ by integrating (120), and $a(u,x)$ by integrating (122).

(iv) The general case $Q \neq 0$, $\xi \neq 0$: In the generic case with both the non-null component of the electromagnetic field $Q = \text{const}$ and its null component $\xi(u,x)$, we obtain the superposition (115). The metric reads (114), with a cosmological constant $\Lambda < 0$ [notice that $\Lambda = 0$ implies $Q = 0$ due to (112), while $\Lambda > 0$ is forbidden]. The metric functions $a$ and $b$ are determined by the differential equations (110) and (111), respectively, and there is also the constraint (108) determining $e$.

This family of Kundt spacetimes in $2+1$ gravity can be interpreted as *mutually coupled exact gravitational and electromagnetic waves* [characterized by the functions $a(u,x)$ and $\xi(u,x)$, respectively] which propagate on the *background with* $\Lambda < 0$ *and uniform Maxwell field* (characterized by the constant $Q$). The simplest such background is

$$\mathrm{d}s^2 = \mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}r + 2\Lambda r^2\mathrm{d}u^2, \tag{123}$$

which is the $2+1$ analog of the exceptional electrovacuum type D metric with $\Lambda < 0$ found by Plebański and Hacyan [23]; see also Eq. (7.20) in [11]. Indeed, introducing $\mathcal{U} = 1/(2\Lambda u)$ and $\mathcal{V} = 2(u + 1/(\Lambda r))$, the metric (123) takes the form $\mathrm{d}s^2 = \mathrm{d}x^2 - 2\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V}/(1-\Lambda\mathcal{U}\mathcal{V})^2$ which is clearly the direct-product $E^1 \times \mathrm{AdS}_2$ spacetime.

### 2. The subclass $f \neq 0$

Recalling $F \equiv P^2 f^2$, cf. (92), in this case $F \neq 0$. The Kundt metric takes the general form (94), the aligned electromagnetic field is (95), and the corresponding Einstein-Maxwell field equations are (102)–(106).

By inspecting this system, it is seen that the first three differential equations (102), (103), (104) relate the metric functions $P$, $e$, $f$ and the electromagnetic field components $Q$, $\xi$. Subsequently, the remaining two equations (105) and (106) can be used to evaluate the metric functions $b$ and $a$, respectively.

Starting with (102), we immediately observe that there are *two distinct subcases*:

(i) The case $Q = 0$: The electromagnetic field is *null* (with $\phi_1 = 0$, $\phi_2 = P\xi$),

$$\mathbf{F} = -\xi\mathrm{d}u \wedge \mathrm{d}x. \tag{124}$$

The field equation (102) is identically satisfied, putting no restriction on the function $f$, while (103), (104) reduce to

$$P\xi = \gamma(u), \tag{125}$$

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}(Pf)^2\right). \tag{126}$$

The first equation determines $\xi$, giving the same expression as (118), i.e., $\xi(u,x) = P^{-1}\gamma(u)$, while the second equation can be integrated for the variable $(Pf)$ in terms of the integral of $P^{-1}$, yielding

$$f(u,x) = -2\sqrt{\Lambda}P^{-1}\tan\left[\sqrt{\Lambda}\int P^{-1}\mathrm{d}x\right] \quad \text{for } \Lambda > 0, \tag{127}$$

and the expression for $\Lambda < 0$ is analogous, replacing tan by tanh.

In the final step, the metric functions $b$ and $a$ are obtained by integrating the field equations (105) and (106), respectively.

(ii) The case $Q \neq 0$: In this generic case, the field equation (102) explicitly determines the metric function $f$ in terms of the electromagnetic field component $Q$, which occurs in

$$\mathbf{F} = Q\mathrm{d}r \wedge \mathrm{d}u + (fQr - \xi)\mathrm{d}u \wedge \mathrm{d}x, \tag{128}$$

as

$$f(u,x) = -(\ln Q)_{,x}. \tag{129}$$

However, there is a further constraint given the field equation (104),

$$P(Pf)_{,x} = -\left(2\Lambda + \frac{1}{2}(Pf)^2 + \kappa_0 Q^2\right). \tag{130}$$

Notice that it can also be rewritten as

$$F_{,x} = -f(F + 4\Lambda + 2\kappa_0 Q^2), \tag{131}$$

or, equivalently,

$$\kappa_0 Q^2 = -\frac{1}{2f}[F_{,x} + (F + 4\Lambda)f]. \tag{132}$$

It remains to be investigated what are the constraints resulting from the simultaneous solution of Eqs. (129) and (132).

## VI. ALL ALIGNED ROBINSON-TRAUTMAN SOLUTIONS

After completing the derivation and preliminary description of the nonexpanding Kundt class, we will now concentrate on systematic integration of the field equations in the *expanding* case $\Theta \neq 0$, which defines the Robinson-Trautman family of spacetimes.

Recall that the field equations (38) take the form

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{133}$$

where $F_a$ are defined by (14)–(16). In this section *we assume that the electromagnetic field is aligned* with $\mathbf{k} = \partial_r$ [see (31)], that is

$$F_{rx} = 0 \Leftrightarrow F_r = 0. \tag{134}$$

This considerably simplifies the field equations (133) whenever at least one of the index $a$, $b$ is $r$.

### A. Integration of $R_{rr} = 0$

From Eq. (A24) we immediately get the constraint

$$\Theta_{,r} + \Theta^2 = 0, \tag{135}$$

which determines the $r$ dependence of the expansion scalar $\Theta$. Its general solution can be written as $\Theta^{-1} = r + r_0(u,x)$. Because the metric (3) is invariant under the gauge transformation $r \to r - r_0(u,x)$, without loss of generality we can set the integration function $r_0(u,x)$ to zero. The expansion thus simplifies to

$$\Theta = \frac{1}{r}. \tag{136}$$

Integrating now the key relation (9) we obtain

$$G(r,u,x) = \frac{P(u,x)}{r}, \tag{137}$$

where $P(u,x)$ is any function independent of $r$. Using (8), we immediately get the generic spatial metric function $g_{xx} \equiv G^{-2}$ in the form

$$g_{xx} = \frac{r^2}{P^2(u,x)}. \tag{138}$$

Of course, by inversion $g^{xx} = P^2 r^{-2}$.

### B. Integration of $R_{rx} = 0$

Using Eqs. (A25) and (135), which implies Eq. (136), the Ricci tensor component $R_{rx}$ becomes

$$R_{rx} = -\frac{1}{2}(g_{ux,rr} - g_{ux,r}r^{-1}). \tag{139}$$

The corresponding field equation $R_{rx} = 0$ can be integrated, yielding a general solution

$$g_{ux} = e(u,x)r^2 + f(u,x), \tag{140}$$

where $e$ and $f$ are arbitrary functions of $u$ and $x$. In view of Eqs. (5) and (138), the contravariant component of the Robinson-Trautman metric is

$$g^{rx} = P^2[e(u,x) + f(u,x)r^{-2}]. \tag{141}$$

### C. Integration of the Maxwell equations

Now, applying the Maxwell equations (40), (41) with $\sqrt{-g} = G^{-1} = r/P$, we will determine the electromagnetic field. There are only four independent Maxwell equations, namely three components of $(\sqrt{-g}F^{ab})_{,b} = 0$ and just one component of $F_{[ab,c]} = 0$. Because (13) with (134) implies

$$F^{ru} = -F_{ru}, \qquad F^{rx} = \frac{P^2}{r^2}(g_{ux}F_{ru} - F_{ux}), \qquad F^{ux} = 0, \tag{142}$$

these four equations for the electromagnetic field take the form

$$(rF_{ru})_{,r} = 0, \tag{143}$$

$$(r^{-1}(g_{ux}F_{ru} - F_{ux}))_{,r} = 0, \tag{144}$$

$$r^2\left(\frac{F_{ru}}{P}\right)_{,u} = (P(g_{ux}F_{ru} - F_{ux}))_{,x}, \tag{145}$$

$$F_{ux,r} + F_{ru,x} = 0. \tag{146}$$

They can be solved for the nontrivial components $F_{ru}$ and $F_{ux}$. From (143) we get

$$F_{ru} = \frac{Q(u,x)}{r}, \tag{147}$$

where $Q(u,x)$ is an arbitrary function of $u$ and $x$. By employing (146), we thus obtain

$$F_{ux} = -Q_{,x}\ln|r| - \xi(u,x), \tag{148}$$

where $\xi(u,x)$ is another arbitrary function. Equation (144) with (140) then reduces to

$$\left(\frac{fQ}{r^2} + Q_{,x}\frac{\ln|r|}{r} + \frac{\xi}{r}\right)_{,r} = 0, \tag{149}$$

which gives the following three independent constraints:

$$fQ = 0, \qquad Q_{,x} = 0, \qquad \xi = Q_{,x}, \tag{150}$$

so that $\xi = 0$ and $Q = Q(u)$ is independent of $x$.

