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Proof Systems: A Study on Form and Complexity

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Abstract: This dissertation includes three parts. The first two parts are related to each other. In [23] and [22], Iemhoff introduced a connection between the existence of a terminating sequent calculus of a certain kind and the uniform interpolation property of the super-intuitionistic logic that the calculus captures. In the second part, we will generalize this relationship to also cover the substructural setting on the one hand and a more powerful type of systems called semi-analytic calculi, on the other. To be more precise, we will show that any sufficiently strong substructural logic with a semi-analytic calculus has Craig interpolation property and in case that the calculus is also terminating, it has uniform interpolation. This relationship then leads to some concrete applications. On the positive side, it provides a uniform method to prove the uniform interpolation property for the logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, **IPC**, **CPC** and some of their **K** and **KD**-type modal extensions. However, on the negative side the relationship finds its more interesting application to show that many sub-structural logics including $Ł_n$, G_n, BL, R and RM^e, almost all super-intutionistic logics (except at most seven of them) and almost all extensions of **S4** (except thirty seven of them) do not have a semi-analytic calculus. It also shows that the logic **K4** and almost all extensions of the logic **S4** (except six of them) do not have a terminating semi-analytic calculus.
Then, in the second part, we pay attention solely to the systems Iemhoff introduced in [23], i.e., focused calculi. She showed almost all super-intuitionistic logics cannot have focused proof systems. In this part, we will provide a complexity theoretic analogue of this negative result to show that even in the cases that these systems exist, their proof-length would computationally explode.
In the third part, we investigate the proof complexity of a wide range of substructural systems. For any proof system **P** at least as strong as Full Lambek calculus, **FL**, and polynomially simulated by the extended Frege system for some infinite branching super-intuitionistic logic, we present an exponential lower bound on the proof lengths. More precisely, we will provide a sequence of **P**-provable formulas $\{A_n\}_{n=1}^{\infty}$ such that the length of the shortest **P**-proof for A_n is exponential in the length of A_n. The lower bound also extends to the number of proof-lines (proof-lengths) in any Frege system (extended Frege system) for a logic between FL and any infinite branching super-intuitionistic logic. We will also prove a similar result for the proof systems and logics extending Visser's basic propositional calculus **BPC** and its logic BPC, respectively. Finally, in the classical substructural setting, we will establish an exponential lower bound on the number of proof-lines in any proof system polynomially simulated by the cut-free version of $\mathbf{CFL_{ew}}$.

Keywords: Propositional proof complexity, Sub-structural logics, Craig Interpolation, Uniform interpolation

Contents

1. Introduction

Proof systems play a crucial role in proof theory, from consistency proofs and proof mining techniques to the characterization of admissible rules. These investigations are based on some *specific* proof systems tailored for *specific* purposes; an approach which we can call the instrumentalist approach to proof theory. However, there can be another approach which studies generic proof systems as the main objects of study as opposed to the technical tools that they always have been. This dissertation belongs to this realm of study in which we are interested in generic proof systems in their most general form.

As a first natural step in this area, in the second chapter of this dissertation we investigate the existence of "*nice*" proof systems for a given class of logics and for some natural interpretation for the adjective term "*nice*". More precisely, we approach this problem by proposing what we mean by nice proof systems and then by finding an invariant property for the logic with such a system. These two steps together provide a machinery to prove some logical properties for any logic with a nice proof system and on the negative side, if we have a property that almost all logics in a certain given class do not enjoy, we can show that almost all logics in the class do not have a nice proof system.
This line of research in its present format was initiated by Iemhoff, who showed that if a super-intuitionistic logic has a terminating proof system consisting of focused rules and focused axioms, it has the uniform interpolation property [23]. In her setting, nice proof systems are focused proof systems; the corresponding class is the class of super-intuitionistic logics; and the invariant is uniform interpolation. Since only seven super-intuitionistic logics have uniform interpolation, she showed that almost all super-intuitionistic logics do not have such a proof system. In our second chapter, we will present a second approximation for the adjective "nice". Our candidate for natural well-behaved sequent-style rules is semi-analytic rules, which is a generalization of Iemhoff's focused rules. Then we show that if a sufficiently strong sub-structural logic has a sequent-style proof system only consisting of semi-analytic rules and focused axioms, it has the Craig interpolation property. As a result, many substructural logics and all super-intuitionistic logics, except seven of them, do not have a sequent calculus of the mentioned form. Moreover, we also show that if a sufficiently strong substructural logic has a *terminating* sequent-style proof system only consisting of semi-analytic rules and focused axioms, it must have the uniform interpolation property. Consequently, **K4** and **S4** do not have a terminating sequent calculus of the mentioned form. The second chapter is a joint work with Amir Akbar Tabatabai. It is submitted to a journal and is currently under review.

In the third chapter, we address the complexity analogue of the previous existence problem, asking whether a given logic has an efficient "*nice*" proof system. We show that any calculus consisting only of two natural subclasses of focused rules is inefficient, meaning that some short statements must have exponentially long proofs in these systems. This result can be interpreted as the complexity counterpart of the negative result in the second chapter, ensuring that even in the cases that Iemhoff's systems exist, their proof-length will computationally explode. This chapter is a paper published in *International Workshop on Logic, Language, Information, and Computation*, [24].

Finally, in the last chapter of this dissertation, we will change our focus to the most general form of proof systems as polynomial time computable functions that are sound and complete with respect to the corresponding logics. Here, the main interesting proof systems are Frege and extended Frege systems for substructural logics and the extensions of Visser's basic logic. In this direction, we will provide an exponential lower bound on the lengths of proofs in extended Frege systems for logics as strong as the basic substructural logic FL or Visser's basic logic BPC and weaker than some super-intuitionistic infinite branching logic. For Frege systems for such logics, we can make the result even stronger by providing an exponential lower bound on the number of proof-lines in the systems. This chapter is currently available in the preprint format.

2. Semi-analytic Rules and Interpolation

2.1 Introduction

Proof systems have the main role in any proof theoretic investigation, from Gentzen's consistency proof and Kreisel's proof mining program to the characterizations of the admissible rules of the logical systems and their decidability problems. In this respect, proof systems are nothing but some technical tools in the study of their corresponding mathematical theories. They are designed and used based on their expected applications and not their inherent mathematical values. They are just the second rank citizens, far from the independent mathematical objects that they could have been.

Fortunately, in the recent years, alongside this instrumentalist approach, another approach has also been emerged; an approach that is more interested in the general behaviour of the proof systems than their possible technical applications (for instance, see [22], [23] and [10]). We call this emerging approach, the *universal proof theory*;[1] a name we hope to be reminiscent of the technical term *universal algebra* used for the theory that is supposed to investigate the generic behaviour of the algebraic structures. This theory is admittedly a hypothetical theory, but whatever it turns out to be, its agenda may include the following fundamental problems:

(*i*) The *existence problem* to investigate the existence of the different sorts of interesting proof systems such as the terminating systems, the normalizable systems, etc.

(*ii*) The *equivalence problem* to investigate the natural notions of equivalence between proof systems. This can be interpreted as an approach to address the so-called Hilbert's twenty fourth problem of studying the equivalence of different mathematical proofs, rigorously.

(*iii*) And finally, the *characterization problem* to investigate the possible characterizations of proof systems via a given equivalence relation as introduced in (*ii*).

As the first step in this so-called universal proof theory and following the spirit of [22] and [23], we begin with the most basic problem of the kind, the *existence problem*, addressing the existence of the natural sequent style proof systems for a given propositional and modal logic. The technique is developing a strong relationship between the existence of some sort of proof systems and some regularity conditions for the logic that it represents. One loose example of such a relationship is the relationship between the existence of a terminating calculus for a logic and its decidability. These relationships are important because they reduce the existence problem partially or completely to the regularity conditions of the

[1] We are grateful to Masoud Memarzadeh for this elegant terminological suggestion.

logic that are calculus-independent and probably more amenable to our technical tools. Again using our loose example, we know that an undecidable logic can not have a terminating calculus; a fact that solves the existence problem negatively.

This paper is devoted to one of these kinds of relationships and to explain how, we have to browse the history a little bit, first. The story begins with Pitts' seminal work, [35], in which he introduced a proof theoretic method to prove the uniform interpolation property for the propositional intuitionistic logic. His technique is built on the following two main ideas: First he extended the notion of uniform interpolation from a logic to its sequent calculus in a way that the uniform p-interpolants for a sequent are roughly the best left and right p-free formulas that if we add them to the left or right side of the sequent, they make the sequent provable. This reduces the task of proving uniform interpolation for the logic, to the task of finding these new uniform interpolants for all sequents. For the latter, he assigned two sets of p-free formulas to any sequent using the structure of the formulas occurred in the sequent itself. To define these sets, though, he needed the second crucial tool of the game namely the terminating calculus for **IPC**, introduced in [13] by Dyckhoff. The terminating calculus provides a well-founded order on sequents on which we can define the sets that we have mentioned before, recursively.

Later, as witnessed in [23], Iemhoff recognized that the main point in the first part of Pitts' argument is flexible enough to apply on any rule with a certain general form. This observation then lets her to lift the technique from the intuitionistic logic to any extension of the intuitionistic logic presented with a generic terminating calculus consisting of the usual axioms of the calculus **LJ** and the above-mentioned rules that she calls focused axioms and focused rules, respectively. These are the rules that are very natural to consider and they are roughly the rules with one main formula in their consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it, i.e. if the main formula is occurred in the left-side (right-side) of the consequence, all non-contextual formulas in the premises should also occur in the left-side (right-side) and any occurrence of any atom in these formulas must also occur in the main formula. The usual conjunction and disjunction rules are the prototype examples of these rules while the implication rules are the non-examples since they clearly do not respect the side of the main formula.

As we explained, the investigations in [23] lead to an exciting relationship between the existence of a terminating calculus consisting only of the focused axioms and focused rules for a logic and the uniform interpolation property of the logic. Iemhoff used this relationship first in a positive manner to prove the uniform interpolation for some well-known super-intuitionistic and super-intuitionistic modal logics including **IPC**, **CPC**, **K** and **KD** and their intuitionistic versions. And then she switched to the negative part to show that no extension of the intuitionistic logic can have a terminating calculus consisting of focused axioms and focused rules unless it has the uniform interpolation property. Since uniform interpolation is a rare property for a logic, it excludes almost all logical systems, including all super-intuitionistic logics, except the seven logics with the uniform interpolation property, from having such a terminating calculus.

Now we are ready to explain what we will pursue in this paper. Our approach is a generalization of the approach in [22] and [23], in the following three aspects: First we use a much more general class of rules that we will call semi-analytic rules. These rules can be defined roughly as the focused rules relaxing the side preserving condition. Therefore, they cover a vast variety of rules including focused rules, implication rules, non-context sharing rules in substructural logics and so many others. Secondly, we generalize the focused axioms of [23] to cover more general forms of axioms. And finally, we lower the base logic from the intuitionistic logic to the basic substructural logic $\mathbf{FL_e}$ to extend the applicability of the final result to cover substructural logics as well.

After these generalizations, as in [23], our main result connects the existence of proof systems consisting of semi-analytic rules and focused axioms to a strong version of Craig interpolation property called the feasible interpolation and in the case that the system is also terminating to an even stronger form of uniform interpolation. As it is expected, this connection also has two sorts of applications. First on the positive side, it says that if we manage to develop a terminating calculus consisting of semi-analytic rules and focused axioms, there is a uniform method to establish the uniform interpolation property. The logics with this property include some substrucutral logics like $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$, $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$ and their **K** and **KD** modal extensions and intuitionistic and classical logics and some of their modal extensions. (For the classical modal case see [7], for the substructural logics see [2] and for intuitionistic and intutionistic modal logics see [35] and [23].) Moreover, note that there is a possibility that we manage to develop a system of the mentioned form that fails to be terminating. In this case the connection is still useful but only to establish the Craig interpolation. The logics in this category include **K4** and **S4**-type of modal extensions of some substructural logics including the intutionistic and classical linear logics in which the exponentials play the role of the **S4**-type modality.

Despite the possible use of the positive applications of the connection, it is fair to say that developing a uniform method to prove interpolation is not very useful. The reason is the common knowledge that it is genuinely rare for a logic to have the interpolation property. To justify this feeling, note that in the substructural setting, there are a lot of relevant and semilinear logics ([43], [32]) that lack this property and as we have already seen in the super-intutionistic case, there is a well-known result by Maksimova [31] stating that among super-intuitionistic logics, there are only seven specific logics that have Craig or uniform interpolation.

Using this insight, we will turn the relationship between the interpolation and the existence of proof systems to its negative side to propose the main contribution of this paper. We will use the connection to show that logics without Craig interpolation do not have a calculus consisting only of semi-analytic rules and focused axioms and if they have Craig interpolation but fail to have uniform interpolation, the proof system if exists will not be terminating. Given the generality of these rules and axioms, this negative application excludes so many logics from having a reasonable proof system. To name a few concrete examples consider the logics $Ł_n$, G_n, BL, R and RM^e in the substructural world, all super-intuitionistic logics except **IPC**, **LC**, **KC**, $\mathbf{Bd_2}$, **Sm**, **GSc** and **CPC** in the super-intuitionistic domain and all extensions of **S4** except at most thirty seven of them in the modal case. In the uniform case, there are also some concrete examples including the logics **K4** and all the extensions of **S4** except at most six of them for which our result shows the non-existence of a terminating calculus consisting only of semi-analytic rules and focused axioms.

2.2 Preliminaries

In this section we will cover some of the preliminaries needed for the following sections. The definitions are similar to the same concepts in [23] and [32], but they have been changed whenever it is needed.

First, note that all of the finite objects that we will use here can be represented by a fixed reasonable binary string code. Therefore, by the length of any object O including formulas, proofs, etc. we mean the length of this string code and we will denote it by $|O|$.
In the following, we define a translation between two arbitrary languages. The reason for using such a notion is that in the upcoming sections we will consider logics with a fixed but an arbitrary language. This is a generalization which makes our results much stronger since their importance is that they are negative results. Therefore, the broader the range of the logics is, the stronger the results will be.

Definition 2.2.1. *Let us denote $p_1, \ldots, p_n$ by $\bar{p}$, where each p_i is an atomic formula. Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages. By a translation $t : \mathcal{L} \to \mathcal{L}'$, we mean an assignment which assigns a formula $\phi_C(\bar{p}) \in \mathcal{L}'$ to any logical connective $C(\bar{p}) \in \mathcal{L}$ such that any p_i has at most one occurrence in $\phi_C(\bar{p})$. It is possible to extend a translation from the basic connectives of the language to all of its formulas in an obvious compositional way. We will denote the translation of a formula ϕ by ϕ^t and the translation of a multiset Γ, by $\Gamma^t = \{\phi^t \mid \phi \in \Gamma\}$.*

Each translation is linearly bounded, i.e., for any translation t there exists a number c such that $|\psi^t| \leq c|\psi|$.
In this paper, we will work with a fixed but arbitrary language $\mathcal{L}$ that is augmented by a translation $t : \{\wedge, \vee, \to, \otimes, 0, 1\} \cup \mathcal{L} \to \mathcal{L}$ in the single-conclusion cases and by $t : \{\wedge, \vee, \to, \otimes, \oplus, 0, 1\} \cup \mathcal{L} \to \mathcal{L}$ in multi-conclusion cases, that fixes all logical connectives in $\mathcal{L}$. For this reason and w.l.o.g, we will assume that the language already includes the connectives $\{\wedge, \vee, \to, \otimes, 0, 1\}$ in single-conclusion cases and $\{\wedge, \vee, \to, \otimes, \oplus, 0, 1\}$ in multi-conclusion ones. In the case of modal logics, the language $\mathcal{L}$ will be extended to contain the modal operator $\Box$, as well.

Example 2.2.2. *The usual language of classical propositional logic is a valid language in our setting. In this case, there is a canonical translation that sends fusion, addition,* 1 *and* 0 *to conjunction, disjunction,* $\top$ *and* $\bot$*, respectively. In this paper, whenever we pick this language, we assume that we are working with this canonical translation.*

2.2.1 Sequent Calculi

We denote atomic formulas by small Roman letters, $p, q, \ldots$. Formulas are defined in the usual way from atomic formulas and atomic constants and connectives in the language, and we denote them by small Greek letters $\phi, \psi, \ldots$ or by capital Roman letters $A, B, \ldots$. We denote multisets of formulas by capital Greek letters $\Gamma, \Delta, \ldots$ and we mean the order does not matter but the multiplicity of formulas is important. However, sometimes we use the bar notation for multisets to make everything simpler. For instance, by $\bar{\phi}$, we mean a multiset consisting of formulas $\phi_1, \ldots, \phi_n$. We denote the number of elements (cardinality) of the multiset Γ by $||\Gamma||$. By $\Gamma \cup \Delta$ or Γ, Δ we mean the multiset containing all the formulas ϕ which is in Γ or in Δ. By a sequent, we mean an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas in the language. By a single-conclusion sequent $\Gamma \Rightarrow \Delta$ we mean that the multiset Δ contains at most one formula, and we call it multi-conclusion otherwise. In the single-conclusion cases a sequent $\Gamma \Rightarrow \Delta$ is interpreted as $\bigotimes \Gamma \to \Delta$, and if $\Delta = \emptyset$ as $\bigotimes \Gamma \to 0$, and in the multi-conclusion cases it is interpreted as $\bigotimes \Gamma \to \bigoplus \Delta$, where by $\bigotimes \Gamma$ we mean the formula $\gamma_1 \otimes \gamma_2 \otimes \ldots \otimes \gamma_n$, where each $\gamma_i \in \Gamma$; the formula $\bigoplus \Delta$ is defined similarly. For a sequent $S = (\Gamma \Rightarrow \Delta)$, by S^a we mean the antecedent of the sequent, which is Γ, and by S^s we mean the succedent of the sequent, which is Δ. The multiplication of two sequents S and T is defined as $S \cdot T = (S^a \cup T^a \Rightarrow S^s \cup T^s)$.
Meta-language, $\widehat{\mathcal{L}}$, is the language in which we define the sequent calculi. It consists of infinitely many formula variables $\hat{\phi}, \hat{\psi}, \ldots$, the logical connectives $\wedge, \vee, \to, \otimes$ (and $\oplus$ in the multi-conclusion cases and $\Box$ in modal cases), and constants $0, 1, \bot, \top$. Meta-formulas are defined as usual: all formula variables, atomic formulas and constants are meta-formulas and if ϕ and ψ are meta-formulas, so is $\phi \circ \psi$ for $\circ \in \{\wedge, \vee, \to, \otimes\}$ (and $\phi \oplus \psi$ in multi-conclusion cases and $\Box\phi$ in modal cases). We have also an infinite number of meta-multiset variables, also called contexts, which are denoted by $\widehat{\Gamma}, \widehat{\Delta}, \ldots$. A meta-sequent is an expression of the form $\widehat{S} = X \Rightarrow Y$ such that X and Y contain finite number of meta-formulas and meta-multisets. We will use multiset variables and contexts interchangeably.
The set of variables of a meta-formula ϕ, $V(\phi)$, is defined inductively. For any constant c in the language, $V(c)$ is defined as the empty set. For an atomic formula p and for a formula variable $\hat{\phi}$ define $V(p) = p$ and $V(\hat{\phi}) = \hat{\phi}$. For a logical connective $\circ \in \{\wedge, \vee, \to, \otimes, \oplus, \backslash, /\}$ define $V(\phi \circ \psi)$ as $V(\phi) \cup V(\psi)$. A meta-formula ϕ is called p-free, for an atomic formula or meta-formula variable p, when $p \notin V(\phi)$.
A substitution σ is a map from the union of meta-multisets and meta-formulas in $\widehat{\mathcal{L}}$ to the union of multisets and formulas in $\mathcal{L}$ that works as follows: constants are mapped to themselves, meta-formulas to formulas, meta-multisets to multisets, and σ commutes with the logical connectives and the modal operator. Therefore, $\sigma(\hat{\phi})$ will be a formula in $\mathcal{L}$, $\sigma(\widehat{\Gamma})$ will be the multiset of formulas $\sigma(\hat{\gamma})$, where $\hat{\gamma} \in \widehat{\Gamma}$, and $\sigma(\widehat{S} = X \Rightarrow Y)$ will be $\sigma(X) \Rightarrow \sigma(Y)$.
A rule is an expression of the form

$$\frac{\widehat{S}_1, \cdots, \widehat{S}_n}{\widehat{S}}$$

where $\widehat{S}, \widehat{S}_1, \ldots, \widehat{S}_n$ are meta-sequents. Meta-sequents above the line are called premisses and below the line, conclusion. In the case the rule has no premises, it is called an axiom. It is called a left (right) rule if $\widehat{S}^a$ ($\widehat{S}^s$) contains a meta-formula. A rule is either a right rule or a left one. An instance of a rule is obtained by using the substitution map on the rule as follows

$$\frac{\sigma(\widehat{S}_1), \cdots, \sigma(\widehat{S}_n)}{\sigma(\widehat{S})}$$

Note that if there is a side condition on the rule, such as the meta-formulas must everywhere be atoms, this condition works as a restriction on the substitution σ. A rule is backward applicable to a sequent S, when there is at least one instance of the rule where S is the conclusion.
By a sequent calculus G, we mean a finite set of rules. A sequent S is derivable from a set of sequents Γ in G, denoted by $\Gamma \vdash_G S$, if there exists a finite tree with sequents as labels of the nodes such that the label of the root is S, labels of the leaves are axioms of G or members of Γ, and in each node the set of the labels of the children of the node together with the label of the node itself, constitute an instance of a rule in G. This finite tree is called the proof of S in G which is sometimes called a tree-like proof to emphasize its tree-like form. If $\Gamma = \emptyset$ then we denote it by $G \vdash S$ and we say S is derivable in G. We will use the same notation for a sequent calculus and its logic, i.e., the set of provable formulas in it, i.e., $\{\phi \mid G \vdash (\Rightarrow \phi)\}$.

As it is usually a convention in proof theory papers, from now on we will not mention "meta" in the meta-language and so on and we will omit the $\widehat{}$ notation. It will be always clear from the context which form we are working with. Therefore, for instance by a meta sequent $\Gamma, \bar{\phi} \Rightarrow \psi$, we mean Γ is a meta-multiset, $\bar{\phi}$ is a possibly empty multiset of meta-formulas and ψ is meta-formulas.

Let us recall some of the notions related to sequent calculi and some of the important systems that we will use throughout the paper. Consider the following set of rules:

Identity:

$$\frac{}{\phi \Rightarrow \phi}$$

Context-free Axioms:

$$\frac{}{\Rightarrow 1} \qquad \frac{}{0 \Rightarrow}$$

Rules for 0 and 1:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\,(1w) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\,(0w)$$

Conjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\,(L\wedge) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\,(L\wedge) \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\,(R\wedge)$$

Disjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}\,(L\vee) \qquad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\,(R\vee) \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\,(R\vee)$$

Fusion Rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \otimes \psi \Rightarrow \Delta}\,(L\otimes) \qquad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi \otimes \psi, \Delta, \Lambda}\,(R\otimes)$$

Implication Rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda}\,(L \to) \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}\,(R \to)$$

The system consisting of the single-conclusion version of all of these rules is $\mathbf{FL_e}^-$. If we also add the single-conclusion version of the following axioms, we will have the system $\mathbf{FL_e}$.

Contextual Axioms:

$$\frac{}{\Gamma \Rightarrow \top, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}$$

In the standard definition of $\mathbf{FL_e}$ the language does not contain the constants $\bot$ and $\top$ and therefore their axioms are not present in the sequent calculus, as well. However, since the presence of $\bot$ and $\top$ is essential in our discussions in the future sections, we allow them in the language and their axioms in the sequent calculus.

In the multi-conclusion case define $\mathbf{CFL_e}^-$ and $\mathbf{CFL_e}$ with the same rules as $\mathbf{FL_e}^-$ and $\mathbf{FL_e}$, this time in their full multi-conclusion version and add $\oplus$ to the language and the following rules to the systems:

Rules for $\oplus$:

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \oplus \psi \Rightarrow \Delta, \Lambda}\,(L\oplus) \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \oplus \psi, \Delta}\,(R\oplus)$$

The system **MALL** is defined as $\mathbf{CFL_e}$ minus the implication rules. Moreover, if we consider the following rules:

$$\frac{!\Gamma \Rightarrow \phi}{!\Gamma \Rightarrow !\phi}\,\dagger \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \qquad \frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

we can define **ILL** as $\mathbf{FL_e}$ plus the single-conclusion version of the above rules and **CLL** as $\mathbf{CFL_e}$ plus the above rules, themselves. In both cases, the rule † is single-conclusion.

We will use later the structural rules given below:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lw) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\,(Rw)$$

Note that in the single-conclusion cases, in the rule (Rw), Δ must be empty.

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lc) \quad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \phi, \Delta}\,(Rc)$$

The rule (Rc) is only allowed in multi-conclusion systems.

If we consider the sequent calculus $\mathbf{FL_e}$ and add the weakening rules (contraction rules), the resulting system is called $\mathbf{FL_{ew}}$ ($\mathbf{FL_{ec}}$). In a similar manner, we define $\mathbf{CFL_{ew}}$ and $\mathbf{CFL_{ec}}$. Finally, adding all the structural rules to $\mathbf{FL_e}$, we obtain the system $\mathbf{FL_{ewc}}$ in which the connectives $\otimes$ and $\wedge$ become equivalent, i.e., $\phi \otimes \psi \Leftrightarrow \phi \wedge \psi$ will become provable in the system. Moreover, $\bot$ and 0, and $\top$ and 1 will become equivalent in $\mathbf{FL_{ewc}}$. Furthermore, in the system $\mathbf{CFL_{ewc}}$, we can also prove that $\oplus$ and $\vee$ are equivalent. Hence, it is possible to define $\mathbf{FL_{ewc}}$ ($\mathbf{CFL_{ewc}}$) even on the restricted language $\{\wedge, \vee, \top, \bot, \to\}$. This system is nothing but the usual sequent calculus **LJ** (**LK**) for the intuitionistic (classical) logic.

We will also use the following rule in the future sections:

Context-sharing left implication:

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \to \psi \Rightarrow \Delta}$$

Finally, note that Γ and Δ are multisets everywhere, therefore the exchange rule is built in and hence admissible in our system. Moreover, note that the calculi defined in this section are written in the given language which can be any extension of the language of the system itself. For instance, $\mathbf{FL_e}$ is the calculus with the mentioned rules on our fixed language that can have more connectives than $\{\wedge, \vee, \otimes, \to, \top, \bot, 1, 0\}$.
By a subsequent of a sequent $\Gamma \Rightarrow \Delta$ we mean a sequent $\Gamma' \Rightarrow \Delta'$. We call it proper if either $\Gamma' \subsetneq \Gamma$ or $\Delta' \subsetneq \Delta$.

Definition 2.2.3. *A calculus is terminating if for any sequent S, the number of rules which are backward applicable to S are finite. Moreover, there is a well-founded order on the sequents such that the order of the following are less than the order of S:*

- *the premises of all instance of a rule whose conclusion is S;*
- *proper subsequents of S, and*
- *any sequent S' of the form $(\Gamma, \Pi \Rightarrow \Delta, \Lambda)$, where S is of the form $(\Gamma, \Box\Pi \Rightarrow \Delta, \Box\Lambda)$. Note that $\Pi \cup \Lambda$ must be non-empty.*

Definition 2.2.4. *We will define the following sequent calculus for intuitionistic logic,* **G4i**, *which was first introduced by Dyckhoff in [13].*

$$\phi \Rightarrow \phi \quad (Id) \qquad , \qquad \Gamma, \bot \Rightarrow \Delta \quad (L\bot) \qquad , \qquad \Gamma \Rightarrow \Delta, \top \quad (R\top)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\,(Lw) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \phi}\,(Rw)$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} (L\wedge) \quad \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} (R\wedge)$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} (L\vee) \quad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi} (R\vee) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi} (R\vee)$$

$$\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi} (R \to)$$

$$\frac{\Gamma, p, \psi \Rightarrow \Delta}{\Gamma, p, p \to \psi \Rightarrow \Delta} (L_1 \to) \quad \frac{\Gamma, \phi \to (\psi \to \gamma) \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \to \gamma \Rightarrow \Delta} (L_2 \to)$$

$$\frac{\Gamma, \phi \to \gamma, \psi \to \gamma \Rightarrow \Delta}{\Gamma, \phi \vee \psi \to \gamma \Rightarrow \Delta} (L_3 \to) \quad \frac{\Gamma, \psi \to \gamma \Rightarrow \phi \to \psi \qquad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \to \psi) \to \gamma \Rightarrow \Delta} (L_4 \to)$$

*where p is an atom. Structural rules and the cut rule are admissible in the system and in each rule Δ has at most one element. Note that this system is slightly different than the usual **G4i** system. The usual definition does not include the explicit weakening rules and the axioms for $\top$ and $\bot$. It also has the axiom $\Gamma, p \Rightarrow p$ only for atomic formula p, instead of the axiom (Id) as we assumed. The system we have introduced is clearly equivalent to the usual one and it is also terminating with the same Dyckhoff order [13] that we will see in a moment. The advantage of the new system, though, is that it is more in line with our later general approach to sequent-style rules.*

Define the rank of a propositional formula as follows:

$$\begin{aligned} r(p) &= r(\bot) = r(\top) = 1 \\ r(\phi \circ \psi) &= r(\phi) + r(\psi) + 1 \quad \circ \in \{\vee, \to\} \\ r(\phi \wedge \psi) &= r(\phi) + r(\psi) + 2 \end{aligned}$$

Then a sequent S is called lower than the sequent T if S is the result of replacing the elements of T with any number of elements with lower ranks. With this order, it is not hard to see that the system **G4i** is terminating. Note that with this order, for any formula ψ and any atom p, the sequent $\Gamma, \psi \Rightarrow \Delta$ is lower than the sequent $\Gamma, p \to \psi \Rightarrow \Delta$. We will use this fact in Corollary 2.5.45.

Definition 2.2.5. *By a logic L in the language $\mathcal{L}$, we mean a subset of the set of all $\mathcal{L}$-formulas that is closed under arbitrary substitution and the following rules:*

(*i*) *The modus ponens rule: If $\phi \to \psi \in L$ and $\phi \in L$ then $\psi \in L$.*

(*ii*) *The adjunction rule: If $\phi \in L$ and $\psi \in L$ then $\phi \wedge \psi \in L$.*

Definition 2.2.6. *Let L and L' be two logics such that $\mathcal{L}_L \subseteq \mathcal{L}_{L'}$. We say L' is an extension of L (or L' extends L) if $L \vdash A$ implies $L' \vdash A$.*

Definition 2.2.7. *Let G and H be two sequent calculi such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. We say H is an extension of G if all the rules of G are admissible in H, i.e., for any instance of a rule $\mathcal{R}$ of G, if the premises are provable in H then so is its consequence. Moreover, H is called an axiomatic extension of G (or H extends G), if the provable sequents of G are considered as axioms of H, to which H adds some rules.*

Definition 2.2.8. *Let G be a sequent calculus and L be a logic with the same language as G's. We say G is a sequent calculus for the logic L when:*

$$G \vdash \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad L \vdash (\bigotimes \Gamma \to \bigoplus \Delta).$$

Note that if the calculus is single-conclusion, by $\bigoplus \Delta$, we mean Δ if Δ is a singleton, and 0 if Δ is empty. Therefore, in this case we do not need the $\oplus$ operator.

Theorem 2.2.9. *Let L be a logic and G a single-conclusion (multi-conclusion) sequent calculus for L. Then, for any logic $M \in \{\mathbf{FL_e}^-, \mathbf{FL_e}, \mathbf{IPC}\}$ ($M \in \{\mathbf{CFL_e}^-, \mathbf{CFL_e}\}$), if we denote the calculus of M, defined previously in this section, by GM, we have:*

- (i) *If L extends $\mathbf{FL_e}^-$ ($\mathbf{CFL_e}^-$), then the cut rule is admissible in G.*
- (ii) *If L extends M, then G extends the calculus GM.*

Proof. First, observe that for any formulas ϕ and ψ, if $L \vdash \phi$ and $L \vdash \psi$ then we have $L \vdash \phi \otimes \psi$. The reason is that L extends $\mathbf{FL_e}^-$ and $\mathbf{FL_e}^- \vdash \phi \to (\psi \to \phi \otimes \psi)$. Therefore, $L \vdash \phi \to (\psi \to \phi \otimes \psi)$. Since L is closed under modus ponens, if $L \vdash \phi$ and $L \vdash \psi$ then $L \vdash \phi \otimes \psi$.

Now let us prove (i). For the single-conclusion case, set $\bigoplus \Delta$ as ϕ when $\Delta = \phi$ and $\bigoplus \Delta = 0$, when Δ is empty. Assume that $G \vdash \Gamma \Rightarrow A, \Delta$ and $G \vdash \Gamma', A \Rightarrow \Delta'$. Hence $L \vdash \bigotimes \Gamma \to A \oplus (\bigoplus \Delta)$ and $L \vdash (\bigotimes \Gamma') \otimes A \to (\bigoplus \Delta')$ by the soundness of G. Therefore, by the previous observation we have

$$L \vdash [\bigotimes \Gamma \to A \oplus (\bigoplus \Delta)] \otimes [(\bigotimes \Gamma') \otimes A \to (\bigoplus \Delta')]$$

Since L extends $\mathbf{FL_e}^-$ ($\mathbf{CFL_e}^-$) and in this logic the previous formula implies the formula

$$[(\bigotimes \Gamma) \otimes (\bigotimes \Gamma') \to (\bigoplus \Delta) \oplus (\bigoplus \Delta')]$$

By modus ponens in L the last formula is also provable in L which implies $G \vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ again by the completeness of G.

For (ii), let R be an instance of a rule in the system M and let $S_1, \cdots, S_n$ and S_0 be the premises and the consequence of R, respectively. Define $F(\Gamma \Rightarrow \Delta) = [\bigotimes \Gamma \to \bigoplus \Delta]$. Then there are three cases to consider:

1. R is an instance of an axiom. Then M proves $F(S_0)$. Since L extends M,we have $L \vdash F(S_0)$ which implies $G \vdash S_0$.

2. R is an instance of the conjunction, the disjunction or the structural rules (in this case $M = \mathbf{IPC}$). Then it is easy to see that the formula

$$\bigwedge_{i=1}^{n} F(S_i) \to F(S_0)$$

is provable in M and hence in L. Now, if $G \vdash S_i$ for all $1 \leq i \leq n$, we have $L \vdash F(S_i)$ which implies $L \vdash \bigwedge_{i=1}^{n} F(S_i)$ by the adjunction rule. Since L is closed under modus ponens, $L \vdash F(S_0)$ which implies $G \vdash S_0$ by the completeness of G.

3. R is an instance of the rules for 0 and 1, the fusion, the addition or the implication rule. Then it is easy to see that the formula

$$\bigotimes_{i=1}^{n} F(S_i) \to F(S_0)$$

is provable in M and hence in L. Now, if $G \vdash S_i$ for all $1 \leq i \leq n$, we have $L \vdash F(S_i)$ which implies $L \vdash \bigotimes_{i=1}^{n} F(S_i)$ by the previous observation. Finally, since L is closed under modus ponens, $L \vdash F(S_0)$ which implies $G \vdash S_0$ by the completeness of G. □

2.2.2 Logical Systems

In this subsection we will recall the Craig interpolation property, the uniform interpolation property and also some useful substructural logics that we will need in the rest of the paper.

Definition 2.2.10. *We say that a logic L has Craig interpolation property if for any formulas ϕ and ψ if $L \vdash \phi \to \psi$, then there exists a formula θ such that $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.*

Definition 2.2.11. *We say a logic L has the uniform interpolation property if for any formulas ϕ and any atomic formula p, there are two p-free formulas, the p-pre-interpolant, $\forall p\phi$ and the p-post-interpolant $\exists p\phi$, such that $V(\exists p\phi) \subseteq V(\phi)$ and $V(\forall p\phi) \subseteq V(\phi)$ and*

- (i) *$L \vdash \forall p\phi \to \phi$,*
- (ii) *For any p-free formula ψ if $L \vdash \psi \to \phi$ then $L \vdash \psi \to \forall p\phi$,*
- (iii) *$L \vdash \phi \to \exists p\phi$, and*
- (iv) *For any p-free formula ψ if $L \vdash \phi \to \psi$ then $L \vdash \exists p\phi \to \psi$.*

To recall some of the well known substructural logics and following [32], we have to introduce the semantical framework, first.

Definition 2.2.12. *By a pointed commutative residuated lattice we mean an algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \otimes, \to, 0, 1\rangle$ where $\wedge, \vee, \otimes, \to$ are binary operations, and $0, 1$ are constants such that $\langle A, \wedge, \vee\rangle$ is a lattice with partial order $\leq$ and $\langle A, \otimes, 1\rangle$ is a commutative monoid. We define for all $x, y, z \in A$, $x \otimes y \leq z$ if and only if $x \leq y \to z$. For a single pointed commutative residuated lattice $\mathbf{A}$ and a class of pointed commutative residuated lattices $\mathbf{K}$, denote $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathbf{K})$ as the varieties generated by $\mathbf{A}$ and $\mathbf{K}$, respectively.*

In the following we will borrow the definitions of some logics from [32]. First, we need the following equational conditions for pointed commutative residuated lattices.

- (prl) prelinearity : $1 \leq (x \to y) \vee (y \to x)$

- (dis) distributivity : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- (inv) involutivity : $\neg\neg x = x$
- (int) integrality : $x \leq 1$
- (bd) boundedness : $0 \leq x$
- (id) idempotence : $x = x \otimes x$
- (fp) fixed point negation : $0 = 1$
- (div) divisibility : $x \otimes (x \to y) = y \otimes (y \to x)$
- (can) cancellation : $x \to (x \otimes y) = y$
- (rcan) restricted cancellation : $1 = \neg x \vee ((x \to (x \otimes y)) \to y)$
- (nc) non-contradiction : $x \wedge \neg x \leq 0$

In the following, we have the definitions of some logics that we are interested in. Note that in all of them, both of the axioms (*prl*) and (*dis*) are present, and hence we just mention the other axioms.

- (UL^-) unbounded uninorm logic
- (IUL^-) unbounded involutive uninorm logic : (inv)
- (MTL) monoidal t-norm logic : $(int), (bd)$
- $(SMTL)$ strict monoidal t-norm logic : $(int), (bd), (nc)$
- $(IMTL)$ involutive monoidal t-norm logic : $(int), (bd), (inv)$
- (BL) basic fuzzy logic : $(int), (bd), (div)$
- (G) Gödel logic : $(int), (bd), (id)$
- (Ł) Łukasiewicz logic : $(int), (bd), (div), (inv)$
- (P) product logic : $(int), (bd), (div), (rcan)$
- (CHL) cancellative hoop logic : $(int), (fp), (div), (can)$
- (UML^-) unbounded uninorm mingle logic : (id)
- (RM^e) R-mingle with unit : $(id), (inv)$
- $(IUML^-)$ unbounded involutive uninorm mingle logic : $(id), (inv), (fp)$
- (A) abelian logic : $(inv), (fp), (can)$

Furthermore, we will define the following important logics, as well.
For $n > 1$ define

$$L_n = \{0, \tfrac{1}{n-1}, \cdots, \tfrac{n-2}{n-1}, 1\} \qquad , \qquad L_\infty = [0, 1]$$

and the pointed commutative residuated lattices (again for $n > 1$)

$$\mathbf{L_n} = \langle L_n, min, max, \otimes_{Ł}, \to_{Ł}, 1, 0\rangle$$

and

$$\mathbf{G_n} = \langle L_n, min, max, min, \to_G, 1, 0\rangle$$

where $x \otimes_{Ł} y = max(0, x+y-1)$, $x \to_{Ł} y = min(1, 1-x+y)$, and $x \to_G y$ is y if $x > y$, otherwise 1. Then, for $n > 1$, $Ł_n$ and G_n are the logics with equivalent algebraic semantics $\mathcal{V}(\mathbf{L_n})$ and $\mathcal{V}(\mathbf{G_n})$, respectively. The logics G_∞ and H_∞ are the Gödel logic and Łukasiewicz logic, as defined before.

R is the logic of a variety consisting of all distributive pointed commutative residuated lattices with the condition that $x \otimes x \leq x$ for all x.

Now consider the following binary functions on the set of integers $\mathbb{Z}$, where $\wedge$ and $\vee$ are min and max, respectively, and $|x|$ is the absolute value of x:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |y| < |x| \end{cases} \qquad x \to y = \begin{cases} -(x) \vee y & \text{if } x \leq y \\ -(x) \wedge y & \text{otherwise} \end{cases}$$

And finally define the following algebras:

$$\mathbf{S_{2m}} = \langle \{-m, -m+1, \cdots, -1, 1, \cdots, m-1, m\}, \wedge, \vee, \otimes, \to, 1, -1\rangle \quad (m \geq 1)$$

$$\mathbf{S_{2m+1}} = \langle \{-m, -m+1, \cdots, -1, 0, 1, \cdots, m-1, m\}, \wedge, \vee, \otimes, \to, 0, 0\rangle \quad (m \geq 0)$$

and define RM^e_n as the logic of $\mathcal{V}(\mathbf{S_n})$.

