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Optimal control of Lévy-driven stochastic equations in Hilbert spaces

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Abstract: Controlled linear stochastic evolution equations driven by Lévy processes are studied in the Hilbert space setting. The control operator may be unbounded which makes the results obtained in the abstract setting applicable to parabolic SPDEs with boundary or point control. The first part contains some preliminary technical results, notably a version of Itô formula which is applicable to weak/mild solutions of controlled equations. In the second part, the ergodic control problem is solved: The feedback form of the optimal control and the formula for the optimal cost are found. The control problem is solved in the mean-value sense and, under selective conditions, in the pathwise sense. As examples, various parabolic type controlled SPDEs are studied.

Keywords: Optimal control Stochastic evolution equations Diffusion processes Lévy processes Ergodic control

Contents

Introduction

Systems with boundary or point control are an important class of controlled distributed parameter systems. They represent, for instance, plate equations with distributed noise which are accessible (controllable) only at certain points in the domain or in terms of boundary conditions, for example

$$w_{tt}(t, x) - \Delta w_t(t, x) + \Delta^2 w(t, x) = \mathbb{I}_{x=x_0} u(t) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \quad (1)$$

$$w(0, x) = w_0, \quad w_t(0, x) = w_1, \quad x \in G,$$

$$w(t, x) = w_t(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial G,$$

where $G \subset \mathbb{R}^n$, $x_0 \in G$, l formally stands for the noise and u for the control. Another example is heat equation

$$y_t(t, x) = \Delta y(t, x) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \quad (2)$$

$$y_v(t, x) + h(x)y(t, x) = u(t, x), \quad (t, x) \in \mathbb{R}_+ \times \partial G,$$

$$y(0, x) = y_0(x), \quad x \in G,$$

where $y_v(t, x)$ is normal derivative of y in $(t, x) \in \mathbb{R}_+ \times \partial G$ in direction v, v being the outward normal to ∂G, $G \subset \mathbb{R}^n$, $x_0 \in G$, l denotes the noise. Ergodic (and adaptive) control for such systems have been studied in numerous papers in case when l is white in time, possibly correlated in space, Gaussian noise. The first results in this direction have been obtained in [5], the main idea of which already appeared in [11] where the (technically easier) distributed control of linear distributed parameter systems has been studied. Adaptive control for such systems dependent also on an unknown parameter was further investigated in [7]. Let us also mention analogous results for the case of semilinear (typically, stochastic reaction-diffusion) systems [12], [8], [14] and [13]. In all these papers the noise term is a (space-dependent) Brownian motion. There are a few papers where other types of noise are considered, like the fractional noise type disturbances in [10] and [9]. All previous control results are proved path-wise sense and some of them are proved in mean-value sense as well.

We are not aware of any earlier result for Lévy noise for this problem. Therefore, the goal of this Thesis is to study the ergodic control of the diffusion processes with the Lévy noise. As these results in the case of the Brownian motion are already known, we focus on the pure jump Lévy noise.

The original results summarized in this Thesis are published in the papers [17], where the square integrable Lévy process is considered, and in [18], where we can find the result for more general cylindrical Lévy process.

The Thesis is divided into five chapters. The original result with proofs are covered by the chapters $2 - 5$.

Chapter 1 contains some preliminaries as

1. strongly continuous and analytic semi-groups, some useful properties of these semi-groups,

2. square integrable Lévy process and its characteristics,

3. square integrable martingales, their basic properties and the stochastic integration with respect to the square integrable martingales,

4. cylindrical Lévy process, its characteristics and stochastic integration with respect to the cylindrical Lévy process.

Chapter 2 is devoted to controlled stochastic evolution equations in the general form, the concepts of the solutions and the hypotheses imposed on the coefficients as well as on the process representing the noise in the controlled stochastic evolution equations.

In chapter 3 the control problem is formulated, including the optimality criterion as well as the assumptions on the corresponding coefficients. Some known results on operator-valued Riccati equations are recalled.

Chapter 4 is devoted to the Itô formula in mean value, applicable to the stochastic evolution equations defined in the chapter 2 and for the quadratic forms with the operators defined in the chapter 3 (Lemma 8). Note that the solutions are not strong in general - the driving Lévy process is merely cylindrical and both the drift term and the control operator are in general unbounded and only densely defined. When we add the assumption on the noise term to be Hilbert-Schmidt, we have the path-wise version of this Itô formula (Lemma 11) as well.

Chapter 5 contains the main results of the paper - the optimal control is found in feedback form in a certain natural class of stabilizing controls, and the formula for the optimal cost is given in terms of the solution to appropriate operator-valued "algebraic" Riccati equation. The control is optimal in the mean for the cylindrical Lévy process (Theorem 12) as we apply the Itô formula in the mean value sense, proved in the previous section in this case. This control is optimal path-wise as well (Theorem 16) if we add some technical assumptions on the diffusion operator as well as on the coefficients of the cost functional. Most importantly, we need that the coefficient in the noise term is Hilbert-Schmidt, to be able to apply the path-wise Itô formula in this particular case. Some applications to ergodic adaptive control problem are summarized in section 5.3 (Theorem 23). The examples of specific controlled SPDEs satisfying the above imposed conditions are given in this chapter as well (section 5.4). These examples are given:

1. boundary control for stochastic heat equation,

2. ergodic point control of stochastic plate equation with structural damping,

3. ergodic point control of stochastic Kelvin-Voigt plate equation.

1. Preliminaries

1.1 Strongly continuous semigroups

In this section, we introduce the theory of the strongly continuous semigroups as well as we summarize basic results related to these semigroups based on [24]. Proofs of the results stated in this section can be found in [24].

Let $\mathbb{H} = (\mathbb{H}, |\cdot|_{\mathbb{H}})$ and $\mathbb{Y} = (\mathbb{Y}, |\cdot|_{\mathbb{Y}})$ be real separable Hilbert spaces. We define the semigroup of bounded linear operators on $\mathbb{H}$ as the family $S(t)$, $t \geq 0$ such that

1. $S(t) \in \mathcal{L}(\mathbb{H})$, $s \geq 0$,

2. $S(0) = \mathbb{I}_{\mathbb{H}}$,

3. $S(s + t) = S(s)S(t)$, $s, t \geq 0$,

where $\mathcal{L}(\mathbb{H})$ denotes the set of bounded linear operators on $\mathbb{H}$ and $\mathbb{I}_{\mathbb{H}}$ denotes the identity operator on $\mathbb{H}$. If there is no danger of confusion, we simply write $\mathbb{I}$ instead of $\mathbb{I}_{\mathbb{H}}$. Note that the condition 3 above is called semigroup property. The semigroup S can be characterized by the infinitesimal generator of S. We say that the operator A is the infinitesimal generator of the semigroup S if

$$Ax = \lim_{t \to 0_+} \frac{(S(t) - S(0))x}{t} = \lim_{t \to 0_+} \frac{S(t)x - x}{t}$$

for all $x \in \mathcal{D}(A)$, where $\mathcal{D}(A)$ denotes the domain of A and in this case

$$\mathcal{D}(A) = \left\{ x \in \mathbb{H}; \exists y \in \mathbb{H} : \lim_{t \to 0_+} \frac{S(t)x - x}{t} = y \right\}.$$

We say that the semigroup S of bounded linear operators on $\mathbb{H}$ is strongly continuous if

$$\lim_{t \to 0_+} S(t)x = x, \quad x \in \mathbb{H}.$$

We state some key properties of the strongly continuous semigroup S with the infinitesimal generator A.

1. There exist $M \geq 1$ and $\omega \geq 0$ such that

$$|S(t)|_{\mathbb{H}} \leq Me^{\omega t}, \quad t \geq 0.$$

2. $S(\cdot)x : \mathbb{R}_+ \to \mathbb{H}$ is continuous, $x \in \mathbb{H}$.

3. For $x \in \mathcal{D}(A)$, $t > s \geq 0$:

$$\int_s^t S(r)x dr \in \mathcal{D}(A),$$

$$A\left(\int_s^t S(r)x dr \right) = S(t)x - S(s)x.$$

4. For $x \in \mathcal{D}(A)$, $t \geq 0$:

$$S(t)x \in \mathcal{D}(A),$$
$$AS(t)x = S(t)Ax.$$

5. A is the closed linear operator and $\overline{\mathcal{D}(A)} = \mathbb{H}$.

6. S is given uniquely by A. It means that if T is a strongly continuous semigroup with the infinitesimal generator A, then $S(t) = T(t)$, $t \geq 0$.

Remark. We can see from the properties above that $S(t)x$ can be interpreted as the solution of the deterministic evolution equation

$$\dot{y} = Ay, \quad y(0) = x \tag{1.1}$$

if $x \in \mathcal{D}(A)$. We show an example of a strongly continuous semigroup on the finite dimensional $\mathbb{H} = \mathbb{R}$ as a motivation of (1.1).

Example. Consider the equation (1.1) with A on $\mathbb{H} = \mathbb{R}$. In this case, the linear operator A can be interpreted as multiplication by $a \in \mathbb{R}$. We can rewrite the equation (1.1) as

$$\dot{y} = ay, \quad y(0) = x, \tag{1.2}$$

where $x \in \mathbb{R}$. It is well known that the solution to the equation 1.2 is the exponential function

$$y(t) = e^{at}x, \quad t \geq 0.$$

Therefore, the strongly continuous semigroup generated by the operator A can be interpreted as the semigroup of linear operators $S(t)$, $t \geq 0$, defined as the multiplication by the numbers e^{at}, $t \geq 0$. This means that

$$S(t)x = e^{at}x, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.3}$$

Remark. Another motivation is the equation (1.1) with the bounded operator A. In this case, the solution is similarly intuitive as in the finite dimensional case (1.2) as we can see in the example below.

Example. Assume the equation (1.1) with $A \in \mathcal{L}(\mathbb{H})$. It is known that:

1. A generates an uniformly continuous semigroup, that means

$$\lim_{t \to 0_+} S(t) = S(0)$$

in $\mathcal{L}(\mathbb{H})$. Note that this property is much stronger than the property of strongly continuous semigroups.

2.

$$S(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad t \geq 0. \tag{1.4}$$

Remark. The sum in (1.4) is defined only for the bounded operator A in general Hilbert spaces. Therefore, to get the correct definition of the more general concept of the evolution equations in the Hilbert spaces, the concept of uniformly continuous semigroups is extended to strongly continuous semigroups.

We define the resolvent set $\rho(A)$ of the operator A as the set of all $\lambda \in \mathbb{C}$ such that

$$(\lambda \mathbb{I}_{\mathbb{H}} - A)^{-1} \in \mathcal{L}(\mathbb{H})$$

and denote

$$R(\lambda, A) = \lambda(\lambda \mathbb{I} - A)^{-1}, \quad \lambda \in \rho(A).$$

The set $\{R(\lambda, A); \ \lambda \in \rho(A)\}$ is called the resolvent of A.

We say that S is a semigroup of contractions iff $M_0 = 1$ in (1.1). This means that there exists $\omega \geq 0$ such that

$$|S(t)|_{\mathbb{H}} \leq e^{\omega t}, \quad t \geq 0.$$

We use later the following properties of the resolvent set and the resolvent of A generating a semigroup of contractions:

1. $\mathbb{R}_+ \subset \rho(A)$,

2. For $\lambda > 0$:

$$|\lambda R(\lambda, A)|_{\mathcal{L}(\mathbb{H})} \leq 1. \tag{1.5}$$

3. For $x \in \mathcal{D}(A)$:

$$\lim_{t \to \infty} \lambda R(\lambda, A)x = x. \tag{1.6}$$

The inequality (1.5) is implied by the Hille-Yosida Theorem:

Theorem 1. *Let A be a linear operator on $\mathbb{H}$. A generates a strongly continuous semigroup of contractions S on $\mathbb{H}$ if and only*

1. A is closed and $\mathcal{D}(\overline{\mathbb{H}}) = \mathbb{H}$,

2. $\mathbb{R}_+ \subset \rho(A)$ and for all $\lambda > 0$:

$$|R(\lambda, A)|_{\mathcal{L}(\mathbb{H})} \leq \frac{1}{\lambda}. \tag{1.7}$$

For us is important the implication that for A a generator of a strongly continupus semigroup of contractions (1.7) holds.

In the definition of the general strongly continuous semigroup, we require the semigroup property only on $\mathbb{R}_+$. We introduce the analytic semigroups as the family of the bounded linear operators with the semigroup property on larger subset of $\mathbb{C}$, a cone around the $\mathbb{R}_+$. More precisely, S is an analytic semigroup on

$$\mathfrak{A}_{\phi_1, \phi_2} = \{z \in \mathbb{C}; \ \arg(z) \in (-\phi_1, \phi_2)\}, \quad \phi_1, \phi_2 > 0,$$

if

1. $S(\cdot)$ is analytic on $\mathfrak{A}_{\phi_1, \phi_2}$,

2. $S(0) = \mathbb{I}$,

3. for each $x \in \mathbb{H}$

$$\lim_{z \to 0, z \in \mathfrak{A}_{\phi_1,\phi_2}} S(z)x = x,$$

4. $S(z_1 + z_2) = S(z_1)S(z_2)$, $z_1, z_2 \in \mathfrak{A}_{\phi_1,\phi_2}$.

The semigroup S is analytic if S is analytic on $\mathfrak{A}_{\phi_1,\phi_2}$ for some $\phi_1, \phi_2 > 0$. The analytic semigroups have some key properties listed below:

It is known that there is a constant $M_0 > 0$ such that for all $t \geq 0$ we have

$$|AS(t)|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_0}{t}.$$

Before further description of the properties of the analytic semigroups, we define the fractional power of the operators. Let A be a densely defined closed linear operator and there exist a neighborhood of zero V and $M > 0$ such that

$$\rho_0 \subset \rho(A),$$

$$|\lambda R(\lambda, A)|_{\mathcal{L}(\mathbb{H})} \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in \rho_0,$$

where

$$\rho_0 = \{\lambda; \ 0 < \omega < |\arg \lambda| \leq \pi\} \cup V.$$

If $M = 1$ and $\omega = \frac{\pi}{2}$, A is the infinitesimal generator of a strongly continuous semigroup. If $\omega < \frac{\pi}{2}$, A is the infinitesimal generator of an analytic semigroup. We further assume $\omega < \frac{\pi}{2}$.

We can define for $\alpha > 0$:

$$(-A)^{-\alpha} = \frac{\int_0^\infty r^{\alpha-1} S(r) dr}{\Gamma(\alpha)}$$

as the integral converges in the uniform operator topology. Γ denotes the Gamma function. Further we define for $\alpha = 0$: $(-A)^{-0} = \mathbb{I}_{\mathbb{H}}$, $(-A)^0 = \mathbb{I}_{\mathbb{H}}$ and for $\alpha > 0$:

$$(-A)^\alpha = ((-A)^{-\alpha})^{-1}.$$

It is known that

1.
$$\mathcal{D}\left((-A)^\alpha\right) = \mathbf{R}\left((-A)^{-\alpha}\right),$$

where $\mathbf{R}\left((-A)^{-\alpha}\right)$ denotes the range of the operator $(-A)^{-\alpha}$,

2.
$$A^{\alpha+\beta} x = A^\alpha A^\beta x, \quad x \in \mathcal{D}\left(A^{\max\{\alpha+\beta,\alpha,\beta\}}\right)$$

for $\alpha, \beta \in \mathbb{R}$.

We continue with the list of properties of the analytic semigroup S with the infinitesimal generator A.

1. There exists a constant c such that

$$\left|(-A)^{-\alpha}\right|_{\mathcal{L}(\mathbb{H})} \leq c, \quad \alpha \in [0, 1].$$

2. If $0 < \alpha \leq 1$ and $B : \mathbb{H} \to \mathbb{H}$ is a closed linear operator such that $\mathcal{D}((-A)^\alpha) \subset \mathcal{D}(B)$, then there is a constant c such that

$$|Bx|_{\mathbb{H}} \leq c|(-A)^\alpha x|_{\mathbb{H}}, \quad x \in \mathcal{D}(A^\alpha).$$

3. $S(t)$ maps $\mathbb{H}$ to $\mathcal{D}((-A)^\alpha)$, $t > 0$, $\alpha \geq 0$.

4. For $x \in \mathcal{D}(A^\alpha)$ and $\alpha \in \mathbb{R}$, we have

$$(-A)^\alpha S(t)x = S(t)(-A)^\alpha x, \quad t \geq 0.$$

5. For $t > 0$ and $\alpha \in \mathbb{R}$, the operator $(-A)^\alpha S(t)$ is bounded on $\mathbb{H}$.

6. For $\alpha \in \mathbb{R}$, we have constants $\beta > 0$ and M_α such that

$$|(-A)^\alpha S(t)|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_\alpha e^{-\beta t}}{t^\alpha}, \quad t > 0.$$

1.2   Square integrable Lévy process

In this chapter, we summarize some properties of the square integrable Lévy process based on [25]. We can find the corresponding proofs in [25].

Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ be a complete filtered probability space. We start with the definition of the $\mathbb{H}$-valued martingale. We say, that the $\mathbb{H}$-valued stochastic process M on I, $I \subset \mathbb{R}$, is $\mathbb{H}$-valued martingale with respect to the filtration $\mathcal{F}$ if

1. M is $\mathcal{F}$-adapted,

2. $M(t)$ is integrable for each $t \in I$,

3. for all $t \geq s$, $s, t \in I$:

$$\mathbf{E}[M(t)|\mathcal{F}_s] = M(s), , \quad \mathbf{P} - a.s.,$$

where $\mathbf{P}$-a.s means almost surely with respect to the probability measure $\mathbf{P}$ (further, we can write a.s. instead of $\mathbf{P}$-a.s.). If there is no danger of confusion, we simply write martingale instead of $\mathbb{H}$-valued martingale. This definition is a straight generalization of the analogous one in finite dimension. Therefore, the properties are similar as in the finite dimension.

1. If we have a $\mathbb{H}$-valued integrable process Z on I, $I \subset \mathbb{R}$, such that $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$, $t \geq s$, $s, t \in I$, then the process M defined as

$$M(t) = Z(t) - \mathbb{E}Z(t), \quad t \in I,$$

is a martingale.

2. Any stochastically continuous square integrable $\mathbb{H}$-valued martingale M has a version with càdlàg trajectories. We denote this version M as well.

Remark. Note that the naming càdlàg is taken from French "continue à droite, limite à gauche" which we can translate as "right continuous with left limits" or as "continuous on (the) right, limit on (the) left". Therefore, the namings "RCLL" ("right continuous with left limits") or sometimes "corlol" ("continuous on (the) right, limit on (the) left") are used in some papers and books as well. Nevertheless, the most popular is the French naming "càdlàg". Therefore, we use it in this Thesis.

We define the Lévy process L in $\mathbb{H}$ as the $\mathbb{H}$-valued stochastic process L on $\mathbb{R}_+$ such that

1. The increments $L(t_i) - L(t_{i-1})$, $i = 1, \ldots, n$, are independent for each $0 \leq t_{i-1} < t_i$, $i = 1, \ldots, n$, $n \in \mathbb{N}$,

2. The increments $L(t) - L(s)$, $0 \leq s < t$, are stationary,

3. L is stochastically continuous.

Example. An important particular case of the Lévy process is the Wiener process. We define the Wiener process W in $\mathbb{H}$ as the Lévy process in $\mathbb{H}$ with continuous trajectories and zero mean. It is known that W is Gaussian and square integrable, which means that there exists a positive semi-definite trace-class operator Q_W on $\mathbb{H}$ such that for all $t_i \geq 0$, $x_i \in \mathbb{H}$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$:

$$(\langle W(t_1), x_1 \rangle_{\mathbb{H}}, \ldots, \langle W(t_n), x_n \rangle_{\mathbb{H}}) \sim \mathbf{N}(0_{\mathbb{R}^n}, Q^n),$$

where Q^n is an $n \times n$ matrix with the cells $Q_{i,j}$ given by

$$Q_{i,j} = \min\{t_i, t_j\} \langle Q x_i, x_j \rangle_{\mathbb{H}}, \quad i, j = 1, \ldots, n$$

and $\langle \cdot \rangle_{\mathbb{H}}$ denotes the scalar product in $\mathbb{H}$.

