"Proof Systems: A Study on Form and Complexity"
by Raheleh Jalali Keshavarz

Doctoral Dissertation Report

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Scope of the Dissertation

This dissertation forms a contribution to the field of structural proof theory, focussing on the existence of proof systems for non-classical logics with certain properties, including Craig interpolation, uniform interpolation, and bounds on proof length. This topic and the specific questions addressed in the dissertation are of interest to a broad community of researchers working in proof theory and non-classical logics. In particular, the question of which logics admit a certain kind of proof system is one that has been studied quite intensively in recent years, both from a proof-theoretic and algebraic perspective.

Apart from a brief introduction, the dissertation consists of three distinct papers, with no notable effort made to remove duplication of definitions or to coordinate terminology. The second chapter (which forms more than half the dissertation) is a joint paper with Amir Akbar Tabatabai currently under review for a journal, the third is a single-author paper that appeared in the (refereed) proceedings of WoLLIC 2019, and the fourth is a single-author work that is currently unpublished. In my reading of the dissertation, I found no mathematical errors in the definitions or proofs that would have a significant bearing on the correctness of the results. The presentation of the material is of a decent standard. Athough there are some grammatical errors and minor technical mistakes, these can easily and quickly be corrected. The dissertation follows a logical and coherent structure and the choice of problems tackled is well motivated. The required background for the presentation of the results is mostly provided in detail, although some references are missing (see below).

Scientific Contribution

As explained in the introduction, the goal of the dissertation is to show that if a logic has a proof system — specifically, a sequent calculus — satisfying a given set of conditions, then it must possess a particular meta-logical property. Since these properties may be quite rare, the results are often used contrapositively to show that many logics do not have a proof system of the described form. This strategy, followed explicitly by Iemhoff in the cited papers [22] and [33], provides an interesting and powerful counterpart to work by Ciabattoni, Galatos, and Terui (cited as [10]) which provides algebraic characterizations of substructural logics admitting a calculus of a certain form. In particular Iemhoff shows that Pitts' proof of uniform interpolation for intuitionistic logic lifts to other intermediate logics given the existence of a terminating sequent calculus that extends Dyckhoff's calculus for intuitionistic logic with "focused" axioms and rules, where formulas do not swap sides in a sequent between premises and conclusion. Only seven intermediate logics admit uniform interpolation, so at most seven of these logics can have such a terminating calculus.

Chapter 2 of the dissertation broadens the scope of Iemhoff's approach to commutative substructural logics, roughly characterized as extensions of the Full Lambek Calculus with exchange $\mathsf{FL_e}$ (possibly with well-behaved modal operators) or, algebraically, as logics given by varieties of commutative pointed residuated lattices. Iemhoff's notion of a "focused" rule is also generalized to allow "semi-analytic" rules where formulas can appear on both sides of a sequent in the premises and conclusion. Moreover, not only uniform interpolation, but also Craig interpolation is considered for these logics, leading to a range of positive results stating that a certain logic has some form of interpolation, and negative results stating that a certain logic does not have a sequent calculus of a certain form.

More precisely, it is shown first that a single-conclusion (multiple-conclusion) calculus extending $\mathsf{FL_e}$ (or its multiple-conclusion variant) and consisting of semi-analytic rules and focused axioms admits Craig interpolation. This result covers a number of important substructural logics that were known to have Craig interpolation (such as $\mathsf{FL_e}$ and $\mathsf{FL_ew}$). But also, using results in the literature on the failure of interpolation in semilinear and relevant logics, it is shown that broad families of logics cannot have a proof system of the given form. The result of Iemhoff for intermediate logics is also strengthened to show that all but seven of these logics cannot have a proof system with semi-analytic rules and focused axioms. A series of uniform interpolation results are then proved for substructural logics broadly following the adaptation of Pitts' proofs of uniform interpolation by Iemhoff in the cited papers [22] and [33], and encompassing results for these logics obtained by Alizadeh, Derakhshan, and Ono in the paper cited as [2]. In the multiple-conclusion setting these results are used to show that there cannot exist terminating proof systems of the given form for a range of modal logics. Taken as a whole, the results of Chapter 2 provide an interesting, novel, and quite powerful methodology for exhibiting the limitations of even quite a flexible sequent calculus framework.

Chapter 3 of the dissertation addresses the question as to whether a given intermediate logic can have an efficient proof system of a certain form. In particular, an exponential lower bound is obtained for proof systems consisting of certain classes of the focussed rules of Iemhoff; that is, it is shown that certain formulas must have exponentially long proofs in these systems. This result is obtained using feasible interpolation, reducing a problem in proof complexity to a problem in circuit complexity by extracting a Boolean circuit for an interpolant from a given proof for an implication, where the size of the circuit is polynomially bounded by the size of the proof. First it is shown that there are classical tautologies with exponential proof lengths in any proof system consisting only of "polarity preserving" focused rules and focused axioms that have polynomial proof length in the sequent calculus LK. Similarly, using the lower bound technique developed by Hrubeš in [20] and [21], it is shown that there are intuitionistic logic tautologies with exponential proof lengths in any proof system consisting only of "monotonicity preserving" focused rules and focused axioms that have polynomial proof length in the sequent calculus LK. I am not an expert in this area of proof theory and cannot judge if these results are particularly surprising, but it seems that the application of these methods to systems with focused rules is new and nicely compliments the methodology applied in Chapter 2.