2.3 Semi-analytic Rules

In this section we will introduce a class of rules which we will investigate in the rest of this paper. We will only consider the rules with exactly one main formula ϕ in the conclusion, i.e., the only active formula in the conclusion, which is different from the contexts (or multiset variables) which are denoted by Γ_i, Π_j or Δ_i.
By the notation $\langle\langle S_{ir}\rangle_r\rangle_i$ we mean first considering the sequents S_{ir} ranging over r and then ranging over i. For instance, $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i$ is short for the following set of sequents where $1 \leq r \leq m_i$ and $1 \leq i \leq n$:

$$\Gamma_1, \bar{\phi}_{11} \Rightarrow \bar{\psi}_{11}, \Delta_1, \cdots, \Gamma_1, \bar{\phi}_{1m_1} \Rightarrow \bar{\psi}_{1m_1}, \Delta_1,$$

$$\Gamma_2, \bar{\phi}_{21} \Rightarrow \bar{\psi}_{21}, \Delta_2, \cdots, \Gamma_2, \bar{\phi}_{2m_2} \Rightarrow \bar{\psi}_{2m_2}, \Delta_2,$$

$$\vdots$$

$$\Gamma_n, \bar{\phi}_{n1} \Rightarrow \bar{\psi}_{n1}, \Delta_n, \cdots, \Gamma_n, \bar{\phi}_{nm_n} \Rightarrow \bar{\psi}_{nm_n}, \Delta_n.$$

where each $\bar{\phi}_{ir}$ is a multiset of meta-formulas $\phi^1_{ir}, \ldots, \phi^{k_{ir}}_{ir}$ or the empty sequence and $\bar{\psi}_{ir}$ is a multiset of meta-formulas $\psi^1_{ir}, \ldots, \psi^{k'_{ir}}_{ir}$ or the empty sequence. The reason for such a complicated notation is that we want to be able to talk about rules as general as possible. The premises in a rule may be made of sequents with the same contexts or/and sequents with different contexts. At a closer look, in the ith horizontal line in the definition above, there are m_i sequents with the same contexts Γ_i and Δ_i and possibly different sequences of meta-formulas $\bar{\phi}_{im_i}$ and $\bar{\psi}_{im_i}$. In each sequent $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$, the multiset variable Γ_i is called the left context and Δ_i the right context. $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i$ and $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j$ are defined similarly. In the former, there are no sequences of meta-formulas in the succedents of sequents and in the latter, there are no contexts in the succedents of sequents.

Definition 2.3.13. *A rule is called occurrence preserving if the set of variables of any meta-formula appeared in any of the premises is a subset of the set of variables of the main formula in the conclusion. For instance for the following rule*

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

the occurrence preserving condition is

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi).$$

Note that the occurrence preserving condition is defined on the form of the rule and not on an instance of a rule. Therefore, when we say a variable is occurred in the premises we mean in $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ or $\bar{\phi}_{ir}$.
In the following we will define semi-analytic rules. Because of the occurrence preserving condition, we call these rules semi-analytic. This occurrence preserving condition is a weaker version of the analycity property in the analytic rules, which demands the formulas in the premises to be subformulas of the formulas in the consequence. Based on a rule being single-conclusion, multi-conclusion, context-sharing or a modal rule, the notion of being semi-analytic is defined as follows.

Definition 2.3.14. *Let Γ_i, Π_j and Δ_i be pairwise distinct multiset variables, $\bar{\psi}_{js}$ and $\bar{\phi}_{ir}$ be multisets of meta-formulas and ϕ be a meta-formula where $i \leq n$ and $j \leq m$. In the left single-conclusion semi-analytic rule, $||\Delta_i|| \leq 1$ and $\bar{\theta}_{js}$ is either one meta-formula or empty for each i, j, and s. Also, in the right single-conclusion semi-analytic rule,$\bar{\psi}_{ir}$ is either one meta-formula or empty for each i and r. A rule is called semi-analytic if it is occurrence preserving and has one of the following forms.*

- *left single-conclusion semi-analytic:*

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *right single-conclusion semi-analytic:*

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi}$$

- *context-sharing semi-analytic:*

$$\frac{\langle\langle \Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *left multi-conclusion semi-analytic:*

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

- *right multi-conclusion semi-analytic:*

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n}$$

- *A rule is called modal semi-analytic if it has one of the following forms:*

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, K \qquad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\, D$$

  with the conditions that whenever the rule (D) *is present, the rule* (K) *must be present.*

Note that in the left single-conclusion semi-analytic rule since the number of elements of the succedent of the conclusion of the rule must be at most 1, it means that at most one of Δ_i's can be non-empty. Whenever it is clear from the context, we will omit the phrase "multi-conclusion".

Moreover, consider the following modal rules that we do *not* consider as semi-analytic but we will address in our investigations. We assume that whenever the rule $(4D)$ is present in a system the modal rule (4) must be present, as well and whenever the rule $(RS4)$ is present in a system, the rule, $(LS4)$, must be present.

$$\frac{\Box\Gamma, \Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, 4 \qquad \frac{\Box\Gamma, \Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\, 4D$$

$$\frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\, RS4 \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\, LS4$$

where Γ and Δ are both multiset variables and we use the convention that $\Box\emptyset = \emptyset$.

Example 2.3.15. *A generic example of a left semi-analytic rule is the following:*

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \qquad \Gamma, \theta \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha).$$

Note that the premises on the left and in the middle of the example have the same context Γ *in the antecedent and have no context in the succedents. Therefore, there should be only one copy of* Γ *in the antecedent of the conclusion. A generic example of a context-sharing left semi-analytic rule is:*

$$\frac{\Gamma, \theta \Rightarrow \eta \qquad \Gamma, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

where

$$V(\theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Moreover, for a generic example of a right semi-analytic rule we can have

$$\frac{\Gamma, \phi \Rightarrow \psi \qquad \Gamma, \theta_1, \theta_2 \Rightarrow \eta \qquad \Pi, \mu_1, \mu_2, \Rightarrow \nu}{\Gamma, \Pi \Rightarrow \alpha}$$

where

$$V(\phi, \psi, \theta_1, \theta_2, \eta, \mu_1, \mu_2, \nu) \subseteq V(\alpha)$$

Here are some remarks. First note that in any left single-conclusion semi-analytic rule there are two types of premises. In the first type, the succedent of the sequent includes only a meta-formula and in the second type the succedent of the sequent includes only a context. This is a crucial point to consider. Any left semi-analytic rule allows any kinds of combination of sharing/combining contexts in any type. However, between two types, we can only combine the contexts in the antecedent. The case in which we can share the contexts of the antecedents of sequents of the two types is called context-sharing semi-analytic rule. This should explain why our second example is called context-sharing left semi-analytic while the first one is not. The reason is the fact that the two types share the same context of the antecedent in the second rule while in the first one this situation happens in just one type. The second point is the presence of contexts. This is very crucial for almost all the arguments in this paper, that any sequent present in a semi-analytic rule must have multiset variables as left contexts and in the case of left rules, at least one multiset variable for the right hand-side must be present.

Example 2.3.16. *Now for more concrete examples, note that all the usual conjunction, disjunction and implication rules for* **IPC** *are semi-analytic. The same also holds for all the rules in substructural logic* $\mathbf{FL_e}$*, the weakening and the contraction rules and some of the well-known restricted versions of them including the following rules for exponentials in linear logic:*

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

For a context-sharing semi-analytic rule, consider the following rule in the Dyckhoff calculus for **IPC** *(see [13]):*

$$\frac{\Gamma, \psi \to \gamma \Rightarrow \phi \to \psi \qquad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \to \psi) \to \gamma \Rightarrow \Delta}$$

Example 2.3.17. *For a concrete non-example consider the cut rule; it is not a semi-analytic rule because it does not preserve the variable occurrence condition. Moreover, the following rule in the calculus of* **KC***:*

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions. The rule of thumb is that any rule in which we have side conditions on the contexts is not semi-analytic.

Definition 2.3.18. *A sequent is called a focused axiom if it has one of the following forms:*

(1) *Identity axiom: ($\phi \Rightarrow \phi$)*

(2) *Context-free right axiom: ($\Rightarrow \bar{\alpha}$)*

(3) *Context-free left axiom: ($\bar{\beta} \Rightarrow$)*

(4) *Contextual left axiom: ($\Gamma, \bar{\phi} \Rightarrow \Delta$)*

(5) *Contextual right axiom: ($\Gamma \Rightarrow \bar{\phi}, \Delta$)*

where Γ and Δ are multiset variables and $\bar{\alpha}, \bar{\beta}, \bar{\phi}$ are multisets of meta-formulas and ϕ is a meta-formul. Moreover, in (2) *the variables in any pair of elements in $\bar{\alpha}$ are equal, or in other words $V(\mu) = V(\nu)$ for any $\mu, \nu \in \bar{\alpha}$. The same condition also holds for any pair of elements in $\bar{\beta}$ in* (3) *or in $\bar{\phi}$ in* (4) *and* (5)*. A sequent is called context-free focused axiom if it has the form* (1)*,* (2) *or* (3)*.*

Example 2.3.19. *It is easy to see that the axioms given in the preliminaries are examples of focused axioms. Here are some more examples:*

$$\neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0$$

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

where the first four are context-free while the last two are contextual. As a non-example consider $p, \neg p, q \Rightarrow$. It is not a focused axiom since the set of variables of p and q (or $\neg p$ and q) are not equivalent.

2.4 Craig Interpolation

In this section we will investigate the relationship between the semi-analytic rules and the Craig interpolation property. Apart from its clear use in proving interpolation for different logics, it has a very interesting application to show that some of the natural substructural and super-intuitionistic logics can not have a calculus consisting only of semi-analytic rules and the focused axioms.

First, let us define the interpolation property for a sequent calculus.

Definition 2.4.20. *(essentially Maehara) Let G and H be sequent calculi. G has H-interpolation if for any sequent $S = (\Sigma, \Lambda \Rightarrow \Delta)$ if S is provable in G by a tree-like proof π, then there exists a formula C such that ($\Sigma \Rightarrow C$) and ($\Lambda, C \Rightarrow \Delta$) are provable in H and $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, where $V(A)$ is the set of the atoms of A. We say G has H-feasible interpolation if we also have the*

bound $|C| \leq |\pi|^{O(1)}$.
Moreover, we say G *has strong* H*-interpolation if for any sequent* $S = (\Sigma, \Lambda \Rightarrow \Theta, \Delta)$ *if* S *is provable in* G *by a tree-like proof* π*, then there exists a formula* C *such that* $(\Sigma \Rightarrow C, \Theta)$ *and* $(\Lambda, C \Rightarrow \Delta)$ *are provable in* H *and* $V(C) \subseteq V(\Sigma \cup \Theta) \cap V(\Lambda \cup \Delta)$*. We say* G *has strong* H*-feasible interpolation if we also have the bound* $|C| \leq |\pi|^{O(1)}$.

The following theorem shows that the interpolation property of a sequent calculus leads to the Craig interpolation of its logic.

Theorem 2.4.21. *If a logic* L *has a complete sequent calculus* G *with the* G*-interpolation property, then* L *has Craig interpolation.*

Proof. Let $L \vdash \phi \to \psi$. Since G is complete for L, we have $G \vdash \phi \Rightarrow \psi$. Since G has the interpolation property, there exists θ such that $G \vdash \phi \Rightarrow \theta$, $G \vdash \theta \Rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$. Again from the completeness of G, $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ which completes the proof. ☐

The following theorem ensures that any set of focused axioms of a sequent calculus H, has H-interpolation property. It can also serve as an example to show how this notion of relative interpolation works.

Theorem 2.4.22. *Let* G *and* H *be two sequent calculi such that every provable sequent in* G *is also provable in* H*, and let* G *consist of only focused (context-free focused) axioms. Then:*

- (i) *If both* G *and* H *are single-conclusion and* H *extends* $\mathbf{FL_e}$ *(*$\mathbf{FL_e}^-$*),* G *has* H*-feasible interpolation.*
- (ii) *If both of* G *and* H *are multi-conclusion and* H *extends* $\mathbf{CFL_e}$ *(*$\mathbf{CFL_e}^-$*),* G *has strong* H*-feasible interpolation.*

Proof. To prove (i), note that a sequent S is provable in G if it is one of the focused axioms. We will check each case separately:

(1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. For any partition Σ and Λ that we have $(\Sigma, \Lambda \Rightarrow \phi)$ in G, we have to find a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \phi)$ are provable in H. There are two cases to consider. First, if $\Sigma = \{\phi\}$ and $\Lambda = \emptyset$. For this case define C to be ϕ. Obviously both conditions hold since we have $(\phi \Rightarrow \phi)$ as an axiom. Second, if $\Sigma = \emptyset$ and $\Lambda = \{\phi\}$ define C as 1. We must have $(\Rightarrow 1)$ and $(1, \phi \Rightarrow \phi)$ in H. The first one is an axiom of G and hence provable in H, and the second is the consequence of an instance of the rule $(1w)$ and the fact that $(\phi \Rightarrow \phi)$ is provable in H.

(2) For the case $(\Rightarrow \bar{\alpha})$, consider C to be 1. Then since both Σ and Λ are empty sequents, we must have $(\Rightarrow 1)$ and $(1 \Rightarrow \bar{\alpha})$ in H. The first one is an axiom of G and hence provable in H, and the second is the consequence of an instance of the rule $(1w)$ and the fact that $(\Rightarrow \bar{\alpha})$ is provable in H.

(3) For the axiom $(\bar{\beta} \Rightarrow)$, there are three cases to consider:

- (i) If $\bar{\beta} \subseteq \Lambda$. Then define $C = 1$. It is clear that $\Sigma = \emptyset$ and hence $\Sigma \Rightarrow 1$. Moreover, since we have $\Lambda = \bar{\beta}$, by the axiom and the rule $(1w)$ we will have $\Lambda, 1 \Rightarrow$.
- (ii) If $\bar{\beta} \subseteq \Sigma$, define $C = 0$. The reasoning is dual of the argument in (i).
- (iii) If none of the above happens, there are at least one element in $\bar{\beta} \cap \Sigma$ and $\bar{\beta} \cap \Lambda$. Define $C = \bigotimes \Sigma$. Then $\Sigma \Rightarrow C$ by $(R\otimes)$ and $\Lambda, C \Rightarrow$ holds by the axiom itself and $(L\otimes)$. For the variables, note that if $p \in V(C)$, then p is clearly occurring in Σ. Moreover, since $\Sigma \cup \Lambda = \bar{\beta}$, we know that p is in one of the members in $\bar{\beta}$. Since there is at least one of $\bar{\beta}$'s in Λ and each pair of the elements of $\bar{\beta}$ have the same variables, $p \in V(\Lambda)$ which completes the proof.

(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$, there are three cases to consider:

- (i) If $\bar{\phi} \subseteq \Lambda$. Then define $C = \top$. It is clear that $\Sigma \Rightarrow \top$. Moreover, if we substitute $\{\top\} \cup \Lambda - \bar{\phi}$ for the left context in the original axiom, we have $\top, \Lambda \Rightarrow \Delta$.
- (ii) If $\bar{\phi} \subseteq \Sigma$, define $C = \bot$. The reasoning is similar to (i).
- (iii) If none of the above happens, there are at least one element in $\bar{\phi} \cap \Sigma$ and $\bar{\phi} \cap \Lambda$. Define $C = \bigotimes(\Sigma \cap \bar{\phi}) \otimes \top^n$ where n is the cardinal of $\Sigma - \Sigma \cap \bar{\phi}$. First we have $\Sigma \Rightarrow C$, simply because for any $\phi_i \in \Sigma \cap \bar{\phi}$, $\phi_i \Rightarrow \phi_i$ and for any $\psi \in \Sigma - \Sigma \cap \bar{\phi}$ we have $\psi \Rightarrow \top$, and at the end we use the rule $(R\otimes)$. Secondly, $\Lambda, C \Rightarrow \Delta$. The reason is that the part of $\bar{\phi}$ which is occurred in Σ (and now in C) together with the part of $\bar{\phi}$ in Λ completes $\bar{\phi}$. Therefore, the left hand-side of $\Lambda, C \Rightarrow \Delta$ contains $\bar{\phi}$ (after suitable partitioning and staring the parts) and hence, the sequent is the consequence of an instance of the axiom and it is valid. Finally, for the variables, note that if $p \in V(C)$ then p is clearly occurring in Σ. Moreover, p is in one of the members in $\bar{\phi}$. Since there is at least one of $\bar{\phi}$'s in Λ and each pair of the elements of $\bar{\phi}$ have the same variables, $p \in V(\Lambda)$ which completes the proof.

(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C = \top$. Note that $\Sigma \Rightarrow \top$ is valid on the one hand and $C, \Lambda \Rightarrow \bar{\phi}, \Delta$ on the other. The latter is an instance of the axiom itself and hence valid.

It is easy to check that in each case the length of C is bounded by the length of the sequent itself. For instance in case (4)(iii), the length of $\bigotimes(\Sigma \cap \bar{\phi}) \otimes \top^n$ is bounded by the length of Σ which is bounded by the length of $(\Gamma, \bar{\phi})$ in the sequent $\Gamma, \bar{\phi} \Rightarrow \Delta$. Hence C is polynomially bounded. Proving (ii) is similar. □

2.4.1 The Single-conclusion Case

Now we are ready to prove that semi-analytic rules respect the interpolation property. First, we will consider the single-conclusion case. More precisely:

Theorem 2.4.23. *Let G and H be two single-conclusion sequent calculi such that H is an axiomatic extension of G with single-conclusion semi-analytic rules or $S4$ rules and H extends $\mathbf{FL_e}^-$. Then if G has H-interpolation (H-feasible* *interpolation), so does H. Moreover, for any rule $\mathcal{R}$ from the following set of rules, there exists a corresponding set of rules $S_{\mathcal{R}}$, presented below, such that if we add the rule $\mathcal{R}$ to H and if we know that all the rules in $S_{\mathcal{R}}$ are admissible in H, then the same claim holds.*

- (i) *If $\mathcal{R}$ is the modal rule 4 or $4D$, then $S_{\mathcal{R}}$ consists of the left weakening rule for boxed formulas.*

- (ii) *If $\mathcal{R}$ is a context-sharing semi-analytic rule then $S_{\mathcal{R}}$ consists of the left weakening, right weakening and left context-sharing implication rules.*

Proof. First we prove the interpolation property and then we will investigate the feasibility case. The proof uses induction on the H-length of π (note that by the H-length we mean counting just the new rules that H adds to the provable sequents in G that H considers as axioms). For the zero H-length, the proof is in G and the existence of the interpolation is proved by the assumption. For the rest, we will consider the last rule used in the proof and there are several cases to investigate.

First we will prove (i).

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ, is in Λ in the Definition 2.4.20 (or informally, ϕ appears in the same sequent as Δ appears). Hence, the sequent S is of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \Pi'\}$ and $\Lambda = \{\Gamma'', \Pi'', \phi\}$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have

$$\Pi'_j \Rightarrow C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rules $(R\wedge)$ and $(L\wedge)$ we have

$$\Pi'_j \Rightarrow \bigwedge_s C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

For the left sequents, using the rule $(R\otimes)$ we have

$$\Pi', \Gamma' \Rightarrow (\bigotimes_j \bigwedge_s C_{js}) \otimes (\bigotimes_i \bigwedge_r D_{ir})$$

And if we substitute the right sequents in the original rule and using the rule $(L\otimes)$, we conclude

$$\Pi'', \Gamma'', (\bigotimes_j \bigwedge_s C_{js}) \otimes (\bigotimes_i \bigwedge_r D_{ir}), \phi \Rightarrow \Delta$$

Therefore, we let C be $(\bigotimes_j \bigwedge_s C_{js}) \otimes (\bigotimes_i \bigwedge_r D_{ir})$ and we have proved $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is in one of C_{js} or D_{ir}. If it is in C_{js}, by induction hypothesis, it is either in Π'_j (which means it is in Σ), or it is in $\{\Pi''_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$. If it is in Π''_j, then it is in Λ and if it is in either $\bar{\psi}_{js}$ or $\bar{\theta}_{js}$, since the rule is occurence preserving, it also appears in ϕ which means it appears in Λ.
If the atom is in D_{ir}, we reason in the similar way, and it either appears in Γ'_i (and hence in Σ) or it appears in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$ and hence in $\Lambda \cup \Delta$.

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ, is this time in Σ in the Definition 2.4.20. Hence, the sequent S is again of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \Pi', \phi \Rightarrow C)$ and $(\Gamma'', \Pi'', C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \Pi', \phi\}$ and $\Lambda = \{\Gamma'', \Pi''\}$. W.l.o.g. suppose that for $i \neq 1$ we have $\Delta_i = \emptyset$ and $\Delta_1 = \Delta$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have (for $i \neq 1$)

$$\Pi'_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Pi'_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow \bigwedge_s C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

If we substitute the left sequents in the original rule, we get (for $i \neq 1$)

$$\Pi', \Gamma', \bigwedge_s C_{js}, \bigwedge_r D_{ir}, \phi \Rightarrow \bigvee_r D_{1r}$$

First, using the rule $(L\otimes)$ and then $(R \rightarrow)$ we get

$$\Pi', \Gamma', \phi \Rightarrow (\bigotimes_{i \neq 1} \bigwedge_r D_{ir}) \otimes (\bigotimes_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r}$$

On the other hand, using the rules $(R\otimes)$ and $(L \rightarrow)$ for the right sequents we have

$$\Pi'', \Gamma'', (\bigotimes_{i \neq 1} \bigwedge_r D_{ir}) \otimes (\bigotimes_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r} \Rightarrow \Delta$$

It is enough to take C as $(\bigotimes_{i \neq 1} \bigwedge_r D_{ir}) \otimes (\bigotimes_j \bigwedge_s C_{js}) \rightarrow \bigvee_r D_{1r}$ to finish the proof of this case.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{js} or D_{ir} for $(i \neq 1)$ or in D_{1r}. By induction hypothesis if it is in C_{js}, it is both in $\{\Pi'_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$ and in Π''_j. If it is in D_{ir} for $(i \neq 1)$, then it is both in $\{\Gamma'_i, \bar{\phi}_{ir}\}$ and in Γ''_i. And if it is in D_{1r}, then it is both in $\{\Gamma'_1, \bar{\phi}_{1r}\}$ and in $\{\Gamma''_1, \Delta\}$. One can easily check that therefore, the atom will be both in $\Sigma = \{\Gamma', \Pi', \phi\}$ and in $\Lambda \cup \Delta = \{\Gamma'', \Pi'', \Delta\}$. Note that in the reasoning we will need the occurrence preserving property, as well.

- Consider the case where the last rule used in the proof is a right semi-analytic rule. Hence, the sequent S is of the form $(\Gamma', \Gamma'' \Rightarrow \phi)$ and we have to find a formula C that satisfies $(\Gamma'' \Rightarrow C)$ and $(\Gamma', C \Rightarrow \phi)$, where $\Sigma = \Gamma''$, $\Lambda = \Gamma'$ and $\Delta = \phi$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma', \Gamma'' \Rightarrow \phi}$$

Using the induction hypothesis we get

$$\Gamma'_i, C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow C_{ir}$$

Using the rules $(L\wedge)$ and $(R\wedge)$ we have

$$\Gamma'_i, \bigwedge_r C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r C_{ir}$$

Substituting the left sequent in the original rule and then using the rule $(L\otimes)$, we conclude

$$\Gamma', \bigotimes_i (\bigwedge_r C_{ir}) \Rightarrow \phi.$$

On the other hand, using the rule $(R\otimes)$ for the sequents $\Gamma''_i \Rightarrow \bigwedge_r C_{ir}$, we get $\Gamma'' \Rightarrow \bigotimes_i(\bigwedge_r C_{ir})$ which means that the sequent $\bigotimes_i(\bigwedge_r C_{ir})$ serves as the formula C.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{ir}. Then by induction hypothesis it is both in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}\}$ and in Γ''_i. It is easy to check that it meets the conditions needed.

- And finally, consider the case where the last rule used in the proof is a modal rule. We will investigate K and D together first, and second 4 and $4D$ together, and at last, we will investigate the rule $RS4$.

  Consider the case where the last rule used in the proof is either K or D. Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta$, where $||\Delta|| \leq 1$ and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\Delta$. Therefore, we must have had the following instance of the rule

  $$\frac{\Gamma', \Gamma'' \Rightarrow \Delta}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta}$$

  Using the induction hypothesis there exists D such that

  $$\Gamma' \Rightarrow D \quad , \quad \Gamma'', D \Rightarrow \Delta$$

  Then, using the rule K for both of them (or if $\Delta = \emptyset$, use the rule D for $(\Gamma'', D \Rightarrow)$), we get

  $$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\Delta$$

  Let $\Box D$ be the formula C and we are done. And since $V(D) \subseteq V(\Gamma') \cap V(\Gamma'' \cup \Delta)$ we have $V(C) \subseteq V(\Box\Gamma') \cap V(\Box\Gamma'' \cup \Box\Delta)$, because the set of atoms of $\Box\Pi$ for a multiset Π is the same as atoms in Π.

  Now, consider the case that the last rule used in the proof is 4. Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, we must have had the following instance of the rule

  $$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

  Using the induction hypothesis there exists D such that

  $$\Gamma', \Box\Gamma' \Rightarrow D \quad , \quad \Gamma'', \Box\Gamma'', D \Rightarrow \phi$$

  If we use the rule 4 on the left sequent and using the left weakening rule on the right sequent (adding $\Box D$ to the left hand side of the sequent) and then using the rule 4, we get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

If we take $C = \Box D$, then the claim follows. Checking the atoms is similar as before.
For the proof of the case $4D$ is identical to the proof of the rule 4, if we ignore ϕ and $\Box\phi$ everywhere.

If the last rule used in the proof is the rule $RS4$, then the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, we must have had the following instance of the rule

$$\frac{\Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis there exists D such that

$$\Box\Gamma' \Rightarrow D \quad , \quad \Box\Gamma'', D \Rightarrow \phi$$

On the left sequent, use the rule $RS4$. On the right sequent, use the rule $LS4$ (since the rule $LS4$ is present in the system, whenever we have $RS4$) and then use the rule $RS4$. We get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

It is easy to see that $C = \Box D$ works in this case.

Now, we will prove part (ii). We have discussed the cases of left and right semi-analytic and modal rules in the previous part. It only remains to investigate the case of context-sharing semi-analytic rules.

- Consider the case where the last rule used in the proof is a context-sharing semi-analytic rule and the main formula, ϕ, is in Λ in the Definition 2.4.20 (or informally, ϕ appears in the same sequent as Δ appears). Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma' \Rightarrow C)$ and $(\Gamma'', \phi, C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma'\}$ and $\Lambda = \{\Gamma'', \phi\}$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have

$$\Gamma'_i \Rightarrow C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rules $(R\wedge)$ and $(L\wedge)$ we have

$$\Gamma'_i \Rightarrow \bigwedge_s C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

We want to the make the contexts of the above sequents in the right the same, so that we can use them in the original rule. Therefore, using the rule $(L\wedge)$ we have

$$\Gamma''_i, \bar{\psi}_{is}, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \Rightarrow \Delta_i$$

Now, we can substitute them in the original rule and conclude

$$\Gamma'', \langle (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \rangle_i, \phi \Rightarrow \Delta$$

And using the rule $(L\otimes)$ we get

$$\Gamma'', \bigotimes_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})], \phi \Rightarrow \Delta$$

On the other hand, considering the sequents $(\Gamma'_i \Rightarrow \bigwedge_s C_{is})$ and $(\Gamma'_i \Rightarrow \bigwedge_r D_{ir})$ and using the rule $(R\wedge)$ for every i, we get

$$\Gamma'_i \Rightarrow (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})$$

and then using the rule $(R\otimes)$ we have

$$\Gamma' \Rightarrow \bigotimes_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})]$$

and we can see that $\bigotimes_i [(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})]$ serves as C.
To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir}. By induction hypothesis, if it is in C_{is}, then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and if it is in D_{ir}, then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$. It is easy to check that it meets the conditions.

- Consider the case where the last rule used in the proof is a context-sharing semi-analytic rule and the main formula, ϕ, is this time in Σ in the Definition 2.4.20. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \phi \Rightarrow C)$ and $(\Gamma'', C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \phi\}$ and $\Lambda = \{\Gamma''\}$. W.l.o.g. suppose that for $i \neq 1$ we have $\Delta_i = \emptyset$ and $\Delta_1 = \Delta$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \qquad \langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \qquad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have (for $i \neq 1$)

$$\Gamma'_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Gamma'_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_s C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

$$\Gamma'_1, \bar{\psi}_{1s}, \bigwedge_s C_{1s} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$$

Now, we want to make the contexts of the sequents in the left the same, so that we can use them in the original rule. For $(i \neq 1)$ use the rule $(L\wedge)$ to make the context $\{\Gamma'_i, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})\}$ and for $(i = 1)$ use the left weakening rule to make the context $\{\Gamma'_1, \bigwedge_s C_{1s}\}$. If we substitute the updated left sequents in the original rule, we get (for $i \neq 1$)

$$\Gamma', \langle(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})\rangle_{i\neq 1}, \bigwedge_s C_{1s}, \phi \Rightarrow \bigvee_r D_{1r}$$

First, using the rule $(L\otimes)$ and then $(R \rightarrow)$ we get

$$\Gamma', \phi \Rightarrow (\bigotimes_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] \otimes \bigwedge_s C_{1s}) \rightarrow \bigvee_r D_{1r}.$$

On the other hand, using the rule $(R\wedge)$ for every $(i \neq 1)$ we have $\Gamma''_i \Rightarrow (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})$. Together with the sequent $\Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$, and using the rule $(R\otimes)$ we get

$$\Gamma'' \Rightarrow (\bigotimes_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] \otimes \bigwedge_s C_{1s}).$$

We have $\Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$. Use the left weakening rule to get $\Gamma'', \bigvee_r D_{1r} \Rightarrow \Delta$. Now, we can use the rule left sharing implication to get

$$\Gamma'', (\bigotimes_{i\neq 1}[(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] \otimes \bigwedge_s C_{1s}) \rightarrow \bigvee_r D_{1r} \Rightarrow \Delta.$$

We can see that $(\bigotimes_{i \neq 1} [(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] \otimes \bigwedge_s C_{1s}) \to \bigvee_r D_{1r}$ serves as C and we are done.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir}. By induction hypothesis, if it is in C_{is}, then it is both in $\{\Gamma'_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and in Γ''_i and if it is in D_{ir} for $(i \neq 1)$, then it is both in $\Gamma'_i, \bar{\phi}_{ir}$, and in $\{\Gamma''_i\}$. If it is in D_{1r}, then it is both in $\Gamma'_1, \bar{\phi}_{1r}$, and in $\{\Gamma''_1, \Delta\}$. It is easy to check that it meets the conditions.

It is easy to check that in both cases of (i) and (ii), if G has H-feasible interpolation, then so does H. By the assumption, we know that there exists a number m (which only depends on the proof system G) such that $|C| \leq |\pi|^m$. Now for the proofs in H we will claim that our previously constructed interpolant C has the property $|C| \leq |\pi|^M$ where $M = max\{m, 2\}$ and we will prove it by induction on the H-length of π.

If the H-length of the proof is 0, then there is no new rule of H in the proof π, and since G has H-feasible interpolation, by Definition $|C| \leq |\pi|^m$ and hence $|C| \leq |\pi|^M$. For the rest, note that in each of the above cases, the number of the formulas which appear in C (we have denoted them by C_{js} and D_{ir}) is equal to the number of premises of the last rule used in the proof. The rest of the symbols appeared in C are connectives, and the number of them is less than or equal to $N_{\mathcal{R}}$, where $N_{\mathcal{R}}$ is the number of the premises of the rule $\mathcal{R}$, which is the last rule used in the proof. Since the sequent S is the conclusion of a rule in H, the H-lengths of the proofs of its premises are less than the H-length of π and we can use the induction hypothesis for them. Then $|C| \leq \Sigma_{j,s}|C_{js}| + \Sigma_{i,r}|D_{ir}| + N_{\mathcal{R}}$. By induction hypothesis we have $|C_{js}| \leq |\pi_{js}|^M$ and $|D_{ir}| \leq |\pi_{ir}|^M$, where π_{js} (or π_{ir}) is the proof of the sequent whose interpolant is C_{js} (or D_{ir}). But since the proof is tree-like, we have $\Sigma_{j,s}|\pi_{j,s}| + \Sigma_{i,r}|\pi_{i,r}| + 1 \leq |\pi|$. It is easy to see that $|C| \leq \Sigma_{j,s}|\pi_{j,s}|^M + \Sigma_{i,r}|\pi_{i,r}|^M + N_{\mathcal{R}} \leq (\Sigma_{j,s}|\pi_{j,s}| + \Sigma_{i,r}|\pi_{i,r}| + 1)^M \leq |\pi|^M$, and the claim follows. The last inequality uses the fact $M \geq 2$ and

$$N_{\mathcal{R}} \leq \Sigma_{j,s}|\pi_{j,s}| + \Sigma_{i,r}|\pi_{i,r}|$$

The latter is an easy consequence of the fact that the number of $\pi_{j,s}$ and $\pi_{i,r}$ in total is $N_{\mathcal{R}}$. □

2.4.2 The Multi-conclusion Case

In this subsection we will generalize the Theorem 2.4.23 to also cover the multi-conclusion case.

Theorem 2.4.24. *Let G and H be two multi-conclusion sequent calculi such that H extends* $\mathbf{CFL_e}^-$. *Suppose H is an axiomatic extension of G with multi-conclusion semi-analytic rules or $S4$ rules and if we also add the modal rule 4 or $4D$ in H, the left weakening rule for boxed formulas is admissible in H. Then if G has strong H-interpolation (strong H-feasible interpolation), so does H.*

Proof. The proof is similar to the proof of Theorem 2.4.23 and again it uses induction on the H-length of π. For the zero H-length, the proof is in G and the existence of the interpolation is proved by the assumption. For the rest, we will consider the last rule used in the proof and there are several cases to investigate. Throughout the proof we use the convention $A = A_1, \cdots, A_k$ for different sequents A and different numbers k, for simplicity.

- Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ, is in Λ in the Definition 2.4.20. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma' \Rightarrow C, \Delta')$ and $(\Gamma'', \phi, C \Rightarrow \Delta'')$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r\rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''}$$

Using the induction hypothesis for the premises we have for every i and r

$$\Gamma'_i \Rightarrow C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i$$

Using the rule $(R\wedge)$ and $(L\wedge)$ we have for every i

$$\Gamma'_i \Rightarrow \bigwedge_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i$$

Using the rule $(R\otimes)$ for the left sequents we get

$$\Gamma' \Rightarrow \bigotimes_i \bigwedge_r C_{ir}, \Delta'$$

and, if we substitute the right sequents in the original rule, and then using the rule $(L\otimes)$, we get

$$\Gamma'', \phi, \bigotimes_i \bigwedge_r C_{ir} \Rightarrow \Delta''$$

Hence, we take C as $\bigotimes_i \bigwedge_r C_{ir}$ and we are done.

To check $V(C) \subseteq V(\Gamma' \cup \Delta') \cap V(\{\Gamma'' \cup \{\phi\}\} \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir}'s. Then, by induction hypothesis, it is in $(\Gamma'_i \cup \Delta'_i)$ and in $\{\Gamma''_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta''_i\}$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ.

- Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ, is in Σ in the Definition 2.4.20. Hence, the sequent S is again of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma', \phi \Rightarrow C, \Delta')$ and $(\Gamma'', C \Rightarrow \Delta'')$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r\rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''}$$

Using the induction hypothesis for the premises we have for every i and r

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, C_{ir} \Rightarrow \Delta''_i$$

Using the rules $(R\vee)$ and $(L\vee)$, we have for every i

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bigvee_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bigvee_r C_{ir} \Rightarrow \Delta''_i$$

If we substitute the left sequents in the original rule, we get

$$\Gamma', \phi \Rightarrow \bigvee_r C_{ir}, \Delta'$$

and, using the rule $(R\oplus)$ we get

$$\Gamma', \phi \Rightarrow \bigoplus_i \bigvee_r C_{ir}, \Delta'$$

On the other hand, using the rule $(L\oplus)$ for the right sequents we have

$$\Gamma'', \bigoplus_i \bigvee_r C_{ir} \Rightarrow \Delta''$$

It is enough to take C as $\bigoplus\limits_i \bigvee\limits_r C_{ir}$ to finish the proof of this case.

To check $V(C) \subseteq V(\{\Gamma' \cup \{\phi\}\} \cup \Delta') \cap V(\Gamma'' \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir}'s. Then, by induction hypothesis, it is in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta'_i\}$ and in $(\Gamma''_i \cup \Delta''_i)$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ.

- Consider the case where the last rule used in the proof is a modal multi-conclusion one. The case where it is the rule D or $4D$ is similar to the proof of the same cases in the Theorem 2.4.23. Let the last rule used in the proof be the rule K. Then, S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, there can be two cases based on the partition of the right side of the sequent. In the first one, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C$ and $\Box\Gamma'', C \Rightarrow \Box\phi$ hold. In the second one, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold. Since the proof of the first case is similar to the proof in Theorem 2.4.23, we will investigate the second case. Hence, we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists D such that we have

$$\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'' \Rightarrow$$

Using the rule $(L \to)$ together with the axiom $(\Rightarrow 0)$ on the one hand and on the other, using the rule $(0w)$ and $(R \to)$ we have

$$\Gamma', \neg D \Rightarrow, \phi \quad , \quad \Gamma'' \Rightarrow \neg D$$

Use the rule K to derive

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D$$

And we can derive

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

which means we have to take $C = \neg\Box\neg D$. The atom check is easy.

Now, consider the case where the last rule used in the proof is the rule 4. Then S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$ and there are the exact two cases as above, in the case of the rule K, and again since the second case is new (the proof of the other one is similar to the proof in Theorem 2.4.23), we will investigate that one. Hence, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold. Therefore we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists D such that we have

$$\Gamma', \Box\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'', \Box\Gamma'' \Rightarrow$$

Using the rule $(L \to)$ together with the axiom $(\Rightarrow 0)$ on the one hand and on the other, using the rule $(0w)$ and $(R \to)$ we have

$$\Gamma', \Box\Gamma', \neg D \Rightarrow \phi \quad , \quad \Gamma'', \Box\Gamma'' \Rightarrow \neg D$$

Use the left weakening rule for the left sequent (to add $\Box\neg D$ to the left side of the sequent) and then apply the rule 4 to get

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D$$

And we can derive

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

If we take $C = \neg\Box\neg D$, we are done. And it is easy to check the condition for atoms.
In the case of the rule $(RS4)$, we have exactly the same cases as in the rule K:

$$\Box\Gamma' \Rightarrow C \quad , \quad \Box\Gamma'', C \Rightarrow \Box\phi$$

and

$$\Box\Gamma' \Rightarrow C, \Box\phi \quad , \quad \Box\Gamma'', C \Rightarrow$$

Only the second case is new (the proof for the first one is the same as the proof of the same case in theorem 2.4.23). The proof of the second case is the same as the case for the rule K in the above, and $C = \neg\Box\neg D$ works here, as well.

The cases where the last rule in the proof is a right multi-conclusion semi-analytic one is similar and we do not investigate them here. The proof for the feasibility part is easy and similar to the proof in the Theorem 2.4.23. □

Therefore, combining the Theorems 2.4.22, 2.4.23 and 2.4.24 we will have:

Theorem 2.4.25. (*i*) *For any calculus H which is an* $\mathbf{FL_e}$*-extension (*$\mathbf{FL_e}^-$*-extension) single-conclusion calculus consisting of semi-analytic rules and focused axioms (context-free focused axioms), H has H-feasible interpolation.*

(*ii*) *For any* **IPC***-extension single-conclusion calculus H consisting of semi-analytic rules, context-sharing semi-analytic rules and focused axioms, H has H-feasible interpolation.*

(*iii*) *For any* $\mathbf{CFL_e}$*-extension (*$\mathbf{CFL_e}^-$*-extension) multi-conclusion H consisting of semi-analytic rules and focused axioms (context-free focused axioms), H has H-feasible interpolation.*

Combining the Theorem 2.4.25, Theorem 2.2.9 and Theorem 2.4.21, we have:

Corollary 2.4.26. (*i*) *If* $\mathbf{FL_e} \subseteq L$*, (*$\mathbf{FL_e}^- \subseteq L$*) and L has a single-conclusion calculus consisting of semi-analytic rules and focused axioms (context-free focused axioms), then L has Craig interpolation.*

(*ii*) *If* $\mathbf{IPC} \subseteq L$ *and L has a single-conclusion sequent calculus consisting of semi-analytic rules, context-sharing semi-analytic rules and focused axioms, then L has Craig interpolation.*

(*iii*) *If* $\mathbf{CFL_e} \subseteq L$*, (*$\mathbf{CFL_e}^- \subseteq L$*) and L has a multi-conclusion sequent calculus consisting of semi-analytic rules and focused axioms (context-free focused axioms), then L has Craig interpolation.*

The following are the applications of the main corollary of this section i.e., Corollary 2.4.26. To begin, let us consider the positive application:

Corollary 2.4.27. *The logics* $\mathbf{FL_e}$*,* $\mathbf{FL_{ec}}$*,* $\mathbf{FL_{ew}}$*,* $\mathbf{CFL_e}$*,* $\mathbf{CFL_{ew}}$*,* $\mathbf{CFL_{ec}}$*,* **ILL***,* **CLL***,* **IPC***,* **CPC** *and their* **K***,* **KD** *and* **S4** *versions have the Craig interpolation property. The same also goes for* **K4** *and* **K4D** *extensions of* **IPC** *and* **CPC***.*

Proof. Note that the usual cut-free sequent calculus for all of these logics consists of semi-analytic rules and focused axioms. Therefore, by the Corollary 2.4.26 we can prove the Craig interpolation property for all of them. □

For the negative applications, we use the results in [16], [32] and [43] to ensure that the following logics do not have Craig interpolation. Then we will use the Corollary 2.4.26 to prove that these logics do not have a semi-analytic calculus consisting only of the focused axioms and semi-analytic rules.