The general Lévy process L has a version with càdlàg trajectories. We consider this version. Denote $\Delta L(t) = L(t) - L(t_-)$. The Poisson random measure corresponding to L is defined as

$$N(t, A) = \#\{s \leq t; \Delta L(s) \in A\}, \quad t \geq 0, \quad A \in \mathcal{B}(\mathbb{H}), \quad \bar{A} \cap 0_{\mathbb{H}} = \emptyset,$$

where $\mathcal{B}(\mathbb{H})$ denotes the Borel σ-algebra of the space $\mathbb{H}$. Since the trajectories are càdlàg, $N(t, A)$ is finite for each $t > 0$ and $A \in \mathcal{B}(\mathbb{H})$, $\bar{A} \cap 0_{\mathbb{H}} = \emptyset$. It is known that

$$\mathbf{E}N(t, A) = t\mathbf{E}N(1, A) = t\nu(A), \quad t \geq 0, \quad A \in \mathcal{B}(\mathbb{H}), \quad \bar{A} \cap 0_{\mathbb{H}} = \emptyset$$

where ν is finite for $A \in \mathcal{B}(\mathbb{H})$, $\bar{A} \cap 0 = \emptyset$. The measure ν is called the Lévy measure or the intensity jump measure of the Lévy process L and satisfies

$$\int_{\mathbb{H}} \left(\min \left\{ |z|_{\mathbb{H}}^2, 1 \right\} \right) d\nu(z) < \infty. \tag{1.8}$$

The compensated Poisson random measure corresponding to L is defined as the compensated measure $N(t, A)$, that is

$$\tilde{N}(t, A) = N(t, A) - t\nu(A), \quad t \geq 0, \quad A \in \mathcal{B}(\mathbb{H}), \quad \bar{A} \cap 0_{\mathbb{H}} = \emptyset.$$

The characteristic function $\phi_{L(t)}$ of the Lévy process L at time $t \geq 0$ is given by

$$\log \phi_{L(t)}(x) = t(i\langle a, x\rangle_{\mathbb{H}} - \frac{1}{2}\langle Q_W x, x\rangle_{\mathbb{H}}$$

$$+ \int_{\mathbb{H}} \left(e^{i\langle z,x\rangle_{\mathbb{H}}} - 1 - i\langle z, x\rangle_{\mathbb{H}} \mathbf{I}_{|x|_{\mathbb{H}}<1}(\langle z, x\rangle_{\mathbb{H}})\right) d\nu(z),$$

where $\mathbf{I}_A(x)$ is defined as 1 if $x \in A$ else 0, log denotes the natural logarithm and

1. $a \in \mathbb{H}$,

2. Q_W is a symmetric positive semi-definite trace-class operator on $\mathbb{H}$ and

3. ν is a non-negative measure concentrated on $\mathbb{H} \setminus \{0_{\mathbb{H}}\}$ such that (1.8) is fulfilled.

The measure ν is the Lévy measure of L defined above and the triple (a, Q_W, ν) is called the characteristics of L. The formula describing the characteristic function of the Lévy process L is known as the Lévy-Khinchin formula. Note that in the Lévy-Khinchin formula, we can use any open bounded set containing $0_{\mathbb{H}}$ instead of the open sphere $|x|_{\mathbb{H}} < 1$ if we modify the vector a accordingly.

Remark. If the integral

$$\int_{|z|_{\mathbb{H}}<1} (e^{i\langle z,x\rangle_{\mathbb{H}}} - 1) d\nu(z)$$

exists, we can use $\emptyset$ instead of $|x|_{\mathbb{H}} < 1$ in the Lévy-Khinchin formula and adjust $\langle a, x\rangle_{\mathbb{H}}$ by

$$- \int_{|z|_{\mathbb{H}}<1} \langle z, x\rangle_{\mathbb{H}} d\nu(z).$$

In this case, we can rewrite the $\log \phi_{L(t)}(x)$ formula as:

$$\log \phi_{L(t)}(x) = t(i\langle \tilde{a}, x\rangle_{\mathbb{H}} - \frac{1}{2}\langle Q_W x, x\rangle_{\mathbb{H}} + \int_{\mathbb{H}} (e^{i\langle z,x\rangle_{\mathbb{H}}} - 1) d\nu(z),$$

where

$$\langle \tilde{a}, x\rangle_{\mathbb{H}} = \langle a, x\rangle_{\mathbb{H}} - \int_{|z|_{\mathbb{H}}<1} \langle z, x\rangle_{\mathbb{H}} d\nu(z).$$

On the other hand, if the integral

$$\int_{|z|_{\mathbb{H}}\geq1} \langle z, x\rangle_{\mathbb{H}} d\nu(z)$$

exists, we can use $\mathbb{R}$ instead of $|x|_{\mathbb{H}} < 1$ in the Lévy-Khinchin formula and adjust $\langle a, x\rangle_{\mathbb{H}}$ by

$$\int_{|z|_{\mathbb{H}}\geq1} \langle z, x\rangle_{\mathbb{H}} d\nu(z).$$

In this case, we can simplify the $\log \phi_{L(t)}(x)$ formula to:

$$\log \phi_{L(t)}(x) = t(i\langle \tilde{a}, x\rangle_{\mathbb{H}} - \frac{1}{2}\langle Q_W x, x\rangle_{\mathbb{H}} + \int_{\mathbb{H}} \left(e^{i\langle z,x\rangle_{\mathbb{H}}} - 1 - i\langle z, x\rangle_{\mathbb{H}}\right) d\nu(z),$$

where

$$\langle \tilde{a}, x\rangle_{\mathbb{H}} = \langle a, x\rangle_{\mathbb{H}} + \int_{|z|_{\mathbb{H}}\geq1} \langle z, x\rangle_{\mathbb{H}} d\nu(z).$$

The Lévy process can be decomposed in distribution ([25], chapters 4.5, 4.6, 6.3) as

$$L(t) = at + W(t) + \int_0^t \int_{|z|_{\mathbb{H}} \geq 1} z N(ds, dz) + \int_0^t \int_{|z|_{\mathbb{H}} < 1} z \tilde{N}(ds, dz), \ \ t \geq 0,$$

where

1. W is a Wiener process in $\mathbb{H}$ with the covariance operator Q_W,

2. N is a Poisson measure with the intensity measure $dt \times d\nu(x)$,

3. L_0 given by
$$L_0(t) = at, \ \ t \geq 0,$$
has the characteristic function
$$\log \phi_{L_0(t)}(x) = it\langle a, x \rangle_{\mathbb{H}}, \ \ x \in \mathbb{H},$$

4. L_1 given by
$$L_1(t) = W(t), \ \ t \geq 0,$$
has the characteristic function
$$\log \phi_{L_1(t)}(x) = -t\frac{1}{2}\langle Q_W x, x \rangle_{\mathbb{H}}, \ \ x \in \mathbb{H},$$
and is a martingale,

5. L_2 given by
$$L_2(t) = \int_0^t \int_{|z|_{\mathbb{H}} \geq 1} z N(ds, dz), \ \ t \geq 0,$$
has the characteristic function
$$\log \phi_{L_2(t)}(x) = \int_{|x|_{\mathbb{H}} \geq 1} \left(e^{i\langle z, x \rangle_{\mathbb{H}}} - 1 \right) d\nu(z), \ \ x \in \mathbb{H},$$

6. L_3 given by
$$L_3(t) = \int_0^t \int_{|z|_{\mathbb{H}} < 1} z \tilde{N}(ds, dz), \ \ t \geq 0,$$
has the characteristic function
$$\log \phi_{L_3(t)}(x) = \int_{|x|_{\mathbb{H}} < 1} \left(e^{i\langle z, x \rangle_{\mathbb{H}}} - 1 - i\langle z, x \rangle_{\mathbb{H}} \right) d\nu(z), \ \ x \in \mathbb{H},$$
and is a martingale.

Note that multiplying the characteristic functions of L_1, L_2 and L_3, we easily obtain the characteristic function described by the Lévy-Khinchin formula.

We consider square integrable Lévy process. We know that we have a vector $\mu \in \mathbb{H}$ and a positive semi-definite operator $\mathcal{Q}$ on $\mathbb{H}$ such that for $s, t \geq 0$, $x, y \in \mathbb{H}$:

1.
$$\mathbf{E}\langle L(t), x \rangle_{\mathbb{H}} = \langle \mu, x \rangle_{\mathbb{H}} t,$$

2.

$$\mathbf{E}\langle L(t) - \mu t, x\rangle_{\mathbb{H}}\langle L(s) - \mu s, y\rangle_{\mathbb{H}} = \min\{s, t\}\langle \mathcal{Q}x, y\rangle_{\mathbb{H}}.$$

We define the mean and the covariance operator of L as μ and $\mathcal{Q}$ with the properties defined above. It is known that

1.

$$\mu = \left(a + \int_{|z|_{\mathbb{H}} \geq 1} z \, d\nu(z)\right) t,$$

2.

$$\int_{\mathbb{H}} |z|_{\mathbb{H}}^2 \, d\nu(z) < \infty$$

and we can decompose the operator $\mathcal{Q}$ to $\mathcal{Q} = Q_W + Q_J$, where

$$\langle Q_J x, y\rangle_{\mathbb{H}} = \int_{\mathbb{H}} \langle x, z\rangle_{\mathbb{H}}\langle y, z\rangle_{\mathbb{H}} \, d\nu(z).$$

We define the stochastic integral with respect to the general $\mathbb{H}$-valued square integrable martingale M as in [25].

Remark. As mentioned in the previous paragraphs, the square integrable $\mathbb{H}$-valued Lévy process in the form

$$L(t) = W(t) + \int_0^t \int_{\mathbb{H}} z \tilde{N}(ds, dz), \quad t \geq 0,$$

is a $\mathbb{H}$-valued square integrable martingale.

We consider the version of M with the càdlàg trajectories. We denote $\langle M\rangle$ the compensator of $\langle M\rangle_{\mathbb{H}}^2$. It means that

$$\langle M(t)\rangle_{\mathbb{H}}^2 - \langle M\rangle_t, \quad t \geq 0,$$

is a martingale. Such process $\langle M\rangle$ exists and is unique in the class of predictable processes with locally bounded variation. There exists so called martingale covariance Q that is a nuclear positive semi-definite operator process and for all $x, y \in \mathbb{H}$, $s, t, u, v \in \mathbb{R}_+$, $s \leq t \leq u \leq v$ we have

$$\mathbf{E}[\langle M(t) - M(s), x\rangle_{\mathbb{H}}\langle M(t) - M(s), y\rangle_{\mathbb{H}}|\mathcal{F}_s] = \mathbf{E}\left[\int_s^t \langle Q_r x, y\rangle_{\mathbb{H}} d\langle M\rangle_r|\mathcal{F}_s\right] \quad (1.9)$$

and

$$\mathbf{E}[\langle M(t) - M(s), x\rangle_{\mathbb{H}}\langle M(v) - M(u), y\rangle_{\mathbb{H}}|\mathcal{F}_u] = 0. \quad (1.10)$$

Based on the equations (1.9) and (1.10), we get an Itô isometry for simple processes and we can follow the standard steps of the construction of the stochastic integral.

We start with the simple Hilbert-Schmidt stochastic process Φ on $[0, T]$, $T > 0$, as

$$\Phi(t) = \sum_{i=1}^n \mathbf{I}_{A_i} \mathbf{I}_{t_{i-1} \leq t \leq t_i} \Phi_i, \quad (1.11)$$

where $T \geq t_i > t_{i-1} \geq 0$, $A_i \in \mathcal{F}_{t_{i-1}}$, $\sigma(\Phi_i) \subset \mathcal{F}_{t_{i-1}}$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. The family of processes in the form (1.11) will be denoted as $\mathcal{S}$. We can define the stochastic integral for $\Phi \in \mathcal{S}$,

$$I_M(\Phi)(T) = \sum_{i=1}^{n} \mathbf{I}_{A_i} \Phi_i (M(t_i) - M(t_{i-1})). \tag{1.12}$$

Applying (1.9) and (1.10), we obtain the isometry

$$\mathbf{E}|I_M(\Phi)(T)|_{\mathbb{H}}^2 = \mathbf{E} \int_0^T \left| \Phi(t) Q_t^{\frac{1}{2}} \right|_{HS}^2 d\langle M \rangle_t, . \tag{1.13}$$

where $|\cdot|_{HS}$ denotes the Hilbert-Schmidt norm.

For the stochastic process of Hilbert-Schmidt operators Y on $[0, T]$, we denote the norm:

$$|Y|_{\mathcal{H}}^2 = \mathbf{E} \int_0^T \left| Y(s) Q_s^{\frac{1}{2}} \right|_{HS}^2 d\langle M \rangle_s. \tag{1.14}$$

Define $\mathcal{H}$ as the space of all $\mathcal{F}$-predictable stochastic process of Hilbert-Schmidt operators Y on $[0, T]$ such that $|\cdot|_{\mathcal{H}} < \infty$. We know by (1.13) that for $\Phi \in \mathcal{S}$, $|\Phi|_{\mathcal{H}} < \infty$. $I_M(\cdot)(T)$ defined on $\mathcal{S}$ by (1.12) can be uniquely extended on $\mathcal{H}$. By this extension, we obtain the stochastic integral for all stochastic process of Hilbert-Schmidt operators $Y \in \mathcal{H}$.

We have for all $Y \in \mathcal{H}$:

1. $I_M(Y)(\cdot)$ is the square integrable martingale which is mean-square continuous and has a càdlàg version.

2. For a bounded operator C on $\mathbb{H}$, $CY \in \mathcal{H}$ and $CI_M(Y) = I_M(CY)$.

1.3 Cylindrical Lévy process

Define the cylindrical subsets of $\mathbb{H}$ as the sets of the form

$$C(\{h_1, \ldots, h_n\}; B) = \{h \in \mathbb{H} : \ (\langle h, h_1 \rangle_{\mathbb{H}}, \ldots, \langle h, h_n \rangle_{\mathbb{H}}) \in B\},$$

where $\{h_1, \ldots, h_n\} \subset \mathbb{H}$, $B \in \mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$. For arbitrary $V \subset \mathbb{H}$,

$$C(V) = \{C(V, B); \ \ B \in \mathcal{B}(\mathbb{R}^n)\}$$

and denote $\mathcal{C}(V) = \sigma(C(V))$. A function $\nu : \mathcal{C}(\mathbb{H}) \to \mathbb{R}$ is a cylindrical measure on $\mathcal{C}(\mathbb{H})$ if it is a measure on $\mathcal{C}(K)$ for each K finite subset of $\mathbb{H}$. Let K be a finite subset of $\mathbb{H}$ and let $f : (\mathbb{H}, \mathcal{C}(K)) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$. The integral

$$\int_{\mathbb{H}} f(z) d\nu(z)$$

is well defined as a complex valued Lebesgue integral if it exists (ν is a measure on $\mathcal{C}(K)$). For more details cf. [26], p. 4.

We define a cylindrical random variable Z on $(\Omega, \mathcal{A}, \mathbf{P})$ in $\mathbb{H}$ as a linear and continuous map $\mathbb{H} \to \mathbb{L}^0(\Omega, \mathcal{A}, \mathbf{P})$, where $\mathbb{L}^0(\Omega, \mathcal{A})$ (or simply $\mathbb{L}^0$) is a set of real

valued random variables on $(\Omega, \mathcal{A})$. Set $Z(u) = \langle Z, u \rangle$. A cylindrical process Z on $(\Omega, \mathcal{A}, \mathbf{P})$ in $\mathbb{H}$ is a family $(Z(t), t \geq 0)$ of cylindrical random variables on $(\Omega, \mathcal{A}, \mathbf{P})$ in $\mathbb{H}$. The characteristic function of cylindrical random variable Z is defined as

$$\phi_Z(h) = \mathbb{E}e^{i\langle Z, h \rangle_{\mathbb{H}}}, \quad h \in \mathbb{H}.$$

A cylindrical process L on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ in $\mathbb{H}$ is a cylindrical Lévy process if the stochastic process

$$((\langle L(t), h_1 \rangle, \ldots, \langle L(t), h_n \rangle), \quad t \geq 0)$$

is a Lévy process on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ in $\mathbb{R}^n$ for each $\{h_1, \ldots, h_n\} \subset \mathbb{H}$, $n \in \mathbb{N}$. We have

$$\log \phi_{L(t)}(h) = t \left(ip(h) - \frac{1}{2} q(h) \right)$$

$$+ \int_{\mathbb{H}} (e^{i\langle z, h \rangle_{\mathbb{H}}} - 1 - i\langle z, h \rangle_{\mathbb{H}} \mathbf{I}_{B_{\mathbb{R}}}(\langle z, h \rangle_{\mathbb{H}})) d\nu(z),$$

where

1. $p : \mathbb{H} \to \mathbb{R}$ is a continuous mapping,

2. $q : \mathbb{H} \to \mathbb{R}$ is a quadratic form and

3. ν is a cylindrical measure on $\mathcal{C}(\mathbb{H})$ such that $\nu \circ \langle h, \cdot \rangle_{\mathbb{H}}^{-1}$ is the Lévy measure on $\mathcal{B}(\mathbb{R})$ of the scalar Lévy process $\langle L, h \rangle = (\langle L(t), h \rangle, t \geq 0)$ for each $h \in \mathbb{H}$.

The cylindrical measure ν is called cylindrical Lévy measure of L and (p, q, ν) are called cylindrical characteristics of L.

The cylindrical process Z has weak second moments if

$$\langle Z, h \rangle = (\langle Z(t), h \rangle, \quad t \geq 0)$$

has finite second moments for all $h \in \mathbb{H}$. The cylindrical process Z is a cylindrical martingale if $\langle Z, h \rangle$ is a martingale for all $h \in \mathbb{H}$.

For each Hilbert-Schmidt operator ϕ on $\mathbb{H}$ and each $t \geq 0$ exists a square integrable random variable L_ϕ such that for all $h \in \mathbb{H}$

$$\langle L(t), \phi^* h \rangle = \langle L_\phi(t), h \rangle_{\mathbb{H}},$$

where ϕ^* denotes the adjoint operator of the operator ϕ .

For simplicity we write $\phi L = L_\phi$.

For each $T : \mathbb{R}_+ \to \mathcal{L}(\mathbb{H})$ and $t \geq 0$ we have that

$$\int_0^t T(s)\phi dL(s) = \int_0^t T(s) d\phi L(s), \tag{1.15}$$

where the first integral is defined as in [26] and the second integral is defined as in [25]. This can be shown by the standard approximation approach as both integrals are defined as limits in L^2 of the same approximating random sequence.

We define the stochastic integral with respect to the cylindrical Lévy process with weak second moments following the steps in [26].

For each $0 \le s \le t$ and Hilbert-Schmidt operator ϕ on $\mathbb{H}$, there exists a square integrable random variable Z_ϕ on $\mathbb{H}$ such that for all $h \in \mathbb{H}$:

$$\langle L(t) - L(s), \phi^* h \rangle = \langle Z_\phi, h \rangle. \tag{1.16}$$

We define the simple random Hilbert-Schmidt operator Φ on $\mathbb{H}$ as

$$\Phi(\omega) = \sum_{i=1}^{n} \mathbf{I}_{A_i}(\omega) \phi_i, \quad \omega \in \Omega, \tag{1.17}$$

where $A_i \in \mathcal{A}$, $i \in \{1, \ldots, n\}$, are disjoint sets, ϕ_i are Hilbert-Schmidt operators on $\mathbb{H}$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. Using (1.16), we can define the square integrable random variable $I_{s,t}(\Phi)$ on $\mathbb{H}$ for Φ in the form (1.17) as

$$I_{s,t}(\Phi) = \sum_{i=1}^{n} \mathbf{I}_{A_i} Z_{\phi_i} \tag{1.18}$$

and combining (1.18) with (1.16), we define

$$\langle L(t) - L(s), \Phi^* h \rangle = \langle I_{s,t}(\Phi), h \rangle,$$

where

$$\langle I_{s,t}(\Phi), h \rangle = \sum_{i=1}^{n} \mathbf{I}_{A_i} \langle Z_{\phi_i}, h \rangle = \sum_{i=1}^{n} \mathbf{I}_{A_i} \langle L(t) - L(s), \phi_i^* h \rangle,$$

for all $h \in \mathbb{H}$.

Note that $I_{s,t}$ is a linear operator from the space of the simple random Hilbert-Schmidt operators on $\mathbb{H}$, $\mathbb{S}_{HS}(\mathbb{H})$, in the form (1.17) to the space of the square integrable random variables $\mathbb{L}^2((\Omega, \mathcal{A}, \mathbf{P}), \mathbb{H})$ (or simply $\mathbb{L}^2(\mathbb{H})$). As the space $\mathbb{S}_{HS}(\mathbb{H})$ is dense in the space of random Hilbert-Schmidt operators on $\mathbb{H}$, $\mathbb{L}_{HS}(\mathbb{H})$, the operator $I_{s,t}$ can be uniquely extended on $\mathbb{L}_{HS}(\mathbb{H})$. Then, we can correctly define for $\Phi \in \mathbb{L}_{HS}(\mathbb{H})$ and $h \in \mathbb{H}$:

$$\langle L(t) - L(s), \Phi^* h \rangle = \langle I_{s,t}(\Phi), h \rangle. \tag{1.19}$$

We define the simple Hilbert-Schmidt stochastic process Y on $[0, T]$, $T > 0$, as

$$J(t) = \sum_{i=1}^{n} \mathbf{I}_{t_{i-1} \le t \le t_i} \Phi_i, \tag{1.20}$$

where $T \ge t_i > t_{i-1} \ge 0$, $\sigma(\Phi_i) \subset \mathcal{F}_{t_{i-1}}$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. Denote the space of the processes in the form (1.20) by $\mathcal{S}$. For $J \in \mathcal{S}$, we define

$$I(J) = \sum_{i=1}^{n} I_{t_{i-1}, t_i}(\Phi_i) \tag{1.21}$$

and using this in combination with (1.19)

$$\langle I(J), h \rangle = \sum_{i=1}^{n} \left\langle I_{t_{i-1}, t_i}(\Phi_i), h \right\rangle = \sum_{i=1}^{n} \mathbf{I}_{A_i} \langle L(t) - L(s), \Phi_i^* h \rangle$$

for all $h \in \mathbb{H}$.

