FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

David Mikšaník

The combinatorics of pattern-avoiding matrices

Computer Science Institute of Charles University

Supervisor of the master thesis: doc. RNDr. Vít Jelínek, Ph.D.
Study programme: Computer Science
Study branch: Discrete Models and Algorithms

Prague 2023

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank my supervisor, doc. RNDr. Vít Jelínek, Ph.D., for all his guidance, feedback, and help with this thesis. I would also like to thank my family for their endless support.

Title: The combinatorics of pattern-avoiding matrices

Author: David Mikšaník

Institute: Computer Science Institute of Charles University

Supervisor: doc. RNDr. Vít Jelínek, Ph.D., Computer Science Institute of Charles University

Abstract: A permutation matrix P partially avoids a quasi-permutation matrix A (i.e., a 01-matrix such that each column and row of A contains at most one nonzero entry) if there is no submatrix P' of P of the same size as A satisfying $A_{i,j} \leq P'_{i,j}$ for all possible indices i and j. Two quasi-permutation matrices A and B are partially Wilf-equivalent if, for every $n \in \mathbb{N}$, the number of permutation matrices of order n partially avoiding A is the same as the number of permutation matrices of order n partially avoiding B. This generalizes the well-known concept of avoidance and Wilf equivalence of permutations. One of the central topics in this area is the classification of permutations of order k into Wilf equivalence classes. The complete classification is known for $k = 1, 2, \ldots, 7$. In our thesis, we study the same problem for quasi-permutation matrices. Namely, we classify all 371 quasi-permutation matrices of size at most 4×4 into partial Wilf equivalence classes (two quasi-permutation matrices belong to the same class if and only if they are partially Wilf-equivalent). Along the way, we prove several general results showing how to construct from one or two quasi-permutation matrices more quasi-permutation matrices that are pairwise partially Wilf-equivalent.

Keywords: forbidden patterns, matrices, permutations, Wilf equivalence

Contents

Introduction

A *quasi-permutation matrix* Q of size $m \times k$ is a 01-matrix (every entry of Q is either zero or one) with m columns and k rows such that each column and row contains at most one nonzero entry. A quasi-permutation matrix Q *exactly contains* (as opposed to "partially" introduced later) a quasi permutation matrix A if A is a submatrix of Q. If Q does not contain A, we say that Q *exactly avoids* A. In this context, the quasi-permutation matrix A is usually called a *pattern*. A *permutation matrix* P of order n is a quasi-permutation matrix of size $n \times n$ with exactly n nonzero entries. Let $\mathcal{P}_n$ be the set of all permutation matrices of order n. Moreover, for a pattern A, let $\mathcal{P}_n^{\mathrm{E}}(A)$ be the set of all permutation matrices of order n that exactly avoid A and $p_n^{\mathrm{E}}(A) := |\mathcal{P}_n^{\mathrm{E}}(A)|$.

If the patterns in the previous paragraph are only permutation matrices, we obtain the well-known concept of avoidance of permutation matrices. One of the central topics in this area is the classification of patterns into exact Wilf equivalence classes. Formally, patterns A and B are *exactly Wilf-equivalent* if $p_n^{\mathrm{E}}(A) = p_n^{\mathrm{E}}(B)$ for every $n \in \mathbb{N}$. The classification is the problem of partitioning the set $\mathcal{Q}$ of all quasi-permutation matrices into *exact Wilf equivalence classes* so that two patterns are in the same class if and only if they are exactly Wilf-equivalent.

We briefly mention the known results on the classification of permutation matrices into exact Wilf equivalence classes. In this paragraph only, a pattern of order k is considered to be a permutation matrix of order k, and the words "exact" or "exactly" are omitted. Since patterns of different order are never Wilf-equivalent, it is sufficient to partition only the set $\mathcal{P}_k$ of patterns of order k into Wilf equivalence classes for every k. The classification is trivial for $k = 1, 2$. Any pattern of order 3 is avoided by c_n permutations of order n, where c_n is the n-th Catalan number. Hence all patterns of order 3 belong to the same Wilf equivalence class (e.g., [1, Section 4.2]). This is no longer true for $k = 4$. The $4! = 24$ patterns of order 4 are partitioned into three Wilf equivalence classes (see [1, Section 4.4] and the references therein). The complete classification of patterns into Wilf equivalence classes is also known for $k = 5, 6, 7$ (see [2] and the references therein).

Classification of permutation matrices into exact Wilf equivalence classes is usually studied in terms of permutations. The definition given in the first and second paragraphs is a straightforward translation from permutations to permutation matrices. However, there is another possible definition of avoidance of quasi-permutation matrices that generalizes the exact avoidance of permutation matrices. For quasi-permutation matrices Q and R of the same size, we write $Q \leq R$ if $Q_{i,j} \leq R_{i,j}$ for all possible column indices i and row indices j. We say that a quasi-permutation matrix Q *partially contains* a quasi-permutation matrix A if there exists a submatrix Q' of Q such that $A \leq Q'$. We remark that if we replace '$\leq$' by '$=$' (the standard matrix equality symbol), we obtained the definition of exact containment introduced earlier. If Q does not partially contain A, we say that Q *partially avoids* A. We give a small example demonstrating the difference between exact and partial avoidance: $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ exactly avoids $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ but Q partially contains A. For a permutation matrix P and a permutation matrix A, we will show that P exactly avoids A if and only if P partially avoids A. In fact, the equivalence still true if A is replaced by a row-permutation matrix (i.e., a quasi-permutation matrix such that each row contains exactly one nonzero entry). The equivalence is also true if A is replaced by a column-permutation matrix, which is defined analogously.

In analogy with the concept of exact Wilf equivalence, we say that A and B are *partially Wilf-equivalent* if the number $p_n^{\mathrm{P}}(A)$ of permutation matrices of order n partially avoiding A is the same as the number $p_n^{\mathrm{P}}(B)$ of permutation matrices of order n partially avoiding B for every n. In our thesis, we classify all 371 patterns of size at most 4×4 into partial Wilf equivalence classes (two patterns belong to the same class if and only if they are partially Wilf -equivalent). Along the way, we prove several general results showing how to construct from one or two quasi-permutation matrices more quasi-permutation matrices that are pairwise partially Wilf-equivalent.

Outline

In the first chapter, we properly define the two types of avoidance (exact and partial) of quasi-permutation matrices. We show that both types generalize the well-known concept of avoidance of permutations. We define three basic symmetry operations that preserve the number of permutation matrices avoiding a given pattern. Finally, we introduce row- and column-permutation matrices on which the exact and partial avoidance also agree.

In the following two chapters, we develop general results that play a crucial role in the classification of patterns of size 4×4 into partial Wilf equivalence classes. Namely, in Chapter 2, we study a connection between patterns obtained from a pattern by appending a zero column or a zero row to any side of the pattern. For example, for a pattern A, we show that $A|0$ (i.e., a pattern obtained from A by adding a zero column after the last column of A) and $0|A$ are partially Wilf-equivalent. If A is in addition a row-permutation matrix, we show that $p_n^{\mathrm{P}}(A|0) = n \cdot p_{n-1}^{\mathrm{P}}(A)$.

In chapter 3, we define shape-Wilf equivalence and provide a short survey of previously-known results about shape-Wilf equivalence between permutation matrices. We observe that one of these results can be generalized to quasi-permutation matrices: if A and B are shape-Wilf-equivalent quasi-permutation matrices and Q is an arbitrary quasi-permutation matrix, then

$$\begin{pmatrix} 0 & Q \\ A & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & Q \\ B & 0 \end{pmatrix}$$

are partially Wilf-equivalent. Since $X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are not shape Wilf-equivalent (we will prove this fact later), the previous theorem says nothing whether the patterns

$$X^+ := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y^+ := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

are partially Wilf-equivalent. In fact, they are not because $p_6^{\mathrm{P}}(X^+) = 434$ but $p_6^{\mathrm{P}}(Y^+) = 430$ (these values were computed by our program [3]). Nevertheless, we prove the following theorem: if Q is a quasi-permutation matrices such that its first column is nonzero, then

$$\begin{pmatrix} 0 & Q \\ X & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & Q \\ Y & 0 \end{pmatrix}$$

are partially Wilf-equivalent.

In Chapter 4, we finally classify all patterns of size at most 4×4 into partial Wilf equivalence classes. We conclude our thesis with possible further directions and open problems in Chapter 5.

1. Basic notation, definitions, and results

We describe the outline of this chapter. In Section 1.1, we introduce basic notation and definitions, which we use repeatedly through this thesis. In Section 1.2, we generalize the concept of permutation avoidance to quasi-permutation matrices (i.e., matrices with at most one non-zero entry in each row and column). In Section 1.3, we define row-permutation matrices (i.e., matrices with exactly one nonzero entry in each row and with at most one nonzero entry in each column). Similarly, we define column-permutation matrices. In the next chapter, we obtain more detailed results when we consider only row-permutation (or column-permutation) matrices.

1.1 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers $\{1, 2, \dots\}$ and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we define $[n]$ to be the set $\{1, 2, \dots, n\}$ and $[0] := \emptyset$. For $n, k \in \mathbb{N}_0$, the *factorial of* n is denoted by $n!$ (we suppose that $0! = 1$) and, if $k \leq n$, the symbol $(n)_k$ is defined as $(n)_k := n!/(n-k)! = \prod_{i=0}^{k-1}(n-i)$. Most of the time we assume that the numbers under consideration are natural numbers.

Convention I: Unless otherwise stated the letters i, j, k, ℓ, m, n and their variants such as $i', i_1, i_2, \dots$ are always assumed to be natural numbers.

A linear ordering $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_n$ of elements of the set $[n]$ is called a *permutation* of order n. Let $\mathcal{S}_n$ be the set of all permutations of order n. Every permutation σ of order n can be represented by a matrix $P^\sigma \in \{0,1\}^{n \times n}$ satisfying

$$\forall i, j \in [n] \ : \ P^\sigma_{i,j} = 1 \iff \sigma_i = j, \qquad (\star)$$

where $P^\sigma_{i,j}$ is the entry in the i-th column and the j-th row of P^σ (see an example in Figure 1.1). Every matrix P^σ has exactly one nonzero entry in each row and column. Such matrices have a special name.

Definition 1. A matrix $P \in \{0,1\}^{n \times n}$ is a *permutation matrix of order* n if every row and every column of P contain exactly one nonzero entry.

Let $\mathcal{P}_n$ be the set of all permutation matrices of order n. Observe that $(\star)$ defines a one-to-one correspondence between the set $\mathcal{S}_n$ of all permutations and the set $\mathcal{P}_n$ of all permutation matrices. Indeed, map a permutation σ to a permutation matrix P^σ.

We prefer to work with permutation matrices rather than permutations for the following reason—the concept of permutation avoidance (we define this notion in the next section) seen in terms of permutation matrices can be straightforwardly generalized to a larger set of matrices. In this thesis, we work only with matrices in which each entry is either 0 (zero) or 1 (nonzero).

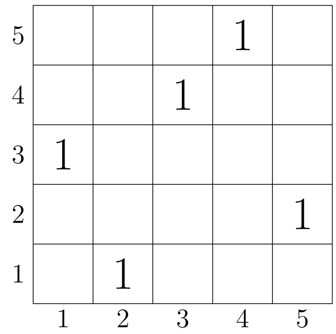

Figure 1.1: Representation of permutation $\sigma = 3, 1, 4, 5, 2$ by a matrix P^σ satisfying ($\star$). For clarity, we do not draw zeros (they are represented by empty entries).

Convention II: All matrices have entries in $\{0, 1\}$.

A matrix A is *of size* $m \times k$ if it has m columns and k rows. Let $\{0, 1\}^{m \times k}$ be the set of all matrices of size $m \times k$. We denote by $A_{i,j}$ the entry in the i-th column and j-th row of a matrix A. For two matrices A and B of the same size, we write $A = B$ (or $A \leq B$) if and only if $A_{i,j} = B_{i,j}$ (or $A_{i,j} \leq B_{i,j}$) for all indices i and j.

For a matrix $A \in \{0, 1\}^{m \times k}$ and two sets of indices $I = \{i_1, i_2, \ldots, i_{m'}\} \subseteq [m]$, $J = \{j_1, j_2, \ldots, j_{k'}\} \subseteq [k]$ such that $i_1 < i_2 < \cdots < i_{m'}$ and $j_1 < j_2 < \cdots < j_{k'}$, a *submatrix of A induced by* (I, J) is the matrix $A[I \times J]$ obtained from A by erasing its i-th column and j-th row for all $i \in [m] \setminus I$ and $j \in [k] \setminus J$. Formally, $A[I \times J]$ is the matrix of size $m' \times k'$ such that

$$\forall a \in [m'],\ \forall b \in [k'] \ :\ A[I \times J]_{a,b} = A_{i_a, i_b}.$$

We implicitly assume that the sets of indices I and J are ordered increasingly. In general, any set of natural numbers in this thesis is ordered increasingly.

Convention III: Every set $N = \{n_1, n_2, \ldots, n_k\}$ of natural numbers is assumed to be ordered increasingly. That is, $n_i < n_j$ whenever $i < j$.

A matrix B is a *submatrix* of $A \in \{0, 1\}^{m \times k}$ if $B = A[I \times J]$ for some set of column indices $I \subseteq [m]$ and some set of row indices $J \subseteq [k]$.

Lastly, we draw matrices such that the columns and rows are indexed from the left to right and from the bottom to top, respectively, by the first natural numbers. Therefore, there is no ambiguity if we draw matrices without explicitly indexing rows and columns. When we write small matrices in the text we replace 1 by $\bullet$ and 0 by $\circ$. For example, for a permutation $\sigma = 2, 3, 1$, we write

$$P^\sigma = \begin{pmatrix} \circ & \bullet & \circ \\ \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{pmatrix} \quad \text{instead of} \quad P^\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, when we draw matrices in figures, we depict only ones, while zeros are represented by empty entries.

1.2 Avoiding quasi-permutation matrices

In this section, we develop the foundations of our work—we introduce quasi-permutation matrices and the concept of avoiding quasi-permutation matrices. We start with recalling a similar concept of avoiding permutations.

Let $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_n$ and $\tau = \tau_1, \tau_2, \cdots, \tau_k$ be two permutations. We say that σ *avoids* τ if there is no sequence of indices

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ such that } \forall a, b \in [k] : \tau_a < \tau_b \iff \sigma_{i_a} < \sigma_{i_b}.$$

In other words, σ avoids τ if there is no linear suborder of σ isomorphic to τ. The definition can be rephrased in the terms of permutation matrices. We say that a permutation matrix P_1 *avoids* a permutation matrix P_2 if P_2 is not a submatrix of P_1. Then σ avoids τ if and only if P^σ avoids P^τ. In this context, the permutation τ and the permutation matrix P^τ to be avoided are usually called *patterns of order* k or simply *patterns*.

Given a pattern A, the ultimate goal is find, for all n, the number $p_n(A)$ of permutation matrices of order n that avoid A. For example, if $A = \begin{pmatrix} \circ & \bullet & \circ \\ \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{pmatrix}$, then $p_n(A)$ is the n-th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. Interestingly, if we consider any other pattern A of order 3, then $p_n(A)$ is still c_n (see for example [1, Section 4.2]). Finding $p_n(A)$ for all patterns of order 4 is an open problem. For example, there is not known exact formula for $p_n(P^\sigma)$, where $\sigma = 1, 3, 2, 4$. A less ambitious goal is to partition the patterns of order 4 into classes such that $p_n(A) = p_n(B)$ if and only if A and B belong to the same class. We know that 3 classes are necessary and also sufficient for patterns of order 4 (see [1, Section 4.4] and the references therein).

We generalize the notion that a permutation matrix P_1 avoids a permutation matrix P_2 to quasi-permutation matrices. Let us start with the definition of a quasi-permutation matrix

Definition 2. A matrix $Q \in \{0, 1\}^{m \times k}$ is a *quasi-permutation matrix* of size $m \times k$ if every row and every column of Q contains at most one nonzero entry.

Two notes about the definition. First, a quasi-permutation matrix may have a different number of rows and columns. Second, every permutation matrix is also a quasi-permutation matrix (but not vice versa). Let us denote by $\mathcal{Q}$ the set of all quasi-permutation matrices and by $\mathcal{Q}_{m,k,\ell}$ the set of all quasi-permutation matrices of size $m \times k$ with exactly ℓ nonzero entries. Note that $\mathcal{P}_n = \mathcal{Q}_{n,n,n}$.

Definition 3. Let $Q \in \mathcal{Q}_{m,k,\ell}$ and A be quasi-permutation matrices. We say that Q *exactly contains* A, written $A \preceq_{\mathrm{E}} Q$, if there exists a set of column indices $I \subseteq [m]$ and a set of row indices $J \subseteq [k]$ such that

$$A = Q[I \times J].$$

The submatrix $Q[I \times J]$ is called an *exact copy of* A *in* Q *induced by* (I, J). If Q does not exactly contain A, we say that Q *exactly avoids* A.

However, there is yet another natural way to define that "Q avoids A".

Definition 4. Let $Q \in \mathcal{Q}_{m,k,\ell}$ and A be quasi-permutation matrices. We say that Q *partially contains* A, written $A \preceq_{\mathrm{P}} Q$, if there exists a set of column indices $I \subseteq [m]$ and a set of row indices $J \subseteq [k]$ such that

$$A \leq Q[I \times J].$$

The submatrix $Q[I \times J]$ is called a *partial copy of* A *in* Q *induced by* (I, J). If Q does not exactly contain A, we say that Q *partially avoids* A.

In the context of Definitions 3 and 4, the quasi-permutation matrix $A \in \mathcal{Q}_{m',k',\ell'}$ is sometimes called a *pattern* or a *pattern of size* $m' \times k'$ *(with exactly* ℓ' *nonzero entries)*.

Clearly, if Q partially avoids A, then also Q exactly avoids A. We remark that the reverse implication is not true in general. For instance, $Q = \left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\right)$ exactly avoids $A = \left(\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix}\right)$ but Q partially contains A. However, the reverse implication is true when Q and A are permutation matrices. We prove this fact in the next section in more general settings (see Lemma 14). Hence, if Q and A are permutation matrices, Q exactly avoids A if and only if Q partially avoids A. This shows that both definitions of avoidance for quasi-permutation matrices (Definitions 3 and 4) generalize the notion of avoidance for permutation matrices.

We introduce few more notions before we state the main goal of this thesis. Let $\mathcal{P}_n^{\mathrm{E}}(A)$ (or $\mathcal{P}_n^{\mathrm{P}}(A)$) denote the set of all permutation matrices of order n that exactly (or partially) avoid a pattern A. Moreover, let

$$p_n^{\mathrm{E}}(A) := |\mathcal{P}_n^{\mathrm{E}}(A)| \text{ and } p_n^{\mathrm{P}}(A) := |\mathcal{P}_n^{\mathrm{P}}(A)|.$$

Let $\overset{\mathrm{E}}{\sim}$ be a relation on $\mathcal{Q}$ given by $A \overset{\mathrm{E}}{\sim} B$ if and only if $p_n^{\mathrm{E}}(A) = p_n^{\mathrm{E}}(B)$. Similarly, let $\overset{\mathrm{P}}{\sim}$ be a relation on $\mathcal{Q}$ given by $A \overset{\mathrm{P}}{\sim} B$ if and only if $p_n^{\mathrm{P}}(A) = p_n^{\mathrm{P}}(B)$. It is easy to see that $\overset{\mathrm{E}}{\sim}$ and $\overset{\mathrm{P}}{\sim}$ are both equivalence relations.

Definition 5. The relation $\overset{\mathrm{E}}{\sim}$ is called *exact Wilf equivalence*. Patterns A and B are *exactly Wilf-equivalent* if $A \overset{\mathrm{E}}{\sim} B$. *Exact Wilf equivalence classes* are the equivalence classes of $\overset{\mathrm{E}}{\sim}$.

Definition 6. The relation $\overset{\mathrm{P}}{\sim}$ is called *partial Wilf equivalence*. Patterns A and B are *partially Wilf-equivalent* if $A \overset{\mathrm{P}}{\sim} B$. *Partial Wilf equivalence classes* are the equivalence classes of $\overset{\mathrm{P}}{\sim}$.

The ultimate goal in pattern avoidance is, given a pattern $A \in \mathcal{Q}_{m,k,\ell}$, determine $p_n^{\mathrm{E}}(A)$ and $p_n^{\mathrm{P}}(A)$ for all n. A more feasible goal is, for small values of m and k is, to determine the exact and partial Wilf equivalence class of every $A \in \mathcal{Q}_{m,k,\ell}$. In Chapter 4, we determine the partial Wilf equivalence classes of every pattern of size at most 4×4.

The first step to determine the equivalence classes of $\overset{\mathrm{E}}{\sim}$ and $\overset{\mathrm{P}}{\sim}$ is to study a symmetry relation on $\mathcal{Q}$ defined below. For that, we introduce the following matrix operation.

Definition 7. Let $Q \in \{0,1\}^{m \times k}$ be a quasi-permutation matrix.

- The *transpose* of Q is a quasi-permutation matrix $Q^\top \in \{0,1\}^{k \times m}$ such that

$$\forall i \in [k], j \in [m] \ : \ Q_{i,j} = (Q^\top)_{j,i}.$$

- The *column reversal* of Q is a quasi-permutation matrix $Q^C \in \{0,1\}^{m\times k}$ obtained by reversing the order of columns.
- The *row reversal* of Q is a matrix $Q^R \in \{0,1\}^{m\times k}$ obtained by reversing the order of rows.

The *symmetry* relation $\approx$ on $\mathcal{Q}$ is defined by $A \approx B$ if and only if B can be obtained from A by a sequence of transpose, column reversal, and row reversal operations. It is not hard to see that $\approx$ is an equivalence relation on $\mathcal{Q}$. The equivalence classes of $\approx$ are called *symmetry classes*. And if $A \approx B$, we say that A is *symmetric* to B or that A and B are *symmetric*.

We show that if two patterns are symmetric, then they are also partially (exactly) Wilf-equivalent. Hence the symmetry relation is a refinement of partial (exact) Wilf equivalence.

Observation 8. *Let Q and A be quasi-permutation matrices. Then*

(i) $A \preceq_P Q$ if and only if $A^\top \preceq_P Q^\top$,

(ii) $A \preceq_P Q$ if and only if $A^C \preceq_P Q^C$, and

(iii) $A \preceq_P Q$ if and only if $A^R \preceq_P Q^R$.

From Observation 8 readily follows the next observation.

Observation 9. *Let $P \in \mathcal{P}_n$ be a permutation matrix and let A be a quasi-permutation matrix. Then*

(i) $P \in \mathcal{P}_n^P(A)$ if and only if $P^\top \in \mathcal{P}_n^P(A^\top)$,

(ii) $P \in \mathcal{P}_n^P(A)$ if and only if $P^C \in \mathcal{P}_n^P(A^C)$, and

(iii) $P \in \mathcal{P}_n^P(A)$ if and only if $P^R \in \mathcal{P}_n^P(A^R)$.

We can finally conclude that two symmetric patterns A and B are also partially Wilf-equivalent.

Observation 10. *Let A and B be quasi-permutation matrices of the same size. If A and B are symmetric, then A and B are partially Wilf-equivalent. In symbols,*

$$A \approx B \implies A \overset{P}{\sim} B.$$

Proof. Since A and B are symmetric, there exists a sequence of matrix operations (see Definition 7) that transform A to B. For any n, take $P \in \mathcal{P}_n^{\mathrm{P}}(A)$ and transform P to P' using the same sequence of matrix operations. By Observation 9, $P \in \mathcal{P}_n^{\mathrm{P}}(A)$ if and only if $P' \in \mathcal{P}_n^{\mathrm{P}}(B)$. This defines a bijection between $\mathcal{P}_n^{\mathrm{P}}(A)$ and $\mathcal{P}_n^{\mathrm{P}}(A)$. Thus, A and B are partially Wilf-equivalent. □

Although we state Observations 8, 9, and 10 for partial Wilf equivalence, they are true for exact Wilf equivalence as well—replace 'P' by 'E' and "partially" by "exactly" in Observations 8, 9, and 10—with the identical proof. Hence if A and B are symmetric, then $A \overset{\mathrm{P}}{\sim} B$ and $A \overset{\mathrm{E}}{\sim} B$.

The next lemma says that, if B partially contains A, every permutation matrix that partially avoids B also partially avoids A.

Lemma 11. *Let A and B be two quasi-permutation matrices. If $A \preceq_P B$, then $\mathcal{P}_n^P(A) \subseteq \mathcal{P}_n^P(B)$.*

Proof. Let $P \in \mathcal{P}_n^{\mathrm{P}}(A)$ be a permutation matrix that partially avoids A. Since $A \preceq_{\mathrm{P}} B$, P also partially avoids B (otherwise $B \preceq_{\mathrm{P}} P$ would imply $A \preceq_{\mathrm{P}} P$). In other words, $P \in \mathcal{P}_n(B)$. Thus, $\mathcal{P}_n^{\mathrm{P}}(A) \subseteq \mathcal{P}_n^{\mathrm{P}}(B)$. □

1.3 Row- and column-permutation matrices

In this section, we introduce two special types of quasi-permutation matrices.

Definition 12. A quasi-permutation matrix $Q \in \{0,1\}^{m\times k}$ is called *row-permutation* matrix if each row of Q contains exactly one nonzero entry.

Definition 13. A quasi-permutation matrix $Q \in \{0,1\}^{m\times k}$ is called *column-permutation* matrix if each column of Q contains exactly one nonzero entry.

Observe that a row-permutation (or column-permutation) matrix with $m = k$ is a permutation matrix. For $m \geq k$, $\mathcal{Q}_{m,k,k}$ is the set of row-permutation matrices of size $m \times k$. On the other hand, for $m \leq k$, $\mathcal{Q}_{m,k,m}$ is the set of column-permutation matrices of size $m \times k$.

