Singular integral operators with rough kernels

Habilitation thesis

Petr Honzík

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University

Chapter 1

Structure of the thesis

1.1 List of included papers

This thesis is based on the following list of papers:

[A] Grafakos, L., Honzík, P., and Ryabogin, D. *On the p-independence boundedness property of Calderón-Zygmund theory*, J. Reine Angew. Math. **602** (2007) 227–234.
`https://doi.org/10.1515/CRELLE.2007.008`

[B] Honzík, P. *On p dependent boundedness of singular integral operators*, Math. Z. **267** (2011) 931–937.
`https://doi.org/10.1007/s00209-009-0654-0`

[C] Honzík, P. *An example of an unbounded maximal singular operator*, J. Geom. Anal. **20**, (2010) 153–167.
`https://doi.org/10.1007/s12220-009-9096-5`

[D] Grafakos, L. and Honzík P. *A weak-type estimate for commutators*, Int. Math. Res. Not. **20** (2012) 4785–4796.
`https://doi.org/10.1093/imrn/rnr193`

[E] Honzík, P. *An endpoint estimate for rough maximal singular integrals* Int. Math. Res. Not. **20** (2020) 6120–6134.

https://doi.org/10.1093/imrn/rny189

[F] Grafakos, L. He, D., and Honzík, P. *Rough bilinear singular integrals* Adv. Math. **326** (2018) 54–78.
https://doi.org/10.1016/j.aim.2017.12.013

[G] Buriánková, E. and Honzík, P. *Rough maximal bilinear singular integrals* Collect. Math. **70** (2019), no. 3, 431–446.
https://doi.org/10.1007/s13348-019-00239-4

1.2 Brief overview

The papers included fall in two groups. Papers [A]–[E] deal with some fine properties of singular integral operators with rough kernel. In [A] and [B], we study the dependence of the boundedness of the rough operator on the space L^p on the properties of the kernel and the index p. In [C] and [E], we study the maximal singular operator with rough kernel. In [C], we give an example of a kernel such that the operator itself is bounded, but its maximal version is not. It is an open question if the maximal operator with rough kernel is of the weak type 1-1. In [E], we provide some weaker endpoint estimate. In [D] we provide a weak type estimate in $\mathbb{R}^2$ for an operator related to commutators of the singular operators. Papers [F]–[G] are about bilinear operators. These papers use a new method developed by the author for proving boundedness of a bilinear multiplier operators using wavelet decomposition, and aply this method to the bilinear singular integral operators with rough kernel in [F] and the bilinear maximal singular integral operators with rough kernel in [G].

Chapter 2

Introduction

We give a brief introduction of the theory of singular integral operators, then we present the results that the author and his collaborators obtained in the field and we also discuss the open questions and the problems remaining.

The singular integral of convolution type is an operator T expressed in the form

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} K(y) f(x-y) dy.$$

Here K is a non-integrable kernel which satisfies suitable set of conditions, and f is a smooth integrable function.

2.1 Hilbert transform

The simplest and most important example of a singular integral operator is the Hilbert transform on $\mathbb{R}$,

$$Hf(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus (-\varepsilon,\varepsilon)} \frac{1}{\pi y} f(x-y) dy. \tag{2.1}$$

We recall quickly the theory of the Hilbert transform, as it is the basis for the theory of singular integrals. Hilbert transform is defined for C^1 integrable function, it is easy to see that the limit then exists for all x real.

The Hilbert transform is one of the principal operators in complex analysis, if $u + iv$ is an analytic function in the upper half-plane, and f is an integrable trace of u on the real axis, then Hf is, up to a constant, the trace of v. This is easily verified in the case that f is also C^1, but much deeper theory is needed in the case when f is not smooth. The situation is still fairly simple if $f \in L^2$. Using the Fourier transform, we arrive for $f \in C^1$ at the formula

$$Hf = \mathcal{F}^{-1}(-i\operatorname{sgn}(\xi)\hat{f}(\xi)).$$

It is then clear from the Plancherel theorem that H is a bounded linear operator on dense subset of L^2 and it may be extended to the entire space. Also, one may show that if f is a trace of u in the sense of L^2 limit

$$f = \lim_{t \to 0^+} u(\cdot, t),$$

then Hf is also a trace of v in L^2 sense.