We define for the stochastic process of Hilbert-Schmidt operators Y on $[0, T]$:

$$|Y|_{\mathcal{H}}^2 = \mathbf{E} \int_0^T |Y(s)|_{HS}^2 ds, \tag{1.22}$$

where $|\cdot|_{HS}$ denotes the Hilbert-Schmidt norm. We define $\mathcal{H}$ as the space of all $\mathcal{F}$-predictable stochastic process of Hilbert-Schmidt operators Y on $[0, T]$ such that $|\cdot|_{\mathcal{H}} < \infty$. Note that for $J \in \mathcal{S}$, $|J|_{\mathcal{H}} < \infty$. It is known from [26] that I is continuous on $\mathcal{S}$ and $\mathcal{S}$ is dense in $\mathcal{H}$ with respect to $|\cdot|_{\mathcal{H}}$. Therefore, I can be uniquely extended on $\mathcal{H}$ and we can correctly define for $Y \in \mathcal{H}$:

$$\int_0^T Y(s)dL(s) = I(Y).$$

It is proved in [26] that for $Y \in \mathcal{H}$:

1. There exists a version with strong second moments and càdlàg trajectories of the process
$$\int_0^t Y(s)dL(s), \quad t \in [0, T].$$

2. If L is a cylindrical martingale, then
$$\int_0^t Y(s)dL(s), \quad t \in [0, T].$$

   is a cylindrical martingale.

2. Controlled stochastic evolution equation

In this chapter, we introduce two different settings describing the controlled dynamic systems we focus on in this Thesis. We summarize basic properties of these dynamic systems as well. We start with the more general one.

We are concerned with the controlled stochastic evolution equation (SEE)

$$dX^U(t) = \left(AX^U(t) + BU(t)\right)dt + \Phi dL(t), \quad X^U(0) = x, \tag{2.1}$$

where $x \in \mathbb{H}$. A is the infinitesimal generator of an analytic semigroup S on $\mathbb{H}$. For some fixed $\beta > 0$ the operator $-A + \beta\mathbb{I}$ is strictly positive (in the sequel, $\beta > 0$ is fixed). For $\alpha > 0$ we denote by $\mathbb{D}_A^\alpha$ the domain of the fractional power $(-A + \beta\mathbb{I})^\alpha$ equipped with the graph norm $|y|_{\mathbb{D}_A^\alpha} = |(-A + \beta\mathbb{I})^\alpha x|_{\mathbb{H}}$ and $\mathbb{D}_{A^*}^\alpha$ the domain of the fractional power $(-A^* + \beta\mathbb{I})^\alpha$ equipped with the graph norm $|y|_{\mathbb{D}_{A^*}^\alpha} = |(-A^* + \beta\mathbb{I})^\alpha x|_{\mathbb{H}}$.

We assume:

(**A1**) $\Phi \in \mathcal{L}(\mathbb{H})$ and there exists $\delta \in (0, \frac{1}{2}]$ such that $\Phi^*(-A^* + \beta\mathbb{I})^{-\frac{1}{2}+\delta}$ is Hilbert-Schmidt.

(**A2**) $B : \mathbb{D}(B) \subset \mathbb{Y} \to \mathbb{D}(A^*)'$, the dual of $\mathbb{D}(A^*)$ with respect to the topology of $\mathbb{H}$, and $(\beta\mathbb{I} - A)^{\epsilon-1}B \in \mathcal{L}(\mathbb{Y}, \mathbb{H})$, for a given $\epsilon \in (\frac{1}{2}, 1]$.

(**A3**) We have that

$$U \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{Y}),$$

where $\mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$ denotes the space of all $\mathcal{F}$-progressively measurable processes from $\mathbb{L}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$, where

$$\mathbb{L}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$$

$$= \left\{ Y : \mathbb{R}_+ \times \Omega \to \mathbb{Y} \text{ measurable}; \ \forall t > 0 : \mathbf{E}\int_0^t |Y(s)|_{\mathbb{Y}}^2 ds < \infty \right\}.$$

U has the meaning of control process and $\mathcal{U} = \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$ is the space of admissible controls.

(**A4**) L is a cylindrical Lévy process on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ in $\mathbb{H}$ with weak second moments and cylindrical characteristics $(p, 0, \nu)$ such that

$$p(h) = -\int_{\mathbb{H}} \left(\langle z, h \rangle_{\mathbb{H}} \mathbf{I}_{B_{\mathbb{R}}^C}(\langle z, h \rangle_{\mathbb{H}}) \right) d\nu(z),$$

where $B_{\mathbb{R}}^C$ denotes the complement of the set $B_{\mathbb{R}}$ in the set $\mathbb{R}$. Note that the integral exists as L has weak second moments (and therefore weak first moments). The characteristic function of L is

$$\mathbf{E}e^{i\langle L(t),h \rangle} = e^{\int_{\mathbb{H}}(e^{i\langle z,h \rangle_{\mathbb{H}}} - 1 - i\langle z,h \rangle_{\mathbb{H}})d\nu(z)}, \quad h \in \mathbb{H}.$$

It means that L has zero Gaussian part and its Lévy measure ν is cylindrical. L is a cylindrical martingale as $\langle L, h \rangle$ is centered for all $h \in \mathbb{H}$ and we can write for all $h \in \mathbb{H}$

$$\langle L(t), h \rangle = \int_0^t \int_{\mathbb{R} \setminus \{0\}} u \tilde{N}_h(ds, du), \ \ t \in \mathbb{R}_+,$$

where $\tilde{N}_h$ is compensated Poisson measure of $\langle L(t), h \rangle$ with Lévy measure $\nu \circ \langle u, \cdot \rangle^{-1}$ (cf. [26], p. 10, 11).

As we mentioned in the previous section, the process $L_\phi = (L_\phi(t), \ t \geq 0)$ is a càdlàg square integrable martingale ([26], Corollary 4.4., p. 16) for a Hilbert-Schmidt operator ϕ. We are using characteristic function of ϕL for each $h \in \mathbb{H}$ and $t \geq 0$:

$$\mathbf{E} e^{i \langle \phi L(t), h \rangle_{\mathbb{H}}} = \mathbf{E} e^{i \langle L(t), \phi^* h \rangle} = e^{\int_{\mathbb{H}} (e^{i \langle z, \phi^* h \rangle_{\mathbb{H}}} - 1 - i \langle z, \phi^* h \rangle_{\mathbb{H}}) d\nu(z)}$$

$$= e^{\int_{\mathbb{H}} (e^{i \langle z, h \rangle_{\mathbb{H}}} - 1 - i \langle z, h \rangle_{\mathbb{H}}) d(\nu \circ \phi^{-1})(z)}$$

(the integrals of cylindrical measurable functions with respect to ν are the standard Lebesgue integrals). It means that ϕL is Lévy process in $\mathbb{H}$ without Gaussian part and with Lévy measure $\nu_\phi = \nu \circ \phi^{-1}$ ($\nu \circ \phi^{-1}$ is indeed a Radon measure with strong second moments, cf. [26], p. 15). We can write

$$L_\phi(t) = \int_0^t \int_{\mathbb{H}} z \tilde{N}_\phi(ds, dz), \ \ t \in \mathbb{R}_+,$$

where $\tilde{N}_\phi$ is a compensated Poisson measure of L_ϕ.

Remark. The most intuitive concept of the solution of (2.1) is the strong solution, which is defined by the equation

$$X^U(t) = x + \int_0^t \left(A X^U(s) + B U(s) \right) ds + \int_0^t \Phi dL(t), \ \ t \geq 0,$$

if we suppose additional assumptions:

1. $X^U(s) \in \mathcal{D}(A)$, $s \geq 0$, a.s.,

2. $U(s) \in \mathcal{D}(B)$, $s \geq 0$, a.s.,

3. $A X^U(s) + B U(s)$ is a.s. integrable:

$$\int_0^t \left| A X^U(s) + B U(s) \right| ds < \infty, \ \ t \geq 0$$

4. Φ is a Hilbert-Schmidt operator on $\mathbb{H}$.

But the assumptions above are fulfilled only for limited set of equations (2.1). Therefore, we work with the concepts of mild or weak solutions as defined below.

The mild solution $X \in \mathbb{L}_{\mathcal{F}}^{p,loc}(\mathbb{R}_+, \mathbb{H})$ of the stochastic evolution equation (2.1) is defined by the equation

$$X^U(t) = S(t)x + \int_0^t S(t-s)BU(s)ds + \int_0^t S(t-s)\Phi dL(s), \quad t \geq 0, \qquad (2.2)$$

where the stochastic integral $\int_0^t S(t-s)\Phi dL(s)$, $t \geq 0$, is defined as in [26]. Due to (**A2**), we may write

$$\int_0^t S(t-s)\Phi dL(s) = \int_0^t S(t-s)(-A^* + \beta\mathbb{I})^{\frac{1}{2}-\delta}d(-A^* + \beta\mathbb{I})^{-\frac{1}{2}+\delta}\Phi L(s), \quad t \geq 0,$$

c.f. [25]. We can easily see that both integrals in (2.2) exist. In fact, we have the following.

Lemma 2. *Let*
$$Z(t) = \int_0^t S(t-r)\Phi dL(r),$$
$$\hat{Z}(t) = \int_0^t S(t-r)BU(r)dr,$$

$t \geq 0$. *Then* $Z, \hat{Z} \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{H})$ *and, consequently,*

$$X^U \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{H}).$$

Proof. The fact that $\hat{Z}(\cdot) = \int_0^{\cdot} S(\cdot - r)BU(r)dr \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{H})$ has been proved in [5], Lemma 2.1.

Let $t \in \mathbb{R}_+$. It follows that

$$\mathbf{E}|Z(t)|^2 = \int_0^t \left| S(t-r)(-A + \beta\mathbb{I})^{\frac{1}{2}-\delta}(-A + \beta\mathbb{I})^{-\frac{1}{2}+\delta}\Phi \right|_{HS}^2 dr$$

$$\leq \left| (-A^* + \beta\mathbb{I})^{-\frac{1}{2}+\delta}\Phi \right|_{HS}^2 \int_0^t \left| S(t-r)(-A^* + \beta\mathbb{I})^{\frac{1}{2}-\delta} \right|_{\mathcal{L}(\mathbb{H})}^2 dr$$

$$\leq c_1 \int_0^t \frac{1}{(t-r)^{1-2\delta}}$$

for each $t \in [0, T]$ and therefore we have $c_2 > 0$ such that

$$\int_0^T \mathbf{E}|Z(t)|^2 dt \leq c_1 \int_0^T \left(\int_0^t \frac{1}{(t-r)^{1-2\delta}}dr \right) dt \leq c_2 < \infty.$$

$\square$

The weak solution of (2.1) is defined by the equation

$$\left\langle a, X^U(t) \right\rangle$$

$$= \langle a, x \rangle_{\mathbb{H}} + \int_0^t \left\langle A^*a, X^U(s) \right\rangle_{\mathbb{H}} ds + \int_0^t \langle B^*a, U(r) \rangle_{\mathbb{Y}} dr + \langle L(t), \Phi^*a \rangle,$$

$a \in \mathcal{D}(A^*)$, $t \in \mathbb{R}_+$. Similarly as in Theorem 9.15 in [25] we obtain the following equivalence:

Proposition 3. X^U *is the mild solution of (2.1) if and only if* X^U *is the weak solution of (2.1).*

Proof. Firstly, we suppose X^U to be the weak solution of (2.1). For all $a \in \mathcal{D}(A^*)$ and $t \in \mathbb{R}_+$, we can approximate $(S(t - \cdot))^* a$ in the space $\mathbb{C}^1(\mathbb{R}_+, \mathbb{H})$ ($\mathbb{C}^1(\mathbb{R}_+, \mathbb{H})$ denotes the space of continuously differentiable functions on $\mathbb{R}_+$ in $\mathbb{H}$) by functions J_n, $n \in \mathbb{N}$, in the form

$$J_n(\cdot) = \sum_{i=1}^{k_n} f_i^{(n)}(\cdot) a_i^{(n)},$$

where $a_i^{(n)} \in \mathcal{D}(A^*)$, $f_i^{(n)} \in \mathbb{C}^1(\mathbb{R}_+)$ ($\mathbb{C}^1(\mathbb{R}_+) = \mathbb{C}^1(\mathbb{R}_+, \mathbb{R})$), $i = 1, \ldots, k_n$, $k_n \in \mathbb{N}$, $n \in \mathbb{N}$. Therefore using Proposition 9.16 in [25] we obtain for all $a \in \mathcal{D}(A^*)$ and $t \in \mathbb{R}_+$

$$\left\langle J_n(t), X^U(t) \right\rangle_{\mathbb{H}} - \left\langle J_n(0), x \right\rangle_{\mathbb{H}}$$

$$= \sum_{i=1}^{k_n} \left(f_i^{(n)}(t) \left\langle a_i^{(n)}, X^U(t) \right\rangle_{\mathbb{H}} - f_i^{(n)}(0) \left\langle a_i^{(n)}, x \right\rangle_{\mathbb{H}} \right)$$

$$= \sum_{i=1}^{k_n} \left(\int_0^t \left(\left(f_i^{(n)} \right)'(s) \left\langle a_i^{(n)}, X^U(s) \right\rangle_{\mathbb{H}} + f_i^{(n)}(s) \left\langle A^* a_i^{(n)}, X^U(s) \right\rangle_{\mathbb{H}} \right) ds \right)$$

$$+ \sum_{i=1}^{k_n} \left(\int_0^t f_i^{(n)}(s) \left\langle B^* a_i^{(n)}, U(s) \right\rangle_{\mathbb{H}} ds + \int_0^t f_i^{(n)}(s) \left\langle a_i^{(n)}, \Phi dL(s) \right\rangle_{\mathbb{H}} \right)$$

Passing to the limit it follows that

$$\langle a, X(t) \rangle_{\mathbb{H}} - \langle a, S(t)x \rangle_{\mathbb{H}} = \langle (S(0))^* a, X(t) \rangle_{\mathbb{H}} - \langle (S(t))^* a, x \rangle_{\mathbb{H}}$$

$$= \int_0^t \left\langle -A^*(S(t - s))^* a + A^*(S(t - s))^* a, X^U(s) \right\rangle_{\mathbb{H}} ds$$

$$+ \int_0^t \langle B^*(S(t - s))^* a, U(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle (S(t - s))^* a, \Phi dL(s) \rangle_{\mathbb{H}}$$

$$= \left\langle a, \int_0^t S(t - s) B U(s) ds \right\rangle_{\mathbb{H}} + \left\langle a, \int_0^t S(t - s) \Phi dL(s) \right\rangle_{\mathbb{H}}.$$

As we obtained a continuous linear functional of a on the right side and $\mathcal{D}(A^*)$ is dense in $\mathbb{H}$, we have for all $a \in \mathbb{H}$

$$\left\langle a, X^U(t) \right\rangle_{\mathbb{H}}$$

$$= \langle a, S(t)x \rangle_{\mathbb{H}} + \left\langle a, \int_0^t S(t - s) B U(s) ds \right\rangle_{\mathbb{H}} + \left\langle a, \int_0^t S(t - s) \Phi dL(s) \right\rangle_{\mathbb{H}}$$

or equivalently

$$X^U(t) = S(t)x + \int_0^t S(t - s) B U(s) ds + \int_0^t S(t - s) \Phi dL(s).$$

To get the second implication, we suppose X^U to be the mild solution of (2.1). We apply the Fubini Theorem on

$$\Lambda(r, s)(\omega) = \mathbf{I}_{[0,s]}(r) S(s - r) B U(r)(\omega), \quad s, r \in [0, t], \quad \omega \in \Omega.$$

As S is strongly continuous, the assumption of Stochastic Fubini Theorem in [19] are fulfilled and we can apply this theorem on

$$\Psi(r, s) = \mathbf{I}_{[0,s]}(r) \Phi^* S^*(s - r) A^* a, \quad s, r \in [0, t].$$

This theorem is formulated in Appendix A. Note that the stochastic integral of $\mathbf{I}_{[0,s]}(r)\Phi^*S^*(s-r)A^*a$, $s, r \in [0,t]$, is defined in [19] in such way that

$$\int_0^t \mathbf{I}_{[0,s]}(r)\Phi^*S^*(s-r)A^*a\,dL(s) = \left\langle A^*a, \int_0^t \mathbf{I}_{[0,s]}(r)S(s-r)\Phi\,dL(s)\right\rangle_{\mathbb{H}}.$$

For all $t \in \mathbb{R}_+$ and $a \in \mathcal{D}(A^*)$, we obtain

$$\int_0^t \langle A^*a, X(s)\rangle_{\mathbb{H}}\,ds$$

$$= \int_0^t \langle A^*a, S(s)x\rangle_{\mathbb{H}}\,ds + \int_0^t \left\langle A^*a, \int_0^t \Lambda(r,s)dr\right\rangle_{\mathbb{H}}\,ds$$

$$+ \int_0^t \left\langle A^*a, \int_0^t \Psi(r,s)\Phi\,dL(r)\right\rangle_{\mathbb{H}}\,ds$$

$$= \left\langle \int_0^t S(s)^*A^*a\,ds, x\right\rangle_{\mathbb{H}} + \int_0^t \left\langle \int_0^t \mathbf{I}_{[r,t]}(s)B^*(S(s-r))^*A^*a\,ds, U(r)\right\rangle_{\mathbb{H}}\,ds$$

$$+ \int_0^t \int_0^t \mathbf{I}_{[r,t]}(s)\Phi^*(S(s-r))^*A^*a\,dL(r)ds$$

$$= \langle (S(t))^*a, x\rangle_{\mathbb{H}} - \langle a, x\rangle_{\mathbb{H}} + \int_0^t \langle B^*(S(t-r))^*a, U(r)\rangle_{\mathbb{H}}\,dr$$

$$- \int_0^t \langle B^*a, U(r)\rangle_{\mathbb{H}}\,dr + \left\langle a, \int_0^t S(t-r)\Phi\,dL(r)\right\rangle_{\mathbb{H}} - \left\langle a, \int_0^t \Phi\,dL(r)\right\rangle_{\mathbb{H}}$$

$$= \langle a, X(t)\rangle_{\mathbb{H}} - \left(\langle a, x\rangle_{\mathbb{H}} + \int_0^t \langle B^*a, U(r)\rangle_{\mathbb{H}}\,dr + \left\langle a, \int_0^t \Phi\,dL(r)\right\rangle_{\mathbb{H}}\right)$$

and X is the weak solution of (2.1). $\qquad\square$

Now we define an alternative set of assumptions on the coefficients of the equation (2.1) which will be used selectively and are more restrictive than the original conditions (**A1**) – (**A4**). As in the previous setting, we suppose that (**A2**) and (**A3**) holds. The main difference is given by replacing the assumption (**A1**) by the stronger alternative

(**A1***) Φ is Hilbert-Schmidt.

However, instead of (**A1***) we will use the conditions (**A1****) and (**A4****) below, which are easier to handle and do not change the type of equation.

(**A1****) $\Phi = \mathbb{I}$,

(**A4****) $L = L_\Phi$ is the square integrable Lévy process defined on a stochastic basis $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ in $\mathbb{H}$ of the form

$$L(t) = \int_{[0,t]}\int_{\mathbb{H}} z\tilde{N}(ds, dz), \ \ t \in \mathbb{R}_+,$$

where $\tilde{N}$ is the compensated Poisson measure with the jump measure ν. We denote the covariance operator of $L(1)$ by $\mathcal{Q} = \Phi\Phi^*$.

Obviously, (**A4****) implies the weaker assumption (**A4**).

It means that we can rewrite the equation (2.1) in the form

$$dX^U(t) = \left(AX^U(t) + BU(t)\right)dt + dL(t), \quad X^U(0) = x. \tag{2.3}$$

The mild solution (2.2) in the case of the stochastic evolution equation (2.3) takes the form

$$X^U(t) = S(t)x + \int_0^t S(t-s)BU(s)ds + \int_0^t S(t-s)dL(s), \quad t \geq 0, \tag{2.4}$$

where the stochastic integral $\int_0^t S(t-s)dL(s)$, $t \geq 0$, is defined as in [25]. We already know from the Lemma 2 that both integrals in (2.4) exist as the assumption of the Proposition 3 are fulfilled for the equation (2.3) as well.

As L is square integrable martingale (with the Hilbert-Schmidt covariance operator $\mathcal{Q}$), the following theorem is a simple consequence of the Theorem 9.24 in [25]. This theorem is formulated in the Appendix B.

Theorem 4. *Assume that S is a strongly continuous semigroup of contractions on $\mathbb{H}$, which means that*

$$|S(t)|_{\mathcal{L}(\mathbb{H})} \leq e^{\omega t}, \quad t \geq 0, \tag{2.5}$$

for $\omega \in \mathbb{R}$. Then there exists a càdlàg version of X.

Therefore, we will impose (2.5) and consider the càdlàg version of the solution X to (2.3) in the following chapters.

The weak solution in the case of the stochastic evolution equation (2.3) takes the form

$$\left\langle a, X^U(t) \right\rangle_{\mathbb{H}}$$

$$= \langle a, x \rangle_{\mathbb{H}} + \int_0^t \left\langle A^*a, X^U(s) \right\rangle_{\mathbb{H}} ds + \int_0^t \langle B^*a, U(r) \rangle_{\mathbb{Y}} dr + \langle a, L(t) \rangle_{\mathbb{H}},$$

$a \in \mathcal{D}(A^*)$, $t \in \mathbb{R}_+$. We already know from the Proposition 3 that X^U is the mild solution of (2.3) if and only if X^U is the weak solution of (2.3) as the assumption of the Proposition 3 are fulfilled for the equation (2.3) as well.

In following chapters, we always start with basic results proved for the stochastic evolution equation in the form (2.1) with the corresponding assumptions (**A1**) – (**A4**). Then, the results will be amplified for the stochastic evolution equation in the form (2.3) with the alternative assumptions (**A1****), (**A2**), (**A3**), (**A4****) and (2.5) as the setting for the equation (2.3) is stronger then the setting for the equation (2.3) and provides us with stronger tools. We specify which form of the stochastic evolution equation we consider at the beginning of each chapter or paragraph.