For a row-permutation matrix A, we prove that the number of permutation matrices exactly avoiding A is the same as the number of permutation matrices partially avoiding A (Corollary 15). The immediate consequence is that two row-permutation matrices are exactly Wilf-equivalent if and only if they are partially Wilf-equivalent (Corollary 16). We start with the following key lemma.

Lemma 14. *Let $A \in \{0,1\}^{m\times k}$ be a row-permutation matrix. For any permutation matrix $P \in \mathcal{P}_n$, P partially avoids A if and only if P exactly avoids A.*

Proof. First, we prove the forward implication by contrapositive. Suppose that P exactly contains A. It implies that there exists an exact copy of A in P induced by (I, J) for some set $I \subseteq [n]$ of column indices and some set $J \subseteq [n]$ of row indices. In other words, $A = P[I \times J]$. Hence $A \leq P[I \times J]$ and so P partially contains A.

Suppose that P partially contains A. It follows that there exists a partial copy of A in P induced by (I, J) for some set $I \subseteq [n]$ of column indices and some set $J \subseteq [n]$ of row indices. In other words, $A \leq P[I \times J]$. We claim that $A = P[I \times J]$. Since A is a row-permutation matrix, for every row index $j \in [k]$, there exists a column index $i(j) \in [m]$ such that $A_{i(j),j} = 1$. Hence $A_{i(j),j} = P[I \times J]_{i(j),j}$. Moreover, for every row index $j \in [k]$ and column index $i \in [m] \setminus \{i(j)\}$, we have $A_{i,j} = P[I \times J]_{i,j} = 0$ because every row of A and $P[I \times J]$ contains at most one nonzero entry. Thus, $A = P[I \times J]$ and so P exactly contains A. □

Corollary 15. *Let $A \in \{0,1\}^{k\times m}$ be a row-permutation matrix. For all $n \in \mathbb{N}$, we have*

$$p_n^E(A) = p_n^P(A).$$

Proof. By Lemma 14 the sets $\mathcal{P}_n^{\mathrm{P}}(A)$ and $\mathcal{P}_n^{\mathrm{E}}(A)$ are identical. Therefore, $p_n^{\mathrm{E}}(A) = |\mathcal{P}_n^{\mathrm{E}}(A)| = |\mathcal{P}_n^{\mathrm{P}}(A)| = p_n^{\mathrm{P}}(A)$. □

Suppose that A and B are exactly Wilf-equivalent row-permutation matrices. It follows that $p_n^{\mathrm{E}}(A) = p_n^{\mathrm{E}}(B)$ for all n. However, by Corollary 15, we have $p_n^{\mathrm{E}}(A) = p_n^{\mathrm{P}}(A)$ and $p_n^{\mathrm{E}}(B) = p_n^{\mathrm{P}}(B)$. Hence A and B are also partially Wilf-equivalent. Analogously, we show that if $A \overset{\mathrm{P}}{\sim} B$, then $A \overset{\mathrm{E}}{\sim} B$. We summarized this result in the following corollary.

Corollary 16. *Let A and B be row-permutation matrices of the same size. Then*

$$A \overset{E}{\sim} B \text{ if and only if } A \overset{P}{\sim} B.$$

We remark that a matrix Q is a column-permutation matrix if and only if a matrix $Q^\top$ is a row-permutation matrix. Since Q contains A if and only if $Q^\top$ contains $A^\top$ by Observation 8, the three previous results for row-permutation matrices holds also for column-permutation matrices.

2. Augmenting patterns by zeros

For a quasi-permutation matrix Q, let $0|Q$ and $Q|0$ be quasi-permutation matrices obtained from Q by adding a zero column before the first column of Q and after the last column of Q, respectively. Similarly, let $\frac{Q}{0}$ and $\frac{0}{Q}$ be quasi-permutation matrices obtained from Q by adding a zero row below the first row of Q and above the last row of Q, respectively.

In Section 2.1, for every quasi-permutation matrix A, we show that $A|0$ and $0|A$ are partially Wilf-equivalent, however, they are not in general exactly Wilf-equivalent. If in addition A is a row-permutation matrix, we show that $p^{\mathrm{P}}_{n+1}(A|0)$ and $p^{\mathrm{P}}_{n+1}(0|A)$ are both exactly $n+1$ times larger than $p^{\mathrm{P}}_n(A)$. Since $A|0$ and $0|A$ are still row-permutation matrices, we can use this result repeatedly. For example, we can deduce that $p^{\mathrm{P}}_{n+2}(0|A|0)$ is exactly $(n+2)(n+1)$ times larger than $p^{\mathrm{P}}_n(A)$.

By symmetry, we conclude that also $\frac{A}{0}$ and $\frac{0}{A}$ are partially Wilf-equivalent. Moreover, if A is a column-permutation matrix, $p^{\mathrm{P}}_{n+1}\left(\frac{A}{0}\right)$ and $p^{\mathrm{P}}_{n+1}\left(\frac{0}{A}\right)$ are both exactly $n+1$ times larger than $p^{\mathrm{P}}_n(A)$.

Let A be a row-permutation matrix. We observe that $A|0$ is a row-permutation matrix but $A|0$ is not a column-permutation matrix (even if A is a column-permutation matrix). Hence we cannot combine the results from the last two paragraphs to say something about $p^{\mathrm{P}}_{n+2}\left(\frac{0}{A|0}\right)$ (i.e., about the number of permutation matrices of order n that partially avoids a pattern A augmented by adding a zero column after the last column and then adding a zero row above the last row). Nonetheless, if A is a permutation matrix, we show in Section 2.2 that

$$p^{\mathrm{P}}_{n+2}\left(\frac{0}{A|0}\right) = p^{\mathrm{P}}_{n+1}(A) + (n+1)^2 \cdot p^{\mathrm{P}}_n(A).$$

2.1 Augmenting patterns by either zero rows or columns

For a permutation matrix $P \in \mathcal{P}_n$, we write $P = (C_1|C_2|\cdots|C_n)$ to denote that the first column of P is C_1, the second column of P is C_2 and so on. This allows us to easily describe matrices obtained from P by manipulating its columns. In particular, $(C_n|C_1|\cdots|C_{n-1}) \in \mathcal{P}_n$ is the permutation matrix obtained from P by cyclically rotating the order of the columns of P and $(C_1|C_2|\cdots|C_{n-1}) \in \mathcal{Q}_{n-1,n,n-1}$ is the quasi-permutation matrix obtained from P by removing its last column. Let A be a pattern (recall that a quasi-permutation matrix is sometimes called a pattern in the context of avoidance). The following lemma plays a crucial role in our proof that $A|0$ and $0|A$ are partially Wilf-equivalent.

Lemma 17. *Let $P = (C_1|C_2|\cdots|C_n)$ be a permutation matrix and $A \in \{0,1\}^{m\times k}$ be a quasi-permutation matrix. The following statements are equivalent:*

(i) $(C_1|C_2|\cdots|C_n)$ *partially contains* $(A|0)$,

(ii) $(C_n|C_1|\cdots|C_{n-1})$ *partially contains* $(0|A)$*, and*

(iii) $(C_1|C_2|\cdots|C_{n-1})$ *partially contains* A*.*

Proof. Since the statements (i), (ii), (iii) are trivially false for $n = 1$, the statements are equivalent for $n = 1$. In the rest of the proof, we assume that $n \geq 2$. Moreover, let $Q := (C_1|C_2|\cdots|C_{n-1})$.

Let us start by proving the equivalence (i) $\iff$ (iii). Suppose that P partially contains $A|0$. Let $P[I\times J]$ be a partial copy of $A|0$ in P induced (I, J) for some set of column indices $I = \{i_1, i_2, \dots, i_m\}$ and some set of row indices J. Recall that we implicitly assume that I and J are ordered increasingly. Let $I' := I \setminus \{i_m\}$. Since $i_m \leq n$, we know that $n \notin I'$. It follows that $Q[I' \times J]$ is a partial copy of A in Q induced by (I', J). Hence Q partially contains A.

On the other hand, suppose that Q partially contains A. Let $Q[I \times J]$ be a partial copy of A in Q induced by (I, J) for some set of column indices I and some set of row indices J. Note that $n \notin I$ because Q has only $n - 1$ columns. Let $I' := I \cup \{n\}$. We claim that $A|0 \leq P[I' \times J]$. Take $i \in [m + 1]$ and $j \in [k]$ arbitrary. If $i < m + 1$, then

$$(A|0)_{i,j} = A_{i,j} \leq Q[I \times J]_{i,j} = P[I' \times J]_{i,j}.$$

If $i = m+1$, then $(A|0)_{i,j} = 0 \leq P[I' \times J]$. Thus, $P[I' \times J]$ is a partial copy of A in Q induced by (I', J) and so P partially contains $A|0$.

The equivalence (ii) $\iff$ (iii) is proved analogously. The last equivalence (i) $\iff$ (ii) follows from the previous two equivalences. $\square$

It is now straightforward to prove the first main result of this section.

Theorem 18. *Let A be a quasi-permutation matrix. Then*

$$A|0 \overset{P}{\sim} 0|A.$$

Proof. For every $n \in \mathbb{N}$, let us choose $P = (C_1|C_2|\cdots|C_n) \in \mathcal{P}_n$ arbitrarily. By Lemma 17, we know that

$$(A|0) \preceq_{\mathrm{P}} (C_1|C_2|\cdots|C_n) \text{ if and only if } (0|A) \preceq_{\mathrm{P}} (C_n|C_1|\cdots|C_{n-1}).$$

This defines a bijection between $\mathcal{P}_n \setminus \mathcal{P}^{\mathrm{P}}_n(A|0)$ and $\mathcal{P}_n \setminus \mathcal{P}^{\mathrm{P}}_n(0|A)$. Thus,

$$p^{\mathrm{P}}_n(A|0) = |\mathcal{P}^{\mathrm{P}}_n(A|0)| = |\mathcal{P}^{\mathrm{P}}_n(0|A)| = p^{\mathrm{P}}_n(0|A).$$

$\square$

We remark that the proof of Lemma 17 is not valid for exact avoidance. The problem is that $P[I' \times J]$ (defined in the third paragraph) might not be an exact copy of $A|0$ in P induced by (I', J) because the last column of $P[I' \times J]$ can contain a nonzero entry. This is not surprising because Theorem 18 is not true in general if we replace 'P' by 'E'. For instance, our computer enumeration (see Table A.2 in Appendix) reveals that the patterns

$$\begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{pmatrix} \text{ and } \begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \end{pmatrix}$$

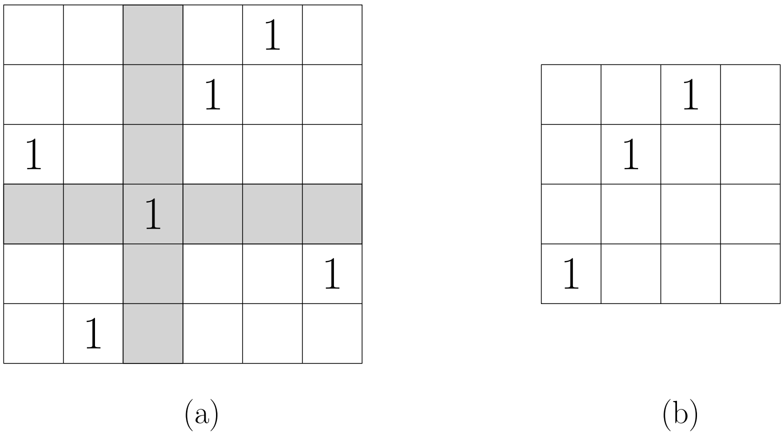

Figure 2.1: Let $P := P^{3,1,4,5,2}$ be a permutation matrix from Figure 1.1. On the left side is a permutation $P[3,3]^+$. The added column and row are shaded. On the right is a permutation $P[1,2]^-$.

are not exactly Wilf-equivalent.

By symmetry, we can conclude that the patterns $\frac{A}{0} \overset{\mathrm{P}}{\sim} \frac{0}{A}$ are also partially Wilf-equivalent.

Corollary 19. *Let A be a quasi-permutation matrix. Then*

$$\frac{A}{0} \overset{P}{\sim} \frac{0}{A}.$$

Proof. By Theorem 18, we know that $A^\top|0 \overset{\mathrm{P}}{\sim} 0|A^\top$. Hence $(A^\top|0)^\top \overset{\mathrm{P}}{\sim} (0|A^\top)^\top$ by Observation 8. Since $(A^\top|0)^\top = \frac{A}{0}$ and $(0|A^\top)^\top = \frac{0}{A}$, we have $\frac{A}{0} \overset{\mathrm{P}}{\sim} \frac{0}{A}$. □

If we concentrate only on row-permutation matrices, we prove a stronger result than Theorem 18. The second main result of this section says that $p_n^{\mathrm{P}}(A|0) = n \cdot p_{n-1}^{\mathrm{P}}(A)$ for every row-permutation matrix A and $n \geq 2$. Before we prove this result, we introduce two operations modifying permutation matrices and prove one technical lemma.

Let P be a permutation matrix of order n. For the indices $i, j \in [n+1]$, let $P[i,j]^+$ be a permutation matrix of order $n+1$ obtained from P by inserting a zero column after the $(i-1)$-th column and a zero row after the $(j-1)$-th row, and then setting the entry lying on the intersection of i-th column and j-th row to 1.[1] On the other hand, we can erase the i-th column and j-th row—let $P[i,j]^-$ be a matrix obtained from P by erasing its i-th column and j-th row. Note that $P[i,j]^-$ is not necessarily a permutation matrix. Indeed, $P[i,j]^-$ is a permutation matrix if and only if $P_{i,j} = 1$. See Figure 2.1 for an illustration.

Lemma 20. *Let P_1 and P_2 be permutation matrices of order $n-1$. For $j, j' \in [n]$, the following holds:*

(i) If $j \neq j'$, then $P_1[n,j]^+ \neq P_2[n,j']^+$.

[1] For $i = 1$, we insert a zero column to the left of the first column. For $j = 1$, we insert a zero row below the first row.

(ii) If $P_1 \neq P_2$, then $P_1[n,j]^+ \neq P_2[n,j']^+$.

Proof. For the sake of the proof, let $P_1^+ := P_1[n,j]^+$ and $P_2^+ := P_2[n,j']^+$. Choose $j, j' \in [n]$ arbitrarily.

We prove the case (i) first. If $j \neq j'$, then $(P_1^+)_{n,j} = 1$ and $(P_2^+)_{n,j} = 0$ because $(P_2^+)_{n,j'} = 1$ and every column contains exactly one nonzero entry. Hence $P_1^+ \neq P_2^+$.

Now we prove the case (ii). Suppose that $P_1 \neq P_2$. If $j \neq j'$, then $P_1^+ \neq P_2^+$ by (i). Otherwise, $j = j'$. Since $P_1 \neq P_2$, there are indices $a, b \in [n-1]$ such that $(P_1)_{a,b} \neq (P_2)_{a,b}$. Hence $(P_1^+)_{a,b'} \neq (P_2^+)_{a,b'}$ for some $b' \in \{b, b+1\}$ depending on $j = j'$. Thus, $P_1^+ \neq P_2^+$. □

Theorem 21. *Let $A \in \{0,1\}^{m \times k}$ be a row-permutation matrix. For $n \geq 2$, we have*

$$p_n^{\mathrm{P}}(A|0) = n \cdot p_{n-1}^{\mathrm{P}}(A).$$

Proof. Let us start by showing that

$$p_n^{\mathrm{P}}(A|0) \geq n \cdot p_{n-1}^{\mathrm{P}}(A).$$

For every $P \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$ and $j' \in [n]$, we claim that $P[n,j']^+ \in \mathcal{P}_n^{\mathrm{P}}(A|0)$. We proceed by contradiction. Suppose that $P[n,j']^+$ contains a partial copy of $A|0$ in $P[n,j']^+$ induced by (I,J) for some set of column indices $I = \{i_1, i_2, \ldots, i_m\}$ and some set of row indices J. Since $(A|0)$ is a row-permutation matrix, for every $j \in J$ there exists $i \in I \setminus i_m$ such that $P[n,j']^+_{i,j} = 1$. This particularly means that $j' \notin J$ because $P[n,j']^+_{n,j'} = 1$ and $n \notin I \setminus i_m$. However, P contains a partial copy of A induced by $(I \setminus \{i_m\}, J)$, a contradiction. By Lemma 20, for every $P \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$ and $j' \in [n]$, the permutation matrices $P[n,j']^+$ of order n are pairwise different. Thus, there are at least $n \cdot p_{n-1}^{\mathrm{P}}(A)$ permutation matrices of order n that partially avoid $A|0$.

On the other hand, we claim that

$$p_n^{\mathrm{P}}(A|0) \leq n \cdot p_{n-1}^{\mathrm{P}}(A).$$

For every $P \in \mathcal{P}_n^{\mathrm{P}}(A|0)$, we show that there is a unique permutation matrix $P' \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$ and a unique row index $j \in [n]$ such that $P = P'[n,j]^+$. Let $j \in [n]$ be an index such that $P_{n,j} = 1$ and let $P' := P[n,j]^-$. Recall that P' is a permutation matrix of order $n-1$ and $P'[n,j]^+ = P$. It remains to show that $P' \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$.

We consider two cases. If $P \in \mathcal{P}_n^{\mathrm{P}}(A)$, then clearly $P' = P[n,j]^- \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$. Otherwise, $P \notin \mathcal{P}_n^{\mathrm{P}}(A)$. Let $P[I \times J]$ be any partial copy of A in P induced by (I,J) for some set of column indices I and some set of row indices J. Observe that $n \in I$. To see this, suppose that $n \notin I$. Then P would contain a partial copy of $A|0$ induced by $(I \cup \{n\}, J)$. It follows that $P' = P[n,j]^- \in \mathcal{P}_{n-1}^{\mathrm{P}}(A)$ because every copy of A in P induced by (I,J) contains the last column of P (i.e., $n \in I$), which has been removed to obtain P'.

The uniqueness of $P' \in \mathcal{P}_{n-1}(A)$ and $j \in [n]$ for $P \in \mathcal{P}_n(A|0)$ follows from Lemma 20. Therefore, we find a one-to-one mapping from $\mathcal{P}_n(A|0)$ to $[n] \times \mathcal{P}_{n-1}(A)$, which completes the proof. □

By symmetry, we obtain the same enumeration result when we add a zero column before the first column. Moreover, analogous results hold also for column-permutation matrices and adding a zero row.

Corollary 22. *Let A be a row-permutation matrix. Moreover, let B be a column-permutation matrix. For $n \geq 2$, we have*

(i) $p_n^P(0|A) = n \cdot p_{n-1}^P(A)$,

(ii) $p_n^P \begin{pmatrix} B \\ \hline 0 \end{pmatrix} = n \cdot p_{n-1}^P(B)$,

(iii) $p_n^P \begin{pmatrix} 0 \\ \hline B \end{pmatrix} = n \cdot p_{n-1}^P(B)$.

Let A be a row-permutation matrix. We observe that $A|0$ and $0|A$ are also row-permutation matrices. Hence $p_n^{\mathrm{P}}(0|A|0) = n \cdot p_{n-1}^{\mathrm{P}}(A|0) = n(n-1) \cdot \mathcal{P}_{n-2}(A)$ by the application of Theorem 21 and Corollary 22. In particular, for $A = \left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\right)$, we get $p_n \left(\begin{smallmatrix} \circ & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \circ \end{smallmatrix}\right) = n(n-1)$. Obviously, we can add more zero columns before the first column or the last column and apply Theorem 21 as many times as the number of added zero columns. We obtain the following result. For $A \in \mathcal{Q}_{m,k,\ell}$, let $(0_{m\times\ell_1}|A|0_{m\times\ell_2}) \in \mathcal{Q}_{m+\ell_1+\ell_2,k,\ell}$ be a quasi-permutation matrix obtained from A by adding ℓ_1 zero columns before the first column of A and adding ℓ_2 zero columns after the last column of A.

Corollary 23. *Let $A \in \{0,1\}^{m\times k}$ be a row-permutation matrix. For all $\ell_1, \ell_2 \in \mathbb{N}_0$, if $n > \ell_1 + \ell_2$, then*

$$p_n^P(0_{m\times\ell_1}|A|0_{m\times\ell_2}) = (n)_{\ell_1+\ell_2} \cdot p_{n-\ell_1-\ell_2}^P(A).$$

Again, by symmetry, we conclude an analogous result for column-permutation matrices. For $B \in \mathcal{Q}_{m,k,\ell}$, let

$$\begin{pmatrix} 0_{\ell_1\times k} \\ \hline B \\ \hline 0_{\ell_2\times k} \end{pmatrix} := (0_{k\times\ell_1}|B^\top|0_{k\times\ell_2})^T.$$

Corollary 24. *Let $B \in \{0,1\}^{m\times k}$ be a column-permutation matrix. For all $\ell_1, \ell_2 \in \mathbb{N}_0$, if $n > \ell_1 + \ell_2$, then*

$$p_n^P \begin{pmatrix} 0_{\ell_1\times k} \\ \hline B \\ \hline 0_{\ell_2\times k} \end{pmatrix} = (n)_{\ell_1+\ell_2} \cdot p_{n-\ell_1-\ell_2}^P(B).$$

2.2 Augmenting permutation matrices by zero row and column

For a quasi-permutation matrix $A \in \mathcal{Q}_{m,k,\ell}$, through this section let

$$\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right) := \begin{pmatrix} 0 \\ \hline A|0 \end{pmatrix} \in \mathcal{Q}_{m+1,k+1,\ell}.$$

Let us mention a limitation of applications of the results (Theorem 21 and Corollary 22) from the previous section. Let A be a row-permutation. Since

$A|0$ and $0|A$ are not column-permutation matrices, we cannot combine Theorem 21 with Corollary 22 to say something about $p_n^{\mathrm{P}}\left(\begin{smallmatrix}0 & 0\\ A & 0\end{smallmatrix}\right)$. Nevertheless, if A is a permutation matrix, we claim that

$$p_{n+2}^{\mathrm{P}}\left(\begin{smallmatrix}0 & 0\\ A & 0\end{smallmatrix}\right) = p_{n+1}^{\mathrm{P}}(A) + (n+1)^2 \cdot p_n^{\mathrm{P}}(A).$$

Before we present a proof of this claim, we prove three auxiliary lemmas. Remember that $P[n,n]^-$ is a submatrix of a permutation matrix P obtained by erasing its n-th row and n-th column.

Lemma 25. *Let $A \in \mathcal{P}_k$ and $P \in \mathcal{P}_n$ be permutation matrices. The permutation matrix P partially contains $\left(\begin{smallmatrix}0 & 0\\ A & 0\end{smallmatrix}\right)$ if and only if $P[n,n]^-$ partially contains A.*

Proof. For the sake of the proof, let $A^0 := \left(\begin{smallmatrix}0 & 0\\ A & 0\end{smallmatrix}\right) \in \mathcal{Q}_{k+1,k+1,k}$. First, suppose that P partially contains A^0. Let $P[I \times J]$ be a partial copy of A^0 in P induced by (I, J) for some set of column indices $I = \{i_1, \dots, i_{k+1}\}$ and some set of row indices $J = \{j_1, \dots, j_{k+1}\}$. Then (I', J'), where $I' := I \setminus \{i_{k+1}\}$ and $J' := J \setminus \{j_{k+1}\}$, induce a partial copy of A in P. Since $n \notin I'$ and $n \notin J'$, (I', J') induce also a partial copy of A in $P[n,n]^-$. Thus, $P[n,n]^-$ partially contains A.

Second, suppose that $P^- := P[n,n]^-$ partially contains A. Let $P^-[I' \times J']$ be a partial copy of A in P^- induced by (I', J') for some set of column indices I' and some set of row indices J'. Then $(I' \cup \{n\},\ J' \cup \{n\})$ induce a partial copy A^0 in P (to see this, proceed similarly as we do in the second part of the proof of Lemma 17.). Thus, P partially contains A^0. We remark that (I, J) does not necessarily induce an exact copy of A^0 in P because the entry $P_{n,n}$ can be equal to 1. ☐

If we erase all zero columns and rows from a quasi-permutation matrix $Q \in \mathcal{Q}_{m,k,\ell}$, we obtain a submatrix $perm(Q)$ of Q such that $perm(Q)$ is a permutation matrix of order ℓ. Formally, let $perm(Q) := Q[I \times J]$, where I is the set of all column indices $i \in [m]$ such that the i-th column of Q is nonzero and J is the set of all row indices $j \in [k]$ such that the j-th row of Q is nonzero.

The second lemma says that whether a quasi-permutation matrix Q partially contains a permutation matrix A only depends on $perm(Q)$.

Lemma 26. *Let A be a permutation matrix and let Q be a quasi-permutation matrix. The quasi-permutation matrix Q partially contains A if and only if $perm(Q)$ partially contains A.*

Proof. Suppose that Q partially contains A. Let $Q[I \times J]$ be a partial copy of A induced by (I, J) for some set of column indices I and some set of row indices J. Since A is a permutation matrix, for every $i \in I$, the i-th column of Q is nonzero because there exists $j \in J$ such that $Q_{i,j} = 1$. Similarly, for every $j \in J$, we show that the j-th row of Q is nonzero. Hence no i-th row and j-th is removed from Q to obtain $perm(Q)$. Thus, $perm(Q)$ partially contains A.

On the other hand, suppose that $perm(Q)$ partially contains A. Inserting zero rows and columns into $perm(Q)$ does not destroy any partial copy of A. Clearly, we can insert back zero rows and columns into $perm(Q)$ to obtain Q. Thus, Q partially contains A. ☐

For a permutation matrix P of order n and the indices $i, j \in [n+1]$, let $P[i,j]^0$ be a quasi-permutation matrix of size $(n+1) \times (n+1)$ with exactly n nonzero entries obtained from P by inserting a zero column above the $(i-1)$-th column and a zero row to the right of the $(j-1)$-th row.[2] Notice the slight difference between $P[i,j]^0$ and $P[i,j]^+$: they agree on all entries but (i,j), where we have $P[i,j]^0_{i,j} = 0$ and $P[i,j]^+_{i,j} = 1$.