Corollary 2.4.28. *None of the logics R, UL^-, IUL^-, MTL, $SMTL$, $IMTL$, BL, $Ł_\infty$, $Ł_n$ for $n \geq 3$, P, CHL and A have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free focused axioms.*

Corollary 2.4.29. *None of the logics IUL^-, $IMTL$, $Ł_\infty$, $Ł_n$ for $n \geq 3$ and A have a multi-conclusion sequent calculus consisting only of multi-conclusion semi-analytic rules and context-free focused axioms.*

Corollary 2.4.30. *Except G, $G3$ and* **CPC***, none of the consistent logics which are BL-extensions have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free focused axioms.*

Corollary 2.4.31. *The only $IMTL$-extension with a calculus consisting of single-conclusion (multi-conclusion) semi-analytic rules and context-free focused axioms, is* **CPC***.*

Corollary 2.4.32. *Except RM^e, $IUML^-$,* **CPC***, RM^e_3 , RM^e_4 ,* **CPC***$\cap IUML^-$, $RM^e_4 \cap IUML^-$, and* **CPC** *$\cap\, RM^e_3$, none of the consistent extensions of RM^e have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free focused axioms. This category includes:*

(i) *RM^e_n for $n \geq 5$,*

(ii) *$RM^e_{2m} \cap RM^e_{2n+1}$ for $n \geq m \geq 1$ with $n \geq 2$.,*

(iii) *$RM^e_{2m} \cap IUML^-$ for $m \geq 3$.*

Corollary 2.4.33. *Except* **IPC***,* **LC***,* **KC***,* **Bd$_2$***,* **Sm***,* **GSc** *and* **CPC***, none of the consistent super-intuitionistic logics have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules, context-sharing semi-analytic rules and focused axioms.*

Corollary 2.4.34. *Except at most thirty seven logics, none of the consistent extensions of* **S4** *have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules, context-sharing semi-analytic rules, the modal rules K, D, 4, $4D$, $RS4$ and focused axioms.*

2.5 Uniform Interpolation

In this section we will generalize the investigations of [23] to also cover the substructural setting and semi-analytic rules. We will show that any extension of a sequent calculus by semi-analytic rules preserves uniform interpolation if the resulted system turns out to be terminating. Our method here is similar to the method used in [23].

As a first step, let us generalize the notion of uniform interpolation from logics to sequent calculi. The following definition offers three versions of such a generalization, each of which suitable for different forms of rules.

Definition 2.5.35. *Let G and H be two sequent calculi. G has H-uniform interpolation if for any sequent S and T where $T^s = \emptyset$ and any atom p, there exist p-free formulas $I(S)$ and $J(T)$ such that $V(I(S)) \subseteq V(S^a \cup S^s)$ and $V(J(T)) \subseteq V(T^a)$ and*

- *(i)* *$S \cdot (I(S) \Rightarrow)$ is derivable in H.*
- *(ii)* *For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $\Gamma \Rightarrow I(S)$ is derivable in H.*
- *(iii)* *$T \cdot (\Rightarrow J(T))$ is derivable in H.*
- *(iv)* *For any p-free multisets Γ and Δ, if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $J(T), \Gamma \Rightarrow \Delta$ is derivable in H.*

Similarly, we say G has weak H-uniform interpolation if instead of (ii) we have

- *(ii')* *For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in H where $\tilde{S} = (S^a \Rightarrow)$.*

We say G has strong H-uniform interpolation if instead of (ii) we have

- *(ii'')* *For any p-free multisets Γ and Δ, if $S \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $\Gamma \Rightarrow I(S), \Delta$ is derivable in H.*

Note that in the case of the strong uniform interpolation, T^s can be non-empty, and we have multi-conclusion rules.

We call $I(S)$ a left p-interpolant (weak p-interpolant, strong p-interpolant) of S and $J(T)$ a right p-interpolant (weak right p-interpolant, strong right p-interpolant) of T in G relative to H. The system H has unifrom interpolation property (weak unifrom interpolation property, strong unifrom interpolation property) if it has H-uniform interpolation (weak H-uniform interpolation, strong H-uniform interpolation).

Theorem 2.5.36. *If G is a sequent calculus with (weak/strong) uniform interpolation and complete for a logic L extending ($\mathbf{FL_e}$/$\mathbf{CFL_e}$) $\mathbf{FL_e}$, L has the uniform interpolation property.*

Proof. First note that since G is complete for L, $L \vdash \phi \rightarrow \psi$ iff $G \vdash \phi \Rightarrow \psi$. Hence we can rewrite the definition of the uniform interpolation using the sequent system G. Now pick $S = (\Rightarrow A)$. By uniform interpolation property of G, there is a p-free formula $I(S)$ such that $S \cdot (I(S) \Rightarrow)$ and for any p-free Σ if $S \cdot (\Sigma \Rightarrow)$, then $\Sigma \Rightarrow I(S)$. It is clear that $I(S)$ works as the p-pre-interpolant of A, because firstly $I(S) \Rightarrow A$ and secondly if $B \Rightarrow A$ then $B \Rightarrow I(S)$ for any p-free B. The same argument also works for the p-post-interpolant. In the case of weak uniform interpolation, first note that by definition if $T = (\Rightarrow)$ then $(\Rightarrow J(T))$. Secondly, note that since G is complete for L, the calculus should admit the cut rule by Theorem 2.2.9. Now we claim that $I(S)$ works again. The reason now is that if $B \Rightarrow A$ for a p-free B, then $J(\tilde{S}), B \Rightarrow I(S)$. Since $\tilde{S} = T$ and we have the cut rule, $B \Rightarrow I(S)$. The case for strong uniform interpolation is similar to the interpolation case. $\square$

In the following theorem, we will check the uniform interpolation property for a set of focused axioms. It can also be considered as an example to show how this notion works in practice.

Theorem 2.5.37. *Let G and H be two sequent calculi such that every provable sequent in G is also provable in H and G consists only of finite focused axioms. Then:*

- (i) *If G and H are single-conclusion and H extends $\mathbf{FL_e}$, then G has H-uniform interpolation.*
- (ii) *If G and H are single-conclusion and H extends $\mathbf{FL_e}$ and has the left weakening rule, then G has weak H-uniform interpolation.*
- (iii) *If G and H are multi-conclusion and H extends $\mathbf{CFL_e}$, then G has strong H-uniform interpolation.*

Proof. To prove part (i) of the theorem, we have to find $I(S)$ and $J(T)$ for given sequents $S = (\Sigma \Rightarrow \Lambda)$ and $T = (\Pi \Rightarrow)$ such that the four conditions in the Definition 2.5.35 hold. We will denote our $I(S)$ and $J(T)$ by $\forall pS$ and $\exists pT$, respectively.

First, we will prove (i) and we will investigate the case $\exists pT$, first. For that purpose, define $\exists pT$ as the following

$$[(\bigotimes \Pi_p) \otimes \top] \wedge 0 \wedge \bot$$

where Π_p is the subset of Π consisting of all p-free formulas and by $\bigotimes \Pi_p$ we mean $\phi_1 \otimes \cdots \otimes \phi_k$, where $\{\phi_1, \cdots, \phi_k\} = \Pi_p$. Note that $\top$ appears in the first conjunct only when $\Pi - \Pi_p$ is non-empty. Moreover, 0 only appears as a conjunct when T is of the form axiom 3 (which is $\bar{\beta} \Rightarrow$) and $\bar{\beta} = \Pi$, and $\bot$ only appears as a conjunction when T is of the form of axiom 4 (which is $\Sigma, \bar{\phi} \Rightarrow \Lambda$) and we have $\bar{\phi} \subseteq \Pi$.

First, we have to show that $\Pi \Rightarrow \exists pT$ holds in H. Note that Π is of the form $\Pi_p \cup (\Pi - \Pi_p)$. By definition, for every $\psi \in \Pi_p$ we have $\psi \Rightarrow \psi$ and hence using the rule $(R\otimes)$ we have $\Pi_p \Rightarrow \bigotimes \Pi_p$ holds in H (note that since H extends $\mathbf{FL_e}$, it has the rule $(R\otimes)$). On the other hand, using the axiom for $\top$ we have $\Pi - \Pi_p \Rightarrow \top$ and then using the rule $(R\otimes)$ we have $\Pi_p, \Pi - \Pi_p \Rightarrow (\bigotimes \Pi_p) \otimes \top$, which is $\Pi \Rightarrow (\bigotimes \Pi_p) \otimes \top$.

The formula 0 appears as a conjunct when T is of the form axiom 3 and $\bar{\beta} = \Pi$, which means that in this case $\Pi \Rightarrow$ is an instance of axiom 3 and it holds in H. Hence, using the rule $(0w)$ we have $\Pi \Rightarrow 0$.

The formula $\bot$ appears as a conjunct when T is of the form axiom 4 and $\bar{\phi} \subseteq \Pi$. Hence, $\Pi \Rightarrow \bot$ is an instance of axiom 4 when we let Δ to be $\bot$.

Now, we have to show that if for p-free sequents $\bar{C}$ and $\bar{D}$ if $\Pi, \bar{C} \Rightarrow \bar{D}$ is provable in G, then $\exists pT, \bar{C} \Rightarrow \bar{D}$ is provable in H. Therefore, $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of one of the focused axioms and we have five cases to consider:

(1) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\phi \Rightarrow \phi$. Then, since $\bar{D} = \phi$, it means that ϕ is p-free. There are two cases; first, if $\Pi = \phi$ and $\bar{C} = \emptyset$, then $\bigotimes \Pi_p = \phi$ and since $\Pi - \Pi_p = \emptyset$, we do not have $\top$ in the conjunct. Hence, $\Pi \Rightarrow \phi$ and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow \bar{D}$. Second, if $\Pi = \emptyset$ and $\bar{C} = \phi$, then $\bigotimes \Pi_p = 1$ and since $\Pi - \Pi_p = \emptyset$, then $\top$ does not appear in the first conjunct in the definition of $\exists pT$. Hence, since $\bar{C} \Rightarrow \bar{D}$ is equal to $\phi \Rightarrow \phi$ and this is of the form of the axiom 1, using the rule $(1w)$ we have $1, \phi \Rightarrow \phi$ and using $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$.

(2) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Rightarrow \bar{\alpha}$. Then, since $\bar{D} = \bar{\alpha}$, it means that $\bar{\alpha}$ is p-free and $\Pi = \bar{C} = \emptyset$. Hence, like the above case $\bigotimes \Pi_p = 1$ and we do not have $\top$ in the definition, either. Again, using the rule $(1w)$ we have $1 \Rightarrow \bar{\alpha}$ and by $(L\wedge)$ we have $\exists pT \Rightarrow \bar{\alpha}$.

(3) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $(\bar{\beta} \Rightarrow)$. Then there are two cases; first if $\bar{\beta} = \Pi$, then we must have 0 as one of the conjuncts in the definition of $\exists pT$. We have $\bar{C} = \bar{D} = \emptyset$ and $0 \Rightarrow$ is an axiom in H and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow$. Second, if $\Pi \subsetneq \bar{\beta}$, since we have $\bar{\beta} = \Pi, \bar{C}$ and $\bar{C}$ is p-free, and we have this condition that for any two formulas in $\bar{\beta}$ they have the same variables, we have Π is p-free, as well, which means every formula in Π is p-free and $\Pi = \Pi_p$ and $\top$ does not appear in the definition of $\exists pT$. Hence, using the rule $(L\otimes)$ on $\Pi, \bar{C} \Rightarrow$, we have $\bigotimes \Pi_p, \bar{C} \Rightarrow$ and by the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow$.

(4) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$, then there are two cases; first if $\bar{\phi} \subseteq \Pi$, then by definition of $\exists pT$, $\bot$ is one of the conjuncts. Therefore, since $\bot, \bar{C} \Rightarrow \bar{D}$ is an instance of an axiom in H, using the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is derivable in H. Second, if $\bar{\phi} \nsubseteq \Pi$, then at least one of the elements in $\bar{\phi}$ is in $\bar{C}$ and hence it is p-free. Therefore, by the condition that for any two formulas in $\bar{\phi}$ they have the same variables, $\bar{\phi}$ is p-free. Hence, there can not be any element of $\bar{\phi}$ present in $\Pi - \Pi_p$ and hence $\bar{\phi} \subseteq \Pi_p, \bar{C}$ and then $\bar{\phi} \subseteq \Pi_p, \bar{C}, \top$. Therefore, we have $\Pi_p, \bar{C} \Rightarrow \bar{D}$ because it is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$ of G and hence it is provable in H. Therefore, using the axiom $(L\otimes)$ we have $(\bigotimes \Pi_p) \otimes \top, \bar{C} \Rightarrow \bar{D}$ and by $(L\wedge)$, $\exists pT, \bar{C} \Rightarrow \bar{D}$. (Note that it is possible that $\Pi - \Pi_p$ is empty. It is easy to show that in this case the claim also holds. It is enough to drop $\top$ in the last part of the proof.)

(5) Consider the case where $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$. Then, since $\bar{\phi} \subseteq \bar{D}$, we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is an instance of the same axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$ when we substitute Γ by $\exists pT, \bar{C}$.

Now, we will investigate the case $\forall pS$ for S of the form $\Sigma \Rightarrow \Lambda$. Define $\forall pS$ as the following

$$[(\bigotimes \Sigma_p \to \bot)] \vee [\bigotimes(\bar{\beta} - \Sigma)] \vee \phi \vee 1 \vee \top$$

where in the first disjunct, Σ_p means the p-free part of Σ, the second disjunct appears whenever there exists an instance of an axiom of the form (3) in G where $\Sigma \subseteq \bar{\beta}$, $\Lambda = \emptyset$ and $\bar{\beta}$ is p-free. The third disjunct appears if $\Sigma = \emptyset$ and $\Lambda = \phi$ where ϕ is p-free. The fourth disjunct appears if $\Sigma \Rightarrow \Lambda$ equals to one of the instances of the axiom (1), (2), or (3) in G. And finally, the fifth disjunct appears when $\bar{\phi} \subseteq \Sigma$ for an instance of $\bar{\phi}$ in axiom (4) in G or $\bar{\phi} \subseteq \Lambda$ for an instance of $\bar{\phi}$ in axiom (5) in G.

First we have to show that $\Sigma, \forall pS \Rightarrow \Lambda$. For this purpose, we have to prove that for any possible disjunct X, we have $\Sigma, X \Rightarrow \Lambda$. For the first disjunct note that $\Sigma_p \Rightarrow \bigotimes \Sigma_p$ and $\Sigma - \Sigma_p, \bot \Rightarrow \Lambda$. Hence, $\Sigma, (\bigotimes \Sigma_p \to \bot) \Rightarrow \Lambda$ using the rule $(\to L)$.
For the second disjucnt, we have $\Sigma \subseteq \bar{\beta}$ and $\Lambda = \emptyset$. Therefore

$$\Sigma, \bigotimes(\bar{\beta} - \Sigma) \Rightarrow \Lambda$$

by the axiom (3) itself. For the third disjunct, note that $\Sigma = \emptyset$ and $\Lambda = \phi$ where ϕ is p-free. Hence $\Sigma, \phi \Rightarrow \Lambda$ by axiom (1). For the fourth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an axiom itself and hence $\Sigma, 1 \Rightarrow \Lambda$. Finally, for the fifth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an instance of the axioms (4) or (5) which means if we also add $\top$ to the left hand-side of the sequent, it remains provable.

Now we have to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda$ in G then $\bar{C} \Rightarrow \forall pS$ in H. For this purpose, we will check all possible axiomatic forms for $\Sigma, \bar{C} \Rightarrow \Lambda$.

(1) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (1), there are two possible cases. First if $\Sigma = \emptyset$ and $\bar{C} = \phi$ and $\Lambda = \phi$. Then ϕ will be p-free and hence ϕ appears in $\forall pS$ as a disjunct. Since $\bar{C} \Rightarrow \phi$, we have $\bar{C} \Rightarrow \forall pS$. For the second case, if $\Sigma = \phi$ and $\bar{C} = \emptyset$ then $\Sigma \Rightarrow \Lambda$ is an instance of the axiom (1) which means that 1 is a disjunct in $\forall pS$. Since $(\Rightarrow 1)$ and $\bar{C} = \emptyset$ we have $\bar{C} \Rightarrow \forall pS$.

(2) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \emptyset$ and $\Lambda = \bar{\alpha}$. Therefore, 1 is a disjunct in $\forall pS$ and since $\bar{C} = \emptyset$ we have $\bar{C} \Rightarrow \forall pS$.

(3) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (3). Then there are two cases to consider. First if $\Sigma = \bar{\beta}$. Then $\bar{C} = \emptyset$ and $\Lambda = \emptyset$. By definition, 1 is a disjunct in $\forall pS$ and again like the previous cases $\bar{C} \Rightarrow \forall pS$. Second if $\Sigma \subsetneq \bar{\beta}$. Then $\bar{\beta} \cap \bar{C}$ is non-empty. Pick $\psi \in \bar{\beta} \cap \bar{C}$. ψ is p-free, since any pair of the elements in $\bar{\beta}$ have the same variables, $\bar{\beta}$ is p-free. Now by definition, $\bigotimes(\bar{\beta} - \Sigma)$ is a disjunct in $\forall pS$. Since $\bar{C} = \beta - \Sigma$, we have $\bar{C} \Rightarrow \forall pS$.

(4) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (4). Similar to the previous case, there are two cases. If $\bar{\phi} \subseteq \Sigma$, then by definition $\top$ is a disjunct in $\forall pS$ and there is nothing to prove. In the second case, at least one the elements of ϕ is in $\bar{C}$ and hence p-free. Since any pair of the elements in $\bar{\phi}$ have the same variables, $\bar{\phi}$ is p-free. We can partition $\Sigma, \bar{C}$ to $\Sigma_p, \bar{C}, (\Sigma - \Sigma_p)$. Since

every element of $(\Sigma - \Sigma_p)$ has p, and $\bar{\phi}$ is p-free, the whole ϕ should belong to $\Sigma_p, \bar{C}$. Therefore, by the axiom (4) itself, $\Sigma_p, \bar{C} \Rightarrow \bot$ which implies $\bar{C} \Rightarrow (\bigotimes \Sigma_p \to \bot)$. By definition $(\bigotimes \Sigma_p) \to \bot$ is a disjunct in $\forall pS$ and hence $\bar{C} \Rightarrow \forall pS$.

(5) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (5). Then $\bar{\phi} \subseteq \Lambda$. By definition $\top$ is a disjunct in $\forall pS$ and therefore, there is nothing to prove.

For (ii), note that using the part (i) we have formulas $\exists pT$ and $\forall pS$ for any sequents S and T ($T^s = \emptyset$) with the conditions of H-uniform interpolation. The conditions for the weak H-uniform interpolation is the same except for the second part of the left weak p-interpolant which demands that if $\Sigma, \bar{C} \Rightarrow \Lambda$, then $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$. If we use the same uniform interpolants, we satisfy all the conditions of weak H-uniform interpolation. The reason is that except the mentioned condition, all of the others are the same as the conditions for H-interpolation and for the other condition, we can argue as follows: By $\Sigma, \bar{C} \Rightarrow \Lambda$, we have $\bar{C} \Rightarrow \forall pS$ and by the left weakening rule we will have $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$.

For (iii), first note that proving the existence of the right interpolants is enough. It is sufficient to define $\forall pS = \neg \exists pS$ and using the assumption that $\mathbf{CFL_e}$ is admissible in H to reduce the conditions of $\forall pS$ to $\exists pS$. Now define $\exists pS$ for any $S = (\Sigma \Rightarrow \Lambda)$ as:

$$[(\bigotimes \Sigma_p) \otimes \top] \wedge [\neg(\bot \oplus (\bigoplus \Lambda_p))] \wedge 0 \wedge \bot$$

where by $\bigotimes \Sigma_p$ we mean $\psi_1 \otimes \cdots \otimes \psi_r$, where $\{\psi_1, \cdots, \psi_r\} = \Sigma_p$ and $\bigoplus \Lambda_p$ is defined similarly. Note that in $[(\bigotimes \Sigma_p) \otimes \top]$ the formula $\top$ appears iff $\Sigma \neq \Sigma_p$, and $\bot$ appears in the second conjunct iff $\Lambda \neq \Lambda_p$. The third conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (1), (2) and (3) in G and the fourth conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (4), (5) in G.

First, we have to show that $\Sigma \Rightarrow \exists pS, \Lambda$. For that purpose, we have to check that for any conjunct X we have $\Sigma \Rightarrow X, \Lambda$. For the first conjunct, if $\Sigma \neq \Sigma_p$ then note that $\Sigma_p \Rightarrow \bigotimes \Sigma_p$ and $\Sigma - \Sigma_p \Rightarrow \top, \Lambda$ therefore

$$\Sigma \Rightarrow [(\bigotimes \Sigma_p) \otimes \top], \Lambda$$

If $\Sigma = \Sigma_p$, then there is no need for $\top$ and the claim is clear by $\Sigma \Rightarrow \bigotimes \Sigma_p$. For the second conjunct, if $\Lambda \neq \Lambda_p$ note that $\bigoplus \Lambda_p \Rightarrow \Lambda_p$ and $\Sigma, \bot \Rightarrow \Lambda - \Lambda_p$, hence

$$\Sigma, [\bot \oplus (\bigoplus \Lambda_p)] \Rightarrow \Lambda$$

hence

$$\Sigma \Rightarrow [\neg(\bot \oplus (\bigoplus \Lambda_p))], \Lambda$$

If $\Lambda = \Lambda_p$, similar to the case before, there is no need for $\bot$.

The cases for the third and the fourth conjuncts are similar to the similar cases in the proof of (i).

Now we want to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ in G, then $\exists pS, \bar{C} \Rightarrow \bar{D}$ in H. For this purpose, we will check all the cases one by one:

(1) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (1), we have four cases to check.

- If $\phi \in \bar{C}$ and $\phi \in \bar{D}$, then $\Sigma = \Lambda = \emptyset$ and $\bar{C} = \bar{D} = \phi$. Hence $\bigotimes \Sigma_p = 1$. Therefore, since $1, \bar{C} \Rightarrow \bar{D}$ we have $\exists pS, \bar{C} \Rightarrow \bar{D}$.
- If $\phi \in \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \emptyset$ and $\Lambda = \phi$. Therefore, ϕ is p-free and hence $\Lambda_p = \phi$. Since $\bar{D} = \emptyset$ and $\Lambda = \phi$, we have $, \neg\phi, \bar{C} \Rightarrow \bar{D}$. Therefore, $\neg(\bigoplus \Lambda_p), \bar{C} \Rightarrow \bar{D}$.
- If $\phi \notin \bar{C}$ and $\phi \in \bar{D}$. This case is similar to the previous case.
- If $\phi \notin \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \Lambda = \phi$ and $\bar{C} = \bar{D} = \emptyset$. Hence, by definition, we have 0 as a conjunct in $\exists pS$. Since $0 \Rightarrow$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$.

(2) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \emptyset$. There are two cases to consider. If $\Lambda = \bar{\alpha}$. Then by definition 0 appears in $\exists pS$. Since $\bar{D} = \emptyset$ and $(0 \Rightarrow)$ we have $\bar{C}, \exists pS \Rightarrow \bar{D}$. If $\Lambda \subsetneq \bar{\alpha}$, then $\bar{D} \cap \bar{\alpha}$ is non empty. Therefore, there exists a p-free formula in $\bar{\alpha}$. Since the variables of any pair in $\bar{\alpha}$ are equal, $\bar{\alpha}$ is p-free. Therefore, $\Lambda \subseteq \bar{\alpha}$ is p-free, hence $\Lambda = \Lambda_p$ (and $\bot$ does not appear in the second conjunct). Since $(\Rightarrow \Lambda, \bar{D})$, we have $(\Rightarrow \bigoplus \Lambda, \bar{D})$ therefore $(\neg(\bigoplus \Lambda_p) \Rightarrow \bar{D})$ which implies $(\exists pS \Rightarrow \bar{D})$.

(3) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (3). This case is similar to the previous case (2).

(4) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (4). There are two cases to consider. If $\bar{\phi} \subseteq \Sigma$. Then by definition $\bot$ is a conjunct in $\exists pS$ and therefore there is nothing to prove. For the second case, if $\bar{\phi} \nsubseteq \Sigma$, then $\bar{\phi} \cap \bar{C}$ is non-empty. Hence, $\bar{\phi}$ has a p-free element. Since the variables of any pair in $\bar{\phi}$ are equal, $\bar{\phi}$ is p-free. Since $\bar{\phi} \subseteq \Sigma_p, \bar{C}, \Sigma - \Sigma_p$ and $\bar{\phi}$ is p-free, we should have $\bar{\phi} \subseteq \Sigma_p, \bar{C}$. Therefore, if $\Sigma \neq \Sigma_p$, by the axiom (4) itself, $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$. Since $(\bigotimes \Sigma_p) \otimes \top$ is a conjunct in $\exists pS$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$. Note that if $\Sigma = \Sigma_p$, then we will use $\Sigma_p, \bar{C} \Rightarrow \bar{D}$ instead of $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$.

(5) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (5). This case is similar to the previous case 4.

□

2.5.1 The Single-conclusion Case

In this section, we assume that for any sequent $S = \Gamma \Rightarrow \Delta$, the nimber of elements of Δ is at most one. We will show how the single-conclusion semi-analytic and context-sharing semi-analytic rules preserve the uniform interpolation property. For this purpose, we will investigate these two kinds of rules separately. First we will study the semi-analytic rules and then we will show in the presence of weakening and context-sharing implication rules, we can also handle the context-sharing semi-analytic rules.

Semi-analytic Case

Let us begin right away with the following theorem which is one of the main theorems of this paper.

Theorem 2.5.38. *Let G and H be two single-conclusion sequent calculi and H extends $\mathbf{FL_e}$. If H is a terminating sequent calculus axiomatically extending G with only single-conclusion semi-analytic rules, then if G has H-uniform interpolation property, then so does H.*

Proof. For any sequent U and V where $V^s = \emptyset$ and any atom p, we define two p-free formulas, denoted by $\forall pU$ and $\exists pV$ and we will prove that they meet the conditions for the left and the right p-interpolants of U and V, respectively. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus H.

If V is the empty sequent we define $\exists pV$ as 1 and otherwise, we define $\exists pV$ as the following

$$(\bigwedge_{par} \bigotimes_{i} \exists pS_i) \wedge (\bigwedge_{L\mathcal{R}} [(\bigotimes_{j} \bigwedge_{s} \forall pT_{js}) \otimes (\bigotimes_{i \neq 1} \bigwedge_{r} \forall pS_{ir}) \to \bigvee_{r} \exists pS_{1r}]) \wedge (\Box \exists pV') \wedge (\exists^G pV).$$

In the first conjunct, the conjunction is over all non-trivial partitions of $V = S_1 \cdot \cdots \cdot S_n$ and i ranges over the number of S_i's, in this case $1 \leq i \leq n$. In the second conjunct, the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Note that picking the block $\langle S_{1r}\rangle$ is arbitrary and we include all conjuncts related to any choice of $\langle S_{1r}\rangle$.) The conjunct $\Box\exists pV'$ appears in the definition whenever V is of the form $(\Box\Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since G has the H-uniform interpolation property, by definition there exists $J(V)$ as right p-interpolant of V. We choose one such $J(V)$ and denote it as $\exists^G pV$ and include it in the definition.

If U is the empty sequent define $\forall pU$ as 0. Otherwise, define $\forall pU$ as the following

$$(\bigvee_{par} (\bigotimes_{i \neq 1} \exists pS_i \to \forall pS_1)) \vee (\bigvee_{L\mathcal{R}} [(\bigotimes_{j} \bigwedge_{s} \forall pT_{js}) \otimes (\bigotimes_{i} \bigwedge_{r} \forall pS_{ir})])$$

$$\vee (\bigvee_{R\mathcal{R}} (\bigotimes_{i} \bigwedge_{r} \forall pS_{ir})) \vee (\Box \forall pU') \vee (\forall^G pU).$$

In the first disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \emptyset$ and $S_1 \neq U$. (Note that in this case, if $S^s = \emptyset$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists pU \to \forall pS_1$ where $\forall pS_1 = 0$. But this does not make any problem, since the definition of $\exists pU$ is prior to the definition of $\forall pU$.) In the second disjunct, the big disjunction is over all left semi-analytic rules that are backward applicable to U in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. In the third disjunct, the big disjunction is over all right semi-analytic rules backward applicable to U in H. The premise of the rule is $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. The fourth disjunct is on all semi-analytic modal rules with the result U and the premise U'. And finally, since G has the H-uniform interpolation property, by definition there exists $I(U)$ as left p-interpolant of U. We choose one such $I(U)$ and denote it as $\forall^G pU$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents and we prove both cases $\forall pU$ and $\exists pV$ simultaneously. First note that both $\forall pU$ and $\exists pV$ are p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pU) \subseteq V(U^a) \cup V(U^s)$ and $V(\exists pV) \subseteq V(V^a)$. Secondly, we have to show that:

(*i*) $V \cdot (\Rightarrow \exists pV)$ is derivable in H.

(*ii*) $U \cdot (\forall pU \Rightarrow)$ is derivable in H.

We show them using induction on the order of the sequents U and V. When proving (*i*), we assume that (*i*) holds for sequents whose succedents are empty and with order less than the order of V and (*ii*) holds for any sequent with order less than the order of V. We have the same condition for U when proving (*ii*).

To prove (*i*), note that if V is the empty sequent, then by definition $\exists pV = 1$ and hence (*i*) holds. For the rest, we have to show that $V \cdot (\Rightarrow X)$ is derivable in H for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule $(R\wedge)$ it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show $\Gamma \Rightarrow X$ is derivable in H.

- In the case that the conjunct is $(\bigwedge_{par} \bigotimes_i \exists pS_i)$, we have to show that for any non-trivial partition $S_1 \cdot \cdots \cdot S_n$ of V we have $\Gamma \Rightarrow \bigotimes_i \exists pS_i$ is derivable in H. Since the order of each S_i is less than the order of V and $S_i^s = (\Gamma_i \Rightarrow)$ for $1 \leq i \leq n$ where $\bigcup_{i=1}^{n} \Gamma_i = \Gamma$, we can use the induction hypothesis and we have $\Gamma_i \Rightarrow \exists pS_i$. Using the right rule for $(\otimes)$ we have $\Gamma_1, \cdots, \Gamma_n \Rightarrow \bigotimes_i \exists pS_i$ which is $\Gamma \Rightarrow \bigotimes_i \exists pS_i$.
- For the second conjunct in the definition of $\exists pV$, we have to check that for every left semi-analytic rule we have

$$V \cdot (\Rightarrow [(\bigotimes_j \bigwedge_s \forall pT_{js}) \otimes (\bigotimes_{i\neq 1} \bigwedge_r \forall pS_{ir}) \to \bigvee_r \exists pS_{1r}]).$$

  is derivable in H. Therefore, V is the conclusion of a left semi-analytic rule such that the premises are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_i$ and $\langle S_{1r}\rangle_r$ and hence the order of all of them are less than the order of V. We can easily see that the claim holds since by induction hypothesis we can add $\forall pT_{js}$ and $\forall pS_{ir}$ to the left side of the sequents T_{js} and S_{ir} for $i \neq 1$. And again by induction hypothesis we can add $\exists pS_{1r}$ to the right side of the sequents S_{1r}. Then using the rules $L\wedge$, $L\otimes$ and $R\vee$ the claim follows. What we have said so far can be seen precisely in the following:

Note that $\langle\langle T_{js}\rangle_s\rangle_j$ is of the form $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ is of the form $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_i$ and V is of the form

$$\Pi_1, \cdots, \Pi_m, \Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow$$

Using induction hypothesis we have for every $1 \leq j \leq m$

$$(\Pi_j, \forall p T_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

for every $1 < i \leq n$ we have

$$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow), \cdots$$

and for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \exists p S_{11}), \cdots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists p S_{1r}), \cdots$$

Hence, using the rule $(L\wedge)$, for every $1 \leq j \leq m$ we have

$$(\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

and for every $1 < i \leq n$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow), \cdots (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow), \cdots$$

and using the rule $(R\vee)$, for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \bigvee_r \exists p S_{1r}), \cdots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}) \cdots$$

Substituting all these three in the original left semi-analytic rule (we can do this, since in the original rule, there are contexts, Δ_i's in the right hand side of the sequents $S'_{ir}s$), we conclude for $\Pi = \Pi_1, \cdots, \Pi_m$ and $\Gamma = \Gamma_1, \cdots, \Gamma_n$

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall p T_{js}\rangle_j, \langle \bigwedge_r \forall p S_{ir}\rangle_{i \neq 1} \Rightarrow \bigvee_r \exists p S_{1r}$$

where we have

$$\langle \bigwedge_s \forall p T_{js}\rangle_j = \bigwedge_s \forall p T_{1s}, \cdots, \bigwedge_s \forall p T_{ms}$$

and

$$\langle \bigwedge_r \forall p S_{ir}\rangle_{i \neq 1} = \bigwedge_r \forall p S_{2r}, \cdots, \bigwedge_r \forall p S_{nr}.$$

Now, using the rule $(L\otimes)$ we have

$$\Pi, \Gamma, \phi, (\bigotimes_j \bigwedge_s \forall p T_{js}) \otimes (\bigotimes_{i \neq 1} \bigwedge_r \forall p S_{ir}) \Rightarrow \bigvee_r \exists p S_{1r}.$$

And finally, using the rule $R \to$ we conclude

$$\Pi, \Gamma, \phi \Rightarrow [(\bigotimes_j \bigwedge_s \forall p T_{js}) \otimes (\bigotimes_{i \neq 1} \bigwedge_r \forall p S_{ir}) \to \bigvee_r \exists p S_{1r}].$$

- Consider the conjunct $\Box\exists pT'$. In this case, T must have been of the form $(\Box\Gamma \Rightarrow)$ and T' of the form $(\Gamma \Rightarrow)$. By definition, the order of T' is less than the order of T. Hence, by induction hypothesis we have $T' \cdot (\Rightarrow \exists pT')$ or in other words $\Gamma \Rightarrow \exists pT'$. Now, we use the rule K and we have $\Box\Gamma \Rightarrow \Box\exists pT'$ which means $T \cdot (\Rightarrow \Box\exists pT')$.

- The last case is $\exists^G pV$. We have to show $V \cdot (\Rightarrow \exists^G pV)$ is provable in H which is the case since G has H-uniform interpolation property and by Definition 2.5.35 part (iii) there exists p-free formula J such that $V \cdot (\Rightarrow J)$ is derivable in H. We chose one such J and call it $\exists^G pV$, hence $V \cdot (\Rightarrow \exists^G pV)$ in H by definition.

To prove (ii), note that if U is the empty sequent, then by definition $\forall pU = 0$ and hence (ii) holds. For the rest, we have to show that $U \cdot (X \Rightarrow)$ is derivable in H for any X that is one of the disjuncts in the definition of $\forall pU$. Then, using the rule $(L\vee)$ it follows that $U \cdot (\forall pU \Rightarrow)$. Since U is of the form $\Gamma \Rightarrow \Delta$, we have to show that $\Gamma, X \Rightarrow \Delta$ is derivable in H.

- In the case that the disjunct is $(\bigvee\limits_{par} (\bigotimes\limits_{i\neq 1} \exists pS_i \to \forall pS_1))$ we have to prove that for any partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that $S_i^s = \emptyset$ for each $i \neq 1$ and $S_1 \neq U$, we have $U \cdot ((\bigotimes\limits_{i\neq 1} \exists pS_i \to \forall pS_1) \Rightarrow)$. First, consider the case that none of S_i's are equal to U (or in other words, $S^s \neq \emptyset$); then the order of each S_i is less than the order of S and we can use the induction hypothesis. Since for $i \neq 1$ the succedent of each S_i is empty, we have $S_i = (\Gamma_i \Rightarrow)$ and $(\Gamma_i \Rightarrow \exists pS_i)$ and using the rule $R\otimes$ we have $(\Gamma_2, \cdots, \Gamma_n \Rightarrow \bigotimes\limits_{i\neq 1} \exists pS_i)$. And for $S_1 = \Gamma_1 \Rightarrow \Delta$ we have $\Gamma_1, \forall pS_1 \Rightarrow \Delta$. Hence using the rule $L \to$ we conclude
$$\Gamma_1, \cdots, \Gamma_n, \bigotimes_{i\neq 1} \exists pS_i \to \forall pS_1 \Rightarrow \Delta$$
and the claim follows.
In the case that $U^s = \emptyset$, it is possible that for $i \neq 1$, one of S_i's is equal to U. In this case what appears in the definition of $\forall pU$ is $\exists pU \to \forall pS_1$ which is equivalent to $\exists pU \to 0$. But, we can do this, since we defined $\exists pU$ prior to the definition of $\forall pU$ and we have proved $U \cdot (\Rightarrow \exists pU)$ prior to the case that we are checking now.

- In the case that the disjunct is $(\bigvee\limits_{L\mathcal{R}} [(\bigotimes\limits_{j} \bigwedge\limits_{s} \forall pT_{js}) \otimes (\bigotimes\limits_{i} \bigwedge\limits_{r} \forall pS_{ir})])$, we have to prove that for any left semi-analytic rule that is backward applicable to U in H we have $U \cdot ((\bigotimes\limits_{j} \bigwedge\limits_{s} \forall pT_{js}) \otimes (\bigotimes\limits_{i} \bigwedge\limits_{r} \forall pS_{ir}) \Rightarrow)$. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$ and $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U. Since the orders of all T_{js}'s and S_{ir}'s are less than the order of U we can use the induction hypothesis and have $T_{js} \cdot (\forall pT_{js} \Rightarrow)$ and $S_{ir} \cdot (\forall pS_{ir} \Rightarrow)$. Using the rule $(L\wedge)$ for context sharing sequents (when j is fixed and i is fixed we have context sharing sequents) and then using the rule $(L\otimes)$ for non context sharing sequents (when s and r are fixed and we are ranging over j and i) and then applying the same left rule we can prove the claim. The proof is similar to the second case of (i) and precisely it goes as the following: Using induction hypothesis we have for every $1 \leq j \leq m$

$$(\Pi_j, \forall p T_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \cdots$$

Hence, using the rule $(L\wedge)$, for every $1 \leq j \leq m$ we have

$$(\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \cdots, (\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \cdots$$

and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \cdots, (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \cdots$$

Substituting these two in the original left semi-analytic rule, we conclude

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall p T_{js} \rangle_j, \langle \bigwedge_r \forall p S_{ir} \rangle_i \Rightarrow \Delta,$$

and using the rule $(L\otimes)$ we have

$$\Pi, \Gamma, \phi, (\bigotimes_j \bigwedge_s \forall p T_{js}) \otimes (\bigotimes_i \bigwedge_r \forall p S_{ir}) \Rightarrow \Delta.$$

- In the case that the disjunt is $(\bigvee_{\mathcal{RR}} (\bigotimes_i \bigwedge_r \forall p S_{ir}))$, we have to prove that for any right semi-analytic rule backward applicable to U in H, we have $U \cdot (\bigotimes_i \bigwedge_r \forall p S_{ir} \Rightarrow)$. In this case the premises of the rule are $\langle \langle S_{ir} \rangle_r \rangle_i$, where $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ and the conclusion is $U = (\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi)$. Since the order of each S_{ir} is less than the order of S, we can use the induction hypothesis and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \forall p S_{i1}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \cdots, (\Gamma_i, \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \cdots$$

Using the rule $L\wedge$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \cdots, (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \cdots$$

and substituting it in the original right rule, we conclude

$$\Gamma, \langle \bigwedge_r \forall p S_{ir} \rangle_i \Rightarrow \phi,$$

and using the rule $(L\otimes)$ we have

$$\Gamma, \bigotimes_i \bigwedge_r \forall p S_{ir} \Rightarrow \phi.$$

- For the case that the disjunct is $\Box\forall pU'$ we have that U is the conclusion of a semi-analytic modal rule and the premise is U'. Hence, U is of the form $(\Box\Gamma \Rightarrow \Box\Delta)$ and U' is of the form $(\Gamma \Rightarrow \Delta)$. Since the order of U' is less than the order of U, we can use the induction hypothesis and we have $(\Gamma, \forall pU' \Rightarrow \Delta)$. Now, using the rule K we can conclude $(\Box\Gamma, \Box\forall pU' \Rightarrow \Box\Delta)$ which is equivalent to $U \cdot (\Box\forall pU' \Rightarrow)$.

- And finally, for the case that the disjunct is $\forall^G pU$ we have to show that $U \cdot (\forall^G pU \Rightarrow)$ holds in H, which does since G has H-uniform interpolation property and by Definition 2.5.35 part (i) there exists p-free formula I such that $U \cdot (I \Rightarrow)$ is derivable in H. We choose one such I and call it $\forall^G pU$ and hence we have $U \cdot (\forall^G pU \Rightarrow)$ in H by definition.

So far we have proved (i) and (ii). We want to show that H has H-uniform interpolation. Therefore, based on the Definition 2.5.35, we have to prove the following, as well:

(iii) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $V \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in H, where $\bar{C} = C_1, \cdots, C_k$ and $|\bar{D}| \leq 1$.

(iv) For any p-free multiset $\bar{C}$, if $U \cdot (\bar{C} \Rightarrow)$ is derivable in G then $\bar{C} \Rightarrow \forall pU$ is derivable in H, where $\bar{C} = C_1, \cdots, C_k$.

Recall that V is of the form $(\Gamma \Rightarrow)$ and U is of the form $(\Gamma \Rightarrow \Delta)$. We will prove (iii) and (iv) simultaneously using induction on the length of the proof and induction on the order of U and V. More precisely, first by induction on the order of U and V and then inside it, by induction on n, we will show:

- For any p-free multisets $\bar{C}$ and $\bar{D}$, if $V \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in G with length less than or equal to n, then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in H.

- For any p-free multiset $\bar{C}$, if $U \cdot (\bar{C} \Rightarrow)$ has a proof in G with length less than or equal to n, then $\bar{C} \Rightarrow \forall pU$ is derivable in H.

Where by the length we mean counting just the new rules that H adds to G.

First note that for the empty sequent and for (iii), we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in G, then $\bar{C}, 1 \Rightarrow \bar{D}$ is valid in H, which is trivial by the rule $(1w)$. Similarly, for (iv), if $\bar{C} \Rightarrow$ is valid in G, then $\bar{C} \Rightarrow 0$ is valid in H, which is trivial by the rule $(0w)$.

For the base of the other induction, note that if $n = 0$, for (iii) it means that $\Gamma, \bar{C} \Rightarrow \bar{D}$ is valid in G. By Definition 2.5.35 part (iv), $\exists^G pV, \bar{C} \Rightarrow \bar{D}$ and hence $\exists pV, \bar{C} \Rightarrow \bar{D}$ is provable in H. For (iv), it means that $\Gamma, \bar{C} \Rightarrow \Delta$ is valid in G. Therefore, again by Definition 2.5.35, $\bar{C} \Rightarrow \forall^G pU$ and hence $\bar{C} \Rightarrow \forall pU$ is provable in H.