3. The control problem

In this chapter, we introduce the control problem to be studied. More precisely, we define the notions of the optimal control as well as the corresponding optimal cost. In fact, we have two concepts of the control problem. We start with the concept in the mean value sense and then, we formulate the concept in the path-wise sense.

Set

$$J(U,t) = \int_0^T \left(\left\langle QX^U(s), X^U(s) \right\rangle_{\mathbb{H}} + \langle RU(s), U(s) \rangle_{\mathbb{Y}} \right) ds, \tag{3.1}$$

where $T > 0$, X^U is the solution of (2.1), $Q \in \mathcal{L}(\mathbb{H})$ is a symmetric positive semi-definite operator and $R \in \mathcal{L}(\mathbb{Y})$ is a symmetric positive definite operator, i. e.

$$\langle Ry, y \rangle \geq r|y|^2, \quad y > 0, \tag{3.2}$$

holds for a constant $r > 0$.

In the case of the mean value control, the ergodic cost functional is defined as the "mean average cost per time unit in long run", that is

$$\tilde{J}_{\mathbf{E}}(U) = \lim_{t \to \infty} \inf \frac{\mathbf{E}J(U,t)}{t}, \quad U \in \mathcal{U}. \tag{3.3}$$

To solve the ergodic control problem is to find $D \in \mathbb{R}$ and $U_0 \in \mathcal{U}$ such that

$$\tilde{J}_{\mathbf{E}}(U) \geq D, \quad U \in \mathcal{U}, \tag{3.4}$$

and

$$\lim_{t \to \infty} \frac{\mathbf{E}J(U_0,t)}{t} = D. \tag{3.5}$$

Then U_0 and D are called the optimal control and the optimal cost, respectively.

Under some stronger assumptions, we are able to prove that D is the optimal cost and U_0 is the optimal control in the path-wise sense. In the case of path-wise control, the ergodic cost functional is defined as the path-wise "average cost per time unit in long run", that is

$$\tilde{J}(U) = \lim_{t \to \infty} \inf \frac{J(U,t)}{t}, \quad U \in \mathcal{U}, \tag{3.6}$$

and $D \in \mathbb{R}$ and $U_0 \in \mathcal{U}$ have to fulfill:

$$\tilde{J}(U) \geq D, \quad a.s., \quad U \in \mathcal{U}, \tag{3.7}$$

and

$$\lim_{t \to \infty} \frac{J(U_0,t)}{t} = D, \quad a.s. \tag{3.8}$$

Consider the stationary Riccati equation

$$\langle a, A^*Vb \rangle_{\mathbb{H}} + \langle A^*Va, b \rangle_{\mathbb{H}} + \langle Qa, b \rangle_{\mathbb{H}} - \left\langle R^{-1}B^*Va, B^*Vb \right\rangle_{\mathbb{H}} = 0, \tag{3.9}$$

$a, b \in \mathcal{D}(A^*)$.

As usual, we impose the standard stabilizability and detectability conditions.

Definition 1. *Let $A \in \mathcal{L}(\mathbb{H})$ and $B \in \mathcal{L}(\mathbb{Y}, \mathbb{H})$. The pair (A, B) is said to be stabilizable iff there exists $H \in \mathcal{L}(\mathbb{H}, \mathbb{Y})$ such that the semigroup generated by the operator $A + BH$ is exponentially stable.*

Definition 2. *Let $A \in \mathcal{L}(\mathbb{H})$ and $Q \in \mathcal{L}(\mathbb{H})$, Q is symmetric positive semi-definite. The pair (A, Q) is said to be detectable iff there exists $K \in \mathcal{L}(\mathbb{H})$ such that the semigroup generated by the operator $A + K\sqrt{Q}$ is exponentially stable.*

Then, as we can see in [20]:

1. The equation (3.9) has a unique solution in the class of non-negative and self-adjoint linear operators on $\mathbb{H}$ and, moreover, $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{1-\epsilon})$.

2. The semigroup S_V generated by $A_K = A - BR^{-1}B^*V$ is exponentially stable, more specifically, there exist constants $M_0 > 0$, $\omega > 0$, such that

$$|S_K(t)|_{\mathcal{L}(\mathbb{H})} \leq M_0 e^{-\omega t}, \tag{3.10}$$

for each $t \geq 0$.

By (3.9) there exists $h(\cdot)$, a continuous extension of $\langle A^*V\cdot, \cdot\rangle_{\mathbb{H}}$ on $\mathbb{H}$ such that for some $c \in \mathbb{R}_+$

$$|h(y)| \leq c|y|_{\mathbb{H}}^2 \tag{3.11}$$

holds for all $y \in \mathbb{H}$.

4. Itô formula

In our general case (**A1**)-(**A4**), we aim at verifying the formula:

$$\mathbf{E}\left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}} - \langle x, Vx\rangle_{\mathbb{H}}$$

$$= 2\mathbf{E}\int_0^t h(X^U(s))ds + \mathbf{E}\int_0^t 2\left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}} ds + t\Pi$$

for all $t \in [0, T]$, where we assume:

$$\Pi = \lim_{\lambda\to\infty} Tr(VR^2(\lambda)\Phi\Phi^*(R^2(\lambda))^*) < \infty, \tag{4.1}$$

where Tr denotes the trace and $h : \mathbb{H} \to \mathbb{R}$ satisfies (3.11) and extends the function $\langle A^*V\cdot, \cdot\rangle_{\mathbb{H}}$.

The standard Itô formula (Theorem D.2 in [25]) may be used for the processes which are strong solutions of Lévy-driven SEEs. However, in the present case

1. the noise in SEE is a general cylindrical Lévy process,

2. operators A and B are not bounded.

Therefore the strong solutions generally do not exist. In this chapter, we prove a modification of Itô formula which fits our needs related to the main results, more specifically:

1. It is applicable to weak/mild solutions.

2. The functional takes the form $\langle \cdot, V\cdot\rangle_{\mathbb{H}}$ where V is the solution to the Riccati equation.

The proof is based on approximation of SEE (2.1) by SEEs which have strong solutions and which fulfill the assumptions of the standard Itô formula. To this end we use the resolvent $R(\lambda, A)$ and define $R(\lambda) = \lambda R(\lambda, A)$ for $\lambda > \beta$. Due to our assumptions, $R(\lambda)^{\frac{1}{2}-\delta}\Phi$ is Hilbert-Schmidt, therefore $R(\lambda)\Phi$ is Hilbert-Schmidt and $L_\lambda(\cdot) = R(\lambda)\Phi L(\cdot)$ has the nuclear incremental covariance $R(\lambda)\Phi\Phi^*(R(\lambda))^*$. Our approximate SEE takes the form:

$$dX_\lambda^U(t) = (AX_\lambda^U(t) + R(\lambda)BU(t))dt + R(\lambda)dL_\lambda(t), \quad X(0) = x_\lambda, \tag{4.2}$$

where $x_\lambda = R(\lambda)x$ and $\lambda > \beta$.

The proof has four steps:

1. We approximate the solution of the equation (2.1) (and specifically (2.3)) by solutions of the equations (4.2) (with $\Phi = \mathbb{I}$ for (2.3)).

2. We state the existence of the strong solution of the equation (4.2) (equation (4.2) with $\Phi = \mathbb{I}$ in the case of (2.3), respectively).

3. We apply the standard Itô formula to the equation (4.2) (equation (4.2) with $\Phi = \mathbb{I}$ in the case of (2.3), respectively).

4. We pass to the limit in the formulae obtained in the step 3.

Firstly, we consider the equation (2.1) with pure jump cylindrical Lévy process with weak second moments to obtain our most general result. In the case of the general cylindrical Lévy process that we consider, we are not able to apply the path-wise limit theorems on the stochastic part of the equation obtained in the step (3). Therefore we have to apply the mean value on the equation obtained from (3) prior to (4) and we obtain only the Itô formula in the the mean value sense.

In this general case, the four steps of the proof are realized by four Lemmas:

1. Lemma 5 allows us to approximate the solution of the equation (2.1) by solutions of the equations (4.2).

2. Lemma 6 states the existence of the strong solution of the equation (4.2).

3. In Lemma 7, we apply the mean value version of the standard Itô formula to the equation (4.2).

4. In Lemma 8, we apply limit theorems to the formula proved in the Lemma 7.

Lemma 5. *1. There exists the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and for all $t \in [0, T]$ $\lim_{n \to \infty} X_{\lambda_n}^U(t) = X^U(t)$ a.s.,*

2. for each $T > 0$ there exists a constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \mathbf{E}|X_\lambda(t)|^2 \leq C. \tag{4.3}$$

Proof. 1. Since $R(\lambda) \to \mathbb{I}$ strongly as $\lambda \to \infty$, we have that $x_\lambda \to x$ and there exists $c > 0$ such that $|R(\lambda)|_{\mathcal{L}(\mathbb{H})} \leq c$, $\lambda > 0$. As in Lemma 3.2. in [5] we have that

$$\lim_{\lambda \to \infty} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds - \int_0^t S(t-s)BU(s)ds \right|_{\mathbb{H}}$$

$$= \lim_{\lambda \to \infty} \left| \int_0^t S(t-s)(R(\lambda) - \mathbb{I})BU(s)ds \right|_{\mathbb{H}} = 0, \quad \text{a.s.}, \quad t \in [0, T],$$

and

$$\sup_{\lambda > \beta, t \in [0,T]} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds \right|_{\mathbb{H}} < \infty \quad \text{a.s.}$$

For fixed $t > 0$ we have :

$$\mathbf{E} \left| \int_0^t S(t-r)\Phi dL(r) - \int_0^t S(t-r)R(\lambda)dL_\lambda(r) \right|_{\mathbb{H}}^2$$

$$= \mathbf{E} \left| \int_0^t S(t-r)\left(\mathbb{I} - R^2(\lambda)\right)\Phi dL(r) \right|_{\mathbb{H}}^2$$

$$\leq c_1 \int_0^t \left| S(t-r)\left(\mathbb{I} - R^2(\lambda)\right)\Phi \right|_{HS}^2 dr$$

$$= c_1 \int_0^t \left| S(t-r)(-A + \beta\mathbb{I})^{\frac{1}{2}-\delta}(-A + \beta\mathbb{I})^{\delta - \frac{1}{2}}\left(\mathbb{I} - R^2(\lambda)\right)\Phi \right|_{HS}^2 dr$$

$$\leq c_2 \int_0^t \left| S(t-r)(-A + \beta\mathbb{I})^{\frac{1}{2}-\delta} \right|_{\mathcal{L}(\mathbb{H})}^2$$

26

$$\left| (-A + \beta \mathbb{I})^{\delta - \frac{1}{2}} (\mathbb{I} - R^2(\lambda)) \Phi \right|^2_{HS} dr$$

$$\leq c_3 \int_0^t \frac{1}{(t-r)^{1-2\delta}} dr \left| (-A + \beta \mathbb{I})^{\delta - \frac{1}{2}} (\mathbb{I} - R^2(\lambda)) \Phi \right|^2_{HS}$$

$$= c_4 Tr \left((-A + \beta \mathbb{I})^{\delta - \frac{1}{2}} \left(\mathbb{I} - R^2(\lambda) \right) \Phi \Phi^* \left(\mathbb{I} - R^2(\lambda) \right)^* ((-A + \beta \mathbb{I})^{\delta - \frac{1}{2}})^* \right),$$

for some constants $c_1, c_2, c_3, c_4 > 0$. This converges to 0 for $\lambda \to \infty$ (cf. Lemma 3.2. in [5]). Therefore, there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \to \infty$, $n \to \infty$, and

$$\lim_{n \to \infty} \int_0^t S(t-r) R(\lambda_n) dL_{\lambda_n}(r) = \int_0^t S(t-r) dL(r), \quad \text{a.s.}$$

2. For $t \in [0, T]$ we have that

$$\mathbf{E} \left| \int_0^t S(t-s) R(\lambda_n) BU(s) ds \right|^2$$

$$\leq c_1 |R(\lambda_n)|^2_{\mathcal{L}(\mathbb{H})} \left| (\beta \mathbb{I} - A)^{\epsilon - 1} B \right|^2_{\mathcal{L}(\mathbb{Y}, \mathbb{H})}$$

$$\times \mathbf{E} \left(\int_0^t \left| S(t-s)(\beta \mathbb{I} - A)^{1-\epsilon} \right|_{\mathcal{L}(\mathbb{H})} |U(s)|_{\mathbb{Y}} ds \right)^2$$

$$\leq c_2 \mathbf{E} \left(\int_0^t \frac{1}{(t-s)^{1-\epsilon}} |U(s)|_{\mathbb{Y}} ds \right)^2$$

$$\leq c_3 \left(\int_0^T \frac{1}{s^{(1-\epsilon)2}} ds \right) \int_0^T \mathbf{E} |U(s)|^2_{\mathbb{Y}} ds \leq c$$

for a universal constant $c > 0$ (which may be different from line to line). Here we used the already mentioned fact that the operators $R(\lambda_n)$ are uniformly bounded in $\mathcal{L}(\mathbb{H})$. Also, similarly as in the proof of lemma 2 we get

$$\mathbf{E} \left| \int_0^t S(t-s) R(\lambda_n) dL_{\lambda_n}(s) \right|^2_{\mathbb{H}}$$

$$\leq c \int_0^t \left| S(t-s)(\beta \mathbb{I} - A)^{\frac{1}{2} - \delta} R^2(\lambda_n)(\beta \mathbb{I} - A)^{\delta - \frac{1}{2}} \Phi \right|^2_{HS} ds$$

$$\leq c \left| (\beta \mathbb{I} - A)^{\delta - \frac{1}{2}} \Phi \right|^2_{HS} |R^2(\lambda_n)|_{\mathcal{L}(\mathbb{H})} \frac{1}{(t-s)^{1-2\delta}} \leq c$$

for a universal constant $c > 0$ independent of $n \in \mathbb{N}$ and $t \in [0, T]$ which completes the proof of (4.3).

$\square$

We can prove the following Lemma similarly as Lemma 3.1 in [5].

Lemma 6. *For $\lambda > \beta$ the equation (4.2) has the unique strong solution.*

Proof. As in Lemma 3.1 in [5] we can prove that a.s.

$$\int_0^T \int_0^t |AS(t-s)R(\lambda)BU(s)|_{\mathbb{H}} ds dt < \infty.$$

Moreover

$$\int_0^T \int_0^t \left| AS(t-s)R^2(\lambda)\Phi^{\frac{1}{2}} \right|_{HS}^2 ds dt$$

$$\leq \int_0^T \int_0^t |AS(t-s)|_{\mathcal{L}(\mathbb{H})}^2 ds dt |R(\lambda)|_{\mathcal{L}(\mathbb{H})}^4 \left| \Phi^{\frac{1}{2}} \right|_{HS}^2 < \infty,$$

therefore we can use the Theorem 8.14 in [25] to obtain

$$\int_0^T AX_\lambda^U(t) dt = \int_0^T AS(t)x dt + \int_0^T A \int_0^t S(t-s)R(\lambda)BU(s) ds dt$$

$$+ \int_0^T A \int_0^t S(t-s)R(\lambda)\Phi dL_\lambda(s) dt$$

$$= S(T)x - x + \int_0^T \int_s^T AS(t-s) dt R(\lambda)BU(s) ds$$

$$+ \int_0^T \int_s^T AS(t-s) dt R(\lambda)\Phi dL_\lambda(s)$$

$$= S(T)x - x + \int_0^T S(T-s)R(\lambda)BU(s) ds - \int_0^T R(\lambda)BU(s) ds$$

$$+ \int_0^T S(T-s)R(\lambda)\Phi dL_\lambda(s) - \int_0^T R(\lambda)\Phi dL_\lambda(s)$$

$$= X_\lambda^U(T) - x - \int_0^T R(\lambda)BU(s) ds - \int_0^T R(\lambda)\Phi dL_\lambda(s).$$

$$\square$$

Now we can apply the standard Itô formula to the strong solutions of (4.2). We use this fact in the following lemma.

Lemma 7. *Assume (A1)-(A4), $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{1-\epsilon})$ is non-negative and self-adjoint operator on $\mathbb{H}$. Then*

$$\mathbf{E}\left\langle X_\lambda^U(t), VX_\lambda^U(t) \right\rangle_{\mathbb{H}} - \left\langle x_\lambda, Vx_\lambda \right\rangle_{\mathbb{H}} = \mathbf{E}\int_0^t 2\left\langle VX_\lambda^U(s), AX_\lambda^U(s) \right\rangle_{\mathbb{H}} ds$$

$$+\mathbf{E}\int_0^t 2\left\langle B^*R^*(\lambda)VX_\lambda^U(s), U(s) \right\rangle_{\mathbb{Y}} ds + Tr(VR^2(\lambda)\Phi\Phi^*(R^2(\lambda))^*).$$

Proof. X_λ is the strong solution of (4.2), therefore using Theorem D.2 in [25] we obtain

$$\left\langle X_\lambda^U(t), VX_\lambda^U(t) \right\rangle_{\mathbb{H}} - \left\langle x_\lambda, Vx_\lambda \right\rangle_{\mathbb{H}} = \int_0^t 2\left\langle VX_\lambda^U(s_-), dX_\lambda^U(s) \right\rangle_{\mathbb{H}}$$

$$+\sum_{s\leq t}\left(\left\langle X_\lambda^U(s), VX_\lambda^U(s) \right\rangle_{\mathbb{H}} - \left\langle X_\lambda^U(s_-), VX_\lambda^U(s_-) \right\rangle_{\mathbb{H}} - 2\left\langle VX_\lambda^U(s_-), \Delta X_\lambda^U(s) \right\rangle_{\mathbb{H}} \right)$$

$$= \int_0^t 2\left\langle VX_\lambda^U(s), AX_\lambda^U(s) \right\rangle_{\mathbb{H}} ds + \int_0^t 2\left\langle VX_\lambda^U(s_-), R^2(\lambda)\Phi dL(s) \right\rangle_{\mathbb{H}}$$

$$+ \int_0^t 2 \left\langle V X_\lambda^U(s_-), R(\lambda)BU(s) \right\rangle_{\mathbb{H}} ds + \sum_{s \leq t} \left\langle \Delta X_\lambda^U(s), V \Delta X_\lambda^U(s) \right\rangle_{\mathbb{H}}$$

$$= \int_0^t 2 \left\langle V X_\lambda^U(s), A X_\lambda^U(s) \right\rangle_{\mathbb{H}} ds + \int_0^t 2 \left\langle \Phi^*(R^2(\lambda))^* V X_\lambda^U(s_-), dL(s) \right\rangle_{\mathbb{H}}$$

$$+ \int_0^t 2 \left\langle B^* R^*(\lambda) V X_\lambda^U(s), U(s) \right\rangle_{\mathbb{Y}} ds + \sum_{s \leq t} \left| \Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) \right|_{\mathbb{H}}^2.$$

where $\Delta X_\lambda^U(s) = X_\lambda^U(s) - X_\lambda^U(s_-)$ and

$$\Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) = V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) - V^{\frac{1}{2}} R^2(\lambda) \Phi L(s_-)$$

$$= V^{\frac{1}{2}}(R^2(\lambda) \Phi L(s) - R^2(\lambda) \Phi L(s_-)) = V^{\frac{1}{2}} \Delta R^2(\lambda) \Phi L(s).$$

Note that $V^{\frac{1}{2}} R^2(\lambda) \Phi L(s)$ is a square integrable Lévy processes. As

$$\mathbf{E} \left| \int_0^t 2 \left\langle \Phi^*(R^2(\lambda))^* V X_\lambda^U(s_-), dL(s) \right\rangle_{\mathbb{H}} \right| < \infty,$$

we obtain

$$\mathbf{E} \int_0^t 2 \left\langle \Phi^*(R^2(\lambda))^* V X_\lambda^U(s_-), dL(s) \right\rangle_{\mathbb{H}} = 0.$$

We can see that

$$\mathbf{E} \sum_{s \leq t} \left| \Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) \right|_{\mathbb{H}}^2 = Tr(V R^2(\lambda) \Phi \Phi^*(R^2(\lambda))^*)$$

and the result follows. $\qquad\square$

In the following Lemma, we approximate this result on the weak/mild solution of (2.1). Here, we use Lemma 5.