We thus conclude that the components of a generic *aligned electromagnetic field in any* $2+1$ *Robinson-Trautman spacetime* can be written as

$$F_{rx} = 0, \qquad F_{ru} = \frac{Q(u)}{r}, \qquad F_{ux} = 0, \tag{151}$$

with the constraint

$$fQ = 0, \tag{152}$$

and the Maxwell equation (145) which reduces to

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{153}$$

Consequently,

$$F_r = 0, \qquad F_x = P^{-2}Qr, \qquad F_u = eQr, \tag{154}$$

and, due to (26)–(28),

$$\phi_0 = 0, \qquad \phi_1 = \frac{Q}{r}, \qquad \phi_2 = ePQ. \tag{155}$$

When $\phi_1 = 0 \Leftrightarrow Q = 0$ then $\phi_2 = 0$. Therefore, there are *no null electromagnetic fields of this type*. When $\phi_2 = 0 \Leftrightarrow eQ = 0$, it is *non-null*, and then $\phi_1 = Q(u)/r$. Notice also, that due to (152), either we have a vacuum solution ($Q = 0$) or a non-null electromagnetic field characterized by $Q(u)$ in the Robinson-Trautman spacetime *without the nondiagonal metric term* ($g_{ux} = 0$).

Now, we will integrate the remaining Einstein's equations which couple the gravitational and electromagnetic fields. In view of (152), there are two cases to consider, namely $Q = 0$ and $f = 0$.

(i) The case $Q = 0$: The electromagnetic field completely vanishes, so that the spacetimes are *vacuum* (with any cosmological constant $\Lambda$). All such Robinson-Trautman solutions in $2+1$ gravity were found and described in our previous work [1]. Interestingly, for these vacuum spacetimes the function $f$ remains nonvanishing (which is not true in $D \geq 4$).

(ii) The case $f = 0$: In this case, the metric component $g_{ux}$ reduces to

$$g_{ux} = er^2 \Leftrightarrow g^{rx} = P^2 e. \tag{156}$$

This simplifies the generic Ricci tensor components in the Appendix, which will now apply.

### D. Integration of $R_{ru} = -2\Lambda$

Using (156), (136), and (138), the Ricci tensor component (A26) becomes

$$R_{ru} = -\frac{1}{2}(r g_{uu,r})_{,r} r^{-1} + \frac{1}{2} c r^{-1} + 2P^2 e^2, \qquad (157)$$

where

$$c \equiv 2P^2 \left( e_{\parallel x} - \frac{1}{2} h_{xx,u} \right), \qquad e_{\parallel x} \equiv e_{,x} + eP_{,x}/P, \qquad (158)$$

from which we obtain useful identities

$$Pe_{\parallel x} = (Pe)_{,x}, \qquad eP^2 e_{\parallel x} = \frac{1}{2}(P^2 e^2)_{,x}, \qquad (159)$$

and thus

$$c = 2[P(Pe)_{,x} + (\ln P)_{,u}]. \qquad (160)$$

With Eq. (157), the Einstein equation $R_{ru} = -2\Lambda$ can now be easily integrated to give

$$g_{uu} = -a - b \ln|r| + cr + (\Lambda + P^2 e^2) r^2, \qquad (161)$$

where $a(u,x)$ and $b(u,x)$ are arbitrary functions. The $r$ dependence of all metric components is thus fully established.

### E. Integration of $R_{xx} = 2\Lambda g_{xx} + \kappa_0 G^2 F_x^2$

Using Eqs. (135)–(138) and (156), the general Ricci tensor component (A27) becomes

$$R_{xx} = -cP^{-2} r - 2e^2 r^2 + P^{-2} r g_{uu,r}. \qquad (162)$$

Substituting now the expression (161), we obtain $R_{xx} = 2\Lambda g_{xx} - b/P^2$. The corresponding Einstein equation with (154) reads $R_{xx} = 2\Lambda g_{xx} + \kappa_0 Q^2/P^2$. It is satisfied if, and only if,

$$b(u) = -\kappa_0 Q^2. \qquad (163)$$

### F. Integration of $R_{ux} = 2\Lambda g_{ux} + \kappa_0 G^2 F_u F_x$

Using Eqs. (136), (138), (156), and (161) with (163), the Ricci tensor component $R_{ux}$ given by Eq. (A28) reads

$$R_{ux} = 2\Lambda g_{ux} + \kappa_0 e Q^2 - \frac{1}{2} a_{,x} r^{-1}. \qquad (164)$$

The field equation with (154) is $R_{ux} = 2\Lambda g_{ux} + \kappa_0 e Q^2$, so that we obtain just one simple constraint:

$$a_{,x} = 0 \Leftrightarrow a = a(u). \qquad (165)$$

The function $a$ can depend only on the coordinate $u$, and the most general Robinson-Trautman aligned electrovacuum solution thus takes the form

$$ds^2 = \frac{r^2}{P^2} dx^2 + 2er^2 du dx - 2du dr + \left( -a(u) + \kappa_0 Q^2(u) \ln|r| + 2[P(Pe)_{,x} + (\ln P)_{,u}] r + (\Lambda + P^2 e^2) r^2 \right) du^2. \qquad (166)$$

### G. Integration of $R_{uu} = 2\Lambda g_{uu} + \kappa_0 G^2 F_u^2$

The Ricci tensor component $R_{uu}$ for the metric (166), given generally by Eq. (A29), becomes

$$R_{uu} = 2\Lambda g_{uu} + A + \frac{1}{2}\left[ a_{,u} - \left( a - \frac{1}{2} b \right) c - \triangle c \right] \frac{1}{r} + \frac{1}{2}[b_{,u} - bc] \frac{\ln r}{r}, \qquad (167)$$

where

$$A = -P^2 e^2 b + \frac{1}{4} c^2 + \frac{1}{2} P^2 e c_{,x} - \frac{1}{2} c_{,u} - \frac{1}{2} \triangle (P^2 e^2) + P(Pe_{,u})_{,x} - 2\frac{P_{,u}^2}{P^2} + \frac{P_{,uu}}{P}, \qquad (168)$$

$c$ is given by Eq. (160), and

$$\triangle c \equiv h^{xx} c_{\parallel xx} = P(Pc_{,x})_{,x} \qquad (169)$$

is the covariant Laplace operator on the one-dimensional transverse Riemannian space spanned by $x$, applied on the function $c$. Remarkably, after substitution from (160) and evaluation, the expression for $A$ enormously simplifies to

$$A = -P^2 e^2 b. \tag{170}$$

Moreover, using the Maxwell equation (153) which can be rewritten as

$$Q_{,u} = \frac{1}{2} c Q, \tag{171}$$

and the relation (163), that is $b = -\kappa_0 Q^2$, we easily prove that $b_{,u} = bc$. The last term in (167) thus always vanishes. To summarize, the last Ricci tensor component takes the form

$$R_{uu} = 2\Lambda g_{uu} + \kappa_0 e^2 P^2 Q^2 + \frac{1}{2}\left[a_{,u} - \left(a - \frac{1}{2}b\right)c - \triangle c\right]\frac{1}{r}. \tag{172}$$

Using (154), the corresponding field equation reads $R_{uu} = 2\Lambda g_{uu} + \kappa_0 e^2 P^2 Q^2$, so that we obtain *only one additional condition* determined by the term proportional to $r^{-1}$, namely

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c + \triangle c. \tag{173}$$

Let us observe that Eq. (171) implies

$$c(u) = 2(\ln Q)_{,u}, \tag{174}$$

i.e., the function $c$ must necessarily be independent of the spatial coordinate $x$. Due to (169), $\triangle c = 0$, and the field equation (173) reduces to

$$a_{,u} = \left(a + \frac{\kappa_0}{2} Q^2\right) c. \tag{175}$$

Its general solution with (174) is

$$a(u) = Q^2(\kappa_0 \ln|Q| - \mu), \tag{176}$$

where $\mu$ is any constant. The metric function $a(u)$ is thus directly related to the electromagnetic field $Q(u)$.

### H. Summary of the aligned Robinson-Trautman solutions

We have solved all the Einstein-Maxwell equations with a cosmological constant $\Lambda$ and *aligned* electromagnetic field in $2+1$ gravity for the Robinson-Trautman family of expanding spacetimes. In the canonical coordinates, the generic gravitational field of this type is

$$\begin{aligned} g_{xx} &= P^{-2}(u,x) r^2, \\ g_{ux} &= e(u,x) r^2, \\ g_{ur} &= -1, \\ g_{uu} &= \mu Q^2(u) - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + (\Lambda + P^2 e^2) r^2, \end{aligned} \tag{177}$$

where $\mu$ is a constant, $Q(u)$ is any function of $u$, and the metric functions $P$, $e$ satisfy the field equation (153), that is

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{178}$$

The corresponding aligned electromagnetic field reads

$$\begin{aligned} F_{rx} &= 0, \\ F_{ru} &= \frac{Q(u)}{r}, \\ F_{ux} &= 0, \end{aligned} \tag{179}$$

see (151); i.e., it has *only one component* $F_{ru}$.