For $n \neq 0$, to prove (iii), we have to consider the following cases:

- The case that the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule,

ϕ, is one of C_i's). Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y}, \phi)$ and we want to prove $(\exists p V, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

where $\bigcup\limits_j \Pi_j = \Pi$, $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_j \bar{X}_j = \bar{X}$, $\bigcup\limits_i \bar{Y}_i = \bar{Y}$ and $\bigcup\limits_i \Delta_i = \Delta$. Consider $T_{js} = (\Pi_j \Rightarrow)$ and $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js}'s do not depend on the suffix s, we have $T_{j1} = \cdots = T_{js}$ and we denote it by T_j. And, since S_{ir}'s do not depend on r, we have $S_{i1} = \cdots = S_{ir}$ and we denote it by S_i. Therefore, $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a partition of V. First, consider the case that it is a non-trivial partition. Then the order of all of them are less than the order of V and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p-free. Hence, we can use the induction hypothesis to get:

$$\exists p T_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists p S_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i$$

If we let $\{\exists p T_j, \bar{X}_j\}$ and $\{\exists p S_i, \bar{Y}_i\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists p T_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \exists p S_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i \rangle_r \rangle_i}{\exists p T_1, \cdots, \exists p T_m, \exists p S_1, \cdots, \exists p S_n, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Using the rule $(L\otimes)$ we have

$$(\bigotimes_j \exists p T_j) \otimes (\bigotimes_i \exists p S_i), \bar{X}, \bar{Y}, \phi \Rightarrow \Delta.$$

Therefore using the rule $(L\wedge)$, we have $(\exists p V, \bar{C} \Rightarrow \bar{D})$.

If $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a trivial partition of V, it means that one of them equals V and all the others are empty sequents. W.l.o.g. suppose $T_1 = V = (\Sigma \Rightarrow)$ and the others are empty. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \Sigma, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i \rangle_r \rangle_i}{\Sigma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $V \cdot (\bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js})$ for every j and s are premises of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$ and $\bar{\theta}_{js}$ are also p-free. Hence, for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $\exists p V, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}$. Substituting $\{\exists p V, \bar{X}_j\}$, $\{\bar{X}_j\}$, $\{\bar{Y}_i\}$ and $\{\Delta\}$ as the contexts of the premises in the original left rule we have

$$\frac{\langle\langle \exists pV, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i\rangle_r\rangle_i}{\exists pV, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

which is $(\exists pV, \bar{C} \Rightarrow \bar{D})$.

- Consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore,

  $$V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$$

  is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma, \phi \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y})$ and we want to prove $(\exists pV, \bar{X}, \bar{Y} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule, which we denote by (†)

  $$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{Y}_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

  where $\bigcup\limits_j \Pi_j = \Pi$, $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_j \bar{X}_j = \bar{X}$ and $\bigcup\limits_i \bar{Y}_i = \bar{Y}$.
  Since, $\bar{X}_j$'s and $\bar{Y}_i$'s are in the context positions in the original rule, we can consider the same substition of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{X}_j$'s and $\bar{Y}_i$'s as contexts. More precisely, we reach the following instance of the original rule:

  $$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

  If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is back ward applicable to V and T_{js}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis for them. Note that we have $V \cdot (\bar{C} \Rightarrow \bar{D})$ is provable in H and from (†) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and for $i \neq 1$, $S_{ir} \cdot (\bar{Y}_i \Rightarrow)$ and $S_{1r} \cdot (\bar{Y}_1 \Rightarrow \Delta)$ are also provable in H. Using the induction hypothesis we get

  $$(\bar{X}_j \Rightarrow \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \forall p S_{ir})_{i\neq 1} \quad , \quad (\bar{Y}_1, \exists p S_{1r} \Rightarrow \Delta)$$

  Note that we were allowed to use the induction hypothesis because for $i \neq 1$ we have $\Delta_i = \emptyset$ and Δ is p-free and T_{js}'s and S_{ir}'s meet the conditions of (iii) and (iv) in the induction step. Now, using the rules $(R\wedge)$ and $(L\vee)$ we have

  $$(\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir})_{i\neq 1} \quad , \quad (\bar{Y}_1, \bigvee_r \exists p S_{1r} \Rightarrow \Delta)$$

  Denote $(\bigwedge\limits_s \forall p T_{js})$ as A_j and $(\bigwedge\limits_r \forall p S_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee\limits_r \exists p S_{1r})$ as C. We have

$$\dfrac{\dfrac{\langle \bar{X}_j \Rightarrow A_j \rangle_j}{\bar{X} \Rightarrow \bigotimes_j A_j}\; R\otimes \qquad \dfrac{\langle \bar{Y}_i \Rightarrow B_i \rangle_{i \neq 1}}{Y_2, \cdots, Y_n \Rightarrow \bigotimes_{i\neq 1} B_i}\; R\otimes}{\bar{X}, Y_2, \cdots, Y_n \Rightarrow (\bigotimes_j A_j) \otimes (\bigotimes_{i\neq 1} B_i)}\; R\otimes \qquad \bar{Y}_1, C \Rightarrow \Delta$$

$$\overline{\bar{X}, \bar{Y}, (\bigotimes_j A_j) \otimes (\bigotimes_{i\neq 1} B_i) \to C \Rightarrow \Delta}\; L \to$$

Note that $(\bigotimes_j A_j) \otimes (\bigotimes_{i\neq 1} B_i) \to C$ is defined as the second conjunct in the definition of $\exists pV$ and hence using the rule $(L\wedge)$ we have $(\exists pV, \bar{C} \Rightarrow \Delta)$.

- Consider the case when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a right semi-analytic rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and V is of the form $(\Gamma \Rightarrow)$ and $\bar{D} = \phi$ and we want to prove $(\exists pV, \bar{C} \Rightarrow \phi)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma, \bar{C} \Rightarrow \phi}$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$. Denote $(\Gamma_i \Rightarrow)$ as S_i. Then we have that $S_1, \cdots, S_n$ is a partition of V. First consider the case where it is a non-trivial partition of V. Therefore, the order of any S_i is less than the order of V and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{ir}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis on the order, and get

$$\exists p S_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}$$

Now, substituting $\{\exists p S_i, \bar{C}_i\}$ as the context in the original rule, we get

$$\exists p S_1, \cdots, \exists p S_n, \bar{C}_1, \cdots, \bar{C}_n \Rightarrow \phi$$

then using the rule $(L\otimes)$ we have

$$\bigotimes_i \exists p S_i, \bar{C} \Rightarrow \phi$$

and since $\bigotimes_i \exists p S_i$ appears as the first conjunct in the definition of $\exists pV$, using the rule $(L\wedge)$ we have $(\exists pV, \bar{C} \Rightarrow \phi)$.
It remains to investigate the case where $S_1, \cdots, S_n$ is a trivial partition of V. W.l.o.g. suppose $S_1 = V$ and all the others are the empty sequents. Hence, we must have had the following instance of the rule

$$\frac{\langle \Gamma, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r} \rangle_r \qquad \langle\langle\ \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_{i \neq 1}}{\Gamma, \bar{C} \Rightarrow \phi}$$

We have, for all r, $V \cdot (\bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r})$ are the premises of $V \cdot (\bar{C} \Rightarrow \phi)$. Hence the length of tree proofs of all of them are less than the length of proof of $V \cdot (\bar{C} \Rightarrow \phi)$ and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{1r}$ and $\bar{\phi}_{1r}$ are also p-free, we can use the induction hypothesis (induction on the length of proof) and get $\exists pV, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r}$. Substituting $\{\exists pV, \bar{C}_1\}$ as the context in the original semi-analytic rule we get

$$\frac{\langle \exists pV, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r}\rangle_r \qquad \langle\langle\ \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_{i\neq 1}}{\exists pV, \bar{C} \Rightarrow \phi}$$

which is what we wanted.

- And the final case is when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta})$ is the conclusion of a semi-analytic modal rule and V is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box\Delta})$, where $||\overline{\Box\Delta}|| \leq 1$ and $V' = (\Gamma \Rightarrow)$. We want to prove $(\exists pV, \bar{C} \Rightarrow \bar{D})$. We must have had the following instance of the rule

  $$\frac{\Gamma, \bar{C'} \Rightarrow \bar{\Delta}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta}}$$

  Since the order of V' is less than the order of V, and C' and Δ are p-free, we can use the induction hypothesis and get

  $$\exists pV', \bar{C'} \Rightarrow \bar{\Delta}$$

  Using the rule K or D (depending on the cardinality of $\overline{\Box\Delta}$) we have $\Box\exists pV', \overline{\Box C'} \Rightarrow \overline{\Box\Delta}$ and since we have $\Box\exists pV'$ as one of the conjuncts in the definition of $\exists pV$, we conclude $\exists pV, \bar{C} \Rightarrow \bar{D}$ using the rule $(L\wedge)$.

Now, we have to prove (iv). Similar to the proof of part (iii), there are several cases to consider.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \in \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}, \phi$ and we want to prove $\bar{X}, \bar{Y}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

  $$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

  where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$. Consider $T_{js} = (\Pi_j \Rightarrow)$, $S_{1r} = \Gamma_1 \Rightarrow \Delta_1$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js}'s do not depend on the suffix s, we have $T_{j1} = \cdots = T_{js}$ and we denote it by T_j. And, since S_{ir}'s do not depend on r for $i \neq 1$, we have $S_{21} = \cdots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1. Therefore, $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a partition of U. First, consider the case that S_1 does not equal U. Then the order of all of them are less than the order of U (or in some cases that the others can be equal to U, the length of their proof in the premises is lower) and since the rule is semi-analytic and ϕ is p-free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis to get (for $i \neq 1$):

  $$\exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \quad , \quad \bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall pS_{1r}$$

If we let $\{\exists p T_j, \bar{X}_j\}$ and $\{\exists p S_i, \bar{Y}_i\}$ and $\{\bar{Y}_1\}$ and $\{\forall p S_{1r}\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists p T_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \exists p S_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall p S_{1r}\rangle_r}{\exists p T_1, \cdots, \exists p T_m, \exists p S_2, \cdots, \exists p S_n, \bar{X}, \bar{Y}, \phi \Rightarrow \forall p S_1}$$

Using the rule $(L\otimes)$ we have

$$(\bigotimes_j \exists p T_j) \otimes (\bigotimes_{i\neq 1} \exists p S_i), \bar{X}, \bar{Y}, \phi \Rightarrow \forall p S_1.$$

Therefore using the rule $(R \rightarrow)$, we have

$$\bar{X}, \bar{Y}, \phi \Rightarrow (\bigotimes_j \exists p T_j) \otimes (\bigotimes_{i\neq 1} \exists p S_i) \rightarrow \forall p S_1.$$

Since the right side of the sequent is a disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C}, \phi \Rightarrow \forall p U$.
In the case that $T_1, \cdots, T_m, S_1, \cdots, S_n$ is a trivial partition of U, it means that either $S_1 = U$ or $U^s = \emptyset$ and one of the others is equal to U. The latter case is investigated in the previous case, so it only remains to consider the first one.
If $S_1 = U = \Gamma \Rightarrow \Delta$, then all the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma, \phi_{1r}, \bar{Y}_1 \Rightarrow \Delta\rangle_r}{\Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $U \cdot (\phi_{1r}, \bar{Y}_1 \Rightarrow)$ for every r are premises of $U \cdot (\bar{C} \Rightarrow)$, and hence the length of their trees are smaller than the length of the proof tree of $U \cdot (\bar{C} \Rightarrow)$ and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{1r}$ are also p-free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{Y}_1 \Rightarrow \forall p U)$. Substituting $\{\forall p U\}$, $\{\bar{X}_i\}$ and $\{\bar{Y}_i\}$ as the contexts of the premises in the original left rule and letting all the other contexts in the original left rule to be empty we have

$$\frac{\langle\langle \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \phi_{1r}, \bar{Y}_1 \Rightarrow \forall p U\rangle_r}{\bar{X}, \bar{Y}, \phi \Rightarrow \forall p U}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma, \phi \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}$ and we want to prove $\bar{X}, \bar{Y} \Rightarrow \forall p U$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta} \quad (\ddagger)$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$.
Since, $\bar{X}_j$'s and $\bar{Y}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{X}_j$'s and $\bar{Y}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$, we can claim that this rule is backward applicable to U and T_{js}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of U and we can use the induction hypothesis for them. Note that we have $U \cdot (\bar{C} \Rightarrow)$ is provable in H and from (‡) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and $S_{ir} \cdot (\bar{Y}_i \Rightarrow)$ are also provable in H. Using the induction hypothesis we get

$$\bar{X}_j \Rightarrow \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \forall p S_{ir}$$

Using the rule $(R\wedge)$ we get

$$\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir}$$

and using the rule $(R\otimes)$ we get

$$\bar{X}, \bar{Y} \Rightarrow (\bigotimes_j \bigwedge_s \forall p T_{js}) \otimes (\bigotimes_r \bigwedge_r \forall p S_{ir}).$$

Since the right side of the sequent is appeared as the second disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall p U$.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a right semi-analytic rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and U is of the form $\Gamma \Rightarrow \phi$ and we want to prove $\bar{C} \Rightarrow \forall p U$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma, \bar{C} \Rightarrow \phi} \quad (\star)$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$.
With the similar reasoning as in the previous case, since $\bar{C}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma \Rightarrow \phi}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ we can claim that this rule is backward applicable to U and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of U and hence we can use the induction hypothesis for them. Using the induction hypothesis we get for every i and r,

$$\bar{C}_i \Rightarrow \forall p S_{ir}.$$

Using the rule $(R\wedge)$ we get $\bar{C}_i \Rightarrow \bigwedge_r \forall p S_{ir}$ and then using the rule $(R\otimes)$ we get $\bar{C}_i \Rightarrow \bigotimes_i \bigwedge_r \forall p S_{ir}$. And since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall pU$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall pU$.

- And the final case is when the last rule used in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a semi-analytic modal rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta})$ is the conclusion of a semi-analytic modal rule and U is of the form $(\Box\Gamma \Rightarrow \overline{\Box\Delta})$ and $\bar{C} = \overline{\Box C'}$, where $||\overline{\Box\Delta}|| \leq 1$ and $U' = (\Gamma \Rightarrow \Delta)$. We want to prove $(\bar{C} \Rightarrow \forall pU)$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{\Delta}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta}}$$

Since the order of U' is less than the order of U and C' is p-free, we can use the induction hypothesis and get

$$\bar{C'} \Rightarrow \forall pU'$$

Using the rule K or D (depending on the cardinality of $\overline{\Box\Delta}$) we have $\overline{\Box C'} \Rightarrow \Box\forall pU'$ and since we have $\Box\forall pU'$ as one of the disjuncts in the definition of $\forall pU$, we conclude $\bar{C} \Rightarrow \forall pU$ using the rule $(R\vee)$.

□

Theorem 2.5.39. *Any terminating single-conclusion sequent calculus H that extends $\mathbf{FL_e}$ and consists of focused axioms and single-conclusion semi-analytic rules, has H-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.37 and Theorem 2.5.38. □

Corollary 2.5.40. *If $\mathbf{FL_e} \subseteq L$ and L has a terminating single-conclusion sequent calculus consisting of focused axioms and single-conclusion semi-analytic rules, then L has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.39 and Theorem 2.5.36. □

In the following application, we will use the Corollary 2.5.40 to generalize the result of [2] to also cover the modal cases:

Corollary 2.5.41. *The logics $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$ and their $\mathbf{K}$ and $\mathbf{KD}$ versions have uniform interpolation.*

Proof. Since all the rules of the usual calculi of these logics are semi-analytic and their axioms are focused and since in the absence of the contraction rule the calculi are clearly terminating, by Corollary 2.5.40, we can prove the claim. □

Context-Sharing Semi-analytic Case

In this subsection we will modify the investigations of the last subsection to also cover the context-sharing semi-analytic rules.

Theorem 2.5.42. *Let G and H be two single-conclusion sequent calculi with the property that the right and left weakening rules and the context-sharing $(L \to)$ rule are admissible in H and H extends $\mathbf{FL_e}$. Then if H is a terminating sequent calculus axiomatically extending G with single-conclusion semi-analytic rules and context-sharing semi-analytic rules and G has weak H-uniform interpolation property, so does H.*

Proof. The proof is similar to the proof of Theorem 2.5.38. For any sequent U and V where $V^s = \emptyset$ and any atom p, we define two p-free formulas, denoted by $\forall pU$ and $\exists pV$ and we will prove that they meet the conditions in the definition of weak H-uniform interpolation. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus H.

If V is the empty sequent we define $\exists pV$ as 1 and otherwise, we define $\exists pV$ as the following:

$$\bigwedge_{L\mathcal{R}_{cs}} [\bigotimes_{i\neq 1}((\bigwedge_r(\exists p\tilde{S}_{ir} \to \forall pS_{ir})) \wedge (\bigwedge_s(\exists p\tilde{T}_{is} \to \forall pT_{is})))$$

$$\otimes((\bigwedge_s \exists p\tilde{T}_{1s} \to \forall pT_{1s}) \to \bigvee_r \exists pS_{1r})]$$

$$\wedge \bigwedge_{L\mathcal{R}_{cs}} (\bigotimes_i \bigwedge_r(\exists p\tilde{S}_{ir} \to \forall pS_{ir}) \otimes (\bigotimes_j \bigwedge_s(\exists p\tilde{T}_{js} \to \forall pT_{js}) \to \bigvee_r \exists pS_{1r})$$

$$\wedge(\bigwedge_{par} \bigotimes_i \exists pS_i) \wedge (\Box\exists pV') \wedge (\exists^G pV).$$

where for any sequent R, by $\tilde{R}$ we mean $R^a \Rightarrow$. In the first conjunct (the first line), the first big conjunction is over all context-sharing semi-analytic rules that are backward applicable to V in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{is}\rangle_s\rangle_i$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the context-sharing semi-analytic rule. (Note that picking the block $\langle S_{1r}\rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r}\rangle$, we add one conjuct to the definition.)

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js}\rangle_s\rangle_j$, $\langle\langle S_{ir}\rangle_r\rangle_{i\neq 1}$ and $\langle S_{1r}\rangle$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir}'s are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Again note that picking the block $\langle S_{1r}\rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r}\rangle$, we add one conjuct to the definition.)

In the third conjunct (first one in the third line), the conjunction is over all non-trivial partitions of $V = S_1 \cdot \cdots \cdot S_n$ and i ranges over the number of S_i's, in this case $1 \leq i \leq n$.

The conjunct $\Box \exists p V'$ appears in the definition whenever V is of the form $(\Box \Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since G has weak H-uniform interpolation property, by definition there exist $J(V)$ as weak right p-interpolant of V. We choose one such $J(V)$ and denote it as $\exists^G p V$ and include it in the definition.

If U is the empty sequent define $\forall p U$ as 0. Otherwise, define $\forall p U$ as the following:

$$\bigvee_{L\mathcal{R}_{cs}} (\bigotimes_i [\bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \to \forall p T_{is})])$$

$$\vee \bigvee_{L\mathcal{R}_{cs}} ([\bigotimes_i \bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir})] \otimes [\bigotimes_j \bigwedge_s (\exists p \tilde{T}_{js} \to \forall p T_{js})])$$

$$\vee (\bigvee_{R\mathcal{R}} (\bigotimes_i \bigwedge_r (\exists p \tilde{S}_{ir} \to \forall p S_{ir})))$$

$$\vee \bigvee_{par} (\bigotimes_{i \neq 1} (\exists p S_i) \to \forall p S_1) \vee (\Box(\exists p \tilde{U'} \to \forall p U')) \vee (\forall^G p U).$$

In the first conjunct (the first line), the first big conjunction is over all context sharing semi-analytic rules that are backward applicable to V in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{is} \rangle_s \rangle_i$, $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is V, where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H. Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{js} \rangle_s \rangle_j$, $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is V, where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the third disjunct (the third line), the big disjunction is over all right semi-analytic rules backward applicable to U in H. The premise of the rule is $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is U.

In the fourth disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \cdots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \emptyset$ and $S_1 \neq U$. (Note that in this case, if $S^s = \emptyset$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists p U \to \forall p S_1$ where $\forall p S_1 = 0$. But this does not make any problem, since the definition of $\exists p U$ is prior to the definition of $\forall p U$.)

The fifth disjunct is on all semi-analytic modal rules with the result U and the premise U'. And finally, since G has weak H-uniform interpolation property, by definition there exist $I(U)$ as left weak p-interpolant of U. We choose one such $I(U)$ and denote it as $\forall^G p U$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents to prove both cases $\forall pU$ and $\exists pV$ simultaneously. First note that both $\forall pU$ and $\exists pV$ are p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pU) \subseteq V(U^a) \cup V(U^s)$ and $V(\exists pV) \subseteq V(V^a)$. Secondly, we have to show that:

(i) $V \cdot (\Rightarrow \exists pV)$ is derivable in H.

(ii) $U \cdot (\forall pU \Rightarrow)$ is derivable in H.

The proof is similar to the proof of the Theorem 2.5.38. Therefore, we will prove two cases, one for (i) and one for (ii), where there is a notable difference.

- In proving (i), we have to show that $V \cdot (\Rightarrow X)$ is derivable in H for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule $(R\wedge)$ it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show that $\Gamma \Rightarrow X$ is derivable in H.
  Consider the case where X is the first conjunct in the definition of $\exists pV$. In this case, we have to prove that for any context-sharing semi-analytic rules that is backward applicable to V in H, we have $V \cdot (\Rightarrow Y)$ in H, where $X = \bigwedge_{L\mathcal{R}_{cs}} Y$. Therefore, V is the conclusion of a context-sharing semi-analytic rule and is of the form $(\Gamma, \phi \Rightarrow)$ such that the premises are $\langle\langle T_{is}\rangle_s\rangle_i$ and $\langle\langle S_{ir}\rangle_r\rangle_i$, where T_{is} is of the form $(\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ and we have $\{\Gamma_1, \cdots, \Gamma_n\} = \Gamma$. Therefore, their orders are less than the order of V. Moreover, since $\tilde{T}_{is} = (T^a_{is} \Rightarrow)$ and $\tilde{S}_{ir} = (T^a_{ir} \Rightarrow)$ and they are subsequents of T_{is} and S_{ir}, their orders are less than or equal to the orders of T_{is} and S_{ir}. Hence, we can use the induction hypothesis for all of them.

  Using the induction hypothesis for T_{is}, $\tilde{T}_{is}$, S_{ir} and $\tilde{S}_{ir}$, for $i \neq 1$, we have the following

  $$\Gamma_i, \bar{\psi}_{is}, \forall pT_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\psi}_{is} \Rightarrow \exists p\tilde{T}_{is},$$

  $$\Gamma_i, \bar{\phi}_{ir}, \forall pS_{ir} \Rightarrow \quad , \quad \Gamma_i, \bar{\phi}_{ir} \Rightarrow \exists p\tilde{S}_{ir}.$$

  And using the induction hypothesis for S_{1r}, T_{1s} and $\tilde{T}_{1s}$ we have

  $$\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists pS_{1r} \quad , \quad \Gamma_1, \bar{\psi}_{1s}, \forall pT_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\psi}_{1s} \Rightarrow \exists p\tilde{T}_{1s}.$$

  Now, using the left context-sharing implication rule, we have

  $$\Gamma_i, \bar{\psi}_{is}, \exists p\tilde{T}_{is} \to \forall pT_{is} \Rightarrow \bar{\theta}_{is}$$

  $$\Gamma_i, \bar{\phi}_{ir}, \exists p\tilde{S}_{ir} \to \forall pS_{ir} \Rightarrow$$

  $$\Gamma_1, \bar{\psi}_{1s}, \exists p\tilde{T}_{1s} \to \forall pT_{1s} \Rightarrow \bar{\theta}_{1s}$$

  Now, first using the rules $(L\wedge)$ and $(R\vee)$, we have

$$\Gamma_i, \bar{\psi}_{is}, \bigwedge_s (\exists p\tilde{T}_{is} \to \forall p T_{is}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, \bigwedge_r (\exists p\tilde{S}_{ir} \to \forall p S_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s (\exists p\tilde{T}_{1s} \to \forall p T_{1s}) \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}.$$

For simplicity, denote $(\exists p\tilde{T}_{is} \to \forall p T_{is})$ as A_{is} and $(\exists p\tilde{S}_{ir} \to \forall p S_{ir})$ as B_{ir}. If we use the rule $(L\wedge)$ again, and the rule left weakening only for S_{1r}, and not changing the rule for T_{1r}, we have

$$\Gamma_i, \bar{\psi}_{is}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s A_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r}, \bigwedge_s A_{1s} \Rightarrow \bigvee_r \exists p S_{1r}.$$

Now, it is easy to see that the contexts are sharing and we can substitute the above sequents in the original rule. More precisely, in the original context-sharing semi-analytic rule consider $(\Gamma_i, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}))$ as the context of the premises (as Γ_i's in definition of a context-sharing semi-analytic rule 2.3.14) for $i \neq 1$ and consider $(\Gamma_1, \bigwedge_s A_{1s})$ as the context of the premises for $i = 1$ (as Γ_1's in definition of a context-sharing semi-analytic rule 2.3.14). Therefore, after substituting the above sequents in the original context-sharing semi-analytic rule, we conclude

$$\Gamma_1, \bigwedge_s A_{1s}, \Gamma_2, \cdots, \Gamma_n, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir})_{i\neq 1}, \phi \Rightarrow \bigvee_r \exists p S_{1r}$$

And finally, using the rule $L\otimes$ and $R\to$ we get

$$\Gamma, \phi \Rightarrow (\bigotimes_{i\neq 1} (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \otimes (\bigwedge_s A_{1s}) \to \bigvee_r \exists p S_{1r})$$

and this is what we wanted.

- To prove (ii), we have to show that $U \cdot (X \Rightarrow)$ is derivable in H for any X that is one of the disjuncts in the definition of $\forall pU$. Then, using the rule $(L\vee)$ it follows that $U \cdot (\forall pU \Rightarrow)$. Since U is of the form $(\Gamma \Rightarrow \Delta)$, we have to show that $(\Gamma, X \Rightarrow \Delta)$ is derivable in H.
  In the case that the disjunt is:

$$\bigvee_{L\mathcal{R}_{cs}} (\bigotimes_i [\bigwedge_r (\exists p\tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_s (\exists p\tilde{T}_{is} \to \forall p T_{is})]),$$

  we have to prove that for any context-sharing semi-analytic rule that is backward applicable to U in H we have

$$U \cdot (\bigotimes_i [\bigwedge_r (\exists p\tilde{S}_{ir} \to \forall p S_{ir}) \wedge \bigwedge_s (\exists p\tilde{T}_{is} \to \forall p T_{is})] \Rightarrow).$$

  The proof goes exactly as in the previous case (in proof of (i) for context-sharing semi-analytic rules), except that this time the succedents of S_{ir}'s and U are not empty and Δ_i's and Δ appear in their positions everywhere. And, we do not separate the cases T_{1s} and S_{1r} and we proceed with the proof considering the induction hypothesis for every i, in a uniform manner.

Note that these two cases were the cases for the only rule that is not considered in the proof of 2.5.38. For the proof of (i) for the other conjuncts and (ii) for the other disjuncts, we proceed with the proof of the corresponding cases as in the proof of 2.5.38, this time substituting $(\exists p\tilde{T}_{js} \to \forall pT_{js})$ for $\forall pT_{js}$ and $(\exists p\tilde{S}_{ir} \to \forall pS_{ir})$ for $\forall pS_{ir}$ wherever it is needed. One can easily see that the proof essentially goes as before, considering this minor change.

Secondly, we have to prove the following, as well.

(iii) For any p-free multisets Γ and Δ, if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $J(T), \Gamma \Rightarrow \Delta$ is derivable in H.

(iv) For any p-free multiset Γ, if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in H.

Again, since the spirit of the proof is the same as the proof of Theorem 2.5.38, we will prove two cases for the context-sharing semi-analytic rule, which were not present in the Theorem 2.5.38. We will prove (iii) and (iv) simultaneously using induction on the length of the proof and induction on the order of U and V as in the Theorem 2.5.38.

- To prove (iii), consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a context-sharing semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \Delta)$ is the conclusion of a context-sharing semi-analytic rule and V is of the form $(\Gamma, \phi \Rightarrow)$ and we want to prove $(\exists pV, \bar{C} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Gamma, \bar{C}, \phi \Rightarrow \Delta}$$

where $\bigcup\limits_j \Pi_j = \Pi$, $\bigcup\limits_i \Gamma_i = \Gamma$ and $\bigcup\limits_i \bar{C}_i = \bar{C}$.

Since, $\bar{C}_i$'s are in the context positions in the original rule, we can consider the same substition of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s as contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is backward applicable to V and T_{is}'s and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis for them. Furthermore, since $\tilde{T}_{is} = (T^a_{is} \Rightarrow)$ and $\tilde{S}_{ir} = (S^a_{ir} \Rightarrow)$, their orders are smaller than or equal to the orders of T_{is} and S_{ir} and we can use the induction hypothesis for them, as well. Using the induction hypothesis (informally speaking, for the first two premises, use the induction hypothesis of $\forall$, and for the last premise use the induction hypothesis of $\exists$) we get

$$(\bar{C}_i, \exists p\tilde{T}_{is} \Rightarrow \forall pT_{is}) \quad , \quad (\bar{C}_i, \exists p\tilde{S}_{ir} \Rightarrow \forall pS_{ir})_{i\neq 1} \quad , \quad (\bar{C}_1, \exists pS_{1r} \Rightarrow \Delta)$$

Now, first using the rules $(R \rightarrow)$ and then using the rule $(R\wedge)$ and $(L\vee)$ we have

$$(\bar{C}_i \Rightarrow \bigwedge_s (\exists p\tilde{T}_{is} \rightarrow \forall pT_{is}))$$

$$(\bar{C}_i \Rightarrow \bigwedge_r (\exists p\tilde{S}_{ir} \rightarrow \forall pS_{ir}))_{i\neq 1}$$

$$(\bar{C}_1, \bigvee_r \exists pS_{1r} \Rightarrow \Delta)$$

Denote $(\bigwedge_s \forall pT_{js})$ as A_j and $(\bigwedge_r \forall pS_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee_r \exists pS_{1r})$ as D. We have for $i \neq 1$

$$\bar{C}_i \Rightarrow A_i \quad , \quad \bar{C}_i \Rightarrow B_i$$

and for $i = 1$ we have

$$\bar{C}_1 \Rightarrow A_1 \quad , \quad \bar{C}_1, D \Rightarrow \Delta.$$

Now, and using the rule $(R\wedge)$ for $i \neq 1$ we get $\bar{C}_i \Rightarrow A_i \wedge B_i$. Together with $\bar{C}_1 \Rightarrow A_1$ and using the rule $(R\otimes)$ we get

$$\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_n \Rightarrow \bigotimes_i (A_i \wedge B_i) \otimes A_1.$$

Consider the sequent $\bar{C}_1, D \Rightarrow \Delta$ and use the left weakening rule to get

$$\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_n, D \Rightarrow \Delta.$$

Now, use the rule left context-sharing implication to reach

$$\bar{C}, (\bigotimes_i (A_i \wedge B_i) \otimes A_1) \rightarrow D \Rightarrow \Delta.$$

And, we are done.

- For the proof of (iv), consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a context-sharing semi-analytic rule and $\phi \in \bar{C}$. Therefore,

$$U \cdot (\bar{C} \Rightarrow) = \Gamma, \bar{X}, \phi \Rightarrow \Delta$$

is the conclusion of a context-sharing semi-analytic rule and U is of the form $\Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \phi$ and we want to prove $\exists p\tilde{U}, \bar{X}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{X}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \Delta}$$

where $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, and $\bigcup_i \Delta_i = \Delta$. Consider $T_{is} = (\Gamma_i \Rightarrow)$, $S_{1r} = (\Gamma_1 \Rightarrow \Delta_1)$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{is}'s do not depend on the suffix s, we have $T_{i1} = \cdots = T_{is}$ and we denote it by T_i. And, since S_{ir}'s do not depend on r for $i \neq 1$, we have $S_{21} = \cdots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1. Therefore, $S_1, \cdots, S_n$ is a partition of U. First, consider the case that $S_1 \neq U$. Then the order of all of them are less than the order of U (or in some cases that one of the others equals to U, the length of the proof is shorter) and since the rule is context sharing semi-analytic and ϕ is p-free then $\bar{\psi}_{is}$ and $\bar{\phi}_{ir}$ are also p-free, we can use the induction hypothesis to get (for $i \neq 1$):

$$\exists p T_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is} \quad , \quad \exists p S_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \quad , \quad \exists p \tilde{S}_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall p S_1$$

Note that for every $i \neq 1$ we have $T_i = S_i$ and for $i = 1$ we have $T_1 = \tilde{S}_1$ and we can rewrite the above sequents according to this new information. Hence, if we let $\{\exists p T_i, \bar{X}_i\}$ and $\{\forall p S_1\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists p T_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \qquad \langle\langle \exists p T_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow\rangle_r\rangle_{i\neq 1} \qquad \langle \exists p T_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall p S_1\rangle_r}{\exists p T_1, \cdots, \exists p T_n, \bar{X}, \phi \Rightarrow \forall p S_1}$$

Using first the rule $(L\otimes)$ and second the rule $R \rightarrow$ we get

$$\exists p T_1, \bar{X}, \phi \Rightarrow \bigotimes_{i\neq 1} \exists p T_i \rightarrow \forall p S_1$$

Since $T_2, \cdots, T_n, S_1$ is a partition of U, the right hand side of the above sequent is appeared as one of the disjuncts in the definition of $\forall p U$. And since $T_1 = \tilde{U}$, we have

$$\exists p \tilde{U}, \bar{C} \Rightarrow \forall p U$$

and we are done.

We have to investigate the case when $S_1 = U$, as well. However, the line of reasoning is as above and as in the case of $\forall p U$, and $\phi \in \bar{C}$ in the proof of the Theorem 2.5.38. The important thing is that in the case where $S_1 = U$, with similar reasoning as above, at the end we get $\exists p \tilde{S}_1, \bar{C} \Rightarrow \forall p S_1$ which solves the problem. Note that this case is one of the main reasons that we have changed uniform interpolation to weak uniform interpolation.

And finally, to prove (iii) and (iv) for the other cases, use similar reasoning as in the proof of Theorem 2.5.38, this time substituting $(\exists p \tilde{T}_{js} \rightarrow \forall p T_{js})$ for $\forall p T_{js}$ and $(\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir})$ for $\forall p S_{ir}$ wherever it is needed, then the proof easily follows. □

Theorem 2.5.43. *Any terminating single-conclusion sequent calculus H that extends* **IPC** *and consists of focused axioms, single-conclusion semi-analytic and context-sharing semi-analytic rules, has weak H-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.37 and the Theorem 2.5.42. □

Corollary 2.5.44. *If* **IPC** $\subseteq L$ *and* L *has a terminating single-conclusion sequent calculus consisting of focused axioms, single-conclusion semi-analytic rules and context-sharing semi-analytic rules, then* L *has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.43 and the Theorem 2.5.36. ☐

Corollary 2.5.45. *[35] The logic* **IPC** *has uniform interpolation.*

Proof. Use **G4i**, the Dyckhoff terminating calculus for **IPC**, introduced in the Preliminaries section. Using the Theorem 2.5.36, it is enough to show that this system has weak **G4i**-uniform interpolation. For this matter, note that all the rules in this calculus, except the rules $(L_4 \to)$ and $(L_1 \to)$ are semi-analytic, while $(L_4 \to)$ is context-sharing semi-analytic and all the axioms are focused. Therefore, the system has only one rule beyond our context-sharing semi-analytic machinery, namely $(L_1 \to)$. However, note that the proof for the Theorem 2.5.42 is pretty modular which addresses any rule separately by adding its corresponding disjunct or conjunct in the recursive definition of $\forall pS$ and $\exists pS$, respectively. Therefore, to prove the claim it is enough to add other disjunct and conjunct terms to also address the rule $(L_1 \to)$. This is what we will implement in the following:

For $\forall pS$ add the following terms as disjuncts to the definition of $\forall pS$ as defined in the proof of the Theorem 2.5.42:

$\forall^1_{at} pS$ For any atom $q \neq p$ if $q \in S^a$ add $q \to (\exists p\tilde{S}' \to \forall pS')$ where S' is S after eliminating one occurrence of q in S^a.

$\forall^2_{at} pS$ For any atom $q \neq p$ if $q \to \psi \in S^a$ for some formula ψ add $(\exists p\tilde{S}' \to \forall pS') \wedge q$ where S' is S after replacing one occurrence of $q \to \psi$ by ψ in S^a.

And for $\exists pS$ add the following terms as conjuncts:

$\exists^1_{at} pS$ For any atom $q \neq p$ if $q \in S^a$ add $q \wedge \exists pS'$ where S' is S after eliminating one occurrence of q in S^a.

$\exists^2_{at} pS$ For any atom $q \neq p$ if $q \to \psi \in S^a$ for some formula ψ add $q \to \exists pS'$ where S' is S after replacing one occurrence of $q \to \psi$ by ψ in S^a.

The first thing to check is that based on the well-founded order on the sequents used for the system **G4i**, the sequent S' in all cases is below the sequent S and hence the recursive step is well-defined. This is clear because in two cases S' is a proper subsequent of S and in two other cases, we are replacing a formula of the form $q \to \psi$ by ψ which has lower rank according to the rank function we introduced in the Preliminaries. Secondly, note that the number of disjuncts or conjuncts that we are adding are clearly finite and hence $\forall pS$ and $\exists pS$ are well-defined as formulas. Finally, note that we are only using $q \neq p$ in the terms and hence $\forall pS$ and $\exists pS$ remain p-free. Moreover, since in all cases $V(S') \subseteq V(S)$, by induction on the Dyckhoff's order we have $V(\forall pS) \subseteq V(S^a) \cup V(S^s)$ and $V(\exists pS) \subseteq V(S^a)$.

Now we have to check that adding these terms respects the properties that we have discussed in the proof of the Theorem 2.5.42. First, let us check that adding the disjuncts $\forall^1_{at} pS$ and $\forall^2_{at} pS$ to $\forall pS$ respects the property (ii) namely $\mathbf{G4i} \vdash S \cdot (\forall pS \Rightarrow)$. We have two cases to check:

For $\forall^1_{at} pS$, let us assume that $S = (\Gamma, q \Rightarrow \Delta)$. Then it is enough to prove that $\Gamma, q, q \to (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$ where $S' = (\Gamma \Rightarrow \Delta)$. Using the rule $(L_1 \to)$, it is enough to prove the sequent $\Gamma, q, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$. But note that by the IH, we have $\Gamma \Rightarrow \exists p\tilde{S}'$ and $\Gamma, \forall pS' \Rightarrow \Delta$. Therefore, by applying $(L \to)$ and weakening by q (both admissible in $\mathbf{G4i}$) we have $\Gamma, q, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$.

For $\forall^2_{at} pS$, let us assume that $S = (\Gamma, q \to \psi \Rightarrow \Delta)$. Then $S' = (\Gamma, \psi \Rightarrow \Delta)$ and we want to prove that $\Gamma, q \to \psi, (\exists p\tilde{S}' \to \forall pS') \wedge q \Rightarrow \Delta$. Again using the rule $(L_1 \to)$ itself, it is enough to prove $\Gamma, q, \psi, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$. By IH we have $\Gamma, \psi \Rightarrow \exists p\tilde{S}'$ and $\Gamma, \psi, \forall pS' \Rightarrow \Delta$. By $(L \to)$ and weakening by q (both admissible in $\mathbf{G4i}$), we can prove $\Gamma, q, \psi, (\exists p\tilde{S}' \to \forall pS') \Rightarrow \Delta$.

Now we will show that adding the conjuncts $\exists^1_{at} pS$ and $\exists^2_{at} pS$ to $\exists pS$ respects the property (i) namely $\mathbf{G4i} \vdash S \cdot (\Rightarrow \exists pS)$ for any S such that $S^s = \emptyset$.

For $\exists^1_{at} pS$, let us assume that $S = (\Gamma, q \Rightarrow)$. Then it is enough to prove that $\Gamma, q \Rightarrow q \wedge \exists pS'$ where $S' = (\Gamma \Rightarrow)$. By the IH, we have $\Gamma \Rightarrow \exists pS'$ and hence we have what we wanted by $(\wedge R)$ and weakening by q.

For $\exists^2_{at} pS$ let us assume that $S = (\Gamma, q \to \psi \Rightarrow)$. Then $S' = (\Gamma, \psi \Rightarrow)$ and we want to prove that $\Gamma, q \to \psi \Rightarrow q \to \exists pS'$. Using the rule $(\to R)$, it is enough to prove $\Gamma, q, q \to \psi \Rightarrow \exists pS'$. By $(L_1 \to)$ itself, it is enough to prove $\Gamma, q, \psi \Rightarrow \exists pS'$. But by IH we have $\Gamma, \psi \Rightarrow \exists pS'$ which implies what we wanted.

Now we are ready to check the other conditions, meaning:

(iii) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in $\mathbf{G4i}$ then $\exists pS, \bar{C} \Rightarrow \bar{D}$ is derivable in $\mathbf{G4i}$ for any S that $S^s = \emptyset$.

(iv) For any p-free multiset $\bar{C}$, if $S \cdot (\bar{C} \Rightarrow)$ is derivable in $\mathbf{G4i}$ then $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$ is derivable in $\mathbf{G4i}$.