These results may be extended for L^p with $1 < p < \infty$, but fail in the case $p = 1$. Instead of L^1 boundedness, it is possible to prove a weak type $1-1$ estimate

$$|\{|Hf| \geq \lambda\}| \leq C\frac{\|f\|_1}{\lambda}.$$

Another problem is the existence of the limit in the formula (2.1). The standard method to address this is to define the maximal Hilbert transform

$$H^*f(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{\pi y} f(x-y)dy\right|. \tag{2.2}$$

There is a general theorem stating that if a maximal operator related to a singular integral operator is of the weak type $p-p$, then for a function $f \in L^p$ the limit exists almost everywhere. In the case of the Hilbert tranform, the maximal version is of the weak type $p-p$ for $1 \leq p < \infty$.

2.2 General singular integrals

For a general singular integral of convolution type T with kernel K, and its maximal version

$$T^*f(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} K(y)f(x-y)dy\right|.$$

we also have the following six natural questions:

- Is T bounded on L^2?
- Is T bounded on L^p, where $1 < p \neq 2 < \infty$?

- Is T of the weak type $1-1$?
- Is T^* bounded on L^2?
- Is T^* of the weak type $p-p$, where $1 < p \neq 2 < \infty$?
- Is T^* of the weak type $1-1$?

All these question are have a positive answer in the case when K is the smooth homogeneous kernel

$$K(x) = \Omega(x/|x|)/|x|^n,$$

where Ω is C^1 on the sphere $\mathbb{S}^{n-1}$. (The smoothness may be even relaxed to only a Dini type condition.) The results for T of this type were proved by Calderón and Zygmund in [4] and the stopping-time argument they used became known as the Calderón-Zygmund decomposition. It has been one of the key methods in the Fourier analysis since.

This thesis is devoted to studying these questions in the case when Ω is non-smooth.

Chapter 3

Description of our results

3.1 L^p estimates for Rough Singular Integrals

The singular integral operator with homogeneous kernel is the operator defined by

$$T_\Omega f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy. \tag{3.1}$$

Ω is an integrable function on the sphere S^{n-1} with a vanishing integral and f is a smooth integrable function. It is easy to check that the integral is well defined, there are, however, some open questions related to its boundedness on L^p spaces.

Let us first explain the utility of this operator. The Riesz transform,

$$R_j f(x) = \lim_{\varepsilon\to 0} C_n \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{y_j}{|y|^{n+1}} f(x-y)dy,$$

where $C_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$, is the most natural example. Riesz transform, for a Schwartz function f, is a composition of a j-th partial derivative with Riesz potential or

$$R_j f = \partial_j(-\Delta)^{-1/2} f = \mathcal{F}^{-1}\left(-\frac{i\xi_j}{|\xi|}\widehat{f}(\xi)\right).$$

Therefore it is a key operator in the theory of partial differential equations, Sobolev spaces, etc. The function Ω in this case is a linear polynomial.

Similar operators of the form $\partial_\alpha(-\Delta)^{-|\alpha|/2}$, where α is a multiindex, also have singular integral representation with polynomial Ω.

There are many methods for proving the L^p boundedness of this basic operator. The situation is fully resolved on L^2. As the operator is of the form of convolution with a distribution, it may be written as a Fourier multiplier. There is an explicit formula for the symbol of the multiplier (see [15] Proposition 5.2.3)

$$m(\xi) = \int_{S^{n-1}} \Omega(\theta) \left(-\log|\xi\cdot\theta| - \frac{i\pi}{2}\mathrm{sgn}(\xi\cdot\theta) \right) d\theta. \tag{3.2}$$

Therefore for given Ω the operator is bounded if and only if $m \in L^\infty$.