Written explicitly in the usual compact form, the solution is

$$\mathrm{d}s^2 = \frac{r^2}{P^2}(\mathrm{d}x + eP^2\mathrm{d}u)^2 - 2\mathrm{d}u\mathrm{d}r + \left(\mu Q^2 - \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u} r + \Lambda r^2\right)\mathrm{d}u^2, \tag{180}$$

with

$$\mathbf{F} = \frac{Q}{r}\mathrm{d}r \wedge \mathrm{d}u \quad \text{equivalent to} \quad {}^*\mathbf{F} = \frac{Q}{P}\mathrm{d}x + ePQ\mathrm{d}u, \tag{181}$$

corresponding to the potential

$$\mathbf{A} = Q \ln\frac{r}{r_0}\mathrm{d}u, \tag{182}$$

and the Maxwell scalars (155)

$$\begin{aligned} \phi_0 &= 0, \\ \phi_1 &= \frac{Q}{r}, \\ \phi_2 &= ePQ. \end{aligned} \tag{183}$$

It follows that there are *no aligned (purely) null electromagnetic fields* in the Robinson-Trautman spacetimes in $2+1$ gravity because $\phi_1 = 0$ implies $\phi_2 = 0$. Moreover,

$\phi_2 = 0 \Leftrightarrow eQ = 0$. Either we have a vacuum solution ($Q = 0$) or a non-null electromagnetic field characterized by $Q(u)$ in the Robinson-Trautman spacetime without the nondiagonal metric term $g_{ux}$ ($e = 0$).

*The simplest* $e \neq 0$ solution of the field equation (178), which can be rewritten as

$$(\ln P)_{,u} + P(eP)_{,x} = (\ln Q)_{,u}, \tag{184}$$

is

$$P = 1, \qquad e = x(\ln Q)_{,u} + \alpha(u), \tag{185}$$

where $\alpha(u)$ is an arbitrary function of $u$, yielding the metric

$$\mathrm{d}s^2 = r^2\Big(\mathrm{d}x + (\alpha + x(\ln Q)_{,u})\mathrm{d}u\Big)^2 - 2\mathrm{d}u\mathrm{d}r + \Big(\mu Q^2 - \kappa_0 Q^2 \ln\Big|\frac{Q}{r}\Big| + 2(\ln Q)_{,u} r + \Lambda r^2\Big)\mathrm{d}u^2. \tag{186}$$

Another interesting subclass of the Robinson–Trautman spacetimes (180) with aligned Maxwell field (181) arises when both sides of the field equation (178) vanish, $(Q/P)_{,u} = 0 \Leftrightarrow (eP)_{,x} = 0$. Then the metric functions $P$ and $e$ are *both factorized* in the coordinates $u$ and $x$ as

$$P = Q(u)\beta(x), \qquad e = \frac{\alpha(u)}{Q(u)\beta(x)}, \tag{187}$$

where $\alpha(u)$, $\beta(x)$ are arbitrary functions of the respective coordinates. Consequently, $eP = \alpha(u)$. [For $\beta = 1$ we obtain simply $P(u) = Q(u)$.] In such a case, the metric (180) takes the form

$$\mathrm{d}s^2 = \frac{r^2}{Q^2}\Big(\frac{\mathrm{d}x}{\beta} + \alpha Q\mathrm{d}u\Big)^2 - 2\mathrm{d}u\mathrm{d}r + \Big(\mu Q^2 - \kappa_0 Q^2 \ln\Big|\frac{Q}{r}\Big| + 2(\ln Q)_{,u} r + \Lambda r^2\Big)\mathrm{d}u^2, \tag{188}$$

and the Maxwell scalars are

$$\phi_0 = 0, \qquad \phi_1 = \frac{Q}{r}, \qquad \phi_2 = \alpha Q.$$

With respect to the natural triad (6), there are thus two components of the admitted Maxwell field, namely non-null component $\phi_1$ and the electromagnetic radiation $\phi_2$ ($\phi_2 \neq 0$ requires $\alpha \neq 0$). However, let us remark that, due to the freedom in the choice of the local null triad, under which the Maxwell scalars transform as (33), at a given point there exists a special triad in which $\phi_2' = 0$.

There is a *special case* $Q =$ const, for which the metric (188) simplifies to

$$\mathrm{d}s^2 = r^2\Big(\mathrm{d}\varphi + \alpha(u)\mathrm{d}u\Big)^2 - 2\mathrm{d}u\mathrm{d}r + \Big(m - \kappa_0 Q^2 \ln\Big|\frac{Q}{r}\Big| + \Lambda r^2\Big)\mathrm{d}u^2, \tag{189}$$

where the rescaled constant reads $m \equiv Q^2\mu$, and the new coordinate is

$$\varphi = \frac{1}{Q}\int \frac{\mathrm{d}x}{\beta(x)}. \tag{190}$$

For $\alpha(u) = 0$ (that is, without the electromagnetic radiation component), and for compact coordinate $\varphi$, this family of spacetimes represents *charged black holes* with any value of the cosmological constant $\Lambda$. Indeed, by introducing the time coordinate $t$ via the transformation

$$\mathrm{d}u = \mathrm{d}t + \Big(m - \kappa_0 Q^2 \ln\Big|\frac{Q}{r}\Big| + \Lambda r^2\Big)^{-1}\mathrm{d}r, \tag{191}$$

we obtain the metric

$$\mathrm{d}s^2 = -\Big(-m + \kappa_0 Q^2 \ln\Big|\frac{Q}{r}\Big| - \Lambda r^2\Big)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{-m + \kappa_0 Q^2 \ln|\frac{Q}{r}| - \Lambda r^2} + r^2\mathrm{d}\varphi^2, \tag{192}$$

with the electromagnetic field

$$\mathbf{F} = \frac{Q}{r}\mathrm{d}r \wedge \mathrm{d}t \quad \text{corresponding to} \quad \mathbf{A} = Q\ln\frac{r}{r_0}\mathrm{d}t. \tag{193}$$

This is the standard form of *cyclic symmetric, electrostatic solution with* $\Lambda$ in polar "Schwarzschild" coordinates found by Peldan in 1993 [24], see Eq. (11.56) in [6], which extended previous solutions by Gott, Simon and Alpert [25,26], Deser and Mazur [27], and Melvin [28] to any cosmological constant; see also García [29]. A thorough review and discussion of this class of solutions is contained in [30] and also Sec. 11.2 of [6].

For $\alpha(u) \neq 0$ the spacetime (189) in general contains *additional* electromagnetic *radiation component* $\phi_2 \neq 0$. It remains to be analyzed in detail if such a situation can be physically interpreted as a charged black hole with a specific radiation, or if the function $\alpha(u)$ is just some kind of a kinematic parameter.

Similarly, the general Robinson-Trautman solution (180) with aligned electromagnetic field (181) needs to be understood and explicitly related to other known solutions summarized in Chapter 11 of [6], in particular the nonstatic ones. This seems to be in principle possible because, e.g., for $e \neq 0$ the transformation (191) introduces the metric component $g_{tx}$ typical for stationary spacetimes.

## VII. ALL NONALIGNED ROBINSON-TRAUTMAN SOLUTIONS

After completing the systematic derivation of all aligned electromagnetic fields in the family of expanding Robinson-Trautman geometries, we now investigate the *possible nonaligned fields*.

The Einstein–Maxwell equations are (133), in which the functions $F_a$ are defined by (14)–(16). The generic nonaligned electromagnetic field has $\phi_0 \neq 0 \Leftrightarrow F_{rx} \neq 0 \Leftrightarrow F_r \neq 0$.

### A. Integration of $R_{rr} = \kappa_0 G^2 F_r^2$

Using Eq. (A24) for the Ricci tensor component $R_{rr}$, we obtain the constraint

$$\kappa_0 F_r^2 = -g_{xx}(\Theta_{,r} + \Theta^2), \tag{194}$$

where $\Theta \neq 0$ is the optical scalar representing the *expansion* of the privileged null congruence generated by $\mathbf{k} = \partial_r$. Let us recall that it is directly related to the spatial metric function $g_{xx}$ via the relations

$$g_{xx} = G^{-2} \quad \text{with} \quad \Theta = -(\ln G)_{,r} \equiv -\frac{G_{,r}}{G}; \tag{195}$$

see (8), (9). Therefore, the metric component $g_{xx}$ must necessarily depend on the coordinate $r$, otherwise $\Theta = 0$.

