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DOCTORAL THESIS

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Spacetimes with black holes

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Abstract: In this thesis, we study exact black hole spacetimes of algebraic type D, which are a part of much wider Plebański–Demiański class of solutions. We reformulate the well-known form of this metric and obtain new improved representation of this black hole family with simplified, explicit and (at least partially) factorized metric functions. This new form of the spacetimes allows us to gain the standard expressions for the well-known solutions such as the Kerr–Newman–NUT–(anti-)de Sitter black hole, accelerating Kerr–Newman–(anti-)de Sitter black hole, (possibly charged) Taub–NUT–(anti-)de Sitter black hole, accelerating Kerr–NUT–(anti-)de Sitter black hole, and their special cases in asymptotically flat universe, just by putting the appropriate parameters to zero. We also provide a thorough physical and geometrical analysis of this new form of spacetimes. Furthermore, we analyze a solution corresponding to the accelerating Taub–NUT black hole, which was originally found by Chng, Mann and Stelea in 2006. We perform an in-depth analysis of this solution, and study its relation to the Plebański–Demiański class.

Keywords: exact spacetimes, accelerating Taub–NUT, Plebański–Demiański metric, type D black holes, algebraic classification

Abstrakt: V práci studujeme přesné prostoročasy představující černé díry algebraického typu D, které jsou součástí mnohem obsáhlejší Plebańského–Demiańského třídy řešení. Přeformulujeme známý tvar této metriky, čímž získáme novou vylepšenou reprezentaci této rodiny řešení se zjednodušenými, explicitními a (alespoň částečně) faktorizovanými metrickými funkcemi. Tento nový tvar prostoročasů nám umožňuje získat standardní výrazy pro známá řešení, jako jsou Kerrova–Newmanova–NUT–(anti-)de Sitterova černá díra, zrychlující Kerrova–Newmanova–(anti-)de Sitterova černá díra, (nabitá) Taub–NUT–(anti-)de Sitterova černá díra, urychlená Kerrova–NUT–(anti-)de Sitterova černá díra a jejich speciální případy v asymptoticky plochém vesmíru, a to pouhým dosazením příslušných parametrů za nulu. Uvádíme také důkladnou fyzikální a geometrickou analýzu tohoto nového tvaru prostoročasů. Dále analyzujeme řešení odpovídající urychlené Taub–NUT černé díře, které původně nalezli Chng, Mann a Stelea v roce 2006. Provádíme důkladnou analýzu tohoto řešení a studujeme jeho vztah k Plebańského–Demiańského třídě.

"Understanding is, after all, what science is all about – and science is a great deal more than mindless computation."

Roger Penrose

Contents

Preface

This thesis starts with an *Initial overview*. Its chapters are labeled by Roman numbers, and equations simply as "(chapter.number)". This is followed by chapters containing our *New results*, divided into 3 main parts. These are labeled by the Arabic numbers, and are linked to the attached publications. The original publications are denoted as P1, P2, and P3, respectively. The corresponding sections and subsections are labeled accordingly to the labeling in the original publications. Each equation, denoted by "(number)" only, refers to the related publication.

Our work uses the convention of standard textbooks, namely *Exact Space-Times in Einstein's General Relativity* by Jerry B. Griffiths and Jiří Podolský [1], and *Exact Solutions of Einstein's Field Equations* by Stephani et al. [2]. In particular, the standard convention of geometrical units $c = G = 1$ is adopted, and we employ the notation for expressing coordinate components of a general tensor as $T^{\mu\nu\ldots}_{\alpha\beta\ldots}$, where the Greek letters span 0,1,2,3. The metric tensor $g_{\mu\nu}$ describing a spacetime is assumed to have the Lorentzian signature $(-,+,+,+)$.

Introduction

More than a century has passed since Albert Einstein introduced his general theory of relativity (see the original paper [3]), ushering in a new era of physics. His original approach brought a completely new perspective into the understanding of the fundamental concepts such as space, time, and gravity. Since then, this theory has proven itself in many different areas of physics and astronomy, and has withstood many attempts to disprove it.

The general theory of relativity has explained a number of problems that remained unresolved until then, namely the principal problems of a non-relativistic (and therefore acausal) behavior of the classical Newtonian theory of gravitation, or the anomalous perihelion advance of Mercury, which the general relativity managed to explain without any arbitrary parameter [4].

Not only has the theory achieved to answer some of the open problems of that time, but it also predicted a completely new and unexpected phenomena. Among the many interesting predictions of general relativity, let us mention especially the (nowadays famous) black holes, gravitational waves, or gravitational lensing.

All of these marvelous predictions emerged from the *Einstein field equations*, the fundamental set of relations for gravitational field of general relativity, which are expressed in a single elegant equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,, \tag{EFE}$$

where $g_{\mu\nu}$ is the spacetime metric, $R_{\mu\nu}$ is the corresponding Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant, and $T_{\mu\nu}$ is the energy-momentum tensor of matter.

The first of these surprising predictions was the black hole solution. This solution of (EFE) describing, in a general case, any *spherically symmetric and static vacuum spacetime* was presented by Karl Schwarzschild already in 1916: it is the famous *Schwarzschild metric* [5].

Finding this solution so early was for Einstein actually quite surprising:

> *"I had not expected that one could formulate the exact solution of the problem in such a simple way."*

This was his reaction, when he received from Schwarzschild this first non-trivial exact solution, less than two months after Einstein's presentation of (EFE) to the Prussian Academy of Science [6].

The gravitational waves were also predicted in the very same year. In June 1916, and two years later in 1918, Albert Einstein published two papers [7, 8] in which he derived and studied the "ripples in spacetime" directly from his (linearized) field equations.

General relativity soon has also found its value in cosmology. In 1917, Albert Einstein published his paper of the *static universe* [9], immediately followed by Willem de Sitter with his fundamental vacuum model with a positive cosmological constant [10].

Nevertheless, what made Albert Einstein instantly famous was the prediction of gravitational deflection of starlight passing near the massive objects [11]. This was confirmed by expedition, led by the British astronomer Arthur Stanley Eddington, which measured deviations in the position of stars near the Sun during the total solar eclipse of May 29, 1919. Still, general relativity remained outside the mainstream of theoretical physics and astrophysics until sometime between 1960 and 1975.

The "golden age of general relativity", as Kip Thorne refers to this era [12], was associated primarily with the general acceptance of phenomena such as the black holes, big bang, their singularities, or gravitational waves – which until then were considered merely as theoretical constructs. Actually, Albert Einstein himself was very sceptical about the physical relevance of all these predictions [6, 13].

The first attempts to prove the real existence of gravitational waves were initiated after 1960 [14]. These were the famous *Weber bars*, large aluminum cylinders constructed by physicist Joseph Weber as resonant antennae for gravitational waves of a specific wave length. However, although his measurements claiming detections have been published [15, 16, 17], it is now generally accepted that these detectors were not efficient enough to be able to find such a small spacetime distortions directly.

The first (somewhat indirect) evidence of their existence was measured after 1974, discovering the first *binary pulsar PSR B1913+16* in the constellation of Aquila. From the timing observations over the subsequent decades there has been a decay of the orbital period corresponding to the loss of energy and angular momentum, as it was predicted by the general relativity [18, 19]. For the first *direct* detection of the gravitational waves we had to wait for the development of highly sensitive laser interferometers LIGO till 2015 (see Fig. 1).

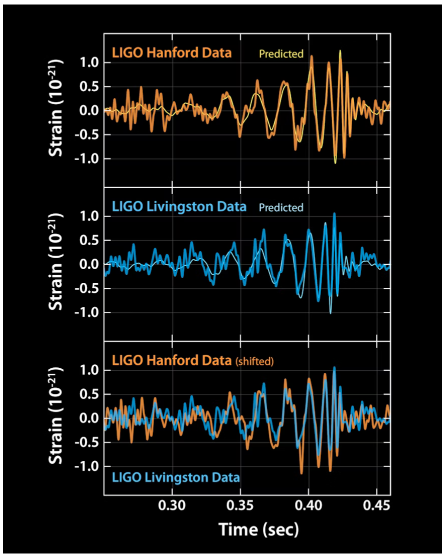

Figure 1: The first detection of gravitational waves observed by both LIGO detectors (Hanford and Livingston) and their comparison. (Image Credit: Caltech/MIT/LIGO Lab. [20])

This era, starting after 1960, also brought a number of *new exact solutions* of Einstein's field equations, representing gravitational waves, inhomogeneous cosmological models, and various black holes. Let us mention just the most important black hole solutions of that time: the Taub–NUT metric (1951, 1963) [21, 22] (an axially symmetric solution with a "specific twist"), the C-metric (named in 1962) [23, 24], see also [25, 26] (an accelerating black hole), or the famous Kerr metric (1963) [27] (a rotating black hole). The whole family of solutions of algebraic type D containing all these black holes was later found by Plebański and Demiański (1971, 1976) [28, 29].