For general $1 < p < \infty$, the situation is more complicated and some questions are still open. Calderón and Zygmund [5] used method of rotations to prove boundedness of the operator under the condition that $\Omega \in L\log L$. To briefly summarize this method, let us remind that it uses the one dimensional result on the boundedness of the Hilbert transform. If $\Omega \in L^1$ is an odd function, the operator generated by it can be written as an integral average of directional Hilbert transforms, and thus it has the same bound on L^p as the Hilbert transform itself. In particular, the constant C_p in the bound

$$\|T_\Omega f\| \le C_p \|f\|_p$$

behaves like $1/(p-1)$ when p approaches 1. This is an optimal result. On the other hand, in case Ω is odd, the idea is to use the Riesz transforms to symmetrize it. The formal identity $\sum_j R_j R_j T_\Omega = T_\Omega$ is combined with the observation that for $\Omega \in L\log L$ the operator $R_j T_\Omega$ can be written as an operator T_{Ω_j} with Ω integrable and odd. (See [15] Section 5.2)

This proof leads to a constant which behaves as $1/(p-1)^2$, which is not optimal. Moreover, there are some even functions Ω which are not in $L\log L$, but the operator is still bounded. The first result of this type was proved by Grafakos and Stefanov in [20] and later improved by Fan, Guo and Pan [14] Let us describe their results. Inspired by the formula (3.2), Grafakos and Stefanov introduced quantity

$$m_\alpha(\Omega)(\xi) = \int_{S^{n-1}} |\Omega|(\theta) \left(\log|\xi\cdot\theta|\right)^{1+\alpha} d\theta, \tag{3.3}$$

and proved that if m_α is bounded on S^{n-1}, then the operator T_Ω is bounded on L^p in the range $|1/p - 1/2| < \alpha/(4+2\alpha)$. This range was improved by Fan, Guo and Pan to $|1/p-1/2| < \alpha/(2+2\alpha)$.

Or contribution to this area is to provide examples to show that the boundedness of the operator T_Ω may be indeed p dependent. In the article [A], we proved the following

Theorem 1. *For every α satisfying $0 \leq \alpha < 1$ there is an even integrable function Ω on S^{n-1} with mean value zero with $m_\alpha(\Omega) \in L^\infty(S^{n-1})$ such that the operator T_Ω is unbounded on L^p whenever*

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{1}{1+\alpha}.$$

In particular, there is a function Ω such that T_Ω is L^p bounded exactly when $p = 2$.

We note that while this example is sharp when $\alpha = 0$ and $p = 2$ for other α there is a gap between the positive and negative result. In effort to close the gap, we proved another theorem in [B]:

Theorem 2. *For every $\alpha > 0$ there is an even integrable function Ω on S^{n-1} with mean value zero with $m_\alpha(\Omega) \in L^\infty(S^{n-1})$ such that the operator T_Ω is unbounded on L^p whenever*

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{3\alpha+1}{6(1+\alpha)}.$$

This theorem does not close the gap completely, but it gives better results for p close to 1.

3.2 Weak type $1-1$ estimates

In general, weak type $1-1$ estimate for an operator T is an estimate

$$|\{|Tf| > \lambda\}| \leq C\|f\|_1/\lambda$$

for all $f \in L^1$ and $\lambda > 0$. While singular integral operators are never bounded on L^1, they often satisfy this weaker estimate. If the operator T is also bounded on some space L^p, $p > 1$, then from the Marcinkiewicz interpolation theorem one gets the bound on all spaces L^q with $1 < q < p$, with a constant that behaves as $1/(q-1)$ when q goes to 1.

The weak space $L^{p,\infty}$ is defined by its quasinorm

$$\|f\|_{p,\infty} = \inf\left\{C > 0 : |\{|f| > \lambda\}| \leq \frac{C^p}{\lambda^p} \quad \text{for all} \quad \lambda > 0\right\}.$$

In this notation the weak type estimate becomes $\|Tf\|_{1,\infty} \leq C\|f\|_1$. Let us note that while the space $L^{p,\infty}$, $p > 1$ may be renormed to become a Banach space, it is not the case with $L^{1,\infty}$. This means that the method of rotations cannot be used to prove the weak type $1-1$ estimates for singular integrals.

Weak type $1-1$ estimates for singular integral operators in $\mathbb{R}^n$ were first proved by Calderón and Zygmund [4]. The method they used involves splitting the function f into a part g, which is in L^2 and a part b, which is in L^1 and has some strong cancellation properties and is supported by a family of disjoint dyadic cubes with measure controlled by $\|f\|_1$.