Lemma 8. *Assume (**A1**)-(**A4**), $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{1-\epsilon})$ is non-negative and self-adjoint on $\mathbb{H}$ and let (4.1) hold. Assume further that there exists a continuous function $h : \mathbb{H} \to \mathbb{R}$ satisfying (3.11) extending the function $\langle A^* V \cdot, \cdot \rangle_{\mathbb{H}}$. Then for all $t \in [0, T]$ we have*

$$\mathbf{E} \left\langle X^U(t), V X^U(t) \right\rangle_{\mathbb{H}} - \langle x, V x \rangle_{\mathbb{H}}$$

$$= 2 \mathbf{E} \int_0^t h(X^U(s)) ds + \mathbf{E} \int_0^t 2 \left\langle B^* V X^U(s), U(s) \right\rangle_{\mathbb{Y}} ds + t \Pi.$$

Proof. By Lemma 7, we have:

$$\mathbf{E} \left\langle X_\lambda^U(t), V X_\lambda^U(t) \right\rangle_{\mathbb{H}} - \langle x_\lambda, V x_\lambda \rangle_{\mathbb{H}} = \mathbf{E} \int_0^t 2 \left\langle V X_\lambda^U(s), A X_\lambda^U(s) \right\rangle_{\mathbb{H}} ds$$

$$+ \mathbf{E} \int_0^t 2 \left\langle B^* R^*(\lambda) V X_\lambda^U(s), U(s) \right\rangle_{\mathbb{Y}} ds + t Tr(V R^2(\lambda) \Phi \Phi^*(R^2(\lambda))^*). \qquad (4.4)$$

Lemma 5 yields a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and for all $t \in [0, T]$ a.s.

$$\lim_{n \to \infty} \left\langle X_{\lambda_n}^U(t), V X_{\lambda_n}^U(t) \right\rangle_{\mathbb{H}} = \left\langle X^U(t), V X^U(t) \right\rangle_{\mathbb{H}}$$

and
$$\lim_{n\to\infty} h(X^U_{\lambda_n}(t)) = h(X^U(t)).$$

Also,
$$\left| \left\langle B^* R^*(\lambda) V X^U_\lambda(s), U(s) \right\rangle_{\mathbb{Y}} - \left\langle B^* V X^U(s), U(s) \right\rangle_{\mathbb{Y}} \right|$$

$$\leq \left| \left\langle B^* R^*(\lambda) V X^U(s), U(s) \right\rangle_{\mathbb{Y}} - \left\langle B^* V X^U(s), U(s) \right\rangle_{\mathbb{Y}} \right|$$

$$+ \left| \left\langle B^* R^*(\lambda) V X^U_\lambda(s), U(s) \right\rangle_{\mathbb{Y}} - \left\langle B^* R^*(\lambda) V X^U(s), U(s) \right\rangle_{\mathbb{Y}} \right|$$

$$= \left| \left\langle B^* (R^*(\lambda) - \mathbb{I}) V X^U(s), U(s) \right\rangle_{\mathbb{Y}} \right| + \left| \left\langle B^* R^*(\lambda) V (X^U_\lambda(s) - X^U(s)), U(s) \right\rangle_{\mathbb{Y}} \right|$$

for $s \in [0, T]$. As
$$\lim_{\lambda\to\infty} |(R^*(\lambda) - \mathbb{I}) V X^U(s)|_{\mathbb{D}^{1-\epsilon}_{A^*}} = 0,$$

and $B^* \in \mathcal{L}(\mathbb{D}^{1-\epsilon}_{A^*}, \mathbb{Y})$, we have

$$\lim_{\lambda\to\infty} \left| \left\langle B^* (R^*(\lambda) - \mathbb{I}) V X^U(s), U(s) \right\rangle_{\mathbb{Y}} \right|$$

$$\leq \lim_{\lambda\to\infty} |B^* (R^*(\lambda) - \mathbb{I}) V X(s)|_{\mathbb{Y}} |U(s)|_{\mathbb{Y}} = 0.$$

It is easy to see that for each $y \in \mathbb{H}$

$$\lim_{z\to y} |\langle B^* R^*(\lambda) V (z - y), U(s) \rangle_{\mathbb{H}}| = 0$$

uniformly with respect to $\lambda > \beta$. Therefore we obtain

$$\lim_{n\to\infty} |\left\langle B^* R^*(\lambda_n) V (X^U_{\lambda_n}(s) - X^U(s)), U(s) \right\rangle_{\mathbb{Y}}| = 0, \text{ a.s.}$$

Furthermore, taking into account (3.11), we may use Lemma 5 (4.3) to conclude that the sequences of functions

$$t \mapsto \left\langle X^U_{\lambda_n}(t), V X^U_{\lambda_n}(t) \right\rangle_{\mathbb{H}}, \quad s \mapsto \left\langle X^U_{\lambda_n}(s), A X^U_{\lambda_n}(s) \right\rangle_{\mathbb{H}}$$

are uniformly integrable on Ω and $(0, t) \times \Omega$, respectively. Let us show that the family

$$s \mapsto 2 \left\langle B^* R^*(\lambda_n) V X^U_{\lambda_n}(s), U(s) \right\rangle_{\mathbb{Y}}$$

is uniformly integrable on $(0, t) \times \Omega$ as well. We have that

$$\left| \left\langle B^* R^*(\lambda_n) V X^U_{\lambda_n}(s), U(s) \right\rangle_{\mathbb{H}} \right|$$

$$\leq |B^*|_{\mathcal{L}(\mathbb{D}^{1-\epsilon}_{A^*}, \mathbb{Y})} |R^*(\lambda_n)|_{\mathcal{L}(\mathbb{D}^{1-\epsilon}_{A^*})} |V|_{\mathcal{L}(\mathbb{H}, \mathbb{D}^{1-\epsilon}_{A^*})} |X^U_{\lambda_n}(s)|_{\mathbb{H}} |U(s)|_{\mathbb{Y}}$$

$$\leq k |X^U_{\lambda_n}(s)|_{\mathbb{H}} |U(s)|_{\mathbb{Y}},$$

$s \in [0, t]$, $n \in \mathbb{N}$, for a constant $k > 0$. Using Hölder inequality we obtain

$$\mathbf{E} \int_0^t \left| X^U_{\lambda_n}(s) \right|_{\mathbb{H}} |U(s)|_{\mathbb{Y}} ds \leq \left(\mathbf{E} \int_0^t |X^U_{\lambda_n}(s)|^2_{\mathbb{H}} ds \right)^{\frac{1}{2}} \left(\mathbf{E} \int_0^t |U(s)|^2_{\mathbb{Y}} ds \right)^{\frac{1}{2}},$$

which is bounded by a constant independent of $n \in \mathbb{N}$ in virtue of Lemma 5 (4.3). Therefore, recalling the assumption (4.1), we may pass to the limit as $\lambda = \lambda_n \to \infty$ in all terms in the formula (4.4), which completes the proof. $\quad\square$

The above Lemma completes the proof of the mean value Itô formula.

In the following proposition, we illustrate some cases in which are the assumption (4.1) of the mean value Itô formula fulfilled.

Proposition 9. *Let $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{1-\epsilon})$ be non-negative and self-adjoint on $\mathbb{H}$ and let one of the following conditions be satisfied.*

1. *Φ is Hilbert-Schmidt,*

2. *V is nuclear,*

3. *$V \in \mathcal{L}(\mathbb{D}_{A^*}^{\delta - \frac{1}{2}}, \mathbb{D}_{A^*}^{\frac{1}{2} - \delta})$.*

Then (4.1) is fulfilled, where $\Pi = Tr(V\Phi\Phi^*)$ in case of (1) and (2) and

$$\Pi = Tr((R^*(\beta))^{\delta - \frac{1}{2}} V\Phi\Phi^*(R^*(\beta))^{\frac{1}{2} - \delta})$$

in case of (3).

Proof. This Proposition is a simple consequence of Proposition 3.4 in [5]. $\square$

Recall that in the case of (1), the equation (2.1) can be rewritten using (1.15) as

$$dX^U(t) = \left(AX^U(t) + BU(t)\right)dt + dL_\Phi(t), \quad X^U(0) = x, \qquad (4.5)$$

where L_Φ is a square integrable Lévy process. It means that the proved mean value Itô formula is applicable to the equation (2.3).

We can obtain stronger result for the equation (2.3) in which we assume (**A1****), (**A4****) and (2.5). The equation (2.3) still does not have strong solution as the operators A and B still are not bounded. Therefore, the standard Itô formula (Theorem D.2 in [25]) is still not applicable. Again, we can use the Yosida approximations (4.2) ($\Phi = \mathbb{I}$) and apply the steps 1-4. But in the case of the square integrable Lévy process, we can apply limit theorems path-wise to the stochastic parts of the equation obtained in the step (3) and we do not need to apply mean value functional to the equation before the step (4). Therefore, we obtain the path-wise result in this case:

$$\left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}} - \left\langle x, Vx\right\rangle_{\mathbb{H}}$$

$$= 2\int_0^t h(X^U(s))ds + \int_0^t 2\left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}} ds$$

$$+ 2\int_0^t \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}} + \sum_{s \leq t}\left|V^{\frac{1}{2}}\Delta L(s)\right|_{\mathbb{H}}^2, \quad \text{a.s.,}$$

for all $t \in [0, T]$, where $\Delta L(s) = L(s) - L(s_-)$, $s \geq 0$.

We already know from the Lemma 5 that for all $t \in [0, T]$

$$\lim_{n \to \infty} X_{\lambda_n}^U(t) = X^U(t) \quad \text{a.s.}$$

and by Lemma 6 for $\lambda > \beta$ (4.2) has the unique strong solution ($\Phi = \mathbb{I}$ and we have square integrable Lévy process in both cases). Therefore, for the steps (1) and (2), it is enough to prove the path-wise version of the upper bound (4.3).

Lemma 10. *There exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and*

$$\sup_{n\in\mathbb{N}, t\in[0,T]} |X_{\lambda_n}^U(t)|_{\mathbb{H}} < \infty, \quad a.s.$$

Proof. Since $R(\lambda) \to \mathbb{I}$ strongly as $\lambda \to \infty$, we have a $c > 0$ such that $|R(\lambda)| \leq c$, $\lambda > 0$. As in Lemma 3.2. in [5] we have that

$$\lim_{\lambda\to\infty} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds - \int_0^t S(t-s)BU(s)ds \right|_{\mathbb{H}}$$

$$= \lim_{\lambda\to\infty} \left| \int_0^t S(t-s)(R(\lambda) - \mathbb{I})BU(s)ds \right|_{\mathbb{H}} = 0, \quad a.s., \quad t \in [0,T],$$

and

$$\sup_{\lambda > \beta, t\in[0,T]} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds \right|_{\mathbb{H}} < \infty \quad a.s.$$

By Corollary 2.14 in [15] we have a $c_1 > 0$ such that

$$\mathbf{E} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)dL(r) - \int_0^t S(t-r)R(\lambda)dL_\lambda(r) \right|_{\mathbb{H}}^2$$

$$= \mathbf{E} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)(\mathbb{I} - R^2(\lambda))dL(r) \right|_{\mathbb{H}}^2$$

$$\leq c_1 \int_{\mathbb{H}} \int_0^T |S(t-r)(\mathbb{I} - R^2(\lambda))z|_{\mathbb{H}}^2 dr d\nu(z).$$

As we have $c_2, c_3 > 0$ such that

$$\int_{\mathbb{H}} \int_0^T |S(t-r)(\mathbb{I} - R^2(\lambda))z|_{\mathbb{H}}^2 dr d\nu(z)$$

$$\leq \int_{\mathbb{H}} \int_0^T |S(t-r)|_{\mathcal{L}(\mathbb{H})}^2 |\mathbb{I} - R^2(\lambda)|_{\mathcal{L}(\mathbb{H})}^2 |z|_{\mathbb{H}}^2 dr d\nu(z)$$

$$\leq c_2 \int_{\mathbb{H}} |z|_{\mathbb{H}}^2 d\nu(z) < \infty$$

for all $\lambda > \beta$ and

$$\lim_{\lambda\to\infty} |S(t-r)(\mathbb{I} - R^2(\lambda))z|_{\mathbb{H}} \leq \lim_{\lambda\to\infty} |S(t-r)|_{\mathcal{L}(\mathbb{H})} |(\mathbb{I} - R^2(\lambda))z|_{\mathbb{H}} = 0,$$

we obtain

$$\lim_{\lambda\to\infty} \mathbf{E} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)dL(r) - \int_0^t S(t-r)R(\lambda)dL_\lambda(r) \right|_{\mathbb{H}}^2 = 0.$$

Therefore, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)R(\lambda_n)dL_\lambda(r) - \int_0^t S(t-r)dL(r) \right|_{\mathbb{H}} = 0, \quad a.s.$$

It follows that

$$\left\{ \int_0^t S(t-r)R(\lambda_n)dL_\lambda(r), \quad t \in [0,T] \right\}_{n\in\mathbb{N}}$$

is a.s. uniformly bounded sequence of càdlàg functions which means

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)R(\lambda_n)dL_\lambda(r) \right|_{\mathbb{H}} < \infty, \quad a.s.$$

$\square$

In the following Lemma, we merge the steps (3) and (4) and prove the pathwise Itô formula for weak solutions.

Lemma 11. *Assume (**A1****), (**A2**)-(**A3**), (**A4****) and let $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{1-\epsilon})$ be non-negative and self-adjoint on $\mathbb{H}$. Assume further that there exists a continuous function $h : \mathbb{H} \to \mathbb{R}$ satisfying (3.11) extending the function $\langle A^*V\cdot, \cdot\rangle_{\mathbb{H}}$. Then for all $t \in [0,T]$ we have*

$$\left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}} - \langle x, Vx\rangle_{\mathbb{H}}$$

$$= 2\int_0^t h(X^U(s))ds + \int_0^t 2\left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}} ds$$

$$+ 2\int_0^t \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}} + \sum_{s \leq t} \left|V^{\frac{1}{2}}\Delta L(s)\right|_{\mathbb{H}}^2, \quad a.s.,$$

where $\Delta L(s) = L(s) - L(s_-)$, $s \geq 0$.

Proof. X_λ is the strong solution of (4.2) in virtue of Theorem 6. Therefore, we obtain (cf. Theorem D.2 in [25])

$$\left\langle X_\lambda^U(t), VX_\lambda^U(t)\right\rangle_{\mathbb{H}} - \langle x_\lambda, Vx_\lambda\rangle_{\mathbb{H}} = \int_0^t 2\left\langle VX_\lambda^U(s_-), dX_\lambda^U(s)\right\rangle_{\mathbb{H}}$$

$$+ \sum_{s \leq t}\left(\left\langle X_\lambda^U(s), VX_\lambda^U(s)\right\rangle_{\mathbb{H}} - \left\langle X_\lambda^U(s_-), VX_\lambda^U(s_-)\right\rangle_{\mathbb{H}} - 2\left\langle VX_\lambda^U(s_-), \Delta X_\lambda^U(s)\right\rangle_{\mathbb{H}}\right)$$

$$= \int_0^t 2\left\langle VX_\lambda^U(s_-), AX_\lambda^U(s)\right\rangle_{\mathbb{H}} ds + \int_0^t 2\left\langle VX_\lambda^U(s_-), R(\lambda)dL_\lambda(s)\right\rangle_{\mathbb{H}}$$

$$+ \int_0^t 2\left\langle VX_\lambda^U(s_-), R(\lambda)BU(s)\right\rangle_{\mathbb{H}} ds + \sum_{s \leq t}\left\langle \Delta X_\lambda^U(s), V\Delta X_\lambda^U(s)\right\rangle_{\mathbb{H}}$$

$$= \int_0^t 2\left\langle VX_\lambda^U(s), AX_\lambda^U(s)\right\rangle_{\mathbb{H}} ds + \int_0^t 2\left\langle \left(R^2(\lambda)\right)^* VX_\lambda^U(s_-), dL(s)\right\rangle_{\mathbb{H}}$$

$$+ \int_0^t 2\left\langle B^*R^*(\lambda)VX_\lambda^U(s), U(s)\right\rangle_{\mathbb{Y}} ds + \sum_{s \leq t}\left|V^{\frac{1}{2}}R^2(\lambda)\Delta L(s)\right|_{\mathbb{H}}^2.$$

We can prove in the same way as in Lemma 8 that there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and for all $t \in [0,T]$

$$\lim_{n\to\infty}\left\langle X_{\lambda_n}^U(t), VX_{\lambda_n}^U(t)\right\rangle_{\mathbb{H}} = \left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}},$$

$$\lim_{n\to\infty} h(X_{\lambda_n}^U(t)) = h(X^U(t)),$$

and

$$\lim_{n\to\infty}\left\langle B^*R^*(\lambda_n)VX_{\lambda_n}^U(s), U(s)\right\rangle_{\mathbb{Y}} = \left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}} \quad a.s..$$

Next we show that

$$\lim_{n\to\infty}\int_0^t h\left(X_{\lambda_n}^U(s)\right) + \left\langle B^*R^*(\lambda_n)VX_{\lambda_n}^U(s), U(s)\right\rangle_{\mathbb{Y}} ds$$

$$= \int_0^t h\left(X^U(s)\right) + \left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}} ds. \tag{4.6}$$

Indeed, by first part of Lemma 5 and continuity of h we have

$$\lim_{n\to\infty} h\left(X^U_{\lambda_n}(s)\right) + \left\langle B^* R^*(\lambda_n) V X^U_{\lambda_n}(s), U(s)\right\rangle_{\mathbb{Y}}$$

$$= h\left(X^U(s)\right) + \left\langle B^* V X^U(s), U(s)\right\rangle_{\mathbb{Y}}, \quad \text{a.s.}$$

for each $s \in (0, t)$. Hence (4.6) follows by the Dominated Convergence Theorem as

$$\left|h(X^U_{\lambda_n}(s))\right| + \left|\left\langle B^* R^*(\lambda_n) V X^U_{\lambda_n}(s), U(s)\right\rangle_{\mathbb{Y}}\right|$$

$$\leq k \sup_{n\in\mathbb{N}, r\in[0,T]} \left|X^U_{\lambda_n}(r)\right|^2_{\mathbb{H}} + |B^*|_{\mathcal{L}(\mathbb{D}^{1-\epsilon}_{A^*}, \mathbb{Y})} |R^*(\lambda_n)|_{\mathcal{L}(\mathbb{H})} |V|_{\mathcal{L}(\mathbb{H}, \mathbb{D}^{1-\epsilon}_{A^*})} |X^U_{\lambda_n}(s)|_{\mathbb{H}} |U(s)|_{\mathbb{Y}}$$

$$\leq c_1 \left(\sup_{n\in\mathbb{N}, r\in[0,T]} \left|X^U_{\lambda_n}(r)\right|^2_{\mathbb{H}} + \sup_{n\in\mathbb{N}, r\in[0,T]} |X^U_{\lambda_n}(r)|_{\mathbb{H}} |U(s)|_{\mathbb{Y}}\right)$$

for a constant $c_1 > 0$, which is an integrable majorant due to second part of Lemma 5.

Next,

$$\mathbf{E}\left|\int_0^t 2\left\langle \left(R^2(\lambda)\right)^* V X^U_\lambda(s_-), dL(s)\right\rangle_{\mathbb{H}} - \int_0^t 2\left\langle V X^U(s_-), dL(s)\right\rangle_{\mathbb{H}}\right|^2$$

$$\leq \mathbf{E}\left|\int_0^t 2\left\langle \left(R^2(\lambda) - \mathbb{I}\right)^* V X^U_\lambda(s_-), dL(s)\right\rangle_{\mathbb{H}}\right|^2$$

$$+ \mathbf{E}\left|\int_0^t 2\left\langle \left(R^2(\lambda)\right)^* V(X^U_\lambda(s_-) - X(s_-)), dL(s)\right\rangle_{\mathbb{H}}\right|^2$$

$$= \mathbf{E}\int_0^t 2\left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* \left(R^2(\lambda) - \mathbb{I}\right)^* V X^U(s)\right|^2_{\mathbb{H}} ds$$

$$+ \mathbf{E}\int_0^t 2\left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* (R(\lambda))^* V(X^U_\lambda(s) - X^U(s))\right|^2_{\mathbb{H}} ds$$

.

Obviously,

$$\lim_{\lambda\to\infty} \left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* \left(R^2(\lambda) - \mathbb{I}\right)^* V X^U(s)\right|_{\mathbb{H}}$$

$$\leq \lim_{\lambda\to\infty} \left|\mathcal{Q}^{\frac{1}{2}}\right|_{\mathcal{L}(\mathbb{H})} \left|\left(R^2(\lambda) - \mathbb{I}\right)^* V X^U(s)\right|_{\mathbb{H}} = 0$$

and

$$2\left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* \left(R^2(\lambda) - \mathbb{I}\right)^* V X^U(s)\right|^2_{\mathbb{H}}$$

$$\leq 4\left|\mathcal{Q}^{\frac{1}{2}}\right|^2_{\mathcal{L}(\mathbb{H})} (c^4 + 1) |V|^2_{\mathcal{L}(\mathbb{H})} |X^U(s)|^2_{\mathbb{H}},$$

therefore,

$$\lim_{\lambda\to\infty} \mathbf{E}\int_0^t 2\left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* \left(R^2(\lambda) - \mathbb{I}\right)^* V X^U(s)\right|^2_{\mathbb{H}} ds = 0.$$

Moreover,

$$\lim_{\lambda\to\infty} \mathbf{E}\int_0^t 2\left|\left(\mathcal{Q}^{\frac{1}{2}}\right)^* \left(R^2(\lambda)\right)^* V(X^U_\lambda(s) - X^U(s))\right|^2_{\mathbb{H}} ds$$

$$\leq \lim_{\lambda\to\infty} 2t \left|\mathcal{Q}^{\frac{1}{2}}\right|^2_{\mathcal{L}(\mathbb{H})} c^4 |V|^2_{\mathcal{L}(\mathbb{H})} \mathbf{E} \sup_{s\in[0,t]} |X^U_\lambda(s) - X^U(s)|^2_{\mathbb{H}} ds = 0$$

as in the proof of Lemma 5. As a consequence we obtain

$$\lim_{\lambda\to\infty} \mathbf{E} \left| \int_0^t 2 \left\langle \left(R^2(\lambda) \right)^* V X_\lambda^U(s_-), dL(s) \right\rangle_{\mathbb{H}} - \int_0^t 2 \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}} \right|^2 = 0.$$

Thus we can find a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and

$$\lim_{n\to\infty} \int_0^t 2 \left\langle \left(R^2(\lambda_n) \right)^* V X_{\lambda_n}^U(s_-), dL(s) \right\rangle_{\mathbb{H}} = \int_0^t 2 \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}}, \quad \text{a.s.}$$

Finally for all $z \in \mathbb{H}$

$$\lim_{\lambda\to\infty} \left| \left\langle V R^2(\lambda) z, R^2(\lambda) z \right\rangle_{\mathbb{H}} - \langle V z, z \rangle_{\mathbb{H}} \right|$$

$$= \lim_{\lambda\to\infty} \left| \left\langle \left(R^2(\lambda) - \mathbb{I} \right) z, V R^2(\lambda) z \right\rangle_{\mathbb{H}} - \left\langle \left(\mathbb{I} - R^2(\lambda) \right) z, V z \right\rangle_{\mathbb{H}} \right|$$

$$\leq \lim_{\lambda\to\infty} \left(\left| \left\langle \left(R^2(\lambda) - \mathbb{I} \right) z, V R^2(\lambda) z \right\rangle_{\mathbb{H}} \right| + \left| \left\langle \left(\mathbb{I} - R^2(\lambda) \right) z, V z \right\rangle_{\mathbb{H}} \right| \right)$$

$$\leq \lim_{\lambda\to\infty} \left| (R^2(\lambda) - \mathbb{I}) z \right|_{\mathbb{H}} |V|_{\mathcal{L}(\mathbb{H})} \left| R^2(\lambda) \right|_{\mathcal{L}(\mathbb{H})} |z|_{\mathbb{H}} + \left| \left(\mathbb{I} - R^2(\lambda) \right) z \right|_{\mathbb{H}} |V|_{\mathcal{L}(\mathbb{H})} |z|_{\mathbb{H}}$$

$$\leq \lim_{\lambda\to\infty} \left| \left(R^2(\lambda) - \mathbb{I} \right) z \right|_{\mathbb{H}} |V|_{\mathcal{L}(\mathbb{H})} (c^2 + 1) |z|_{\mathbb{H}} = 0$$

and

$$\lim_{\lambda\to\infty} \mathbf{E} \left| \int_0^t \int_{\mathbb{H}} \left\langle V R^2(\lambda) z, R^2(\lambda) z \right\rangle_{\mathbb{H}} N(ds, dz) - \int_0^t \int_{\mathbb{H}} \langle V z, z \rangle_{\mathbb{H}} N(ds, dz) \right|$$

$$\leq \lim_{n\to\infty} t \int_{\mathbb{H}} \left| \left\langle V R^2(\lambda) z, R^2(\lambda) z \right\rangle_{\mathbb{H}} - \langle V z, z \rangle_{\mathbb{H}} \right| d\nu(z) = 0$$

by Dominated Convergence Theorem.

