Proof Systems: A Study on Form and Complexity

This dissertation includes three parts. The first two parts are related to each other. In [2] and [1], Iemhoff introduced a connection between the existence of a terminating sequent calculus of a certain kind and the uniform interpolation property of the super-intuitionistic logic that the calculus captures. In the second part, we will generalize this relationship to also cover the substructural setting on the one hand and a more powerful type of systems called semi-analytic calculi, on the other. To be more precise, we will show that any sufficiently strong substructural logic with a semi-analytic calculus has Craig interpolation property and in case that the calculus is also terminating, it has uniform interpolation. This relationship then leads to some concrete applications. On the positive side, it provides a uniform method to prove the uniform interpolation property for the logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, **IPC**, **CPC** and some of their **K** and **KD**-type modal extensions. However, on the negative side the relationship finds its more interesting application to show that many sub-structural logics including $Ł_n$, G_n, BL, R and RM^e, almost all super-intutionistic logics (except at most seven of them) and almost all extensions of **S4** (except thirty seven of them) do not have a semi-analytic calculus. It also shows that the logic **K4** and almost all extensions of the logic **S4** (except six of them) do not have a terminating semi-analytic calculus.

Then, in the second part, we pay attention solely to the systems Iemhoff introduced in [2], i.e., focused calculi. She showed almost all super-intuitionistic logics cannot have focused proof systems. In this part, we will provide a complexity theoretic analogue of this negative result to show that even in the cases that these systems exist, their proof-length would computationally explode. More precisely, we will first introduce two natural subclasses of focused rules, called PPF and MPF rules. Then, we will introduce some **CPC**-valid (**IPC**-valid) sequents with polynomially short tree-like proofs in the usual Hilbert-style proof system for classical logic, or equivalently $\mathbf{LK} + \mathbf{Cut}$, that have exponentially long proofs in the systems only consisting of PPF (MPF) rules.

In the third part, we investigate the proof complexity of a wide range of substructural systems. For any proof system **P** at least as strong as Full Lambek calculus, **FL**, and polynomially simulated by the extended Frege system for some infinite branching super-intuitionistic logic, we present an exponential lower bound on the proof lengths. More precisely, we will provide a sequence of **P**-provable formulas $\{A_n\}_{n=1}^{\infty}$ such that the length of the shortest **P**-proof for A_n is exponential in the length of A_n. The lower bound also extends to the number of proof-lines (proof-lengths) in any Frege system (extended Frege system) for a logic between FL and any infinite branching super-intuitionistic logic. We will also prove a similar result for the proof systems and logics extending Visser's basic propositional calculus **BPC** and its logic BPC, respectively. Finally, in the classical substructural setting, we will establish an exponential lower bound on the number of proof-lines in any proof system polynomially simulated by the cut-free version of $\mathbf{CFL_{ew}}$.

Keywords: Propositional proof complexity, Super-intuitionistic logics, Substructural logics, Modal logics, Craig Interpolation, Uniform interpolation, Feasible interpolation, Focused calculi.

References

[1] Rosalie Iemhoff. Uniform interpolation and sequent calculi in modal logic. Archive for Mathematical Logic, pages 1-27, 2015.

[2] Rosalie Iemhoff. Uniform interpolation and the existence of sequent calculi. 2017.