In recent years, we have witnessed several breakthrough observations, confirming general relativity. The first is the rapid development of the new scientific discipline called *gravitational-wave astronomy.* In fact, the first detection of gravitational waves was achieved only 7 years ago [30] (see Fig. 1). Now we detect a surprising number of gravitational wave sources in the *LIGO* and *VIRGO* detectors. The black hole mergers are among the most significant phenomena.

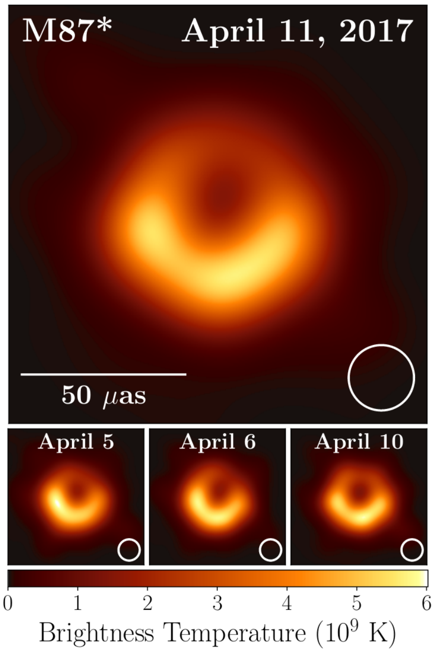

Figure 2: The first images of the supermassive black hole at the core of the galaxy Messier 87 produced by the Event Horizon Telescope collaboration. (Image Credit: [33])

In addition, the launch of the European *LISA satellites* is planned in the next decade. This should open the gravitational-wave window to space even further at different frequencies (see for example [31], summarizing the current topics to which LISA observations can make an essential contribution). This is also why the description and study of the properties of various black holes, albeit at a purely theoretical level, is still a very important topic in Einstein's theory of gravity and its generalizations.

Another recent observational breakthrough is the first-ever image of the shadow of the supermassive black hole in the center of the galaxy Messier 87 taken by the *Event Horizon Telescope collaboration* in April, 2017. This was analyzed and announced in 2019 [32, 33]. In March 2021, the collaboration team of EHT has revealed a first polarized-based image of M87* [34], and subsequently, also an image of Sagittarius A* (the supermassive black hole at the center of the Milky Way), was made public in May 2022 [35].

All these recent achievements indicate that we may be at the beginning of an era that will bring us new and unexpected discoveries. This, of course, would not be possible without a proper understanding of the physics behind all these observations, which puts even more emphasis on the study of exact solutions of Einstein's theory. These are a main topic of this Doctoral Thesis.

Conception and contents of the Doctoral Thesis

This work focuses on exact four-dimensional black hole solutions with a high degree of symmetry. More precisely, the entire thesis is (either directly or indirectly) linked to the Plebański–Demiański family of type D black hole solutions [28, 29].

Actually, it is the result of my long-lasting personal journey which I have started already nine years ago. It began with a study of an article *"Accelerating Taub–NUT and Eguchi-Hanson solitons in four dimensions"* published by Brenda Chng, Robert Mann and Cristian Stelea in 2006 [37]. In this paper, a brand new solution, which seemed to represent *an accelerating Taub–NUT* solution, was introduced. This was surprising, since there was no such a solution found in the large Plebański–Demiański class of black hole solutions [1], [38]–[40], although the *rotating accelerating Taub–NUT solution* was included.

First, it was necessary to explicitly calculate all components of the Ricci tensor to verify that the new solution is indeed a *vacuum solution* of the field equations. However, this was difficult due to the complexity of the problem. Therefore, we reformulated the metric into another form, more liable for our computations. In fact, we developed two independent methods (algorithms) to compute the Ricci tensor – one based on the direct computation, and the second utilizing the relations of the curvature tensors of mutually conformal metrics.

From the Weyl tensor we then computed its Newman–Penrose scalars, and using the scalar invariants I and J we have identified *the general algebraic type* of this solution, with four *distinct* principal null directions.

These results were important. Not only we verified that this new metric *is a vacuum solution*, but we also confirmed that it *does not belong to the Plebański–Demiański family of type D solutions*. It thus turned out to *deviate* from it.

Then, we introduced a new representation of this new metric in "spherical-type" coordinates, which is more convenient for any physical analysis. Explicitly depending on three physical parameters – namely the mass m, acceleration α and the NUT parameter l – this new representation makes possible to recover the well-known spacetimes in the standard coordinates (that is the C-metric, and the Taub–NUT metric in "spherical-like" coordinates) by switching off the parameters l and α, respectively.

Using this new convenient metric, we performed a thorough physical and geometrical analysis of such accelerating NUT black hole. In particular, we localized and study its *four Killing horizons*. Employing the scalar invariants, we investigated the curvature. Interestingly, no curvature singularities occur while keeping non-zero NUT parameter l. We provided a complete understanding of the global structure by identifying the asymptotically flat regions and by relating them to the conformal infinities.

We also proved that the solution can be analytically extended, so that it corresponds to a *pair* of such black holes uniformly accelerating in opposite directions. The source of this acceleration comes from the *rotating cosmic strings* (or struts) located along the axes. The rotation is caused exclusively by the NUT parameter l. Of course, similarly as in the Taub–NUT case without acceleration, there occur a pathological regions with *closed timelike curves* in the vicinity of these strings (or struts).

I addressed this topic already in my Bachelor Thesis [41], in which I verified the "vacuumness" of the original solution, computed the NP scalars, determined the algebraic type of this spacetime, and outlined a new better representation of this metric. I followed up on this work in my Diploma Thesis [42], reformulating the key scalars and invariants in a new metric representation, verifying earlier results, and providing the principal null directions. We carefully analyzed the pathological regions and visualized their localization according to the concrete values of the mass, acceleration and the NUT parameter. We also performed some investigations of the non-accelerating Taub–NUT metric.

The outputs of the Diploma Thesis were presented as an article in the proceedings of Week of Doctoral Students organised by the Charles University [43].

My first two years of PhD studies focused, among other things, on completion of open questions concerning this topic, and on writing them up (during the COVID lockdown) in an exhaustive publication *Accelerating NUT black holes* [44]. This article is summarized and attached in **Chapter 1** of this thesis.

Another topic was outlined in my Diploma Thesis [42], namely the reinvestigation of the whole Plebański–Demiański metric. We started from the convenient representation of this family found by Griffiths and Podolský in 2005 [38]–[40], and using a suitable redefinition of the physical parameters we managed to considerably simplify and fully factorize the metric functions in the case of vanishing cosmological constant Λ.

Our new metric depends on 6 physical parameters, namely the mass m, acceleration α, rotational parameter a, NUT parameter l, and electric and magnetic charges e, g, whereby no other free parameter was left undetermined.

The main advantage of this new representation is that it is possible to simply set an appropriate physical parameters to zero, thus obtaining the standard forms of the simpler black hole solutions such as the Kerr–Newman–NUT black hole, accelerating Kerr–Newman black hole, (possibly charged) Taub–NUT black hole, or accelerating Kerr–NUT black hole, respectively. Extreme and hyperextreme cases can also be discussed.

Moreover, it explicitly demonstrates that no accelerating Taub–NUT black hole is included in this large family, which further confirms conclusions of our previous work [44].