First let us prove (iv). It is enough to address the case that the last rule in the proof of $S \cdot (\bar{C} \Rightarrow)$ is the rule $(L_1 \to)$. There are four cases to consider:

- Both q and $q \to \psi$ are in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 2.5.42 where the main formula is in $\bar{C}$.
- Both q and $q \to \psi$ are not in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 2.5.42 where the main formula is not in $\bar{C}$.
- $q \to \psi \in \bar{C}$ and $q \notin \bar{C}$. Since $q \to \psi$ is in $\bar{C}$, it is p-free and hence $q \neq p$ and ψ is p-free. We have

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \to \psi \Rightarrow \Delta}$$

Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C'} = \bar{C} - \{q \to \psi\}$. Therefore, $S = (\Gamma', q \Rightarrow \Delta)$. Define $S' = (\Gamma' \Rightarrow \Delta)$. Since both q and ψ are p-free and S' is a proper subsequent of S and hence lower than S in the Dyckhoff's order, by IH, $\exists p\tilde{S'}, \bar{C'}, q, \psi \Rightarrow \forall pS'$. By $(L_1 \to)$ we have $\exists p\tilde{S'}, \bar{C'}, q, q \to \psi \Rightarrow \forall pS'$ Hence, $\bar{C'}, q \to \psi \Rightarrow q \to (\exists p\tilde{S'} \to \forall pS')$. Since the right hand-side is a disjunct in $\forall pS$, we have $q \to \psi, \bar{C'} \Rightarrow \forall pS$ and by weakening $\exists p\tilde{S}, q \to \psi, \bar{C'} \Rightarrow \forall pS$.

- $q \to \psi \notin \bar{C}$ and $q \in \bar{C}$. Since $q \in \bar{C}$, it is not p itself. Again, we have

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \to \psi \Rightarrow \Delta}$$

Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C'} = \bar{C} - \{q\}$. Therefore, $S = (\Gamma', q \to \psi \Rightarrow \Delta)$. Define $S' = (\Gamma', \psi \Rightarrow \Delta)$. Since q is p-free and S' is lower than S in the Dyckhoff's order, by IH, $\exists p\tilde{S'}, \bar{C'}, q \Rightarrow \forall pS'$. Hence, $\bar{C'}, q \Rightarrow (\exists p\tilde{S'} \to \forall pS') \wedge q$. Since the right hand-side is a disjunct in $\forall pS$, we have $\bar{C'}, q \Rightarrow \forall pS$ and by weakening $\exists p\tilde{S}, q, \bar{C'} \Rightarrow \forall pS$. .

For (iii), again there are four cases:

- Both q and $q \to \psi$ are in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 2.5.42 where the main formula is in $\bar{C}$.
- Both q and $q \to \psi$ are not in $\bar{C}$. This case is similar to the left semi-analytic case in the proof of the Theorem 2.5.42 where the main formula is not in $\bar{C}$.
- $q \to \psi \in \bar{C}$ and $q \notin \bar{C}$. Since $q \to \psi$ is in $\bar{C}$, it is p-free and hence $q \neq p$ and ψ is p-free. We have

$$\frac{\Gamma, q, \psi \Rightarrow \bar{D}}{\Gamma, q, q \to \psi \Rightarrow \bar{D}}$$

Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C'} = \bar{C} - \{q \to \psi\}$. Therefore, $S = (\Gamma', q \Rightarrow)$. Define $S' = (\Gamma' \Rightarrow)$. Since both q and ψ are p-free and S' is a proper subsequent of S and hence lower than S in the Dyckhoff's order, by IH, $\exists pS', \bar{C'}, q, \psi \Rightarrow \bar{D}$. By $(L_1 \to)$ we have $\exists pS', \bar{C'}, q, q \to \psi \Rightarrow \bar{D}$ Hence, $(\exists pS' \wedge q), \bar{C'}, q \to \psi \Rightarrow \bar{D}$. Since $(\exists pS' \wedge q)$ is a conjuct in $\exists pS$, we have $\exists pS, q \to \psi, \bar{C'} \Rightarrow \bar{D}$.

- $q \to \psi \notin \bar{C}$ and $q \in \bar{C}$. Since $q \in \bar{C}$, it is not p itself. again, we have

$$\frac{\Gamma, q, \psi \Rightarrow \bar{D}}{\Gamma, q, q \to \psi \Rightarrow \bar{D}}$$

Define $\Gamma' = \Gamma - \bar{C}$ and $\bar{C'} = \bar{C} - \{q\}$. Therefore, $S = (\Gamma', q \to \psi \Rightarrow)$. Define $S' = (\Gamma', \psi \Rightarrow)$. Since q is p-free and S' is lower than S in the Dyckhoff's order, by IH, $\exists pS', \bar{C'}, q \Rightarrow \bar{D}$. Hence by $(L_1 \to)$, we have $\bar{C'}, q \to \exists pS', q \Rightarrow \bar{D}$. Since $q \to \exists pS'$ is a conjunct in $\exists pS$, we have $\exists pS, \bar{C'}, q \Rightarrow \bar{D}$.

□

2.5.2 The Multi-conclusion Case

Finally we will move to the multi-conclusion case to handle the more general form of semi-analytic rules.

Theorem 2.5.46. *Let G and H be two multi-conclusion sequent calculi and H extends $\mathbf{CFL_e}$. Then if H is a terminating sequent calculus axiomatically extending G with multi-conclusion semi-analytic rules and G has strong H-uniform interpolation property, so does H.*

Proof. For a given sequent $S = (\Gamma \Rightarrow \Delta)$ and an atom p, we define a p-free formula, denoted by $\forall pS$ and we will prove that it meets the conditions for the strong left and right p-interpolants of S, respectively.

If S is the empty sequent define $\forall pS$ as 0. Otherwise, define $\forall pS$ as

$$\bigvee_{\mathcal{R}} (\bigotimes_{i} \bigwedge_{r} \forall pS_{ir}) \vee \bigvee_{par} (\bigoplus_{i} \forall pS_i) \vee (\Box \forall pS') \vee (\neg \Box \neg \forall pS'') \vee (\forall^G pS)$$

where the first disjunction is over all multi-conclusion semi-analytic rules backward applicable to S in H, which means the result is S and the premises are S_{ir}. Since H is terminating, there are finitely many of such rules. The second disjunction is over all non-trivial partitions of S. The third disjunction is over all semi-analytic modal rules with the result S and the premise S'. Moreover, If S is of the form $\Box\Gamma \Rightarrow$, then we consider S'' to be $\Gamma \Rightarrow$ and $\neg\Box\neg\forall pS''$ must be appeared in the definition of $\forall pS$. And finally $\forall^G pS$ is the strong left p-interpolant of a sequent S in G relative to H.

We define the strong right p-interpolant of S as $\neg\forall pS$ and we denote it by $\exists pS$. Note that if we prove $\forall pS$ is the strong left p-interpolant, it is easy to show that $\exists pS$ meets the conditions for the strong right p-interpolant. The reason is the following: First we have to show that $\Gamma \Rightarrow \Delta, \exists pS$ is provable in H. But we have $\Gamma, \forall pS \Rightarrow \Delta$ is provable in H and using the rule $(0w)$, we have $\Gamma, \forall pS \Rightarrow \Delta, 0$ which means $\Gamma \Rightarrow \Delta, \neg\forall pS$ is provable in H.
Secondly, we have to show that if for p-free multisets Σ and Λ, if $\Gamma, \Sigma \Rightarrow \Lambda, \Delta$ is derivable in G, then $\exists pS, \Sigma \Rightarrow \Lambda$ is derivable in H. However, we have $\Sigma \Rightarrow \Lambda, \forall pS$ is derivable in H and using the axiom $0 \Rightarrow$ we can use the rule $(L \to)$ to get $\Sigma, \neg\forall pS \Rightarrow \Lambda$ in H.

Now, let us prove that $\forall pS$ meets all of the conditions of a strong left p-interpolant. The proof is similar to the proofs of the Theorems 2.5.38 and 2.5.42. To prove the theorem we use induction on the order of the sequents. First note that $\forall pS$ is p-free by construction and since in all the rules the variables in the premises also occurs in the consequence, we have $V(\forall pS) \subseteq V(S^a) \cup V(S^s)$. Secondly, we have to show that:

(i) $S \cdot (\forall pS \Rightarrow)$ is provable in H.

We have to show that $\Gamma, X \Rightarrow \Delta$ is derivable in H for every disjunct X in the definition of $\forall pS$.

- In the case that the disjunct is $\bigvee_{\mathcal{R}}(\bigotimes_i \bigwedge_r \forall p S_{ir})$, we have to show that for any multi-conclusion semi-analytic rule $\mathcal{R}$ with the premises S_{ir} we have

$$S \cdot (\bigotimes_i \bigwedge_r \forall p S_{ir} \Rightarrow)$$

where S is of the form $(\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n)$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$. Note that since $S'_{ir}s$ are the premises of the rule, the order of all of them are less than the order of S and we can use the induction hypothesis for them. We have for every i and r

$$\Gamma_i, \bar{\phi}_{ir}, \forall p S_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$

Using the rule $(L\wedge)$ we have for every i

$$\Gamma_i, \bar{\phi}_{ir}, \bigwedge_r \forall p S_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$

Using $\Gamma_i, \bigwedge_r \forall p S_{ir}$ as the left context in the original rule (we can do this, since $\bigwedge_r \forall p S_{ir}$ does not depend on r and it only ranges over i), we have

$$\Gamma_1, \cdots, \Gamma_n, \langle \bigwedge_r \forall p S_{ir} \rangle_i, \phi \Rightarrow \Delta_1, \cdots, \Delta_n$$

and then using the rule $(L\otimes)$, we have

$$\Gamma_1, \cdots, \Gamma_n, (\bigotimes_i \bigwedge_r \forall p S_{ir}), \phi \Rightarrow \Delta_1, \cdots, \Delta_n.$$

- In the case that the disjunct is $\bigvee_{par} \bigoplus_i \forall p S_i$, we have to show that for any non-trivial partition $S_1, \cdots, S_n$ of S we have$S \cdot (\bigoplus_i \forall p S_i \Rightarrow)$ is derivable in H. Since the order of each S_i is less than the order of S, we can use the induction hypothesis for them and get $(\Gamma_i, \forall p S_i \Rightarrow \Delta_i)$. Using the rule $(L\oplus)$ we get $\Gamma_1, \cdots, \Gamma_n, (\bigoplus_i \forall p S_i) \Rightarrow \Delta_1, \cdots, \Delta_n$.

- The proof of case that the disjunct is $\Box \forall p S'$ is exactly the same as the similar case in the proof of the Theorem 2.5.38.

- In the case that the disjunct is $\neg\Box\neg\forall p S''$, the sequent S must have been of the form $(\Box\Gamma \Rightarrow)$ and S'' is of the form $(\Gamma \Rightarrow)$. Since the order of S'' is less than the order of S, we can use the induction hypothesis and get $(\Gamma, \forall p S'' \Rightarrow)$ is derivable in H. Using the rule $(0w)$ and then the rule $(R \to)$ we have $(\Gamma \Rightarrow \neg\forall p S'')$. Using the rule (K) we have $(\Box\Gamma \Rightarrow \Box\neg\forall p S'')$ and together with the axiom $(0 \Rightarrow)$ we can use the rule $(L \to)$ and we have $(\Box\Gamma, \neg\Box\neg\forall p S'' \Rightarrow)$ is derivable in H.

- The case for $\forall^G p S$, holds trivially by definition.

Second, we have to show that

(ii) For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\bar{C} \Rightarrow \forall p S, \bar{D}$ is derivable in H.

We will prove it using induction on the length of the proof and induction on the order of S. More precisely, first by induction on the order of S and then inside it, by induction on n, we will show:

- For any p-free multisets $\bar{C}$ and $\bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in G with length less than or equal to n, then $\bar{C} \Rightarrow \forall pS, \bar{D}$ is derivable in H.

First note that for the empty sequent, we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in G, then $\bar{C} \Rightarrow 0, \bar{D}$ is valid in H, which is trivial by the rule $(0w)$.

For the base of the other induction, note that if $n = 0$, it means that $\Gamma, \bar{C} \Rightarrow \bar{D}, \Delta$ is valid in G. Therefore, by Definition 2.5.35, $\bar{C} \Rightarrow \forall^G pS, \bar{D}$ and hence $\bar{C} \Rightarrow \forall pS, \bar{D}$ is valid in H.

For $n \neq 0$ we have to consider the following cases:

◦ Consider the case that the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule, ϕ, is one of C_i's). Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $(\Gamma \Rightarrow \Delta)$ and $\bar{C} = (\bar{X}, \phi)$ and we want to prove $(\bar{X}, \phi \Rightarrow \forall pS, \bar{D})$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i \rangle_r \rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_i \bar{X}_i = \bar{X}$, $\bigcup\limits_i \bar{D}_i = \bar{D}$ and $\bigcup\limits_i \Delta_i = \Delta$. Consider $S_{ir} = (\Gamma_i \Rightarrow \Delta_i)$. Since S_{ir}'s do not depend on the suffix r, all of them are equal and we denote it by S_i. Therefore, $S_1, \cdots, S_n$ is a partition of S. First, consider that it is a non-trivial partition of S. Then the order of all of them are less than the order of S and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{ir}$ and $\bar{\psi}_{ir}$ are also p-free, we can use the induction hypothesis to get for every i and r:

$$\bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \forall pS_i$$

If we let $\bar{X}_i$ and $\bar{D}_i, \forall pS_i$ be the contexts in the left side and right side in the original rule, respectively, we have the following

$$\bar{X}, \phi \Rightarrow \bar{D}, \forall pS_1, \cdots, \forall pS_n$$

Using the rule $(R\oplus)$ we have

$$\bar{X}, \phi \Rightarrow \bar{D}, \bigoplus_i \forall pS_i$$

Since the right side of the sequent is a disjunct in the definition of $\forall pU$, using the rule $(R\vee)$ we have $\bar{C}, \phi \Rightarrow \forall pS, \bar{D}$.

In the case that $S_1, \cdots, S_n$ is a trivial partition of S, it means that one of them equals S. W.l.o.g. suppose $S_1 = S$ and all of the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle\langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i\rangle_r\rangle_{i\neq 1} \qquad \langle \Gamma, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1, \Delta\rangle_r}{\Gamma, \phi, \bar{X} \Rightarrow \bar{D}, \Delta}$$

Therefore, $S \cdot (\phi_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1)$ for every r are premises of $S \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $S \cdot (\bar{C} \Rightarrow \bar{D})$ and since the rule is semi-analytic and ϕ is p-free then $\bar{\phi}_{1r}$ and $\bar{\psi}_{1r}$ are also p-free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{X}_1 \Rightarrow \forall pS, \bar{\psi}_{1r}, \bar{D}_1)$. Substituting $\{\bar{X}_j\}$ and $\{\forall pS, \bar{D}_1\}$ as the contexts of the premises in the original rule we have

$$\frac{\langle\langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i\rangle_r\rangle_{i\neq 1} \qquad \langle \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall pS, \bar{\psi}_{1r}, \bar{D}_1\rangle_r}{\bar{X}, \phi \Rightarrow \forall pS, \bar{D}}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $\Gamma, \phi \Rightarrow \Delta$ and we want to prove $\bar{C} \Rightarrow \forall pS, \bar{D}$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i\rangle_r\rangle_i}{\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup\limits_i \Gamma_i = \Gamma$, $\bigcup\limits_i \bar{C}_i = \bar{C}$, $\bigcup\limits_i \bar{D}_i = \bar{D}$ and $\bigcup\limits_i \Delta_i = \Delta$.

Since, $\bar{C}_i$'s and $\bar{D}_i$'s are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take $\bar{C}_i$'s and $\bar{D}_i$'s in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma, \phi \Rightarrow \Delta}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$, we can claim that this rule is backward applicable to S and S_{ir}'s are the premises of the rule. Hence, their orders are less than the order of S and we can use the induction hypothesis for them. Using the induction hypothesis we get for every i and r

$$\bar{C}_i \Rightarrow \forall pS_{ir}, \bar{D}_i$$

Using the rule $(R\wedge)$ we get for every i

$$\bar{C}_i \Rightarrow \bigwedge_r \forall pS_{ir}, \bar{D}_i$$

and using the rule $(R\otimes)$ we get

$$\bar{C} \Rightarrow \bigotimes_i \bigwedge_r \forall pS_{ir}, \bar{D}.$$

Since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall pS$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall pS, \bar{D}$.

- Consider the case when the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'})$ is the conclusion of a semi-analytic modal rule. Hence, there are two cases to consider.

  The first one is the case where S is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box D'}$, where $||\overline{\Box D'}|| \leq 1$ and $S'' = (\Gamma \Rightarrow)$. We want to prove $(\bar{C} \Rightarrow \forall p S, \bar{D})$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{D'}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

  Since the order of S'' is less than the order of S and C' and D' are p-free, we can use the induction hypothesis and get

$$\bar{C'} \Rightarrow \forall p S'', \bar{D'}$$

  Using the axiom $(0 \Rightarrow)$ and the rule $(L \to)$ we have

$$\bar{C'}, \neg \forall p S'' \Rightarrow \bar{D'}$$

  Now, using the rule K or D (depending on the cardinality of $\bar{D'}$) we have

$$\overline{\Box C'}, \Box \neg \forall p S'' \Rightarrow \overline{\Box D'}$$

  and using the rule $(0w)$ and $(R \to)$ we get

$$\overline{\Box C'} \Rightarrow \neg \Box \neg \forall p S'', \overline{\Box D'}$$

  since we have $\neg \Box \neg \forall p S''$ as one of the disjuncts in the definition of $\forall p S$, we conclude $\bar{C} \Rightarrow \forall p S, \bar{D}$ using the rule $(R\vee)$.

  The second case is when S is of the form $\Box\Gamma \Rightarrow \Box D'$, where D' is a p-free formula and S' is of the form $\Gamma \Rightarrow D$. We want to prove $\bar{C} \Rightarrow \forall p S$. Then we must have had the following instance of the rule

$$\frac{\Gamma, \bar{C'} \Rightarrow \bar{D'}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

  Since $\bar{C'}$ is in the context position of the original rule, we can consider the same substitution of meta-sequents as above in the original rule, except that we do not take $\bar{C'}$ in the context. More precisely, we reach the following instance of the original rule:

$$\frac{\Gamma \Rightarrow \bar{D'}}{\Box\Gamma \Rightarrow \overline{\Box D'}}$$

  Therefore, this rule is backward applicable to S and the order of the premise, S', is less than the order of S and we can use the induction hypothesis for that to reach $C' \Rightarrow \forall p S'$. Then we can use the rule K and we get $\overline{\Box C'} \Rightarrow \Box \forall p S'$, which is a disjunct in the definition of $\forall p S$ and we have $\bar{C} \Rightarrow \forall p S$.

- ◦ The case for the right multi-conclusion semi-analytic rules is similar to the cases for the left ones disccused in this proof, and the proof of other two cases are similar to the proof of the same cases in the Theorem 2.5.38.

□

Theorem 2.5.47. *Any terminating multi-conclusion sequent calculus H that extends* $\mathbf{CFL_e}$ *and consists of focused axioms and multi-conclusion semi-analytic rules, has strong H-uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.37 and Theorem 2.5.46. □

Corollary 2.5.48. *If* $\mathbf{CFL_e} \subseteq L$ *and L has a terminating multi-conclusion sequent calculus consisting of focused axioms and multi-conclusion semi-analytic rules, then L has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 2.5.47 and Theorem 2.5.36. □

Using the Theorem 2.5.48, we can extend the results of [2] and [7] to:

Corollary 2.5.49. *The logics* $\mathbf{CFL_e}$*,* $\mathbf{CFL_{ew}}$ *and* **CPC** *and their* **K** *and* **KD** *modal versions have uniform interpolation property.*

Proof. For $\mathbf{CFL_e}$, $\mathbf{CFL_{ew}}$, since all the rules of the usual calculus of these logics are semi-analytic and their axioms are focused and since in the absence of the contraction rule the calculus is clearly terminating, by Theorem 2.5.48, we can prove the claim. For **CPC** use the contraction-free calculus for which the proof goes as the other cases. □

In the negative side we use the negative results in [7], [16] and [17] to ensure that the following logics do not have uniform interpolation. Then we will use the Theorems 2.5.40, 2.5.44 and 2.5.48 to the non-existence of terminating calculus consisting only of semi-analytic and context-sharing semi-analytic rules together with focused axioms.

Corollary 2.5.50. *The logic* **K4** *does not have a terminating single-conclusion (multi-conclusion) sequent calculus consisting only of single conclusion (multi-conclusion) semi-analytic and context-sharing semi-analytic rules plus some focused axioms.*

Corollary 2.5.51. *Except the logics* **IPC***,* **LC***,* **KC***,* $\mathbf{Bd_2}$*,* **Sm***,* **GSc** *and* **CPC***, none of the super-intutionistic logics have a terminating single-conclusion sequent calculus consisting only of single conclusion semi-analytic rules and context-sharing semi-analytic rules plus some focused axioms.*

Corollary 2.5.52. *Except at most six logics, none of the extensions of* **S4** *have a terminating single-conclusion (multi-conclusion) sequent calculus consisting only of single conclusion (multi-conclusion) semi-analytic rules and context-sharing semi-analytic rules plus some focused axioms.*

3. Proof Complexity of Focussed Calculi

3.1 Introduction

In the field of proof theory, proof systems, as the main players of the game, deserve to be considered as the objects of the study themselves. Regarding this matter, there are various problems to attack. One of them is investigating whether some special kinds of proof systems exist and if they do, what properties they or their corresponding logics posses, including the Craig or uniform interpolation of the corresponding logic, or the complexity of proofs in the given proof system.

These problems have been studied by many researchers (for instance [10], [22] and [23]). In [22] and [23], Iemhoff inspected the relationship between a specific kind of proof system and the uniform interpolation property of the logic that the proof system captures. She introduced the so-called focused rules and axioms, and studied the sequent calculi only consisting of these rules and axioms, which she named focused calculi. Roughly speaking, a focused axiom is just a modest generalization of the axioms of the classical sequent-style proof system, **LK**. A focused rule is a rule where only one side of its sequents, either left or right, is active in all the premises and in the conclusion and also all the variables in its premises occur in its conclusion. For instance, the usual conjunction and disjunction rules in **LK**, are focused, while the cut rule is not. After her formalization of the focused rules and focused axioms, she provided a method to prove that a super-intuitionistic logic enjoys the uniform interpolation property if it has a terminating focused proof system. Since there are only seven super-intuitionistic logics with the uniform interpolation property, she finally excluded almost all the super-intuitionistic logics (except at most seven of them) from having a focused proof system.

Inspired by Iemhoff's work, [1] proposed a generalization of focused rules, called semi-analytic rules, to cover a wider range of proof systems for a wider range of logics. Stated informally, in a semi-analytic rule, the side condition is relaxed and the formulas can appear freely in any side of the sequents in the premises and the conclusion. Iemhoff's results in [22] and [23] are then strengthened to also hold for these rules. It implies that many substructural logics and almost all super-intuitionistic logics (except at most seven of them) do not have a sequent style proof system only consisting of semi-analytic rules and focused axioms.

This paper is a sequel of [1] in its extension of the negative results of [22] and [23] to the remaining cases in which the interpolation property exists. For this purpose, we change our focus from the existence of a proof system of some kind to its efficiency to show an exponential lower bound for the focused proof systems of a certain sort. Beside the clear impacts in the study of focused rules, these lower bounds can also be considered as the basic steps in a universal approach to the proof complexity of the propositional proof systems. In such an approach, we are interested in investigating the proof lengths of a given sequence of tautologies in a generically given proof system with a certain form of axioms and rules. The method we use here is the well-known technique in proof complexity called the feasible interpolation. It reduces a problem in proof complexity to a problem in circuit complexity by extracting a Boolean circuit for an interpolant from a given proof for an implication, where the size of the circuit is polynomially bounded by the size of the proof. The feasible interpolation property for various classical calculi has been studied by Krajíček[27], Pudlák[36], and Pudlák and Sgall[40]. For the intuitionistic calculus, the feasible interpolation theorem was proved by Pudlák in [39] based on the feasible witnessing of the disjunction property developed in [8]. Buss and Pudlák in [9] and Buss and Mints in [8] studied the connection between intuitionistic propositional proof lengths and Boolean circuits. In [21], Hrubeš showed the connection is tighter in the sense that the circuit in question in [9] and [8] is monotone. Here we will use the technique of [21] as we will explain in a moment. For more information on feasible interpolation and its role in proof complexity, the reader is refered to [38].

In this paper, we will prove two lower bounds, one for the classical logic and the other for super-intuitionistic logics. For the first one, we will define a natural subclass of the focused rules, which we will call *polarity preserving focused*, PPF, rules. Then, we show that there are **CPC**-tautologies with exponential proof lengths in any proof system only consisting of PPF rules and focused axioms, which we call PPF calculi, while they have polynomial proof lengths in **LK**. This shows an exponential speed-up of the Frege-style proof system for classical logic with respect to any PPF calculus. To prove the similar exponential lower bound for intuitionistically valid formulas, we first define *monotonicity preserving focused*, MPF, rules and subsequently MPF calculi. Then, we will use the mentioned lower bound technique developed by Hrubeš in [20] and [21] to obtain an exponential lower bound for the lengths of proofs of particular **IPC**-tautologies in MPF calculi, while they have polynomial length proofs in **LK**.

3.2 Preliminaries

In this section, we will present some definitions and notions that will be needed in the rest of the paper.

Note that any finite object O that we use here, such as a formula or a proof, can be represented by a fixed suitable binary string and by $|O|$ we mean the length of the string representing the object.

In this paper, we work with the usual propositional language $\{\wedge, \vee, \neg, \rightarrow, \bot, \top\}$. By **IPC** and **CPC** we mean intuitionistic and classical propositional logics, respectively. By meta-language, we mean the language in which we define the sequent calculi. A meta-formula is defined inductively; an atom and a formula symbol are meta-formulas and we can construct new meta formulas using the existing ones and the connectives of the language. A meta-multiset is a set of meta-formulas and meta-multiset variables. By $V(A)$, we mean the atoms and meta-formula variables of the meta-formula A.

By a sequent $\Gamma \Rightarrow \Delta$, we mean an expression where Γ and Δ are multisets and it is interpreted as $\bigwedge \Gamma \to \bigvee \Delta$. A meta-sequent is essentially a sequent defined by meta-multisets. A rule is an expression of the form:

$$\frac{T_1, \cdots, T_n}{T}$$

where T_i's and T are meta-sequents. A sequent calculus is a set of rules.

By monotone **LK**, **mLK**, we mean the sequent calculus consisting of the axioms of **LK**, the structural rules (exchange, weakening, contraction), and its usual conjunction and disjunction rules.

A calculus G is *sound* for logic L, if $G \vdash \Gamma \Rightarrow \Delta$ implies $L \vdash \bigwedge \Gamma \to \bigvee \Delta$. It is called *complete* if $L \vdash \bigwedge \Gamma \to \bigvee \Delta$ implies $G \vdash \Gamma \Rightarrow \Delta$ and *feasibly complete* if the length of the tree-like proof is polynomially bounded by the sequent, i.e., there exists a tree-like proof π of $\Gamma \Rightarrow \Delta$ in G such that $|\pi| \leq |\Gamma \Rightarrow \Delta|^{O(1)}$. We say that logic M is an *extension of logic* L, if $L \vdash A$ implies $M \vdash A$. We say a calculus H is an *extension of a calculus* G, if for any rule of G, if all the premises are provable in H, then the consequence is also provable in H. Moreover, H is called an *axiomatic extension* of G, when all the provable sequents of G are considered as axioms of H, and H can add some rules to them.

A logic L is called *sub-classical* if **CPC** extends L. In the same way, a calculus G is called sub-classical if **LK** extends G.

A logic L (calculus G) has the *Craig interpolation* property when for any formula $\phi \to \psi$ (sequent $\Gamma \Rightarrow \Delta$), if $L \vdash \phi \to \psi$ ($G \vdash \Gamma \Rightarrow \Delta$) then there exists a formula θ such that $V(\theta) \subseteq V(\phi) \cap V(\psi)$ ($V(\theta) \subseteq V(\Gamma) \cap V(\Delta)$) and $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ ($G \vdash \Gamma \Rightarrow \theta$ and $G \vdash \theta \Rightarrow \Delta$). The calculus G has *feasible interpolation* if for any tree-like proof π of $\Gamma \Rightarrow \Delta$, there exists an interpolant θ such that $|\theta| \leq |\pi|^{O(1)}$.

3.3 Focused Calculi

In this section we will give the definitions of the focused axioms, rules and calculi, which are the building blocks of the rest of the paper.

Definition 3.3.1. *A rule is called focused (a left focused rule, L, or a right focused rule, R) if it has one of the following forms:*

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_{r=1}^{m_i} \rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n} L \qquad \frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_{r=1}^{m_i} \rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n} R$$

where Γ_i's and Δ_i's are meta-multiset variables, $\bar{\phi}_{ir}$ is a multi-set of formulas, and $\bigcup_{i,r} V(\phi_{ir}) \subseteq V(\phi)$. By the notation $\langle\langle \cdot \rangle_r \rangle_i$, we mean the sequents first range over $1 \leq r \leq m_i$ and then over $1 \leq i \leq n$.

Example 3.3.2. *The usual conjunction and disjunction rules in **LK** are focused. On the other hand, the implication rules:*

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda} \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

are not focused, simply because both sides of the sequents are active.

Definition 3.3.3. *A sequent is called a focused axiom if it is of the following form:*

(1) *Identity axiom:* $(\phi \Rightarrow \phi)$

(2) *Context-free right axiom:* $(\Rightarrow \bar{\alpha})$

(3) *Context-free left axiom:* $(\bar{\beta} \Rightarrow)$

(4) *Contextual left axiom:* $(\Gamma, \bar{\phi} \Rightarrow \Delta)$

(5) *Contextual right axiom:* $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ *and* Δ *are meta-multiset variables and in* $2-5$*, the set of the variables of any two elements of* $\bar{\alpha}$*,* $\bar{\beta}$ *and* $\bar{\phi}$ *must be the same.*

Example 3.3.4. *It is easy to see that the axioms of **LK**,* $(\phi \Rightarrow \phi)$*,* $(\Gamma \Rightarrow \top, \Delta)$ *and* $(\Gamma, \bot \Rightarrow \Delta)$ *are focused. Here are some more examples which are not in* **LK***:*

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

First let us investigate the power of focused rules and focused axioms. The natural question to ask is whether it is possible to have a calculus consisting only of these rules and axioms, that is complete for some given logic. For **CPC** the answer is yes, and the following theorem can be considered as a witness of the power and naturalness of focused axioms and rules.

Theorem 3.3.5. **CPC** *has a sequent calculus consisting only of focused rules and focused axioms.*

Proof. Consider a sequent calculus containing the usual axioms of **CPC** and the following axioms:

Axioms:

$$\frac{}{\phi \Rightarrow \phi} \qquad \frac{}{\phi, \neg\phi \Rightarrow} \qquad \frac{}{\Rightarrow \phi, \neg\phi}$$

$$\frac{}{\Gamma \Rightarrow \neg\bot, \Delta} \qquad \frac{}{\Gamma, \neg\top \Rightarrow \Delta}$$

The usual left and right rules for disjunction and conjunction and the following rules for implication:

$$\frac{\Gamma \Rightarrow \neg\phi, \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta} \qquad \frac{\Gamma_1, \neg\phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \to \psi \Rightarrow \Delta_1, \Delta_2}$$

And finally, for any combination $\neg\vee$, $\neg\wedge$, and $\neg\neg$ we have the corresponding right and left rules, using De Morgan's laws. For instance, we have

$$\frac{\Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\phi \wedge \psi), \Delta} R\neg\wedge$$

It is easy to check that all the rules of this sequent calculus are focused and the system is sound and complete for **CPC**. The proof of the completeness part is based on the observation that if $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable in the usual calculus for classical logic, then $\Gamma, \neg\Delta \Rightarrow \neg\Gamma', \Delta'$ is provable in the new calculus. The proof is an easy application of induction on the length of the usual **LK** proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. □

So far, we have seen some definitions and a sequent calculus consisting only of focused axioms and rules. Now, it is time to examine how effective such a characterization can be. For this purpose, from now on we will restrict our investigations to two natural sub-classes of focused rules, polarity preserving focused, PPF rules, and monotonicity preserving focused, MPF rules.

Definition 3.3.6. *Let $\mathcal{P}$ be a set of meta-formula variables or atomic constants. A meta-formula ψ is called $\mathcal{P}$-monotone if for any $\phi \in \mathcal{P}$, all occurrences of ϕ in ψ are positive, i.e., ϕ does not occur in the scope of negations or in the antecedents of implications. A multiset Γ of meta-formulas is called $\mathcal{P}$-monotone if all of its elements are $\mathcal{P}$-monotone.*
A meta-formula is called monotone if it is constructed by conjunctions and disjunctions on meta-formula variables, atomic constants and variable-free formulas.

Remark 3.3.7. *Note that since any variable-free formula is classically equivalent to $\top$ or $\bot$, then any monotone formula in our sense is classically equivalent to the usual monotone formulas i.e., the formulas constructed from atomic formulas by applying conjunctions and disjunctions. Therefore, from now on, in the classical settings, we always assume that a monotone formula has the mentioned simpler form.*

Definition 3.3.8. *A focused rule is called polarity preserving,* PPF*, if it preserves $\mathcal{P}$-monotonicity backwardly for any $\mathcal{P}$, i.e., if the antecedent of the consequence is $\mathcal{P}$-monotone, then the antecedents of all the premises are also $\mathcal{P}$-monotone. It is monotonicity preserving,* MPF*, if it is focused and preserves monotonicity backwardly, in the same way.*

Example 3.3.9. *All analytic focused rules in the language of* **CPC***, the focused rules in which any formula in the premises is a subformula of a formula in the consequence, are both* PPF *and* MPF.

3.3.1 The Classical Case

Let us first see a relationship between focused calculi and the Craig interpolation property.

Theorem 3.3.10. *Let G be a sequent calculus extending* **mLK** *and only consisting of focused rules and focused axioms. Then, G has feasible interpolation property. Moreover, if the rules are also* PPF *and Γ is $\mathcal{P}$-monotone, then $\Gamma \Rightarrow \Delta$ has a feasible $\mathcal{P}$-monotone interpolant.*

Proof. We need to prove that to any provable sequent $\Gamma \Rightarrow \Delta$, we can assign a formula C such that $G \vdash \Gamma \Rightarrow C$ and $G \vdash C \Rightarrow \Delta$ and $V(C) \subseteq V(\Gamma) \cap V(\Delta)$. Use induction on the length of the proof π of the sequent $\Gamma \Rightarrow \Delta$ in G. If $\Gamma \Rightarrow \Delta$ is a focused axiom, it is easy to see that in different cases of the focused axioms, the interpolant C is either ϕ or $\bot$ or $\top$. We check the case 4 of the focused axioms. The rest are similar. In this case, we have to find C such that $\Gamma, \bar{\phi} \Rightarrow C$ and $C \Rightarrow \Delta$. We claim that $C = \bot$ works here. Note that in the focused axioms, since Γ and Δ are meta-multiset variables, we can substitute anything for them. Hence, we have $\Gamma, \bar{\phi} \Rightarrow \bot$, since it is an instance of the axiom 4 when Δ is substituted by $\bot$. And $\bot \Rightarrow \Delta$ is an instance of the axiom $\bot$ in **mLK** which is weaker than the system G by assumption.

For the rules, suppose the last rule used in the proof π is the following left focused rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

Then, by induction, there are formulas C_{ir} such that $\Gamma_i, \bar{\phi}_{ir} \Rightarrow C_{ir}$ and $C_{ir} \Rightarrow \Delta_i$. Using the right and left disjunction rules we have $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ and $\bigvee_r C_{ir} \Rightarrow \Delta_i$. By the left disjunction rule we have $\bigvee_i \bigvee_r C_{ir} \Rightarrow \Delta_1, \cdots, \Delta_n$. And if we substitute the sequents $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ in the original left focused rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_r C_{1r}, \cdots, \bigvee_r C_{nr}$ and then using the right disjunction rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_i \bigvee_r C_{ir}$.

Note that for any i and r, by induction we have $V(C_{ir}) \subseteq V(\Gamma_i \cup \{\phi_{ir}\}) \cap V(\Delta_i)$. Using this and the fact that for focused rules $\bigcup_{ir} V(\phi_{ir}) \subseteq V(\phi)$, we can easily show that $V(\bigvee_i \bigvee_r C_{ir}) \subseteq V(\Gamma \cup \{\phi\}) \cap V(\Delta)$, where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. Therefore, we have shown that $\bigvee_i \bigvee_r C_{ir}$ is the interpolant.

The case for a right focused rule is dual to the previous case.

The proof for the upper bound for the length of the interpolant goes as follows. We claim that our previously constructed interpolant C has the property $|C| \leq |\pi|^2$ and we will prove it by induction on π.

For the axioms, as we have seen, the interpolant is either ϕ (in the case that the sequent is of the form of the first axiom $(\phi \Rightarrow \phi)$) or $\bot$ or $\top$ (in other cases). In these cases, we have $|C| \leq |\pi|$.

For the left focused rules, we have shown that $C = \bigvee_i \bigvee_r C_{ir}$. Let $N_{\mathcal{R}}$ be the number of the premises of the rule $\mathcal{R}$, which is the last rule used in the proof. We have that the number of the formulas which appear in C , i.e. C_{ir}, is equal to $N_{\mathcal{R}}$. The rest of the symbols appeared in C are connectives, and the number of them is again equal to $N_{\mathcal{R}}$. Since the sequent $\Gamma \Rightarrow \Delta$ is the conclusion of a rule in G, the lengths of the proofs of its premises are less than the length of π and we can use the induction hypothesis for them. Then $|C| \leq \Sigma_{i,r}|C_{ir}| + N_{\mathcal{R}}$. By induction hypothesis we have $|C_{ir}| \leq |\pi_{ir}|^2$, where π_{ir} is the proof of the sequent whose interpolant is C_{ir}. But since the proof is tree-like, we have $\Sigma_{ir}|\pi_{ir}| \leq |\pi|$. It is easy to see that $|C| \leq \Sigma_{i,r}|\pi_{i,r}|^2 + N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|^2 + \Sigma_{i,r}|\pi_{i,r}| \leq (\Sigma_{i,r}|\pi_{i,r}|)^2 \leq |\pi|^2$, and the claim follows. We have usesd the fact that $N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|$. The latter is an easy consequence of the fact that the number of $\pi_{i,r}$ in total is $N_{\mathcal{R}}$.

Finally, for $\mathcal{P}$-monotonicity note that since Γ is $\mathcal{P}$-monotone and all the rules are PPF, all the antecedents in the proof must be $\mathcal{P}$-monotone, as well. Therefore, the interpolants of the axioms are $\mathcal{P}$-monotone. Because, for the axioms, except for the axiom $\phi \Rightarrow \phi$, the interpolants are variable-free and hence $\mathcal{P}$-monotone. And for the identity axiom $\phi \Rightarrow \phi$, the interpolant is ϕ itself which is also $\mathcal{P}$-monotone. Finally, since the interpolants are constructed by the interpolants of the axioms via disjunctions and conjunctions, the interpolant for $\Gamma \Rightarrow \Delta$ is also $\mathcal{P}$-monotone. □

The following theorem is our first example of the mentioned ineffectiveness of the combination of focused axioms and PPF rules. It shows that none of the combinations of focused axioms and PPF rules can simulate the cut rule in a feasible way.

Corollary 3.3.11. *There is no calculus G consisting of only focused axioms and* PPF *rules, sound and feasibly complete for* **CPC**. *More precisely, if G is a complete calculus for* **CPC**, *then there exists a sequence of* **CPC**-*valid sequents $\phi_n \Rightarrow \psi_n$, with polynomially short tree-like proofs in the Hilbert-style system or equivalently in* **LK** + **Cut** *such that $||\phi_n \Rightarrow \psi_n||_G$, the length of the shortest tree-like G-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* PPF *rules together with focused axioms are either incomplete or feasibly incomplete for* **CPC**.

Proof. Assume that G is a calculus for **CPC** consisting of PPF rules and focused axioms. In the following, we bring the definitions for clique and coloring formulas from [28]. Note that we use $[n]$ to denote $\{1, 2, \cdots, n\}$. Let $Clique^k_n(\bar{p}, \bar{q})$ be the proposition asserting that $\bar{q}$ is a clique of size at least k on a graph with vertices $[n]$. There are $\binom{n}{2}$ atoms p_{ij} where $p_{ij} = 1$ if and only if there is an edge between nodes $\{i, j\} \in \binom{n}{2}$. There are also $k.n$ atoms q_{ui} where their intended meaning is to describe a mapping from $[k]$ to $[n]$. $Clique^k_n(\bar{p}, \bar{q})$ is the conjunction of the following clauses:

- $\bigvee_{i \in [n]} q_{ui}$, all $u \leq k$,
- $\neg q_{ui} \vee \neg q_{uj}$, all $u \in [k]$ and $i \neq j \in [n]$,
- $\neg q_{ui} \vee \neg q_{vi}$, all $u \neq v \in [k]$ and $i \in [n]$,
- $\neg q_{ui} \vee \neg q_{vj} \vee p_{ij}$, all $u \neq v \in [k]$ and $\{i, j\} \in \binom{n}{2}$.

The proposition $Color^m_n(\bar{p}, \bar{r})$ asserts that $\bar{r}$ is an m-coloring of the same graph represented by $\bar{p}$ and also uses $n.m$ atoms r_{ia} where $i \in [n]$ and $a \in [m]$. $Color^m_n(\bar{p}, \bar{r})$ is the conjunction of the following clauses:

- $\bigvee_{a \in [m]} r_{ia}$, all $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ib}$, all $a \neq b \in [m]$ and $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ja} \vee \neg p_{ij}$, all $a \in [m]$ and $\{i, j\} \in \binom{n}{2}$.

Note that by the formalization of the Clique formula, every occurrence of $\bar{p}$ in $Clique^k_n(\bar{p}, \bar{q})$ is positive (which means it is monotone in $\bar{p}$). We know that for $m < k$, the formula $\neg Clique^k_n(\bar{p}, \bar{q}) \vee \neg Color^m_n(\bar{p}, \bar{r})$ is a tautology in classical logic which implies that

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

is **CPC**-valid.