Hence there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and a.s.

$$\lim_{n\to\infty} \sum_{s\leq t} \left| V^{\frac{1}{2}} R^2(\lambda_n) \Delta L(s) \right|_{\mathbb{H}}^2 = \lim_{n\to\infty} \int_0^t \int_{\mathbb{H}} \left\langle V R^2(\lambda_n) z, R^2(\lambda_n) z \right\rangle_{\mathbb{H}} N(ds, dz)$$

$$= \int_0^t \int_{\mathbb{H}} \langle V z, z \rangle_{\mathbb{H}} N(ds, dz) = \sum_{s\leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 \quad \text{a.s.,}$$

which completes the proof. $\qquad\square$

5. Optimal control results

5.1 Mean value optimal control result

As in the previous chapter, we start with the equation (2.1) with pure jump cylindrical Lévy process with weak second moments. As we mentioned, the key tool in the proofs is the Itô formula. In the case of cylindrical Lévy process, we have only the Itô formula in mean value sense (Lemma 8) and we prove only the mean value optimal control for the ergodic problem (3.4)-(3.5). For such ergodic problem, we assume that (**A1**)-(**A4**), the standard stabilizability and detectability conditions (Definitions 1 and 2) and the assumptions of Proposition 9 are satisfied.

We consider a process

$$K(\omega, \cdot) : \mathbb{R} \to \mathcal{L}(\mathbb{H}, \mathbb{Y})$$

that is progresively measurable, **P**-a.s. continuous in the operator topology and such that

$$\lim_{t \to \infty} K(t) = K_0 = -R^{-1}B^*V$$

P-a.s. in $\mathcal{L}(\mathbb{H}, \mathbb{Y})$, where V is the solution to the Riccati equation (3.9).

The main result for the equation (2.1) and the ergodic problem (3.7) and (3.8) is stated below.

Theorem 12. *Assume (**A1**)-(**A4**), (4.1) and stabilizability and detectability conditions from Definitions 1, 2, let K be deterministic. Then the feedback control*

$$U_0(t) = K(t)X(t), \quad t \geq 0,$$

is an optimal control for the ergodic problem (3.4)-(3.5) in the class of all controls from the space $\mathcal{U}$ satisfying

$$\frac{\mathbf{E}\left\langle VX^U(t), X^U(t)\right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty. \tag{5.1}$$

The optimal cost is $D = \Pi$.

The three steps of the proof of the theorem Theorem 12 can be reformulated as follows:

1. We have that

$$\liminf_{t \to \infty} \frac{\mathbf{E}J(U, t)}{t} \geq \Pi \tag{5.2}$$

    for all controls U from the space $\mathcal{U}$ satisfying (5.1).

2. The feedback control U_0 satisfies (5.1).

3. The feedback control U_0 is an optimal control for the ergodic problem (3.4) - (3.5), that is

$$\lim_{t \to \infty} \frac{\mathbf{E}J(U_0, t)}{t} = \Pi. \tag{5.3}$$

We prove these three facts in three separate lemmas.

Lemma 13. *Let V be the solution of the Riccati equation (3.9). Then (5.2) holds true.*

Proof. Using Lemma 8 we obtain

$$\mathbf{E}\left\langle X^U(t), VX^U(t)\right\rangle_{\mathbb{H}} - \langle x, Vx\rangle_{\mathbb{H}}$$

$$= 2\mathbf{E}\int_0^t h(X^U(s))ds + \mathbf{E}\int_0^t 2\left\langle B^*VX^U(s), U(s)\right\rangle_{\mathbb{Y}}ds + t\Pi.$$

We have, using (3.9),

$$\mathbf{E}\frac{\left\langle X^U(s), VX^U(s)\right\rangle_{\mathbb{H}} - \langle x, Vx\rangle_{\mathbb{H}}}{t}$$

$$= -\mathbf{E}\frac{\int_0^t \left(\left\langle X^U(s), QX^U(s)\right\rangle_{\mathbb{H}} + \langle U(s), RU(s)\rangle_{\mathbb{Y}}\right)ds}{t} + \Pi$$

$$+\mathbf{E}\frac{\int_0^t \left\langle X^U(s), VBR^{-1}B^*VX^U(s)\right\rangle_{\mathbb{H}}ds}{t}$$

$$+\frac{2\int_0^t \left(\left\langle X^U(s), VBU(s)\right\rangle_{\mathbb{H}} + \langle U(s), RU(s)\rangle_{\mathbb{Y}}\right)ds}{t}$$

$$= -\mathbf{E}\frac{J(U,t)}{t} + \Pi$$

$$+\mathbf{E}\frac{\int_0^t \left\langle R^{-1}B^*VX^U(s) + U(s), R(R^{-1}B^*VX^U(s) + U(s))\right\rangle_{\mathbb{Y}}ds}{t}.$$

Since

$$\frac{\mathbf{E}\int_0^t \left\langle R^{-1}B^*VX^U(s) + U(s), R(R^{-1}B^*VX^U(s) + U(s))\right\rangle_{\mathbb{Y}}ds}{t} \geq 0, \quad t \geq 0,$$

we obtain (5.2) by (5.1). $\qquad\square$

As the next step, in the following lemma we prove that the feedback control U_0 satisfies (5.1), that is U_0 is in the set of controls where we want to find the optimal one.

Lemma 14. *We have that*

$$\frac{\mathbf{E}\left\langle X^{U_0}(t), VX^{U_0}(t)\right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty. \tag{5.4}$$

Proof. Take $\epsilon^* > 0$ such that

$$\theta = \theta_{\epsilon^*} = (3\epsilon^*\Gamma(\epsilon))^{\frac{1}{\epsilon}} < \omega.$$

Since $K(t)$ converges to K_0 in the uniform operator topology as $t \to \infty$, we can find $r > 0$ such that for all $s \geq r$:

$$|K(s) - K_0|^2_{\mathcal{L}(\mathbb{H},\mathbb{Y})} < \epsilon^*. \tag{5.5}$$

Using the Proposition 3, we can see that X^{U_0} is the mild solution of

$$dX^{U_0}(t) = \left(AX^{U_0}(t) + BK(t)X^{U_0}(t)\right)dt + \Phi dL(t), \quad X^{U_0}(0) = x, \qquad (5.6)$$

iff X^{U_0} satisfies the mild formula

$$X^{U_0}(t) = S_V(t-r)X^{U_0}(r)$$

$$+ \int_r^t S_V(t-s)B(K(t)-K_0)X^{U_0}(s)ds + \int_r^t S_V(t-s)\Phi dL(s), \quad t \geq r, \qquad (5.7)$$

(recall the notation in (3.10)). By (3.10) we obtain

$$\mathbf{E}\left\langle X^{U_0}(t), VX^{U_0}(t)\right\rangle_{\mathbb{H}} \leq \mathbf{E}|X^{U_0}(t)|_{\mathbb{H}}^2 |V|_{\mathcal{L}(\mathbb{H})}$$

$$\leq 3|V|_{\mathcal{L}(\mathbb{H})}|S_V(t-r)|_{\mathcal{L}(\mathbb{H})}^2 \mathbf{E}|X^{U_0}(r)|_{\mathbb{H}}^2$$

$$+3|V|_{\mathcal{L}(\mathbb{H})}\mathbf{E}\left(\int_r^t |S_V(t-s)B|_{\mathcal{L}(\mathbb{Y},\mathbb{H})}|K(s)-K_0|_{\mathcal{L}(\mathbb{H},\mathbb{Y})}\left|X^{U_0}(s)\right|_{\mathbb{H}}ds\right)^2$$

$$+3\mathbf{E}\left|\int_r^t S_V(t-s)\Phi dL(s)\right|_{\mathbb{H}}^2$$

$$\leq 3M_0 e^{-\omega(t-r)}\mathbf{E}\left|X^{U_0}(r)\right|_{\mathbb{H}}^2 + 3\int_r^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\epsilon}}|K(s)-K_0|_{\mathcal{L}(\mathbb{H},\mathbb{Y})}^2 \mathbf{E}\left|X^{U_0}(s)\right|_{\mathbb{H}}^2 ds + c_1$$

$$\leq c_2 + 3\epsilon^* \int_r^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\epsilon}}\mathbf{E}\left|X^{U_0}(s)\right|_{\mathbb{H}}^2 ds + c_1$$

for some constants c_1 and c_2 dependent only on r.

Following the same steps of as in the proof of Lemma 7.1.1. in [16], we obtain:

$$\mathbf{E}\left|X^{U_0}(t)\right|_{\mathbb{H}}^2 \leq c_2 + c_3 \int_r^t E'(\theta(t-s))e^{-\omega(t-s)}ds + c_1,$$

for a constant c_3, where $E'(z)$ is asymptotically

$$\frac{z^{\epsilon-1}}{\Gamma(\epsilon)}, \quad z \to 0_+,$$

$$\frac{e^z}{\epsilon}, \quad z \to \infty.$$

Therefore we can find $t_0 > r$ such that for all $t > t_0$:

$$\mathbf{E}\left|X^{U_0}(t)\right|_{\mathbb{H}}^2 \leq c_4 \int_r^t \frac{e^{(\theta-\omega)(t-s)}}{(t-s)^{1-\epsilon}}ds + c_5 < c_6 \qquad (5.8)$$

for some constants c_4, c_5 and c_6 dependent only on r. Therefore

$$\frac{\mathbf{E}\left\langle X^{U_0}(t), VX^{U_0}(t)\right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty,$$

which concludes the proof. $\qquad \square$

It remains to show that U_0 is the optimal control for this ergodic problem. This last step is formulated now.

Lemma 15. *Let the assumptions of Theorem 12 be satisfied. Then*

$$\lim_{t \to \infty} \frac{\mathbf{E} J(U_0, t)}{t} = \Pi. \tag{5.9}$$

Proof. Recall that $U_0(t) = K(t)X(t)$ where $K(t) \to K_0$ in $\mathcal{L}(\mathbb{H}, \mathbb{Y})$. Following the steps of the proof of Lemma 13, we obtain

$$\mathbf{E} \frac{\left\langle X^{U_0}(s), VX^{U_0}(s) \right\rangle_{\mathbb{H}} - \left\langle x, Vx \right\rangle_{\mathbb{H}}}{t}$$

$$= -\mathbf{E} \frac{J(U_0, t)}{t} + \Pi +$$

$$\frac{\mathbf{E} \int_0^t \left\langle -KX^{U_0}(s) + K(s)X^{U_0}(s), R(-KX^{U_0}(s) + K(s)X^{U_0}(s)) \right\rangle_{\mathbb{H}} ds}{t}.$$

By (5.8) it is easy to see that

$$\lim_{t \to \infty} \frac{\mathbf{E} \int_0^t \left\langle -KX^{U_0}(s) + K(s)X^{U_0}(s), R(-KX^{U_0}(s) + K(s)X^{U_0}(s)) \right\rangle_{\mathbb{H}} ds}{t}$$

$$= \lim_{t \to \infty} c_1 \frac{\mathbf{E} \int_0^t |K - K(s)|^2_{\mathcal{L}(\mathbb{H}, \mathbb{Y})} \left| X^{U_0}(s) \right|^2_{\mathbb{H}} ds}{t} = 0.$$

Hence in virtue of Lemma 14 we obtain (5.3). $\qquad\square$

Let us remind that in the case of the equation (2.3), the assumptions (**A1**)-(**A4**) as well as the assumptions of Proposition 9 are satisfied. Therefore, for the equation (2.3), the optimal cost is $\Pi = Tr(V\Phi\Phi^*)$ and the optimal control is U_0 for the ergodic problem (3.4)-(3.5).

5.2 Path-wise optimal control result

In the previous chapter, we proved the path-wise Itô lemma 11 for the equation (2.3). Using this version of the Itô lemma, we can obtain a path-wise result for the equation (2.3) if we add the assumption of strict positivity of the operator Q in (3.6). Therefore, in the rest if this chapter, we assume Q in (3.6) to be the positive operator, which means that

$$\langle Qy, y \rangle \geq s|y|^2, \quad y > 0, \tag{5.10}$$

holds for a constant $s > 0$. Moreover, suppose (**A1****), (**A2**)-(**A3**) and (**A4****) are fulfilled. Recall the operator-valued process K introduced in the section 5.1.

The main result for the equation (2.3) and the control problem (3.7)-(3.8) follows.

Theorem 16. *Assume ($\boldsymbol{A1}$)-($\boldsymbol{A4}$), (4.1) and stabilizability and detectability conditions from Definitions 1, 2. Then the feedback control $U_0(t) = KX(t)$, $t \geq 0$, is an optimal control for the ergodic problem (3.7), (3.8) in the class of all controls from the space $\mathcal{U}$ satisfying*

$$\limsup_{t \to \infty} \frac{\int_0^t \left| X^U(s) \right|_{\mathbb{H}}^2 ds}{t} < \infty, \quad a.s. \tag{5.11}$$

and

$$\frac{\left\langle V X^U(t), X^U(t) \right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \quad a.s. \tag{5.12}$$

The optimal cost is $D = Tr(V\mathcal{Q})$.

In this case, the three steps of the proof of the theorem 12 have the following form:

1. We have that

$$\liminf_{t \to \infty} \frac{\mathbf{E}J(U,t)}{t} \geq Tr(V\mathcal{Q}) \tag{5.13}$$

   for all controls U from the space $\mathcal{U}$ satisfying (5.11) and (5.12). This will be proved in the Lemma 19.

2. The feedback control $U_0(t)$, $t \geq 0$, satisfies (5.11) and (5.12). These two assumptions will be verified in the Lemmas 20 and 21 respectively.

3. The feedback control $U_0(t)$, $t \geq 0$, is an optimal control for the ergodic problem (3.7) - (3.8), that is

$$\lim_{t \to \infty} \frac{J(U_0,t)}{t} = Tr(V\mathcal{Q}). \tag{5.14}$$

   This step will be proved in the Lemma 22.

In the rest of the section 5.2 the assumptions of Theorem 16 are supposed to hold.

Before proving this final result, the following two Lemmas will be useful.

Lemma 17. *Suppose that (5.11) holds. Then we have a.s.*

$$\frac{\int_0^t \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty. \tag{5.15}$$

Proof.

$$\frac{\int_0^t \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t}$$

$$= \frac{\int_0^t \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}}}{\left\langle \int_0^{\cdot} \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}} \right\rangle_t} \frac{\left\langle \int_0^{\cdot} \left\langle V X^U(s_-), dL(s) \right\rangle_{\mathbb{H}} \right\rangle_t}{t}$$

$$\times \mathbf{I}_{\left[\lim_{t \to \infty} \left\langle \int_0^t \langle V X^U(s_-), dL(s) \rangle_{\mathbb{H}} \right\rangle_t = \infty \right]}$$

$$+\frac{\int_0^t \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}}}{t}\mathbf{I}_{[\forall t\in\mathbb{R}_+:\left\langle\int_0^\cdot \langle VX^U(s_-),dL(s)\rangle_{\mathbb{H}}\right\rangle_t\leq C]},$$

where C depends only on $\omega \in \Omega$ and $\langle\cdot\rangle$ denotes the quadratic variation. Now we compute

$$\frac{\left\langle\int_0^\cdot \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}}\right\rangle_t}{t} = \frac{\int_0^t \left|(\mathcal{Q}^{\frac{1}{2}})^* VX^U(s_-)\right|_{\mathbb{H}}^2 ds}{t}$$

$$\leq \left|(\mathcal{Q}^{\frac{1}{2}})^*\right|_{\mathcal{L}(\mathbb{H})} |V|_{\mathcal{L}(\mathbb{H})} \frac{\int_0^t \left|X^U(s)\right|_{\mathbb{H}}^2 ds}{t},$$

which converges by (5.11). Since $\int_0^\cdot \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}}$ is a càdlàg martingale, we obtain

$$\lim_{t\to\infty} \frac{\int_0^t \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}}}{\left\langle\int_0^\cdot \langle VX^U(s_-), dL(s)\rangle_{\mathbb{H}}\right\rangle_t}\mathbf{I}_{[\lim_{t\to\infty}\left\langle\int_0^\cdot\langle VX^U(s_-),dL(s)\rangle_{\mathbb{H}}\right\rangle_t=\infty]} = 0$$

a.s. (cf. [28] or [22]). Moreover, it is well known that

$$\limsup_{t\to\infty} \frac{\int_0^t \left\langle VX^U(s_-), dL(s)\right\rangle_{\mathbb{H}}\mathbf{I}_{[\forall t\in\mathbb{R}_+:\left\langle\int_0^\cdot\langle VX^U(s_-),dL(s)\rangle_{\mathbb{H}}\right\rangle_t\leq C]}}{t} = 0,$$

which concludes the proof. $\qquad\square$

Lemma 18. *We have*

$$\frac{\sum_{s\leq t}\left|V^{\frac{1}{2}}\Delta L(s)\right|_{\mathbb{H}}^2}{t} - Tr(V\mathcal{Q}) \to 0, \quad t\to\infty, \quad a.s. \qquad (5.16)$$

Proof. We can easily see that for all $s,t\in\mathbb{R}_+$

$$\sum_{r\leq s}\left|V^{\frac{1}{2}}\Delta L(r)\right|_{\mathbb{H}}^2$$

and

$$\sum_{s<r\leq s+t}\left|V^{\frac{1}{2}}\Delta L(r)\right|_{\mathbb{H}}^2$$

are independent,

$$\sum_{r\leq t}\left|V^{\frac{1}{2}}\Delta L(r)\right|_{\mathbb{H}}^2$$

and

$$\sum_{s\leq r\leq s+t}\left|V^{\frac{1}{2}}\Delta L(r)\right|_{\mathbb{H}}^2$$

are equally distributed and

$$\mathbf{E}\sum_{r\leq t}\left|V^{\frac{1}{2}}\Delta L(r)\right|_{\mathbb{H}}^2 = tTr(V\mathcal{Q}) < \infty.$$

Therefore, using the law of large numbers for processes with stationary independent increments (c.f. Theorem 2.1 in [27]), we obtain (5.16). $\qquad\square$

Lemma 19. *Let V be the solution of the Riccati equation (3.9). Let $U \in \mathcal{U}$ be arbitrary such that (5.11)-(5.12) are satisfied. Then we have*

$$\liminf_{t \to \infty} \frac{J(U,t)}{t} \geq Tr(V\mathcal{Q}), \quad a.s. \tag{5.17}$$

Proof. By Lemma 11,

$$\left\langle X^U(t), VX^U(t) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}$$

$$= 2 \int_0^t h(X^U(s)) + \int_0^t 2 \left\langle B^*VX^U(s), U(s) \right\rangle_{\mathbb{Y}}$$

$$+ 2 \int_0^t \left\langle VX^U(s_-), dL(t) \right\rangle_{\mathbb{H}}$$

$$+ \left(\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - t Tr(V\mathcal{Q}) \right) + t Tr(V\mathcal{Q})$$