The new improved metric also enabled us to investigate various physical and geometrical properties, such as the location and the nature of the *horizons*, or the character of *singularities*. We also studied and visualized the *ergoregions*, and the *global structure* of the solution including the *Penrose conformal diagrams*. We analyzed the nature of the axes, namely their rotational character, *conicity* of the cosmic strings or struts causing the acceleration of the black hole, and the *pathological behavior* caused by the presence of the parameter NUT. Additionally, we calculated *the area* and *the surface gravity* of the horizons from which we provided basic *thermodynamic quantities*.

This was studied and published in a comprehensive publication *New improved form of black holes of type D* [45] in 2021. The main results of this publication are presented in **Chapter 2**.

However, this paper did not describe black holes in the (anti-)de Sitter background. Taking into the consideration a non-zero Λ causes multiple problems. For example, it is not possible to fully factorize *both* key metric functions $P(\theta)$ and $Q(r)$. Actually, this problem occurs already in the most simplest subcase of the Schwarzschild–de Sitter solution (see [1] for more details). Therefore, the general analysis of the horizons is not as clear as in the $\Lambda = 0$ case.

Even though, we managed to simplify the metric functions, and to factorize the function $P(\theta)$. We introduced a new representation for a *fully general black hole of type D* determined by the mass m, acceleration α, Kerr-like rotation a, the NUT parameter l, the electric and magnetic charges e, g, and the cosmological constant Λ, respectively.

This new metric reduces to the standard forms of the well-known black holes, namely to the Kerr–Newman–NUT–(anti-)de Sitter black hole ($\alpha = 0$), accelerating Kerr–Newman–(anti-)de Sitter black hole ($l = 0$), charged Taub–NUT–(anti-)de Sitter black hole ($a = 0$), accelerating Kerr–NUT–(anti-)de Sitter black hole ($e = g = 0$) and their analogies in the flat universe ($\Lambda = 0$) just by switching off the appropriate parameters. Even for $\Lambda \neq 0$ we explicitly observe that no accelerating Taub–NUT–(anti-)de Sitter solution exists in this wide class.

We were able to fully analyze and explicitly evaluate physically and geometrically relevant entities. We did localize all the *horizons* and classify generally their multiplicity. We investigated the location of *ergoregions*, the character of *singularities*, as well as the *global structure* including the *Penrose conformal diagrams*.

Moreover, we investigated the cosmic *strings or struts* along the axes of symmetry $\theta = 0$, or $\theta = \pi$, respectively. Their conicity causing the acceleration of the black hole was explicitly determined, and it can be regularized for a specific combination of the parameters. Both axes are twisting, and one of them is encircled by a pathological region with *closed timelike curves* caused by the presence of the NUT parameter. Explicit thermodynamic properties, such as the entropy or temperature of the horizons, were also evaluated.

We summarized and presented all these original results in the publication *New form of all black holes of type D with a cosmological constant* [46]. This is the basis of **Chapter 3** of this Doctoral Thesis.

Initial overview

I. Some basic tools

General relativity enables us to express physical variables for specific different observers by using different coordinates and frames. In particular, we express the key tensors in the most convenient "directions". For that reason, we introduce *orthonormal frames* $(\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ and *null tetrads*

$$\mathbf{k} = \tfrac{1}{\sqrt{2}}(\mathbf{t}+\mathbf{z})\,, \qquad \mathbf{l} = \tfrac{1}{\sqrt{2}}(\mathbf{t}-\mathbf{z})\,, \qquad \mathbf{m} = \tfrac{1}{\sqrt{2}}(\mathbf{x}-\mathrm{i}\mathbf{y})\,, \qquad \bar{\mathbf{m}} = \tfrac{1}{\sqrt{2}}(\mathbf{x}+\mathrm{i}\mathbf{y})\,.$$

This null tetrad is normalized as $\mathbf{k}\cdot\mathbf{l} = -1$ and $\mathbf{m}\cdot\bar{\mathbf{m}} = 1$. Its four vectors can be transformed via the following relations:

$$\begin{aligned}
&\mathbf{k}' = \mathbf{k}\,, && \mathbf{l}' = \mathbf{l} + L\bar{\mathbf{m}} + \bar{L}\mathbf{m} + L\bar{L}\,\mathbf{l}\,, && \mathbf{m}' = \mathbf{m} + L\mathbf{k}\,,\\
&\mathbf{k}' = \mathbf{k} + K\bar{\mathbf{m}} + \bar{K}\mathbf{m} + K\bar{K}\,\mathbf{l}, && \mathbf{l}' = \mathbf{l}\,, && \mathbf{m}' = \mathbf{m} + K\mathbf{l},\\
&\mathbf{k}' = B\,\mathbf{k}\,, && \mathbf{l}' = B^{-1}\,\mathbf{l}\,, && \mathbf{m}' = e^{\mathrm{i}\phi}\,\mathbf{m}\,,
\end{aligned} \tag{I.1}$$

where K and L are any complex parameters, while B, ϕ are any real parameters. Together these six transformations generate the whole Lorentz group.

I.1 Curvature

The fundamental geometrical object describing curvature of the spacetime is the *Riemann tensor* $R_{\mu\nu\rho\sigma}$. Its contraction $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$ is the *Ricci tensor*. It is usually expressed in the *Newman–Penrose formalism*. This means that the components are projected to the appropriate null tetrad $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$, namely

$$\begin{aligned}
\Phi_{00} &= \frac{1}{2}R_{\mu\nu}k^{\mu}k^{\nu}\,, & \Phi_{22} &= \frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu}\,,\\
\Phi_{01} &= \frac{1}{2}R_{\mu\nu}k^{\mu}m^{\nu}\,, & \Phi_{12} &= \frac{1}{2}R_{\mu\nu}l^{\mu}m^{\mu}\,,\\
\Phi_{02} &= \frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu}\,, & \Phi_{11} &= \frac{1}{2}R_{\mu\nu}\left(k^{\mu}l^{\nu} - m^{\mu}\bar{m}^{\nu}\right)\,,
\end{aligned} \tag{I.2}$$

where Φ_{01}, Φ_{02}, Φ_{12} are complex. The trace of the Ricci tensor $R = R^{\mu}{}_{\mu}$ is the *Ricci scalar*.

Remaining 10 independent components of the Riemann tensor form the *Weyl tensor* defined by the expression

$$\begin{aligned}
C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} &- \frac{1}{2}\left(R_{\lambda\mu}g_{\kappa\nu} + R_{\kappa\nu}g_{\lambda\mu} - R_{\lambda\nu}g_{\kappa\mu} - R_{\kappa\mu}g_{\lambda\nu}\right)\\
&+ \frac{1}{6}R\left(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}\right)\,.
\end{aligned} \tag{I.3}$$

While the Ricci tensor is *directly* connected to the *stress-energy tensor* $T_{\mu\nu}$ of matter via the Einstein field equations (EFE), the Weyl tensor corresponds to the curvature components representing a "free gravitational energy". For vacuum solutions the Riemann tensor is fully determined by the Weyl tensor, $C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu}$.

All 10 independent components of the Weyl tensor are encoded in 5 complex Newman–Penrose scalars Ψ_A given by the expressions

$$\begin{aligned}
\Psi_0 &= C_{\kappa\lambda\mu\nu}\, k^\kappa m^\lambda k^\mu m^\nu \,, \\
\Psi_1 &= C_{\kappa\lambda\mu\nu}\, k^\kappa l^\lambda k^\mu m^\nu \,, \\
\Psi_2 &= C_{\kappa\lambda\mu\nu}\, k^\kappa m^\lambda \bar{m}^\mu l^\nu \,, \\
\Psi_3 &= C_{\kappa\lambda\mu\nu}\, l^\kappa k^\lambda l^\mu \bar{m}^\nu \,, \\
\Psi_4 &= C_{\kappa\lambda\mu\nu}\, l^\kappa \bar{m}^\lambda l^\mu \bar{m}^\nu \,.
\end{aligned} \tag{I.4}$$

The definitions (I.2) and (I.4) depend on the chosen null tetrad $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$ as there exists the freedom due to the Lorentz transformations (I.1).

I.2 Algebraic classification

A null vector k^μ is called the *principal null direction* if it satisfies the condition $k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0$, which is equivalent to

$$\Psi_0 = 0 \,,$$

see [47, 2].