Lemma 27. *Let P_1 and P_2 be permutation matrices of order $n-2$. For $i, i', j, j' \in [n-1]$, the following holds:*

(i) If $i \neq i'$ or $j \neq j'$, then $P_1[i,j]^0 \neq P_2[i',j']^0$.

(ii) If $P_1 \neq P_2$, then $P_1[i,j]^0 \neq P_2[i',j']^0$.

Proof. For the sake of the proof, let $P_1^0 := P_1[i,j]^0$ and $P_2^0 := P_2[i',j']^0$.

If $i \neq i'$, then the i-th column is the only zero column of P_1^0 and i'-th column is the only zero column of P_2^0. Hence $P_1^0 \neq P_2^0$. We proceed analogously if $j \neq j'$.

Now we prove the case (ii). Suppose that $P_1 \neq P_2$. If $i \neq i'$ or $j \neq j'$, then $P_1^0 \neq P_2^0$ by (i). Otherwise, $i = i'$ and $j = j'$. Since $P_1 \neq P_2$, there are indices $a, b \in [n-2]$ such that $(P_1)_{a,b} \neq (P_2)_{a,b}$. Hence $(P_1^0)_{a',b'} \neq (P_2^0)_{a',b'}$ for some $a' \in \{a, a+1\}$ and $b' \in \{b, b+1\}$ depending on i, j. Thus, $P_1^0 \neq P_2^0$. □

Finally, in order to prove the main result of this section, we extend the notion of $\mathcal{P}_n^{\mathrm{P}}(A)$ and $p_n^{\mathrm{P}}(A)$ to quasi-permutation matrices as follows. For a pattern A, let $\mathcal{Q}^{\mathrm{P}}_{m,k,\ell}(A)$ be the set of all quasi-permutation matrices of size $m \times k$ with exactly ℓ nonzero entries that partially avoid A and let $q^{\mathrm{P}}_{m,k,\ell}(A) := |\mathcal{Q}^{\mathrm{P}}_{m,k,\ell}(A)|$.

Theorem 28. *Let A be a permutation matrix of order k. For every $n \geq 3$, we have*

$$p_n^P \left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix} \right) = p_{n-1}(A) + (n-1)^2 \cdot p_{n-2}(A).$$

Proof. For the sake of the proof, let $A^0 := \left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix} \right)$. For a permutation matrix $P \in \mathcal{P}_n$, P partially contains A^0 if and only if $P[n,n]^-$ partially contains A by Lemma 25. Recall that $P[n,n]^-$ is either a permutation matrix of order $n-1$ or a quasi-permutation matrix of size $(n-1) \times (n-1)$ with exactly $n-2$ nonzero entries. Hence Lemma 25 defines a one-to-one correspondence between the set $\mathcal{P}_n^{\mathrm{P}}(A^0)$ and the set $\mathcal{P}^{\mathrm{P}}_{n-1}(A) \cup \mathcal{Q}^{\mathrm{P}}_{n-1,n-1,n-2}(A)$. Thus,

$$p_n^{\mathrm{P}}(A^0) = p^{\mathrm{P}}_{n-1}(A) + q^{\mathrm{P}}_{n-1,n-1,n-2}(A).$$

In the rest of the proof, we show that $q^{\mathrm{P}}_{n-1,n-1,n-2}(A) = (n-1)^2 \cdot p^{\mathrm{P}}_{n-2}(A)$.

For every $P \in \mathcal{P}^{\mathrm{P}}_{n-2}(A)$ and $i, j \in [n-1]$, the quasi-permutation matrix $P[i,j]^0$ partially avoids A because A is a permutation matrix. In other words, $P[i,j]^0 \in \mathcal{Q}^{\mathrm{P}}_{n-1,n-1,n-2}(A)$. Moreover, by Lemma 27, if $P_1, P_2 \in \mathcal{P}_{n-2}(A)$ are permutation matrices, then $P_1[i,j]^0 \neq P_2[i',j']^0$ for every $i, i', j, j' \in [n-1]$. It follows that $(n-1)^2 \cdot p_{n-2}(A) \leq q^{\mathrm{P}}_{n-1,n-1,n-2}(A)$.

On the other hand, every $Q \in \mathcal{Q}^{\mathrm{P}}_{n-1,n-1,n-2}(A)$ is constructed at least once in the last paragraph. Indeed, let $i, j \in [n-1]$ such that the i-th row of Q is zero and j-th column of Q is zero. Observe that $perm(Q) \in \mathcal{P}^{\mathrm{P}}_{n-2}(A)$ by Lemma 26 and $Q = perm(Q)[i,j]^0$. Therefore, $(n-1)^2 \cdot p_{n-2}(A) \geq q^{\mathrm{P}}_{n-1,n-1,n-2}(A)$, which finishes the proof. □

[2]For $i = 1$, we insert a zero column to the left of the first column. For $j = 1$, we insert a zero row below the first row.

Using this theorem and symmetry we conclude that $p_n^{\mathrm{P}}\left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$, $p_n^{\mathrm{P}}\left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$, and $p_n^{\mathrm{P}}\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are given by the same recurrence as $p_n^{\mathrm{P}}\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right)$. It implies that these patterns belong to the same partial Wilf-equivalence class.

Corollary 29. *Let A be a permutation matrix of order k. For all $n \geq 3$, we have*

$$p_n^P\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right) = p_n^P\left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right) = p_n^P\left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right) = p_n^P\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) = p_{n-1}(A) + (n-1)^2 \cdot p_{n-2}(A).$$

Proof. By Theorem 28 and Observation 10, we have

$$\begin{aligned} p_n^{\mathrm{P}}\begin{pmatrix} 0 & 0 \\ A^R & 0 \end{pmatrix} &= p_{n-1}(A^R) + (n-1)^2 \cdot p_{n-2}(A^R) \\ &= p_{n-1}(A) + (n-1)^2 \cdot p_{n-2}(A). \end{aligned}$$

Since $\begin{pmatrix} 0 & 0 \\ A^R & 0 \end{pmatrix}^R = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$, we can write

$$p_n^{\mathrm{P}}\left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right) = p_n^{\mathrm{P}}\begin{pmatrix} 0 & 0 \\ A^R & 0 \end{pmatrix} = p_{n-1}(A) + (n-1)^2 \cdot p_{n-2}(A) = p_n^{\mathrm{P}}\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right).$$

Analogously, we prove the remaining two equalities. □

We mention at least one nice consequence of the previous result. Recall that $p_n^{\mathrm{P}}(A) = c_n$ for any permutation matrix of order 3, where c_n is the n-th Catalan number. Using Corollary 29 we obtain the following enumerating result.

Corollary 30. *Let A be a permutation matrices of order 3. For $n \geq 3$ and all $i, j \in \{1, 4\}$, we have*

$$p_n^P\Big(A[i,j]^0\Big) = c_{n-1} + (n-1)^2 \cdot c_{n-2}.$$

In particular, all patterns of the form $\begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix} \in \mathcal{Q}_{4,4,3}$ belong to the same partial Wilf equivalence class, where A_i is a permutation matrix of order 3 and the other A_j are 0.

The reader may pose a question if there exists an analogy of Corollary 29 for the exact avoidance. The answer is no. Indeed, let $A = \left(\begin{smallmatrix} \circ & \bullet & \circ \\ \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{smallmatrix}\right)$. The computer enumeration shows that $\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are not exactly Wilf-equivalent (see Table A.2 in Appendix).

3. Shape-Wilf equivalence and nice filled Ferrers diagrams

For quasi-permutation matrices $A \in \mathcal{Q}_{m,k,\ell}$ and $B \in \mathcal{Q}_{m',k',\ell'}$, let

$$\left(\begin{smallmatrix} 0 & B \\ A & 0 \end{smallmatrix}\right) \in \mathcal{Q}_{m+m',k+k',\ell+\ell'}$$

be a quasi-permutation matrix whose bottom-left corner contains the matrix A, its top-right corner contains the matrix B, and the remaining entries are equal to zero. We note that if A and B are permutation matrices, then $\left(\begin{smallmatrix} 0 & B \\ A & 0 \end{smallmatrix}\right)$ is also a permutation matrix.

A pattern A is said to be *small* if it has at most 4 columns and at most 4 rows. By the results obtained so far, we can classify the majority of the small patterns into partial Wilf equivalence classes. After this chapter, we will be able to resolve the remaining small patterns.

This chapter is motivated by the result by Backelin, West, and Xin [4]: if A and B are shape-Wilf-equivalent permutation matrices and C is an arbitrary permutation matrix, then

$$\left(\begin{smallmatrix} 0 & C \\ A & 0 \end{smallmatrix}\right) \quad \text{and} \quad \left(\begin{smallmatrix} 0 & C \\ B & 0 \end{smallmatrix}\right),$$

are shape-Wilf-equivalent and hence also partial Wilf-equivalent. In Section 3.1, we provide a necessary background for the definition of shape-Wilf equivalence, which is given in Section 3.2. For now, think of shape-Wilf equivalence as some refinement of partial Wilf equivalence. Careful examination of the proof of this result shows that we can replace the permutation matrices A, B, and A by quasi-permutation matrices. For example, we can deduce that the patterns

$$\begin{pmatrix} \circ & \circ & \circ & \bullet \\ \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{pmatrix}$$

are partial Wilf-equivalent because it is well known that $\left(\begin{smallmatrix} \circ & \bullet \\ \bullet & \circ \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}\right)$ are shape-Wilf-equivalent. We summarize the known results about shape-Wilf equivalence in Section 3.2.

In Sections 3.3 and 3.4, we prove that

$$\left(\begin{smallmatrix} 0 & A \\ X & 0 \end{smallmatrix}\right) \quad \text{and} \quad \left(\begin{smallmatrix} 0 & A \\ Y & 0 \end{smallmatrix}\right),$$

are partial Wilf-equivalent, where $X = \left(\begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{smallmatrix}\right)$ and A is any quasi-permutation matrix whose first column is nonzero. We remark that X and Y are not shape-Wilf-equivalent as we will see in Section 3.2 and the assumption that the first column of C is nonzero cannot be omitted.

3.1 Ferrers diagrams

The main objective here is to introduce the necessary background for the following three sections. Let us start with the definitions of a diagram and a Ferrers diagram.

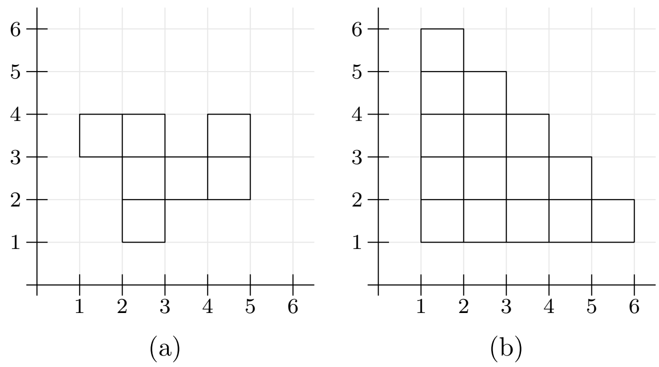

Figure 3.1: On the left side is a diagram D represented by the set $\{(1,3),(2,1),(2,2),(2,3),\ (3,2),(4,2),(4,3)\}$. On the right side is a diagram D' represented by the set $\{(i,j) \mid i,j \in [5],\ i+j \leq 6\}$. Only the diagram D' is a Ferrers diagram.

Definition 31. A *diagram* is a finite set of cells in the plane, where each *cell* is a square of unit size whose vertices have positive integer coordinates.

We remark that each cell C in the plane is uniquely identified by a pair $(i,j) \in \mathbb{N} \times \mathbb{N}$, where i is the x-coordinate and j is the y-coordinate of bottom-left vertex of C. Thus, there is a one-to-one correspondence between the set of all diagrams and the set of all finite subset of $\mathbb{N} \times \mathbb{N}$. We will use both representations interchangeably (see the examples in Figure 3.1).

Let D be a diagram. The *j-th row* $R_j(D)$ of D is the set of all cells of D which are identified by pairs (ℓ, j) for some $\ell \in \mathbb{N}$ and the *i-th column* $C_i(D)$ of D is the set of all cells of D which are identified by pairs (i, ℓ) for some $\ell \in \mathbb{N}$. In other words,

$$R_j(D) = \{(\ell, j) \in D \mid \ell \in \mathbb{N}\} \quad \text{and} \quad C_i(D) = \{(i, \ell) \in D \mid \ell \in \mathbb{N}\}.$$

The *size* of j-th row $R_j(D)$ is $r_j(D) := |R_j(D)|$ and the *size* of i-th column $C_i(D)$ is $c_i(D) := |C_i(D)|$. We write R_j, C_i, r_j, c_i instead of $R_j(D), C_i(D), r_j(D), c_i(D)$, respectively, if it is clear from the context which diagram is meant.

Definition 32. A *Ferrers diagram*[1] F is a diagram satisfying, for every $j \in \mathbb{N}$, the following two conditions:

(i) $R_j(F) = \Big\{(\ell, j) \in F \mid \ell \in [r_j(F)]\Big\}$ and

(ii) $r_j(F) \geq r_{j+1}(F)$.

Remember that $[r_j(F)] = \{1, 2, \dots, r_i(F)\}$ is the empty set if $r_i(F) = 0$. The first condition says that the cells of every nonempty row $R_j(F)$ are contiguous and the leftmost cell of $R_j(F)$ is $(1, j)$. The second condition says that the size of $(j+1)$-th row is not larger than the size of j-th row. In Figure 3.1 (b), we see an example of a Ferrers diagram, however, the diagram in Figure 3.1 (a) is not a Ferrers diagram.

[1]Sometimes Ferrers diagram is called *Ferrers shape*.

0	0	1	0	0
1	0	0	0	0
0	0	0	0	1

Figure 3.2: Quasi-permutation matrix $\begin{pmatrix} \circ & \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \bullet \end{pmatrix}$ considered as a filling of a rectangular diagram.

0	0	0	1	0
0	0	1	0	0
1	0	0	0	0
0	0	0	0	1
0	1	0	0	0

Figure 3.3: Permutation matrix from Figure 1.1 considered as a transversal.

Since $(1,1)$ is the most bottom-left cell of any nonempty Ferrers diagram, henceforth, as no confusion can arise, we do not depict the x-axis and y-axis when we draw a Ferrers diagram F (as we do in Figure 3.1 (b)).

For a Ferrers diagram F, let $r(F)$ denote the number of nonempty rows of F and let $c(F)$ denote the number of nonempty columns of F. Observe that the rows $R_1, R_2, \dots, R_{r(F)}$ are exactly the nonempty rows in F. Similarly, the columns $C_1, C_2 \dots, C_{c(F)}$ are exactly the nonempty columns in F. We remark that $r(F) = c_1$ and $c(F) = r_1$. Despite these simple equalities, it is usually more transparent to write $r(F)$ instead of c_1 and $c(F)$ instead of r_1. A Ferrers diagram F is called *rectangular* if $r_1 = r_{r(F)}$. In other words, F is rectangular if all rows of F have the same size (and hence all columns have the same size). If, in addition to $r_1(F) = r_{r(F)}$ it holds $r(F) = c(F)$, the Ferrers diagram F is called *square*. We simply write a rectangular (square) diagram instead of a rectangular (square) Ferrers diagram.

Our intention is to generalize quasi-permutation matrices so that their rows (or columns) may consist of different number of entries. For this purpose, we define diagrams. It remains to fill each cell by a number 0 or 1 in such a way that every row and every column contain at most one nonzero cell. This is captured in the following definition.

Definition 33. A *filled diagram* is a pair (D, φ), where D is diagram endowed with a mapping $\varphi\colon D \to \{0,1\}$ such that

- the number of cells C in any row R_j of D with $\varphi(C) = 1$ is at most one,
- the number of cells C in any column C_i of D with $\varphi(C) = 1$ is at most one.

The *underlying diagram* and *underlying mapping* of a filled diagram (D, φ) is the diagram D and the mapping φ, respectively. Instead of $\varphi((i,j))$, although it is formally correct, we write $\varphi(i,j)$. If no confusion can arise, we use the same symbol D to denote both the filled diagram (D, φ) and the underlying diagram D. In this case, we write $D(i,j)$ to denote $\varphi(i,j)$. A filled diagram (D, φ) is called a *filled Ferrers diagram* if the underlying diagram D is a Ferrers diagram. Similarly, we define a *filled rectangular (square) diagram*. Similarly as for matrices, when we draw filled diagrams, we usually depict only ones, while zeros are represented by empty cells.

Let (D, φ) be a filled diagram. A cell C in D is called *zero* (with respect to φ) if $\varphi(C) = 0$ and *nonzero* if $\varphi(C) = 1$. We extend this definition to rows and columns—a nonempty row R_i of D is called *zero* (with respect to φ) if $\varphi(R_i) = \{0\}$ and *nonzero* if it is not zero. Analogously, we define zero and nonzero columns.

Definition 34. A *transversal* is a filled Ferrers diagram (F, φ) such that no row or no column of F is zero.

Every quasi-permutation matrix naturally corresponds to a filled rectangular diagram and vice versa (see Figure 3.2). Moreover, there is also a straightforward one-to-one correspondence between permutation matrices and transversals (see Figure 3.3). All definitions for filled diagrams are consistent with the definitions for quasi-permutation matrices. Hence filled diagrams may be regarded as generalization of quasi-permutation matrices. From this point forward, we move from quasi-permutation matrices to filled rectangular diagrams and vice versa without usually saying it explicitly.

Finally, we define a notion of avoidance of filled diagrams, which may be viewed as a generalization of partial avoidance of quasi-permutation matrices (Definition 4). For two filled diagrams D and D' with the same underlying diagram, we write $D \leq D'$ if $\forall (a, b) \in D\colon D(a, b) \leq D'(a, b)$. Moreover, we write $D = D'$ if $D \leq D'$ and $D' \leq D$.

Definition 35. Let D be a filled diagram. For a set of column indices $I = \{i_1, i_2, \ldots, i_m\}$ and a set of row indices $J = \{j_1, j_2, \ldots, j_k\}$, *a filled subdiagram of D induced by* (I, J), denoted $D[I \times J]$, is the filled diagram with cells in $[m] \times [k]$ such that, for every $a \in [m]$ and $b \in [k]$,

- $(a, b) \in D[I \times J]$ if and only if $(i_a, j_b) \in D$
- if $(a, b) \in D[I \times J]$, then $D[I \times J](a, b) = D(i_a, i_b)$.

It is not hard to see that if F is a filled Ferrers diagram, then $F[I \times J]$ is also a filled Ferrers diagram for any set of column indices I and any set of row indices J.

Definition 36. Let A be a filled Ferrers diagram. We say that a filled diagram D *contains* A if there exists a set of column indices I and a set of row indices J such that

$$A \leq D[I \times J].$$

The filled subdiagram $D[I \times J]$ is called a *copy of A in D induced by* (I, J). If D does not contain A, we say that D *avoids* A.

More specifically, we say that D *contains a copy of A induced by* (I, J) or (I, J) *induced a copy of A in D* if $D[I \times J]$ is a copy of A in D induced by (I, J). Intuitively, D contains a copy of A induced by (I, J) if we can erase all columns from D except those whose indices are in I and all rows from D except those whose indices are in J to obtain a diagram D' of the same "shape" as the "shape" of A and, moreover, every cell in D' contains a number that is not smaller than the number in the corresponding cell in A. For example, the transversal in Figure 3.3 contains a copy of the rectangular diagram in Figure 3.2 induced by $(\{1, 2, 3, 4, 5\}, \{2, 3, 4\})$ (i.e., we erase the first and last rows). In our settings, a filled Ferrers diagram A in Definition 36 is always a quasi-permutation matrix considered as a filled rectangular diagram and it is sometimes called a *pattern*.

Later it will be useful to construct from a filled diagram smaller filled diagrams by removing cells from the diagram that do not satisfy certain conditions. For two filled diagrams D and D', we say that D' is a *filled subdiagram* of D if

$$D' \subseteq D \text{ and } \forall (a, b) \in D' \colon D'(i, j) = D(i, j).$$

We use this notion to introduce new filled diagrams—for a diagram D, by saying "let D' be a filled subdiagram of D with the underlying diagram C" (we assume that $C \subseteq D$) we define the unique filled diagram D' such that D' is a filled subdiagram of D and the underlying diagram of D' is C.

3.2 Shape-Wilf equivalence

In this section, we define shape-Wilf equivalence as an analogy of Wilf equivalence for transversals. Then we state known results about shape-Wilf equivalence that we use to classify small patterns into partial Wilf equivalence classes in Chapter 4.

For a Ferrers diagram F and a pattern A, let $\mathcal{T}_F(A)$ be the set of all transversals with the underlying diagram F that avoid A. We say that two quasi-permutation matrices A and B considered as filled rectangular diagrams are shape-Wilf-equivalent, written $A \overset{\mathrm{s}}{\sim} B$, if for every Ferrers diagram F, we have $|\mathcal{T}_F(A)| = |\mathcal{T}_F(B)|$

Since the transversals of square Ferrers diagram F are exactly permutation matrices of order $c(F) = r(F)$, if two patterns are shape-Wilf-equivalent, then they are also partially Wilf-equivalent.

Observation 37. *Let A and B be quasi-permutation matrices. Then*

$$A \overset{s}{\sim} B \implies A \overset{P}{\sim} B.$$

Theorem 38 ([4])**.** *Let A and B be two shape-Wilf equivalent permutation matrices, and let C be an arbitrary permutation matrix. Then*

$$\left(\begin{smallmatrix} 0 & C \\ A & 0 \end{smallmatrix}\right) \overset{s}{\sim} \left(\begin{smallmatrix} 0 & C \\ B & 0 \end{smallmatrix}\right).$$

A careful examination of the proof of Theorem 38 by Jelínek in [5, Proof of Proposition 1] shows that we can replace the permutation matrices A, B, and C by quasi-permutation matrices to obtain the following theorem.

Theorem 39. *Let A and B be two shape-Wilf-equivalent quasi-permutation matrices, and let C be an arbitrary quasi-permutation matrix. Then*

$$\left(\begin{smallmatrix} 0 & C \\ A & 0 \end{smallmatrix}\right) \overset{P}{\sim} \left(\begin{smallmatrix} 0 & C \\ B & 0 \end{smallmatrix}\right).$$

We remark that in the previous theorem, we can strengthen the conclusion by replacing $\overset{\mathrm{P}}{\sim}$ by $\overset{\mathrm{s}}{\sim}$. However, we do not need this added generality. The proofs of Theorems 38 and 39 are omitted because they follow similar ideas as the proof of Theorem 42 proved in Section 3.4.

Let I_k be the identity matrix of order k and J_k be the anti-identity matrix of order k. Formally,

$$(I_k)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad (J_k)_{i,j} = \begin{cases} 1 & \text{if } i + j = k + 1 \\ 0 & \text{otherwise.} \end{cases}$$

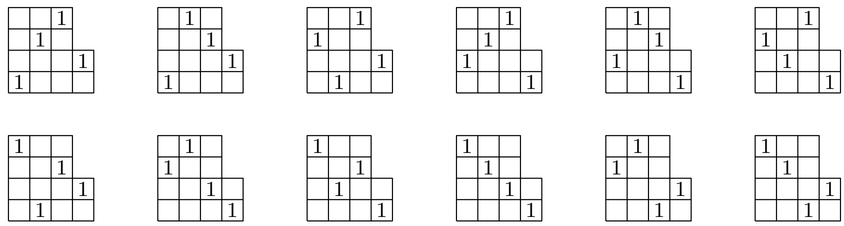

Figure 3.4: The first 6 transversals contain $X = \left(\begin{smallmatrix}\circ&\circ&\bullet\\\bullet&\circ&\circ\end{smallmatrix}\right)$ and the last 9 transversals contain $Y = \left(\begin{smallmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{smallmatrix}\right)$. Since we consider all transversals of the underlying diagram, it implies that X and Y are not shape Wilf-equivalent.

The theorem by Backelin, West, and Xin [4] says that I_k and J_k are shape Wilf-equivalent.

Theorem 40 ([4])**.** *For all $k \in \mathbb{N}$, we have*

$$I_k \overset{s}{\sim} J_k.$$

For future reference, we prove here that $X = \left(\begin{smallmatrix}\circ&\circ&\bullet\\\bullet&\circ&\circ\end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{smallmatrix}\right)$ are not shape Wilf-equivalent. It is sufficient to find a single Ferrers diagram F for which $|\mathcal{F}_F(X)| \neq |\mathcal{F}_F(Y)|$. See Figure 3.4 for a proof.

Observation 41. *The patterns $\left(\begin{smallmatrix}\circ&\circ&\bullet\\\bullet&\circ&\circ\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{smallmatrix}\right)$ are not shape Wilf-equivalent.*

3.3 Nice filled Ferrers diagrams

Through this section, let $X := \left(\begin{smallmatrix}\circ&\circ&\bullet\\\bullet&\circ&\circ\end{smallmatrix}\right)$ and $Y := \left(\begin{smallmatrix}\bullet&\circ&\circ\\\circ&\circ&\bullet\end{smallmatrix}\right)$. We aim to prove the following theorem.