In order to use the cancellation of b, the singular kernel needs to have some minimal smoothness. Therefore, while it was possible to use the Calderón and Zygmund method to prove the boundedness of the operator T_Ω when the function Ω satisfies some type of Hölder smoothness or a Dini condition, it is no longer possible if the Ω is only in L^∞ or some bigger space. It has been an open question for quite some time if the weak type estimate may be proved in the non-smooth case.

The question was first solved by Christ and Rubio de Francia in [7] and also Hofmann [21] for operators on $\mathbb{R}^2$. The method they use is, roughly speaking, based on estimating the L^2 norm of the operator T_Ω away from the support of the function b. This allows the use of some L^2 methods specific to the convolution operators. In higher dimension, the question was solved by Seeger in [25]. In particular, he showed that if $\Omega \in L\log L(\mathbb{S}^{n-1})$, the T_Ω is of the weak type $1-1$. This result was generalized by Tao in [28] to more general underlying spaces.

Our results form [B] show that there is an operator T_Ω bounded on all L^p, $1<p<\infty$, which is not of the weak type $1-1$.

The method of Christ and Rubio de Francia works even for more complicated operators, which are not of the convolution type. In the article [D], we studied the commutator operators introduced by Christ and Journé in [6]. Let us describe the operators briefly.

Suppose that Ω is $C^1(S^{n-1})$ and

$$K(x) = \Omega(x/|x|)/|x|^n,$$

The operator T_Ω is the bounded on L^p $1<p<\infty$ and of the weak type $1-1$. Chris and Journé introduced operator with kernel

$$L_a(x,y) = K(x-y)\int_0^1 a((1-t)x+ty)dt,$$

where a is a function from L^∞. The operator is then defined for smooth integrable functions f as

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} L(x,y)f(y)dy.$$

This operator is called commutator operator of Christ and Journé, which is a bit of misnomer, because the operator is not directly an algebraic commutator, however, it does play very important role in the study of commutator operators.

Christ and Journé proved that this operator is bounded on L^p for $1 < p < \infty$, see [6]. While the kernel K is smooth, the new kernel L is in general not smooth, but it does have some smoothness properties, in particular, for fixed x and y the function $\tau(t) = L(x, ty)$ is smooth for $t > 0$. We observed that this smoothness enables the use of the L^2 methods from [7] and in [D] we proved the following theorem:

Theorem 3. *The Christ and Journé commutator T on $\mathbb{R}^2$ is of the weak type $1-1$.*

This result was later extended to all dimensions by Seeger [26] and generalized for a wider class of operators by Ding and Lai in [12].

3.3 Estimates for Maximal Rough Singular Integrals

The singular integral operator is initially defined on smooth functions, and if it is proved to be apriori bounded on some function space such as L^p into a complete space, such as L^p or $L^{p,\infty}$, it may be then extended to the entire space using the density. Therefore,if the operator T_Ω is apriori bounded on smooth L^p functions, we get a method to assign a value $T_\Omega f$ for a function $f \in L^p$ even if the function f is not smooth or even continuous. It is, however, an interesting question if the limit in the formula (3.1) converges to the same function almost everywhere.

This problem is solved using a maximal operator. We define the maximal singular operator

$$T^*_\Omega f(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)dy \right|. \tag{3.4}$$

Now, it is a well known theorem that if for $1 \le p < \infty$ the operator $T^*_\Omega f$ is bounded on L^p, or even if it is only bounded from L^p to the weak space $L^{p,\infty}$, then the limit in the formula (3.1) converges almost everywhere. (See Theorem 2.1.14 of [15].)

The supremum in the definition of T^*_Ω means that the operator is well defined pointwise for any locally integrable f. It is also bigger than T_Ω in the points, where T_Ω is defined. In general, T^*_Ω is not directly controlled pointwise or in norm by T_Ω, however, many important cases are covered by the Cotlar inequality. For example, if the function Ω is smooth, then

$$T^*_\Omega(f)(x) \leq C(M(T_\Omega(f))(x) + M(f)(x)),$$

where M is the Hardy-Littlewood maximal function and f is a Schwartz functions. This was first obtained by Cotlar [10] for Hilbert transform, and later extended to many other settings. (See also Theorem 5.3.4 of [15].)