The scalar Ψ_0 can be expressed in a different null tetrad using the Lorentz transformation (I.1) by rotating the vector $\mathbf{k}$ while keeping $\mathbf{l}$ fixed. The null rotation yields

$$\Psi_0' = \Psi_0 - 4K\,\Psi_1 + 6K^2\,\Psi_2 - 4K^3\,\Psi_3 + K^4\,\Psi_4 \overset{!}{=} 0 \tag{I.5}$$

where K is the complex parameter from (I.1), see [1, 2]. In every event of the spacetime there thus exist 4 principal null directions. Depending on their multiplicity we distinguish *the algebraic type* of the given metric.

More specifically, if k^μ is the *double* degenerate principal null direction, an appropriate rotation achieving $\Psi_0 = \Psi_1 = 0$ can be found, and the spacetime is of algebraic type II. If k^μ is a *triple* degenerate principal null direction, a transformation for getting $\Psi_0 = \Psi_1 = \Psi_2 = 0$ exists, and the spacetime is of *type III*. In the case of quadruply degenerate principal null direction k^μ, so called *type N*, the tetrad can be found for which $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$.

Similarly, we can rotate the vector l^μ while having k^μ fixed. In that matter we proceed "backwardly", i.e. we are looking for such L from (I.1) which could satisfy that $\Psi_4 = 0$. When the direction which is aligned with the null vector l^μ is double aligned, we can found a transformation which satisfies that $\Psi_4 = \Psi_3 = 0$ etc.

Depending on the existence of *two double degenerate directions* (corresponding to both k^μ and l^μ) we distinguish the *type D* (i.e. $\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4$). If not, we have the *type II*.

The remaining two types are the general algebraic *type I* with *all distinct* PNDs, and a trivial *conformally flat* solution (*type O*) for which all NP scalars are zero, $\Psi_A \equiv 0$, and for which does not make sense to define PNDs.

This algebraic classification is summarized in the following table:

Type	**Multiplicity**	Ψ_A **in an appropriate null tetrad**
I	1 1 1 1	$\Psi_0 = 0$, other components are nonzero
II	2 1 1	$\Psi_0 = \Psi_1 = 0$, other components are nonzero
D	2 2	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4,\ \Psi_2 \neq 0$
III	3 1	$\Psi_0 = \Psi_1 = \Psi_2 = 0,\ \Psi_3 \neq 0$
N	4	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,\ \Psi_4 \neq 0$
O	no PND	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$

There exists an *invariant way*, how to determine the algebraic type of any solution. Originally, the algorithm was presented by d'Inverno and Russell-Clark in 1971 [48], but we employ the notation from the textbook Stephani et al., 2003 [2].

This approach to classification is based on the scalar invariants I, J, K, L, N, defined using the NP scalars of the Weyl tensor (I.4) explicitly as:

$$\mathrm{I} \equiv \Psi_0\Psi_4 - 4\,\Psi_1\Psi_3 + 3\,\Psi_2^2\,, \qquad \mathrm{J} \equiv \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}\,, \tag{I.6}$$

$$\mathrm{K} \equiv \Psi_1\Psi_4^2 - 3\,\Psi_4\Psi_3\Psi_2 + 2\,\Psi_3^3\,, \qquad \mathrm{L} \equiv \Psi_2\Psi_4 - \Psi_3^2\,, \qquad \mathrm{N} \equiv 12\,\mathrm{L}^2 - \Psi_4^2\,\mathrm{I}\,.$$

Interestingly in vacuum, the real part of I is proportional to the *Kretschmann scalar*. This relation will be extended in the next section.

The advantage of these scalar invariants is that they can easily be used to determine the *algebraic structure* of a given metric. More precisely, only *an algebraically special spacetimes* (all types except the trivial type O and a general type I) comply the equation

$$\mathrm{I}^3 = 27\mathrm{J}^2\,. \tag{I.7}$$

Moreover, we distinguish whether the condition $\mathrm{I} = 0 = \mathrm{J}$ holds. If it holds, then we further need to investigate whether the equation $\mathrm{K} = 0 = \mathrm{N}$ holds as well. If yes, then the spacetime is of algebraic type II. If not the solution has the type D structure.

If the condition $\mathrm{I} = 0 = \mathrm{J}$ does not hold, then we further verify the condition $\mathrm{K} = 0 = \mathrm{L}$. If this condition does not hold then the solution is of algebraic type III, otherwise we have the type N metric.

All these possibilities with an appropriate structure of the NP scalars (I.4) are illustrated in the schematic diagram in Fig. I.1. For further details please see Griffiths and Podolský, 2009 [1], or Stephani et al., 2003 [2].

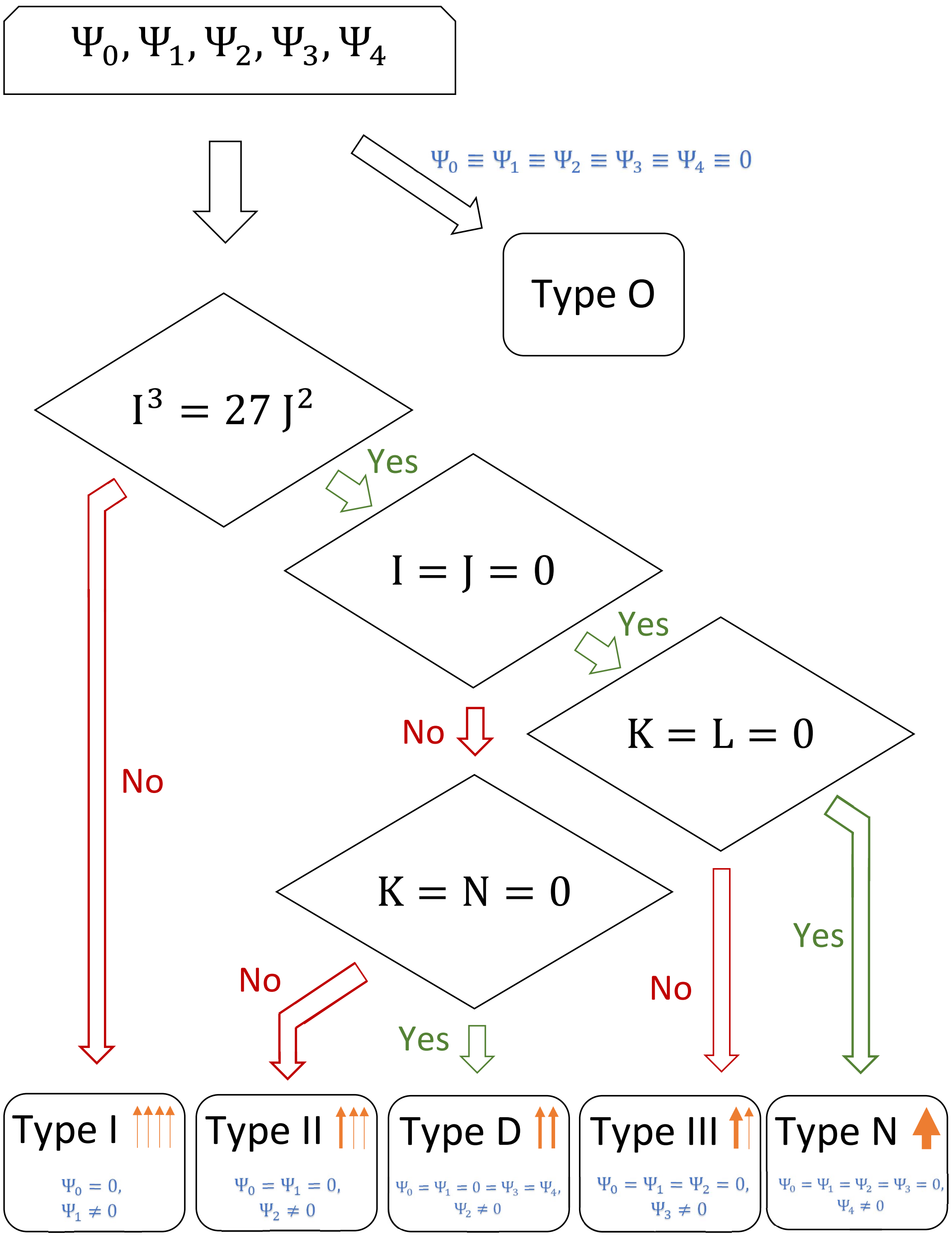

Figure I.1: A schematic diagram of the algebraic classification of a general metric, using the scalar invariants I, J, K, L and N computed from the NP scalars of the Weyl tensor (I.4) via the relations (I.6).