Theorem 42. *Let $A \in \mathcal{Q}_{m,k,\ell}$ be a quasi-permutation matrix. If there there exists $t \in [k]$ such that $A_{1,t} = 1$, then*

$$\left(\begin{smallmatrix}0&A\\X&0\end{smallmatrix}\right) \overset{P}{\sim} \left(\begin{smallmatrix}0&A\\Y&0\end{smallmatrix}\right).$$

We remark that this theorem does not follow from Theorem 38 because X and Y are not shape-Wilf-equivalent as we have seen in the last section. Moreover, the assumption that the first column of A is nonzero cannot be ommited because the patterns

$$X' := \begin{pmatrix}\circ&\circ&\circ&\circ&\bullet\\\circ&\circ&\bullet&\circ&\circ\\\bullet&\circ&\circ&\circ&\circ\end{pmatrix} \quad\text{and}\quad Y' := \begin{pmatrix}\circ&\circ&\circ&\circ&\bullet\\\bullet&\circ&\circ&\circ&\circ\\\circ&\circ&\bullet&\circ&\circ\end{pmatrix}$$

are not partially Wilf-equivalent since $p_6^{\mathrm{P}}(X') = 434$ and $p_6^{\mathrm{P}}(Y') = 430$ (these values were computed by our program [3]).

Let us first prove Thereom 42 for $A = (\bullet)$. In other words, we want to prove that

$$X^+ := \begin{pmatrix}\circ&\circ&\circ&\bullet\\\circ&\circ&\bullet&\circ\\\bullet&\circ&\circ&\circ\end{pmatrix} \quad\text{and}\quad Y^+ := \begin{pmatrix}\circ&\circ&\circ&\bullet\\\bullet&\circ&\circ&\circ\\\circ&\circ&\bullet&\circ\end{pmatrix}$$

are partially Wilf-equivalent. The first step is to give a right characterization of permutation matrices avoiding X^+ and Y^+, respectively (see Observation 44). For that, we restate the definition of right-to-left maxima of permutation in terms of permutation matrices.

Definition 43. Let $P \in \mathcal{P}_n$ be a permutation matrix. A pair (a, b) is a *right-to-left maximum* of P if $P_{a,b} = 1$ and $P_{i,j} = 0$ for all $a < i \leq n$ and $b < j \leq n$.

We denote by $\mathcal{RL}(P)$ the set of all right-to-left maxima of P. Observe that $(n, j) \in \mathcal{RL}(P)$ for any permutation matrix $P \in \mathcal{P}_n$, where $j \in [n]$ is an index such that $P_{n,j} = 1$. For a permutation matrix P, let $F(P)$ be a filled subdiagram of P with the underlying diagram

$$\{(i, j) \mid \exists (a, b) \in \mathcal{RL}(P) \ : \ i < a \text{ and } j < b\}.$$

We claim that $F(P)$ is a filled Ferrers diagram. Indeed, if $(i, j + 1) \in F(P)$, then $(i, j) \in F(P)$. And if $(i + 1, j) \in F(P)$, then $(i, j) \in F(P)$. It implies that conditions (i) and (ii) in Definition 32 are satisfied. We have the following simple characterization of permutation matrices avoiding the patterns X^+ and Y^+, respectively.

Observation 44. *A permutation matrix P partially avoids X^+ (or Y^+) if and only if $F(P)$ avoids X (or Y).*

Let P be a permutation matrix of order n and let $F := F(P)$. We claim that if $c_i(F) > c_{i+1}(F)$, then the column $C_{i+1}(F)$ is zero. It is enough to show that $(i + 1, c_i + 1)$ is a right-to-left maximum of P, where $c_i := c_i(F)$. Since $(i, c_i) \in F(P)$ and $(i, c_i + 1) \notin F(P)$, there exists a right-to-left maximum (a, b) of P such that $a > i$ and $b = c_i + 1$. Moreover, since $(i + 1, c_i) \notin F$, we have $a = i + 1$. Hence $(a, b) = (i + 1, c_i + 1)$ is a right-to-left of maximum of P. It follows that $C_{i+1}(F)$ is zero because $(i + 1, c_i + 1) \notin F$ is the only nonzero entry in $C_{i+1}(P)$. In general, we call any filled diagram with such property "nice".

Definition 45. A filled Ferrers diagram F is called *nice* if, for every $i \in [c(F)-1]$, the column $C_{i+1}(F)$ is zero whenever $c_i(F) > c_{i+1}(F)$.

For a Ferrers diagram G, let $\mathcal{N}_G$ be the set of all nice filled Ferrers diagrams with the underlying diagram G. And for a pattern A, let $\mathcal{N}_G(A)$ be the set of all nice filled diagrams from $\mathcal{N}_G$ that avoid A. Although the patterns X and Y are not shape-Wilf-equivalent, they are "nice-Wilf-equivalent" if we consider nice filled Ferrers diagrams. More precisely, we prove the following lemma.

Lemma 46. *Let G be a Ferrers diagram. Then there exists a bijection*

$$\Psi \colon \mathcal{N}_G(X) \longrightarrow \mathcal{N}_G(Y)$$

preserving the position of zero columns and zero rows.

We are now ready to sketch the proof that X^+ and Y^+ are partially Wilf-equivalent. We use Lemma 46 to find a bijection $\Phi \colon \mathcal{P}_n^{\mathrm{P}}(X^+) \to \mathcal{P}_n^{\mathrm{P}}(Y^+)$. Take $P \in \mathcal{P}_n^{\mathrm{P}}(X^+)$ and construct $F(P)$. Let G be the underlying Ferrers diagram of $F(P)$. Since $F(P) \in \mathcal{N}_G(X)$ by Observation 44, we replace $F(P)$ in P by $\Psi(F(P))$ to obtain a permutation matrix P'. It can be shown that $F(P') = \Psi(F(P))$. Since $F(P') \in \mathcal{N}_G(Y)$, we know that P' partially avoids Y^+ by Observation 44. Using a similar idea we prove Theorem 42 in full generality in the next section. The rest of this section is devoted to the proof of Lemma 46.

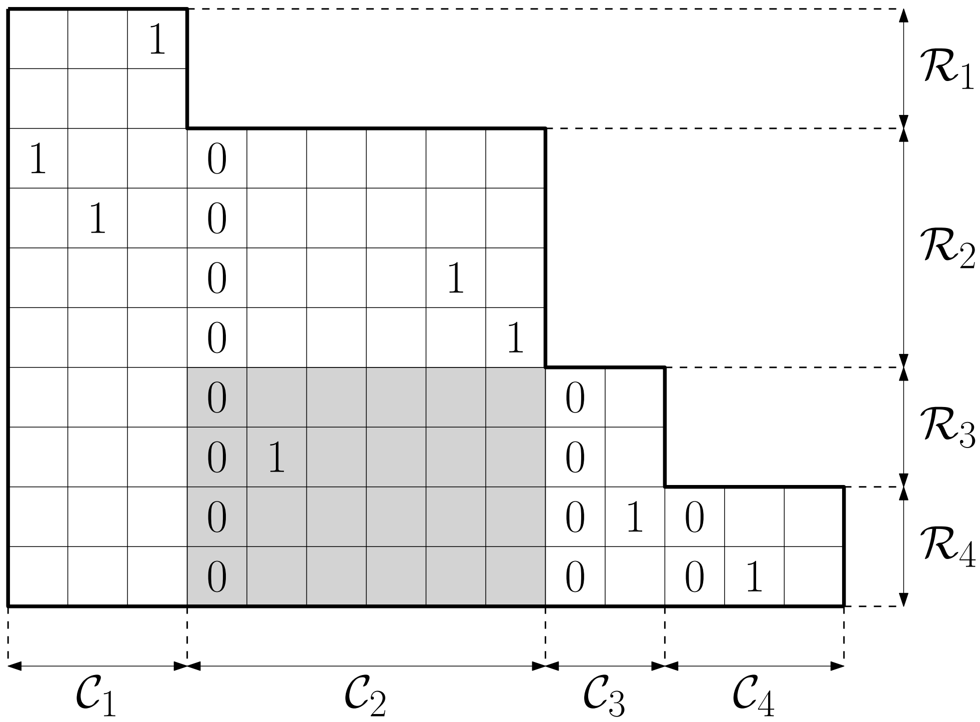

Figure 3.5: The partition of columns and partition rows of given filled nice Ferrers diagram H by their sizes. The shaded cells form a filled subdiagram $H|_{2\times\geq 3}$.

The crucial step in our proof of Lemma 46 is to find an appropriate characterization of nice filled Ferrers diagrams avoiding X and Y, respectively (see Lemma 47 and Lemma 48). For that, we need to partition the set of columns and the set of rows of a (filled) Ferrers diagram by their sizes.

For a Ferrers diagram G, we define an equivalence relation $\sqcap$ on the set of all nonempty columns of G as follows: $C_i \sqcap C_j$ if and only if $c_i = c_j$ (i.e., C_i and C_j have the same size). We denote by $\mathcal{C}_1(G), \dots, \mathcal{C}_\ell(G)$ the partition of the set of all nonempty columns of G induced by $\sqcap$. Moreover, we assume that this partition is ordered decreasingly; that is, if $C_i \in \mathcal{C}_a(G)$ and $C_j \in \mathcal{C}_b(G)$ for $a < b$, then $c_i > c_j$. We refer to this ordered partition simply as the *partition of columns of G by their sizes.* In a similar way, we define the *partition of rows of G by their sizes* into $\mathcal{R}_1(G), \dots, \mathcal{R}_{\ell'}(G)$. However, contrary to the order of partition of columns, we assume that the partition of rows is ordered increasingly; that is, if $R_i \in \mathcal{R}_a(G)$ and $R_j \in \mathcal{R}_b(G)$ for $a < b$, then $r_i < r_j$. Moreover, for $a \in [\ell]$, let $I(\mathcal{C}_a(G))$ be the set of all indices of columns in $\mathcal{C}_a(G)$. Formally,

$$I(\mathcal{C}_a(G)) := \{i \mid C_i \in \mathcal{C}_a(G)\}.$$

For a filled Ferrers diagram F, we define the *partition of columns (or rows) of F by their sizes* to be the partition of columns (or rows) of the underlying diagram of F by their sizes. See Figure 3.5 for an illustration.

Two following remarks are worth mentioning. First, we have $\ell = \ell'$ for every Ferrers diagram G. Second, if F and F' are two filled Ferrers diagrams with the same underlying diagram, the partition of columns (rows) of F by their sizes is the same as the partition of columns (rows) of F' by their sizes.

Unless otherwise stated, till the beginning of the proof of Lemma 46, let F be an arbitrary but fixed nice filled Ferrers diagram with the partition $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ of columns of F by their sizes and partition $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$ of rows of F by their sizes.

We characterize nice filled Ferrers diagrams avoiding X by looking only at columns in $\mathcal{C}_{a-1}$ and $\mathcal{C}_a$ for every $a \in [\ell] \setminus \{1\}$. Before that, we introduce a few more definitions. For $a, b \in [\ell]$, we denote by $F|_{a\times \geq b}$ a filled subdiagram of F with the underlying diagram

$$F \cap \mathcal{C}_a \cap \bigcup_{k=b}^{\ell} \mathcal{R}_k.$$

See Figure 3.5 for an example. We say that $F|_{a\times \geq b}$ is *zero* if $F|_{a\times \geq b}(i,j) = 0$ for every cell $(i,j) \in F|_{a\times \geq b}$ and $F|_{a\times \geq b}$ is *nonzero* if it is not zero. Moreover, let $H_{a,b}(F)$ and $L_{a,b}(F)$ be the highest and lowest index of nonempty row of $F|_{a\times \geq b}$, respectively. Formally,

$$H_{a,b}(F) := \max\{j \mid (i,j) \in F|_{a\times \geq b} \text{ and } F(i,j) \neq 0\}$$

and

$$L_{a,b}(F) := \min\{j \mid (i,j) \in F|_{a\times \geq b} \text{ and } F(i,j) \neq 0\}.$$

If $F|_{a\times \geq b}$ is zero, then $H_{a,b}(F)$ and $L_{a,b}(F)$ are undefined.

Unless otherwise stated, till the beginning of the proof of Lemma 46, we assume that for every $a \in [\ell]$, $F|_{a\times \geq a}$ is nonzero.

Note that by the assumption $H_{a,a}(F)$ and $L_{a,a}(F)$ are always well-defined for every $a \in [\ell]$. Moreover, it also says that every column partition $\mathcal{C}_a$ contain at least one column that is nonzero in F. We are now ready to give a characterization of nice filled Ferrers diagrams avoiding X.

Lemma 47. *The nice filled Ferrers diagram F avoids X if and only if, for all $a \in [\ell]$, the following two conditions hold*

(i) $F|_{a\times \geq a}$ avoids X and

(ii) if $a \geq 2$, then either $F|_{a-1\times \geq a}$ is zero or $L_{a-1,a}(F) > H_{a,a}(F)$.

Proof. Suppose first that F avoids X. For all $a \in [\ell]$, $F|_{a\times \geq a}$ avoids X because $F|_{a\times \geq a}$ is a filled subdiagram of F. For the sake of contradiction, assume that there exists $a \in [\ell] \setminus \{1\}$ such that $F|_{a-1\times \geq a}$ is nonzero and $L_{a-1,a}(F) < H_{a,a}(F)$. It follows that there exist two nonzero cells $(i_1, j_1) \in F|_{a-1\times \geq a}$ and $(i_2, j_2) \in F|_{a\times \geq a}$ such that $i_1 < i_2$ and $j_1 < j_2$. Let m be the index of the leftmost nonempty column in $F|_{a\times \geq a}$. Since F is a nice filled diagram, the column $C_m(F)$ is zero. Hence $i_1 < m < i_2$. It implies that F contains a copy of X induced by $(\{i_1, m, i_2\}, \{j_1, j_2\})$, a contradiction. Thus, the conditions (i) and (ii) are true for all $a \in [\ell]$.

On the other hand, suppose that F contains X. Let $F[I \times J]$ be a copy of X in F induced by (I, J) for some set of column indices $I = \{i_1, i_2, i_3\}$ and some set of row indices $J = \{j_1, j_2\}$ such that the difference $i_3 - i_1$ is as small as possible. If $I \subseteq I(\mathcal{C}_a)$ for some $a \in [\ell]$, then also $F|_{a\times \geq a}$ contains X. Otherwise, $i_1 \in I(\mathcal{C}_b)$

and $i_3 \in I(\mathcal{C}_a)$ for $1 \le b < a \le \ell$. We consider two cases. First, suppose that $b = a - 1$. Since $F(i_1, j_1) = 1$ and $F(i_3, j_2) = 1$, we have $L_{a-1,a}(F) \le j_1$ and $H_{a,a}(F) \ge j_2$. Hence

$$L_{a-1,a}(F) \le j_1 < j_2 \le H_{a,a}(F)$$

and so the condition (ii) is not true for some $a \in [\ell]$.

Second, suppose that $b < a - 1$. If $F|_{a-1 \times \ge a}$ is zero, then let (i_3', j_2') be a nonzero cell in $F|_{a-1 \times \ge a-1}$ (remember that $F|_{a-1 \times \ge a-1}$ is nonzero). Observe that $j_2' > j_2$. Moreover, let i_2' be the index of the leftmost nonempty column in $F|_{a-1 \times \ge a-1}$. Since $i_1 < i_2' < i_3' < i_3$ and $j_1 < j_2 < j_2'$, $F[\{i_1, i_2', i_3'\} \times \{j_1, j_2'\}]$ is a copy of X in F such that $i_1 - i_3' < i_1 - i_3$, a contradiction with the choice of I.

We can now assume that $F|_{a-1 \times \ge a}$ is nonzero. If there exists a nonzero cell $(i_3', j_2') \in F|_{a-1 \times \ge a}$ such that $j_2' > j_2$, then we reach a contradiction in the same way as in the previous paragraph. Otherwise, all cells $(i, j) \in F|_{a-1 \times \ge a}$ such that $j > j_2$ are zero. In particular, $L_{a-1,a}(F) < j_2$. On the other hand, clearly $H_{a,a}(F) \ge j_2$. Hence $L_{a-1,a}(F) < H_{a,a}(F)$. Thus, at least one condition (i) or (ii) is not true for some $a \in [\ell]$ as required. □

Along the same line, we characterize nice filed Ferrers diagrams avoiding Y, however, it is not sufficient to look only at the columns in $\mathcal{C}_{a-1}$ and $\mathcal{C}_a$ for every $a \in [\ell] \setminus \{1\}$. For $a \in [\ell] \setminus \{1\}$, let $b(F, a) \in [a - 1]$ be the largest number such that $F|_{b(F,a) \times \ge a}$ is nonzero. If no such number exists, we set $b(F, a) := 0$.

Lemma 48. *The nice filled Ferrers diagram F avoids Y if and only if, for all $a \in [\ell]$, the following two conditions hold*

(i) $F|_{a \times \ge a}$ avoids Y and

(ii) if $a \ge 2$ and $b(F, a) > 0$, then $H_{b(F,a),a}(F) < L_{a,a}(F)$.

Proof. We omit the proof of the forward implication since it is similar to the proof of forward implication of Lemma 47.

For the backward implication, suppose that F contains Y. Let $F[I \times J]$ be a copy of Y in F induced by (I, J) for some set of column indices $I = \{i_1, i_2, i_3\}$ and some set of row indices $J = \{j_1, j_2\}$ such that the difference $i_3 - i_1$ is as small as possible. If $I \subseteq I(\mathcal{C}_a)$ for some $a \in [\ell]$, then $F|_{a \times \ge a}$ contains Y. Otherwise, $i_1 \in I(\mathcal{C}_b)$ and $i_3 \in I(\mathcal{C}_a)$ from some $a, b \in [\ell]$ such that $b < a$. By the choice of I, we have $b = b(F, a)$. Observe that $H_{b,a}(F) \ge j_2$ and $L_{a,a}(F) \le j_1$. Hence

$$H_{b,a}(F) \ge j_2 > j_1 \ge L_{a,a}(F).$$

Thus, the condition (i) or the condition (ii) is not true for some $a \in [\ell]$. □

Suppose for a moment that F avoids X. In other words, F satisfies the conditions (i) and (ii) in Lemma 47 for every $a \in [\ell]$. Roughly speaking, the main idea of proof of Lemma 46 is transform F into a nice filled Ferrers diagram F' in ℓ successive steps while keeping the following invariant: after each step $t \in [\ell]$, F' satisfies the conditions (i) and (ii) in Lemma 48 for every $a \in [t]$ and the conditions (i) and (ii) in Lemma 47 for every $a \in \{t+1, t+2 \ldots, \ell\}$. After the last step, notice that F' avoids Y by Lemma 48. We now describe the transforming operations.

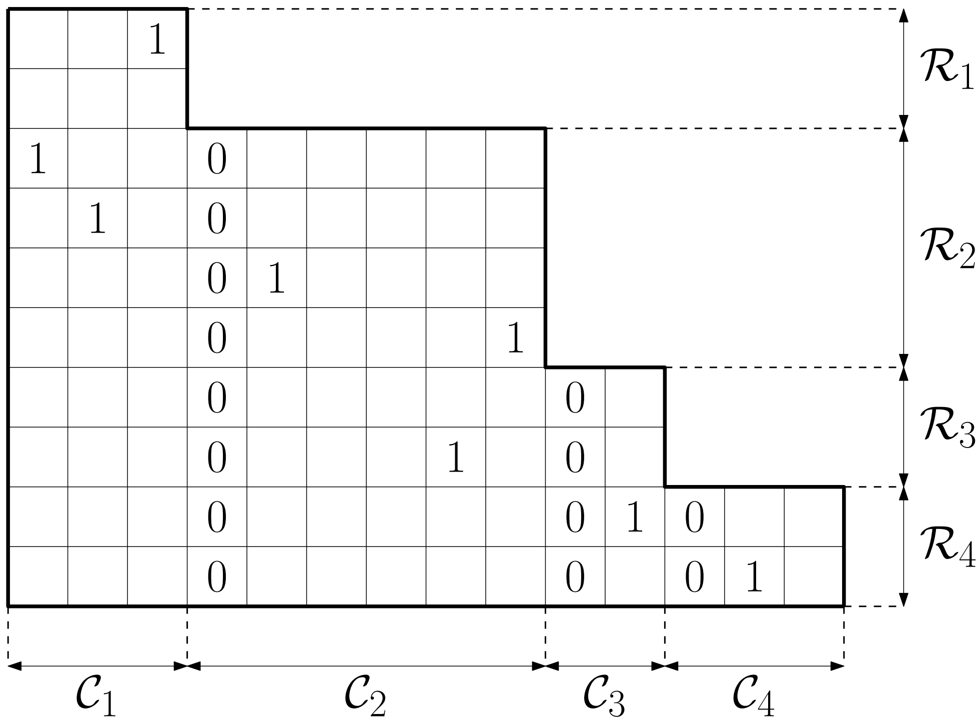

Figure 3.6: Recall the nice filled diagram H from Figure 3.5. In this figure, we see the nice filled diagram $\varphi_2(H)$ obtained from H by reverting the ordered of nonzero row in $H|_{2\times\geq 2}$.

Fix $a, b \in [\ell]$. We denote by $F|_{\leq a\times\geq b}$ a filled subdiagram of F with the underlying diagram

$$F \cap \bigcup_{k=1}^{a} \mathcal{C}_k \cap \bigcup_{k=b}^{\ell} \mathcal{R}_k.$$

A row $R_j(F)$ of F is said to be *zero in* $F|_{a\times\geq b}$ if $R_j(F|_{a\times\geq b})$ is a zero row, and *nonzero in* $F|_{a\times\geq b}$ otherwise. Similarly, a row $R_j(F)$ of F is said to be *zero in* $F|_{\leq a\times\geq b}$ if $R_j(F|_{\leq a\times\geq b})$ is a zero row, and *nonzero in* $F|_{\leq a\times\geq b}$ otherwise. Let

$$\mathcal{E}_a(F) := \{j \in [r(F)] \mid \text{the } j\text{-th row } R_j(F) \text{ is nonzero in } F|_{a\times\geq a}\}$$

and

$$\mathcal{E}_{\leq a}(F) := \{j \in [r(F)] \mid \text{the } j\text{-th row } R_j(F) \text{ is nonzero in } F|_{\leq a\times\geq a}\}.$$

Let G be the underlying diagram of F. Note that the partition of column of G by their sizes is $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ and the partition of rows of G by their sizes is $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$. For $a \in [\ell]$, the first transforming operation

$$\varphi_a \colon \mathcal{N}_G \to \mathcal{N}_G$$

is defined as follows. Given a nice filled Ferrers diagram $F \in \mathcal{N}_G$, the operation φ_a reverts the order of nonzero rows in $F|_{a\times\geq a}$. Suppose that $\mathcal{E}_a(F) = \{j_1, j_2, \ldots, j_k\}$. Formally, $F' = \varphi_a(F)$ is a nice filled Ferrers diagram with the underlying G such that

- $\forall i \in I(\mathcal{C}_a),\ \forall j_t \in \mathcal{E}_a(F)\ :\ F'(i, j_t) = F(i, j_{k-t+1})$ and

- $\forall (i,j) \in F \ : \ i \notin I(\mathcal{C}_a) \vee j \notin \mathcal{E}_a(F) \implies F'(i,j) = F(i,j)$.

See Figure 3.6 for an illustration. Note that this transforming operation preserves the positions of zero columns and rows. Since $R_{j_t}(F)$ is nonzero in $F|_{a\times \geq a}$ if and only if $R_{j_{k-t+1}}(F')$ is nonzero in $F'|_{a\times\geq a}$, we have $\mathcal{E}_a(F') = \mathcal{E}_a(F)$. It implies that

$$F(i, j_t) = F'(i, j_{k-t+1}) = F''(i, j_{k-(k-t+1)+1}) = F''(i, j_t),$$

where $F'' := \varphi_a(F') = \varphi_a(\varphi_a(F))$. Hence $F = \varphi_a(\varphi_a(F))$ and so φ_a is a bijection. We summarize this results in the following observation.

Observation 49. *Let $a \in [\ell]$. The transforming operation $\varphi_a \colon \mathcal{N}_G \to \mathcal{N}_G$ is a bijection preserving the positions of zero columns and zero rows. Moreover, for every $F \in \mathcal{N}_G$ and $b \in [\ell]$, the following holds:*

(i) if $b \neq a$, then $F'|_{b\times\geq b} = F|_{b\times\geq b}$,

(ii) $\mathcal{E}_b(F') = \mathcal{E}_b(F)$,

(iii) $F|_{b\times\geq a}$ is zero if and only if $F'|_{b\times\geq a}$ is zero,

(iv) $L_{b,b}(F) = L_{b,b}(F')$ and $H_{b,b}(F) = H_{b,b}(F')$.

where $F' = \varphi_a(F)$.

Proof. To see the case (i), take $(i,j) \in F$ such that $i \in I(\mathcal{C}_b)$ for some $b \neq a$ and observe that $F'(i,j) = F(i,j)$ by definition.

For $a = b$, we have already observed that $\mathcal{E}_b(F') = \mathcal{E}_b(F)$. For $b \neq a$, this equality holds by the case (i). The cases (iii) and (iv) directly follow from (ii). □

By reverting the order of nonzero rows in X we obtain Y and vice versa. In other words, we have $\varphi_1(X) = Y$ and hence the following observation.

Observation 50. *Let F be a nice filled Ferrers diagram. For $a \in [\ell]$, $F|_{a\times\geq a}$ avoids X if and only if $\varphi_a(F)|_{a\times\geq a}$ avoids Y.*

For $a \in [\ell]$ and $m \in \{0, 1, \dots r(G)\}$, the second transforming operation

$$\psi_{a,m} \colon \mathcal{N}_G \to \mathcal{N}_G$$

is defined as follows. Given a nice filled Ferrers diagram $F \in \mathcal{N}_G$ with $\mathcal{E}_{\leq a}(F) = \{j_1, j_2, \dots, j_k\}$. For $0 \leq m \leq k$, the operation $\psi_{a,m}$ swaps the block of nonzero rows $\{R_{j_1}(F), \dots, R_{j_m}(F)\}$ and the block of nonzero rows $\{R_{j_{m+1}}, \dots, R_{j_k}(F)\}$ while keeping the order inside of each block. Formally, $F' := \psi_{a,m}(F)$ is a nice filled Ferrers diagram with the underlying diagram G such that

- $\forall i \in \bigcup_{s=1}^{a} I(\mathcal{C}_s), \ \forall t \in \{1, \dots, m\} \ : \ F'(i, j_t) = F(i, j_{t+k-m})$,
- $\forall i \in \bigcup_{s=1}^{a} I(\mathcal{C}_s), \ \forall t \in \{m+1, \dots, k\} \ : \ F'(i, j_t) = F(i, j_{t-m})$,
- $\forall (i,j) \in F \ : \ i \notin \bigcup_{t=1}^{a} I(\mathcal{C}_t) \vee j \notin \mathcal{E}_{\leq a}(F) \implies F'(i,j) = F(i,j)$.