First observe that by the Craig interpolation theorem for **CPC** and the fact that the antecedent is monotone in $\bar{p}$, there exists a monotone interpolant $I(\bar{p})$ such that

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow I(\bar{p}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

which means that if the graph H represented by $\bar{p}$ has a k-clique then $I(\bar{p}) = 1$ and if H has an m-coloring then $I(\bar{p}) = 0$. In other words, if $I(\bar{p}) \neq 0$ then H does not have an m-coloring and if $I(\bar{p}) \neq 1$ then H does not have a k-clique. By the result in [3], every such monotone interpolant I must have exponential length in n for suitable polynomially bounded choices for k and m.

Secondly, define $\phi_n(\bar{p}, \bar{q}) = Clique^k_n(\bar{p}, \bar{q})$ and $\psi_n(\bar{p}, \bar{r}) = \neg Color^m_n(\bar{p}, \bar{r})$. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$, serve as the **CPC**-valid sequents mentioned in the theorem. The idea is simple. First note that the fact that the sequent

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

has a tree-like proof of the size $n^{O(1)}$ in the classical Hilbert-style proof system or equivalently **LK** + **Cut** is a folklore well-known fact in the proof complexity community. Now pick π_n as the shortest tree-like proof of the sequent in G. Note that the antecedent of our sequent, $Clique^k_n(\bar{p}, \bar{q})$, is $\bar{p}$-monotone. Hence, by Lemma 3.3.10, the interpolant for the sequent $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$ will be $\bar{p}$-monotone, as well. And since $\bar{p}$ are the only common variables and hence the only variables in the interpolant, the interpolant is monotone. However, G captures **CPC**. Therefore, the whole process provides a classical monotone interpolant for the sequent

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

which we will call C_n. By Lemma 3.3.10, we have $|C_n| \leq |\pi_n|^2$. However, any such C_n should be exponentially long in n as we explained before. Therefore, the shortest proof π_n for our sequent is exponentially long. □

3.3.2 The Intuitionistic Case

It is also possible to lower down the previous exponential lower bound to the level of the **IPC**-valid sequents. For that purpose we need a new form of interpolation and its preservation theorem.

Definition 3.3.12. *A sequent is called a strongly focused axiom if it has one of the following forms:*

(1) $\phi \Rightarrow \phi$

(2) $\Rightarrow \bar{\alpha}$

(3) $\bar{\beta} \Rightarrow$

(4) $\Gamma, \bar{\phi} \Rightarrow \Delta$

(5) $\Gamma \Rightarrow \bar{\phi}, \Delta$

where in (2) *and* (5)*,* $\bar{\alpha}$ *and* $\bar{\phi}$ *have no variable and* Γ *and* Δ *are meta-multiset variables.*

Example 3.3.13. *For the strongly focused axioms, note that all the axioms of* **LK** *are strongly focused. An example of a focused axiom which is not strongly focused is* $(\Rightarrow \phi, \neg\phi)$*. Since otherwise it would have been an instance of either* 2 *or* 5*, which is not possible. The reason is that* ϕ *can have a variable which must not appear in the right side of the sequent.*

Definition 3.3.14. *Let* G *and* H *be two sequent calculi.* G *has* H*-monotone feasible interpolation with the exponent* $m \geq 1$ *if for any* k *and any sequent* $S = (\Sigma \Rightarrow \Lambda_1, \cdots, \Lambda_k)$ *if* S *is provable in* G *by a tree-like proof* π *and for any* $1 \leq j \leq k$*,* $\Lambda_j \neq \emptyset$*, then there exist formulas* $|C_j| \leq |\pi|^m$ *for* $1 \leq j \leq k$ *such that* $(\Sigma \Rightarrow C_1, \cdots, C_k)$ *and* $(C_j \Rightarrow \Lambda_j)$ *are provable in* H *and* $V(C_j) \subseteq V(\Sigma) \cap V(\Lambda_j)$*, where* $V(A)$ *is the set of the atoms of* A*. Moreover, if* Σ *is monotone, then* C_j *is also monotone for all* $1 \leq j \leq k$*. We call* C_j*'s, the interpolants of the partition* $\Lambda_1, \cdots, \Lambda_k$ *of the succedent of the sequent* S*. The system* G *has* H*-monotone feasible interpolation if it has* H*-monotone feasible interpolation with some exponent* $m \geq 1$*.*

Theorem 3.3.15. *Let* G *and* H *be two sequent calculi such that* G *is a set of strongly focused axioms,* H *extends* **mLK** *and any sequent in* G *is provable in* H*. Then* G *has* H*-monotone feasible interpolation with the exponent one.*

Proof. We will consider the strongly focused axioms one by one:

(1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. Therefore, $\Lambda_1 = \phi$. Pick $C_1 = \phi$. It is easy to see that this C_1 works and since ϕ is monotone, C_1 is also monotone.

(2) For the case $(\Rightarrow \bar{\alpha})$, consider C_j to be $\bigvee \Lambda_j$. We can easily see that these C_j's work, using the left and right disjunction rules. For the variables, since $V(\bar{\alpha}) = \emptyset$, we have $V(C_j) \subseteq V(\emptyset) \cap V(\Lambda_j)$. And for the monotonicity, since $V(C_j) = \emptyset$, then C_j is monotone.

(3) The case $(\bar{\beta} \Rightarrow)$ does not happen.

(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$ define $C_j = \bot$. First note that we have $\Gamma, \bar{\phi} \Rightarrow \bot, \bot, \cdots, \bot$ where in the right hand-side we have k many $\bot$'s. The reason is that this sequent is an instance of the axiom (4) itself. Moreover, for every j we have $\bot \Rightarrow \Lambda_j$ since it is an instance of the axiom $\bot$. And again $V(C_j) = \emptyset$.

(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C_j = \bigvee(\Lambda_j \cap \bar{\phi})$. It is easy to see that this C_j works. Because, $C_j \Rightarrow \Lambda_j$ is an instance of an axiom. We also have $\Gamma \Rightarrow C_1, \cdots, C_k$, since in the right hand-side we will have the formula $\bar{\phi}$ (together with some other formulas which we will treat as the context) and it will become an instance of the same axiom. Note that since $V(\bar{\phi}) = \emptyset$, there is nothing to check for the variables. For the monotonicity, note that $V(C_j) = \emptyset$, therefore C_j is monotone.

Note that in all cases and for all $1 \leq j \leq k$, $|C_j| \leq |\pi|$. □

The next theorem shows that MPF rules preserve the monotone feasible interpolation property. We will use this theorem later in the lower bound result that we have promised before.

Theorem 3.3.16. *(monotone feasible interpolation) Let G and H be two sequent calculi such that H extends* **mLK** *and axiomatically extends G by* MPF *rules. Then if G has H-monotone feasible interpolation property, so does H.*

Proof. To prove the theorem, we will prove the following claim:

Claim. Let G and H be two sequent calculi such that H extends **mLK** and axiomatically extends G by MPF rules and G has H-monotone feasible interpolation with the exponent m. Then for any H-provable sequent $\Gamma \Rightarrow \Delta$ and any non-trivial partition of Δ as $\Lambda_1, \cdots, \Lambda_k$ (non-trivial means that none of the Λ_j's are empty), there exist the required interpolants C_j as in the Definition 3.3.14 such that $\Sigma_j |C_j| \leq |\pi|^M$ where $M = m + 1$.

The proof uses induction on the H-length of π (the number of the rules of H in the proof π). First we will explain how to construct C_j's. Then we will prove the bound for the given construction.

If the H-length of π is zero, it means that the proof is in G. Hence the claim is clear by the assumption. There are two cases to consider based on the last rule of the proof.

- If the last rule used in the proof is a right focused one, then it is of the following form:

$$\frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_r\rangle_i}{\Gamma \Rightarrow \phi, \Delta}$$

  where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. And, again $\Lambda_1, \cdots, \Lambda_k$ are given such that they are non-empty and $\bigcup_{j=1}^{k} \Lambda_j = \Delta \cup \{\phi\}$. W.l.o.g. suppose $\phi \in \Lambda_1$ and we denote $\Lambda_1 - \{\phi\}$ by Λ'_1. Consider the case that all of the $\Lambda_{ij} = \Delta_i \cap \Lambda_j$ and $\bar{\phi}_{ir} \cup \Lambda'_{i1}$ are non-empty where $\Lambda'_{i1} = \Delta_i \cap \Lambda'_1$. By induction hypothesis for the premises, there exist formulas $D_{ir1}, \cdots, D_{irk}$ such that for every i, r and $j \neq 1$

$$D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \quad , \quad D_{irj} \Rightarrow \Lambda_{ij} \quad , \quad \Gamma_i \Rightarrow D_{ir1}, \cdots, D_{irk}$$

Again, note that if some of Λ_{ij}'s or $\bar{\phi}_{ir}, \Lambda'_{i1}$ are empty, we can eliminate them from the partition to have a non-trivial partition and hence to apply the IH. Then in these cases, we can simply pick D_{irj} as $\bot$. Now using the rules $(R\vee)$, $(L\vee)$, $(R\wedge)$ and $(L\wedge)$, we get for every i and $j \neq 1$

$$\bigwedge_r D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \ , \ \bigvee_r D_{irj} \Rightarrow \Lambda_{ij} \ , \ \Gamma_i \Rightarrow \bigwedge_r D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$$

Note that in the right sequent, we first use $(R\vee)$ to get
$\Gamma_i \Rightarrow D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$, and then we can use the rule $(R\wedge)$. Now, we can substitute the left sequents in the original rule to get

$$\bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

and using the rule $(L\wedge)$ we have

$$\bigwedge_i \bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

We denote $\bigwedge_i \bigwedge_r D_{ir1}$ by C_1. Using the rule $(L\vee)$ for the sequents $\bigvee_r D_{irj} \Rightarrow \Lambda_{ij}$ we get

$$\bigvee_i \bigvee_r D_{irj} \Rightarrow \Lambda_j$$

and we denote $\bigvee_i \bigvee_r D_{irj}$ by C_j for $j \neq 1$. We can see that first using the rule $(R\vee)$ and after that using the rule $(R\wedge)$ we get

$$\Gamma \Rightarrow \bigwedge_i \bigwedge_r D_{ir1}, \bigvee_i \bigvee_r D_{ir2}, \cdots, \bigvee_i \bigvee_r D_{irk}$$

which is

$$\Gamma \Rightarrow C_1, \cdots, C_k$$

It only remains to check the variables. If a variable is in C_j, then it is in one of D_{irj}'s. By induction hypothesis we have $V(D_{ir1}) \subseteq V(\Gamma_1) \cap V(\{\{\bar{\phi}_{ir}\} \cup \Lambda'_{i1}\}) \subseteq V(\Gamma) \cap V(\{\{\phi\} \cup \Lambda'_1\})$ and $V(D_{irj}) \subseteq V(\{\Gamma_i\}) \cap V(\Lambda_{ij}) \subseteq V(\Gamma) \cap V(\Lambda_j)$, since the rule is occurrence preserving, and this is what we wanted.

- The case of the left focused rule is similar to the case for right.

For the monotonicity part, since the extending rules are MPF, it is easy to prove that if the antecedent of the consequence is monotone, then all the antecedents, everywhere in the proof up to the sequents in G, are also monotone. Since G has H-monotone feasible interpolation property, the interpolants in the base case are monotone. Finally, since the conjunctions and disjunctions do not change monotonicity, our constructed interpolants are also monotone.

For the upper bound part, use a similar proof to the corresponding part in Lemma 3.3.10, this time using the induction on π to show that $\Sigma_j |C_j| \leq |\pi|^M$. For the axioms note $|C_j| \leq |\pi|^m$ for $1 \leq j \leq k$ by the assumption that G has H-monotone feasible interpolation with the exponent m. Since the partition is non-trivial $k \leq |S| \leq |\pi|$, hence $\Sigma_{j=1}^k |C_j| \leq k|\pi|^m \leq |\pi|^{m+1} = |\pi|^M$.
For the rules, define X as the set of all (i, r, j)'s where D_{irj} is $\bot$ coming from handling the empty cases. It is clear that X has at most $N_{\mathcal{R}}$ elements, the number of the premises of the rule $\mathcal{R}$. We have $\Sigma_j |C_j| \leq \Sigma_{(i,r,j)\notin X} |D_{irj}| + |X| + N_{\mathcal{R}} \leq \Sigma_{ir} |\pi_{ir}|^M + 2N_{\mathcal{R}} \leq (\Sigma_{ir} |\pi_{ir}| + 1)^M \leq |\pi|^M$. The second inequality holds using the induction hypothesis and the third inequality holds because $N_{\mathcal{R}} \leq \Sigma_{ir} |\pi_{ir}|$ and $M \geq 2$.

Finally, the theorem is a clear consequence of the Claim. It is enough to apply the Claim to provide the formulas C_j such that $\Sigma_j |C_j| \leq |\pi|^M$ which implies $|C_j| \leq |\pi|^M$. □

Lemma 3.3.17. *[21] Let $A(\bar{p}, \bar{r_1})$ and $B(\bar{q}, \bar{r_2})$ be propositional formulas and $\bar{p}$, $\bar{q}$, $\bar{r_1}$ and $\bar{r_2}$ be mutually disjoint. Let $\bar{p} = p_1, \cdots, p_n$ and $\bar{q} = q_1, \cdots, q_n$. Assume that A is monotone in $\bar{p}$ or B is monotone in $\bar{q}$ and $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Then*

$$\bigwedge_{i=1}^n (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

is **IPC***-valid.*

Proof. For the details, the reader is referred to [21]. □

Theorem 3.3.18. *Let G and H be two sequent calculi such that H is sub-classical, extends* **mLK***, axiomatically extends G by* MPF *rules and G has H-monotone feasible interpolation property. Then there exists a family of* **IPC***-valid sequents $\phi_n \Rightarrow \psi_n$ with the length of $\phi_n \Rightarrow \psi_n$ bounded by a polynomial in n such that either there exists some n such that $H \nvdash \phi_n \Rightarrow \psi_n$ or $||\phi_n \Rightarrow \psi_n||_H$, the shortest tree-like H-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* MPF *rules together with strongly focused axioms are either incomplete or feasibly incomplete for* **IPC**.

Proof. The proof is similar and also inspired by the lower bound proof given in [21]. Similar to the proof of Theorem 3.3.11, consider the **CPC**-valid sequent

$$Clique_n^k(\bar{p}, \bar{r_2}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r_1})$$

which is equivalent to

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

Then, using the Lemma 3.3.17, if we rewrite $\neg Clique_n^k(\bar{p}, \bar{r_2})$ as $B(\neg\bar{p}, \bar{r_2})$ and $\neg Color_n^m(\bar{p}, \bar{r_1})$ as $A(\bar{p}, \bar{r_1})$, we can easily see that A is monotone in $\bar{p}$ and the formula $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Hence, we can transfer the **CPC**-valid sequent

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

to a sequent of the form

$$\bigwedge_i (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

valid in **IPC**. Now, let

$$\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r_1}), \theta_n(\bar{q}, \bar{r_2})$$

be this sequent. We will show that this family of sequents

$$\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r_1}), \theta_n(\bar{q}, \bar{r_2})$$

serve as the **IPC**-valid sequents mentioned in the theorem.

If for some n we have $H \nvdash \phi_n \Rightarrow \psi_n, \theta_n$, then the claim follows. Therefore, suppose that for every n we have $H \vdash \phi_n \Rightarrow \psi_n, \theta_n$. Let π_n be the shortest tree-like proof of the sequent $\phi_n \Rightarrow \psi_n, \theta_n$ in H. By Theorem 3.3.16, for every n, there exist monotone formulas $C_n(\bar{p})$ and $D_n(\bar{q})$ such that $|C_n| \leq |\pi_n|^{O(1)}$ and $|D_n| \leq |\pi_n|^{O(1)}$ and the followings are provable in H: $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$. Since H captures a sub-classical logic we have $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$ in **CPC**. Since $(\phi_n \Rightarrow C_n, D_n)$ is valid in classical logic, we have $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$. On the other hand, since A_n is classically equivalent to ψ_n we know that $C_n(\bar{p}) = 1$ implies $A(\bar{p}, \bar{r_1}) = 1$. Similarly, we have that $D_n(\bar{q}) = 1$ implies $B(\bar{q}, \bar{r_2}) = 1$. We Claim that $C_n(\bar{p})$ interpolates $\neg B(\neg\bar{p}, \bar{r_2}) \Rightarrow A(\bar{p}, \bar{r_1})$. One direction is proved. For the other direction, note that if $B(\neg\bar{p}, \bar{r_2}) = 0$ then $D_n(\neg\bar{p}) = 0$ and since $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$ we have $C_n(\bar{p}) = 1$. Hence the monotone formula C_n interpolates $\neg B(\neg\bar{p}, \bar{r_2}) \Rightarrow A(\bar{p}, \bar{r_1})$ or equivalently the sequent

$$Clique^k_n(\bar{p}, \bar{r_2}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r_1})$$

However, in the proof of the Theorem 3.3.11, we mentioned that any such monotone interpolant must have exponential length. Together with the fact that $|C_n(\bar{p})| \leq |\pi_n|^{O(1)}$, we can conclude that $||\phi_n \Rightarrow \psi_n, \theta_n||_H$ is exponential in n which implies the claim. □

Corollary 3.3.19. *There is no calculus consisting only of strongly focused axioms and* MPF *rules, sound and feasibly complete for super-intuitionistic logics.*

Proof. This is an obvious consequence of Theorem 3.3.16, Theorem 3.3.15 and Theorem 3.3.18 . The only point that we have to explain is that if a calculus G consisting only of strongly focused axioms and MPF rules is sound and complete for a super-intuitionistic logic, then G extends **mLK**. The reason is that G is complete for a super-intuitionistic logic and any calculus complete even for **IPC** extends **mLK**. □

4. Proof Complexity of Substructural Calculi

4.1 Introduction

Propositional proof complexity, as a new independent field, was established predominantly to address the fundamental unsolved problems in computational complexity. Starting steps in this systematic study were taken by Cook and Reckhow. In their seminal paper [11], they defined a propositional proof system, PPS, as a polynomial-time computable function whose range is the set of all classical propositional tautologies. Then, they defined a polynomially bounded proof system as a PPS having a short proof for any tautology, i.e., a proof whose length is polynomially bounded by the length of the tautology itself. They proved that the existence of a polynomially-bounded proof system for the classical logic is equivalent to NP = coNP. Accordingly, if for any PPS there are super-polynomial lower bounds on the lengths of proofs, as a result NP will be different from coNP and consequently, P will be different from NP. Since these are considered to be major open problems in computational complexity, providing super-polynomial lower bounds for all PPS's gained momentum in the field of proof complexity of classical proof systems. Thus far, exponential lower bounds on proof lengths have been established in many different propositional proof systems, including resolution [18], cutting planes [37], and bounded-depth Frege systems [34]. For more on the lengths of proofs, see [28].

Aside from the extensive study of some well-known classical proof systems, recently there have been some investigations into the complexity of proofs in non-classical logics on account of their various applications, their power in expressibility and their essential role in computer science. Therefore, it is important to fully understand the inherent complexity of proofs in non-classical logics, considering specially the impact that lower bounds on lengths of proofs will have on the performance of the proof search algorithms. Moreover, from the computational complexity perspective, the study of complexity of proofs in non-classical logics is associated with another major computational complexity problem, namely the NP vs. PSPACE problem. Various results have been acheived in this area, for instance exponential lower bounds for the intuitionistic and modal logics [21], and for modal and intuitionistic Frege and extended Frege systems [25]. A comprehensive overview of results concerning proof complexity of non-classical logics can be found in [6].

In the realm of non-classical logics, substructural ones are logics originally defined by the systems where some or all of the usual structural rules are absent. These logics include relevant logics, linear logic, fuzzy logics, and many-valued logics. However, the field is more ambitious than any limited investigation of possible effects of the structural rules. The purpose of the study of substructural logics is to uniformly investigate the non-classical logics that originated from different motivations. Complexity-theoretically, several substructural logics are PSPACE-complete, for instance the multiplicative-additive fragment of linear logic, **MALL** [30], and full Lambek calculus, **FL** [26]. Check also the PSPACE-hardness for a wide range of substructural logics and PSPACE-completeness for a class of extensions of **FL** in [19]. Some complexity results about the decision problem of some fragments of Visser's basic propositional logic, BPC, and formal propositional logic, FPL are also studied in [42].

In this paper, we will study the proof complexity of proof systems for substructural logics and basic logic, and hence a wide-range class of proof systems. More precisely, we will start with an arbitrary proof system **P** at least as strong as **FL** (or **BPC**) and polynomially simulated by an extended Frege system for some super-intuitionistic infinite branching logic L, denoted by $\mathsf{L}-\mathbf{EF}$. For such a **P**, we will provide a sequence of hard **P**-tautologies, namely a sequence of **P**-provable formulas $\{A_n\}_{n=1}^{\infty}$ with length polynomial in n such that their shortest **P**-proofs are exponentially long in n. Our method is using a sequence of intuitionistic tautologies for which we know there exists an exponential lower bound on the length of their proofs in any $\mathsf{L}-\mathbf{EF}$, where L is infinite branching. Since these formulas are not necessarily provable in **P**, the essential step is their modification so that they become provable in **FL** (or **BPC**) and hence in **P**, while they remain hard for $\mathsf{L}-\mathbf{EF}$. Finally, since $\mathsf{L}-\mathbf{EF}$ is shown to be polynomially as strong as **P**, the length of any **P**-proofs of the **P**-tautologies must be exponential in n. Furthermore, using the same **FL**-tautologies, one can infer an exponential lower bound also for proof systems polynomially simulated by $\mathbf{CFL}_{\mathbf{ew}}^{-}$, where the superscript "$-$" means the sequent calculus does not have the cut rule.

4.2 Preliminaries

In this section we provide some background and also some new notions needed in the future sections. Throughout the paper we mainly work with substructural logics and we follow [14] as the canonical source for the study of the theory of such logics. Nevertheless, to make the paper as self-contained as possible, we include all necessary background information.

4.2.1 Substructural logics

Consider the propositional language $\{\wedge, \vee, \otimes, \top, \bot, 1, 0, /, \backslash, \rightarrow\}$. The logical connective $\otimes$ is called fusion and the connectives $/$ and $\backslash$ are called left and right residuals, respectively. Throughout the paper, small Roman letters, $p, q, \ldots$, are reserved for propositional variables, Greek small letters ϕ, ψ, $\ldots$, and Roman capital letters A, B, $\ldots$, are meta-variables for formulas and Greek capital letters Γ, Σ, $\ldots$, are meta-variables for (possibly empty) finite sequences of formulas, separated by commas (unless specified otherwise).

Consider the following set of rules over sequents of the form $\Gamma \Rightarrow \Delta$. The meta-variable Γ is called the antecedent of the sequent and Δ its succedent. All the rules are presented in the form of schemes. Therefore, an instance of a rule is obtained by substituting formulas for lower case letters and finite (possibly empty) sequences of formulas for upper case letters.

Initial sequents:

$$\phi \Rightarrow \phi \qquad \Gamma \Rightarrow \Delta, \top, \Lambda \qquad \Gamma, \bot, \Sigma \Rightarrow \Delta \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

Structural rules:

Weakening rules:

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, \phi, \Sigma \Rightarrow \Delta}\,(Lw) \qquad \frac{\Gamma \Rightarrow \Delta, \Lambda}{\Gamma \Rightarrow \Delta, \phi, \Lambda}\,(Rw)$$

Contraction rules:

$$\frac{\Gamma, \phi, \phi, \Sigma \Rightarrow \Delta}{\Gamma, \phi, \Sigma \Rightarrow \Delta}\,(Lc) \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi, \Lambda}{\Gamma \Rightarrow \Delta, \phi, \Lambda}\,(Rc)$$

Exchange rules:

$$\frac{\Gamma, \phi, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \psi, \phi, \Sigma \Rightarrow \Delta}\,(Le) \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \phi, \Lambda}\,(Re)$$

The cut rule:

$$\frac{\Gamma \Rightarrow \phi, \Lambda \qquad \Sigma, \phi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,(cut)$$

The logical rules:

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, 1, \Sigma \Rightarrow \Delta}\,(1w) \qquad \frac{\Gamma \Rightarrow \Delta, \Lambda}{\Gamma \Rightarrow \Delta, 0, \Lambda}\,(0w)$$

$$\frac{\Gamma, \phi, \Sigma \Rightarrow \Delta}{\Gamma, \phi \wedge \psi, \Sigma \Rightarrow \Delta}\,(L\wedge_1) \qquad \frac{\Gamma, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \phi \wedge \psi, \Sigma \Rightarrow \Delta}\,(L\wedge_2)$$

$$\frac{\Gamma \Rightarrow \Delta, \phi, \Lambda \qquad \Gamma \Rightarrow \Delta, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \phi \wedge \psi, \Lambda}\,(R\wedge)$$

$$\frac{\Gamma, \phi, \Sigma \Rightarrow \Delta \qquad \Gamma, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \phi \vee \psi, \Sigma \Rightarrow \Delta}\,(L\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, \phi, \Lambda}{\Gamma \Rightarrow \Delta, \phi \vee \psi, \Lambda}\,(R\vee_1) \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \phi \vee \psi, \Lambda}\,(R\vee_2)$$

$$\frac{\Gamma, \phi, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \phi \otimes \psi, \Sigma \Rightarrow \Delta}\,(L\otimes) \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \Lambda \qquad \Sigma \Rightarrow \Delta, \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \Delta, \phi \otimes \psi, \Lambda}\,(R\otimes)$$

The non-commutative implications rules:

$$\frac{\Gamma \Rightarrow \phi \qquad \Pi, \psi, \Sigma \Rightarrow \Delta}{\Pi, \psi / \phi, \Gamma, \Sigma \Rightarrow \Delta}\,(L/) \qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \psi / \phi}\,(R/)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Pi, \psi, \Sigma \Rightarrow \Delta}{\Pi, \Gamma, \phi \setminus \psi, \Sigma \Rightarrow \Delta}\,(L\setminus) \qquad \frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \setminus \psi}\,(R\setminus)$$

The commutative implication rules:

$$\frac{\Gamma \Rightarrow \phi, \Lambda \qquad \Pi, \psi, \Sigma \Rightarrow \Delta}{\Pi, \phi \to \psi, \Gamma, \Sigma \Rightarrow \Delta, \Lambda}\,(L \to) \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}\,(R \to)$$

Using these rules, we define two families of sequent-style systems in the following. By a single-conclusion sequent we mean the succedent of the sequent is empty or there is at most one formula. Otherwise, we call it multi-conclusion. Let (e), (c), (i), (o), and $(w) = (i+o)$ stand for exchange, contraction, left-weakening, right-weakening and weakening, respectively:

Single-conclusion. By a single-conclusion version of any of the the aforementioned rules, we mean one of its instances where both the premisses and the conclusion sequents are single-conclusion. Notice that the rules (Rc) and (Re) do not have a single-conclusion instance. The meta-variables Δ and Λ are schematic variables to be replaced by the empty set or a single formula so that all the sequents remain single-conclusion. For instance, in the rule (Rw) both Δ and Λ must be empty. We will use the convention that $\otimes$ more strongly than $\backslash$ and $/$. The interpretation of any single-conclusion sequent $\Gamma \Rightarrow \phi$ is defined as $I(\Gamma \Rightarrow \phi) = \bigotimes \Gamma \backslash \phi$ and for the sequent $(\Gamma \Rightarrow)$ as $I(\Gamma \Rightarrow) = \bigotimes \Gamma \backslash 0$, where by $\bigotimes \Gamma$ for $\Gamma = \gamma_1, \ldots, \gamma_n$ we mean $\gamma_1 \otimes \ldots \otimes \gamma_n$, and for $\Gamma = \emptyset$, we have $\bigotimes \Gamma = 1$. Set $\mathcal{L}^{\otimes} = \{\wedge, \vee, \otimes, \backslash, /, 1, 0\}$. For any $S \subseteq \{e, i, o, c\}$, define $\mathbf{FL_S}$ over the language $\mathcal{L}^{\otimes}$ as the system consisting of the single-conclusion version of the previous rules except for: the commutative implication rules, the structural rules out of the set S, and the initial sequents for $\bot$ and $\top$. Define $\mathbf{FL}_\bot$ over the language $\mathcal{L}^{\otimes} \cup \{\bot\}$ as $\mathbf{FL}$ with the initial sequent for $\bot$. Figure 4.2.1, which is adapted from [14], shows the relationship between these sequent calculi. Moreover, define the system *weak Lambek*, denoted by $\mathbf{WL}$, over the language $\{1, \bot, \wedge, \vee, \otimes, \backslash\}$ similar to $\mathbf{FL}_\bot$, excluding the following rules: $(L/), (R/)$, and $(L\backslash)$. Some other useful calculi are introduced in Table 4.1. For a sequent calculus $\mathbf{S}$ and a set of sequents Γ by the notation $\mathbf{S} + \Gamma$ we mean the sequent calculus obtained from adding the elements of Γ as initial sequents to $\mathbf{S}$. By the notation $\phi \Leftrightarrow \psi$ we mean both $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$. The formula ϕ^n is defined inductively. ϕ^1 is ϕ and by ϕ^{n+1}, we mean $\phi \otimes \phi^n$.

Table 4.1: Some sequent calculi with their definitions.

Logic	Definition
RL	$\mathbf{FL} + (0 \Leftrightarrow 1)$
CyFL	$\mathbf{FL} + (\phi \backslash 0 \Leftrightarrow 0/\phi)$
DFL	$\mathbf{FL} + (\phi \wedge (\psi \vee \theta) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \theta))$
$\mathbf{P_nFL}$	$\mathbf{FL} + (\phi^n \Leftrightarrow \phi^{n+1})$
psBL	$\mathbf{FL_w} + \{(\phi \wedge \psi \Leftrightarrow \phi \otimes (\phi \backslash \psi)), (\phi \wedge \psi \Leftrightarrow (\psi/\phi) \otimes \phi)\}$
DRL	$\mathbf{RL} + (\phi \wedge (\psi \vee \theta) \Rightarrow (\phi \wedge \psi) \vee (\phi \wedge \theta))$
IRL	$\mathbf{RL} + (\phi \Rightarrow 1)$
CRL	$\mathbf{RL} + (\phi \otimes \psi \Leftrightarrow \psi \otimes \phi)$
GBH	$\mathbf{RL} + \{(\phi \wedge \psi \Leftrightarrow \phi \otimes (\phi \backslash \psi)), (\phi \wedge \psi \Leftrightarrow (\psi/\phi) \otimes \phi)\}$
Br	$\mathbf{RL} + (\phi \wedge \psi \Leftrightarrow \phi \otimes \psi)$

Multi-conclusion. In the absence of the exchange rules, there are many possible ways to define the multi-conclusion rules for fusion and implications and the systems are in some respects more difficult than the commutative case. In this paper, we only consider the commutative case and hence we will use the language $\{\wedge, \vee, \otimes, \to, 0, 1\}$, assuming only one implication. The interpretation of any sequent $\Gamma \Rightarrow \Delta$ is defined as $I(\Gamma \Rightarrow \Delta) = \bigotimes \Gamma \to \neg(\bigotimes \neg \Delta)$, where $\neg \phi$ is an abbreviation for $\phi \to 0$.
Let $S \subseteq \{e, i, o, c\}$ such that $e \in S$. By $\mathbf{CFL_S}$, we mean the system consisting of the multi-conclusion version of the previous rules except for: the structural rules out of the set S, the non-commutative implication rules, and the initial sequent for $\bot$. By $\mathbf{CFL_S}^-$, we mean $\mathbf{CFL_S}$ without the cut rule.
For a sequent calculus $\mathbf{S}$, proofs and provability of formulas are defined in the usual way, and by its logic, S, we mean the set of provable formulas in it, i.e., all formulas ϕ such that $(\Rightarrow \phi)$ is provable in $\mathbf{S}$.

Remark 4.2.1. *Note that if $e \in S$, it is easy to show that in the system $\mathbf{FL_S}$ the two connectives ψ/ϕ and $\phi\backslash\psi$ are provably equivalent and we can denote them by the usual connective $\phi \to \psi$. Moreover, it is also possible to axiomatize the system $\mathbf{FL_S}$ over the language $\mathcal{L}^{\otimes} - \{/, \backslash\} \cup \{\to\}$, using all the rules in $\mathbf{FL_S}$, replacing the non-commutative implication rules with the commutative ones. Similarly, in the sequent calculus $\mathbf{FL_{ecw}}$, the formulas $\phi \otimes \psi$ and $\phi \wedge \psi$ become equivalent and 0 and 1 will be equivalent to $\bot$ and $\top$, respectively. Hence, it is possible to axiomatize $\mathbf{FL_{ecw}}$ over the language $\mathcal{L} = \{\wedge, \vee, \to, \top, \bot\}$, using all the initial sequents and rules for the corresponding connectives. This is nothing but the usual system $\mathbf{LJ}$, for the intuitionistic logic, IPC. A similar type of argument also applies on $\mathbf{CFL_S}$ when $e \in S$ and for $\mathbf{CFL_{ecw}} = \mathbf{LK}$, where $\mathbf{LK}$ is the sequent calculus for the classical logic, CPC. Finally, it is worth mentioning that the logic $\mathsf{CFL_e}$ is essentially equivalent to the multiplicative additive linear logic, MALL, introduced by Girard [15] and the logic $\mathsf{FL_e}$ is known as its intuitionistic version, called IMALL. $\mathsf{CFL_{ew}}$ is sometimes called the monoidal logic and $\mathsf{CFL_{ec}}$ is essentially equivalent to the relevant logic R without the distributive law. For more details, see [14].*

The sequent calculi $\mathbf{FL_S}$ and $\mathbf{CFL_S}$ enjoy cut elimination. This fact has been shown independently by several authors. For instance, see [15], [29], and [33].

Definition 4.2.2. *We say a formula ϕ is provable from a set of formulas Γ in the logic FL and we write it as $\Gamma \vdash_{\mathsf{FL}} \phi$ when the sequent $\Rightarrow \phi$ is provable in the sequent calculus $\mathbf{FL}$ by adding all $\Rightarrow \gamma$ for $\gamma \in \Gamma$ as initial sequents, i.e., $\{\Rightarrow \gamma\}_{\gamma \in \Gamma} \vdash_{\mathbf{FL}} \Rightarrow \phi$. When Γ is the empty set we sometimes write $\mathsf{FL} \vdash \phi$ for $\vdash_{\mathsf{FL}} \phi$.*

We will use a similar convention that for S a logic or a proof system or a sequent calculus, $\vdash_S \phi$ and $S \vdash \phi$ are used interchangeably.
If the sequent $\phi_1, \ldots, \phi_n \Rightarrow \psi$ is provable in the sequent calculus $\mathbf{FL}$, then we have $\{\phi_1, \ldots, \phi_n\} \vdash_{\mathsf{FL}} \psi$. However, the converse, which is the deduction theorem, does not hold. In fact, unlike the classical and intuitionistic logics, most other substructural logics, including FL, do not have a deduction theorem. We will see in Theorem 4.2.5 that only a restricted version of the deduction theorem (called

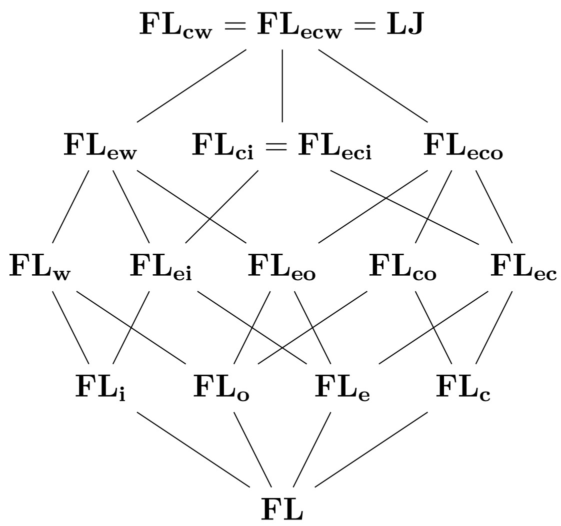

Figure 4.1: Basic substructural calculi

parametrized local deduction theorem) holds for $\vdash_{\mathsf{FL}}$. However, note that by definition for a formula ϕ we have $\vdash_{\mathsf{FL}} \phi$ if and only if $\Rightarrow \phi$ is provable in the sequent calculus **FL**.

So far, we have defined some basic substructural logics with their sequent calculi. Now, it is a good point to introduce a substructural logic in a general sense. From now on, when no confusion occurs, we will write the fusion $\phi \otimes \psi$ as $\phi\psi$.

Definition 4.2.3. *Let* L *be a set of* $\mathcal{L}^{\otimes}$*-formulas.* L *is a substructural logic (over* FL*) if it is closed under substitution and satisfies the following conditions:*

(i) L *includes all formulas in* FL*,*

(ii) if $\phi, \psi \in \mathsf{L}$*, then* $\phi \wedge \psi \in \mathsf{L}$*,*

(iii) if $\phi, \phi \setminus \psi \in \mathsf{L}$*, then* $\phi \in \mathsf{L}$*,*

(iv) if $\phi \in \mathsf{L}$ *and* ψ *is an arbitrary formula, then* $\psi \setminus \phi\psi, \psi\phi/\psi \in \mathsf{L}$*.*

For a set of formulas $\Gamma \cup \{\phi\}$*, define* $\Gamma \vdash_{\mathsf{L}} \phi$ *as* $\Gamma \cup \mathsf{L} \vdash_{\mathsf{FL}} \phi$*. We have* $\vdash_{\mathsf{L}} \phi$ *is equivalent to* $\phi \in \mathsf{L}$*.*

When L is the logic FL, then $\vdash_{\mathsf{FL}}$ defined above will be the same as the one defined in Definition 4.2.2. Therefore, there will be no ambiguity. As a corollary of Theorem [14, 2.16], it is shown that the above definition can be replaced by the following: a substructural logic over FL is a set of formulas closed under both substitution and $\vdash_{\mathsf{FL}}$.

It is easy to see that for any subset S of $\{e, i, o, c\}$, the logic $\mathsf{FL_S}$ is a substructural logic. We can see that if $\vdash_{\mathbf{FL_S}} \Gamma \Rightarrow \phi$, then $\Gamma \vdash_{\mathsf{FL_S}} \phi$. This can be easily shown since we can simulate each rule in $\{e, i, o, c\}$ by the corresponding axiom below and using the cut rule:

$$(e) : (\phi \otimes \psi) \setminus (\psi \otimes \phi) \ \ , \ \ (c) : \phi \setminus (\phi \otimes \phi) \ \ , \ \ (i) : \phi \setminus 1 \ \ , \ \ (o) : 0 \setminus \phi$$

Moreover, note that for all the sequent calculi in Table 4.1, the sequent calculus **FL** is present and hence all their corresponding logics are closed under the conditions in Definition 4.2.3. Therefore, they are substructural logics.

Definition 4.2.4. *Let ϕ and α be formulas. Define*

$$\lambda_\alpha(\phi) = (\alpha \setminus (\phi\alpha)) \wedge 1 \quad \textit{and} \quad \rho_\alpha(\phi) = ((\alpha\phi)/\alpha) \wedge 1.$$

We call $\lambda_\alpha(\phi)$ and $\rho_\alpha(\phi)$ the left and right conjugate of ϕ with respect to α, respectively. An iterated conjugate of ϕ is a composition $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\dots\gamma_{\alpha_n}(\phi)))$, for formulas $\alpha_1, \dots, \alpha_n$ where $n \geq 0$ and $\gamma_{\alpha_i} \in \{\lambda_{\alpha_i}, \rho_{\alpha_i}\}$.

It can be easily shown ([14, Lemma 2.13.]) that if a sequent $\Gamma, \alpha, \beta, \Sigma \Rightarrow \phi$ is provable in **FL**, then the following sequents are also provable in **FL**:

$$\Gamma, \beta, \lambda_\beta(\alpha), \Sigma \Rightarrow \phi \quad \textit{and} \quad \Gamma, \rho_\alpha(\beta), \alpha, \Sigma \Rightarrow \phi.$$

The following theorem states the parametrized local deduction theorem for FL.

Theorem 4.2.5. *[14, Theorem 2.14.] Let L be a substructural logic and $\Phi \cup \Psi \cup \{\phi\}$ be a set of formulas. Then,*

$$\Phi, \Psi \vdash_{\mathsf{L}} \phi \quad \textit{iff} \quad \Phi \vdash_{\mathsf{L}} (\bigotimes_{i=1}^{n} \gamma_i(\psi_i)) \setminus \phi$$

for some n, where each $\gamma_i(\psi_i)$ is an iterated conjugate of a formula $\psi_i \in \Psi$.

Remark 4.2.6. *Note that the definition of $\vdash_{\mathsf{L}}$ in Definition 4.2.3 depends on the sequent calculus* **FL** *and not the mere logic FL. The reason is that $\vdash_{\mathsf{FL}}$, which is defined in Definition 4.2.2, uses the sequent calculus* **FL**. *It is possible to use Theorem 4.2.5 to provide the following proof system-independent definition of $\vdash_{\mathsf{L}}$:*

$$\Gamma \vdash_{\mathsf{L}} \phi \quad \textit{iff} \quad (\bigotimes_{i=1}^{n} \gamma_i(A_i)) \setminus \phi \in \mathsf{L} \quad \textit{iff} \quad (\bigotimes_{i=1}^{m} \gamma_i(B_i)) \setminus \phi \in \mathsf{FL}$$

for some n and m and some $A_i \in \Gamma$ and $B_i \in \Gamma \cup \{\mathsf{L}\}$.