In case of the non-smooth Ω, the Cotlar inequality is no longer valid. Also, while for $p > 1$ it follows from the Cotlar inequality that if T_Ω is L^p bounded, then T^*_Ω is L^p bounded, this line of reasoning cannot be used for the weak type $1-1$ estimate. For a function f in L^1, the $T_\Omega f$ is only in $L^{1,\infty}$, and therefore $M(T_\Omega f)$ may not be even well defined.

For operators with smooth kernel, such as T^*_Ω when Ω is smooth, it is possible to use the Calderón-Zygmund method to obtain the weak type $1-1$ estimate in very similar way it is used for the operator T_Ω itself. This is no longer true for a rough function Ω. Interestingly, the new take on the Calderón-Zygmund method, used by Christ and Rubio de Francia in [7] and Seeger [25] to prove weak type $1-1$ estimate for rough T_Ω, does not work for the maximal version T^*_Ω, even if $\Omega \in L^\infty$. It is a famous open question if the operator is of the weak type $1-1$ under this condition. Also it is open question if the limit in the definition of T_Ω converges almost everywhere for $f \in L^1$.

Let us now describe our contribution to this area. In [E], we proved the following:

Theorem 4. *Let $\Omega \in L^\infty(S^{n-1})$ with mean value 0. Then the operator T^*_Ω is bounded from $L(\log\log L)^{2+\varepsilon}(B(0,1))$ to $L^{1,\infty}$ for any $\varepsilon > 0$.*

This theorem replaces the space L^1 in the weak type estimate by the slightly smaller Orlicz space $L(\log\log L)^{2+\varepsilon}$. Because of scaling considerations, such result may only be local, and therefore we restrict the space to the unit ball. An obvious corollary to this result is that for $f \in L(\log\log L)^{2+\varepsilon}(\mathbb{R}^n)$ the limit in (3.1) converges almost everywhere, since the maximal operator may be easily localized.

The difference in the boundedness of T_Ω and T^*_Ω is rather subtle. While some examples are known for more general kernels, the only example where T_Ω is bounded on some space and T^*_Ω is not is due to us in [C]. We proved:

Theorem 5. *There is $\Omega \in L^1(S^1)$ with mean value 0 such that the operator T^*_Ω is not bounded on L^2, while the operator T_Ω is bounded on L^2.*

3.4 Estimates for Bilinear Rough Singular Integrals

Let us have a tempered distribution ω on $\mathbb{R}^{2n}$. We define a convolution operator $T_\omega = \omega * \psi$ for a Schwartz function ψ on $\mathbb{R}^{2n}$. Related *bilinear* operator is defined for φ_1, φ_2 from Schwartz space on $\mathbb{R}^n$ as $T_\omega(\varphi_1, \varphi_2)(x) = T_\omega(\varphi_1 \times \varphi_2)(x,x)$, where $\varphi_1 \times \varphi_2$ is tensor product of functions and (x,x) is the member of $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, x_1, \ldots, x_n)$.

Since the bilinear operators of this type are not as well known as the linear convolution operators, we will make here few basic observations from the theory. First, in linear case, the simplest convolution operator is the identity, represented by the convolution with the Dirac measure. In the bilinear case, the same role is played by the product operator. Thus, in linear case $T_\delta(f)(x) = (f * \delta)(x) = f(x)$, in the bilinear case we have $T_\delta(f,g)(x) = f(x)g(x)$. In a similar fashion, if the tempered distribution ω on $\mathbb{R}^{2n}$ is a tensor product of two tempered distributions ω_1, ω_2 on $\mathbb{R}^n$, we see that $T_\omega(f,g) = T_{\omega_1}(f)T_{\omega_2}(g)$.

A convolution operator may be expressed as a Fourier multiplier. We have

$$T_\omega(f) = \mathcal{F}^{-1}(\widehat{\omega}\widehat{f}).$$

The function $m = \widehat{\omega}$ is then called the symbol of the multiplier. In similar fashion for a bilinear operator, we have bilinear multiplier

$$T_\omega(f,g)(x) = \mathcal{F}^{-1}(\widehat{\omega}(\widehat{f} \times \widehat{g}))(x,x).$$

We can write it in this form, since Fourier transform of a tensor product is a tensor product of Fourier transforms.