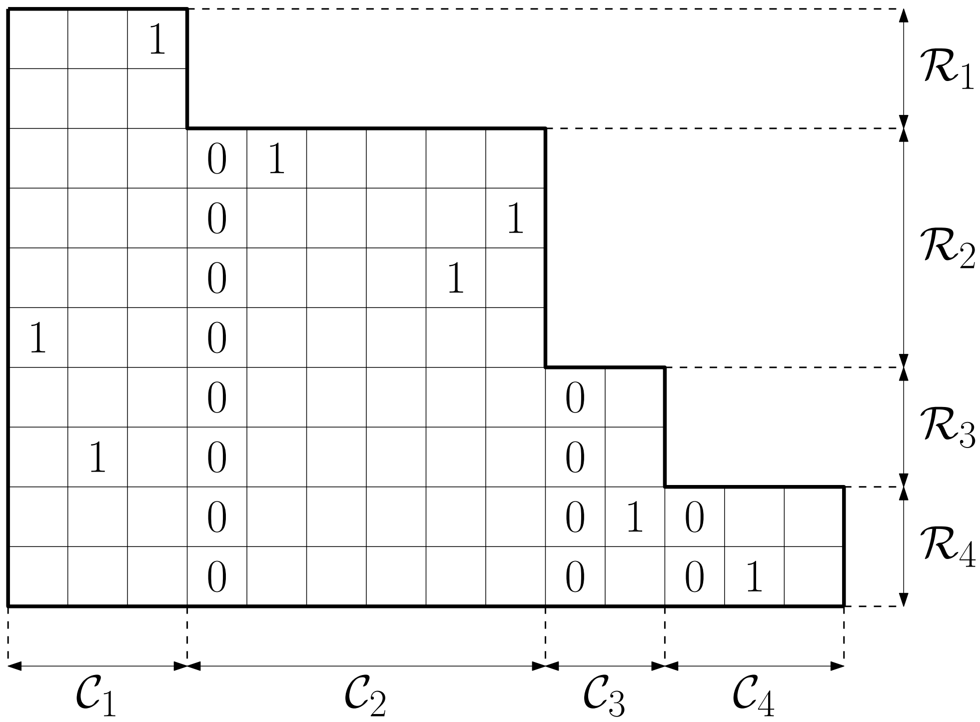

Figure 3.7: Recall the nice filled diagram $\varphi_2(H)$ from Figure 3.6. In this figure, we see the nice filled diagram $\psi_{2,3}(\varphi_2(H))$.

For $m > k$, we define $\psi_{a,m}(F) = F$.

See Figure 3.7 for an illustration. Observe that also the second transforming operation preserves the position of zero columns and rows. Moreover, we claim that $\mathcal{E}_{\leq a}(\varphi_{a,m}(F)) = \mathcal{E}_{\leq a}(F)$. For $m = 0$ or $m > k$, the equality holds trivially. For $0 < m \leq k$, it follows from the fact that the function $f\colon [m] \to [m]$ defined as

$$f(t) = \begin{cases} t + k - m & \text{if } t \leq m, \\ t - m & \text{if } t > m \end{cases}$$

is a bijection.

Finally, we prove that $\psi_{a,m}$ is a bijection. It is sufficient to show that $\psi_{a,m}$ is injective. Let F_1 and F_2 be different nice filled Ferrers diagrams with the same underlying diagram G. If $\mathcal{E}_{\leq a}(F_1) = \mathcal{E}_{\leq a}(F_2)$, then $\psi_{a,m}(F_1) \neq \psi_{a,m}(F_2)$ because $\psi_{a,m}$ swaps the same two blocks of rows in both nice filled Ferrers diagrams. Otherwise, $\mathcal{E}_{\leq a}(F_1) \neq \mathcal{E}_{\leq a}(F_2)$. In this case, we also have $\psi_{a,m}(F_1) \neq \psi_{a,m}(F_2)$ because $\mathcal{E}_{\leq a}(\psi_{a,m}(F_i)) = \mathcal{E}_{\leq a}(F_i)$ for $i \in \{1, 2\}$.

Observation 51. *For $a \in [\ell]$ and $m \in \{0, 1, \ldots, r(G)\}$, the transforming operation $\psi_{a,m}\colon \mathcal{N}_G \longrightarrow \mathcal{N}_G$ is a bijection preserving the position of zero columns and rows.*

Proof of Lemma 46. Let G be a Ferrers diagram with the partition $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ of columns by their sizes and the partition $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_\ell$ of rows by their sizes. For $k \in [\ell + 1]$, let $\mathcal{N}_G^k$ be the set of all nice filled diagrams F with the underlying diagram G such that $F|_{a \times \geq a}$ is nonzero for every $a \in [\ell]$ and satisfies the following three conditions:

(i) $\forall a \in \{1, 2, \ldots k-1\} : F|_{a \times \geq a}$ avoids Y and if $a \geq 2$ and $b(F, a) > 0$, then $H_{b(F,a),a}(F) < L_{a,a}(F)$,

(ii) $\forall a \in \{k, k+1, \ldots \ell\} : F|_{a \times \geq a}$ avoids X and if $a \geq 2$, then either $F|_{a-1 \times \geq a}$ is zero or $L_{a-1,a}(F) > H_{a,a}(F)$,

(iii) let $a(F, k)$ be the smallest number such that $F|_{a(F,k) \times \geq k}$ is nonzero; if $a(F, k) < k \leq \ell$, then $L_{a(F,k),k}(F) > H_{k,k}(F)$.

For every $k \in [\ell]$, $a(F, k)$ is well-defined because $F|_{k \times \geq k}$ is nonzero, and it satisfies $1 \leq a(F, k) \leq k$. Note that the condition (i) above is equivalent to the assumption that F has no copy of Y in $F|_{\leq k-1 \times \geq 1}$, while the conditions (ii) and (iii) together guarantee that F has no copy of X whose rightmost column is in $\bigcup_{t=k}^{\ell} \mathcal{C}_t$.

The main step of the proof is to construct a bijection $\chi_G^k : \mathcal{N}_G^k \to \mathcal{N}_G^{k+1}$ preserving the position of zero columns and zero rows for every $k \in \ell$. If no confusion can arise, we drop the lower index in $\mathcal{N}_G^k$ and χ_G^k.

To properly refer to the conditions (i), we say that a nice filled Ferrers diagram F *satisfies the condition (i) for k' (or $k = k'$)* if the condition (i) is true for F and $k = k'$. Similarly, F *satisfies the conditions (ii) and (iii) for k' (or $k = k'$).*

If $k = 1$, we define $\chi_1 : \mathcal{N}^1 \to \mathcal{N}^2$ as follows

$$\forall F \in \mathcal{N}^1 \ : \ \chi_1(F) := \varphi_1(F).$$

By Observation 49, we know that χ_1 is injective and preserves the position of zero columns and rows. We claim that $\chi_1(F) \in \mathcal{N}^2$ for every $F \in \mathcal{N}^1$. Let $F \in \mathcal{N}^1$ be arbitrary and let $F' := \chi_1(F)$. Since $F|_{1 \times \geq 1}$ avoids X, by Observation 50, $F'|_{1 \times \geq 1}$ avoids Y. Hence F' satisfies the condition (i) for $k = 2$. By Observation 49, F' satisfies the condition (ii) for $k = 2$. Moreover, the condition (ii) and (iii) are equivalent for $k = 2$. Hence F' satisfies the condition (iii) for $k = 2$ and so $F' \in \mathcal{N}^2$. It remains to show that χ_1 is surjective. Take $F' \in \mathcal{N}^2$ and let $F := \varphi_1(F')$. We claim that $F \in \mathcal{N}^1$. Note that F trivially satisfies the conditions (i) and (iii) for $k = 1$. Moreover, it satisfies the condition (ii) for $k = 1$ by Observation 49. Finally, recall that $\chi_1(F) = \varphi_1(F) = \varphi_1(\varphi_1(F')) = F'$. Thus, χ_1 is a bijection from $\mathcal{N}^1$ to $\mathcal{N}^2$ preserving the position of zero columns and zero rows.

Let $k \in [\ell] \setminus \{1\}$ be arbitrary. We first explore the structure of nice filled Ferrers diagrams in $\mathcal{N}^k$. Take an arbitrary nice filled Ferrers diagram F from $F \in \mathcal{N}^k$.

Claim 1. *Let $\{a_1, a_2, \ldots, a_h\}$ be the set of all numbers such that $F|_{a_i \times \geq k}$ is nonzero and $a_i < k$. Then*

$$L_{a_1,k}(F) < L_{a_2,k}(F) < \cdots < L_{a_h,k}(F).$$

Proof. There is nothing to prove for $h \leq 1$. If $h \geq 2$, take an arbitrary number i such that $2 \leq i \leq h$. Since $a_i < k$ and F satisfies the condition (i) for k, we know that $b(F, a_i) = a_{i-1}$ and so $H_{a_{i-1},a_i}(F) < L_{a_i,a_i}(F)$. Hence $H_{a_{i-1},k}(F) < L_{a_i,k}(F)$ because $F|_{a_{i-1}\times\geq k}$ and $F|_{a_i\times\geq k}$ are both nonzero. Thus,

$$L_{a_{i-1},k}(F) \leq H_{a_{i-1},k}(F) < L_{a_i,k}(F)$$

as required. ☐

If $a(F, k) < k \leq \ell$, then $a(F, k) = a_1$ and

$$H_{k,k}(F) < L_{a(F,k),k}(F) < L_{a_2,k}(F) < \cdots < L_{a_h,k}(F)$$

by previous claim. Suppose that $\mathcal{E}_{\leq k}(F) = \{j_1, j_2, \ldots j_t\}$. The direct consequence of the chain of inequalities is that there exists a unique $m \in [t]$ such that $\mathcal{E}_k(F) = \{j_1, j_2, \ldots, j_m\}$. For future reference, let $\mathcal{E} := \mathcal{E}_{\leq k}(F) \setminus \mathcal{E}_k(F) = \{j_{m+1}, j_{m+2}, \ldots, j_t\}$. We construct $\chi^k \colon \mathcal{N}^k \to \mathcal{N}^{k+1}$ as follows

$$\forall F \in \mathcal{N}^k \ : \ \chi^k(F) := \psi_{k,m}(\varphi_k(F)).$$

We remark that the exact value of 'm' depends on F and k. We claim that $\chi^k(F) \in \mathcal{N}^{k+1}$. We first discuss the properties of $F' := \varphi_k(F)$.

Claim 2. *The nice filled Ferrers diagram F' satisfies the condition (i) and (iii) for k and satisfies the condition (ii) for $k+1$. Moreover, $\mathcal{E}_k(F') = \mathcal{E}_k(F)$ and $\mathcal{E}_{\leq k}(F') = \mathcal{E}_{\leq k}(F)$. In particular, $\mathcal{E} = \mathcal{E}_{\leq k}(F') \setminus \mathcal{E}_k(F')$.*

Proof. Directly follows from Observation 49. ☐

Second, we show that $F'' := \psi_{k,m}(F') = \chi^k(F)$ satisfies the conditions (i),(ii), and (iii) for $k+1$. Recall that $\mathcal{E}_{\leq k}(F'') = \mathcal{E}_{\leq k}(F')$. Moreover, we have

$$\mathcal{E}_k(F'') = \{j_{t-m+1}, j_{t-m+2}, \ldots, j_t\}.$$

Let $\mathcal{E}'' := \mathcal{E}_k(F'') \setminus \mathcal{E}_k(F'') = \{j_1, j_2, \ldots j_{t-m}\}$.

Claim 3. *For every $a \in [k] \setminus \{1\}$, if $b(F'', a) > 0$, then $H_{b(F'',a),a}(F'') < L_{a,a}(F'')$.*

Proof. First, take $a \in [k-1] \setminus \{1\}$ arbitrary such that $b(F'', a) > 0$. Observe that $b := b(F'', a) = b(F', a)$. Suppose that $L_{a,a}(F') = j$ and $H_{b,a}(F) = j'$. Since F' satisfies the condition (i) for k, we know that $j' < j$. We consider three cases:

- $j, j' \in \mathcal{E}$; let $s, s' \in \ \{m+1, m+2, \ldots, t\}$ be such that $j = j_s$ and $j' = j_{s'}$. Clearly, $s' < s$. Notice that $L_{a,a}(F'') = j_{s-m}$ and $H_{b,a}(F'') = j_{s'-m}$. Since $s' - m < s - m$, we have $H_{b,a}(F'') < L_{a,a}(F'')$.

- $j \notin \mathcal{E}$ and $j' \in \mathcal{E}$; from the previous case we know that $H_{b,a}(F'') = j_{s'-m}$. Moreover, $L_{a,a}(F'') = L_{a,a}(F')$ because nonzero rows in $F'|_{a\times\geq a}$ are not reordered. Hence

$$H_{b,a}(F'') = j_{s'-m} < j_t < L_{a,a}(F') = L_{a,a}(F'').$$

- $j, j' \notin \mathcal{E}$; from the previous case we know that $L_{a,a}(F'') = L_{a,a}(F')$. For all $i \in [r(G)] \setminus \mathcal{E}$, the i-th row of F' is nonzero in $F'|_{a\times\geq a}$ if and only if the i-th row of F'' is nonzero in $F''|_{a\times\geq a}$ Moreover, since $\mathcal{E}_{\leq a}(F'') = \mathcal{E}_{\leq a}(F')$, we have $H_{b,a}(F'') = H_{b,a}(F')$. Hence

$$H_{b,a}(F'') = H_{b,a}(F') < L_{a,a}(F') = L_{a,a}(F'').$$

Second, suppose that $a = k$ and $b(F'', k) > 0$. Again observe that $b := b(F'', k) = b(F', k)$. Hence $a(F', k) < k$. Since F' satisfies the condition (iii) for k, we have $L_{a(F',k),k}(F') > H_{k,k}(F)$. By Claim 1 and Observation 49,

$$H_{k,k}(F') < L_{a_1,k}(F') < L_{a_2,k}(F') < \cdots < L_{a_h,k}(F').$$

Observe that $a_1 = a(F', k)$ and $a_h = b$. In particular,

$$L_{k,k}(F') \leq H_{k,k}(F') < L_{b,k}(F') \leq H_{b,k}(F').$$

Hence $H_{b,k}(F'') < L_{k,k}(F'')$. □

Claim 4. *For every $a \in [k]$, $F''|_{a\times\geq a}$ avoids Y.*

Proof. Since F' satisfies the condition (i) for k, $F'|_{a\times\geq a}$ avoids Y for every $a \in [k-1]$. Moreover, $F'|_{k\times\geq k}$ avoids Y by Observation 50 because $F|_{k\times\geq k}$ avoids X. Let $a \in [k]$ be arbitrary. Note that $F''|_{a\times\geq a}$ also avoids Y because only the positions of nonzero rows in $F'|_{a\times\geq a}$ are changed but not their order. □

Claim 5. *The nice filled Ferrers diagram F'' satisfies the condition (i) for $k+1$.*

Proof. Combine Claim 3 and Claim 4. □

Claim 6. *The nice filled Ferrers diagram F'' satisfies the condition (ii) for $k+1$.*

Proof. Since $F''|_{a\times\geq a} = F'|_{a\times\geq a}$ for every $a \in \{k+1, \ldots, \ell\}$ and F' satisfies condition (ii) for $k+1$, we know that $F''|_{a\times\geq a}$ avoids X for every $a \in \{k+1, \ldots, \ell\}$ and, if $F|_{a-1\times\geq a}$ is nonzero, $L_{a-1,a}(F'') > H_{a,a}(F'')$ for every $a \in \{k+2, \ldots, \ell\}$.

It remains to verify that

$$L_{k,k+1}(F'') > H_{k+1,k+1}(F'') = H_{k+1,k+1}(F')$$

if $F''|_{k\times\geq k+1}$ is nonzero. Suppose that $F''|_{k\times\geq k+1}$ is nonzero. Observe that $F'|_{k\times\geq k+1}$ is nonzero. Hence $L_{k,k+1}(F') > H_{k+1,k+1}(F')$ because F' satisfies the condition (ii) for $k+1$. Clearly, $L_{k,k+1}(F'') \geq L_{k,k+1}(F')$, which completes the proof. □

Claim 7. *The nice filled Ferrers diagram F'' satisfies the condition (iii) for $k+1$.*

Proof. Let $a := a(F'', k+1)$ be the smallest number such that $F''|_{a\times\geq k+1}$ is nonzero and $a < k+1 \leq \ell$. We claim that

$$L_{a,k}(F'') > H_{k+1,k+1}(F'').$$

Observe that $a = a(F', k)$ because $F''|_{b\times\geq k}$ is zero for all $b < a$ by Claims 1 and 2. Hence $L_{a,k}(F'') = L_{k,k}(F')$. It follows that $F'|_{k\times\geq k+1}$ is nonzero. Since F' satisfies the condition (ii) for $k+1$, we know that $L_{k,k+1}(F') > H_{k+1,k+1}(F')$. Hence

$$L_{a,k}(F'') = L_{k,k}(F') = L_{k,k+1}(F') > H_{k+1,k+1}(F') = H_{k+1,k+1}(F'').$$

□

Thus, χ^k is a mapping from $\mathcal{N}^k$ to $\mathcal{N}^{k+1}$. Moreover, it preserves the position of zero columns and rows because φ_m and $\psi_{k,m}$ also preserve the position of zero columns and rows. We claim that χ^k is injective. Let $F, H \in \mathcal{N}^k$ be two distinct nice filled Ferrers diagrams. Moreover, let $m := |\mathcal{E}_k(F)|$ and $m' := |\mathcal{E}_k(H)|$. Recall that

$$\chi^k(F) = \psi_{k,m}(\varphi_k(F)) \text{ and } \chi^k(H) = \psi_{k,m'}(\varphi_k(H)).$$

Since φ_k is a bijection, we know that $\varphi_k(F) \neq \varphi_k(H)$. If $m = m'$, then $\chi^k(F) \neq \chi^k(H)$ because $\psi_{k,m}$ is a bijection. If $m \neq m'$, then $\chi^k(F) \neq \chi^k(H)$ because both $\varphi_k, \psi_{k,m}$ preserve the number of nonzero rows in $F|_{k\times \geq k}$ and both $\varphi_k, \psi_{k,m'}$ preserve the number of nonzero rows in $H|_{k\times\geq k}$. Hence χ^k is injective as claimed.

The next part of the proof shows that χ^k is surjective. This part should be considered as a new proof and, unless noted otherwise, the objects introduced here have no connection to the object introduced so far. Let $F'' \in \mathcal{N}^{k+1}$. Our goal is to find $F \in \mathcal{N}^k$ such that $\chi^k(F) = F''$.

Claim 8. *Let $\{a_1, a_2, \ldots, a_h\}$ be the set of all numbers such that $F''|_{a_i\times\geq k}$ is nonzero and $a_i \leq k$. Then*

$$L_{a_1,k}(F'') < L_{a_2,k}(F'') < \cdots < L_{a_h,k}(F'').$$

Proof. Almost identical to the proof of Claim 1. □

Since $F''|_{k\times\geq k}$ is nonzero, we have $a_h = k$. Suppose that

$$\mathcal{E}_{\leq k}(F'') = \{j_1, j_2, \ldots, j_t\}.$$

It follows that there exists a unique $m \in [t]$ such that $\mathcal{E}_k(F'') = \{j_m, j_{m+1}, \ldots, j_t\}$. For further reference, let $\mathcal{E}'' = \mathcal{E}_{\leq k}(F'') \setminus \mathcal{E}_k(F'') = \{j_1, j_2, \ldots, j_{m-1}\}$. We claim that

$$F := \varphi_k(\psi_{k,m-1}(F''))$$

belongs to $\mathcal{N}^k$. Observe that $\chi^k(F) = F''$. Let us first consider $F' := \psi_{k,m-1}(F'')$.

Claim 9. *The filled nice Ferrers diagram F' satisfies the condition (iii) for k.*

Proof. Suppose that $a(F'', k) < k$. Since $\mathcal{E}_{\leq k}(F'') = \mathcal{E}_{\leq k}(F')$, we know that $a := a(F', k) = a(F'', k)$. Clearly, $a_1 = a$ and $a_h = k$. Hence $L_{a,k}(F'') < L_{k,k}(F'') \leq H_{k,k}(F'')$ by Claim 8. Thus, $L_{a,k}(F') > H_{k,k}(F')$ as claimed. □

Claim 10. *The filled nice Ferrers diagram F' satisfies the condition (ii) for $k+1$.*

Proof. By slightly adjusting the first paragraph in the proof of Claim 6, it is enough to verify that

$$L_{k,k+1}(F') > H_{k+1,k+1}(F'),$$

if $F'|_{k\times\geq k+1}$ is nonzero. Recall that $H_{k+1,k+1}(F') = H_{k+1,k+1}(F'')$.

Suppose that $F'|_{k\times\geq k+1}$ is nonzero. It follows that $a(F'', k+1) < k+1$. Since F'' satisfies the condition (iii) for $k+1$, we have $L_{a(F'',k+1),k+1}(F'') > H_{k+1,k+1}(F'')$.

Note that $L_{k,k}(F') = L_{a(F'',k),k}(F'')$. Moreover, $L_{k,k+1}(F') = L_{k,k}(F')$ because $F'|_{k\times\geq k+1}$ is nonzero. Hence

$$L_{k,k+1}(F') = L_{k,k}(F') = L_{a(F'',k),k}(F'') > H_{k+1,k+1}(F'') = H_{k+1,k+1}(F').$$

□

Claim 11. *For every $a \in [k] \setminus \{1\}$, if $b(F, a) > 0$, then $H_{b(F',a),a}(F') < L_{a,a}(F')$.*

Proof. Analogous to the first paragraph in the proof of Claim 3. The only difference is that the nonzero rows in $F''|_{\leq k-1\times\geq k}$ are moved up and not down. □

Claim 12. *For every $a \in [k]$, $F'|_{a\times\geq a}$ avoids Y.*

Proof. Let $a \in [k]$ be arbitrary. Since $F''|_{a\times\geq a}$ avoids Y, $F'|_{a\times\geq a}$ also avoids Y because only the positions of nonzero rows in $F''|_{a\times\geq a}$ is changed but not their order. □

Claim 13. *The filled nice Ferrers diagram F' satisfies the condition (i) for k.*

Proof. Combine Claim 11 and Claim 12. □

Finally, we show that F satisfies the conditions (i), (ii), and (iii) for k.

Claim 14. *The filled nice Ferrers diagram F satisfies the condition (i) and (iii) for k.*

Proof. Since F' satisfies the condition (i) and (iii) for k, the claim directly follows from Observation 49. □

Claim 15. *The filled nice Ferrers diagram F satisfies the condition (ii) for k.*

Proof. Since $F'|_{k\times\geq k}$ avoids Y, $F|_{k\times\geq k}$ avoids X by Observation 50. Recall that $F|_{a\times\geq a} = F'|_{a\times\geq a}$ for every $a \in \{k+1, k+2, \ldots, \ell\}$. Hence $F|_{a\times\geq a}$ avoids X for every $a \in \{k, k+1, \ldots, \ell\}$ because F' satisfies the condition (ii) for $k+1$. Moreover, since F' satisfies the condition (ii) for $k+1$, we have $L_{a-1,a}(F) > H_{a,a}(F)$ for every $a \in \{k+1, k+2, \ldots, \ell\}$ whenever $F|_{a-1\times\geq a}$ is nonzero.

It remains to verify that if $F|_{k-1\times\geq k}$ is nonzero, then

$$L_{k-1,k}(F) > H_{k,k}(F).$$

Suppose that $F|_{k-1\times\geq k}$ is nonzero. It implies that $F''|_{k-1\times\geq k}$ is nonzero. Hence $L_{k-1,k}(F'') < L_{k,k}(F'') \leq H_{k,k}(F'')$ by Claim 8. Hence $L_{k-1,k}(F') > H_{k,k}(F')$ and so $L_{k-1,k}(F) > H_{k,k}(F)$ as required. □

Thus, χ_G^k is a bijection from $\mathcal{N}_G^k$ to $\mathcal{N}_G^{k+1}$ preserving the position of nonzero columns and rows. Hence, for every Ferrers diagram G, there exists a bijection χ_G^* from $\mathcal{N}_G^1$ to $\mathcal{N}_G^{\ell+1}$ preserving the position of nonzero columns and rows of G. Observe that $\mathcal{N}_G^1 \subseteq \mathcal{N}_G(X)$ and $\mathcal{N}_G^{\ell+1} \subseteq \mathcal{N}_G(Y)$ by Lemma 47 and Lemma 48, respectively. However, both sets $\mathcal{N}_G^1$ and $\mathcal{N}_G^{\ell+1}$ do not contain nice filled Ferrers diagrams F (with the underlying diagram G) such that $F|_{a\times\geq a}$ is zero for some $a \in [\ell]$. Let us define the *size* of the partition $\mathcal{C}_1(G), \mathcal{C}_2(G), \ldots, \mathcal{C}_\ell(G)$ of columns of G by their sizes to be ℓ. For a Ferrers diagram G, we define a bijection

$\chi_G \colon \mathcal{N}_G(X) \to \mathcal{N}_G(Y)$ by induction on the size ℓ of the partition of columns of G by their sizes.