Chapter 4 focuses on the proof complexity of proof systems for substructural logics, specifically a proof system extending either the Full Lambek Calculus or Visser's basic propositional logic that is polynomially simulated by an extended Frege system for some infinite-branching intermediate logic. A sequence of provable formulas with polynomial length is provided for these systems such that their shortest proofs are exponentially long, obtained by modifying a sequence of tautologies of intuitionistic logic, for which there is already known to exist an exponential lower bound on the length of their proofs in any extended Frege system for the infinite branching intermediate logic. Again, I am not an expert in this area of proof theory, but, as far as I know, these are the first results on proof complexity for substructural logics, and provide a new perspective on the development of proof systems for these logics.

Further Comments and Questions

- The author describes *universal proof theory* as a recent development in proof theory involving the study of the general behaviour of proof systems (existence of systems with certain properties, equivalence of systems, etc.). In my opinion such an approach is not at all new. It reaches back at least to the 1982 paper of Belnap on Display Logic; also the work of Guglielmi and others on Deep Inference from the early 2000s onwards also fits into this perspective.

- A general theorem is given that establishes Craig interpolation for a number of well-known substructural and modal logics such as $\mathsf{FL_e}$, $\mathsf{FL_{ew}}$, $\mathsf{K4}$, etc. (Corollary 2.4.27). However, most (perhaps all) of these results were already known and appropriate references should be given.

- The negative applications of the general uniform interpolation results presented in Corollaries 2.5.50, 2.5.51, and 2.5.52 are all for intermediate and modal logics. Can these results also be used to establish the failure of uniform interpolation for some substructural logics that admit Craig interpolation? Perhaps the very general results on failures of uniform deductive interpolation in the following paper could be used:

  T. Kowalski and G. Metcalfe. Uniform interpolation and coherence.
  Annals of Pure and Applied Logic 170(7) (2019), 825–841.

- Craig interpolation was recently proved for bi-intuitionistic logic using a calculus with an analytic cut rule in

  T. Kowalski and H. Ono. Analytic cut and interpolation for bi-intuitionistic logic.
  Review of Symbolic Logic 10(2) (2017), 259–283.

  Could the approach described in Chapter 2 be applied to proof systems that allow analytic applications of the cut rule?

- Although sequent calculi are the most suitable formalism, there has a been a lot of work recently on developing general methods for proving interpolation for modal and other logics using nested sequents, hypersequents, etc.; see, e.g.,

  R. Kuznets. Multicomponent Proof-theoretic Method for Proving Interpolation Property. *Annals of Pure and Applied Logic* 169(12) (2018), 1369–1418.

Could the methods described in Chapter 2 extend to these more general frameworks?

- The results of the dissertation suggest a close connection between the existence of proof systems of a certain form and (uniform) interpolation; is there some way to derive the existence of a proof system using the fact that a logic admits uniform interpolation?

- The first part of Chapter 2 is not very well-written, compared with the rest of the thesis; for example:

  1. The cut rule is mentioned and used several times without definition (this is important because cut rules can take different forms).
  2. There is some confusion about whether FL_e is a logic or a proof system.
  3. Some definitions are incorrectly formulated; e.g., in Definition 2.2.12, the second sentence, expressing the residuation property, should be part of the definition of a pointed commutative residuated lattice, not an additional definition.
  4. The symbol $\neg$ is used without a definition on page 15.
  5. On pages 15-16, different symbols are used for Łukasiewicz logic and Gödel logic.
  6. In Chapter 2, it is not explained what it means for a logic to be the logic of a class of algebras.
  7. As mentioned in the dissertation, FL_e is not formulated with the constants $\bot$ and $\top$; instead of writing FL_e^- for the usual calculus, it would be better to write $\mathsf{FL}_\mathsf{e}^\mathsf{b}$ (or something similar) for the extended version.

- The definitions of an extension and an axiomatic extension of a sequent calculus (Definition 2.2.7) using admissible rules is non-standard (usually, derivable rules are considered); does it follow that every axiomatic extension is an extension? Is a sequent calculus even an axiomatic extension of itself?

- The definition of a terminating calculus (Definition 2.2.3) appears to capture only a special notion of termination of a proof system; could there be a sequent calculus that is terminating in the usual sense, but for which no suitable well-founded order on sequents exists? In that case, it would be better to use different terminology.

- There are some issues in the bibliography with repetition of entries, capitalization of names, different spellings of the same name, etc.

Recommendation

In my opinion, the dissertation of Raheleh Jalali Keshavarz provides an original and distinct contribution to the study of proof systems, demonstrating a deep understanding of the subject matter and awareness of the relevant literature. I am therefore satisfied that the dissertation is of sufficient quality for the degree of PhD.

Prof. Dr. George Metcalfe, 29. September 2020