I.3 The scalar invariant I

There exist relations between the *Newman–Penrose scalars* Ψ_A, the *Weyl scalar*

$$\mathcal{C} \equiv C_{abcd}C^{abcd}$$

and the *Kretschmann scalar*

$$\mathcal{K} \equiv R_{abcd}R^{abcd}.$$

Let us assume a general Weyl tensor C_{abcd} and its components Ψ_A with respect to its *null directions* $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$.

Now, by introducing a *complex Weyl tensor*

$$C^*_{abcd} \equiv C_{abcd} + \mathrm{i}\, C^{\sim}_{abcd}\,, \tag{I.8}$$

where $C^{\sim}_{abcd}$ is the *right dual*

$$C^{\sim}_{abcd} \equiv \frac{1}{2}\epsilon_{cdef}\, C_{ab}{}^{ef}\,, \tag{I.9}$$

we find out that

$$C^*_{abcd}C^{*abcd} = (C_{abcd}C^{abcd} - C^{\sim}_{abcd}C^{\sim abcd}) + 2\mathrm{i}\, C^{abcd}C^{\sim}_{abcd}\,, \tag{I.10}$$

and $C^{\sim}_{abcd}C^{\sim abcd} = -C_{abcd}C^{abcd}$, so that

$$C^*_{abcd}C^{*abcd} = 2(C_{abcd}C^{abcd} + \mathrm{i}\, C^{abcd}C^{\sim}_{abcd})\,. \tag{I.11}$$

The complex Weyl tensor C^*_{abcd} can be also expressed in the bivector base $U_{ab} \equiv -l_a\bar{m}_b + l_b\bar{m}_a$, $V_{ab} \equiv k_a m_b - k_b m_a$, $W_{ab} \equiv m_a\bar{m}_b - m_b\bar{m}_a - k_a l_b + k_b l_a$, constructed from the null tetrad, as

$$\begin{aligned}\frac{1}{2}C^*_{abcd} &= \Psi_0\, U_{ab}U_{cd} + \Psi_1\Big(U_{ab}W_{cd} + W_{ab}U_{cd}\Big) \\ &\quad + \Psi_2\Big(V_{ab}U_{cd} + U_{ab}V_{cd} + W_{ab}W_{cd}\Big) + \Psi_3\Big(V_{ab}W_{cd} + W_{ab}V_{cd}\Big) + \Psi_4\, V_{ab}V_{cd}\,,\end{aligned} \tag{I.12}$$

see eq. (3.58) of [2].

Using the fact that all contractions vanish except $U_{ab}V^{ab} = 2$ and $W_{ab}W^{ab} = -4$, we obtain

$$C^*_{abcd}C^{*abcd} = 32\Big(\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2\Big) \equiv 32\mathrm{I}\,. \tag{I.13}$$

From (I.11) it follows that

$$C_{abcd}C^{abcd} = 16\ \mathcal{R}e\,(\mathrm{I})\,. \tag{I.14}$$

For type D spacetimes, for which the only non-vanishing component of the Weyl tensor is Ψ_2, we simply get

$$\mathcal{C} \equiv C_{abcd}C^{abcd} = 48\ \mathcal{R}e\,(\Psi_2^2)\,. \tag{I.15}$$

I.4 Relation to the Kretschmann scalar

From the definition of the Weyl tensor (I.3), we easily prove

$$\mathcal{C} = \mathcal{K} - 2R_{ab}R^{ab} + \frac{1}{3}R^2\,. \tag{I.16}$$

Splitting up the Ricci tensor R_{ab} into its *trace-less* tensor S_{ab} and the Ricci scalar R, we get, in terms of the NP scalars Φ_{ab},

$$
\begin{aligned}
R_{ab} \;\equiv\; & S_{ab} + \frac{1}{4} R g_{ab} \\
= & \, 2\Phi_{00} l_a l_b - 2\Phi_{01}(l_a \bar{m}_b + \bar{m}_a l_b) \\
& -2\bar{\Phi}_{01}(l_a m_b + m_a l_b) + 2\Phi_{02}\bar{m}_a \bar{m}_b + 2\bar{\Phi}_{02} m_a m_b \\
& +2\Phi_{11}(k_a l_b + l_a k_b + \bar{m}_a m_b + m_a \bar{m}_b) - 2\Phi_{12}(k_a \bar{m}_b + \bar{m}_a k_b) \\
& -2\bar{\Phi}_{12}(k_a m_b + m_a k_b) + 2\Phi_{22} k_a k_b + \frac{1}{4} R g_{ab} \,. \qquad \text{(I.17)}
\end{aligned}
$$

By contraction

$$
R_{ab}R^{ab} = 8(\Phi_{00}\Phi_{22} - 2\Phi_{01}\bar{\Phi}_{12} - 2\bar{\Phi}_{01}\Phi_{12} + \bar{\Phi}_{02}\Phi_{02} + 2\Phi_{11}^2) + \frac{1}{4}R^2 \qquad \text{(I.18)}
$$

(see also the equation (19) of [49]). With the expressions (I.14), and (I.16), we get

$$
\mathcal{K} = 16 \; \mathcal{R}e\,(\mathrm{I}) + 16(\Phi_{00}\Phi_{22} - 2\Phi_{01}\bar{\Phi}_{12} - 2\bar{\Phi}_{01}\Phi_{12} + \bar{\Phi}_{02}\Phi_{02} + 2\Phi_{11}^2) + \frac{1}{6}R^2. \qquad \text{(I.19)}
$$

Specially *for the Plebański–Demiański metric*, which is of type D solution with the only non-zero component of the Ricci tensor Φ_{11} and the Ricci scalar equal to 4Λ (see the following section, eq. (II.16), we obtain

$$
\mathcal{K} = 48 \; \mathcal{R}e\,(\Psi_2^2) + 32\,\Phi_{11}^2 + \frac{8}{3}\,\Lambda^2 \,. \qquad \text{(I.20)}
$$

This relation will be importing in the arguments contained in the following chapters.

II. Plebański–Demiański class of solutions

This Doctoral Thesis mainly concentrates on the Plebański–Demiański family of exact solutions – a general class of type D spacetimes with a double-aligned non-null electromagnetic field and (possibly) non-zero cosmological constant. This large class of solutions was originally found by Debever in 1971 [28] and later reformulated in a more convenient form by Plebański and Demiański in 1976 [29], namely

$$\begin{aligned}\mathrm{d}\hat{s}^2 = \frac{1}{(1-\hat{p}\hat{r})^2}\bigg[&-\frac{\hat{Q}(\hat{r})}{\hat{r}^2+\hat{p}^2}\,(\mathrm{d}\hat{\tau}-\hat{p}^2\mathrm{d}\hat{\sigma})^2+\frac{\hat{r}^2+\hat{p}^2}{\hat{Q}(\hat{r})}\,\mathrm{d}\hat{r}^2\\ &+\frac{\hat{P}(\hat{p})}{\hat{r}^2+\hat{p}^2}\,(\mathrm{d}\hat{\tau}+\hat{r}^2\mathrm{d}\hat{\sigma})^2+\frac{\hat{r}^2+\hat{p}^2}{\hat{P}(\hat{p})}\,\mathrm{d}\hat{p}^2\bigg]\,, \end{aligned} \tag{II.1}$$

with the metric functions

$$\hat{P}(\hat{p}) = \hat{k} + 2\hat{n}\hat{p} - \hat{\epsilon}\hat{p}^2 + 2\hat{m}\hat{p}^3 - (\hat{k} + \hat{e}^2 + \hat{g}^2 + \tilde{\Lambda}/3)\,\hat{p}^4\,, \tag{II.2}$$

$$\hat{Q}(\hat{r}) = (\hat{k} + \hat{e}^2 + \hat{g}^2) - 2\hat{m}\hat{r} + \hat{\epsilon}\hat{r}^2 - 2\hat{n}\hat{r}^3 - (\hat{k} + \tilde{\Lambda}/3)\,\hat{r}^4\,. \tag{II.3}$$

This metric contains seven arbitrary real parameters $\hat{m}$, $\hat{n}$, $\hat{e}$, $\hat{g}$, $\hat{\epsilon}$, $\hat{k}$, and $\tilde{\Lambda}$, of which the first four are, according to Plebański and Demiański [29], somehow connected to the *mass*, *NUT parameter*, *electric and magnetic charges*, respectively, while $\tilde{\Lambda}$ is the *cosmological constant*.[1] The meaning of $\hat{\epsilon}$ and $\hat{k}$ was unclear.