Let G be a Ferrers diagram and let $\mathcal{C}_1(G), \mathcal{C}_2(G), \ldots, \mathcal{C}_\ell(G)$ be the partition of columns of G by their sizes. If $\ell = 1$, we define χ_G as follows

$$\forall F \in \mathcal{N}_G(X) \ : \ \chi_G(F) := \varphi_1(F).$$

Note that χ_G has the required properties by Observations 49 and 50.

Suppose that $\ell \geq 2$ and take $F \in \mathcal{N}_G(X)$ arbitrary. If $F|_{a\times \geq a}$ is nonzero for every $a \in \ell$, we define $\chi_G(F) := \chi^*_G(F)$. Otherwise, let $a \in [\ell]$ be the smallest index such that $F|_{a\times\geq a}$ is zero. We temporarily erase the zero columns in $F|_{a\times\geq a}$. Formally, we consider the following nice filled Ferrers diagram

$$F' := F[[c(G)] \setminus I(\mathcal{C}_a(G)) \times [r(G)]].$$

Let G' be the underlying diagram of F'. Note that the partition of columns of G' by their sizes is

$$\mathcal{C}_1(G), \ldots, \mathcal{C}_{a-1}(G), \mathcal{C}_{a+1}(G), \ldots, \mathcal{C}_\ell(G).$$

Hence, by induction hypothesis, there exists a bijection $\chi_{G'} \colon \mathcal{N}_{G'}(X) \to \mathcal{N}_{G'}(Y)$ preserving the position of zero columns and rows. Let $\chi_G(F)$ be a nice filled Ferrers diagram obtained from $\chi_{G'}(F')$ by inserting back the zero columns so that its underlying diagram is G. Lemma 52 and Lemma 53, which are stated and proved after the end of this proof, ensure that this construction is correct: F' avoids X and $\chi_G(F)$ avoids Y. Moreover, notice that χ_G preserves the position of zero columns and rows by the construction.

Finally, we claim that χ_G is invertible. Take $F \in \mathcal{N}_G(Y)$ arbitrary. If $F|_{a\times\geq a}$ is nonzero for every $a \in [\ell]$, then $\chi_G^{-1}(F) = (\chi^*_G)^{-1}(F)$. Otherwise, let $a \in [\ell]$ be the smallest index such that $F|_{a\times\geq a}$ is zero. Again, consider

$$F' := F[[c(G)] \setminus I(\mathcal{C}_a(G)) \times [r(G)]].$$

Let G' be the underlying diagram of F'. Then $\chi_G^{-1}(F)$ is obtained from $\chi_{G'}^{-1}(F')$ by inserting back the zero columns so that its underlying diagram is G. Lemma 53 ensures that F' avoids Y and Lemma 52 ensures that $\chi_G^{-1}(F)$ avoids X. Therefore, χ_G is a bijection from $\mathcal{N}_G(X)$ to $\mathcal{N}_G(Y)$ preserving the position of zero columns and zero rows. □

Finally, we state and prove two lemmas used in the proof of Lemma 46. For a filled Ferrers diagram F with the partition $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ of columns of F by their sizes, let $F \setminus \mathcal{C}_a$ be a shorthand for $F[[c(F)] \setminus I(\mathcal{C}_a) \times [r(F)]]$. Intuitively, $F \setminus \mathcal{C}_a$ is a filled Ferrers diagram obtained from F by erasing all its columns in $\mathcal{C}_a$.

Lemma 52. *Let F be a nice filled Ferrers diagram and let $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ be partition of columns of F by their sizes. If $F|_{a\times\geq a}$ is zero for some $a \in [\ell]$, F avoids X if and only if $F \setminus \mathcal{C}_a$ avoids X.*

Proof. If F avoids X, then $F \setminus \mathcal{C}_a$ avoids X as well.

On the other hand, suppose that F contains X. We claim that $F \setminus \mathcal{C}_a$ also contains X. To see this, it is sufficient to find a copy of X in F induced by (I, J)

for some set of column indices I and some set of row indices J such that $i \notin I(\mathcal{C}_a)$ for every $i \in I$.

Let $F[I \times J]$ be a copy of X in F induced by (I, J), where $I = \{i_1, i_2, i_3\}$ is a set of columns indices and $J = \{j_1, j_2\}$ is a set of rows indices such that the difference $i_3 - i_2$ is as small as possible. We claim that $i \notin I(\mathcal{C}_a)$ for every $i \in I$. Since $F|_{a \times \geq a}$ is zero, we know that $i_1, i_3 \notin I(\mathcal{C}_a)$. For the sake of contradiction, suppose that $i_2 \in I(\mathcal{C}_a)$. Let $b > a$ be an index such that $i_3 \in I(\mathcal{C}_b)$ and let $i'_2 \in I(\mathcal{C}_b)$ be the index of the leftmost nonempty column in $F|_{b \times \geq b}$. Since F is a nice filled Ferrers diagram, we have $F(i'_2, j) = 0$ for every j such that $(i'_2, j) \in F$. Hence $i'_2 < i_3$. Let $I' := \{i_1, i'_2, i_3\}$. Thus, $F[I' \times J]$ is a copy of X in F induced by (I', J). This contradicts the choice of I and J because $i_2 < i'_2$. □

The same lemma is true if we replace X by Y. The proof is a verbatim copy of the previous proof and it is omitted.

Lemma 53. *Let F be a nice filled Ferrers diagram $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ be partition of columns of F by their sizes. If $F|_{a \times \geq a}$ is zero for some $a \in [\ell]$, then F avoids Y if and only if $F \setminus \mathcal{C}_a$ avoids Y.*

3.4 Matrix bottom-left corner extension

As in the previous section, let $X := \left(\begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{smallmatrix}\right)$ and $Y := \left(\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{smallmatrix}\right)$. Our aim is to prove Theorem 42, which says that if A is a quasi-permutation matrix whose first column is nonzero, then

$$\left(\begin{smallmatrix} 0 & A \\ X & 0 \end{smallmatrix}\right) \overset{\mathrm{P}}{\sim} \left(\begin{smallmatrix} 0 & A \\ Y & 0 \end{smallmatrix}\right).$$

For a permutation matrix P considered as a transversal and a cell $(a, b) \in P$, let $F_{a,b}(P)$ be a filled subdiagram of P with the underlying diagram

$$\{(i, j) \mid a < i \leq n \text{ and } b < j \leq n\}.$$

Proof of Theorem 42. Let $A \in \mathcal{Q}_{m,k,\ell}$ be a quasi-permutation matrix whose first column is nonzero. It follows that there exists a row index $t \in [k]$ such that $A_{1,t} = 1$. Moreover, for the rest of the proof, let

$$X^+ := \left(\begin{smallmatrix} 0 & A \\ X & 0 \end{smallmatrix}\right) \quad \text{and } Y^+ := \left(\begin{smallmatrix} 0 & A \\ Y & 0 \end{smallmatrix}\right).$$

Note that $X^+, Y^+ \in \mathcal{Q}_{m+3,k+2,\ell+2}$.

For every $n \in \mathbb{N}$, our goal is to show that $p_n^{\mathrm{P}}(X^+) = p_n^{\mathrm{P}}(Y^+)$. For that, we fix $n \in \mathbb{N}$ and construct a bijection

$$\Phi \colon \mathcal{P}_n^{\mathrm{P}}(X^+) \longrightarrow \mathcal{P}_n^{\mathrm{P}}(Y^+).$$

Let $P \in \mathcal{P}_n^{\mathrm{P}}(X^+)$ be an arbitrary permutation matrix of order n that partially avoids X^+. Recall that we look at permutation matrices as transversals and vice versa.

The initial step is to extract a filled subdiagram of P that avoids X. Let us color the cells of P by either green or red as follows—a cell $(a, b) \in P$ is colored green in P if $F_{a,b}(P)$ contains A; otherwise, the cell (a, b) is colored red. We say that a cell (a, b) is green (or red) in P if it is colored green (or red) in P. Let $F_G := F_G(P)$ be a filled subdiagram of P with the underlying diagram

$$\{(a, b) \in P \mid (a, b) \text{ is colored green}\}$$

and let $F_R := F_R(P)$ be a filled subdiagram of P with the underlying diagram

$$\{(a, b) \in P \mid (a, b) \text{ is colored red}\}.$$

The notions defined for P in this paragraph are defined in the same way for any other permutation matrix of order n. We remark that if $A = (\bullet)$, then $F_G = F(P)$, where $F(P)$ is defined in the previous section.

We now prove several claims about the coloring and the filled diagrams F_G and F_R. The following claim is a simple observation.

Claim 0. If $(a+1, b+1)$ is a green cell in P, then $(a+1, b)$ and $(a, b+1)$ are green cells in P.

Claim 1. The filled diagram F_G is a nice filled Ferrers diagram.

Proof of Claim 1. By Claim 0, F_G is a filled Ferrers diagram. It remains to prove that F_G is nice. Let $i \in [c(F_G)-1]$ be a column index such that $c_i(F_G) > c_{i+1}(F_G)$. We claim that $C_{i+1}(F_G)$ is a zero column. Let $c_i := c_i(F_G)$. Since $(i, c_i) \in F_G$ and $(i+1, c_i) \notin F_G$, every copy of A in $F_{i,c_i}(P)$ induced by (I, J) contains $(i+1)$-th column of P (i.e., $i+1 \in I$). Since the leftmost column of A is nonzero, there must exist a row index $j > c_i$ such that $P(i+1, j) = 1$. In particular, $P(i+1, j') = 0$ for every $j' < j$. Hence $C_{i+1}(F_G)$ is a zero column. □

Claim 2. *The nice filled Ferrers diagram F_G avoids X.*

Proof of Claim 2. We proceed by contradiction. Suppose that F_G contains a copy of X induced by (I, J) for some $I = \{i_1, i_2, i_3\}$ and $J = \{j_1, j_2\}$. Since (i_3, j_2) is a green cell in P, the filled diagram $F_{i_3,j_2}(P)$ contains a copy of A induced by (I', J'). Observe that $i_3 < i'$ for every $i' \in I'$ and $j_2 < j'$ for every J'. Hence P contains a copy of X^+ induced by $(I \cup I', J \cup J')$, a contradiction (here it is crucial that we require only that $X^+ \leq P[I \cup I' \times J \cup J']$ and not equality). □

Claim 3. *If $(a, b) \in P$ is a green cell in P, then there is a copy of A in $F_{a,b}(P)$ induced by (I, J) such that, for every $i \in I$ and $j \in J$, the cell (i, j) is red in P.*

Proof of Claim 3. Let $F_{a,b}(P)[I \times J]$ be a copy of A induced by (I, J) for some $I = \{i_1, i_2, \ldots, i_m\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ such that i_1 is a large as possible. We claim that (i_1, j_1) is a red cell in P. If not, we can find a copy of A in $F_{i_1,j_1}(P)$ induced by (I', J'), a contradiction because $i_1 < i'$ for all $i' \in I'$. Moreover, for every $i \in I$ and $j \in J$, the cell (i, j) is red in P by Claim 0 because (i_1, j_1) is red in P and $i_1 \leq i, j_1 \leq j$, which finishes the proof of this claim. □

Let G be the underlying Ferrers diagram of F_G. Note that $F_G \in \mathcal{N}_G(X)$ by Claims 1 and 2. Let P' be a permutation matrix of order n obtained from P

by replacing $F_G(P)$ by $\Psi(F_G(P))$, where $\Psi\colon \mathcal{N}_G(X) \longrightarrow \mathcal{N}_G(Y)$ is the bijection from Lemma 46. Formally,

$$P'(a,b) = \begin{cases} P(a,b) & \text{if } (a,b) \in F_R(P) \\ \Psi(F_G(P))(a,b) & \text{if } (a,b) \in F_G(P). \end{cases}$$

Observe that P' is indeed a permutation matrix because Ψ preserves the position of zero rows and zero columns.

Let us color the cell of P' by either green or red using the same rule as we color the cells of P. Moreover, define $F'_G := F_G(P')$ and $F'_R := F_R(P')$ in the same way as we define $F_G(P)$ and $F_R(P)$.

Claim 4. $F'_R = F_R$.

Proof of Claim 4. We need to only verify that the underlying diagram of F'_R is the same as the underlying diagram of F_R. If (a,b) is a red cell in P, then (a,b) is also a red cell in P' because $F_{a,b}(P) = F_{a,b}(P')$ by definition. On the other hand, if (a,b) is a green cell in P, there is a copy of A in $F_{a,b}(P)$ induced by (I,J) such that, for every $i \in I$ and $j \in J$, the cell (i,j) is red in P by Claim 3. Since red cells in P are also red in P', $F_{a,b}(P')$ contains a copy of A induced by (I,J). Hence (a,b) is a green cell in P'. ☐

It implies that the underlying diagram of F'_G is the same as the underlying diagram of F_G. Hence, $\Psi(F_G) = F'_G$.

Claim 5. $P' \in \mathcal{P}_n^P(Y^+)$.

Proof of Claim 5. We proceed by contradiction. Suppose that P' contains a copy of Y^+ induced by (I,J) for some $I = \{i_1, i_2, \ldots, i_{m+3}\}$ and $J = \{j_1, j_2, \ldots, i_{k+2}\}$. In particular, P' contains a copy of Y induced by $(\{i_1, i_2, i_3\}, \{j_1, j_2\})$. Since $\Psi(F_G) = F'_G$ and F_G avoids X by Claim 2, we know that F'_G avoids Y by Lemma 46. It implies that (a,b) is a red cell in P' for some $a \in \{i_1, i_2, i_3\}$ and $b \in \{j_1, j_2\}$. However, $F_{a,b}(P')$ contains a copy of A induced by $(\{i_4, i_5, \ldots, i_{m+3}\}, \{j_3, j_4, \ldots, j_{k+2}\})$, a contradiction because (a,b) is a red cell in P'. ☐

We define the bijection $\Phi\colon \mathcal{P}_n^{\mathrm{P}}(X^+) \longrightarrow \mathcal{P}_n^{\mathrm{P}}(Y^+)$ by letting $\Phi(P) := P'$. It remains to show that Φ is indeed a bijection or, equivalently, that Φ is invertible. The latter one follows from two facts: first that Ψ is invertible and second that Φ preserves the color of cells (see Claim 4). Our proof of Theorem 42 is now complete. ☐

4. Classification of small patterns

Recall that $\overset{\mathrm{P}}{\sim}$ is an equivalence on the set $\mathcal{Q}$ of all quasi-permutation matrices given by $A \overset{\mathrm{P}}{\sim} B$ if and only if $p_n^{\mathrm{P}}(A) = p_n^{\mathrm{P}}(A)$ for every n. The equivalence $\overset{\mathrm{P}}{\sim}$ is called partial Wilf equivalence and A and B are said to be partially Wilf-equivalent if $A \overset{\mathrm{P}}{\sim} B$. For a pattern $A \in \mathcal{Q}$, we denote by

$$[A]^{\mathrm{P}} := \{B \in \mathcal{Q} \mid A \overset{\mathrm{P}}{\sim} B\}$$

the partial Wilf equivalence class of A.

In this chapter, we utilize the results from previous chapters to classify patterns of size at most 4×4 into partial Wilf equivalence classes—for every pattern $A \in \mathcal{Q}_{m,k,\ell}$, where $1 \leq m, k \leq 4$ and $0 \leq \ell \leq \min\{m, k\}$, we determine its partial Wilf equivalence class $[A]^{\mathrm{P}}$. Since A and $A^\top$ are symmetric, their partial Wilf equivalence classes are the same.

Observation 54. *Let $A \in \mathcal{Q}_{m,k,\ell}$ be a quasi-permutation matrix. Then*

$$[A]^P = [A^\top]^P.$$

Hence it is sufficient to determine only the partial Wilf equivalence classes of patterns that have at least as many columns as rows. In the following observation, we reduce the number of patterns that can belong to the partial Wilf equivalence class of A.

Observation 55. *Let $A \in \mathcal{Q}_{m,k,\ell}$ and $B \in \mathcal{Q}_{m',k',\ell'}$ be quasi-permutation matrices. If $\max\{m, k\} \neq \max\{m', k'\}$, then A and B are not partially Wilf-equivalent.*

Proof. Without loss of generality, we can assume that $\max\{m, k\} = m$ and $\max\{m', k\} = m'$ (otherwise we consider $A^\top$ or $B^\top$). Moreover, we can also assume that $m < m'$ (otherwise we interchange the role of A and B).

Every permutation matrix P of order m partially avoids B because B has too many columns. On the other hand, there exists a permutation matrix P' of order m that partially contains A. We construct P' from A by adding $m - k$ zero rows and after that $m - \ell$ ones so that every column and row contains exactly one nonzero entry. Hence $p_m^{\mathrm{P}}(A) < p_m^{\mathrm{P}}(B) = m!$. □

Let $\mathcal{Q}^*_{\leq m, \leq k}$ be the set of all quasi-permutation matrices of size at most $m \times k$ that have at least as many columns as rows. Our goal is to determine the partial Wilf equivalence class of every pattern $A \in \mathcal{Q}^*_{\leq 4, \leq 4}$. By Observation 55, no pattern with more than 4 columns or more than 4 rows is partially Wilf-equivalent to a pattern of size at most 4×4. Thus, it remains to decide which patterns in $\mathcal{Q}^*_{\leq 4, \leq 4}$ are partially Wilf-equivalent to A.

We have created a computer program [3] that enumerates the number $p_n^{\mathrm{P}}(A)$ of permutation matrices of order n that partially avoid a given pattern $A \in \mathcal{Q}^*_{\leq 4, \leq 4}$ for every $n \in [8]$. With this data (see Table A.1 in Appendix), we easily identify pairs of patterns of size at most 4×4 that are not partially Wilf-equivalent. On the other hand, the data suggest which patterns might be partially Wilf-equivalent. Trivially, symmetric patterns are partially Wilf-equivalent (see Observation 10).

The interesting part is to prove or disprove that nonsymmetric patterns A and B are partially Wilf-equivalent if $p_n^{\mathrm{P}}(A) = p_n^{\mathrm{P}}(B)$ for every $n \in [8]$. Using the results from the two previous chapters, we prove that all such patterns A and B are indeed partially Wilf-equivalent.

The plan of this chapter is as follows. In Section 4.1, we introduce a "one-line" representation of quasi-permutation matrices, which is a more compact representation than a 2-dimensional array for listing a large number of patterns. In Section 4.2, we determine the partial Wilf equivalence class of *zero* pattern $0_{m\times k}$ for arbitrary m and k, where $0_{m\times k}$ is the unique pattern in $\mathcal{Q}_{m,k,0}$. Moreover, we determine the partial Wilf equivalence classes of patterns of size at most 4×4 with exactly one nonzero entry. Finally, in Section 4.3, we determine the partial Wilf equivalence classes of the remaining patterns of size at most 4×4.

4.1 Linear representation of quasi-permutation matrices

It is not practical to draw hundreds of patterns as 2-dimensional arrays because each of them occupies a nontrivial space on a printed page. For this reason, we introduce a "one-line" representation of quasi-permutation matrices. This representation is used in Section 4.3 and Appendix. We represent a quasi-permutation matrix $Q \in \mathcal{Q}_{m,k,\ell}$ as a sequence of numbers $(a_1, a_2, \dots, a_m)|_k$, where

$$a_i = \begin{cases} j & \text{if there exists } j \in [k] \text{ such that } Q_{i,j} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If k is clear from the context, we write $(a_1, a_2, \dots, a_m)$. Observe that the sequence $(a_1, a_2, \dots, a_m)|_k$ contains $m - \ell$ zeros and ℓ distinct numbers from $[k]$. On the other hand, every such sequence represents a unique quasi-permutation matrix. For example, $\left(\begin{smallmatrix} \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{smallmatrix}\right)$ is represented by $(2,0,0,1)|_2$ but $(2,0,0,1)|_3$ represents

$$\begin{pmatrix} \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{pmatrix}.$$

If we write a sequence $(a_1, a_2, \dots, a_m)|_k$, we automatically assume that $0 \le a_i \le k$ for every $i \in [m]$. Since every sequence $(a_1, a_2, \dots, a_m)|_k$ denotes a unique quasi-permutation matrix Q, we can write $(a_1, a_2, \dots, a_m)|_k$ anywhere we can write Q. For example, we say $(a_1, a_2, \dots, a_m)|_k$ and $(b_1, b_2, \dots, b_m)|_k$ are partially Wilf-equivalent meaning that the quasi-permutation matrices represented by the sequences are partially Wilf-equivalent.

Reformulation of theorems

For the reader's convenience, we restate some theorems from previous chapters in the "one-line" representation.

Theorem 56 (Theorem 18 restated)**.** *Let*

$$(a_1, a_2, \dots, a_m)|_k$$

be a quasi-permutation matrix. Then

$$(0, a_1, a_2, \dots, a_m)|_k \overset{P}{\sim} (a_1, a_2, \dots, a_m, 0).$$

For the next set of theorems, it is useful the introduce the following notion. The *direct sum* of quasi-permutation matrices

$$(a_1, a_2, \ldots, a_m)|_k \quad \text{and} \quad (b_1, b_2, \ldots, b_{m'})|_{k'}$$

is the quasi-permutation matrix

$$(a_1, a_2, \ldots, a_m)|_k \oplus (b_1, b_2, \ldots, b_{m'})|_{k'} = (c_1, c_2, \ldots, c_{m+m'})|_{k+k'},$$

where

$$c_i = \begin{cases} a_i & \text{if } i \leq m \\ b_{i-m} + k & \text{if } i > m \text{ and } b_{i-m} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 57 (Theorem 38 restated)**.** *Let $(a_1, a_2, \ldots, a_m)|_m$ and $(b_1, b_2, \ldots, b_m)|_m$ be shape-Wilf-equivalent permutation matrices. For any quasi-permutation matrix $(c_1, c_2, \ldots, c_{m'})|_{k'}$, we have*

$$(a_1, a_2, \ldots, a_m)|_m \oplus (c_1, c_2, \ldots, c_{m'})_{k'} \overset{s}{\sim} (b_1, b_2, \ldots, b_m)|_m \oplus (c_1, c_2, \ldots, c_{m'})_{k'}.$$

Theorem 58 (Theorem 42 restated)**.** *Let*

$$(a_1, a_2, \ldots, a_m)|_k$$

be a quasi-permutation matrix such that $a_1 \neq 0$. Then

$$(1, 0, 2) \oplus (a_1, a_2, \ldots, a_m)|_k \overset{P}{\sim} (2, 0, 1) \oplus (a_1, a_2, \ldots, a_m)|_k.$$

Theorem 59 (Theorem 40 restated)**.** *For any $k \in \mathbb{N}$, we have*

$$(1, 2, \ldots, k)_k \overset{s}{\sim} (k, k-1, \ldots, 1)_k.$$

4.2 Zero and single-one patterns

Let us denote by $0_{m\times k}$ the unique pattern in $\mathcal{Q}_{m,k,0}$. It is easy to compute the number of permutation matrices partially avoiding $0_{m\times k}$ and $A \in \mathcal{Q}_{m,1,1}$, respectively.

Observation 60. *If $k \leq m$, then*

$$p_n^P(0_{m\times k}) = \begin{cases} n! & \text{if } n < m \\ 0 & \text{otherwise.} \end{cases}$$

Observation 61. *Let $A \in \mathcal{Q}_{m,1,1}$ be a quasi-permutation matrix. Then*

$$p_n^P(A) = \begin{cases} n! & \text{if } n < m \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $i \in [m]$ be a column index such that $A_{i,1} = 1$. We claim that any permutation matrix P of order $n \geq m$ partially contains A. Indeed, let $j \in [n]$ be a row index such that $P_{i,j} = 1$. Then P contains a partial copy of A induced by $([m], \{j\})$. ☐

We show that patterns of the same shape with exactly one nonzero entry are partially Wilf-equivalent.

Lemma 62. *Let $A, B \in \mathcal{Q}_{m,k,1}$ be quasi-permutation matrices. Then $A \overset{P}{\sim} B$.*

Proof. Let $i \in [m]$ and $j \in [k]$ be such that $A_{i,j} = 1$. The first $i-1$ columns of A and the first $j-1$ rows of A are zero. Let $C \in \mathcal{Q}_{m,1,1}$ be a quasi-permutation matrix obtained from A by moving the first $i-1$ columns of A after the last column and moving the first $j-1$ rows of A after the last row. Note that $C_{1,1} = 1$. By Theorem 18 and Corollary 19, we have $A \overset{\mathrm{P}}{\sim} C$.

Analogously, we deduce that $B \overset{\mathrm{P}}{\sim} C$. Thus, $A \overset{\mathrm{P}}{\sim} B$ as required. □

Next, we claim that if a nonzero pattern has at least two columns and two rows, then the pattern is avoided by a sufficiently large permutation matrix. Let $1_{m\times k} \in \mathcal{Q}_{m,k,1}$ be a quasi-permutation matrix such that $(1_{m\times k})_{1,k} = 1$.

Observation 63. *Let $A \in \mathcal{Q}_{m,k,\ell}$ be a quasi-permutation matrix with $\ell \geq 1$. If $2 \leq k \leq m$, then $p_m^P(A) > 0$.*

Proof. Let $B \in \mathcal{Q}_{m,k,1}$ be a partial copy of A with exactly one nonzero entry (B is obtained from A by changing some ones to zeros). By Lemma 11, we know that $p_m^{\mathrm{P}}(B) \leq p_m^{\mathrm{P}}(A)$. And by Lemma 62, we know that $B \overset{\mathrm{P}}{\sim} 1_{m\times k}$. Moreover, $p_m^{\mathrm{P}}(1_{m\times k}) > 0$ because any permutation matrix P of size $m \times m$ with $P_{1,1} = 1$ partially avoids $1_{m\times k}$. Hence

$$0 < p_m^{\mathrm{P}}(1_{m\times k}) = p_m^{\mathrm{P}}(B) \leq p_m^{\mathrm{P}}(A).$$

□

We are finally prepared to determine the partial Wilf equivalence classes of zero patterns. Let $\mathcal{Q}_{m,\leq m,0} := \bigcup_{b=1}^{m} \mathcal{Q}_{m,b,0}$ and $\mathcal{Q}_{\leq m,m,0} := \bigcup_{a=1}^{m} \mathcal{Q}_{a,m,0}$.