Then we have, using (3.9),

$$\frac{\left\langle X^U(t), VX^U(t) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}}{t}$$

$$= -\frac{\int_0^t \left(\left\langle X^U(s), QX^U(s) \right\rangle_{\mathbb{H}} + \langle U(s), RU(s) \rangle_{\mathbb{Y}} \right) ds}{t} + Tr(V\mathcal{Q})$$

$$+ \frac{\int_0^t \left\langle X^U(s), VBR^{-1}B^*VX^U(s) \right\rangle_{\mathbb{H}} ds}{t}$$

$$+ \frac{2 \int_0^t \left(\left\langle B^*VX^U(s), U(s) \right\rangle_{\mathbb{Y}} + \langle U(s), RU(s) \rangle_{\mathbb{Y}} \right) ds}{t}$$

$$+ \frac{\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - t Tr(V\mathcal{Q})}{t}$$

$$+ 2 \frac{\int_0^t \left\langle VX^U(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t}.$$

Hence,

$$\frac{\left\langle X^U(t), VX^U(t) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}}{t}$$

$$= -\frac{J(U,t)}{t} + Tr(V\mathcal{Q})$$

$$+ \frac{\int_0^t \left\langle R^{-1}B^*VX^U(s) + U(s), R(R^{-1}B^*VX^U(s) + U(s)) \right\rangle_{\mathbb{Y}} ds}{t}$$

$$+ \frac{\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - t Tr(V\mathcal{Q})}{t}$$

$$+ \frac{\int_0^t \left\langle VX^U(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t}. \tag{5.18}$$

Since

$$\frac{\int_0^t \left\langle R^{-1}B^*VX(s) - U(s), R(R^{-1}B^*VX^U(s) - U(s)) \right\rangle_{\mathbb{Y}} ds}{t} \geq 0, \quad t \geq 0,$$

it suffices to use (5.12) on the left side and Lemmas 17 and 18 on the right side of the equality (5.18). □

Lemma 20. *We have that*

$$\limsup_{t \to \infty} \frac{\int_0^t \left| X^{U_0}(s) \right|_{\mathbb{H}}^2 ds}{t} < \infty, \quad a.s. \tag{5.19}$$

Proof. Since Q and R^{-1} are strictly positive definite and

$$\left(\mathcal{Q}^{\frac{1}{2}} \right)^* \text{ and } V$$

are bounded, Lemma 11 yields

$$\left\langle X^{U_0}(t), VX^{U_0}(t) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}$$

$$= 2 \int_0^t h(X^{U_0}(s)) ds + 2 \int_0^t \left\langle B^*VX^{U_0}(s), U_0(s) \right\rangle_{\mathbb{Y}} ds + Tr(V\mathcal{Q})$$

$$+ \left(\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - tTr(V\mathcal{Q}) \right)$$

$$+ \int_0^t \left\langle VX^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}}$$

$$= 2 \int_0^t \left\langle B^*VX^{U_0}(s), (K(s) - K_0)X^{U_0}(s) \right\rangle_{\mathbb{Y}} ds - \int_0^t \left\langle X^{U_0}(s), QX^{U_0}(s) \right\rangle_{\mathbb{H}} ds$$

$$- \int_0^t \left\langle B^*VX^{U_0}(s), R^{-1}B^*VX^{U_0}(s) \right\rangle_{\mathbb{Y}} ds + tTr(V\mathcal{Q})$$

$$+ \left(\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - tTr(V\mathcal{Q}) \right)$$

$$+ \int_0^t \left\langle VX^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}}$$

$$\leq -c_0 \int_0^t \left| X^{U_0}(s) \right|_{\mathbb{H}}^2 ds + \int_0^t \left(-c_1 + c_2 |K(s) - K_0|_{\mathcal{L}(\mathbb{Y}, \mathbb{H})} \right) \left| X^{U_0}(s) \right|_{\mathbb{H}}^2 ds$$

$$- 2 \int_0^t \left| \left(\mathcal{Q}^{\frac{1}{2}} \right)^* VX^{U_0}(s) \right|_{\mathbb{H}}^2 ds \left(c_2 - \frac{\int_0^t \left\langle VX^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}}}{\langle \int_0^{\cdot} \left\langle VX^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}} \rangle_t} \right)$$

$$+ \left(\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - tTr(V\mathcal{Q}) \right) + tTr(V\mathcal{Q}),$$

where

$$\limsup_{t\to\infty} \frac{c_1 \int_0^t \left(-c_1 + c_2|K(s) - K_0|_{\mathcal{L}(\mathbb{Y},\mathbb{H})}\right) \left|X^{U_0}(s)\right|_{\mathbb{H}}^2 ds}{t} \leq 0, \quad \text{a.s..}$$

Similarly as in the proof of Lemma 17 we obtain

$$\limsup_{t\to\infty} \frac{\int_0^t \left|\left(Q^{\frac{1}{2}}\right)^* V X^{U_0}(s)\right|_{\mathbb{H}} ds}{t} \left(-c_2 + \frac{\int_0^t \left\langle V X^{U_0}(s_-), dL(s)\right\rangle_{\mathbb{H}}}{\langle \int_0^{\cdot} \left\langle V X^{U_0}(s_-), dL(s)\right\rangle_{\mathbb{H}}\rangle_t}\right) \leq 0, \quad \text{a.s.}$$

Therefore by Lemma 18 we arrive at

$$c_1 \limsup_{t\to\infty} \frac{\int_0^t \left|X^{U_0}(s)\right|_{\mathbb{H}}^2 ds}{t}$$

$$\leq Tr(VQ) - \liminf_{t\to\infty} \frac{\left\langle X^{U_0}(s), V X^{U_0}(s)\right\rangle_{\mathbb{H}}}{t} \leq Tr(VQ), \quad \text{a.s.,}$$

which completes the proof since V is non-negative. $\qquad\square$

Lemma 21. *We have that*

$$\frac{\left\langle X^{U_0}(t), V X^{U_0}(t)\right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \quad a.s. \tag{5.20}$$

Proof. In virtue of Lemma 11, positivity of Q and R^{-1} and boundedness of V, we obtain for all $t \geq 0$:

$$\left\langle X^{U_0}(t), V X^{U_0}(t)\right\rangle_{\mathbb{H}} - \left\langle X^{U_0}(r), V X^{U_0}(r)\right\rangle_{\mathbb{H}}$$

$$= 2\int_r^t \left\langle B^* V X^{U_0}(s), (K(s) - K_0)X^{U_0}(s)\right\rangle_{\mathbb{Y}} ds - \int_r^t \left\langle X^{U_0}(s), Q X^{U_0}(s)\right\rangle_{\mathbb{H}} ds$$

$$- \int_r^t \left\langle B^* V X^{U_0}(s), R^{-1}B^* V X^{U_0}(s)\right\rangle_{\mathbb{Y}} ds + t Tr(VQ)$$

$$+ \left(\sum_{s\leq t}\left|V^{\frac{1}{2}}\Delta L(s)\right|_{\mathbb{H}}^2 - t Tr(VQ)\right)$$

$$+ \int_r^t \left\langle V X^{U_0}(s_-), dL(s)\right\rangle_{\mathbb{H}}$$

$$\leq \int_r^t \left(-c_1 + c_2|K(s) - K_0|_{\mathcal{L}(\mathbb{Y},\mathbb{H})}\right)|X^{U_0}(s)|_{\mathbb{H}}^2 ds + c_3(t - r) + M_r(t)$$

$$\leq -c_3 \int_r^t \left\langle X^{U_0}(s), V X^{U_0}(s)\right\rangle_{\mathbb{H}} ds + c_3(t - r) + M_r(t),$$

$r \geq t_0$, for some $t_0 > 0$, where

$$M_r(t) = \sum_{r\leq s\leq t}\left|V^{\frac{1}{2}}\Delta L(s)\right|_{\mathbb{H}}^2 - (t - r)Tr(VQ) + \int_r^t \left\langle V X^{U_0}(s_-), dL(s)\right\rangle_{\mathbb{H}}, \quad t \geq t_0,$$

has càdlàg trajectories, $c_1, c_2, c_3, c_4 > 0$. We fix $\omega \in \Omega_0$ where

$$\Omega_0 = \left\{ \omega \in \Omega; \ \lim_{t \to \infty} \frac{M_{t_0}(t)(\omega)}{t} = 0 \right\} \tag{5.21}$$

(note that due to Lemmas 20, 17 and 18 we have that $\mathbf{P}(\Omega_0) = 1$) and define

$$y_\omega(t) = \left\langle X^{U_0}(t)(\omega), V X^{U_0}(t)(\omega) \right\rangle_{\mathbb{H}}$$

for $t \geq t_0$. By (5.21), we can prove similarly as in [5] that

$$\lim_{t \to \infty} \frac{y_\omega(t)}{t} = 0$$

and (5.20) follows. $\qquad \square$

Lemma 22. *We have that (5.14) holds.*

Proof. As

$$K_0 = -R^{-1} B^* V,$$

we obtain by Lemma 11 and (3.9):

$$\frac{\left\langle X^{U_0}(s), V X^{U_0}(s) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}}{t}$$

$$= 2 \frac{\int_0^t \left\langle B^* V X^{U_0}(s), K(s) X^{U_0}(s) \right\rangle_{\mathbb{Y}} ds}{t}$$

$$- \frac{\int_0^t \left\langle X^{U_0}(s), Q X^{U_0}(s) \right\rangle_{\mathbb{H}} ds + \int_0^t \langle K(s) X(s), R K(s) X(s) \rangle_{\mathbb{Y}} ds}{t}$$

$$+ \frac{\int_0^t \left\langle K(s) X^{U_0}(s), R K(s) X^{U_0}(s)(s) \right\rangle_{\mathbb{Y}} ds}{t} + \frac{\int_0^t \left\langle K_0 X^{U_0}(s), R K_0 X^{U_0}(s)(s) \right\rangle_{\mathbb{Y}} ds}{t}$$

$$+ Tr(V\mathcal{Q}) + \frac{\left(\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - t Tr(V\mathcal{Q}) \right)}{t}$$

$$+ 2 \frac{\int_0^t \left\langle V X^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t}$$

$$= -\frac{J(U_0, t)}{t} + Tr(V\mathcal{Q})$$

$$+ \frac{\int_0^t \langle (K(s) - K_0) X(s), R (K(s) - K_0) X(s) \rangle_{\mathbb{Y}} ds}{t}$$

$$+ \frac{\sum_{s \leq t} \left| V^{\frac{1}{2}} \Delta L(s) \right|_{\mathbb{H}}^2 - t Tr(V\mathcal{Q})}{t}$$

$$+ 2 \frac{\int_0^t \left\langle V X^{U_0}(s_-), dL(s) \right\rangle_{\mathbb{H}}}{t}.$$

We can simply prove that

$$\lim t \to \infty \frac{\int_0^t \left\langle (K(s) - K_0)X^{U_0}(s), R(K(s) - K_0)X^{U_0}(s) \right\rangle_{\mathbb{Y}} ds}{t}$$

$$\leq c_1 \lim t \to \infty \frac{\int_0^t |K(s) - K_0|^2_{\mathcal{L}(\mathbb{H}, \mathbb{Y})} \left| X^{U_0}(s) \right|^2_{\mathbb{H}} ds}{t} = 0.$$

Due to Lemma 21 left-hand side of the equality tends to zero a.s. as $t \to \infty$. Similarly, the last two terms on the right-hand side converge to zero in virtue of Lemmas 20, 17 and 18, and (5.13) follows. $\qquad \square$

Remark. This section is focused on the square integrable Lévy proces. If we assume the Lévy proces with finite p-th moments, $p > 2$, we can extend the path-wise result in Theorem 16 to the equations with $\epsilon \in (\frac{1}{p}, 1]$ in the assumption (**A2**) on the operator B. But the optimality is achieved on the smaller set of admissible controls which are in addition in $\mathbb{L}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$.

More precisely, as in Theorem 16 assume (**A1****), (**A2**)-(**A3**) and (**A4****). Moreover suppose that L has finite p-th moments, $p > 2$, and replace the assumptions (**A2**)-(**A3**) of Theorem 16 by the assumptions

(**A2*****) $B : \mathbb{D}(B) \subset \mathbb{Y} \to \mathbb{D}(A^*)'$, the dual of $\mathbb{D}(A^*)$ with respect to the topology of $\mathbb{H}$, and $(\beta \mathbb{I} - A)^{\epsilon-1} B \in \mathcal{L}(\mathbb{Y}, \mathbb{H})$, for a given $\epsilon \in (\frac{1}{p}, 1]$.

(**A3*****) We have that
$$U \in \mathbb{L}^{p,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{Y}),$$
where $\mathbb{L}^{p,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{Y})$ is the space of all $\mathcal{F}$-progressively measurable processes from $\mathbb{L}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$, where
$$\mathbb{L}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$$
$$= \left\{ Y : \mathbb{R}_+ \times \Omega \to \mathbb{Y} \text{ measurable}; \ \forall t > 0 : \mathbf{E} \int_0^t |Y(s)|^p_{\mathbb{Y}} ds < \infty \right\}.$$
and denote $\mathcal{U}_p = \mathbb{L}^{p,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{Y})$ the space of admissible controls.

Then the feedback control $U_0(t) = KX(t)$, $t \geq 0$, is an optimal control for the ergodic problem (3.7), (3.8) in the class of all controls from the space $\mathcal{U}_p$ satisfying

$$\limsup_{t \to \infty} \frac{\int_0^t \left| X^U(s) \right|^2_{\mathbb{H}} ds}{t} < \infty, \quad \text{a.s.} \tag{5.22}$$

and

$$\frac{\left\langle V X^U(t), X^U(t) \right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \quad a.s. \tag{5.23}$$

The optimal cost is $D = Tr(V\mathcal{Q})$.

Indeed, following the steps of the proof of Lemma 2.1. in [5] and applying the assumptions ((**A2*****)) and ((**A3*****)), we obtain

$$\int_0^t S(t - r)BU(r)dr \in \mathbb{L}^{p,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{H}).$$

Furthermore, we can prove for the $\mathbb{L}^p$ Lévy proces L that

$$\int_0^t S(t-r)dL(r) \in \mathbb{L}_\mathcal{F}^{p,loc}(\mathbb{R}_+, \mathbb{H}).$$

Therefore,

$$X^U \in \mathbb{L}_\mathcal{F}^{p,loc}(\mathbb{R}_+, \mathbb{H}).$$

Then we can repeat the steps of the proof of Theorem 16.

5.3 Path-wise adaptive control

In this section, we apply the above path-wise control result to adaptive control of parameter-dependent systems with an unknown parameter $\alpha \in \mathbb{K}$, where $\mathbb{K}$ is a compact metric space. We assume that we have

$$\hat{\alpha} = (\hat{\alpha}(t), t \geq 0) \subset \mathbb{K},$$

a strongly consistent family of estimators of $\alpha = \alpha_0 \in \mathbb{K}$, that is, a $\mathbb{K}$-valued progressive process such that

$$\lim_{t \to \infty} \hat{\alpha}(t) = \alpha_0, \quad a.s.$$

We prove that the path-wise adaptive control corresponding to $\hat{\alpha}$ is optimal.

The controlled stochastic evolution equation dependends on $\alpha \in \mathbb{K}$, that is

$$dX_\alpha^U(t) = \left(A_\alpha X_\alpha^U(t) + B_\alpha U(t)\right) dt + dL(t), \quad X_\alpha^U(0) = x, \tag{5.24}$$

where $x \in \mathbb{H}$, $\alpha \in \mathbb{K}$. A_α, $\alpha \in \mathbb{K}$, are the infinitesimal generators of analytic semigroups S_α on $\mathbb{H}$. For some $\beta > 0$ the operators $-A_\alpha + \beta \mathbb{I}$, $\alpha \in \mathbb{K}$, are strictly positive (in the sequel, $\beta > 0$ is fixed).

As in the previous section, we suppose (**A1****), (**A3**), (**A4****). Moreover, we add the assumptions

(**B1**) For each $\alpha \in \mathbb{K}$: $B : \mathbb{D}(B_\alpha) \subset \mathbb{Y} \to \mathbb{D}(A_\alpha^*)'$, the dual of $\mathbb{D}(A_\alpha^*)$ with respect to the topology of $\mathbb{H}$, and $(\beta \mathbb{I} - A_\alpha)^{\epsilon-1} B_\alpha \in \mathcal{L}(\mathbb{Y}, \mathbb{H})$, for a given $\epsilon \in (\frac{1}{2}, 1]$.

(**B2**) For each $\alpha_1, \alpha_2 \in \mathbb{K}$: $\mathbb{D}(A_{\alpha_1}) = \mathbb{D}(A_{\alpha_2})$, $\mathbb{D}_{A_{\alpha_1}}^\gamma = \mathbb{D}_{A_{\alpha_2}}^\gamma$ and $\mathbb{D}_{A_{\alpha_1}^*}^{1-\epsilon} = \mathbb{D}_{A_{\alpha_2}^*}^{1-\epsilon}$.

Similarly as in the Chapter 2, we set

$$J(U,t) = \int_0^T \left(\left\langle QX_\alpha^U(s), X_\alpha^U(s) \right\rangle_{\mathbb{H}} + \left\langle RU(s), U(s) \right\rangle_{\mathbb{Y}} \right) ds, \tag{5.25}$$

where $T > 0$, X_α^U is the solution of (5.24), $Q \in \mathcal{L}(\mathbb{H})$ and $R \in \mathcal{L}(\mathbb{Y})$ are symmetric positive definite operators, i. e.

$$\langle Ry, y \rangle \geq r|y|^2, \quad y > 0, \tag{5.26}$$

$$\langle Qy, y \rangle \geq r|y|^2, \quad y > 0, \tag{5.27}$$

holds for a constant $r > 0$.

The corresponding Riccati equation takes the form

$$\langle a, A_\alpha^* V_\alpha b\rangle_\mathbb{H} + \langle A_\alpha^* V_\alpha a, b\rangle_\mathbb{H} + \langle Qa, b\rangle_\mathbb{H} - \left\langle R^{-1} B_\alpha^* V_\alpha a, B_\alpha^* V_\alpha b\right\rangle_\mathbb{H} = 0, \qquad (5.28)$$

$a, b \in \mathcal{D}(A_\alpha^*)$.

For a fixed $d > 0$ we consider the adaptive feedback control in the form $U_{\hat\alpha}(t) = K_{\hat\alpha}(t)X(t)$, $t > 0$, where

$$K_{\hat\alpha}(t) = -R^{-1} B_{\hat\alpha(t-d)}^* V_{\hat\alpha(t-d)} \mathbf{I}_{[t>d]}, \quad t > 0, \qquad (5.29)$$

The result for such adaptive control follows:

Theorem 23. *Assume (A1**), (A3), (A4**), (B1), (B2) and stabilizability and detectability for all $\alpha \in \mathbb{K}$. Let*

$$\lim_{\alpha \to \alpha_0, \alpha \in \mathbb{K}} \left| B_\alpha^* - B_{\alpha_0}^* \right|_{\mathcal{L}\left(\mathbb{D}_{A_{\alpha_0}}^{1-\epsilon}, \mathbb{Y}\right)} = 0, \qquad (5.30)$$

$$\lim_{\alpha \to \alpha_0, \alpha \in \mathbb{K}} \left| S_\alpha(t) - S_{\alpha_0}(t) \right|_{\mathcal{L}\left(\mathbb{D}_{A_{\alpha_0}}^{\epsilon-1}, \mathbb{H}\right)} = 0 \qquad (5.31)$$

for all $t > 0$. Then

$$\lim_{t \to \infty} |K_{\hat\alpha}(t) - K_{\alpha_0}|_{\mathcal{L}(\mathbb{H}, \mathbb{Y})} = 0, \quad a.s., \qquad (5.32)$$

where $K_{\alpha_0} = -R^{-1} B_{\alpha_0}^* V_{\alpha_0}$, and

$$\lim_{t \to \infty} \frac{J_{\alpha_0}(U_{\hat\alpha}, t)}{t} = Tr(V_{\alpha_0} \mathcal{Q}), \quad a.s. \qquad (5.33)$$

Proof. Applying the Lemma 5.3 in [5], we obtain from (5.30) and (5.31) that

$$\lim_{\alpha \to \alpha_0, \alpha \in \mathbb{K}} |V_\alpha - V_{\alpha_0}|_{\mathcal{L}\left(\mathbb{H}, \mathbb{D}_{A_{\alpha_0}}^{1-\epsilon}\right)} = 0. \qquad (5.34)$$

If we combine (5.34) with (5.30), we obtain (5.32). Therefore, we can apply the Theorem 22 with $A = A_{\alpha_0}$, $B = B_{\alpha_0}$ and

$$K(t) = K_{\hat\alpha(t)}, \quad t > 0.$$

$\square$

Remark. Similarly as in the Remark 5.2 in the previous section, we can consider $\mathbb{L}^p$-Lévy process in $\mathbb{H}$, $p > 2$, and we can extend the path-wise result in Theorem 23 for the equations with $\epsilon \in (\frac{1}{p}, 1]$ in the assumption (**B1**) on the operator B.

5.4 Examples

Example (bundary control for stochastic heat equation). We consider the equation

$$b_t(t, x) = \Delta_x b(t, x) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \qquad (5.35)$$

$$b_v(t, x) + h(x)b(t, x) = u(t, x), \quad (t, x) \in \mathbb{R}_+ \times \partial G,$$

$$b(0, x) = b_0(x), \quad x \in G,$$

where $b_v(t, x)$ is normal derivative of b in $(t, x) \in \mathbb{R}_+ \times \partial G$ in direction v, v being the outward normal to ∂G, Δ_x is the Laplace operator, ∂G denotes the boundary of the set G, $G \subset \mathbb{R}^n$ is an open bounded domain with $\mathbb{C}^\infty$ boundary, $h \in \mathbb{C}^\infty(\partial G)$, $h \geq 0$ and l formally represents Lévy noise on $\mathbb{L}^2(\partial G)$.