It is possible to substitute from (195) into (194), but we found it more convenient to keep the expansion scalar $\Theta$ in (194). This equation *explicitly expresses the nonaligned Maxwell field component* $F_{rx} \equiv F_r$ *in terms of the metric component* $g_{xx}$ (and its $r$ derivatives via $G$). This relation can be rewritten as

$$\kappa_0 F_{rx}^2 = G^{-2}\Theta^2\left((\Theta^{-1})_{,r} - 1\right). \tag{196}$$

Notice that (in the Robinson-Trautman family) $F_{rx} = 0 \Leftrightarrow \Theta^{-1} = r + r_0(u,x)$. This fully corresponds to the previously studied aligned case, for which (136) applies.

### B. Integration of $R_{rx} = \kappa_0 G^2 F_r F_x$

Using Eq. (A25) for the Ricci tensor component $R_{rx}$ and (194), we get the relation

$$\frac{1}{2}(\Theta g_{ux,r} - g_{ux,rr}) = \kappa_0 G^2 F_r (F_x + g_{ux} F_r). \tag{197}$$

In view of (14), (15), this is equivalent to

$$\kappa_0 F_{ru} F_{rx} = \frac{1}{2}(\Theta g_{ux,r} - g_{ux,rr}). \tag{198}$$

Therefore, *by prescribing any metric function* $g_{ux}$, *the electromagnetic field component* $F_{ru}$ *is explicitly determined*.

Notice that it *admits a special solution* $F_{ru} = 0 \Leftrightarrow \Theta g_{ux,r} = g_{ux,rr}$. This occurs *either* when $g_{ux}$ is independent of the coordinate $r$,

$$g_{ux} = B(u,x), \tag{199}$$

*or*, using (195), when $\Theta = (\ln G^{-1})_{,r} = (\ln g_{ux,r})_{,r}$ which can be completely integrated as

$$g_{xx} = A(u,x)(g_{ux,r})^2, \tag{200}$$

where $A > 0$ is any function independent of $r$.

### C. Integration of $R_{ru} = -2\Lambda + \kappa_0 G^2 F_r F_u$

The generic Ricci tensor component $R_{ru}$ is given by (A26), so that the corresponding Einstein-Maxwell field equation becomes

$$\begin{aligned}
&-\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{rx}g_{ux,rr} + \frac{1}{2}g^{xx}(g_{ux,r||x} + (g_{ux,r})^2) \\
&\quad - \Theta_{,u} - \frac{1}{2}\Theta(g^{xx}g_{xx,u} + g^{rx}g_{ux,r} + g_{uu,r}) \\
&= -2\Lambda + \kappa_0 G^2 F_r F_u.
\end{aligned} \tag{201}$$

This uniquely *determines the third electromagnetic field component* (16) represented by $F_u$. Using (14)–(16) and then (194), (198), the last term on the right-hand side can be expressed as

$$\begin{aligned}
&\kappa_0 G^2 F_{rx}(g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}) \\
&= \kappa_0 g^{rx}F_{ru}F_{rx} - \kappa_0 g^{xx}F_{ux}F_{rx} - \kappa_0 g^{xx}g_{uu}F_{rx}^2 \\
&= -\kappa_0 g^{xx}F_{ux}F_{rx} + \frac{1}{2}g^{rx}(\Theta g_{ux,r} - g_{ux,rr}) + g_{uu}(\Theta_{,r} + \Theta^2).
\end{aligned} \tag{202}$$

The field equation (201) thus reads

$$\begin{aligned}
\kappa_0 F_{ux}F_{rx} = {}& \frac{1}{2}g_{xx}(g_{uu,rr} + \Theta g_{uu,r} + 2(\Theta_{,r} + \Theta^2)g_{uu} - 4\Lambda) \\
& + g_{ux}(\Theta g_{ux,r} - g_{ux,rr}) - \frac{1}{2}(g_{ux,r||x} + (g_{ux,r})^2) \\
& + \frac{1}{2}\Theta g_{xx,u} + g_{xx}\Theta_{,u}.
\end{aligned} \tag{203}$$

*By prescribing any metric function* $g_{uu}$, *the third electromagnetic field component* $F_{ux}$ *is thus explicitly determined.*

To summarize, by employing three (out of six) independent components of the Einstein field equations, we have now derived explicit expressions (196), (198), and (203) which *determine all three components of the* *electromagnetic field*, namely $F_{rx}$, $F_{ru}$, and $F_{ux}$, respectively, in terms of the three (so far) independent metric components $g_{xx}$, $g_{ux}$, and $g_{uu}$.

These three expressions are equivalent to Eqs. (194), (197), (201) for the three dual electromagnetic functions $F_a \equiv {}^*F_a/G$. They can be written in a very compact form:

$$\kappa_0 F_r^2 = \alpha, \tag{204}$$

$$\kappa_0 F_r F_x = \beta - \alpha g_{ux}, \tag{205}$$

$$\kappa_0 F_r F_u = \gamma, \tag{206}$$

where the functions $\alpha$, $\beta$, $\gamma$ are useful shorthand for the combination of the three metric functions:

$$\alpha \equiv -g_{xx}(\Theta_{,r} + \Theta^2), \tag{207}$$

$$\beta \equiv \frac{1}{2} g_{xx}(\Theta g_{ux,r} - g_{ux,rr}), \tag{208}$$

$$\gamma \equiv \frac{1}{2}\Big[g_{xx}(4\Lambda - g_{uu,rr}) + g_{ux}g_{ux,rr} + g_{ux,r\|x} + (g_{ux,r})^2 - 2g_{xx}\Theta_{,u} - \Theta(g_{xx,u} + g_{ux}g_{ux,r} + g_{xx}g_{uu,r})\Big]. \tag{209}$$

Consequently,

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}}, \qquad F_x = \left(\frac{\beta}{\alpha} - g_{ux}\right) F_r, \qquad F_u = \frac{\gamma}{\alpha} F_r. \tag{210}$$

Let us recall that $\alpha$ is fully determined by $g_{xx}$, the function $\beta$ is determined by $g_{xx}$ and $g_{ux}$, while the third metric component $g_{uu}$ enters only $\gamma$.

### D. The Maxwell equations

As the next step, we apply the four independent Maxwell equations in the form (43) and (41), namely

$$(GF_a)_{,b} = (GF_b)_{,a} \quad \text{and} \quad F_{ux,r} + F_{ru,x} - F_{rx,u} = 0, \tag{211}$$

which restrict the possible electromagnetic field and its coupling to the gravitational field. For explicit evaluation of the partial derivatives with respect to $a, b = \{r, u, x\}$ we employ the expressions directly following from (195) and (204)–(206), implying (210), namely

$$G_{,a} = -\frac{1}{2} G^3 g_{xx,a}, \tag{212}$$

$$F_{r,a} = \frac{1}{\kappa_0 F_r}\left(\frac{1}{2}\alpha_{,a}\right), \tag{213}$$

$$F_{x,a} = \frac{1}{\kappa_0 F_r}\left((\beta - \alpha g_{ux})_{,a} - \frac{1}{2}(\beta - \alpha g_{ux})\frac{\alpha_{,a}}{\alpha}\right), \tag{214}$$

$$F_{u,a} = \frac{1}{\kappa_0 F_r}\left(\gamma_{,a} - \frac{1}{2}\gamma\frac{\alpha_{,a}}{\alpha}\right). \tag{215}$$

Using these relations in calculating $(GF_a)_{,b} = (GF_b)_{,a}$ for $ab = rx, ru, ux$ we obtain

$$\left(\alpha_{,x} + 2\alpha\frac{G_{,x}}{G}\right) - 2(\beta - \alpha g_{ux})_{,r} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,r}}{\alpha} + 2\Theta\right) = 0, \tag{216}$$

$$\left(\alpha_{,u} + 2\alpha\frac{G_{,u}}{G}\right) - 2\gamma_{,r} + \gamma\left(\frac{\alpha_{,r}}{\alpha} + 2\Theta\right) = 0, \tag{217}$$

$$\gamma_{,x} - \gamma\left(\frac{\alpha_{,x}}{2\alpha} - \frac{G_{,x}}{G}\right) - (\beta - \alpha g_{ux})_{,u} + (\beta - \alpha g_{ux})\left(\frac{\alpha_{,u}}{2\alpha} - \frac{G_{,u}}{G}\right) = 0, \tag{218}$$

respectively. Notice that the terms in the large brackets depend only on $g_{xx} \equiv G^{-2}$ and their derivatives. The last Maxwell equation (211), using the inversion of (14)–(16),

$$F_{rx} = F_r, \tag{219}$$

$$F_{ru} = G^2(F_x + g_{ux}F_r), \tag{220}$$

$$F_{ux} = g_{ux}G^2(F_x + g_{ux}F_r) - F_u - g_{uu}F_r, \tag{221}$$

reads

$$\beta_{,x} + \beta_{,r}g_{ux} + \beta\left[g_{ux,r} - g_{ux}\left(\frac{\alpha_{,r}}{2\alpha} + 2\Theta\right) - \left(\frac{\alpha_{,x}}{2\alpha} - 2\frac{G_{,x}}{G}\right)\right] - \frac{1}{2G^2}\left[2\gamma_{,r} - \alpha_{,r}\left(\frac{\gamma}{\alpha} - g_{uu}\right) + \alpha_{,u} + 2\alpha g_{uu,r}\right] = 0. \tag{222}$$

The four equations (216)–(218) and (222) put restrictions on the metric functions, encoded in $G$, $\alpha$, $\beta$, $\gamma$.