4.2.2 Super-basic logics

In [44], Visser introduced basic propositional logic, BPC, and formal propositional logic, FPL, to interpret implication as formal provability. In [41], Ruitenberg reintroduced BPC via philosophical reasons and produced its predicate version, BQC. In the following, we present the sequent calculus introduced in [4] for the logic BPC, denoted by **BPC**. It was shown that this proof system is complete with respect to transitive persistent Kripke models. Since formulas $(A \to (A \to B)) \to (A \to B)$ and $(A \to (B \to C)) \to (B \to (A \to C))$ are not always true in transitive models (the former formula corresponds to the contraction rule and the latter to the exchange rule), one may view BPC as a substructural logic. In this logic modus ponens is weakened and hence BPC is weaker than the intuitionistic logic. BPC is also connected with the modal logic **K4** via Gödel's translation T, as shown in [44].

The language of **BPC** is $\mathcal{L} = \{\wedge, \vee, \top, \bot, \rightarrow\}$ and negation is defined as the abbreviation for $\neg\phi = \phi \rightarrow \bot$. In this subsection capital Greek letters denote (possibly empty) multisets of $\mathcal{L}$-formulas. By Γ, ϕ or ϕ, Γ, we mean the multiset $\Gamma \cup \{\phi\}$. Sequents of **BPC** are of the same form of the sequents of **LK** and they are interpreted in the same way, i.e., $I(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$ where $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. The initial sequent and rules of **BPC** are as follows:

$$\Gamma, \phi \Rightarrow \phi, \Delta \qquad \Gamma \Rightarrow \top, \Delta \qquad \Gamma, \bot \Rightarrow \Delta$$

$$\frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}\,(L\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}\,(R\wedge)$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}\,(L\vee) \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}\,(R\vee)$$

$$\frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi}\,(R \rightarrow)$$

$$\frac{\phi \wedge \psi, \Gamma \Rightarrow \Delta \qquad \phi \wedge \theta, \Gamma \Rightarrow \Delta}{\phi \wedge (\psi \vee \theta), \Gamma \Rightarrow \Delta}\,(D) \qquad \frac{\Gamma \Rightarrow \phi \rightarrow \psi \qquad \Gamma \Rightarrow \psi \rightarrow \theta}{\Gamma \Rightarrow \Delta, \phi \rightarrow \theta}\,(Tr)$$

$$\frac{\Gamma \Rightarrow \phi \rightarrow \psi \qquad \Gamma \Rightarrow \phi \rightarrow \theta}{\Gamma \Rightarrow \Delta, \phi \rightarrow (\psi \wedge \theta)}\,(F\wedge) \qquad \frac{\Gamma \Rightarrow \phi \rightarrow \theta \qquad \Gamma \Rightarrow \psi \rightarrow \theta}{\Gamma \Rightarrow \Delta, (\phi \vee \psi) \rightarrow \theta}\,(F\vee)$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \phi \Rightarrow \Lambda}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}\,(cut)$$

Note that since we are assuming multisets of formulas, in this proof system the exchange rules are built in. Moreover, the left and right weakening and contraction rules are admissible in this proof system and it enjoys the cut elimination (see [4] Lemma 2.2, Lemma 2.12, Lemma 2.14, and Theorem 2.17, respectively). It can be easily seen that if **BPC** $\vdash\Rightarrow \phi$ and **BPC** $\vdash\Rightarrow \phi \rightarrow \psi$ then since **BPC** enjoys cut elimination we also have **BPC** $\vdash \phi \Rightarrow \psi$. Then using the cut rule we have **BPC** $\vdash\Rightarrow \psi$, which means that the modus ponens rule is admissible in **BPC**.
An extension of **BPC** augmented by the axiom $\top \rightarrow \bot \Rightarrow \bot$ is given in [5], denoted by **EBPC**. It is shown that this proof system is complete with respect to transitive persistent Kripke models that are serial [5]. Logic of the sequent calculus **BPC** is defined as the set of all formulas ϕ such that **BPC** $\vdash (\Rightarrow \phi)$, and is denoted by BPC. In a similar way, we can define the logic of the sequent calculus **EBPC** which we denote by EBPC. It is shown that $\mathsf{BPC} \subsetneq \mathsf{EBPC} \subsetneq \mathsf{IPC}$ [5].

Definition 4.2.7. *We say a formula ϕ is provable from a set of formulas Γ in the logic* BPC *and we write it as $\Gamma \vdash_{\mathsf{BPC}} \phi$ when the sequent $\Rightarrow \phi$ is provable in the sequent calculus* **BPC** *by adding all $\Rightarrow \gamma$ for $\gamma \in \Gamma$ as initial sequents.*

Remark 4.2.8. *Note that although the modus ponens rule in the form*

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \phi \rightarrow \psi}{\Gamma \Rightarrow \psi}$$

is neither present nor admissible in the sequent calculus **BPC***, a simplified version of it, when* Γ *is the empty sequence, is admissible (but not provable) in* **BPC***, namely*

$$\frac{\Rightarrow \phi \qquad \Rightarrow \phi \to \psi}{\Rightarrow \psi}$$

Therefore, the logic BPC *admits the modus ponens rule, i.e., if* $\phi \in \mathsf{BPC}$ *and* $\phi \to \psi \in \mathsf{BPC}$*, then* $\psi \in \mathsf{BPC}$*. The reason is that if* $\phi \to \psi \in \mathsf{BPC}$ *then* $\mathbf{BPC} \vdash (\Rightarrow \phi \to \psi)$*. By cut elimination, there exists a cut-free proof of* $(\Rightarrow \phi \to \psi)$ *in* **BPC***. Then by induction on the structure of this cut-free proof we can show that* $\mathbf{BPC} \vdash \phi \Rightarrow \psi$*. Finally, since* $\phi \in \mathsf{BPC}$*, we have* $\mathbf{BPC} \vdash (\Rightarrow \phi)$*, and then using the cut rule we get* $\mathbf{BPC} \vdash (\Rightarrow \psi)$ *which means* $\psi \in \mathsf{BPC}$*. However, we have* $\Rightarrow \phi, \Rightarrow \phi \to \psi \nvdash_{\mathbf{BPC}} \Rightarrow \psi$*, which means modus ponens is not provable in* **BPC***. The same property also holds for the logic* EBPC*. The proof is an easy consequence of the completeness of* EBPC *with respect to serial transitive persistent Kripke models.*

Definition 4.2.9. *Let* L *be a set of* $\mathcal{L}$*-formulas.* L *is a super-basic logic (over* BPC*) if it is closed under substitution and satisfies the following conditions:*

(*i*) L *includes all formulas in* BPC*,*

(*ii*) *if* $\phi, \phi \to \psi \in \mathsf{L}$*, then* $\phi \in \mathsf{L}$*.*

For a set of formulas $\Gamma \cup \{\phi\}$*, define* $\Gamma \vdash_{\mathsf{L}} \phi$ *as* $\Gamma \cup \mathsf{L} \vdash_{\mathsf{BPC}} \phi$*.*

Note that $\vdash_{\mathsf{L}} \phi$ is equivalent to $\phi \in \mathsf{L}$. One direction is obvious; if $\phi \in \mathsf{L}$ then $\vdash_{\mathsf{L}} \phi$. For the other direction, we will prove a stronger result that if $\Gamma \vdash_{\mathsf{L}} \phi$ then $\bigwedge \Gamma \to \phi \in \mathsf{L}$. This can be proved using induction on the structure of the proof. For this matter, we transform every rule of **BPC** into a BPC-provable formula. To complete the proof of the other direction, since $\bigwedge \Gamma = \top$ for $\Gamma = \emptyset$, we have $\top \to \phi \in \mathsf{L}$, which by modus ponens implies $\phi \in \mathsf{L}$.

As an example, using Remark 4.2.8, both BPC and EBPC are super-basic logics. Moreover, super-intuitionistic logics (changing the first condition by including all formulas in IPC) are also super-basic, since $\mathsf{BPC} \subset \mathsf{IPC}$ and they are closed under modus ponens.

For a logic L and a set of formulas Γ, by $\mathsf{L} + \Gamma$ we mean the smallest logic containing L and all the substitutions of formulas in Γ. We can define Jankov's logic, KC, as follows: it is the smallest logic containing IPC and the weak excluded middle formula, i.e., $\mathsf{KC} = \mathsf{IPC} + \neg p \vee \neg\neg p$. The condition on the Kripke models for this logic is being directed. The axioms BD_n are defined in the following way:

$$BD_0 := \bot \quad , \quad BD_{n+1} := p_n \vee (p_n \to BD_n).$$

The logic of bounded depth $\mathsf{BD_n}$ is then defined as $\mathsf{IPC}+BD_n$. Define logic $\mathsf{T_k}$ as

$$\mathsf{IPC} + \bigwedge_{i=0}^{k}((p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j} p_j) \to \bigvee_{i} p_i.$$

A super-intuitionistic logic L has branching k if $\mathsf{T_k} \subseteq \mathsf{L}$. We say a super-intuitionistic logic L has finite branching if there exists a number k such that L has branching less than or equal to k, otherwise we call it infinite branching. We will not use the following theorem by Jeřábek in our future discussions. However, it is worth mentioning since it presents a nice characterization of super-intuitionistic infinite branching logics.

Theorem 4.2.10. *[25, Theorem 6.9] Let L be a super-intuitionistic logic. Then, L has infinite branching if and only if $\mathsf{L} \subseteq \mathsf{BD_2}$ or $\mathsf{L} \subseteq \mathsf{KC} + \mathsf{BD_3}$.*

4.3 Frege and extended Frege systems

The purpose of this section is to introduce Frege and extended Frege systems for substructural and super-basic logics. For that matter, we will recall or generalize some basic concepts in proof complexity. For more background the reader may consult [28].

Definition 4.3.11. *Let L be a set of finite strings over a finite alphabet. A (propositional) proof system for L is a polynomial-time function $\mathbf{P}$ with the range L. Any string π such that $\mathbf{P}(\pi) = \phi$ is a $\mathbf{P}$-proof of the string ϕ, sometimes written as $\mathbf{P} \vdash^{\pi} \phi$. We denote proof systems by bold-face capital Roman letters.*

By length of a formula ϕ, or a proof π, we mean the number of symbols it contains and we denote it by $|\phi|$ and $|\pi|$, respectively. We usually consider proof systems for a logic L. The usual Hilbert-style systems with finitely many axiom schemes and Gentzen's sequent calculi are instances of propositional proof systems, because they are complete and in polynomial time one can decide whether a finite string is a proof in the system or not.

Definition 4.3.12. *Let $\mathbf{P}$ and $\mathbf{Q}$ be two proof systems with the languages $\mathcal{L}_{\mathbf{P}}$ and $\mathcal{L}_{\mathbf{Q}}$, respectively. Let tr be a polynomial-time translation function from the strings in the language $\mathcal{L}_{\mathbf{P}}$ to the strings in the language $\mathcal{L}_{\mathbf{Q}}$. We will denote it by $tr : \mathcal{L}_{\mathbf{P}} \to \mathcal{L}_{\mathbf{Q}}$.*
We say that the proof system $\mathbf{Q}$ simulates the proof system $\mathbf{P}$ (or $\mathbf{P}$ is simulated by $\mathbf{Q}$, or $\mathbf{Q}$ is at least as strong as $\mathbf{P}$) with respect to tr, if there is a function f such that $\mathbf{Q}(f(\pi)) = tr(\mathbf{P}(\pi))$ and we denote it by $\mathbf{P} \leq^{tr} \mathbf{Q}$. We say that the proof system $\mathbf{Q}$ polynomially simulates (p-simulates) the proof system $\mathbf{P}$ (or $\mathbf{P}$ is polynomially simulated by $\mathbf{Q}$) with respect to tr, if the function f is also polynomially bounded in length, i.e., there exists a polynomial $q(n)$ such that $|f(\pi)| \leq q(|\pi|)$. We denote this reduction by $\mathbf{P} \leq^{tr}_{p} \mathbf{Q}$.
In the simpler case that $\mathcal{L}_{\mathbf{P}} \subseteq \mathcal{L}_{\mathbf{Q}}$ and the translation function is the inclusion function, we say $\mathbf{Q}$ simulates (p-simulates) $\mathbf{P}$ and denote it by $\mathbf{P} \leq \mathbf{Q}$ ($\mathbf{P} \leq_p \mathbf{Q}$).
If $\mathcal{L}_{\mathbf{P}} = \mathcal{L}_{\mathbf{Q}}$ and the translation function is the identity function, we say that the proof system P and Q are polynomially equivalent when they p-simulate each other.
Finally, in a similar manner, for two logics L and M and a translation function $tr : \mathcal{L}_{\mathsf{L}} \to \mathcal{L}_{\mathsf{M}}$, by $\mathsf{L} \subseteq^{tr} \mathsf{M}$, we mean that for any $\phi \in \mathcal{L}_{\mathsf{L}}$, if $\phi \in \mathsf{L}$ then $tr(\phi) \in \mathsf{M}$.

Note that if we take $\mathsf{L} = \mathsf{CPC}$ to be the range of both proof systems $\mathbf{P}$ and $\mathbf{Q}$, and let the translation function tr to be the identity function, we reach Cook and Reckhow's original definition of p-simulation in [11].

In the following we present a translation function t that enables us to carry out results in systems with the language $\mathcal{L}$ to systems with the language $\mathcal{L}^{\otimes}$. This translation function is nothing but bringing back the structural rules:

Definition 4.3.13. *Define the function $t : \mathcal{L}^{\otimes} \to \mathcal{L}$ as follows:*

- $p^t = p$, *where p is a propositional variable;*
- $0^t = \bot$, $1^t = \top$;
- $(\phi \circ \psi)^t = \phi^t \circ \psi^t$, *where* $\circ \in \{\wedge, \vee\}$;
- $(\phi \otimes \psi)^t = \phi^t \wedge \psi^t$;
- $(\psi / \phi)^t = (\phi \setminus \psi)^t = \phi^t \to \psi^t$.

For Γ, a finite sequence of formulas $\gamma_1, \gamma_2, \ldots, \gamma_n$, by Γ^t we mean the sequence of formulas $\gamma_1^t, \gamma_2^t \ldots, \gamma_n^t$. It is easy to see that $|\phi^t| = |\phi|$.

The following lemma, which will be used in the future sections, is an example of how the translation t works. It expresses the relation between sequents provable in the sequent calculus **WL** and the translated version of the sequents in the system **BPC**.

Lemma 4.3.14. *Let Γ be a sequence of formulas and A be a formula. Then*

$$\mathbf{WL} \vdash \Gamma \Rightarrow A \;\; \textit{implies} \;\; \mathbf{BPC} \vdash \Gamma^t \Rightarrow A^t.$$

Proof. It can be shown by an easy induction on the structure of the proof. Note that as mentioned earlier, the left contraction rule and both right and left weakening rules are derivable in **BPC** and exchange rules are built in. As an example, suppose the last rule in the proof of $\Gamma \Rightarrow A$ is $(R\otimes)$:

$$\frac{\Sigma \Rightarrow \phi \qquad \Pi \Rightarrow \psi}{\Sigma, \Pi \Rightarrow \phi \otimes \psi}$$

By induction hypothesis we have $\mathbf{BPC} \vdash \Sigma^t \Rightarrow \phi^t$ and $\mathbf{BPC} \vdash \Pi^t \Rightarrow \psi^t$. Since the left weakening rule is admissible in **BPC**, we can have both $\mathbf{BPC} \vdash \Sigma^t, \Pi^t \Rightarrow \phi^t$ and $\mathbf{BPC} \vdash \Sigma^t, \Pi^t \Rightarrow \psi^t$. Using the rule $(R\wedge)$ we obtain $\mathbf{BPC} \vdash \Sigma^t, \Pi^t \Rightarrow \phi^t \wedge \psi^t$, which is what we wanted. □

Remark 4.3.15. *For any substructural logic L and any super-intuitionistic logic M, it is easy to see that $\mathsf{L} \subseteq^t \mathsf{M}$ implies the stronger form:*

$$\phi_1, \ldots, \phi_n \vdash_{\mathsf{L}} \phi \quad \textit{implies} \quad \phi_1^t, \ldots, \phi_n^t \vdash_{\mathsf{M}} \phi^t.$$

The reason lies in the definition of $\vdash_{\mathsf{L}}$ and $\vdash_{\mathsf{M}}$. The proof is similar to the proof of Lemma 4.3.14.

In the following we will define Frege and extended Frege systems for substructural and super-basic logics.

Definition 4.3.16. *An inference system* **P** *is defined by a set of rules of the form*

$$\frac{\phi_1 \quad \dots \quad \phi_m}{\phi}$$

where ϕ_i and ϕ are formulas. ϕ_i's are called the premises and ϕ the conclusion. A rule with no premise is called an axiom. A $\mathbf{P}$-proof, π, of a formula ϕ from a set of formulas X is defined as a sequence of formulas $\phi_1, \dots, \phi_n = \phi$, where $\phi_i \in X$ or ϕ_i is obtained by substituting some ϕ_j's, $j < i$, in a rule of the system $\mathbf{P}$. If the set X is empty, then we say that the formula ϕ is provable in $\mathbf{P}$. Each ϕ_i is called a step or a line in the proof π. The number of lines of a proof π is denoted by $\lambda(\pi)$ and it is clear that it is less than or equal to the length of the proof (the number of symbols in the proof). The set of all provable formulas in $\mathbf{P}$ is called its logic. If there is a $\mathbf{P}$-proof for ϕ from assumptions $\phi_1, \dots, \phi_n$, we write $\phi_1, \dots, \phi_n \vdash_{\mathbf{P}} \phi$. Specially, for every rule of the above form we have $\phi_1, \dots, \phi_m \vdash_{\mathbf{P}} \phi$. Finally, the number of lines of the proof π is defined as the number of formulas in the proof π.

Definition 4.3.17. *In a sequent calculus a line in a proof is a sequent of the form $\Gamma \Rightarrow \Delta$. We denote the number of proof-lines in a proof π in a sequent calculus by $\lambda(\pi)$, as in an inference systems. It is obvious that the number of proof-lines of a sequent is less than or equal to the length of the proof.*

There are two measures for the complexity of proofs in proof systems. The first one is the length of the proof and the other is the number of proof steps (also called proof-lines). This only makes sense for proof systems in which the proofs consist of lines containing formulas or sequents. Hilbert-style proof systems, Gentzen's sequent calculi, and Frege systems are examples of such proof systems.

Definition 4.3.18. *Let L and M be two substructural or two super-basic logics such that $\mathsf{L} \subseteq \mathsf{M}$. The inference system $\mathbf{P}$ is called a Frege system for L with respect to M, for short an $\mathsf{L} - \mathbf{F}$ system wrt M, if it satisfies the following conditions:*

(1) $\mathbf{P}$ *has finitely many rules,*

(2) $\mathbf{P}$ *is sound: if $\vdash_{\mathbf{P}} \phi$, then $\phi \in \mathsf{L}$,*

(3) $\mathbf{P}$ *is strongly complete: if $\phi_1, \dots, \phi_n \vdash_{\mathsf{L}} \phi$, then $\phi_1, \dots, \phi_n \vdash_{\mathbf{P}} \phi$.*

(4) *every rule in $\mathbf{P}$ is M-standard: if $\dfrac{\phi_1 \quad \dots \quad \phi_m}{\phi}$ is a rule in $\mathbf{P}$, then $\phi_1, \dots, \phi_m \vdash_{\mathsf{M}} \phi$.*

In the case that $\mathsf{L} = \mathsf{M}$, we simply call this system a Frege system for L.

Here are some remarks. It is easy to see that any Frege system $\mathbf{P}$ for L wrt M has the property that if $\phi_1, \dots, \phi_n \vdash_{\mathbf{P}} \phi$, then $\phi_1, \dots, \phi_n \vdash_{\mathsf{M}} \phi$. This can be shown using induction on the structure of the proof and the condition 4. For a substructural logic L, we will only consider Frege systems for L wrt L, i.e., $\mathsf{M} = \mathsf{L}$. For S a subset of $\{e, i, o, c\}$, the Hilbert-style proof system $\mathbf{HFL_S}$ is an example of a Frege system for the basic substructural logic $\mathsf{FL_S}$ (see [14, Section 2.5]). The usual Hilbert-style systems for classical and intuitionistic logics, **HK** and **HJ**, are also examples of Frege systems for CPC and IPC, respectively; see [14, Sections 1.3.1 and 1.3.3].

Usually, a Frege system for a logic L is defined by some L-standard rules in the sense of the condition (4) in Definition 4.3.18. This condition is useful to establish the uniqueness of these systems up to p-equivalence (see 4.3.21). However, in this paper we generalize the usual definition to add another and possibly stronger logic M as a parameter to control the derivability of the rules. The logic M is not necessarily equal to L. The reason for this choice is the somehow strange behaviour of some Hilbert-style proof systems for some super-basic logics. For instance, any natural Hilbert-style system for BPC includes the modus pones rule. (See for instance Theorem 4.3.19 below). While this rule is admissible in BPC and hence harmless to the soundness of the system, it can not be derivable inside the logic BPC itself, i.e, $\phi, \phi \to \psi \nvdash_{\mathsf{BPC}} \psi$. Therefore, the modus ponens rule violates the BPC-standradness condition. To address such systems, it may be reasonable to relax the BPC-standardness condition a bit to also include the modus ponense rule. The smallest logic containing BPC and modus ponens is IPC and hence we have to pick $\mathsf{M} = \mathsf{IPC}$ as our controlling parameter. Although this choice of definition may seem a bit artificial, it actually serves our goal better than the usual systems. The aim of the present paper is establishing a lower bound for any possible Frege system for some classes of logics and addressing a larger class of Frege systems with a possibly stronger parameters M is admittedly a stronger result. Moreover, later in the last section we will even use the mentioned strange system for BPC to provide a lower bound for the usual natural sequent-style proof system for BPC. Therefore, investigating this larger class of systems is both strengthening and useful.

Theorem 4.3.19. *There exists a Frege system* $\mathbf{P}$ *for* BPC *wrt* IPC *such that for* Γ *and* Δ *sequences of formulas* $\gamma_1, \ldots, \gamma_m$*, and* $\delta_1, \ldots, \delta_n$*, if* $\mathbf{BPC} \vdash^{\pi} \Gamma \Rightarrow \Delta$ *then*

$$\mathbf{P} \vdash^{\pi'} \bigwedge_{i=1}^{m} \gamma_i \to \bigvee_{j=1}^{n} \delta_j$$

and $\lambda(\pi') = \lambda(\pi)$*.*

Proof. Consider the sequent calculus $\mathbf{BPC}$. Recall that for any sequent $S = \Gamma \Rightarrow \Delta$, the formula $I(S)$ is defined as $\bigwedge \Gamma \to \bigvee \Delta$, and if $\Gamma = \emptyset$ then $\top \to \bigvee \Delta$ and if $\Delta = \emptyset$ then $\bigwedge \Gamma \to \bot$. Define a system $\mathbf{P}$ for BPC as the following. For any initial sequent T of $\mathbf{BPC}$ add $I(T)$ to $\mathbf{P}$, and for any rule of the form

$$\frac{T_1 \qquad \ldots \qquad T_m}{T}$$

in $\mathbf{BPC}$ add the following rule to $\mathbf{P}$

$$\frac{I(T_1) \qquad \ldots \qquad I(T_m)}{I(T)}$$

Moreover, add the following two rules to $\mathbf{P}$:

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,(mp) \qquad \frac{\phi \qquad \psi}{\phi \wedge \psi}\,(adj)$$

We will prove that $\mathbf{P}$ is a Frege system for BPC wrt IPC. We have to check all the conditions of Definition 4.3.18. First, it is an inference system with finitely many rules. Second, we have to show that $\mathbf{P}$ is sound, i.e., if $\mathbf{P} \vdash \phi$ then $\phi \in \mathsf{BPC}$. This can be proved using induction on the structure of the proof. As an example, suppose the last rule used in the proof is

$$\frac{\gamma \to (\phi \to \psi) \qquad \gamma \to (\psi \to \theta)}{\gamma \to (\delta \vee (\phi \to \theta))}$$

By IH, $\gamma \to (\phi \to \psi) \in \mathsf{BPC}$ and $\gamma \to (\psi \to \theta) \in \mathsf{BPC}$. Therefore, $\mathbf{BPC} \vdash \Rightarrow \gamma \to (\phi \to \psi)$ and $\mathbf{BPC} \vdash \Rightarrow \gamma \to (\psi \to \theta)$. Since the cut elimination theorem holds in **BPC**, we obtain $\mathbf{BPC} \vdash \gamma \Rightarrow (\phi \to \psi)$ and $\mathbf{BPC} \vdash \gamma \Rightarrow (\psi \to \theta)$. Using the rule (Tr) in **BPC** we get $\mathbf{BPC} \vdash (\gamma \Rightarrow \delta, \phi \to \theta)$ which implies $\mathbf{BPC} \vdash \Rightarrow \gamma \to (\delta \vee (\phi \to \theta))$ by the rules $(R\vee)$ and $(R \to)$, hence $\gamma \to (\delta \vee (\phi \to \theta)) \in \mathsf{BPC}$. The cases for the other rules are similar.
Third, we have to show that **P** is strongly complete, i.e., if $\phi_1, \ldots, \phi_n \vdash_{\mathsf{BPC}} \phi$, then $\phi_1, \ldots, \phi_n \vdash_{\mathbf{P}} \phi$. It can be derived by showing the following:

- If $\Gamma \vdash_{\mathsf{BPC}} A$ then $\mathbf{BPC} \vdash \Gamma \Rightarrow A$;
- if $\mathbf{BPC} \vdash \Gamma \Rightarrow \Delta$ then $\mathbf{P} \vdash \bigwedge \Gamma \to \bigvee \Delta$;
- $\phi_1, \ldots, \phi_n \vdash_{\mathbf{P}} \bigwedge_{i=1}^n \phi_i$.

The sketch of the proof for each follows.

- Observe that each rule in **BPC** has a context on the antecedent of the premises and the conclusion. Therefore, for every rule of the form

  $$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \ldots \qquad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

  the following is also a rule in **BPC**

  $$\frac{\Sigma, \Gamma_1 \Rightarrow \Delta_1 \qquad \ldots \qquad \Sigma, \Gamma_n \Rightarrow \Delta_n}{\Sigma, \Gamma \Rightarrow \Delta}$$

  It means that if we add a context Σ to the antecedents of all sequents in a proof, the result is also a proof in **BPC**. Now, suppose $\Gamma \vdash_{\mathsf{BPC}} A$ where $\Gamma = \gamma_1, \ldots, \gamma_n$. Therefore, there exists a proof for $\Rightarrow A$ with $\Rightarrow \gamma_1, \ldots, \Rightarrow \gamma_n$ as initial sequents in **BPC**. Based on the observation, we can add Γ to the antecedent of each sequent in the proof and get a proof for $\Gamma \Rightarrow A$ in **BPC** from the initial sequents $\Gamma \Rightarrow \gamma_1, \ldots, \Gamma \Rightarrow \gamma_n$. However, these initial sequents are instances of the initial sequent in **BPC** and hence we get $\vdash_{\mathbf{BPC}} \Gamma \Rightarrow A$.

- It can be easily derived using induction on the structure of the proof. Suppose the premises of a rule are of the form $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ and the conclusion $\Gamma \Rightarrow \Delta$. Then by IH we get $\mathbf{P} \vdash \bigwedge \Gamma_1 \to \bigvee \Delta_1$ and $\mathbf{P} \vdash \bigwedge \Gamma_2 \to \bigvee \Delta_2$. Using the corresponding rule in **P** we get $\mathbf{P} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

- This can be derived using the rule (adj) in **P** for $n - 1$ times.

Then, using these facts, we get $\phi_1, \ldots, \phi_n \vdash_{\mathbf{P}} \bigwedge_{i=1}^n \phi_i$ and $\vdash_{\mathbf{P}} \bigwedge_{i=1}^n \phi_i \to \phi$. Using the modus ponens rule we get $\phi_1, \ldots, \phi_n \vdash_{\mathbf{P}} \phi$. Note that if $\vdash_{\mathbf{BPC}} \Rightarrow \phi$ we obtain $\vdash_{\mathbf{P}} \top \to \phi$ and by modus ponens we get $\vdash_{\mathbf{P}} \phi$.
Finally, we have to show that each rule in **P** is IPC-standard. Let us investigate the following rule of **P**, which corresponds to the rule $(F\vee)$ in **BPC**, as an example.

$$\frac{\gamma \to (\phi \to \theta) \qquad \gamma \to (\psi \to \theta)}{\gamma \to (\delta \vee (\phi \vee \psi \to \theta))}$$

We have to show $\gamma \to (\phi \to \theta), \gamma \to (\psi \to \theta) \vdash_{\mathsf{IPC}} \gamma \to (\delta \vee (\phi \vee \psi \to \theta))$. By definition, we have to show

$$\Rightarrow \gamma \to (\phi \to \theta), \Rightarrow \gamma \to (\psi \to \theta) \vdash_{\mathbf{LJ}} \Rightarrow \gamma \to (\delta \vee (\phi \vee \psi \to \theta)).$$

However, for any formulas A and B we have $\vdash_{\mathbf{LJ}} A, A \to B \Rightarrow B$. Using this fact, the premises will become $\gamma, \phi \Rightarrow \theta$ and $\gamma, \psi \Rightarrow \theta$. And then, we easily get the conclusion in **LJ**. In a similar manner, all the other rules of **P**, and especially the rule (mp), are IPC-standard.
So far, we proved **P** is a Frege system for BPC wrt IPC. Let $\pi = T_1, \dots, T_n = (\Gamma \Rightarrow \Delta)$ be a proof in **BPC** written in a linear fashion. Then it is enough to take π' as $I(T_1), \dots, I(T_n) = I(\Gamma \Rightarrow \Delta)$. The new proof π' is a proof in **P**, since **P** is defined by imitating the rules of **BPC**. □

Definition 4.3.20. *An extended Frege system for a substructural logic* L *is a Frege system for* L *together with the extension axiom which allows formulas of the form* $p \equiv \phi := (p \setminus \phi \wedge \phi \setminus p)$ *to be added to a derivation with the following conditions:* p *is a new variable not occurring in* ϕ*, in any lines before* $p \equiv \phi$*, or in any hypotheses to the derivation. It can however appear in later lines, but not in the last line. An extended Frege system for a super-basic logic* L *wrt* M *is defined similarly, where* $\mathsf{L} \subseteq \mathsf{M}$*, with the extension axiom being* $p \equiv \phi := (p \to \phi \wedge \phi \to p)$*.*

It is easy to check that the definition of equivalence introduced in Definition 4.3.20 is closed under substitution, i.e., if $A \equiv B$ then for any formula $\phi(p, \bar{q})$ we have $\phi(A, \bar{q}) \equiv \phi(B, \bar{q})$.

Lemma 4.3.21. *For any two Frege system* **P** *and* **Q** *for a substructural logic* L*, there exists a number* c *such that for any formula* ϕ *and any proof* π*, there exists a proof* π' *such that*

$$\mathbf{P} \vdash^{\pi} \phi \quad \textit{implies} \quad \mathbf{Q} \vdash^{\pi'} \phi$$

and $\lambda(\pi') \leq c\lambda(\pi)$*. In the case that* **P** *and* **Q** *are extended Frege systems, they are polynomially equivalent.*

Proof. The proof is easy and originally shown in [11]. The reason is that any instance of a rule in **P** can be replaced by its proof in **Q**, which has a fixed number of lines. Take c as the largest number of proof-lines of these proofs. Since there are finite many rules in **P**, finding c is possible. Therefore, $\lambda(\pi') \leq c\lambda(\pi)$. A similar argument also works for the lengths of the proofs when **P** and **Q** are extended Frege systems. □

As a result of Lemma 4.3.21, since we are concerned with the number of proof-lines and lengths of proofs, we can talk about "the" Frege (extended Frege) system for the substructural logic L and denote it by $\mathsf{L} - \mathbf{F}$ ($\mathsf{L} - \mathbf{EF}$). Note that Lemma 4.3.21 cannot be proved for any two Frege systems for L wrt M for super-basic logics $\mathsf{L} \subseteq \mathsf{M}$. For this to hold, we need an $\mathsf{L} - \mathbf{F}$ system wrt M to be strongly sound, i.e., if $\phi_1, \dots, \phi_n \vdash_{\mathbf{P}} \phi$ then $\phi_1, \dots, \phi_n \vdash_{\mathbf{L}} \phi$, which does not hold because of the condition 4 in Definition 4.3.18.

Definition 4.3.22. *A proof in a Frege (extended Frege, Hilbert-style, Gentzen-style) system is called tree-like if every step of the proof is used at most once as a hypothesis of a rule in the proof. It is called a general (or dag-like) proof, otherwise.*

In this paper we will not use this distinction, because throughout the paper all the proofs are considered to be dag-like, which is the more general notion.

4.4 A descent into the substructural world

In this section, we will present a sequence of tautologies and then we show they are exponentially hard for any system $\mathsf{L} - \mathbf{EF}$ for any substructural and super-basic logics. In order to do so, we first provide some sentences provable in the weak system **WL**. This uniformly provides two sequence of formulas provable in $\mathbf{FL}_\bot$ and **BPC**. In the case of $\mathbf{FL}_\bot$, since the system $\mathbf{FL}_\bot$ is conservative over **FL** and the formulas we are interested in do not contain $\bot$, we will automatically have a proof in **FL**.

To provide tautologies in **WL**, we pursue the following strategy: First, using the representations $\{\bot, 1\}$ for true and false, we encode every binary evaluation of an **LK**-formula by a suitable **WL**-proof. Then, using this encoding, we map a certain fragment of **LK** into the system **WL**, without any essential change into the original sequent. Finally, applying this map on a certain hard intuitionistic tautology provides the intended hard **WL**-tautology that we are looking for.

Definition 4.4.23. *Let v be a Boolean valuation assigning truth values $\{t, f\}$ to the propositional variables. For a formula A in the language $\mathcal{L}$, by $v(A)$ we mean the Boolean valuation of A by v, defined in the usual way. The substitution σ_v for a formula A is defined in the following way: if an atom is assigned "t" in the valuation v, substitute 1 for this atom in A and if an atom is assigned "f" in v then substitute $\bot$ for this atom in A. We write A^{σ_v} for the formula obtained from this substitution.*

Lemma 4.4.24. *For any formula A constructed from atoms and $\{\wedge, \vee\}$ and for any valuation v we have*

$$\textit{if } \ v(A) = t, \ \textit{then } \mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow 1,$$

$$\textit{if } \ v(A) = f, \ \textit{then } \mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow \bot.$$

Proof. The proof is simple and uses induction on the structure of the formula A. If it is an atom, then the claim is clear by the definition of A^{σ_v}. If $A = B \wedge C$ then if $v(A) = t$ we have $V(B) = v(C) = t$. Therefore, by induction hypothesis we have

$$\mathbf{WL} \vdash B^{\sigma_v} \Leftrightarrow 1 \ \textit{ and } \ \mathbf{WL} \vdash C^{\sigma_v} \Leftrightarrow 1$$

Using the following proof-trees in **WL**

$$\frac{1 \Rightarrow B^{\sigma_v} \qquad 1 \Rightarrow C^{\sigma_v}}{1 \Rightarrow B^{\sigma_v} \wedge C^{\sigma_v}} R\wedge \qquad \frac{B^{\sigma_v} \Rightarrow 1}{B^{\sigma_v} \wedge C^{\sigma_v} \Rightarrow 1} L\wedge_1$$

we obtain $\mathbf{WL} \vdash B^{\sigma_v} \wedge C^{\sigma_v} \Leftrightarrow 1$, which is $\mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow 1$.
If $A = B \wedge C$ and $v(A) = f$, then one of the following happens

$$v(B) = t, v(C) = f \quad or \quad v(B) = f, v(C) = t \quad or \quad v(B) = v(C) = f$$

We investigate the first case, the other cases are similar. If $v(B) = t$ and $v(C) = f$, by induction hypothesis we get

$$\mathbf{WL} \vdash B^{\sigma_v} \Leftrightarrow 1 \ \ and \ \ \mathbf{WL} \vdash C^{\sigma_v} \Leftrightarrow \bot$$

Therefore, the following are provable in **WL**

$$\frac{C^{\sigma_v} \Rightarrow \bot}{B^{\sigma_v} \wedge C^{\sigma_v} \Rightarrow \bot} (L\wedge_2) \qquad \bot \Rightarrow B^{\sigma_v} \wedge C^{\sigma_v}$$

where the right sequent is an instance of the axiom for $\bot$. Hence, we get $\mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow \bot$.
Finally, if $A = B \vee C$, based on whether $v(A) = t$ or $v(A) = f$ we proceed as before. All the cases are simple, therefore here we only investigate the case where $v(A) = v(B \vee C) = t$ and $v(B) = f$ and $v(C) = t$, as an example. Using the induction hypothesis for B and C, consider the following proof-trees in **WL**:

$$\frac{1 \Rightarrow C^{\sigma_v}}{1 \Rightarrow B^{\sigma_v} \vee C^{\sigma_v}} (R\vee_2) \qquad \frac{\dfrac{B^{\sigma_v} \Rightarrow \bot \quad \bot \Rightarrow 1}{B^{\sigma_v} \Rightarrow 1} (cut) \qquad C^{\sigma_v} \Rightarrow 1}{B^{\sigma_v} \vee C^{\sigma_v} \Rightarrow 1} (L\vee)$$

□

The following theorem is our main tool in proving the lower bound and it provides a method to convert classical tautologies to tautologies in **WL**.

Theorem 4.4.25. *If* $\bigwedge_{i_j \in I} p_{i_j} \to A(\bar{p})$ *is a classical tautology, then we have*

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k} (p_{i_j} \wedge 1) \Rightarrow A(\bar{p})$$

where $A(\bar{p})$ *is a formula only consisting of* $\bar{p} = p_1, \ldots, p_n$ *and connectives* $\{\wedge, \vee\}$ *and* $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

Proof. The theorem states that due to the commutativity of conjunction in classical logic, any order on the elements of I, i.e. the sequence $i_1, \ldots, i_k$, can be used and $\bigotimes_{j=1}^{k} (p_{i_j} \wedge 1) \Rightarrow A(\bar{p})$ is provable in **WL**. However, the order must be fixed throughout the proof.
Since $\bigwedge_{i_j \in I} p_{i_j} \to A(\bar{p})$ is a classical tautology, it will be true under any assignment of truth values to the propositional variables, especially the valuation v assigning truth to every p_i, for $i \in I$, and falsity to the rest. It is easy to see that under this valuation we have $v(\bigwedge_{i_j \in I} p_{i_j}) = t$ and since we also have $v(\bigwedge_{i_j \in I} p_{i_j} \to A(\bar{p})) = t$ (because the formula is a classical tautology), we get as a result $v(A) = t$. Therefore, using Lemma 4.4.24 we obtain $\mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow 1$ and since $\mathbf{WL} \vdash \Rightarrow 1$, using the cut rule we get

$$\mathbf{WL} \vdash \Rightarrow A^{\sigma_v} \qquad (\star)$$

On the other hand if we show

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k} (p_{i_j} \wedge 1), A^{\sigma_v} \Rightarrow A \ \ (\dagger)$$

then using the cut rule on the sequents in $(\star)$ and $(\dagger)$ we get

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1) \Rightarrow A.$$

We will prove $(\dagger)$ by induction on the structure of the formula A. If A is equal to p_{i_j}, for some j where $i_j \in I$, then since $v(p_{i_j}) = t$, we have $A^{\sigma_v} = p_{i_j}^{\sigma_v} = 1$. Therefore, the following proof-tree represents a proof in **WL**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_{i_j} \Rightarrow p_{i_j}}{p_{i_j} \wedge 1 \Rightarrow p_{i_j}}\,(L\wedge_1)}{p_{i_j} \wedge 1, 1 \Rightarrow p_{i_j}}\,(1w)}{1, p_{i_j} \wedge 1, 1 \Rightarrow p_{i_j}}\,(1w)}{p_{i_{j-1}} \wedge 1, p_{i_j} \wedge 1, 1 \Rightarrow p_{i_j}}\,(L\wedge_2)}{\vdots}$$

$$\dfrac{\dfrac{\vdots}{p_{i_1} \wedge 1, \ldots, p_{i_{j-1}} \wedge 1, p_{i_j} \wedge 1, \ldots, p_{i_k} \wedge 1, 1 \Rightarrow p_{i_j}}}{p_{i_1} \wedge 1 \otimes p_{i_2} \wedge 1, \ldots, p_{i_{j-1}} \wedge 1, p_{i_j} \wedge 1, \ldots, p_{i_k} \wedge 1, 1 \Rightarrow p_{i_j}}\,(L\otimes)$$

$$\dfrac{\vdots}{\bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), 1 \Rightarrow p_{i_j}}\,(L\otimes)$$

where the first vertical dots means using the rules $(1w)$ and $(L\wedge_2)$ consecutively. Note that based on the rule $(1w)$, we can add 1 in any position on the left hand-side of the sequents. Using this fact together with the rule $(L\wedge_2)$ we obtain all formulas in the appropriate order. The second vertical dots represents applications of the rule $(L\otimes)$ consecutively until one reaches the conclusion. Therefore, we have proved

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), A^{\sigma_v} \Rightarrow A.$$

The case where $A = p_{i_j}$ where $i_j \notin I$ is easier. Since for such j we have $v(p_{i_j}) = f$, using Lemma 4.4.24 we get $\mathbf{WL} \vdash A^{\sigma_v} \Leftrightarrow \bot$. Using the initial sequent for $\bot$ we have $\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), \bot \Rightarrow A$ and using the cut rule we get $(\dagger)$.