II.1 The Griffiths–Podolský representation

Our study is based on another representation of this metric introduced by Griffiths and Podolský in 2005 [38]–[40]. We will briefly describe the coordinate and the parametric transformation they have used.

First of all, we rescale the original coordinates and the parameters, see [1],

$$\begin{aligned}&\hat{p} = \sqrt{\alpha\omega}\,p\,,\quad \hat{r} = \sqrt{\frac{\alpha}{\omega}}\,r\,,\quad \hat{\sigma} = \sqrt{\frac{\omega}{\alpha^3}}\,\sigma\,,\quad \hat{\tau} = \sqrt{\frac{\omega}{\alpha}}\,\tau\,,\\ &\hat{m}+\mathrm{i}\hat{n} = \Big(\frac{\alpha}{\omega}\Big)^{3/2}(\tilde{m}+\mathrm{i}\tilde{n})\,,\quad \hat{e}+\mathrm{i}\hat{g} = \frac{\alpha}{\omega}(\tilde{e}+\mathrm{i}\tilde{g})\,,\quad \hat{\epsilon} = \frac{\alpha}{\omega}\epsilon\,,\quad \hat{k} = \alpha^2 k\,,\end{aligned} \tag{II.4}$$

where $\tilde{m}$, $\tilde{n}$, $\tilde{e}$, $\tilde{g}$, ϵ, α, k, ω, and $\tilde{\Lambda}$ is a new set of arbitrary real parameters, and two of them can be set for convenience. An appropriate choice of the twist parameter ω is a mainstay of our papers [45, 46], and will be discussed in chapters 2 and 3.

[1] Note that in this thesis we use both Λ and $\tilde{\Lambda}$ as cosmological constants. The different labeling is only to emphasize that the solutions are related to the Einstein field equations (EFE) with *different* values of cosmological constant. This is caused by the conformal rescaling of the metric $\mathrm{d}\tilde{s}^2 \to \mathrm{d}s^2$, which also changes the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R.

The metric (II.1)–(II.3) transforms into

$$\mathrm{d}\tilde{s}^2 = \frac{1}{(1-\alpha pr)^2}\Bigg[-\frac{\mathrm{Q}(r)}{r^2+\omega^2p^2}\,(\mathrm{d}\tau-\omega p^2\mathrm{d}\sigma)^2+\frac{r^2+\omega^2p^2}{\mathrm{Q}(r)}\,\mathrm{d}r^2 +\frac{\mathrm{P}(p)}{r^2+\omega^2p^2}\,(\omega\mathrm{d}\tau+r^2\mathrm{d}\sigma)^2+\frac{r^2+\omega^2p^2}{\mathrm{P}(p)}\,\mathrm{d}p^2\Bigg]\,, \tag{II.5}$$

where

$$\mathrm{P}(p)=k+2\omega^{-1}\tilde{n}p-\epsilon p^2+2\alpha\tilde{m}p^3-\Big[\alpha^2(\omega^2k+\tilde{e}^2+\tilde{g}^2)+\omega^2\tilde{\Lambda}/3\Big]\,p^4\,, \tag{II.6}$$

$$\mathrm{Q}(r)=(\omega^2k+\tilde{e}^2+\tilde{g}^2)-2\tilde{m}r+\epsilon r^2-2\alpha\omega^{-1}\tilde{n}r^3-(\alpha^2k+\tilde{\Lambda}/3)r^4\,. \tag{II.7}$$

Let us also mention an interesting relation between these metric functions (see the unnumbered equation on page 340 of [40]), namely

$$\mathrm{Q}(r) = -\alpha^2r^4\,\mathrm{P}\Big(\frac{1}{\alpha r}\Big)-\frac{\tilde{\Lambda}}{3}\Big(\frac{\omega^2}{\alpha^2}+r^4\Big)\,. \tag{II.8}$$

The most convenient null tetrad naturally adapted to the metric (II.1)–(II.3) is given by the vectors

$$\begin{aligned}
\mathbf{k} &= \frac{1-\alpha pr}{\sqrt{2(r^2+\omega^2p^2)}}\Bigg[\frac{1}{\sqrt{\mathrm{Q}(r)}}\Big(r^2\,\partial_\tau-\omega\,\partial_\sigma\Big)-\sqrt{\mathrm{Q}(r)}\,\partial_r\Bigg]\,,\\
\mathbf{l} &= \frac{1-\alpha pr}{\sqrt{2(r^2+\omega^2p^2)}}\Bigg[\frac{1}{\sqrt{\mathrm{Q}(r)}}\Big(r^2\,\partial_\tau-\omega\,\partial_\sigma\Big)+\sqrt{\mathrm{Q}(r)}\,\partial_r\Bigg]\,,\\
\mathbf{m} &= \frac{1-\alpha pr}{\sqrt{2(r^2+\omega^2p^2)}}\Bigg[-\frac{1}{\sqrt{\mathrm{P}(p)}}\Big(\omega p^2\,\partial_\tau+\partial_\sigma\Big)+\mathrm{i}\sqrt{\mathrm{P}(p)}\,\partial_p\Bigg]\,.
\end{aligned} \tag{II.9}$$

In terms of this null tetrad, the spin coefficients are given by

$$\kappa=\sigma=\lambda=\nu=0\,, \tag{II.10}$$

$$\varrho=\mu=\sqrt{\frac{Q}{2(r^2+\omega^2p^2)}}\,\frac{1+\mathrm{i}\,\alpha\omega p^2}{r+\mathrm{i}\,\omega p}\,, \tag{II.11}$$

$$\tau=\pi=\sqrt{\frac{P}{2(r^2+\omega^2p^2)}}\,\frac{\omega-\mathrm{i}\,\alpha r^2}{r+\mathrm{i}\,\omega p}\,, \tag{II.12}$$

$$\epsilon=\gamma=\frac{1}{4}\sqrt{\frac{Q}{2(r^2+\omega^2p^2)}}\,\Big[2\,\frac{1-\alpha pr}{r+\mathrm{i}\,\omega p}-2\alpha p-(1-\alpha pr)\,\frac{Q'}{Q}\Big]\,, \tag{II.13}$$

$$\alpha=\beta=\frac{1}{4}\sqrt{\frac{P}{2(r^2+\omega^2p^2)}}\,\Big[2\omega\,\frac{1-\alpha pr}{r+\mathrm{i}\,\omega p}+2\mathrm{i}\,\alpha r+\mathrm{i}\,(1-\alpha pr)\,\frac{P'}{P}\Big]\,, \tag{II.14}$$

which indicates that the null congruences tangent to the vectors k^μ and l^μ are both *geodesic*, *shear-free* but with *non-zero expansion* and possibly *twist*. Moreover, the twist of each congruence is proportional to the parameter ω, which gives us a hint for fixing it.

Utilizing the null tetrad (II.9), the only non-zero component of the Weyl tensor in the Newman–Penrose formalism equals to

$$\Psi_2=-(\tilde{m}+\mathrm{i}\,\tilde{n})\left(\frac{1-\alpha pr}{r+\mathrm{i}\,\omega p}\right)^3+(\tilde{e}^2+\tilde{g}^2)\left(\frac{1-\alpha pr}{r+\mathrm{i}\,\omega p}\right)^3\frac{1+\alpha pr}{r-\mathrm{i}\,\omega p}\,, \tag{II.15}$$

confirming that the metric is of algebraic type D. The corresponding projection of the Ricci tensor onto the null tetrad (II.9) simply yields

$$\Phi_{11} = \frac{1}{2}\,(\tilde{e}^2 + \tilde{g}^2)\,\frac{(1-\alpha p r)^4}{(r^2+\omega^2 p^2)^2}\,, \qquad \Lambda_{NP} \equiv \tfrac{1}{24}\,R = \tfrac{1}{6}\,\tilde{\Lambda}\,. \tag{II.16}$$

Both relations (II.15) and (II.16) indicate the presence of the curvature singularity at $p = 0 = r$.