### E. Remaining Einstein equations $R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b$

Finally, it is necessary to solve the remaining three Einstein equations (38) for the components $ab = xx, ux, uu$. Using (210) we immediately derive their form:

$$R_{xx} = 2\Lambda g_{xx} + \frac{G^2}{\alpha}(\beta - \alpha g_{ux})^2, \tag{223}$$

$$R_{ux} = 2\Lambda g_{ux} + \frac{G^2}{\alpha}(\beta - \alpha g_{ux})\gamma, \quad (224)$$

$$R_{uu} = 2\Lambda g_{uu} + \frac{G^2}{\alpha}\gamma^2. \quad (225)$$

Substituting the explicit expressions for the corresponding Ricci tensor components (A27)–(A29) reveals a rather complicated system of partial differential equations for the metric functions which must be solved together with (216)–(218) and (222).

At this stage, it does not seem possible to find a *general* solution of these equations. However, *we have achieved a separation of the variables representing the gravitational and the electromagnetic field*. Indeed, the system of seven equations (216)–(218), (222), and (223)–(225) with (A27)–(A29) *involves only the three metric functions* $g_{xx}$, $g_{ux}$, $g_{uu}$, encoded also in the functions $G$ and $\alpha, \beta, \gamma$ defined in (195) and (207)–(209). After their solution is found, the corresponding three (dual) components of the electromagnetic field $F_r$, $F_x$, $F_u$ are easily obtained by applying the relations (210). The components $F_{rx}$, $F_{ru}$, $F_{ux}$ are then their simple combinations (219)–(221).

### F. A simple particular solution

To demonstrate the usefulness of our formulation of the most general Einstein-Maxwell field equations and also *to show that the class of Robinson-Trautman* $2+1$ *spacetimes with nonaligned electromagnetic field is not empty*, we will now derive a special solution of the above system of equations.

Let us *assume* that *only the nonaligned component* $F_r$ of the electromagnetic field is nontrivial, i.e.,

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}} \neq 0, \qquad F_x = 0, \qquad F_u = 0. \quad (226)$$

The field equations (204)–(206) then imply

$$\beta - \alpha g_{ux} = 0, \quad (227)$$

$$\gamma = 0. \quad (228)$$

Further simplification is achieved by *assuming*

$$g_{ux} = 0. \quad (229)$$

In such a case the condition (227) $\beta = 0$ is satisfied due to (208), while (228) gives

$$g_{uu,rr} - 4\Lambda + 2\Theta_{,u} + \Theta\left(g_{uu,r} - 2\frac{G_{,u}}{G}\right) = 0. \quad (230)$$

The Maxwell equations (216)–(218), (222) reduce to

$$\frac{\alpha_{,x}}{\alpha} + 2\frac{G_{,x}}{G} = 0, \quad (231)$$

$$\frac{\alpha_{,u}}{\alpha} + 2\frac{G_{,u}}{G} = 0, \quad (232)$$

$$\alpha_{,r}g_{uu} + \alpha_{,u} + 2\alpha g_{uu,r} = 0, \quad (233)$$

and the final three Einstein equations simplify as

$$R_{xx} = 2\Lambda g_{xx}, \quad (234)$$

$$R_{ux} = 0, \quad (235)$$

$$R_{uu} = 2\Lambda g_{uu}, \quad (236)$$

where

$$R_{xx} = g_{xx}g_{uu}(\Theta_{,r} + \Theta^2) + 2g_{xx}\Theta_{,u} + \Theta(g_{xx}g_{uu,r} + g_{xx,u}), \quad (237)$$

$$R_{ux} = -\frac{1}{2}g_{uu,xr} + \frac{1}{2}\Theta g_{uu,x}, \quad (238)$$

$$R_{uu} = \frac{1}{2}g_{uu}g_{uu,rr} + \frac{1}{4}g^{xx}g_{xx,u}g_{uu,r} - \frac{1}{2}g^{xx}g_{xx,uu} - \frac{1}{2}g^{xx}g_{uu||xx} + \frac{1}{4}(g^{xx}g_{xx,u})^2 + \frac{1}{2}\Theta(g_{uu}g_{uu,r} - g_{uu,u}). \quad (239)$$

Equations (231) and (232) can be easily integrated, yielding

$$\alpha = f(r)G^{-2} \equiv f(r)g_{xx}, \quad (240)$$

where $f(r)$ is any function of the coordinate $r$. Equation (233) gives the constraint

$$g_{uu,r} + \left(\frac{f'}{2f} + \Theta\right)g_{uu} - \frac{G_{,u}}{G} = 0, \quad (241)$$

in which $f'$ is the derivative of $f$. It thus remains to solve (230), (241), and (234)–(236).

Now, combining (240) with the definition (207) we obtain

$$\Theta_{,r} + \Theta^2 = -f(r),$$

which is the Ricatti-type equation for the expansion $\Theta$. Using the substitution $\Theta = z_{,r}/z$, it can be rewritten as the linear equation $z_{,rr} + f(r)z = 0$. Let us consider here only *the simplest case of a constant* $f$,

$$f \equiv C^2. \quad (242)$$

By applying (195) we obtain the explicit solution

$$\Theta(r) = C\cot(Cr), \tag{243}$$

$$G = \frac{P(u,x)}{\sin(Cr)}, \tag{244}$$

$$g_{xx} = \frac{\sin^2(Cr)}{P^2(u,x)}. \tag{245}$$

(We have applied the coordinate freedom, namely a trivial constant shift in the coordinate $r$, to simplify the expressions.) It is now easily seen that for the particular choice

$$P = 1, \tag{246}$$

$$g_{uu} = 0, \tag{247}$$

$$\Lambda = 0, \tag{248}$$

all the remaining field equations (230), (241), and (234)–(236) are satisfied because $R_{xx} = 0$, $R_{ux} = 0$, and $R_{uu} = 0$. We have thus obtained a *special Robinson-Trautman solution*,

$$\mathrm{d}s^2 = \sin^2(Cr)\mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}r, \tag{249}$$

*with a nonaligned electromagnetic field*:

$$F_r = \frac{C}{\sqrt{\kappa_0}G} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr), \qquad F_x = 0, \qquad F_u = 0, \tag{250}$$

that is,

$$^*\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\mathrm{d}r. \tag{251}$$

Using (219)–(221), this is equivalent to

$$\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr)\mathrm{d}r\wedge\mathrm{d}x, \tag{252}$$

corresponding to the potential

$$\mathbf{A} = -\frac{1}{\sqrt{\kappa_0}}\cos(Cr)\mathrm{d}x. \tag{253}$$

By rescaling the coordinates $r$ and $u$ the constant $C$ can be set to $C = 1$, but we prefer to keep it free because it represents the value of the electromagnetic field and $r$ is not dimensionless.

Actually, (249) is the metric 3) on page 133 of [31] for $q = 0$, which admits four Killing vectors [see also the metric (4.1) in [32]].

## VIII. FINAL SUMMARY AND REMARKS

In this paper we systematically solved the Einstein-Maxwell equations with $\Lambda$, obtaining all electrovacuum $2+1$ spacetimes. We identified main geometrically distinct subclasses, and we explicitly derived the corresponding metrics and electromagnetic fields. In particular:

(1) The metric of *any* such spacetime can be written in canonical coordinates in the form (3)

$$\mathrm{d}s^2 = G^{-2}\mathrm{d}x^2 + 2g_{ux}\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + g_{uu}\mathrm{d}u^2. \tag{254}$$

(2) The *generic* electromagnetic Maxwell 2-form field and its dual 1-form have three independent components (11) and (21), namely

$$\mathbf{F} = F_{ru}\mathrm{d}r\wedge\mathrm{d}u + F_{rx}\mathrm{d}r\wedge\mathrm{d}x + F_{ux}\mathrm{d}u\wedge\mathrm{d}x, \tag{255}$$

$$^*\mathbf{F} = G(F_r\mathrm{d}r + F_u\mathrm{d}u + F_x\mathrm{d}x), \tag{256}$$

where $F_r = F_{rx}$, $F_x = g_{xx}F_{ru} - g_{ux}F_{rx}$, $F_u = g_{ux}F_{ru} - F_{ux} - g_{uu}F_{rx}$.