If $A(\bar{p}) = B(\bar{p}) \wedge C(\bar{p})$, and the induction hypothesis holds for $B(\bar{p})$ and $C(\bar{p})$, i.e.,

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), B^{\sigma_v} \Rightarrow B \quad , \quad \mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), C^{\sigma_v} \Rightarrow C \quad (\ddagger)$$

then first using the rule $(L\wedge_1)$ for the left sequent and rule $(L\wedge_2)$ for the right sequent, and then using the rule $(R\wedge)$ we get

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), B^{\sigma_v} \wedge C^{\sigma_v} \Rightarrow B \wedge C.$$

If $A(\bar{p}) = B(\bar{p}) \vee C(\bar{p})$ then first using the rule $(R\vee_1)$ for the left sequent in $(\ddagger)$ and rule $(R\vee_2)$ for the right sequent in $(\ddagger)$, and then using $(L\vee)$ we get

$$\mathbf{WL} \vdash \bigotimes_{j=1}^{k}(p_{i_j} \wedge 1), B^{\sigma_v} \vee C^{\sigma_v} \Rightarrow B \vee C.$$

$\square$

4.4.1 A brief digression into hard tautologies

The formulas we are going to introduce as our hard tautologies for the system $\mathsf{FL}-\mathbf{EF}$ and $\mathsf{BPC}-\mathbf{EF}$ are inspired by the hard formulas for $\mathsf{IPC}-\mathbf{F}$ introduced by Hrubeš [21] and their negation-free version introduced by Jeřábek [25]. In this subsection, we briefly explain these formulas and what combinatorial facts they represent.

Let us first define formulas $Clique_{n,k}$ and $Color_{n,m}$ which will be used in Hrubeš's formulas.

Definition 4.4.26. *[28, Section 13.5] Let $n, k, m \geq 1$. By an undirected simple graph on $[n]$ we mean the set of strings of length $\binom{n}{2}$. We say a graph has a clique when there exists a complete subgraph, which is a subgraph with all possible edges among its vertices. Define $Clique_{n,k}$ to be the set of undirected simple graphs on $[n]$ that have a clique of size at least k, and define $Color_{n,m}$ to be the set of garphs on $[n]$ that are m-colorable, and they are defined by the following two sets.*
The set of clauses denoted by $Clique^k_n(\bar{p}, \bar{q})$ uses $\binom{n}{2}$ atoms p_{ij}, $\{i, j\} \in \binom{n}{2}$, one for each potential edge in a graph on $[n]$, and $k.n$ atoms q_{ui} intended to describe a mapping from $[k]$ to $[n]$. It consists of the following clauses:

- $\bigvee_{i\in[n]} q_{ui}$*, all* $u \leq k$,
- $\neg q_{ui} \vee \neg q_{uj}$*, all* $u \in [k]$ *and* $i \neq j \in [n]$,
- $\neg q_{ui} \vee \neg q_{vi}$*, all* $u \neq v \in [k]$ *and* $i \in [n]$,
- $\neg q_{ui} \vee \neg q_{vj} \vee p_{ij}$*, all* $u \neq v \in [k]$ *and* $\{i, j\} \in \binom{n}{2}$.

The set of clauses $Color^m_n(\bar{p}, \bar{r})$ uses atoms $\bar{p}$ and $n.m$ more atoms r_{ia} where $i \in [n]$ and $a \in [m]$, intended to describe an m-coloring of the graph. It consists of the following clauses:

- $\bigvee_{a\in[m]} r_{ia}$*, all* $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ib}$*, all* $a \neq b \in [m]$ *and* $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ja} \vee \neg p_{ij}$*, all* $a \in [m]$ *and* $\{i, j\} \in \binom{n}{2}$.

Note that every occurrence of atoms p_{ij} in $Clique^k_n(\bar{p}, \bar{q})$ is positive, or in other words it is monotone in $\bar{p}$.

The exponential lower bound for intuitionistic logic is demonstrated in the following theorem due to P. Hrubeš. The main idea is that any short proof for the hard tautology provides a small monotone circuit to decide whether a given graph is a clique or colorable, which we know is a hard problem to decide [3].

Theorem 4.4.27. *[21] Let $\bar{p} = p_1, \cdots, p_n$ and $\bar{q} = q_1, \cdots, q_n$ and $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ be disjoint variables, $\bar{v} = \{\bar{p}, \bar{q}, \bar{r}, \bar{s}\}$, and $k = \lfloor \sqrt{n} \rfloor$. Then the formulas*

$$\Theta^{\perp}_n := \bigwedge_{i=1,\cdots,n} (p_i \vee q_i) \to \neg Color^k_n(\bar{p}, \bar{s}) \vee \neg Clique^{k+1}_n(\neg\bar{q}, \bar{r})$$

are intuitionistic tautologies. Moreover, every $\mathsf{IPC}-\mathbf{F}$-proof of $\Theta^{\perp}_n$ contains at least $2^{\Omega(n^{1/4})}$ proof-lines.

We refer to the formulas $\Theta_n^{\perp}$ as Hrubeš's formulas. The superscript $\perp$ in $\Theta_n^{\perp}$ stresses that the formulas contain negations. For our purposes, we need to use a negation-free version of Hrubeš's formulas.

Definition 4.4.28. *[25, Definition 6.28] For $k \leq n$ define:*

$$\alpha_n^k(\bar{p}, \bar{s}, \bar{s'}) := \bigvee_{i<n} \bigwedge_{l<k} s'_{i,l} \vee \bigvee_{i,j<n} \bigvee_{l<k} (s_{i,l} \wedge s_{j,l} \wedge p_{i,j}),$$

$$\beta_n^k(\bar{q}, \bar{r}, \bar{r'}) := \bigvee_{l<k} \bigwedge_{i<n} r'_{i,l} \vee \bigvee_{i,j<n} \bigvee_{l<m<k} (r_{i,l} \wedge r_{j,m} \wedge q_{i,j}).$$

Define the negation-free Hrubeš formulas for $k = \lfloor \sqrt{n} \rfloor$ as follows:

$$\Theta_n := \bigwedge_{i,j}(p_{i,j} \vee q_{i,j}) \to [(\bigwedge_{i,l}(s_{i,l} \vee s'_{i,l}) \to \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})) \vee (\bigwedge_{i,l}(r_{i,l} \vee r'_{i,l}) \to \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'}))].$$

Notice that $Color_n^k(\bar{p}, \bar{s}) = \neg\alpha_n^k(\bar{p}, \bar{s}, \neg\bar{s})$ and $Clique_n^k(\bar{p}, \bar{r}) = \neg\beta_n^k(\neg\bar{p}, \bar{r}, \neg\bar{r})$. The lower bound of Theorem 4.4.27 also applies to Θ_n [25].

To make Hrubeš's formulas negation-free, Jeřábek introduced new propositional variables $s'_{i,l}$ and $r'_{i,l}$ to play the role of $\neg s_{i,l}$ and $\neg r_{i,l}$, respectively. This trick provides some implication-free formulas α_n^k and β_n^k in the definition 4.4.28 to make the formulas Θ_n more amenable to the technique that we provided in Section 4.4.

Theorem 4.4.29. *([25, Theorem 6.37]) Let L be a super-intuitionistic logic with infinite branching. Then the formulas Θ_n are intuitionistic tautologies and they require $\mathsf{L} - \mathbf{EF}$-proofs of length $2^{n^{\Omega(1)}}$, and $\mathsf{L} - \mathbf{F}$-proofs with at least $2^{n^{\Omega(1)}}$ lines.*

4.4.2 Weak hard tautologies

The following lemmas are easy observations. The first one states that fusion distributes over disjunction in substructural logics. The second one presents a property of the sequent calculus **LK**.

Lemma 4.4.30. *In the sequent calculus* **WL** *we have the following:*

$$\mathbf{WL} \vdash \bigotimes_{i=1}^{n}(A_i \vee B_i) \Leftrightarrow \bigvee_{I}(\bigotimes_{i=1}^{n} D_i^I)$$

where $I \subseteq \{1, 2, \cdots, n\}$ and $D_i^I = \begin{cases} A_i & , \ i \in I \\ B_i & , \ i \notin I \end{cases}$.

Proof. The proof is easy and uses induction on n. Note that in each disjunct in the right hand-side, D_i^I is either A_i or B_i, according to the subset I. However, the order of the subscripts must be increasing. For instance, for the case $n = 2$ we have

$$\mathbf{WL} \vdash (A_1 \vee B_1) \otimes (A_2 \vee B_2) \Leftrightarrow (A_1 \otimes B_2) \vee (A_1 \otimes A_2) \vee (B_1 \otimes A_2) \vee (B_1 \otimes B_2).$$

□

Lemma 4.4.31. *Suppose $\alpha_1 \to \alpha_2$ and $\beta_1 \to \beta_2$ have no propositional variables in common. If the formula $\alpha_1 \wedge \alpha_2 \to \beta_1 \vee \beta_2$ is provable in* **LK**, *then either $\alpha_1 \to \alpha_2$ or $\beta_1 \to \beta_2$ is provable in* **LK**.

Proof. It is an easy corollary of Craig's interpolation theorem. $\square$

We are now ready to formulate hard tautologies in **WL** and prove the lower bound. By $\bigotimes_{i=1}^{n-1} \bigotimes_{j=1}^{n-1} A_{i,j}$, we mean that the indices first range over j and then over i, which will result in the lexicographic order, i.e., it has the following form

$$A_{1,1} \otimes A_{1,2} \otimes \cdots \otimes A_{1,n-1} \otimes A_{2,1} \otimes \cdots \otimes A_{n-1,n-1}.$$

For a set (sequence of formulas) Γ, by $\parallel \Gamma \parallel$ we mean the number of elements of the set (the number of formulas the sequence contains).

Theorem 4.4.32.

$$\Theta_n^{\otimes} := [\bigotimes_{i=1}^{n-1} \bigotimes_{j=1}^{n-1} ((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1))] \ \backslash$$

$$[\bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1)) \backslash \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})] \vee [\bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1)) \backslash \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'})]$$

are provable in **WL**, *where $k = \lfloor \sqrt{n} \rfloor$.*

Proof. Let us denote the following formula by A:

$$[\bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1)) \backslash \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})] \vee [\bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1)) \backslash \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'})].$$

First, we show for any $I \subseteq \{(i,j) \mid i,j \in \{1, \cdots, n-1\}\}$

$$\mathbf{WL} \vdash \bigotimes_{i=1}^{n-1} \bigotimes_{j=1}^{n-1} Q_{i,j}^I \Rightarrow A \qquad (\dagger)$$

such that $Q_{i,j} = \begin{cases} p_{i,j} \wedge 1 \ , & (i,j) \in I \\ q_{i,j} \wedge 1 \ , & (i,j) \notin I \end{cases}$.

For simplicity from now on, unless specified otherwise, we will delete the ranges of i, j and l, which are indicated in $\Theta_n^{\otimes}$.

It is easy to see how proving (†) will result in proving the theorem. The reason is the following. Since (†) is provable for any $I \subseteq \{(i,j) \mid i,j \in \{1, \cdots, n-1\}\}$, using the left disjunction rule for $2^{(n-1)^2} - 1$ many times on (†), we get

$$\mathbf{WL} \vdash \bigvee_I \bigotimes_i \bigotimes_j Q_{i,j}^I \Rightarrow A.$$

Furthermore, Lemma 4.4.30 allows us to obtain

$$\mathbf{WL} \vdash \bigotimes_i \bigotimes_j ((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1)) \Rightarrow \bigvee_I \bigotimes_i \bigotimes_j Q_{i,j}^I,$$

and using the cut rule and the rule $(R\backslash)$, we conclude

$$\mathbf{WL} \vdash \Rightarrow \Theta_n^{\otimes}.$$

On the other hand, Θ_n presented in Definition 4.4.28, is provable in **LJ**, and therefore also provable in **LK**. Using the distributivity of conjunction over disjunction we obtain the following sequent

$$\bigwedge_{(i,j)\in M} p_{i,j} \wedge \bigwedge_{(i,j)\in N} q_{i,j} \Rightarrow [\bigwedge_{i,l}(s_{i,l} \vee s'_{i,l}) \rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'})] \vee [\bigwedge_{i,l}(r_{i,l} \vee r'_{i,l}) \rightarrow \beta_n^{k+1}(\bar{q},\bar{r},\bar{r'})]$$

is provable in **LK** for any M and N such that $M \cup N = \{(i,j) \mid i,j \in \{1,\cdots,n-1\}\}$. For such M and N, using Lemma 4.4.31 we have either

$$\mathbf{LK} \vdash \bigwedge_{(i,j)\in M} p_{i,j} \Rightarrow (\bigwedge_{i,l}(s_{i,l} \vee s'_{i,l}) \rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'})),$$

or

$$\mathbf{LK} \vdash \bigwedge_{(i,j)\in N} q_{i,j} \Rightarrow (\bigwedge_{i,l}(r_{i,l} \vee r'_{i,l}) \rightarrow \beta_n^{k+1}(\bar{q},\bar{r},\bar{r'})).$$

We consider the first case, the second one being similar. Therefore, suppose the first case holds. Using the cut rule, we have

$$\mathbf{LK} \vdash \bigwedge_{(i,j)\in M} p_{i,j}\ ,\ \bigwedge_{i,l}(s_{i,l} \vee s'_{i,l}) \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}),$$

and using the left exchange rule we obtain

$$\mathbf{LK} \vdash \bigwedge_{i,l}(s_{i,l} \vee s'_{i,l})\ ,\ \bigwedge_{(i,j)\in M} p_{i,j} \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}).$$

Now, using the distributivity of conjunction over disjunction in **LK** we have for any U and V such that $U \cup V = \{(i,l) \mid i < n, l < k\}$

$$\mathbf{LK} \vdash (\bigwedge_{(i,l)\in U} s_{i,l} \wedge \bigwedge_{(i,l)\in V} s'_{i,l})\ ,\ \bigwedge_{(i,j)\in M} p_{i,j} \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}),$$

or equivalently (using the rules $(L\wedge_1), (L\wedge_2)$, and left contraction),

$$\mathbf{LK} \vdash \bigwedge_{(i,l)\in U} s_{i,l} \wedge \bigwedge_{(i,l)\in V} s'_{i,l} \wedge \bigwedge_{(i,j)\in M} p_{i,j} \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}).$$

Now, using Theorem 4.4.25

$$\mathbf{WL} \vdash (\bigotimes_{i=1}^{n-1}\bigotimes_{l=1}^{k-1} S_{i,l}^{U,V})\ \otimes\ (\bigotimes_{(i,j)\in M}(p_{i,j} \wedge 1)) \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}),$$

where $S_{i,l}^{U,V} = \begin{cases} s_{i,l} \wedge 1 & , \quad (i,j) \in U \\ s'_{i,l} \wedge 1 & , \quad (i,j) \in V \end{cases}$.

Note that by Theorem 4.4.25, we can choose any order on $\bigotimes_{(i,j)\in M}(p_{i,j} \wedge 1)$ provided we do not change it throughout the proof. Since the order is arbitrary, for simplicity we do not explicitly write it down.

Equivalently (using the fact that for any formulas A and B, we have $\mathbf{WL} \vdash A, B \Rightarrow A \otimes B$ and then using the cut rule), we have

$$\mathbf{WL} \vdash (\bigotimes_{i=1}^{n-1}\bigotimes_{l=1}^{k-1} S_{i,l}^{U,V})\ ,\ (\bigotimes_{(i,j)\in M}(p_{i,j} \wedge 1)) \Rightarrow \alpha_n^k(\bar{p},\bar{s},\bar{s'}).$$

Since this sequent is provable for any U and V such that $U \cup V = \{(i,l) \mid i < n, l < k\}$, using the left disjunction rule for $2^{(n-1)(k-1)} - 1$ many times we get

$$\mathbf{WL} \vdash \bigvee_{U,V} (\bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} S_{i,l}^{U,V}) \ , \ (\bigotimes_{(i,j) \in M} (p_{i,j} \wedge 1)) \Rightarrow \alpha_n^k(\bar{p}, \bar{s}, \bar{s'}).$$

Using Theorem 4.4.30 and the cut rule we have

$$\mathbf{WL} \vdash [\bigotimes_i \bigotimes_l ((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1))] \ , \ (\bigotimes_{(i,j) \in M} (p_{i,j} \wedge 1)) \Rightarrow \alpha_n^k(\bar{p}, \bar{s}, \bar{s'}),$$

and using the rule $(R\backslash)$ we get

$$\mathbf{WL} \vdash \bigotimes_{(i,j) \in M} (p_{i,j} \wedge 1) \Rightarrow [\bigotimes_i \bigotimes_l ((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1))] \ \backslash \ \alpha_n^k(\bar{p}, \bar{s}, \bar{s'}).$$

Now, using the rules $(L1)$ and $(L\wedge_2)$ consecutively for $\| N \|$-many times (each time we produce $q_{i,j} \wedge 1$ for each element of N, in the same manner as in the proof of Theorem 4.4.25) and then using the rule $(L\otimes)$ for $\| N \|$-many times and in the end using the rule $(R\vee_1)$ we prove (†). □

Note that the formulas $\Theta_n^{\otimes}$ are provable for any choice of $1 \leq k \leq n$, as well. The reason for the restriction to $k = \lfloor \sqrt{n} \rfloor$ is that we are only concerned with this case, when it comes to the proof of the lower bound.

Remark 4.4.33. *It is worth noting that the system* **WL** *could have been defined in an alternative way by deleting / instead of \ from the language, and having the same initial sequents and rules as* **FL** *and leaving the rules* $(R/), (L/)$, *and* $(L\backslash)$ *out. Then, in a similar manner, the following formulas would be provable in this alternative calculus:*

$$[\alpha_n^k / \bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1))] \ \vee \ [\beta_n^{k+1} / \bigotimes_{i=1}^{n-1} \bigotimes_{l=1}^{k-1} ((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1))] \ /$$

$$[\bigotimes_{i=1}^{n-1} \bigotimes_{j=1}^{n-1} ((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1)].$$

where $\alpha_n^k = \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})$ *and* $\beta_n^{k+1} = \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'})$.

Now, we are ready to present tautologies in FL and BPC. It is easy to see that the tautologies introduced in Theorem 4.4.32 are provable in basic substructural logics.

Corollary 4.4.34. *The formulas* $\Theta_n^{\otimes}$ *are provable in the logic* FL.

Proof. Clearly, **WL** is a subsystem of the sequent calculus $\mathbf{FL}_\bot$. Then, using the cut elimination theorem [14, Theorem 7.8] for $\mathbf{FL}_\bot$, and the fact that $\Theta_n^{\otimes}$ do not contain $\bot$, we obtain the result. □

To provide tautologies in BPC, we need the translation function, t, defined in Section 4.3.

Corollary 4.4.35. *The formulas* $(\Theta_n^{\otimes})^t$ *are provable in* BPC.

Proof. The provability of $(\Theta_n^{\otimes})^t$ is a consequence of Theorem 4.4.25 and Lemma 4.3.14. □

4.5 The main theorem

In this section we will present the main result of the paper. We will prove that there exists an exponential lower bound on the lengths of proofs in proof systems for a wide range of logics. Furthermore, we will obtain an exponential lower bound on the number of proof-lines in a broad range of Frege systems.

Theorem 4.5.36. *Let* L *be a super-intitionistic logic with infinite branching.*

(i) *Let* $\mathbf{P}$ *be a proof system for a logic with the language* $\mathcal{L}^{\otimes}$ *such that* $\mathbf{FL} \leq \mathbf{P} \leq_p^t \mathsf{L}-\mathbf{EF}$. *Then* $\mathbf{P} \vdash \Theta_n^{\otimes}$ *and the length of any such proof is exponential in* n.

(ii) *Let* $\mathbf{P}$ *be a proof system for a logic with the language* $\mathcal{L}$ *such that* $\mathbf{BPC} \leq \mathbf{P} \leq_p \mathsf{L}-\mathbf{EF}$. *Then* $\mathbf{P} \vdash (\Theta_n^{\otimes})^t$ *and the length of any such proof is exponential in* n.

Proof. (i) Since $\mathbf{FL} \vdash \Theta_n^{\otimes}$ by 4.4.32, and $\mathbf{FL} \leq \mathbf{P}$, the formulas $\Theta_n^{\otimes}$ are also provable in $\mathbf{P}$. Take such a proof π, i.e., $\mathbf{P} \vdash^{\pi} \Theta_n^{\otimes}$. Since $\mathbf{P} \leq_p^t \mathsf{L}-\mathbf{EF}$, there exists a proof π_1 and a polynomial p, such that $\mathsf{L}-\mathbf{EF} \vdash^{\pi_1} (\Theta_n^{\otimes})^t$ and $|\pi_1| = p(|\pi|)$. We want to prove $\mathsf{L}-\mathbf{EF} \vdash (\Theta_n^{\otimes})^t \to \Theta_n$ by a proof whose length is polynomial in n. First, since L is a super-intuitionistic logic, we have $\mathsf{L}-\mathbf{EF} \vdash u \wedge \top \leftrightarrow u$, where u is any of the atoms present in the formulas Θ_n. This proof has a fix number of proof-lines in $\mathsf{L}-\mathbf{EF}$. The claim then easily follows from the fact that the length of the formula Θ_n is also polynomial in n. Therefore, Θ_n is provable in $\mathsf{L}-\mathbf{EF}$ with a proof polynomially long in n and π_1. By 4.4.29, any $\mathsf{L}-\mathbf{EF}$-proof of Θ_n has length at least $2^{\Omega(n^{1/4})}$. Therefore, the length of π must be exponential in n.
(ii) The proof for this part is similar to that of (i). Here, the formulas $(\Theta_n^{\otimes})^t$ are provable in $\mathbf{BPC}$ and hence in $\mathbf{P}$. Since $\mathsf{L}-\mathbf{EF}$ polynomially simulates $\mathbf{P}$, we obtain the exponential lower bound using the fact that $\mathsf{L}-\mathbf{EF} \vdash (\Theta_n^{\otimes})^t \to \Theta_n$. □

The following theorem states an exponential lower bound on the number of proof-lines in a wide range of Frege systems.

Theorem 4.5.37. *Let* M *be a super-intitionistic logic with infinite branching.*

(i) *Let* L *be a logic with the language* $\mathcal{L}^{\otimes}$ *such that* $\mathsf{FL} \subseteq \mathsf{L} \subseteq^t \mathsf{M}$. *Then, the number of lines of every proof of* $\Theta_n^{\otimes}$ *in* $\mathsf{L}-\mathbf{F}$ *is exponential in* n *and every proof of* $\Theta_n^{\otimes}$ *in* $\mathsf{L}-\mathbf{EF}$ *has length exponential in* n.

(ii) *Let* L *be a logic with the language* $\mathcal{L}$ *such that* $\mathsf{BPC} \subseteq \mathsf{L} \subseteq \mathsf{M}$. *Then, the number of lines of every proof of* $(\Theta_n^{\otimes})^t$ *in any* $\mathsf{L}-\mathbf{F}$ *system wrt* M *is exponential in* n *and every proof of* $(\Theta_n^{\otimes})^t$ *in any* $\mathsf{L}-\mathbf{EF}$ *system wrt* M *has length exponential in* n.

Proof. To prove (i), note that since $\mathsf{FL} \vdash \Theta_n^{\otimes}$ by 4.4.34, and $\mathsf{FL} \subseteq \mathsf{L}$, we have $\mathsf{L} \vdash \Theta_n^{\otimes}$. Let π be a proof of $\Theta_n^{\otimes}$ in $\mathsf{L}-\mathbf{EF}$. Using the assumption we will provide a proof π' of $(\Theta_n^{\otimes})^t$ in $\mathsf{M}-\mathbf{EF}$ such that $\lambda(\pi') \leq c\lambda(\pi)$. Fix an extended Frege system $\mathbf{Q}$ for the logic M. Define the system $\mathbf{P}$ as the system consisting of all the rules in $\mathbf{Q}$ plus the rules:

$$\frac{A_1^t \quad \dots \quad A_l^t}{A^t}$$

for any rule of $\mathsf{L}-\mathbf{EF}$ of the form:

$$\frac{A_1 \quad \dots \quad A_l}{A}$$

We have to show that $\mathbf{P}$ is an extended Frege system for the logic M. To be more precise, we have to show that $\mathbf{P}$ is strongly sound and strongly complete with respect to M, because the other conditions (1 and 2 in Definition 4.3.18) are obvious. $\mathbf{P}$ is strongly complete wrt M, since it contains $\mathbf{Q}$ and $\mathbf{Q}$ is strongly complete wrt M. For strongly soundness, note that for any rule in $\mathsf{L}-\mathbf{EF}$ of the form:

$$\frac{A_1 \quad \dots \quad A_l}{A}$$

since all the rules in $\mathsf{L}-\mathbf{EF}$ are standard, we have $A_1, \dots A_l \vdash_{\mathsf{L}} A$. By Remark 4.3.15, $A_1^t, \dots A_l^t \vdash_{\mathsf{M}} A^t$. Hence, all the new rules in $\mathbf{P}$ are standard with respect to M.

To bound the number of proof-lines, let $\pi = \phi_1, \dots, \phi_m$ be a proof for $\Theta_n^{\otimes}$ in $\mathsf{L}-\mathbf{EF}$. Then, each ϕ_i is either an extension axiom, or it is derived from $\{\phi_{j_1}, \dots, \phi_{j_l}\}$ such that all j_r's are less than i. It is clear that $\pi' = \pi^t = \phi_1^t, \dots, \phi_m^t$ is a proof in $\mathbf{P}$, since the translation t of the extension axiom of $\mathsf{L}-\mathbf{EF}$ will be the extension axiom of $\mathsf{M}-\mathbf{EF}$ and moreover,

$$\frac{\phi_{j_1}^t \quad \dots \quad \phi_{j_l}^t}{\phi_i^t}$$

is an instance of a rule in $\mathbf{P}$. Note that the number of proof-lines stay the same, i.e., $\lambda(\pi) = \lambda(\pi')$.

Therefore, the formula $(\Theta_n^{\otimes})^t$ has a proof in $\mathbf{P}$ whose number of lines is the same as the number of lines of the proof of $\Theta_n^{\otimes}$ in $\mathsf{L}-\mathbf{EF}$. Since for any formula ϕ in the language $\mathcal{L}^{\otimes}$ we have $|\phi| = |\phi^t|$, therefore the length of π is the same as the length of π'. On the other hand, as we observed in the proof of Theorem 4.5.36, we can show that $(\Theta_n^{\otimes})^t \to \Theta_n$ has a proof in $\mathbf{P}$ with polynomial number of lines. Gluing these proofs together, we will obtain a proof for Θ_n in $\mathbf{P}$. Since any proof for Θ_n in $\mathbf{P}$ has exponential length (Theorem 4.4.29), any proof for $\Theta_n^{\otimes}$ in $\mathsf{L}-\mathbf{EF}$ will also have exponential length.

Note that the above construction also works for the case of considering Frege systems. It is easy to see that the translation of every proof in $\mathsf{L}-\mathbf{F}$ will be a proof in $\mathsf{M}-\mathbf{F}$, and the number of proof-lines stay the same. Therefore, the bound on the number of proof-lines follows.

For part (ii), using Corollary 4.4.35, $(\Theta_n^{\otimes})^t$ is provable in BPC and hence in L. Fix an extended Frege system $\mathbf{Q}$ for the logic M. Add the rules of the $\mathsf{L}-\mathbf{EF}$ system wrt to M to $\mathbf{Q}$. The resulting system, which we denote by $\mathbf{P}$, is an extended Frege system for the logic M. The reason is similar to the argument in the part (i), using the facts that $\mathsf{L} \subseteq \mathsf{M}$ and all the rules of $\mathsf{L}-\mathbf{EF}$ system wrt M are M-standard. Let π be a proof for $(\Theta_n^{\otimes})^t$ in $\mathsf{L}-\mathbf{EF}$ system wrt M, therefore, it will also be a proof in $\mathbf{P}$ with the same number of lines and same length. Again by gluing the short proof of $\mathbf{P} \vdash (\Theta_n^{\otimes})^t \to \Theta_n$ to π, we reach the result as in the proof for part (i). □

Corollary 4.5.38. • *Let S be any subset of $\{e, c, i, o\}$, and* L *be* $\mathsf{FL_S}$*, or any of the logics of the sequent calculi in Table 4.1. Then, the number of lines of every proof of $\Theta_n^{\otimes}$ in* $\mathsf{L} - \mathbf{F}$ *is exponential in n and every proof of $\Theta_n^{\otimes}$ in* $\mathsf{L} - \mathbf{EF}$ *has length exponential in n.*

- *Let* L *be* BPC *or* EBPC *and* M *a super-intuitionistic infinite branching logic. Then, the number of lines of every proof of $(\Theta_n^{\otimes})^t$ in any* $\mathsf{L} - \mathbf{F}$ *system wrt* M *is exponential in n and every proof of $\Theta_n^{\otimes}$ in any* $\mathsf{L} - \mathbf{EF}$ *system wrt* M *has length exponential in n.*

4.6 The lower bound for sequent calculi

So far, we have provided a lower bound for proof systems for logics as least as strong as FL and polynomially simulated by an extended Frege system for an infinite branching super-intuitionistic logic. It is very desirable to see if the lower bound also applies to proof systems for logics outside this range, for instance their classical counterparts. The result in this section is an attempt in this direction and we reach a positive answer for any proof system polynomially weaker than $\mathbf{CFL_{ew}^-}$, which is the system $\mathbf{CFL_{ew}}$ without the cut rule. For that matter, we first transfer the lower bound from the previous section to the sequent-style proof system $\mathbf{FL_S}$ for any $S \subseteq \{e, c, i, o\}$. Then we use the observation that any cut-free proof of a single-conclusion sequent in the 0-free fragment of $\mathbf{CFL_{ew}}$ is also an $\mathbf{FL_{ew}}$-proof.

Theorem 4.6.39. *Let Γ be a sequence of formulas $\gamma_1, \ldots, \gamma_m$, A a formula and S any subset of $\{e, c, i, o\}$. If $\mathbf{FL_S} \vdash^{\pi} \Gamma \Rightarrow A$ then there exists a Frege system $\mathbf{P}$ for* $\mathsf{FL_S}$ *such that*

$$\mathbf{P} \vdash^{\pi'} \bigotimes_{i=1}^{m} \gamma_i \setminus A$$

such that $\lambda(\pi') = \lambda(\pi)$.

Proof. The proof is similar to the proof of Theorem 4.5.37. As noted in the discussion after Definition 4.2.3, since $\mathbf{FL_S} \vdash \Gamma \Rightarrow A$, we have $\Gamma \vdash_{\mathsf{FL_S}} A$. Therefore, for any Frege system $\mathbf{Q}$ for the logic $\mathsf{FL_S}$, by strong completeness in Definition 4.3.18, we have $\Gamma \vdash_{\mathbf{Q}} A$. Fix such $\mathbf{Q}$. The method is developing a Frege system $\mathbf{P}$ for $\mathsf{FL_S}$ by transforming all the axioms and rules of the sequent calculus $\mathbf{FL_S}$ to Frege rules in the new system. For the sake of completeness, we also add $\mathbf{Q}$ to the resulting system.
Recall that for $\Gamma = \emptyset$, the formula $\bigotimes \Gamma$ is defined as 1 and for any single-conclusion sequent $T = (\Gamma \Rightarrow \Delta)$ by $I(T)$, the interpretation of the sequent T, we meant $\bigotimes \Gamma \setminus \Delta$, if Δ is non-empty, and $\bigotimes \Gamma \setminus 0$ for $\Delta = \emptyset$. Now, define $\mathbf{P}$ as the system consisting of the rules of $\mathbf{Q}$ plus the following rules: for the axiom T in the sequent calculus $\mathbf{FL_S}$ add

$$\overline{I(T)}$$

and for any rule in the sequent calculus $\mathbf{FL_S}$ of the form

$$\frac{T_1 \quad \ldots \quad T_m}{T}$$

add the following rule

$$\frac{I(T_1) \qquad \dots \qquad I(T_m)}{I(T)}$$

where $m = 1$ or $m = 2$. We have to show that $\mathbf{P}$ is a Frege system for the logic $\mathsf{FL_S}$, i.e., $\mathbf{P}$ is strongly sound and strongly complete wrt the logic $\mathsf{FL_S}$. First, since $\mathbf{Q}$ is strongly complete wrt $\mathsf{FL_S}$, then so is $\mathbf{P}$. Now for strongly soundness, we have to show that the new rules are standard wrt $\mathsf{FL_S}$. I.e., for any rule of the form

$$\frac{I(T_1) \qquad \dots \qquad I(T_m)}{I(T)}$$

in $\mathbf{P}$ we have to show $I(T_1), \dots, I(T_m) \vdash_{\mathsf{FL_S}} I(T)$. However, it is not hard to show that, since in the sequent calculus $\mathbf{FL_S}$ the cut rule exists, we have $\Rightarrow I(T_i) \vdash_{\mathbf{FL_S}} T_i$ using

$$\Rightarrow \bigotimes \Gamma \setminus \phi \vdash_{\mathbf{FL_S}} \Gamma \Rightarrow \phi$$

and the fact that for any two formulas ϕ and ψ, we have $\mathbf{FL_S} \vdash \phi, \phi \setminus \psi \Rightarrow \psi$. Using the corresponding rule, $T_1, \dots, T_m \vdash_{\mathbf{FL_S}} T$, the fact that $T \vdash_{\mathbf{FL_S}} \Rightarrow I(T)$, and the cut rule we have $\Rightarrow I(T_1), \dots, \Rightarrow I(T_m) \vdash_{\mathbf{FL_S}} \Rightarrow I(T)$. Therefore, by definition, $I(T_1), \dots, I(T_m) \vdash_{\mathsf{FL_S}} I(T)$. Therefore, $\mathbf{P}$ is a Frege system for $\mathsf{FL_S}$. For the number of proof-lines, note that if $\pi = T_1, \dots, T_n$ is a proof for $T_n = (\Gamma \Rightarrow A)$ in $\mathbf{FL_S}$, then it is easy to see that $I(T_1), \dots, I(T_n)$ will be a proof for $\bigotimes_{i=1}^{m} \gamma_i \setminus A$ in $\mathbf{P}$. Therefore, $\lambda(\pi') = \lambda(\pi)$. □

Corollary 4.6.40. *For any $S \subseteq \{e, i, o, c\}$ we have $\mathbf{FL_S} \vdash \Rightarrow \Theta_n^{\otimes}$ and the number of lines of any proof of this sequent is exponential in n.*

By a 0-free formula in $\mathsf{CFL_{ew}}$, we mean a formula only consisting of propositional variables, the constant 1, and the connectives $\{\wedge, \vee, \to, \otimes\}$.

Lemma 4.6.41. *If Γ is a sequence of 0-free formulas, then $\mathbf{CFL_{ew}^-} \nvdash \Gamma \Rightarrow$.*

Proof. Suppose $(\Gamma \Rightarrow)$ has a proof in $\mathbf{CFL_{ew}^-}$. Since the proof is cut-free and Γ is 0-free, by the subformula property of $\mathbf{CFL_{ew}^-}$, the whole proof is also 0-free. Therefore, there is no axiom in the proof with an empty succedent, because such an axiom must be in the form $(0 \Rightarrow)$, which is not 0-free. Moreover, if the succedent of the conclusion of any rule is empty, then the succedent of at least one of its premises must be empty, as well. The reason is the following. First, note that the last rule is not an axiom, as stated. It cannot be a right rule either, because they always have at least one formula in the succedent of their conclusion. And for the left rules, the claim is evident by a simple case checking. The only non-trivial case to check is $(L \to)$ which also has such a premise:

$$\frac{\Upsilon \Rightarrow \phi \qquad \Pi, \psi, \Sigma \Rightarrow}{\Pi, \phi \to \psi, \Upsilon, \Sigma \Rightarrow} (L \to)$$

Therefore, any sequent in the proof with an empty succedent has also a premise with an empty succedent. This is clearly a contradiction. □

The following theorem, which is of independent interest, states that for positive formulas, a cut-free proof for a single-conclusion sequent in $\mathbf{CFL_{ew}}$ is also a proof for the same sequent in $\mathbf{FL_{ew}}$.

Theorem 4.6.42. *Suppose* Γ *is a sequence of* 0*-free formulas and* A *is a* 0*-free formula. Then any proof* π *for* $\Gamma \Rightarrow A$ *in* $\mathbf{CFL_{ew}^-}$ *is also a proof in* $\mathbf{FL_{ew}}$.

Proof. The sketch of the proof is the following: suppose π is a cut-free proof in $\mathbf{CFL_{ew}}$ such that all the formulas in the proof are 0-free. Then, along the proof, the number of formulas in the succedent of the sequents does not decrease. The reason lies in the fact that neither the cut rule nor the contraction rules are present. Hence, in the special case that the sequent is also single-conclusion, the succedents of all the sequents in the whole proof will contain exactly one formula. Therefore, the proof is in $\mathbf{FL_{ew}}$.
Let π be a proof for $\Gamma \Rightarrow A$ in $\mathbf{CFL_{ew}^-}$. By induction on the structure of π we will show it is also a proof for the same sequent in $\mathbf{FL_{ew}}$. As stated in the proof of Lemma 4.6.41, every formula in the proof must be 0-free.
If $\Gamma \Rightarrow A$ is an instance of an axiom in $\mathbf{CFL_{ew}^-}$, then it is either $\Rightarrow 1$ or an instance of the axiom $\phi \Rightarrow \phi$, which are both also axioms in the sequent calculus $\mathbf{FL_e}$. For the induction step, note that the last rule in the proof cannot be $(0w)$. For all the other rules (except for the rule $(L \to)$), it is easy to see that since the conclusion of the rule is single-conclusion, then every premise must also be single-conclusion. It remains to investigate the case where the last rule used in the proof is $(L \to)$:

$$\frac{\overset{\pi_1}{\Upsilon \Rightarrow \phi, \Lambda} \qquad \overset{\pi_2}{\Pi, \psi, \Sigma \Rightarrow \Delta}}{\Pi, \phi \to \psi, \Upsilon, \Sigma \Rightarrow \Delta, \Lambda} (L \to)$$

There are two possibilities; either Λ is empty and Δ is equal to A

$$\frac{\overset{\pi_1}{\Upsilon \Rightarrow \phi} \qquad \overset{\pi_2}{\Pi, \psi, \Sigma \Rightarrow A}}{\Pi, \phi \to \psi, \Upsilon, \Sigma \Rightarrow A} (L \to)$$

or Δ is empty and Λ is equal to A

$$\frac{\overset{\pi_1}{\Upsilon \Rightarrow \phi, A} \qquad \overset{\pi_2}{\Pi, \psi, \Sigma \Rightarrow}}{\Pi, \phi \to \psi, \Upsilon, \Sigma \Rightarrow A} (L \to)$$

In the former since both premises are single-conclusion, by induction hypothesis, π_1 and π_2 are proofs in $\mathbf{FL_{ew}}$ and by applying the rule $(L \to)$ we obtain a proof for $\Gamma \Rightarrow A$. On the other hand, the latter cannot happen since the right premise is of the form $\Pi, \psi, \Sigma \Rightarrow$ and the antecedent of this sequent is 0-free. Therefore, Lemma 4.6.41 implies that it is not provable in $\mathbf{CFL_{ew}^-}$. □

Theorem 4.6.43. *The formulas*

$$\tilde{\Theta}_n^{\otimes} := [\bigotimes_{i,j}((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1))] \;\; \to$$

$$[\bigotimes_{i,l}((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1)) \to \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})] \vee [\bigotimes_{i,l}((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1)) \to \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'})].$$

are provable in $\mathbf{CFL_e^-}$*, where* $k = \lfloor \sqrt{n} \rfloor$*. Moreover, every* $\mathbf{CFL_e^-}$*-proof of* $\tilde{\Theta}_n^{\otimes}$ *contains at least* $2^{\Omega(n^{1/4})}$ *proof-lines and hence has length exponential in terms of the length of* $\tilde{\Theta}_n^{\otimes}$.

Proof. Since formulas $\Theta_n^{\otimes}$ are provable in **FL** 4.4.32 and therefore in $\mathbf{FL_{ew}}$, they are provable in $\mathbf{CFL_{ew}}$. However, since in $\mathbf{FL_{ew}}$ and $\mathbf{CFL_{ew}}$ the exchange rules are present, as stated in the preliminaries the connectives \ and / are substituted by $\to$. Therefore, the tautologies $\Theta_n^{\otimes}$ will have the more recognizable form $\tilde{\Theta}_n^{\otimes}$. Using the cut elimination theorem for $\mathbf{CFL_{ew}}$, formulas $\tilde{\Theta}_n^{\otimes}$ are also provable in $\mathbf{CFL_{ew}^-}$. By Theorem 4.6.42, since $\tilde{\Theta}_n^{\otimes}$ are 0-free any cut-free proof for these formulas in $\mathbf{CFL_{ew}^-}$ is also a proof in $\mathbf{FL_{ew}}$. However, Theorem 4.6.39 guaranties these proofs contain at least $2^{\Omega(n^{1/4})}$ proof-lines and hence the lengths of these proofs are exponential in terms of the length of $\tilde{\Theta}_n^{\otimes}$. □

Remark 4.6.44. *So far, we do not have any method to extend the lower bound to the calculus* $\mathbf{CFL_e}$*, where the cut rule is present. Note that since there are no non-trivial lower bounds for the sequent calculus* **LK***, we can not use a similar argument as that in the proof of Theorem 4.4.32.*

Corollary 4.6.45. *For any proof system* $\mathbf{P}$ *such that* $\mathbf{CFL_e^-} \leq \mathbf{P} \leq_p \mathbf{CFL_{ew}^-}$*, there is an exponential lower bound on the length of proofs in* $\mathbf{P}$*. As a result, there are exponential lower bounds on the length of proofs in sequent calculi* $\mathbf{CFL_e^-}$*,* $\mathbf{CFL_{ei}^-}$*, and* $\mathbf{CFL_{eo}^-}$*.*

Proof. It follows from Theorem 4.6.43. □

In the end, we will provide a result similar to Theorem 4.6.39 for the sequent calculus **BPC**.

Corollary 4.6.46. *We have* $\mathbf{BPC} \vdash \Rightarrow (\Theta_n^{\otimes})^t$ *and the number of lines of any proof of this sequent is exponential in* n*.*

Proof. Take the Frege system $\mathbf{P}$ for BPC wrt IPC. Using Theorem 4.3.19, since $\mathbf{BPC} \vdash \Rightarrow (\Theta_n^{\otimes})^t$ then $\mathbf{P} \top \to (\Theta_n^{\otimes})^t$. By the modus ponens rule we get $\mathbf{P}(\Theta_n^{\otimes})^t$. Using the second part of Theorem 4.5.37, we get the lower bound. □

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