II.2 Black hole solutions of type D

The character of the spacetime (II.5) is determined by the metric functions $\mathrm{Q}(r)$ and $\mathrm{P}(p)$. We focus the function $\mathrm{P}(p)$. It turns out to be appropriate to change the coordinates by the following transformation

$$p = \frac{l}{\omega} + \frac{a}{\omega}\,\tilde{p}\,, \qquad \tau = t - \frac{(l+a)^2}{a}\,\varphi\,, \qquad \sigma = -\frac{\omega}{a}\,\varphi\,, \tag{II.17}$$

where a and l are new arbitrary parameters, later interpreted as the *Kerr-like rotation* and the *NUT parameter*. With these changes, the metric (II.5)–(II.7) becomes, see [1],

$$\begin{aligned} \mathrm{d}\tilde{s}^2 \;=\; & \frac{1}{\Omega^2}\Bigg[-\frac{\mathrm{Q}(r)}{\rho^2}\Big[\mathrm{d}t - \Big(a(1-\tilde{p}^2) + 2l(1-\tilde{p})\Big)\,\mathrm{d}\varphi\Big]^2 + \frac{\rho^2}{\mathrm{Q}(r)}\,\mathrm{d}r^2 \\ & + \frac{\rho^2}{\tilde{\mathrm{P}}(\tilde{p})}\,\mathrm{d}\tilde{p}^2 + \frac{\tilde{\mathrm{P}}(\tilde{p})}{\rho^2}\Big[a\,\mathrm{d}t - \Big(r^2 + (a+l)^2\Big)\,\mathrm{d}\varphi\Big]^2\Bigg]\,, \end{aligned} \tag{II.18}$$

where

$$\Omega = 1 - \frac{\alpha}{\omega}(l + a\tilde{p})\,r\,, \tag{II.19}$$

$$\rho^2 = r^2 + (l + a\tilde{p})^2\,, \tag{II.20}$$

$$\tilde{\mathrm{P}}(\tilde{p}) = a_0 + a_1\tilde{p} + a_2\tilde{p}^2 + a_3\tilde{p}^3 + a_4\tilde{p}^4\,, \tag{II.21}$$

$$\mathrm{Q}(r) = (\omega^2 k + \tilde{e}^2 + \tilde{g}^2) - 2\tilde{m}r + \epsilon r^2 - 2\alpha\omega^{-1} n r^3 - (\alpha^2 k + \tilde{\Lambda}/3)r^4\,, \tag{II.22}$$

and

$$\begin{aligned} a_0 &= \frac{1}{a^2}\left(\omega^2 k + 2\tilde{n}l - \epsilon l^2 + 2\alpha\frac{l^3}{\omega}\tilde{m} - \left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right] l^4\right)\,, \\ a_1 &= \frac{2}{a}\left(\tilde{n} - \epsilon l + 3\alpha\frac{l^2}{\omega}\tilde{m} - 2\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right] l^3\right)\,, \\ a_2 &= -\epsilon + 6\alpha\frac{l}{\omega}\tilde{m} - 6\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right] l^2\,, \\ a_3 &= 2\alpha\frac{a}{\omega}\tilde{m} - 4\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right] a l\,, \\ a_4 &= -\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right] a^2\,. \end{aligned} \tag{II.23}$$

Concentrating on the metric function $\tilde{\mathrm{P}}(\tilde{p})$, the most physically relevant case corresponds to the situation of *at least two distinct roots*. In such a case, we can utilize the coordinate freedom to set these roots at $\tilde{p} = \pm 1$, that is

$$\tilde{\mathrm{P}}(\tilde{p}) = (1 - \tilde{p}^2)(a_0 - a_3\tilde{p} - a_4\tilde{p}^2)\,. \tag{II.24}$$

Comparing (II.21) and (II.24) we get two constraints, namely $a_1 = -a_3$ and $a_2 = -a_0 - a_4$, leading directly to the expressions for ϵ and $\tilde{n}$, namely

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}\tilde{m} - (a^2 + 3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right], \tag{II.25}$$

$$\tilde{n} = \frac{\omega^2 k l}{a^2 - l^2} - \alpha\frac{(a^2 - l^2)}{\omega}\tilde{m} + (a^2 - l^2)\, l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right]. \tag{II.26}$$

Furthermore, in terms of these expressions involving the parameter a_0 (II.23) can be reexpressed as

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right)k = a_0 + 2\alpha\frac{l}{\omega}\tilde{m} - 3\alpha^2\frac{l^2}{\omega^2}(\tilde{e}^2 + \tilde{g}^2) - l^2\tilde{\Lambda}\,. \tag{II.27}$$

We have to impose a condition for positive value of the parameter a_0 to preserve that $\tilde{\mathrm{P}}(\tilde{p})$ is positive for $\tilde{p} \in [-1, 1]$. Using the freedom, we can consider only the

$$a_0 = 1$$

case.

Using the natural substitution $\tilde{p} = \cos\theta$, where $\theta \in [0, \pi]$, we finally obtain a class of solutions describing a *general Plebański–Demiański black holes* of type D in a simple form (c.f. eq. (16.18) of [1]):

$$\begin{aligned} \mathrm{d}\tilde{s}^2 \;=\; & \frac{1}{\Omega^2}\Bigg[-\frac{\mathrm{Q}(r)}{\rho^2}\Big[\mathrm{d}t - \Big(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\Big)\mathrm{d}\varphi\Big]^2 + \frac{\rho^2}{\mathrm{Q}(r)}\mathrm{d}r^2 \\ & + \frac{\rho^2}{\mathcal{P}(\theta)}\mathrm{d}\theta^2 + \frac{\mathcal{P}(\theta)}{\rho^2}\sin^2\theta\Big[a\,\mathrm{d}t - \Big(r^2 + (a+l)^2\Big)\mathrm{d}\varphi\Big]^2\Bigg], \end{aligned} \tag{II.28}$$

where

$$\Omega = 1 - \frac{\alpha}{\omega}(l + a\cos\theta)\, r\,, \tag{II.29}$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2\,, \tag{II.30}$$

$$\mathcal{P}(\theta) \;\Big(\equiv \sin^2\theta\,\tilde{\mathrm{P}}(\cos\theta)\Big) = 1 - a_3\cos\theta - a_4\cos^2\theta\,, \tag{II.31}$$

$$\mathrm{Q}(r) = (\omega^2 k + \tilde{e}^2 + \tilde{g}^2) - 2\tilde{m}r + \epsilon r^2 - 2\alpha\omega^{-1} n r^3 - (\alpha^2 k + \tilde{\Lambda}/3)r^4\,. \tag{II.32}$$

This metric simplifies the Plebański–Demiański solution to a very convenient form, and presents a clear and direct generalization of the well-known metrics (see the scheme on Fig. II.1). The only remaining problem is a rather complex form of the explicit metric functions $\mathcal{P}(\theta)$ and $\mathrm{Q}(r)$, and the ambiguity of the parameter ω. In this work we have succeeded in solving these open problems, and we offer a new useful reparametrization of this metric. More specifically, we use the metric form (II.28), which is already in the most convenient form, and we *simplify the metric functions* (II.31), (II.32). We also conveniently fix the twist parameter ω. All this is done in Chapters 2 and 3.