(3) In terms of the Newman-Penrose scalars (25) of distinct boost weights $+1, 0, -1$, the Maxwell field *invariants* $F^2 \equiv F_{ab}F^{ab}$ and $^*F^2 \equiv {}^*F_a{}^*F^a$ are

$$\frac{1}{2}F^2 = -^*F^2 = 2\phi_0\phi_2 - \phi_1^2. \tag{257}$$

The electromagnetic field is *aligned with* $\mathbf{k} = \partial_r \Leftrightarrow \phi_0 = 0 \Leftrightarrow F_{rx} = 0 \Leftrightarrow F_r = 0$.

Such an aligned field has only two components, namely $\phi_2 = GF_u \equiv G(g_{ux}F_{ru} - F_{ux})$ and $\phi_1 = G^2F_x \equiv F_{ru}$. In the case when $\phi_2 = 0 \Leftrightarrow F_u = 0$, the electromagnetic field is *non-null*, characterized just by $\phi_1 = F_{ru}$. Contrarily, when $\phi_1 = 0 \Leftrightarrow F_x = 0$, it is *null (radiative)*, characterized just by $\phi_2 = -GF_{ux}$.

(4) Evaluating the energy-momentum tensor (34) we derived that, in terms of these quantities, the *Einstein-Maxwell field equations take a simple form* (38),

$$R_{ab} = 2\Lambda g_{ab} + \kappa_0 G^2 F_a F_b, \tag{258}$$

(equivalent to $R_{ab} = 2\Lambda g_{ab} + \kappa_0{}^*F_a{}^*F_b$) and (43), (41),

$$(GF_a)_{,b} = (GF_b)_{,a}, \qquad F_{[ab,c]} = 0. \tag{259}$$

(5) In the triad (6) of the metric (254), all *optical scalars* of a congruence generated by the privileged null vector field $\mathbf{k} = \partial_r$ *vanish except*, possibly, *expansion:*

$$\Theta = -(\ln G)_{,r}. \tag{260}$$

There are thus *two geometrically distinct classes of spacetimes* to be investigated:

(a) $\Theta = 0$, defining the nonexpanding *Kundt class*, with the metric function

$$G \equiv P(u,x), \tag{261}$$

(b) $\Theta \neq 0$, defining the expanding *Robinson-Trautman class*, with the metric function

$$G \equiv G(r,u,x). \tag{262}$$

(6) Keeping the full generality, we explicitly integrated the coupled system of the field equations (258) and (259) *both* for the Kundt and the Robinson-Trautman spacetimes. It turned out that, as in standard $3+1$ general relativity, the *Kundt class only admits aligned electromagnetic fields* while the *Robinson-Trautman class admits both aligned and nonaligned electromagnetic fields*. Therefore, we treated these three distinct families of spacetimes in three separate sections of our paper, namely Sec. V, Sec. VI, and Sec. VII, respectively.

(7) All *Kundt* spacetimes (Sec. V) with *necessarily aligned electromagnetic fields* have the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2(e+fr)\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + \left(a + br + \left(\Lambda + \frac{1}{4}P^2f^2 - \frac{\kappa_0}{2}Q^2\right)r^2\right)\mathrm{d}u^2, \tag{263}$$

and

$$\mathbf{F} = Q\mathrm{d}r\wedge\mathrm{d}u + (fQr - \xi)\mathrm{d}u\wedge\mathrm{d}x, \tag{264}$$

corresponding to the potential

$$\mathbf{A} = A_r\mathrm{d}r + A_x\mathrm{d}x, \tag{265}$$

where $A_r = -\int Q\mathrm{d}u$ and $A_x = r\int fQ\mathrm{d}u - \int \xi\mathrm{d}u$; see Eqs. (94)–(97). As summarized in Sec. V H, the function $Q(u,x)$ represents the *non-null* component, while the function $\xi(u,x)$ represents the *null* component of the Maxwell field. Their relation to the metric functions $P$, $e$, $f$ and $a$, $b$ is explicitly given by the Einstein-Maxwell equations (102)–(106). In Sec. V H we presented a basic description of these solutions, separately for *two geometrically distinct subclasses* $f = 0$ and $f \neq 0$.

This large family of nonexpanding Kundt spacetimes contains many interesting subclasses which represent electrovacuum universes and also waves on these cosmological backgrounds. The simplest of them are gravitational and electromagnetic *pp waves* with $\Lambda = 0$. These are defined by the condition $k_{a;b} = \frac{1}{2}g_{ab,r} = 0$ which requires $f = 0$, $b = 0$, $Q = 0$. The field equations (107)–(111) then yield the explicit metric in the Brinkmann form [33]:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{P^2} + 2e\mathrm{d}u\mathrm{d}x - 2\mathrm{d}u\mathrm{d}r + a\mathrm{d}u^2, \tag{266}$$

and the coupled electromagnetic wave:

$$\mathbf{F} = -\frac{\gamma(u)}{P(u,x)}\mathrm{d}u\wedge\mathrm{d}x, \tag{267}$$

corresponding to

$$\mathbf{A} = A_x\mathrm{d}x \quad \text{where} \quad A_x = -\int\frac{\gamma(u)}{P(u,x)}\mathrm{d}u. \tag{268}$$

Here $\gamma(u)$ is an *arbitrary profile function* of the retarded time $u$, while the metric function $a(u,x)$ is obtained by integrating the only remaining field equation (111).

(8) All *Robinson-Trautman* spacetimes (Sec. VI) with *aligned electromagnetic fields* [for which the metric function $G$ simplifies to $G = P(u,x)/r$] can be written as

$$\mathrm{d}s^2 = \frac{r^2}{P^2}(\mathrm{d}x + eP^2\mathrm{d}u)^2 - 2\mathrm{d}u\mathrm{d}r + \left(\mu Q^2 - \kappa_0 Q^2\ln\left|\frac{Q}{r}\right| + 2(\ln Q)_{,u}r + \Lambda r^2\right)\mathrm{d}u^2, \tag{269}$$

with

$$\mathbf{F} = \frac{Q(u)}{r}\mathrm{d}r\wedge\mathrm{d}u \quad \text{corresponding to}$$

$$\mathbf{A} = Q(u)\ln\frac{r}{r_0}\mathrm{d}u; \tag{270}$$

see Eqs. (180)–(182). Here $\mu$ is a constant while the metric functions $P$ and $e$ satisfy the field equation (178), that is

$$\left(\frac{Q}{P}\right)_{,u} = Q(eP)_{,x}. \tag{271}$$

The dual 1-form Maxwell field reads

$$^{*}\mathbf{F} = \frac{Q}{P}\mathrm{d}x + ePQ\mathrm{d}u. \tag{272}$$

As summarized in Sec. VI H, the function $Q(u)$ gives the *non-null* component $\phi_1 = Q(u)/r$ of the

Maxwell field. Somewhat surprisingly, there is also an additional *null* (radiative) component $\phi_2 = ePQ$ when $e \neq 0$. However, such Maxwell fields *cannot be purely null* because $\phi_1 = 0$ implies $\phi_2 = 0$.

The simplest $e \neq 0$ solution of the field equation (271) is $P = 1$, $e = x(\ln Q)_{,u} + \alpha(u)$, which yields the metric (186).

Another interesting subclass (188) arises for factorized $P$ such that $P = Q(u)\beta(x)$ and $eP = \alpha(u)$. The special case $\alpha = 0$ and $Q = \text{const}$ of these expanding Robinson-Trautman spacetimes is equivalent to the solution (192), (193),

$$\mathrm{d}s^2 = -\Phi(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi(r)} + r^2\mathrm{d}\varphi^2,$$

$$\Phi(r) = -m + \kappa_0 Q^2 \ln\left|\frac{Q}{r}\right| - \Lambda r^2, \qquad (273)$$

which is the family of *cyclic symmetric, electrostatic black holes with* $\Lambda$ found in [24] and discussed in Sec. 11.2 of [6].

(9) The complementary class of *Robinson-Trautman* spacetimes with *nonaligned electromagnetic fields* is presented in Sec. VII. In this more complex case, the metric has the form (254) with a general function $G(r, u, x)$; cf. (262). Moreover, the electromagnetic field now has a nontrivial component $\phi_0 \neq 0 \Leftrightarrow F_{rx} \neq 0 \Leftrightarrow F_r \neq 0$, which considerably complicates the solution of the Einstein-Maxwell equations.