Theorem 64. *For every $m, k \in \mathbb{N}$ such that $k \leq m$, we have*

$$[0_{m\times k}]^P = \mathcal{Q}_{m,\leq m,0} \cup \mathcal{Q}_{\leq m,m,0} \cup \mathcal{Q}_{m,1,1} \cup \mathcal{Q}_{1,m,1}.$$

Proof. By Observations 60 and 61, we have

$$\mathcal{Q}_{m,\leq m,0} \cup \mathcal{Q}_{m,1,1} \subseteq [0_{m\times k}]^{\mathrm{P}}.$$

Moreover,

$$\mathcal{Q}_{\leq m,m,0} \cup \mathcal{Q}_{1,m,1} \subseteq [0_{m\times k}]^{\mathrm{P}}$$

because symmetric patterns belong to the same partial Wilf-equivalence class.

Let $A \in \mathcal{Q}_{m',k',\ell'}$ be a pattern. Without loss of generality, we can assume that $k' \leq m'$. If $m' \neq m$, then A and $0_{m\times k}$ are not partially Wilf-equivalent by Observation 55. From now on, we assume that that $m' = m$. If $\ell' = 0$, or if $\ell' = 1$ and $k' = 1$, we know that $A \in \mathcal{Q}_{m,\leq m,0}$ or $A \in \mathcal{Q}_{m,1,1}$. Otherwise, $\ell' \geq 1$ and $k' \geq 2$. In this case, A and $0_{m\times k}$ are not partially Wilf-equivalent by Observation 63. Therefore,

$$[0_{m\times k}]^{\mathrm{P}} = \mathcal{Q}_{m,\leq m,0} \cup \mathcal{Q}_{\leq m,m,0} \cup \mathcal{Q}_{m,1,1} \cup \mathcal{Q}_{1,m,1}.$$

□

Recall that a pattern $A \in \mathcal{Q}_{m,k,1}$ is partially Wilf-equivalent to every pattern $B \in \mathcal{Q}_{m,k,1}$. Hence A is also partially Wilf-equivalent to every pattern $B \in \mathcal{Q}_{k,m,1}$. In general, we do not know whether A is partially Wilf-equivalent to some pattern with at least two nonzero entries. Nevertheless, for $2 \leq m \leq 4$ and $2 \leq k \leq 4$, we prove that $[A]^{\mathrm{P}} = \mathcal{Q}_{m,k,1} \cup \mathcal{Q}_{k,m,1}$.

Theorem 65. *Let $A \in \mathcal{Q}_{m,k,1}$ be a quasi-permutation matrix. If $2 \leq m \leq 4$ and $2 \leq k \leq 4$, then*

$$[A]^P = \mathcal{Q}_{m,k,1} \cup \mathcal{Q}_{k,m,1}.$$

Proof. It follows from Observation 55, Lemma 62 and the computer enumeration (see Table A.1 in Appendix). □

By Theorem 64, we have $[A]^{\mathrm{P}} = [0_{m\times m}]^{\mathrm{P}}$ for every pattern $A \in \mathcal{Q}_{m,1,1}$. Thus, we determine the partial Wilf-equivalence classes of every pattern of size at most 4×4 with exactly one nonzero entry.

4.3 Small patterns with at least two ones

For every $m, k, \ell \in \mathbb{N}$ such that $2 \leq \ell \leq k \leq m \leq 4$, the computer enumeration (see Table A.1 in Appendix) shows that the partial Wilf equivalence class of $A \in \mathcal{Q}_{m,k,\ell}$ satisfies

$$[A]^{\mathrm{P}} \subseteq \mathcal{Q}_{m,k,\ell} \cup \mathcal{Q}_{k,m,\ell}.$$

We have seen in the previous section that the inclusion is not true for all patterns. Hence, for each $m, k, \ell \in \mathbb{N}$ such that $2 \leq \ell \leq k \leq m \leq 4$, we describe all partial Wilf equivalence classes on $\mathcal{Q}_{m,k,\ell} \cup \mathcal{Q}_{k,m,\ell}$ as follows. Since a partial Wilf equivalence class consists of more than one symmetry class in general (see Observation 10), we describe the partial Wilf equivalence class by listing the representatives of these symmetry classes. It is tedious but not hard to verify that every pattern of size at most 4×4 with at least two nonzero entries belongs to one of the listed symmetry classes. Moreover, for each partial Wilf equivalence, we give eight numbers

$$p_1^{\mathrm{P}}(A), p_2^{\mathrm{P}}(A), \ldots, p_8^{\mathrm{P}}(A),$$

where A is an arbitrary pattern from the partial Wilf equivalence class. Using this sequence, the reader can easily verify that every two partial Wilf equivalence classes are different. For each partial Wilf equivalence class, we only verify that the listed representatives of symmetry classes indeed belong to this partial Wilf equivalence class. For a partial Wilf equivalence class $\mathcal{A}$, let $p_n^{\mathrm{P}}(\mathcal{A}) := p_n^{\mathrm{P}}(A)$, where $A \in \mathcal{A}$ is chosen arbitrarily.

More exactly, we describe all partial Wilf-equivalence classes on $\mathcal{Q}_{m,k,\ell} \cup \mathcal{Q}_{k,m,\ell}$ by a table that has as many rows as the number of partial Wilf equivalence classes on $\mathcal{Q}_{m,k,\ell} \cup \mathcal{Q}_{k,m,\ell}$. For each partial Wilf equivalence class, we describe its elements by listing the representatives of the symmetry classes, which form the partial Wilf equivalence class. Moreover, we give a reference to the theorem from which it follows that the representatives are indeed partially Wilf-equivalent.

Recall that $\mathcal{Q}_{m,m,m}$ are exactly the permutation matrices of order m. Since $[A]^{\mathrm{P}} \subseteq \mathcal{Q}_{m,m,m}$ for any permutation matrix A of order $2 \leq m \leq 4$, the partial Wilf equivalence class of any permutation matrix of order m contains only permutation matrices and hence $[A]^{\mathrm{P}}$ is well-known (for example, see [1, Chapter 4]). From now on, we only deal with patterns that contain at least one zero column or zero row.

In Tables 4.1, 4.2, 4.3, and 4.4, we see partial Wilf equivalence classes of

$$\mathcal{Q}_{3,2,2} \cup \mathcal{Q}_{3,2,2},\ \ \mathcal{Q}_{3,3,2}\ ,\ \mathcal{Q}_{4,2,2} \cup \mathcal{Q}_{4,2,2},\ \text{and}\ \mathcal{Q}_{4,3,2} \cup \mathcal{Q}_{3,4,2},$$

respectively. We remark that $p_n^{\mathrm{P}}(\mathcal{B}_{3,2,2}) = F_{n+1}$, where F_{n+1} is the $(n+1)$-th Fibonacci number[1] Hence $p_n^{\mathrm{P}}(\mathcal{B}_{4,2,2}) = n \cdot F_n$ by Corollary 24. Since these results are not necessary to determine the partial Wilf equivalence classes, we do not prove them.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{3,2,2}$	$\begin{pmatrix} \circ & \bullet & \circ \\ \bullet & \circ & \circ \end{pmatrix}$		$1, 2, 3, 4, 5, 6$ $7, 8, \dots$
$\mathcal{B}_{3,2,2}$	$\begin{pmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{pmatrix}$		$1, 2, 3, 5, 8, 13$ $21, 34, \dots$

Table 4.1: Partition of $\mathcal{Q}_{3,2,2} \cup \mathcal{Q}_{2,3,2}$ into partial Wilf equivalence classes. By Theorem 21, we have $p_n^{\mathrm{P}}(\mathcal{A}_{3,2,2}) = n$ for every $n \in \mathbb{N}$.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{3,3,2}$	$\begin{pmatrix} \circ & \circ & \circ \\ \bullet & \circ & \circ \\ \circ & \bullet & \circ \end{pmatrix}$ $\begin{pmatrix} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{pmatrix}$	Thm 18	$1, 2, 5, 10, 17, 26$ $37, 50, \dots$
$\mathcal{B}_{3,3,2}$	$\begin{pmatrix} \circ & \bullet & \circ \\ \circ & \circ & \circ \\ \bullet & \circ & \circ \end{pmatrix}$		$1, 2, 5, 10, 20, 38$ $71, 130, \dots$
$\mathcal{C}_{3,3,2}$	$\begin{pmatrix} \circ & \circ & \bullet \\ \circ & \circ & \circ \\ \bullet & \circ & \circ \end{pmatrix}$		$1, 2, 5, 11, 24, 53$ $117, 258, \dots$

Table 4.2: Partition of $\mathcal{Q}_{3,3,2}$ into partial Wilf equivalence classes. By Theorem 28, we have $p_n^{\mathrm{P}}(\mathcal{A}_{3,3,2}) = 1 + (n-1)^2$ for every $n \in \mathbb{N}$.

[1] The Fibonacci numbers F_n are defined as follows: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{4,2,2}$	$\begin{pmatrix}\circ&\bullet&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$	Thm 18	$1, 2, 6, 12, 20, 30$ $42, 56, \dots$
$\mathcal{B}_{4,2,2}$	$\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$		$1, 2, 6, 12, 25, 48$ $91, 168, \dots$
$\mathcal{C}_{4,2,2}$	$\begin{pmatrix}\circ&\circ&\circ&\bullet\\ \bullet&\circ&\circ&\circ\end{pmatrix}$		$1, 2, 6, 12, 25, 57$ $124, 268, \dots$

Table 4.3: Partition of $\mathcal{Q}_{4,2,2} \cup \mathcal{Q}_{2,4,2}$ into partial Wilf equivalence classes. By Corollary 24, we have $p_n^{\mathrm{P}}(\mathcal{A}_{4,2,2}) = n \cdot (n-1)$ for $n \geq 2$.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{4,3,2}$	$\begin{pmatrix}\circ&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\circ&\circ\\ \bullet&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\circ&\circ\\ \circ&\circ&\bullet&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$	Thm 18	$1, 2, 6, 18, 44, 90$ $192, 266, \dots$
$\mathcal{B}_{4,3,2}$	$\begin{pmatrix}\circ&\bullet&\circ&\circ\\ \circ&\circ&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \circ&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$	Thm 18	$1, 2, 6, 18, 44, 102$ $222, 466, \dots$
$\mathcal{C}_{4,3,2}$	$\begin{pmatrix}\circ&\circ&\circ&\circ\\ \bullet&\circ&\circ&\circ\\ \circ&\circ&\bullet&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\\ \circ&\circ&\circ&\bullet\end{pmatrix}$	Thm 18	$1, 2, 6, 18, 45, 108$ $241, 518, \dots$
$\mathcal{D}_{4,3,2}$	$\begin{pmatrix}\circ&\circ&\circ&\circ\\ \circ&\circ&\circ&\bullet\\ \bullet&\circ&\circ&\circ\end{pmatrix}$		$1, 2, 6, 18, 45, 114$ $288, 704, \dots$
$\mathcal{E}_{4,3,2}$	$\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \circ&\circ&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$		$1, 2, 6, 18, 48, 124$ $315, 786, \dots$
$\mathcal{F}_{4,3,2}$	$\begin{pmatrix}\circ&\circ&\circ&\bullet\\ \circ&\circ&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$		$1, 2, 6, 18, 48, 129$ $352, 960, \dots$

Table 4.4: Partition of $\mathcal{Q}_{4,3,2} \cup \mathcal{Q}_{3,4,2}$ into partial Wilf equivalence classes.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{4,3,3}$	$\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \bullet&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\bullet&\circ\\ \circ&\bullet&\circ&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\bullet&\circ&\circ\\ \circ&\circ&\bullet&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$	Thm 21	$1, 2, 6, 20, 70, 252$ $924, 3432, \dots$
$\mathcal{B}_{4,3,3}$	$\begin{pmatrix}\circ&\circ&\circ&\bullet\\ \bullet&\circ&\circ&\circ\\ \circ&\circ&\bullet&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\circ&\bullet\\ \circ&\circ&\bullet&\circ\\ \bullet&\circ&\circ&\circ\end{pmatrix}$ $\begin{pmatrix}\circ&\circ&\circ&\bullet\\ \bullet&\circ&\circ&\circ\\ \circ&\bullet&\circ&\circ\end{pmatrix}$	Thm 66	$1, 2, 6, 20, 71, 264$ $1015, 4002, \dots$

Table 4.5: Partition of $\mathcal{Q}_{4,3,3} \cup \mathcal{Q}_{3,4,3}$ into partial Wilf equivalence classes. By Theorem 21, we have $p_n^{\mathrm{P}}(\mathcal{A}_{4,3,3}) = (n+1) \cdot c_n = \binom{2n-2}{n-1}$ for every $n \in \mathbb{N}$, where c_n is the n-th Catalan numer.

In Tables 4.5, 4.6, and 4.7, we see partial Wilf equivalence classes of

$$\mathcal{Q}_{4,3,3} \cup \mathcal{Q}_{3,4,3},\ \mathcal{Q}_{4,4,2}\ ,\ \text{and}\ \mathcal{Q}_{4,4,3} \cup \mathcal{Q}_{3,4,2},$$

respectively. The following four theorems clarify that the representatives listed in $\mathcal{B}_{4,3,3}$, $\mathcal{A}_{4,4,2}$, $\mathcal{B}_{4,4,2}$, and $\mathcal{B}_{4,4,3}$, respectively, are partially Wilf-equivalent.

Theorem 66. *The following patterns*

$$B_1 = (2,0,1,3)|_3,\ B_2 = (1,0,2,3)|_3,\ B_3 = (2,1,0,3)|_3$$

belong to the partial Wilf equivalence class $\mathcal{B}_{4,3,3}$.

Proof. By Theorem 58, $B_1 \overset{\mathrm{P}}{\sim} B_2$. Moreover, note that $B_2 \approx (1,2,0,3)|_3$. Since $(1,2,0,3)|_3 \overset{\mathrm{P}}{\sim} B_3$ by Theorem 57, we have $B_2 \overset{\mathrm{P}}{\sim} B_3$. □

Theorem 67. *The following patterns*

$$A_1 = (0,1,2,0)|_4,\ A_2 = (1,2,0,0)|_4,\ A_3 = (0,3,2,0)|_4,\ A_4 = (0,0,1,2)|_4$$

belong to the partial Wilf equivalence class $\mathcal{A}_{4,4,2}$.

Proof. By Theorem 56, we have $A_1 \overset{\mathrm{P}}{\sim} A_2$ and $A_1 \overset{\mathrm{P}}{\sim} A_4$. Observe that $A_1 \approx (3,2,0,0)|_4$. Hence $(3,2,0,0)|_4 \overset{\mathrm{P}}{\sim} A_3$ by Theorem 56 and so $A_1 \overset{\mathrm{P}}{\sim} A_3$. □

Theorem 68. *The following patterns*

$$B_1 = (2,4,0,0)|_4,\ B_2 = (0,2,4,0)|_4,\ B_3 = (1,3,0,0)|_4,$$

belong to the partial Wilf equivalence class $\mathcal{B}_{4,4,2}$.

Proof. By Theorem 56, we have $A_1 \overset{\mathrm{P}}{\sim} A_2$. Observe that $A_2 \approx (0,1,3,0)|_4$. Hence $(0,1,3,0)|_4 \overset{\mathrm{P}}{\sim} A_3$ by Theorem 56 and so $A_2 \overset{\mathrm{P}}{\sim} A_3$. □

Theorem 69. *The following patterns*

$$B_1 = (1,2,4,0)|_4,\ B_2 = (0,1,2,4)|_4,\ B_3 = (0,1,4,3)|_4,$$
$$B_4 = (2,4,1,0)|_4,\ B_5 = (1,4,3,0)|_4,\ B_6 = (0,2,4,1)|_4$$

belong to the partial Wilf equivalence class $\mathcal{B}_{4,4,3}$.

Proof. It is sufficient to show that the given patterns are pairwise partially Wilf-equivalent. By Theorem 18, we have

$$B_1 \overset{\mathrm{P}}{\sim} B_2,\ B_3 \overset{\mathrm{P}}{\sim} B_5,\ \text{ and }\ B_4 \overset{\mathrm{P}}{\sim} B_6.$$

Note that $B_3 \approx (2,1,4,0)|_4$. Moreover, $B_1 \overset{\mathrm{P}}{\sim} (2,1,4,0)|_4$ by Theorem 38. Hence $B_1 \overset{\mathrm{P}}{\sim} B_3$. Finally, observe that $B_2 \approx (1,0,2,3)|_4$ and $B_4 \approx (2,0,1,3)|_4$. Since $(1,0,2,3)|_4 \overset{\mathrm{P}}{\sim} (2,0,1,3)|_4$ by Theorem 42, we have $B_2 \overset{\mathrm{P}}{\sim} B_4$, which completes the proof. □

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{4,4,2}$	$\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\bullet \\ \circ&\circ&\bullet&\circ \end{pmatrix}$	Thm 67	$1, 2, 6, 22, 74, 210$ $502, 1046, \ldots$
$\mathcal{B}_{4,4,2}$	$\begin{pmatrix} \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$	Thm 68	$1, 2, 6, 22, 74, 216$ $586, 1474, \ldots$
$\mathcal{C}_{4,4,2}$	$\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \end{pmatrix}$	Thm 18	$1, 2, 6, 22, 74, 218$ $628, 1756, \ldots$
$\mathcal{D}_{4,4,2}$	$\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$	Thm 18	$1, 2, 6, 22, 75, 232$ $689, 1978, \ldots$
$\mathcal{E}_{4,4,2}$	$\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$		$1, 2, 6, 22, 75, 236$ $728, 2228, \ldots$
$\mathcal{F}_{4,4,2}$	$\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$		$1, 2, 6, 22, 75, 241$ $772, 2488, \ldots$

Table 4.6: Partition of $\mathcal{Q}_{4,4,2}$ into partial Wilf equivalence classes.

Wilf class	Representatives of symmetry classes	Ref	$p_n^{\mathrm{P}}(\cdot)$
$\mathcal{A}_{4,4,3}$	$\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\bullet&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\bullet \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\bullet&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\bullet&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \end{pmatrix}$	Cor 30	$1, 2, 6, 23, 94, 392$ $1644, 6897, \ldots$
$\mathcal{B}_{4,4,3}$	$\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\bullet&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\bullet&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\bullet \end{pmatrix}$	Thm 69	$1, 2, 6, 23, 94, 396$ $1704, 7442, \ldots$
$\mathcal{C}_{4,4,3}$	$\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\circ&\bullet&\circ \end{pmatrix}$	Thm 42	$1, 2, 6, 23, 94, 401$ $1764, 7951, \ldots$
$\mathcal{D}_{4,4,3}$	$\begin{pmatrix} \circ&\bullet&\circ&\circ \\ \circ&\circ&\circ&\circ \\ \circ&\circ&\circ&\bullet \\ \bullet&\circ&\circ&\circ \end{pmatrix}$		$1, 2, 6, 23, 95, 407$ $1795, 8109, \ldots$
$\mathcal{E}_{4,4,3}$	$\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \\ \bullet&\circ&\circ&\circ \end{pmatrix}$ $\begin{pmatrix} \circ&\circ&\circ&\bullet \\ \circ&\circ&\circ&\circ \\ \bullet&\circ&\circ&\circ \\ \circ&\bullet&\circ&\circ \end{pmatrix}$	Thm 38	$1, 2, 6, 23, 95, 407$ $1797, 8135, \ldots$

Table 4.7: Partition of $\mathcal{Q}_{4,4,3}$ into partial Wilf equivalence classes.

5. Conclusion

In this thesis, we studied two generalizations of the concept of avoidance of permutation matrices: a permutation matrix P partially (or exactly) avoids a quasi-permutation matrix A if there is no submatrix P' of P such that $A \leq P'$ (or $A = P'$). We showed that partial avoidance and exact avoidance agree not only on the set of permutation matrices but also on the set of row- or column-permutation matrices. From this point forward, we worked only with partial avoidance of quasi-permutation matrices. Our main motivation was to classify all 371 patterns (i.e., quasi-permutation matrices) of size at most 4×4 into partial Wilf equivalence classes, which we did successfully in Chapter 4. During this journey, we developed some general results showing how to create from one or two quasi-permutation matrices more quasi-permutation matrices that are pairwise partially Wilf-equivalent. In Chapter 2, we proved that patterns obtained from a pattern by appending a zero column or a zero row (to any side of the pattern) are partially Wilf-equivalent. Next, in Chapter 3, we showed that the direct sum of X and Q is partially Wilf-equivalent to the direct sum of Y and Q, where $X = \left(\begin{smallmatrix} \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \end{smallmatrix}\right)$, and Q is any quasi-permutation matrix such that its first column is nonzero (and this assumption cannot be omitted). A straightforward continuation of our work is to determine the partial Wilf-equivalence classes of patterns that have at least 5 columns or 5 rows.

By looking at the Tables 4.1, 4.2, 4.3, 4.4, and 4.6, it seems that a pattern avoids more permutation matrices if the ones are "far" from each other than a pattern whose ones are "close" to each other. Formally, for a quasi-permutation matrix $\mathcal{Q}_{m,k,2}$, we define a *distance* of A as

$$d(A) := |i - i'| + |j - j'|,$$

where (i, j) and (i', j') are two distinct entries such that $A_{i,j} = 1$ and $A_{i',j'} = 1$.

Question 1. *Let $A, B \in \mathcal{Q}_{m,k,2}$ be quasi-permutation matrices. It is true that if $d(A) < d(B)$, then $p^P_n(A) \leq p^P_n(B)$ for all $n \in \mathbb{N}$?*

Moreover, we can ask the same question for patterns of different shape. The answer will somewhat depend not only on the distance but also on the shape. The first step might be to study patterns whose nonzero entries are in the corners. Formally, for $m, k \in \mathbb{N}$, let $A_{m,k} \in \mathcal{Q}_{m,k,2}$ be a quasi-permutation matrix such that $A_{1,1} = 1$ and $A_{m,k} = 1$.

Question 2. *It is true that if $d(A_{m,k}) < d(A_{m',k'})$, then there exists $N \in \mathbb{N}$ such that $p^P_n(A_{m,k}) \leq p^P_n(_{m',k'})$ for all $n \geq N$?*

Bibliography

[1] Miklós Bóna. *Combinatorics of Permutations*. Second eddition. CRC Press, Inc., USA, 2012.

[2] Zvezdelina Stankova and Julian West. A new class of wilf-equivalent permutations. *Journal of Algebraic Combinatorics*, 15(3):271–290, 2002.

[3] David Mikšaník. Avoidance of quasi-permutation matrices. `https://github.com/KapurD/Avoidance-of-quasi-permutation-matrices`, January 2023.

[4] Jörgen Backelin, Julian West, and Guoce Xin. Wilf-equivalence for singleton classes. *Advances in Applied Mathematics*, 38(2):133–148, 2007.

[5] Vít Jelínek. *Wilf-Type Classifications, Extremal and Enumerative Theory of Ordered Structures*. Doctoral thesis, Charles University, Faculty of Mathematics and Physics, 2009.

A. Appendix

The *partial avoidance sequence* of a pattern A is a sequence of numbers

$$p_1^{\mathrm{P}}(A),\ p_2^{\mathrm{P}}(A),\ p_3^{\mathrm{P}}(A),\ \ldots$$

and the *exact avoidance sequence* of a pattern A is a sequence of numbers

$$p_1^{\mathrm{E}}(A),\ p_2^{\mathrm{E}}(A),\ p_3^{\mathrm{E}}(A),\ \ldots\ .$$

In order to classify patterns of size at most 4×4 into partial Wilf equivalence classes, we created a program [3] that, for each pattern A of size at most $4{\times}4$, computes either the partial or exact avoidance sequence of A up to a given number n (the choice of partial or exact avoidance is fixed throughout the computation). We print both outputs of the program for $n = 8$ (see Table A.1 and Table A.2). Both outputs are sorted lexicographically according to the partial/exact avoidance sequences and the patterns are written in the "one-line" notation introduced in Section 4.1.

Table A.1: Partial avoidance sequence of each pattern of size at most 4×4.