Bilinear singular integral is then defined as the bilinear p.v. convolution with a singular kernel K. In the included papers, we have the kernel of the rough type

$$K(y) = \frac{\Omega(y/|y|)}{|y|^{2n}},$$

with $\Omega \in L^p(\mathcal{S}^{2n-1})$ for suitable $p > 1$ and with mean value 0. We denote such operator T_Ω.

The study of bilinear operators of this type goes back to Calderón, who introduced the bilinear Hilbert transform as a tool to study certain commutator operators in [2], [3]. Coifmann and Meyer proved the boundedness of the bilinear singular integral with smooth kernel, such as T_Ω, where Ω is a smooth, from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for $1 < p_1, p_2 < \infty$, $1/q = 1/p_1 + 1/p_2$ and $q > 1$ in [8], [9]. This result was later extended by Grafakos and Torres in [19] and also independently by Kenig and Stein in [22] to $q > 1/2$.

Bilinear Hilbert transform is much more complex operator than the Hilbert transform itself. The operator is defined for f, g in the Schwartz space on $\mathbb{R}$ and unit vector $(\alpha, \beta) \in S^1$ as

$$H_{\alpha,\beta}(f,g)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} f(x-\alpha t) g(x-\beta t) \frac{dt}{t}.$$

This operator was shown to be bounded form $L^{p_1} \times L^{p_2} \to L^q$ for $1/q = 1/p_2 + 1/p_2$ with $q > 2/3$ for $\alpha \neq 0 \neq \beta$ and $\alpha \neq \beta$ by Lacey and Thiele in [23], [24].

The bilinear Hilbert transform is generated by the directional Hilbert transform on $\mathbb{R}^2$ in the direction (α, β). It is an open problem if a similar operator generated by directional Hilbert transform on $\mathbb{R}^{2n}$ is bounded for any p_1, p_2. We note that even the proof in one dimension is extremely difficult.

In order to prove boundedness of the bilinear singular integral with rough kernel, several strategies may be employed. First, just as in the linear case, it is possible to use the method of rotations. This relies on the uniform boundedness of the directional bilinear Hilbert transform, proved by Grafakos and Li [18] and therefore it may be only used in dimesion one. The author and some coauthors summarized this approach in [11]. As this paper contains mostly simple observations, we do not include it in this thesis.

The method of rotations fails in higher dimensions, because the estimates for bilinear Hilbert transforms are not available. Therefore, we decided for the approach used in Duoandikoetxea and Rubio de Francia [13]. To briefly summarize this method, the operator T_Ω is written as a series of singular integral operators T_j, where the operators T_j have increasingly rough kernels, but the L^2 norm of the operators decreases as $2^{-\varepsilon j}$. Duoandikoetxea and Rubio de Francia then use a bootstrapping argument to extend this to all $1 < p < \infty$. Alternatively, the weak type estimate for the operators T_j increases as j and it is possible to use interpolation and duality to show that T_j form convergent series for all $1 < p < \infty$.

The critical point is the L^2 estimate for the operator T_j. In the linear case it is obtained simply by noting that the operator T_j may be written as a Fourier multiplier with symbol m_j and $\|m_j\|_\infty \leq 2^{-\varepsilon j}$. There is no direct analog of this in the bilinear case. It is possible to show that there is a bounded symbol m such that for any combination of p_1, p_2 the related bilinear Fourier multiplier operator is not bounded.

Or efforts therefore concentrated on improving the understanding of the bilinear Fourier multipliers. We achieved a breakthrough by using a wavelet decomposition of the multiplier symbol. This is the central idea of the papers [F] and [G].

First, we obtained the following theorem for bilinear singular integrals with rough kernel in [F]:

Theorem 6. *For all $n \geq 1$, if $\Omega \in L^2(\mathbb{S}^{2n-1})$, then for T_Ω, we have*

$$\|T_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty.$$

Using the usual decompositions of the kernel and interpolating with known results for smooth kernels, we also obtained the following:

Theorem 7. *For all $n \geq 1$, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, then for T_Ω, we have*

$$\|T_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty$$

whenever $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.