Nevertheless, we were able to explicitly express the generic three components of the Maxwell field *separately* in terms (of the combination) of the *metric functions* as

$$F_r = \sqrt{\frac{\alpha}{\kappa_0}}, \qquad F_x = \left(\frac{\beta}{\alpha} - g_{ux}\right)F_r, \qquad F_u = \frac{\gamma}{\alpha}F_r, \qquad (274)$$

where the functions $\alpha, \beta, \gamma$ are defined in (207)–(209). Interestingly, $\alpha$ is determined only by $g_{xx}$, $\beta$ is determined by $g_{xx}$ and $g_{ux}$, while the third metric component $g_{uu}$ enters only $\gamma$.

We also derived a fully explicit form (216)–(218), (222) of all four Maxwell equations (259). Finally, there are three remaining Einstein equations (223)–(225). This system of seven equations involves only three *metric* functions. After their solution is found, all components $F_r$, $F_x$, $F_u$ of the corresponding electromagnetic field are easily obtained using (274). In this sense, *we have achieved a separation of the variables representing the gravitational and the electromagnetic field*.

Although at present it is not possible for us to find a *general* solution to these seven equations, the formulation of the problem presented here seems to be useful. This fact has been demonstrated in Sec. VII F, where we have explicitly identified a *particular solution with nonaligned electromagnetic field*

$$\mathrm{d}s^2 = \sin^2(Cr)\mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}r, \qquad (275)$$

with

$$\mathbf{F} = \frac{C}{\sqrt{\kappa_0}}\sin(Cr)\mathrm{d}r \wedge \mathrm{d}x \quad \text{corresponding to}$$

$$\mathbf{A} = -\frac{1}{\sqrt{\kappa_0}}\cos(Cr)\mathrm{d}x. \qquad (276)$$

This special exact Robinson-Trautman spacetime contains electromagnetic field which has *only* the nonaligned component $F_r = (C/\sqrt{\kappa_0})\sin(Cr)$. It admits four Killing vectors [31,32,34].

Of course, many questions have remained open. First of all, it is necessary to find explicit relations to already known solutions summarized in [6]. Some basic identifications have already been presented here, namely:

(i) Maximally symmetric backgrounds (Minkowski, de Sitter, AdS) are contained both in the Kundt and Robinson-Trautman class of spacetimes (263) and (269), respectively.

(ii) There are electrovacuum backgrounds in the form of direct-product geometries, such as the $2+1$ analog of the exceptional Plebański-Hacyan metric with $\Lambda < 0$ and uniform Maxwell field (123).

(iii) We identified the complete family of *pp* waves in flat space, which are spacetimes admitting a covariantly constant null vecor field. In the Brinkmann form (266) they include the off-diagonal metric terms.

(iv) Within the Robinson-Trautman class with aligned fields we explicitly identified the cyclic symmetric charged black holes with any cosmological constant and electrostatic field (273).

Our main problem now is to identify all other known classes of solutions in $2+1$ dimensions by using specific invariant geometrical characterizations (such as an algebraic structure, symmetries, identification of rotation, and acceleration of the sources, etc.). Subsequently, explicit coordinate transformation must be found to relate our form of the solutions to those derived previously.

After identification of new spacetimes, their geometrical and physical analysis should be performed. Also, a systematic integration of the field equations for nonaligned Maxwell fields in the Robinson-Trautman class is desirable. However, these tasks are left for future works.

## ACKNOWLEDGMENTS

This work was supported by the Czech Science Foundation Grant No. GAČR 20-05421S. We are also grateful to the referee for many useful comments and suggestions which helped us to improve this paper.

## APPENDIX: CONNECTIONS AND CURVATURE COMPONENTS IN CANONICAL COORDINATES

The Christoffel symbols for the general nontwisting spacetime (3) after applying the condition (7) are

$$\Gamma^r_{rr} = 0, \tag{A1}$$

$$\Gamma^r_{ru} = -\frac{1}{2} g_{uu,r} + \frac{1}{2} g^{rx} g_{ux,r}, \tag{A2}$$

$$\Gamma^r_{rx} = -\frac{1}{2} g_{ux,r} + \Theta g_{ux}, \tag{A3}$$

$$\Gamma^r_{uu} = \frac{1}{2}\left[-g^{rr} g_{uu,r} - g_{uu,u} + g^{rx}(2g_{ux,u} - g_{uu,x})\right], \tag{A4}$$

$$\Gamma^r_{ux} = \frac{1}{2}\left[-g^{rr} g_{ux,r} - g_{uu,x} + g^{rx} g_{xx,u}\right], \tag{A5}$$

$$\Gamma^r_{xx} = -\Theta g^{rr} g_{xx} - g_{ux||x} + \frac{1}{2} g_{xx,u}, \tag{A6}$$

$$\Gamma^u_{rr} = \Gamma^u_{ru} = \Gamma^u_{rx} = 0, \tag{A7}$$

$$\Gamma^u_{uu} = \frac{1}{2} g_{uu,r}, \tag{A8}$$

$$\Gamma^u_{ux} = \frac{1}{2} g_{ux,r}, \tag{A9}$$

$$\Gamma^u_{xx} = \Theta g_{xx}, \tag{A10}$$

$$\Gamma^x_{rr} = 0, \tag{A11}$$

$$\Gamma^x_{ru} = \frac{1}{2} g^{xx} g_{ux,r}, \tag{A12}$$

$$\Gamma^x_{rx} = \Theta, \tag{A13}$$

$$\Gamma^x_{uu} = \frac{1}{2}\left[-g^{rx} g_{uu,r} + g^{xx}(2g_{ux,u} - g_{uu,x})\right], \tag{A14}$$

$$\Gamma^x_{ux} = \frac{1}{2}\left[-g^{rx} g_{ux,r} + g^{xx} g_{xx,u}\right], \tag{A15}$$

$$\Gamma^x_{xx} = -\Theta g^{rx} g_{xx} + {}^S\Gamma^x_{xx}, \tag{A16}$$

where

$${}^S\Gamma^x_{xx} \equiv \frac{1}{2} g^{xx} g_{xx,x} = -\frac{G_{,x}}{G} \tag{A17}$$

is the Christoffel symbol with respect to the only spatial coordinate $x$, i.e., coefficient of the covariant derivative on the transverse one-dimensional space spanned by $x$.

The nonvanishing Riemann curvature tensor components are then

$$R_{rxrx} = -(\Theta_{,r} + \Theta^2) g_{xx}, \tag{A18}$$

$$R_{rxru} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r}, \tag{A19}$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} g^{xx}(g_{ux,r})^2, \tag{A20}$$

$$R_{rxux} = \frac{1}{2} g_{ux,r||x} + \frac{1}{4}(g_{ux,r})^2 - g_{xx}\Theta_{,u} - \frac{1}{2}\Theta(g_{xx,u} + g_{xx} g_{uu,r}), \tag{A21}$$

$$R_{ruux} = g_{u[u,x],r} + \frac{1}{4} g^{rx}(g_{ux,r})^2 - \frac{1}{4} g^{xx} g_{xx,u} g_{ux,r} + \Theta\left(g_{ux,u} - \frac{1}{2} g_{uu,x} - \frac{1}{2} g_{ux} g_{uu,r}\right), \tag{A22}$$

$$\begin{aligned} R_{uxux} = {} & -\frac{1}{2}(g_{uu})_{||xx} + g_{ux,u||x} - \frac{1}{2} g_{xx,uu} + \frac{1}{4} g^{rr}(g_{ux,r})^2 - \frac{1}{2} g_{uu,r} e_{xx} + \frac{1}{2} g_{uu,x} g_{ux,r} - \frac{1}{2} g^{rx} g_{xx,u} g_{ux,r} + \frac{1}{4} g^{xx}(g_{xx,u})^2 \\ & -\frac{1}{2}\Theta g_{xx}[g^{rr} g_{uu,r} + g_{uu,u} - g^{rx}(2g_{ux,u} - g_{uu,x})]. \end{aligned} \tag{A23}$$