Set $H = \mathbb{L}^2(G)$, $Y = \mathbb{L}^2(\partial G)$, $A = \Delta_x$ on the domain

$$\mathcal{D}(A) = \left\{ f \in H^2(G) : f_v(x) + h(x)f(x) = 0, \quad x \in \partial G \right\},$$

where $H^2(G)$ denotes the Sobolev space. As well known, A generates an analytic semigroup and there exists $\beta \geq 0$ such that $(A - \beta \mathbb{I})$ is strictly negative. To obtain the infinite-dimensional form (2.1) of the system (5.35) we follow the lines of the standard approach developed for deterministic equations in [2] (see also [23], [9] for various modifications in stochastic cases). Consider the elliptic problem

$$\Delta_x z - \beta z = 0$$

on G and

$$z_v + hz = -g$$

on ∂G. The solution map $M : g \mapsto -z$ belongs to $\mathcal{L}(\mathbb{L}^2(\partial G), \mathbb{D}_A^\epsilon)$ for $\epsilon < \frac{3}{4}$. The operator B is obtained as the composition $B = \hat{A}M$, where $\hat{A}$ is the isomorphic extension of the operator A into $\mathbb{D}_A^\epsilon$.

Then $B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_A^{\epsilon-1})$, $\epsilon \in (\frac{1}{2}, \frac{3}{4})$, and the pairs (A, B), (A, Q) satisfy the stabilizability and detectability considions, respectively. Therefore, all conditions of Theorems 12 and 16 are satisfied for $\epsilon \in (\frac{1}{2}, \frac{3}{4})$ and the optimal mean value as well as path-wise ergodic control may be obtained following these Theorems. If we take into account Remark 5.2, we have the path-wise result for $\epsilon \in (\frac{1}{p}, \frac{3}{4})$.

Now consider the equation (5.35) dependent on a parameter α in a compact set $\mathbb{K} \subset \mathbb{R}_+$:

$$b_t(t, x) = \alpha \Delta_x b(t, x) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \tag{5.36}$$

$$b_v(t, x) + h(x)b(t, x) = u(t, x), \quad (t, x) \in \mathbb{R}_+ \times \partial G,$$

$$b(0, x) = b_0(x), \quad x \in G,$$

where $\alpha \in \mathbb{K}$, $\mathbb{K} \subset \mathbb{R}_+$ is compact, $G \subset \mathbb{R}^n$ is an open bounded domain with $\mathbb{C}^\infty$ boundary, $h \geq 0$, l formally represents Lévy noise on $\mathbb{L}^2(\partial G)$.

In this case $H = \mathbb{L}^2(G)$, $Y = \mathbb{L}^2(\partial G)$, $A_\alpha = \alpha \Delta_x$, $\alpha \in \mathbb{K}$, on the domain

$$\mathcal{D}(A_\alpha) = \left\{ f \in H^2(G) : h(x) = f_v(x) + h(x)f(x) = 0, \quad x \in \partial G \right\},$$

which does not depend on $\alpha \in \mathbb{K}$. Also, B_α, $\alpha \in \mathbb{K}$, are obtained from A_α, $\alpha \in \mathbb{K}$, by the same approach as above for $\epsilon < \frac{3}{4}$.

Then

$$B_\alpha \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_{A_\alpha}^{\epsilon-1}) = \mathcal{L}(\mathbb{Y}, \mathbb{D}_{A_{\alpha_0}}^{\epsilon-1})$$

and all conditions of Theorem 23 are satisfied for $\epsilon \in (\frac{1}{2}, \frac{3}{4})$. Therefore, the adaptive feedback control defined by (5.29) is optimal. We have the path-wise result for $\epsilon \in (\frac{1}{p}, \frac{3}{4})$ applying Remark 5.3.

Example (ergodic point control of stochastic plate equation with structural damping). Consider the problem

$$p_{tt}(t,x) - \Delta_x p_t(t,x) + \Delta_x^2 p(t,x) = \mathbb{I}_{x=x_0} u(t) + l(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \quad (5.37)$$

$$p(0,x) = p_0, \quad p_t(0,x) = p_1, \quad x \in G,$$

$$p(t,x) = p_t(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial G,$$

where $G \subset \mathbb{R}^n$ is an open and bounded domain with a sufficiently smooth boundary ∂G, $n \in \{1,2,3\}$, $x_0 \in G$, l formally represents a (space-dependent) Lévy noise. Define the cost functional

$$J(u,T) = \int_0^T (|p(t)|_{H^2(G)}^2 + |p_t(t)|_{\mathbb{L}(G)}^2 + |u(t)|^2) dt,$$

where H^2 denotes the Sobolev space $\{y \in \mathbb{L}^2(G) : D^\alpha y \in \mathbb{L}^2(G), |\alpha| \leq 2\}$.

The deterministic case ($l \equiv 0$) is analyzed in [3] and [20]. Let $\mathbb{A} = \Delta_x^2$ on the domain

$$\mathcal{D}(\mathbb{A}) = \left\{ h \in H^4(G) : h(x) = \Delta_x h(x) = 0, \quad x \in \partial G \right\}.$$

We rewrite the equation (5.37) in the form (2.1),

$$dX^U(t) = \left(AX^U(t) + BU(t) \right) dt + dL(t), \quad X^U(0) = x.$$

Here

$$\mathbb{H} = \mathcal{D}\left(\mathbb{A}^{\frac{1}{2}}\right) \times \mathbb{L}^2(G) = \left(H^2(G) \cap H_0^1(G) \right) \times \mathbb{L}^2(G), \quad \mathbb{Y} = \mathbb{R},$$

$$A = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{A} & \mathbb{A}^{\frac{1}{2}} \end{pmatrix},$$

$$Bu = \begin{pmatrix} 0 \\ \mathbb{I}_{x=x_0} u \end{pmatrix},$$

$$\mathcal{Q}^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix},$$

$$R = Q = \mathbb{I},$$

where $\Phi_1 \in \mathcal{L}(\mathbb{L}^2(G))$ is Hilbert-Schmidt, $X_0 = (v_0, v_1)^T$ and L is a cylindrical Lévy proces on $\mathbb{H}$.

It is well known ([3]) that A generates an exponentially stable analytic semigroup. In [20] it is shown that $B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_A^{\epsilon-1})$ for $\epsilon \in (0, 1 - \frac{n}{4})$. In the case of $n = 1$, all conditions of the Theorems 12 and 16 are satisfied for $\epsilon \in (\frac{1}{2}, \frac{3}{4})$ and the optimal mean value as well as path-wise ergodic control may be obtained following these theorems. In the case of $n = 2$ we may choose $\epsilon \in (\frac{1}{p}, \frac{1}{2})$, and follow Remark 5.2 to find the path-wise ergodic control. Moreover, if we assume $p > 4$ in Remark 5.2, we obtain the path-wise result for $n = 3$ by the choice $\epsilon \in (\frac{1}{p}, \frac{1}{4})$.

Consider parameter dependent equation (5.37). That is:

$$p_{tt}(t,x) - \alpha \Delta_x p(t,x) + \Delta_x^2 p(t,x) = \mathbb{I}_{x=x_0} u(t) + l(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \quad (5.38)$$

$$p(0,x) = p_0, \quad p_t(0,x) = p_1, \quad x \in G,$$

$$p(t,x) = p_t(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial G,$$

where $\alpha \in \mathbb{K}$, $\mathbb{K} \subset \mathbb{R}_+$ is compact, $G \subset \mathbb{R}^n$ is an open and bounded domain with a sufficiently smooth boundary ∂G, $n \in \{1,2,3\}$, $x_0 \in G$, l formally represents a (space-dependent) Lévy noise.

Now for all $\alpha \in \mathbb{K}$

$$dX_\alpha^U(t) = \left(A_\alpha X_\alpha^U(t) + B_\alpha U(t) \right) dt + dL(t)$$

$$= \left(A_\alpha X_\alpha^U(t) + BU(t) \right) dt + dL(t), \quad X^U(0) = x,$$

where

$$\mathbb{H} = \mathcal{D}\left(\mathbb{A}^{\frac{1}{2}} \right) \times \mathbb{L}^2(G) = \left(H^2(G) \cap H_0^1(G) \right) \times \mathbb{L}^2(G), \quad \mathbb{Y} = \mathbb{R},$$

$$A_\alpha = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{A} & \alpha \mathbb{A}^{\frac{1}{2}} \end{pmatrix},$$

$$B_\alpha u = Bu = \begin{pmatrix} 0 \\ \mathbb{I}_{x=x_0} u \end{pmatrix},$$

$$\mathcal{Q}^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix},$$

where $\Phi_1 \in \mathcal{L}(\mathbb{L}^2(G))$ is Hilbert-Schmidt, $X_0 = (v_0, v_1)^T$ and L is a cylindrical Lévy proces on $\mathbb{H}$.

For all $\alpha \in \mathbb{K}$, is well known ([3]) that A_α generates a stable analytic semigroup. As above

$$B_\alpha = B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_{A_{\alpha_0}}^{\epsilon-1})$$

for $\epsilon \in (0, 1 - \frac{n}{4})$. In the case of $n = 1$, all conditions of the Theorem 23 are satisfied for $\epsilon \in (\frac{1}{2}, \frac{3}{4})$ and the adaptive feedback control defined by (5.29) is optimal. For $n = 2$ and $n = 3$ we again may follow Remark 5.3 and take p large enough so that $\epsilon \in (\frac{1}{p}, 1 - \frac{n}{4})$.

Example (ergodic point control of stochastic Kelvin-Voigt plate equation). Consider the problem

$$w_{tt}(t,x) + \Delta_x^2 w(t,x) + \rho \Delta_x^2 w_t(t,x) = \mathbb{I}_{x=x_0} u(t) + l(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \tag{5.39}$$

$$w(0,x) = w_0, \quad w_t(0,x) = w_1, \quad x \in G,$$

$$\Delta_x w(t,x) + (1-\mu) B_1 w(t,x) = \frac{\partial \Delta_x w(t,x)}{\partial \nu} + (1-\mu) B_1 w(t,x) = 0,$$

$$(t,x) \in \mathbb{R}_+ \times \partial G,$$

where $\mu \in (0, \frac{1}{2})$, $\rho > 0$, $x_0 \in G \subset \mathbb{R}^n$, $n \le 2$. Note that the boundary operators B_1 and B_2 are given by

1. In the case of $n = 1$: $B_1 = B_2 = 0$,

2. In the case of $n = 2$:

$$B_1 w = 2\nu_1\nu_2 w_{x,y} - \nu_1^2 w_{y,y} - \nu_2^2 w_{x,x},$$

$$B_2 w = \frac{\partial}{\partial\tau}\left(\nu_1^2 - \nu_2^2\right) w_{x,y} + \nu_1\nu_2(w_{y,y} - w_{x,x}),$$

where $\frac{\partial}{\partial\tau}$ is tangential derivative.

l formally represents a (space-dependent) Lévy noise. Define the cost functional

$$J(u,T) = \int_0^T \left(|w(t)|_{H^2(G)}^2 + |w_t(t)|_{\mathbb{L}(G)}^2 + |u(t)|^2\right) dt,$$

where H^2 denotes the Sobolev space $\{y \in \mathbb{L}^2(G) : D^\alpha y \in \mathbb{L}^2(G), |\alpha| \le 2\}$.

The deterministic case ($l \equiv 0$) is analyzed in [20]. Let $\mathbb{A} = \Delta_x^2$ on the domain

$$\mathcal{D}(\mathbb{A}) = \{h \in H^4(G):$$

$$h(x) + (1-\mu)B_1 h(x) = \frac{\partial \Delta_x h(x)}{\partial\nu} + (1-\mu)B_1 h(x) = 0, \quad x \in \partial G\}.$$

We rewrite the equation (5.39) in the form (2.1),

$$dX^U(t) = \left(AX^U(t) + BU(t)\right)dt + dL(t), \quad X^U(0) = x.$$

Here

$$\mathbb{H} = \mathcal{D}\left(\mathbb{A}^{\frac{1}{2}}\right) \times \mathbb{L}^2(G) = \left(H^2(G) \cap H_0^1(G)\right) \times \mathbb{L}^2(G), \quad \mathbb{Y} = \mathbb{R},$$

$$A = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{A} & -\nu\mathbb{A} \end{pmatrix},$$

$$Bu = \begin{pmatrix} 0 \\ \mathbb{I}_{x=x_0} u \end{pmatrix},$$

$$Q^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix},$$

$$R = Q = \mathbb{I},$$

where $\Phi_1 \in \mathcal{L}(\mathbb{L}^2(G))$ is Hilbert-Schmidt, $X_0 = (w_0, w_1)^T$ and L is a cylindrical Lévy proces on $\mathbb{H}$.

It is well known that A generates an exponentially stable analytic semigroup. In [20] it is shown that $B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_A^{\epsilon-1})$ for $\epsilon \in [\frac{1}{2}, 1 - \frac{n}{8})$, $n \le 3$. Thus all conditions imposed in the paper are satisfied and the optimal ergodic control may be obtained following Theorems 12 and 16.

Conclusion

In the Thesis, we are focused on the optimal ergodic control of processes in the Hilbert space $\mathbb{H}$ described by the controlled stochastic evolution equation

$$dX^U(t) = \left(AX^U(t) + BU(t)\right)dt + \Phi dL(t), \quad X^U(0) = x, \qquad (5.40)$$

$x \in \mathbb{H}$, where A is the infinitesimal generator of an analytic semigroup S on $\mathbb{H}$. The control of the equation (5.40) was set up by the control process U on $(\Omega, \mathcal{A}, \mathbf{P})$ in the Hilbert space Y with the coefficient operator B acting from $\mathbb{Y}$ to $\mathbb{H}$. As results related to ergodic control were already known in the case of the continuous Gaussian noise, we assumed $L(t)$ in (5.40) to be a pure jump noise $(\Omega, \mathcal{A}, \mathbf{P})$ in $\mathbb{H}$. The coefficient Φ is assumed to be, in general, a bounded linear operator on $\mathbb{H}$. The optimality is defined by asymptotics of the cost in the form

$$J(U, t) = \int_0^T \left(\left\langle QX^U(s), X^U(s)\right\rangle_{\mathbb{H}} + \left\langle RU(s), U(s)\right\rangle_{\mathbb{Y}}\right) ds \qquad (5.41)$$

giving the weights to the value of the controlled process X^U as well as to the control process U by the operators $Q \in \mathcal{L}(\mathbb{H})$ and $R \in \mathcal{L}(\mathbb{Y})$. Roughly speaking, the aim was to find the solution to the optimal control problem with the following two particular goals:

1. Optimal control as the control process U in the set of the admissible controls which minimize the asymptotic cost functional.

2. Optimal cost as the formula for the minimal value of the asymptotic cost functional.

An important part of the work was the proof of the Itô formula and as the consequence, the assumptions on the coefficients were driven by the assumptions needed for proving the Itô formula.

We found two sets of meaningful assumptions for the ergodic control problem and we found the optimal control as well as the optimal cost for both of them.

In both cases, we were able to cover the point and boundary control problems. This means that the assumptions on the operator B in (5.40) are the same:

($\mathbf{A2}$) $B : \mathbb{D}(B) \subset \mathbb{Y} \to \mathbb{D}(A^*)'$, the dual of $\mathbb{D}(A^*)$ with respect to the topology of $\mathbb{H}$, and $(\beta\mathbb{I} - A)^{\epsilon-1}B \in \mathcal{L}(\mathbb{Y}, \mathbb{H})$, for a given $\epsilon \in (0, 1]$.

In both cases, the coefficiens Q in (5.41) was needed to be positive semi-definite and the coefficient R in (5.41) was needed to be positive definite. In both cases, the set of admissible controls has to be the subset of $\mathbb{L}_{\mathcal{F}}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$.

The case of cylindrical Lévy processes

In the case of setting L in (5.40) as the cylindrical Lévy processes, we had to consider the ergodic cost functional as the "mean average cost per time unit in long run", that is

$$\tilde{J}_{\mathbf{E}}(U) = \lim_{t \to \infty} \inf \frac{\mathbf{E}J(U, t)}{t}. \qquad (5.42)$$

We were not able to prove the path-wise version of the Itô formula as the diffusion parts in the proof did not converge path-wise in the case of the cylindrical Lévy processes. Therefore, we were not able to work with the path-wise version of the ergodic cost functional. Note that the problematic diffusion part disappeared in the case of the mean value versions of the ergodic cost functional and the corresponding Itô formula.

Moreover, we had to add the following assumption on the diffusion coefficient Φ in (5.40) and on the coefficient Q in (5.41):

1. There exists $\delta \in (0, \frac{1}{2}]$ such that $\Phi^*(-A^* + \beta\mathbb{I})^{-\frac{1}{2}+\delta}$ is Hilbert-Schmidt.

The set of the admissible controls is the set of $U \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$ such that

$$\frac{\mathbb{E}\left\langle VX_t^U, X_t^U \right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty. \tag{5.43}$$

For the cylindrical Lévy processes and ergodic control functional in the form (5.42) with the added assumptions on Φ and Q mentioned above, we proved that (on the given set of the admissible controls):

1. The optimal cost equals to Π given by the formula

$$\Pi = \lim_{\lambda \to \infty} Tr\left(VR^2(\lambda)\Phi\Phi^* \left(R^2(\lambda)\right)^*\right).$$

2. The optimal control is every feedback control process $U(t) = K(t)X(t)$, $t \in \mathbb{R}_+$, such that $K(t) \to -R^{-1}B^*V$, $t \to \infty$, is deterministic.

The case of the square integrable Lévy processes

When we wanted to consider the path-wise "average cost per time unit in long run", we had to strengthen the assumptions on L in (5.40) to have the square integrable Lévy processes as the noise or, equivalently, the assumptions on Φ in (5.40) to have the Hilbert-Schmidt diffusion operator. The path-wise "average cost per time unit in long run" is defined as

$$\tilde{J}(U) = \liminf_{t \to \infty} \frac{J(U, t)}{t}. \tag{5.44}$$

We had to strengthen the assumptions to have Q in (5.41) positive definite. The set of the admissible controls is the set of $U \in \mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{Y})$ such that

$$\frac{\langle VX_t^U, X_t^U \rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \quad a.s., \tag{5.45}$$

and

$$\limsup_{t \to \infty} \frac{\int_0^t \left|X_s^U\right|_{\mathbb{H}}^2 ds}{t} < \infty \quad a.s. \tag{5.46}$$

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List of Abbreviations

List of publications

1. KADLEC, K. and MASLOWSKI, B. *Ergodic Control for Lévy-driven linear stochastic equations in Hilbert spaces.* Applied Mathematics & Optimization, 79, 547–565, 2017.

2. KADLEC, K. and MASLOWSKI, B. *Ergodic boundary and point control for linear stochastic PDEs driven by a cylindrical Lévy processes.* Discrete and Continuous Dynamic Systems - B, doi: 10.3934/dcdsb.2020137, 2020.

A. Stochastic Fubini theorem

In this Appendix, we state the Stochastic Fubini Theorem for the case of the stochastic integration with respect to the Cylindrical Lévy process L. This Theorem was proved in [19] for more general integrators than we need, for so called regulated functions. Therefore, we start with the definition of the regulated functions.

We say that the function g on $[0, T]$ in $\mathbb{H}$, $T \geq 0$, is regulated if g has only discontinuities of the first kind. We denote the space of all regulated functions on $[0, T]$ in $\mathbb{H}$ as $\mathcal{R}([0, T], \mathbb{H})$.

Theorem 24. *We assume a finite μ-measure space $(M, \mathcal{M}, \mu)$, a Bochner space $L^2_\mu(M, \mathbb{H})$ and $T \geq 0$. Let g be a function from $M \times [0, T]$ to $\mathbb{H}$ such that*

1. *g is measureble with respect to $\mathcal{M} \times \mathcal{B}$,*

2. *for μ-almost all $x \in M$, $g(x, \cdot)$ is regulated,*

3. *$g \in \mathcal{R}([0, T], L^2_\mu(M, \mathbb{H}))$.*

Then

1. *$\int_M g(x, \cdot) d\mu(x)$ is regulated,*

2. *$\int_0^T g(\cdot, t) dL(t)$ is a random variable in $L^2_\mu(M, \mathbb{R})$,*

3. *we have $\mathbf{P}$-a.s.*

$$\int_M \int_0^T g(x, t) dL(t) d\mu(s) = \int_0^T \int_M g(x, t) d\mu(s) dL(t).$$

Proof. This theorem is proved in [19]. □

B. Regular modification

In this Appendix, we state the regular modification theorem for the case of the contraction semi-group in the integrand. This theorem in [25] is more general as it is proved for integrals with respect to square integrable martingales.

Theorem 25. *We assume*

1. *S is a strongly continuous semi-group of contractions,*

2. *Φ is $\mathcal{L}(\mathbb{H})$-valued process such that $\Phi(\cdot)z$ is $\mathcal{F}$-predictable $\mathbb{H}$-valued process for each $z \in \mathbb{H}$,*

3. *M is a square integrable martingale with the martingale covariance $\mathcal{Q}$,*

4.
$$\mathbf{E} \int_0^t |\Phi(s)Q(s)|_{HS}^2 ds < \infty, \ \ t \geq 0.$$

Then the process
$$\int_0^\cdot S(t-s)\Phi(s)dM(s)$$

has a càdlàg version in $\mathbb{H}$.

Proof. This theorem is proved in [25]. □