Patterns	**Partial avoidance sequence (from 1 to 8)**							
$(0)_1$	0,	0,	0,	0,	0,	0,	0,	0
$(1)_1$	0,	0,	0,	0,	0,	0,	0,	0
$(0{,}0)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(0{,}0)_2$	1,	0,	0,	0,	0,	0,	0,	0
$(0{,}1)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(1{,}0)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(0{,}1)_2$	1,	1,	0,	0,	0,	0,	0,	0
$(0{,}2)_2$	1,	1,	0,	0,	0,	0,	0,	0
$(1{,}0)_2$	1,	1,	0,	0,	0,	0,	0,	0
$(2{,}0)_2$	1,	1,	0,	0,	0,	0,	0,	0
$(1{,}2)_2$	1,	1,	1,	1,	1,	1,	1,	1
$(2{,}1)_2$	1,	1,	1,	1,	1,	1,	1,	1
$(0{,}0{,}0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0{,}0{,}0)_2$	1,	2,	0,	0,	0,	0,	0,	0
$(0{,}0{,}0)_3$	1,	2,	0,	0,	0,	0,	0,	0
$(0{,}0{,}1)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0{,}1{,}0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(1{,}0{,}0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0{,}0{,}1)_2$	1,	2,	2,	0,	0,	0,	0,	0
$(0{,}0{,}2)_2$	1,	2,	2,	0,	0,	0,	0,	0

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(0,1,0)_2$	1,	2,	2,	0,	0,	0,	0,	0
$(0,2,0)_2$	1,	2,	2,	0,	0,	0,	0,	0
$(1,0,0)_2$	1,	2,	2,	0,	0,	0,	0,	0
$(2,0,0)_2$	1,	2,	2,	0,	0,	0,	0,	0
$(0,1,2)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(0,2,1)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(1,2,0)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(2,1,0)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(1,0,2)_2$	1,	2,	3,	5,	8,	13,	21,	34
$(2,0,1)_2$	1,	2,	3,	5,	8,	13,	21,	34
$(0,0,1)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,0,2)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,0,3)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,1,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,2,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,3,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(1,0,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(2,0,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(3,0,0)_3$	1,	2,	4,	4,	0,	0,	0,	0
$(0,1,2)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(0,2,1)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(0,2,3)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(0,3,2)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(1,2,0)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(2,1,0)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(2,3,0)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(3,2,0)_3$	1,	2,	5,	10,	17,	26,	37,	50
$(0,1,3)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(0,3,1)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(1,0,2)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(1,3,0)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(2,0,1)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(2,0,3)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(3,0,2)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(3,1,0)_3$	1,	2,	5,	10,	20,	38,	71,	130
$(1,0,3)_3$	1,	2,	5,	11,	24,	53,	117,	258

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(3,0,1)_3$	1,	2,	5,	11,	24,	53,	117,	258
$(1,2,3)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(1,3,2)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(2,1,3)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(2,3,1)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(3,1,2)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(3,2,1)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(0,0,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0,0)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0,0)_3$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0,0)_4$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0,1)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,1,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,1,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(1,0,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0,1)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,0,0,2)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,0,1,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,0,2,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,1,0,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,2,0,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(1,0,0,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(2,0,0,0)_2$	1,	2,	6,	6,	0,	0,	0,	0
$(0,0,0,1)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,0,2)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,0,3)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,1,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,2,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,3,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,1,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,2,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,3,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(1,0,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(2,0,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(3,0,0,0)_3$	1,	2,	6,	12,	12,	0,	0,	0
$(0,0,1,2)_2$	1,	2,	6,	12,	20,	30,	42,	56

Continued on next page

Patterns	**Partial avoidance sequence (from 1 to 8)**							
$(0,0,2,1)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,1,2,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,2,1,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(1,2,0,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(2,1,0,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,1,0,2)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(0,2,0,1)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(1,0,2,0)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(2,0,1,0)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(1,0,0,2)_2$	1,	2,	6,	12,	25,	57,	124,	268
$(2,0,0,1)_2$	1,	2,	6,	12,	25,	57,	124,	268
$(0,0,0,1)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,0,2)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,0,3)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,0,4)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,1,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,2,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,3,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,4,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,1,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,2,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,3,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,4,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(1,0,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(2,0,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(3,0,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(4,0,0,0)_4$	1,	2,	6,	18,	36,	36,	0,	0
$(0,0,1,2)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,0,2,1)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,0,2,3)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,0,3,2)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,1,2,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,2,1,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,2,3,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,3,2,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(1,2,0,0)_3$	1,	2,	6,	18,	44,	90,	162,	266

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(2,1,0,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(2,3,0,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(3,2,0,0)_3$	1,	2,	6,	18,	44,	90,	162,	266
$(0,0,1,3)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(0,0,3,1)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(0,1,3,0)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(0,3,1,0)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(1,3,0,0)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(3,1,0,0)_3$	1,	2,	6,	18,	44,	102,	222,	466
$(0,1,0,2)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(0,2,0,1)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(0,2,0,3)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(0,3,0,2)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(1,0,2,0)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(2,0,1,0)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(2,0,3,0)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(3,0,2,0)_3$	1,	2,	6,	18,	45,	108,	241,	518
$(1,0,0,2)_3$	1,	2,	6,	18,	45,	114,	288,	704
$(2,0,0,1)_3$	1,	2,	6,	18,	45,	114,	288,	704
$(2,0,0,3)_3$	1,	2,	6,	18,	45,	114,	288,	704
$(3,0,0,2)_3$	1,	2,	6,	18,	45,	114,	288,	704
$(0,1,0,3)_3$	1,	2,	6,	18,	48,	124,	315,	786
$(0,3,0,1)_3$	1,	2,	6,	18,	48,	124,	315,	786
$(1,0,3,0)_3$	1,	2,	6,	18,	48,	124,	315,	786
$(3,0,1,0)_3$	1,	2,	6,	18,	48,	124,	315,	786
$(1,0,0,3)_3$	1,	2,	6,	18,	48,	129,	352,	960
$(3,0,0,1)_3$	1,	2,	6,	18,	48,	129,	352,	960
$(0,1,2,3)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,1,3,2)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,2,1,3)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,2,3,1)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,3,1,2)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,3,2,1)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,2,3,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,3,2,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(2,1,3,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(2,3,1,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(3,1,2,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(3,2,1,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,0,2,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,0,3,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,2,0,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,3,0,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,0,1,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,0,3,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,1,0,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,3,0,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,0,1,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,0,2,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,1,0,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,2,0,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(0,0,1,2)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,2,1)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,2,3)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,3,2)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,3,4)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,4,3)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,1,2,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,2,1,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,2,3,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,3,2,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,3,4,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,4,3,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(1,2,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(2,1,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(2,3,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(3,2,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(3,4,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(4,3,0,0)_4$	1,	2,	6,	22,	74,	210,	502,	1046
$(0,0,1,3)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,0,2,4)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,0,3,1)_4$	1,	2,	6,	22,	74,	216,	586,	1474

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(0,0,4,2)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,1,0,2)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,1,3,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,2,0,1)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,2,0,3)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,2,4,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,3,0,2)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,3,0,4)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,3,1,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,4,0,3)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,4,2,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(1,0,2,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(1,3,0,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(2,0,1,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(2,0,3,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(2,4,0,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(3,0,2,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(3,0,4,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(3,1,0,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(4,0,3,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(4,2,0,0)_4$	1,	2,	6,	22,	74,	216,	586,	1474
$(0,0,1,4)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(0,0,4,1)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(0,1,4,0)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(0,4,1,0)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(1,0,0,2)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(1,4,0,0)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(2,0,0,1)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(2,0,0,3)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(3,0,0,2)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(3,0,0,4)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(4,0,0,3)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(4,1,0,0)_4$	1,	2,	6,	22,	74,	218,	628,	1756
$(0,1,0,3)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(0,2,0,4)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(0,3,0,1)_4$	1,	2,	6,	22,	75,	232,	689,	1978

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(0,4,0,2)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(1,0,3,0)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(2,0,4,0)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(3,0,1,0)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(4,0,2,0)_4$	1,	2,	6,	22,	75,	232,	689,	1978
$(0,1,0,4)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(0,4,0,1)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(1,0,0,3)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(1,0,4,0)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(2,0,0,4)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(3,0,0,1)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(4,0,0,2)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(4,0,1,0)_4$	1,	2,	6,	22,	75,	236,	728,	2228
$(1,0,0,4)_4$	1,	2,	6,	22,	75,	241,	772,	2488
$(4,0,0,1)_4$	1,	2,	6,	22,	75,	241,	772,	2488
$(0,1,2,3)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,1,3,2)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,2,1,3)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,2,3,1)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,2,3,4)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,2,4,3)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,3,1,2)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,3,2,1)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,3,2,4)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,3,4,2)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,4,2,3)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,4,3,2)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(1,2,3,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(1,3,2,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(2,1,3,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(2,3,1,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(2,3,4,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(2,4,3,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(3,1,2,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(3,2,1,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(3,2,4,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(3,4,2,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(4,2,3,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(4,3,2,0)_4$	1,	2,	6,	23,	94,	392,	1644,	6897
$(0,1,2,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,1,3,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,1,4,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,1,4,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,2,1,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,2,4,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,3,1,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,3,4,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,4,1,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,4,1,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,4,2,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(0,4,3,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,0,2,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,0,3,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,2,0,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,2,4,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,3,0,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,3,4,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,4,2,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,4,3,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,0,1,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,0,3,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,0,3,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,0,4,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,1,0,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,1,4,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,3,0,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,3,0,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,4,0,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(2,4,1,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,0,1,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,0,2,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,0,2,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442

Continued on next page

Patterns	**Partial avoidance sequence (from 1 to 8)**							
$(3,0,4,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,1,0,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,1,4,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,2,0,1)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,2,0,4)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,4,0,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(3,4,1,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,0,2,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,0,3,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,1,2,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,1,3,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,2,0,3)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,2,1,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,3,0,2)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(4,3,1,0)_4$	1,	2,	6,	23,	94,	396,	1704,	7442
$(1,0,2,4)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(1,0,4,2)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(1,3,0,4)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(1,4,0,3)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(2,0,1,4)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(2,4,0,1)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(3,0,4,1)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(3,1,0,4)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(4,0,1,3)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(4,0,3,1)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(4,1,0,2)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(4,2,0,1)_4$	1,	2,	6,	23,	94,	401,	1764,	7951
$(1,4,0,2)_4$	1,	2,	6,	23,	95,	407,	1795,	8109
$(2,0,4,1)_4$	1,	2,	6,	23,	95,	407,	1795,	8109
$(3,0,1,4)_4$	1,	2,	6,	23,	95,	407,	1795,	8109
$(4,1,0,3)_4$	1,	2,	6,	23,	95,	407,	1795,	8109
$(1,0,3,4)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(1,0,4,3)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(1,2,0,4)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(2,1,0,4)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(3,4,0,1)_4$	1,	2,	6,	23,	95,	407,	1797,	8135

Continued on next page

Patterns	Partial avoidance sequence (from 1 to 8)							
$(4,0,1,2)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(4,0,2,1)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(4,3,0,1)_4$	1,	2,	6,	23,	95,	407,	1797,	8135
$(1,3,4,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(1,4,2,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,3,1,4)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,4,1,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,4,3,1)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,1,2,4)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,1,4,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,2,4,1)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(4,1,3,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(4,2,1,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(1,2,3,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,2,4,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,4,3,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(2,1,3,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(2,1,4,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(2,3,4,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,2,1,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,4,1,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,4,2,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,1,2,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,3,1,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,3,2,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,3,2,4)_4$	1,	2,	6,	23,	103,	513,	2762,	15793
$(4,2,3,1)_4$	1,	2,	6,	23,	103,	513,	2762,	15793

Table A.2: Exact avoidance sequence of each pattern of size at most 4×4.

Patterns	**Exact avoidance sequence (from 1 to 8)**							
$(1)_1$	0,	0,	0,	0,	0,	0,	0,	0
$(0)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(0,1)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(1,0)_1$	1,	0,	0,	0,	0,	0,	0,	0
$(1,2)_2$	1,	1,	1,	1,	1,	1,	1,	1
$(2,1)_2$	1,	1,	1,	1,	1,	1,	1,	1
$(0,0,1)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0,0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0,1,0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(0,1)_2$	1,	2,	0,	0,	0,	0,	0,	0
$(0,2)_2$	1,	2,	0,	0,	0,	0,	0,	0
$(1,0,0)_1$	1,	2,	0,	0,	0,	0,	0,	0
$(1,0)_2$	1,	2,	0,	0,	0,	0,	0,	0
$(2,0)_2$	1,	2,	0,	0,	0,	0,	0,	0
$(0,1,2)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(0,2,1)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(1,2,0)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(2,1,0)_2$	1,	2,	3,	4,	5,	6,	7,	8
$(1,0,2)_2$	1,	2,	3,	5,	8,	13,	21,	34
$(2,0,1)_2$	1,	2,	3,	5,	8,	13,	21,	34
$(1,2,3)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(1,3,2)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(2,1,3)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(2,3,1)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(3,1,2)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(3,2,1)_3$	1,	2,	5,	14,	42,	132,	429,	1430
$(0,0,0,1)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,1,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,1)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(0,0,2)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(0,1,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(0,1,0)_2$	1,	2,	6,	0,	0,	0,	0,	0

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(0,2,0)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(1,0,0,0)_1$	1,	2,	6,	0,	0,	0,	0,	0
$(1,0,0)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(2,0,0)_2$	1,	2,	6,	0,	0,	0,	0,	0
$(0,1,2)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(0,2,1)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(0,2,3)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(0,3,2)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(1,2,0)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(2,1,0)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(2,3,0)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(3,2,0)_3$	1,	2,	6,	11,	18,	27,	38,	51
$(0,1,3)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(0,3,1)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(1,0,2)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(1,3,0)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(2,0,1)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(2,0,3)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(3,0,2)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(3,1,0)_3$	1,	2,	6,	11,	22,	41,	76,	138
$(0,0,1,2)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,0,2,1)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,1,2,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,2,1,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(1,2,0,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(2,1,0,0)_2$	1,	2,	6,	12,	20,	30,	42,	56
$(0,1,0,2)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(0,2,0,1)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(1,0,2,0)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(2,0,1,0)_2$	1,	2,	6,	12,	25,	48,	91,	168
$(1,0,0,2)_2$	1,	2,	6,	12,	25,	57,	124,	268
$(2,0,0,1)_2$	1,	2,	6,	12,	25,	57,	124,	268
$(1,0,3)_3$	1,	2,	6,	13,	29,	68,	156,	357
$(3,0,1)_3$	1,	2,	6,	13,	29,	68,	156,	357
$(0,1,2,3)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,1,3,2)_3$	1,	2,	6,	20,	70,	252,	924,	3432

Continued on next page

Patterns	**Exact avoidance sequence (from 1 to 8)**							
$(0,2,1,3)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,2,3,1)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,3,1,2)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(0,3,2,1)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,2,3,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,3,2,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(2,1,3,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(2,3,1,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(3,1,2,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(3,2,1,0)_3$	1,	2,	6,	20,	70,	252,	924,	3432
$(1,0,2,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,0,3,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,2,0,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,3,0,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,0,1,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,0,3,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,1,0,3)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(2,3,0,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,0,1,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,0,2,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,1,0,2)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(3,2,0,1)_3$	1,	2,	6,	20,	71,	264,	1015,	4002
$(1,3,4,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(1,4,2,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,3,1,4)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,4,1,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(2,4,3,1)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,1,2,4)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,1,4,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(3,2,4,1)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(4,1,3,2)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(4,2,1,3)_4$	1,	2,	6,	23,	103,	512,	2740,	15485
$(1,2,3,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,2,4,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,4,3,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(2,1,3,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(2,1,4,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(2,3,4,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,2,1,4)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,4,1,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(3,4,2,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,1,2,3)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,3,1,2)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(4,3,2,1)_4$	1,	2,	6,	23,	103,	513,	2761,	15767
$(1,3,2,4)_4$	1,	2,	6,	23,	103,	513,	2762,	15793
$(4,2,3,1)_4$	1,	2,	6,	23,	103,	513,	2762,	15793
$(0,0,0,0)_1$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,0,1)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,0,2)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,1,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,1)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,2,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,2)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,0,3)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,1,0,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,1,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,2,0,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(0,2,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,3,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(1,0,0,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(1,0,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(2,0,0,0)_2$	1,	2,	6,	24,	0,	0,	0,	0
$(2,0,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(3,0,0)_3$	1,	2,	6,	24,	0,	0,	0,	0
$(0,1,2,0)_3$	1,	2,	6,	24,	50,	98,	172,	278
$(0,2,1,0)_3$	1,	2,	6,	24,	50,	98,	172,	278
$(0,2,3,0)_3$	1,	2,	6,	24,	50,	98,	172,	278
$(0,3,2,0)_3$	1,	2,	6,	24,	50,	98,	172,	278
$(0,1,3,0)_3$	1,	2,	6,	24,	50,	117,	249,	518
$(0,3,1,0)_3$	1,	2,	6,	24,	50,	117,	249,	518
$(0,2,0,1)_3$	1,	2,	6,	24,	51,	121,	264,	561

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(0,2,0,3)_3$	1,	2,	6,	24,	51,	121,	264,	561
$(1,0,2,0)_3$	1,	2,	6,	24,	51,	121,	264,	561
$(3,0,2,0)_3$	1,	2,	6,	24,	51,	121,	264,	561
$(0,0,1,2)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(0,0,2,1)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(0,0,2,3)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(0,0,3,2)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(1,2,0,0)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(2,1,0,0)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(2,3,0,0)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(3,2,0,0)_3$	1,	2,	6,	24,	52,	100,	174,	280
$(0,0,1,3)_3$	1,	2,	6,	24,	52,	120,	254,	526
$(0,0,3,1)_3$	1,	2,	6,	24,	52,	120,	254,	526
$(1,3,0,0)_3$	1,	2,	6,	24,	52,	120,	254,	526
$(3,1,0,0)_3$	1,	2,	6,	24,	52,	120,	254,	526
$(0,1,0,2)_3$	1,	2,	6,	24,	52,	124,	268,	568
$(0,3,0,2)_3$	1,	2,	6,	24,	52,	124,	268,	568
$(2,0,1,0)_3$	1,	2,	6,	24,	52,	124,	268,	568
$(2,0,3,0)_3$	1,	2,	6,	24,	52,	124,	268,	568
$(1,0,0,2)_3$	1,	2,	6,	24,	52,	127,	326,	782
$(2,0,0,1)_3$	1,	2,	6,	24,	52,	127,	326,	782
$(2,0,0,3)_3$	1,	2,	6,	24,	52,	127,	326,	782
$(3,0,0,2)_3$	1,	2,	6,	24,	52,	127,	326,	782
$(0,1,0,3)_3$	1,	2,	6,	24,	60,	156,	416,	1068
$(0,3,0,1)_3$	1,	2,	6,	24,	60,	156,	416,	1068
$(1,0,3,0)_3$	1,	2,	6,	24,	60,	156,	416,	1068
$(3,0,1,0)_3$	1,	2,	6,	24,	60,	156,	416,	1068
$(1,0,0,3)_3$	1,	2,	6,	24,	64,	174,	496,	1465
$(3,0,0,1)_3$	1,	2,	6,	24,	64,	174,	496,	1465
$(0,1,3,2)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(0,2,1,3)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(0,3,4,2)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(0,4,2,3)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(2,3,1,0)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(2,4,3,0)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(3,1,2,0)_4$	1,	2,	6,	24,	97,	401,	1672,	6987

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(3,2,4,0)_4$	1,	2,	6,	24,	97,	401,	1672,	6987
$(0,1,4,2)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(0,4,1,3)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(2,0,1,3)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(2,4,0,3)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(2,4,1,0)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(3,0,4,2)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(3,1,0,2)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(3,1,4,0)_4$	1,	2,	6,	24,	97,	406,	1740,	7577
$(0,1,2,3)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,2,3,4)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,2,4,3)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,3,1,2)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,3,2,1)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,4,3,2)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(1,2,3,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(2,1,3,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(2,3,4,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(3,2,1,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(3,4,2,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(4,3,2,0)_4$	1,	2,	6,	24,	98,	405,	1685,	7028
$(0,2,3,1)_4$	1,	2,	6,	24,	98,	406,	1692,	7062
$(0,3,2,4)_4$	1,	2,	6,	24,	98,	406,	1692,	7062
$(1,3,2,0)_4$	1,	2,	6,	24,	98,	406,	1692,	7062
$(4,2,3,0)_4$	1,	2,	6,	24,	98,	406,	1692,	7062
$(0,1,3,4)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(0,1,4,3)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(0,4,1,2)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(0,4,2,1)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(1,2,0,3)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(1,2,4,0)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(2,0,3,4)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(2,0,4,3)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(2,1,0,3)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(2,1,4,0)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(3,0,1,2)_4$	1,	2,	6,	24,	98,	410,	1755,	7635

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(3,0,2,1)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(3,4,0,2)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(3,4,1,0)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(4,3,0,2)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(4,3,1,0)_4$	1,	2,	6,	24,	98,	410,	1755,	7635
$(0,1,2,4)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(0,4,3,1)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(1,0,2,3)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(1,3,4,0)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(2,3,0,4)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(3,2,0,1)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(4,0,3,2)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(4,2,1,0)_4$	1,	2,	6,	24,	98,	411,	1759,	7647
$(0,2,1,4)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(0,2,4,1)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(0,3,1,4)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(0,3,4,1)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(1,0,3,2)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(1,3,0,2)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(1,4,2,0)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(1,4,3,0)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(2,0,3,1)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(2,3,0,1)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(3,0,2,4)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(3,2,0,4)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(4,0,2,3)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(4,1,2,0)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(4,1,3,0)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(4,2,0,3)_4$	1,	2,	6,	24,	98,	411,	1762,	7671
$(1,0,2,4)_4$	1,	2,	6,	24,	98,	417,	1830,	8238
$(1,3,0,4)_4$	1,	2,	6,	24,	98,	417,	1830,	8238
$(4,0,3,1)_4$	1,	2,	6,	24,	98,	417,	1830,	8238
$(4,2,0,1)_4$	1,	2,	6,	24,	98,	417,	1830,	8238
$(1,0,4,2)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(1,4,0,3)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(2,0,1,4)_4$	1,	2,	6,	24,	98,	418,	1840,	8312

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(2,4,0,1)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(3,0,4,1)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(3,1,0,4)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(4,0,1,3)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(4,1,0,2)_4$	1,	2,	6,	24,	98,	418,	1840,	8312
$(1,4,0,2)_4$	1,	2,	6,	24,	100,	430,	1908,	8682
$(2,0,4,1)_4$	1,	2,	6,	24,	100,	430,	1908,	8682
$(3,0,1,4)_4$	1,	2,	6,	24,	100,	430,	1908,	8682
$(4,1,0,3)_4$	1,	2,	6,	24,	100,	430,	1908,	8682
$(1,0,3,4)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(1,0,4,3)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(1,2,0,4)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(2,1,0,4)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(3,4,0,1)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(4,0,1,2)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(4,0,2,1)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(4,3,0,1)_4$	1,	2,	6,	24,	100,	430,	1910,	8707
$(0,0,0,0)_2$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,0,1)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,0,2)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,0,3)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,1,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,2,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,0,3,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,1,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,2,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,3,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(1,0,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(2,0,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(3,0,0,0)_3$	1,	2,	6,	24,	120,	0,	0,	0
$(0,2,3,0)_4$	1,	2,	6,	24,	120,	252,	572,	1152
$(0,3,2,0)_4$	1,	2,	6,	24,	120,	252,	572,	1152
$(0,1,3,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(0,2,0,3)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(0,2,4,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(0,3,0,2)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(0,3,1,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(0,4,2,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(2,0,3,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(3,0,2,0)_4$	1,	2,	6,	24,	120,	260,	702,	1711
$(0,0,2,3)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,0,3,2)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,1,2,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,2,1,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,3,4,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,4,3,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(2,3,0,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(3,2,0,0)_4$	1,	2,	6,	24,	120,	262,	586,	1170
$(0,1,4,0)_4$	1,	2,	6,	24,	120,	267,	754,	2144
$(0,4,1,0)_4$	1,	2,	6,	24,	120,	267,	754,	2144
$(2,0,0,3)_4$	1,	2,	6,	24,	120,	267,	754,	2144
$(3,0,0,2)_4$	1,	2,	6,	24,	120,	267,	754,	2144
$(0,0,2,4)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(0,0,3,1)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(0,2,0,1)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(0,3,0,4)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(1,0,2,0)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(1,3,0,0)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(4,0,3,0)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(4,2,0,0)_4$	1,	2,	6,	24,	120,	270,	714,	1736
$(0,0,1,2)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(0,0,2,1)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(0,0,3,4)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(0,0,4,3)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(1,2,0,0)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(2,1,0,0)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(3,4,0,0)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(4,3,0,0)_4$	1,	2,	6,	24,	120,	284,	612,	1200
$(0,0,1,3)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(0,0,4,2)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(0,1,0,2)_4$	1,	2,	6,	24,	120,	284,	752,	1796

Continued on next page

Patterns	**Exact avoidance sequence (from 1 to 8)**							
$(0,4,0,3)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(2,0,1,0)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(2,4,0,0)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(3,0,4,0)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(3,1,0,0)_4$	1,	2,	6,	24,	120,	284,	752,	1796
$(0,0,1,4)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(0,0,4,1)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(1,0,0,2)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(1,4,0,0)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(2,0,0,1)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(3,0,0,4)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(4,0,0,3)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(4,1,0,0)_4$	1,	2,	6,	24,	120,	284,	784,	2208
$(0,2,0,4)_4$	1,	2,	6,	24,	120,	304,	907,	2696
$(0,3,0,1)_4$	1,	2,	6,	24,	120,	304,	907,	2696
$(1,0,3,0)_4$	1,	2,	6,	24,	120,	304,	907,	2696
$(4,0,2,0)_4$	1,	2,	6,	24,	120,	304,	907,	2696
$(0,1,0,3)_4$	1,	2,	6,	24,	120,	321,	938,	2785
$(0,4,0,2)_4$	1,	2,	6,	24,	120,	321,	938,	2785
$(2,0,4,0)_4$	1,	2,	6,	24,	120,	321,	938,	2785
$(3,0,1,0)_4$	1,	2,	6,	24,	120,	321,	938,	2785
$(0,1,0,4)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(0,4,0,1)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(1,0,0,3)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(1,0,4,0)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(2,0,0,4)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(3,0,0,1)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(4,0,0,2)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(4,0,1,0)_4$	1,	2,	6,	24,	120,	341,	1044,	3339
$(1,0,0,4)_4$	1,	2,	6,	24,	120,	374,	1219,	4121
$(4,0,0,1)_4$	1,	2,	6,	24,	120,	374,	1219,	4121
$(0,0,0,0)_3$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,0,1)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,0,2)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,0,3)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,0,4)_4$	1,	2,	6,	24,	120,	720,	0,	0

Continued on next page

Patterns	Exact avoidance sequence (from 1 to 8)							
$(0,0,1,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,2,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,3,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,4,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,1,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,2,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,3,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,4,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(1,0,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(2,0,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(3,0,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(4,0,0,0)_4$	1,	2,	6,	24,	120,	720,	0,	0
$(0,0,0,0)_4$	1,	2,	6,	24,	120,	720,	5040,	0