Finally, the components of the Ricci tensor are

$$R_{rr} = -(\Theta_{,r} + \Theta^2), \tag{A24}$$

$$R_{rx} = -\frac{1}{2} g_{ux,rr} + \frac{1}{2}\Theta g_{ux,r} + (\Theta_{,r} + \Theta^2) g_{ux}, \tag{A25}$$

$$R_{ru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{2} g^{rx} g_{ux,rr} + \frac{1}{2} g^{xx}(g_{ux,r||x} + (g_{ux,r})^2) - \Theta_{,u} - \frac{1}{2}\Theta(g^{xx} g_{xx,u} + g^{rx} g_{ux,r} + g_{uu,r}), \tag{A26}$$

$$R_{xx} = -g_{xx}g^{rr}(\Theta_{,r}+\Theta^2) + 2g_{xx}(\Theta_{,u} - g^{rx}\Theta_{,x}) + 2g_{ux}\Theta_{,x} - f_{xx} + \Theta[2g_{ux\|x} + 2g_{ux,r}g_{ux} + g_{xx}(g_{uu,r} - 2g^{rx}g_{ux,r}) - 2e_{xx}], \tag{A27}$$

$$R_{ux} = -\frac{1}{2}g^{rr}g_{ux,rr} - \frac{1}{2}g_{uu,rx} + \frac{1}{2}g_{ux,ru} - \frac{1}{2}g^{rx}\left[g_{ux,r\|x} + (g_{ux,r})^2\right] + g^{xx}\left(\frac{1}{2}g_{ux,r}g_{ux\|x} - \frac{1}{2}e_{xx}g_{ux,r}\right) + g_{ux}\Theta_{,u}$$
$$+\Theta\left[g_{ux}g_{uu,r} - \frac{1}{2}(g_{uu}g_{ux,r} - g_{uu,x}) - g_{ux,u} + \frac{1}{2}g^{rx}g_{ux,r}g_{ux} + \frac{1}{2}g^{rx}g_{xx,u}\right], \tag{A28}$$

$$R_{uu} = -\frac{1}{2}g^{rr}g_{uu,rr} - g^{rx}g_{uu,rx} - \frac{1}{2}g^{xx}e_{xx}g_{uu,r} + g^{rx}g_{ux,ru} - \frac{1}{2}g^{xx}g_{xx,uu}$$
$$+g^{xx}\left(g_{ux,u\|x} - \frac{1}{2}g_{uu\|xx}\right) + \frac{1}{2}(g^{rr}g^{xx} - g^{rx}g^{rx})(g_{ux,r})^2 + \frac{1}{2}g^{xx}g_{ux,r}g_{uu,x} + \frac{1}{4}(g^{xx}g_{xx,u})^2$$
$$+\frac{1}{2}\Theta\left[-g^{rx}(2g_{ux,u} - g_{uu,x} - g_{ux}g_{uu,r}) + g_{uu}g_{uu,r} - g_{uu,u}\right], \tag{A29}$$

and the Ricci scalar is

$$R = g_{uu,rr} - 2g^{rx}g_{ux,rr} - 2g^{xx}g_{ux,r\|x} - \frac{3}{2}g^{xx}(g_{ux,r})^2 + 2\Theta_{,r}g_{uu} + 4\Theta_{,u} + 2\Theta^2 g_{uu} + \Theta(2g_{uu,r} + 2g^{rx}g_{ux,r} + 2g^{xx}g_{xx,u}). \tag{A30}$$

The symbol $_\|$ denotes the covariant derivative with respect to $g_{xx}$:

$$g_{ux\|x} = g_{ux,x} - g_{ux}{}^S\Gamma^x_{xx}, \tag{A31}$$

$$g_{ux,r\|x} = g_{ux,rx} - g_{ux,r}{}^S\Gamma^x_{xx}, \tag{A32}$$

$$g_{ux,u\|x} = g_{ux,ux} - g_{ux,u}{}^S\Gamma^x_{xx}, \tag{A33}$$

$$(g_{uu})_{\|xx} = g_{uu,xx} - g_{uu,x}{}^S\Gamma^x_{xx}, \tag{A34}$$

where $e_{xx}$ and $f_{xx}$ are convenient shorthand defined as

$$e_{xx} \equiv g_{ux\|x} - \frac{1}{2}g_{xx,u}, \tag{A35}$$

$$f_{xx} \equiv g_{ux,r\|x} + \frac{1}{2}(g_{ux,r})^2. \tag{A36}$$

The expressions (A24)–(A29) of the Ricci tensor enable us to write explicitly the gravitational field equations for any $D = 3$ Kundt or Robinson-Trautman spacetime.


[1] J. Podolský, R. Švarc, and H. Maeda, All solutions of Einstein's equations in $2+1$ dimensions: Λ-vacuum, pure radiation, or gyratons, Classical Quantum Gravity **36**, 015009 (2019).

[2] M. Bañados, C. Teitelboim, and J. Zanelli, The Black Hole in Three-Dimensional Space-Time, Phys. Rev. Lett. **69**, 1849 (1992).

[3] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the $(2+1)$ black hole, Phys. Rev. D **48**, 1506 (1993).

[4] C. Martinez, C. Teitelboim, and J. Zanelli, Charged rotating black hole in three space-time dimensions, Phys. Rev. D **61**, 104013 (2000).

[5] S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, Cambridge, 2003).

[6] A. A. García-Díaz, *Exact Solutions in Three-Dimensional Gravity* (Cambridge University Press, Cambridge, 2017).

[7] D. S. Krongos and C. G. Torre, Rainich conditions in $(2+1)$-dimensional gravity, J. Math. Phys. (N.Y.) **58**, 012501 (2017).

[8] D. D. K. Chow, C. N. Pope, and E. Sezgin, Kundt spacetimes as solutions of topological massive gravity, Classical Quantum Gravity **27**, 105002 (2010).

[9] W. Kundt, The plane-fronted gravitational waves, Z. Phys. **163**, 77 (1961).

[10] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003).

[11] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, 2009).

[12] J. Podolský and M. Žofka, General Kundt spacetimes in higher dimensions, Classical Quantum Gravity **26**, 105008 (2009).
[13] M. Ortaggio, V. Pravda, and A. Pravdová, Algebraic classification of higher dimensional spacetimes based on null alignment, Classical Quantum Gravity **30**, 013001 (2013).
[14] I. Robinson and A. Trautman, Spherical Gravitational Waves, Phys. Rev. Lett. **4**, 431 (1960).
[15] I. Robinson and A. Trautman, Some spherical gravitational waves in general relativity, Proc. R. Soc. A **265**, 463 (1962).
[16] J. Podolský and M. Ortaggio, Robinson-Trautman spacetimes in higher dimensions, Classical Quantum Gravity **23**, 5785 (2006).
[17] M. Ortaggio, J. Podolský, and M. Žofka, Robinson-Trautman spacetimes with an electromagnetic field in higher dimensions, Classical Quantum Gravity **25**, 025006 (2008).
[18] M. Ortaggio, J. Podolský, and M. Žofka, Static and radiating p-form black holes in the higher dimensional Robinson-Trautman class, J. High Energy Phys. 02 (2015) 045.
[19] H. Maeda and C. Martínez, Energy conditions in arbitrary dimensions, Prog. Theor. Exp. Phys. **2020**, 043E02 (2020).
[20] J. Podolský and R. Švarc, Explicit algebraic classification of Kundt geometries in any dimension, Classical Quantum Gravity **30**, 125007 (2013).
[21] J. Podolský and R. Švarc, Algebraic structure of Robinson-Trautman and Kundt geometries in arbitrary dimension, Classical Quantum Gravity **32**, 015001 (2015).
[22] J. Podolský and R. Švarc, Algebraic classification of Robinson-Trautman spacetimes, Phys. Rev. D **94**, 064043 (2016).
[23] J. F. Plebański and S. Hacyan, Some exceptional electrovac type D metrics with cosmological constant, J. Math. Phys. (N.Y.) **20**, 1004 (1979).
[24] P. Peldan, Unification of gravity and Yang-Mills theory in $(2+1)$-dimensions, Nucl. Phys. **B395**, 239 (1993).
[25] J. R. Gott and M. Alpert, General relativity in a $(2+1)$-dimensional spacetime, Gen. Relativ. Gravit. **16**, 243 (1984).
[26] J. R. Gott, J. Z. Simon, and M. Alpert, General relativity in a $(2+1)$-dimensional spacetime: An electrically charged solution, Gen. Relativ. Gravit. **18**, 1019 (1986).
[27] S. Deser and P. O. Mazur, Static solutions in $D=3$ Einstein-Maxwell theory, Classical Quantum Gravity **2**, L51 (1985).
[28] M. A. Melvin, Exterior solutions for electric and magnetic stars in $2+1$ dimensions, Classical Quantum Gravity **3**, 117 (1986).
[29] A. A. García, Three-dimensional stationary cyclic symmetric Einstein-Maxwell solutions; black holes, Ann. Phys. (Amsterdam) **324**, 2004 (2009).
[30] A. A. García-Díaz, Three dimensional stationary cyclic symmetric Einstein-Maxwell solutions; energy, mass, momentum, and algebraic tensors characteristics, arXiv: 1307.6652.
[31] G. I. Kruchkovich, Invariant criteria of spaces $V_3$ with the group of motions $G_4$, Usp. Mat. Nauk, **10**, 129 (1955).
[32] D. D. K. Chow, Characterization of three-dimensional Lorentzian metrics that admit four Killing vectors, arXiv: 1903.10496.
[33] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. **94**, 119 (1925).
[34] G. Clément, Classical solutions in three-dimensional Einstein-Maxwell cosmological gravity, Classical Quantum Gravity **10**, L49 (1993).