A maximal version of the bilinear theorem was proved in [G]. The maximal singular operator is defined in the same way as in the linear case, by replacing the principal value by supremum. We again denote such operator T^*_Ω. The main result of [G] is:

Theorem 8. *For all $n \geq 1$, if $\Omega \in L^2(\mathbb{S}^{2n-1})$, then for T^*_Ω, we have*

$$\|T^*_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty.$$

These result were since improved and extended several times. For example, Grafakos, He, and Slavíková extended the result for Ω with weaker integrability, see [17]. Also, a multilinear version of the theorem is currently in preparation. The wavelet argument was applied to other bilinear operators, such as bilinear Hörmander multipliers, see [16] or bilinear spherical maximal function[1].

Bibliography

[1] J. A. Barrionuevo, L. Grafakos, D. He, P. Honzík, and L. Oliveira, *Bilinear spherical maximal function*, Math. Res. Lett. **25** (2018), no. 5, 1369–1388.

[2] A. P. Calderón. *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1092–1099.

[3] C. Calderón. *On commutators of singular integrals*, Studia Math. **53** (1975), 139–174.

[4] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[5] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[6] M. Christ and J. Journé, *Polynomial growth estimates for multilinear singular integral operators* Acta Math. **159** (1987), no. 1-2, 51–80.

[7] M. Christ and J.-L. Rubio de Francia, *Weak type* $(1,1)$ *bounds for rough operators II,* Invent. Math. **93** (1988), 225–237.

[8] R. R. Coifman, Y. Meyer, *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[9] R. R. Coifman, Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[10] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Rev. Mat. Cuyana **1** (1955), 105–167.

[11] G. Diestel, L. Grafakos, P. Honzik, Z. Si, and E. Terwilleger, *Method of rotations for bilinear singular integrals*. Communications in Mathematical Analysis, Conference **3** (2011), 99–107.

[12] Y. Ding and X. Lai, *Weak type $(1,1)$ bound criterion for singular integrals with rough kernel and its applications*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 1649–1675.

[13] J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Inv. Math. **84** (1986), 541–561.

[14] D. Fan, K. Guo, and Y. Pan, *A note of a rough singular integral operator*, (English summary) Math. Inequal. Appl. **2** (1999), no. 1, 73–81.

[15] L. Grafakos, *Classical Fourier analysis*, Third edition, Graduate Texts in Mathematics, **249**, Springer, New York, 2014.

[16] L. Grafakos, D. He, and P. Honzík *The Hörmander multiplier theorem, II: The bilinear local L^2 case* Math. Z. **289** (2018), no. 3-4, 875–887.

[17] L. Grafakos, D. He, and L. Slavíková, $L^2 \times L^2 \to L^1$ *boundedness criteria* Math. Ann. **376** (2020), no. 1–2, 431–455.

[18] L. Grafakos and X. Li, *Uniform bounds for the bilinear Hilbert transforms I*, Ann. of Math. (Ser. 2) **159** (2004), 889–933.

[19] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[20] L. Grafakos and A. Stefanov, L^p *bounds for singular integrals and maximal singular integrals*, Indiana Univ. Math. J., **47** (1998), 455–469.

[21] S. Hofmann, *Weak type $(1,1)$ boundedness of singular integrals with nonsmooth kernels,* Proc. Amer. Math. Soc. **103** (1988), 260–264.

[22] C. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.

[23] M. T. Lacey and C. M. Thiele, L^p *bounds for the bilinear Hilbert transform,* $2 < p < \infty$, Ann. Math. **146** (1997), 693–724.

[24] M. T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. Math. **149** (1999), 475–496.

[25] A. Seeger, *Singular integral operators with rough convolution kernels*, Jour. Amer. Math. Soc., **9** (1996), 95–105.

[26] A. Seeger, *A weak type bound for a singular integral*, Rev. Mat. Iberoam. **30** (2014), no. 3, 961–978.

[27] A. Stefanov, *Weak type estimates for certain Calderón-Zygmund singular integral operators*, Studia Math. **147** (2001), 1–13.

[28] T. Tao, *The weak type (1,1) of* $L\log L$ *homogeneous convolution operators*, Indiana Univ. Math. J. **48** (1999), 1547–1584.