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Dear committee members:

Review of the habilitation thesis by Martin Balko

This is a review of the habilitation thesis written by Martin Balko. Balko is a prolific researcher, producing exciting results in Ramsey theory, discrete geometry, and graph theory. I am very familiar with his work as his area of research is very close to mine. Below, I will highlight some of his results from his thesis.

Ordered Ramsey numbers

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is that "Every structure of a given kind contains a large well-organized substructure." This is a very popular area within combinatorics, in which a great variety of techniques from many branches of mathematics is used. In the last 10 years or so, there has been a lot of exciting developments in *Ordered Ramsey theory* and Martin Balko is a leading expert in this area. A large portion of his thesis is dedicated to this area. Below, I describe some of these results in detail.

Given a vertex-ordered graph G, the ordered Ramsey number $R(G)$ is the minimum number N such that every ordered complete graph with N vertices and with edges colored by two colors contains a monochromatic copy of the G (with the correct ordering). A very fundamental problem in this area is to decide for which graphs G is the ordered Ramsey number $R(G)$ polynomial or superpolynomial in $|G|$. Moreover, how does the Ramsey number change when the ordering of G changes? One of the earliest results in this direction was obtained by Balko, Cibulka, Král, and Kynčl in their paper "Ramsey numbers of ordered graphs," where the authors showed that for certain ordered matchings, the ordered Ramsey number can differ substantially from the classical Ramsey number. In particular, they showed that there is an ordered matching M_n on n vertices for which the ordered Ramsey number $R(M_n)$ is superpolynomial in n, while it is well-known to be polynomial in the unordered setting. This implies that ordered Ramsey numbers of the same graph can grow superpolynomially in the size of the graph in one ordering and remain linear in another ordering (which was very surprising). Answering a question of Conlon, Fox, Lee, and Sudakov, Balko, Cibulka, Král, and Kynčl proved that the ordered Ramsey number $R(G)$ is polynomial in the number of vertices of G if the bandwidth of G is constant or if G is an ordered graph of constant degeneracy and constant interval chromatic number. For so-called monotone cycles, they compute their ordered Ramsey numbers exactly. This last result implies exact formulas for geometric Ramsey numbers of cycles, a problem introducd by Károlyi, Pach, Tóth, and Valtr. Their paper appeared in *The Electronic Journal of Combinatorics*.

The paper ”On ordered Ramsey numbers of bounded-degree graphs” of Balko, with Jelínek and Valtr, shows that for every integer $d \geq 3$, almost every d-regular graph G satisfies $R(G) \geq n^{3/2-1/d-o(1)}$ for every ordering of G. In particular, there are 3-regular graphs G on n vertices for which the numbers $R(G)$ are superlinear in n, regardless of the ordering of G. This is a very important and significant result that answers a question of Conlon, Fox, Lee, and Sudakov. On the other hand, they proved that every graph G on n vertices with maximum degree 2 admits an ordering G of G such that $R(G)$ is linear in n. Their paper appeared in *Journal of Combinatorial Theory, Series B*.

Ordered Ramsey theory is an exciting area of research, and as stated in the thesis, there are many interesting open problems in this area. In my opinion, Martin Balko is the leading expert in this area. I look forward to seeing new results and applications from him in this area.

Counting Holes

In 1935, Erdős and Szekeres proved that every sufficiently large finite point set in the plane in general position contains a convex n-gon. Shortly after their seminal result, Harborth showed that the analogous statement for ”holes” is false, that is, he constructed arbitrarily large finite point sets in the plane with no 7-hole (7 points in convex position with no other point from P inside). For $k \leq 6$, the existence of a k-hole was known for quite some time, but counting them seems extremely hard. For $k \leq 4$, it was known that the number of k-holes in an n-element planar point set in general position is $\Theta(n^2)$, but for 5 and 6 holes, it was somewhere between linear and quadratic. For decades, there was no progress on estimating the number of 5-holes in planar point sets, until Balko and his coauthors (O. Aichholzer, T. Hackl, J. Kynčl, I. Parada, M. Scheucher, P. Valtr, and B. Vogtenhuber) established a breakthrough by showing that every n-element planar point set in the plane in general position contains a super linear number of 5-holes. The proof combines a computed assisted case analysis along with a divide and conquer approach. Their paper appeared in the *Journal of Combinatorial Theory Series A.*

Martin Balko has published many *original* papers in high-ranking mathematical journals as well as computer science conferences. Any coincidences detected by the Turnitin system is due to the fact that Dr. Balko’s dissertation is a collection these papers and his commentary. I give you my strongest support for this promotion.

With kind regards,

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