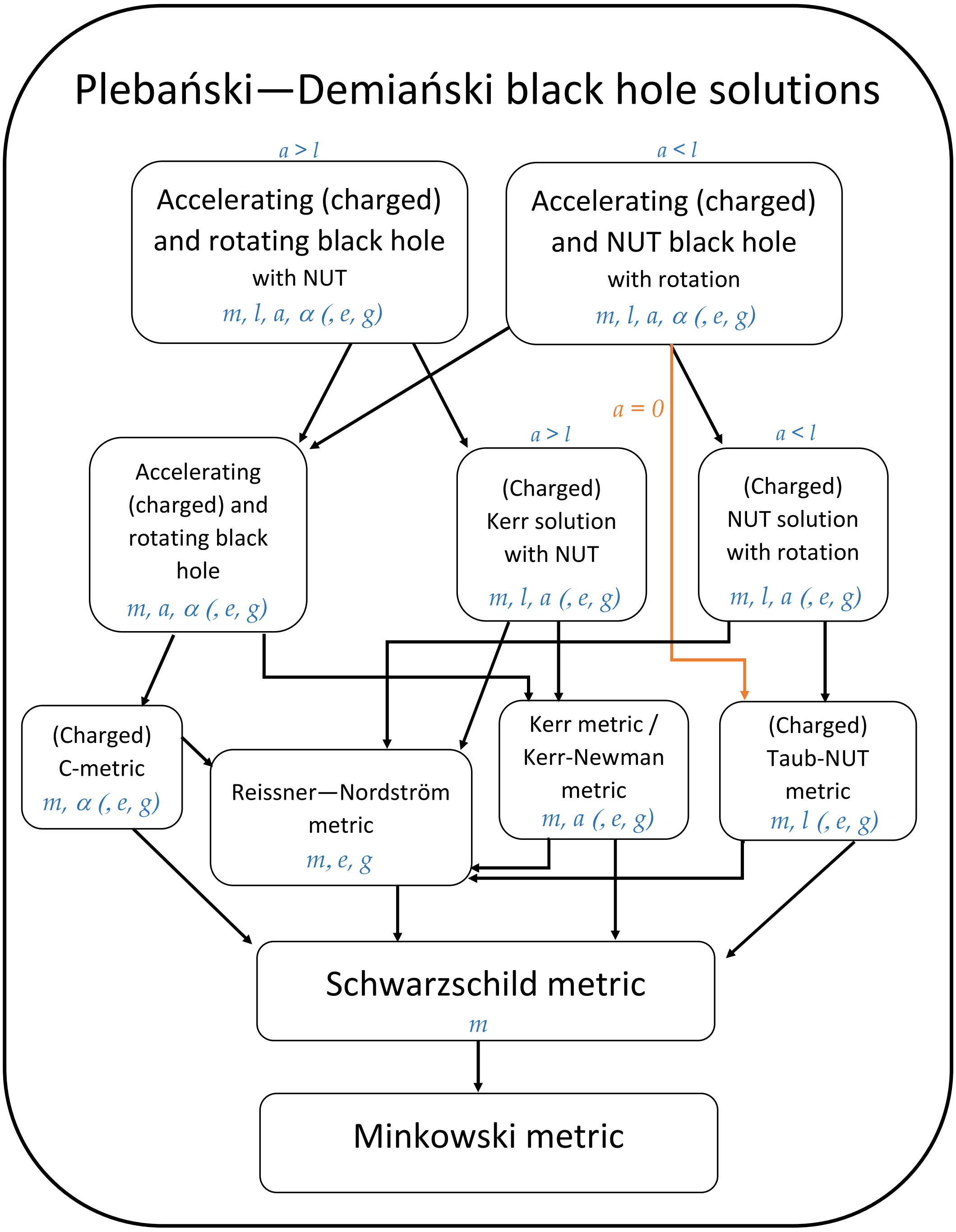

Figure II.1: Schematic diagram of a Plebański–Demiański class of type D black hole solutions (II.28)–(II.32) in a flat universe background ($\tilde{\Lambda} = 0$), see [1, 40] for a similar diagram. A general metric solution can be separated into two cases – when the Kerr-like rotation parameter is bigger than the NUT parameter $a > l$, or when the NUT parameter exceeds the Kerr-like rotation parameter, $l > a$. The diagram illustrates all relevant transitions to simpler black hole solutions, including the C-metric, the Taub–NUT metric, Reissner–Nordström metric, Kerr–Newman metric, or Schwarzschild metric and Minkowski flat space, respectively. We can see that no (possibly charged) accelerating Taub–NUT is included, because setting $a = 0$ simplifies the metric directly to the (possibly charged) Taub–NUT metric.

III. Deviating solution – accelerating Taub–NUT black holes

Now, we come to an important observation. Although the Plebański–Demiański metric (II.28) generally includes an accelerating and rotating Taub–NUT black hole, its *non-rotating* version ($a = 0$) turns out not to be possible, see the Figure II.1. This can be seen from (II.28)–(II.32) by fixing the twist parameter as $\omega = a^{-1}$.

Firstly, we express the parameter k given by (II.27) for such a fixed ω, and expand it in the powers of the Kerr-like rotation parameter a as

$$k|_{\omega=a^{-1}} = -a^2 l^2 \left(1 - l^2 \tilde{\Lambda}\right) + \mathcal{O}(a^3)\,. \tag{III.1}$$

Now we put this into the expressions (II.25), (II.26), for the parameters ϵ, $\tilde{n}$, that is

$$\epsilon|_{\omega=a^{-1}} = 1 - 2l^2\tilde{\Lambda} + \mathcal{O}(a)\,, \tag{III.2}$$

$$\tilde{n}|_{\omega=a^{-1}} = \left(1 - \frac{4}{3}\, l^2\,\tilde{\Lambda}\right) l + \mathcal{O}(a)\,, \tag{III.3}$$

which simplifies the metric function $\mathrm{Q}(r)$ to

$$\mathrm{Q}(r)|_{\omega=a^{-1}} = -l^2\left(1 - l^2\tilde{\Lambda}\right) + \tilde{e}^2 + \tilde{g}^2 - 2\tilde{m}r + (1 - 2l^2\tilde{\Lambda})r^2 - \frac{\tilde{\Lambda}}{3}r^4 + \mathcal{O}(a)\,.$$

We also substitute these expansions into (II.23) for the parameters a_3, a_4 defining the metric function $\mathcal{P}(\theta)$ via (II.31), and we obtain

$$a_3|_{\omega=a^{-1}} = \mathcal{O}(a) \qquad \text{and} \qquad a_4|_{\omega=a^{-1}} = \mathcal{O}(a^2)\,. \tag{III.4}$$

If we now set $a = 0$, we obtain

$$\Omega = 1 \qquad \text{and} \qquad \mathcal{P}(\theta) = 1\,. \tag{III.5}$$

The resulting solution can thus be expressed as

$$\mathrm{d}\tilde{s}^2 \;=\; -f(r)\left(\mathrm{d}t - 4l\sin^2\frac{\theta}{2}\,\mathrm{d}\varphi\right)^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + \left(r^2 + l^2\right)\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\varphi^2\right),$$

where

$$f(r) \;=\; \frac{1}{r^2 + l^2}\left[r^2 - l^2 - 2\tilde{m}r + \tilde{e}^2 + \tilde{g}^2 - \tilde{\Lambda}\left(\frac{1}{3}r^4 + 2l^2r^2 + l^4\right)\right].$$

This is the non-accelerating charged Taub–NUT black hole in the (anti-)de Sitter background (see eq. (12.19) in [1] and [36]), therefore we do not expect any solution describing the accelerating Taub–NUT in the Plebański–Demiański class of metrics.

III.1 The Chng–Mann–Stelea solution

It was thus a surprise when such a solution was found in a work of Brenda Chng, Robert Mann and Cristian Stelea in 2006 [37]. It was constructed via the generating method which utilizes the SL(2, R) symmetry of the reduced Lagrangian (for more details see the second section of [37]). This method has been applied on the accelerating version of the Zippoy–Vorhees metric (on the metric form presented by Teo in 2006 [50]). The metric generated by this method was presented in the form

$$\begin{aligned} \mathrm{d}\bar{s}^2 = &-\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2}\frac{C^2\delta}{\bar{H}(x,y)}\left[\mathrm{d}\bar{t}+\frac{1}{C}\left(\frac{(1-x^2)F(x)}{\alpha^2(x-y)^2}+\frac{2Mx}{\alpha}\right)\mathrm{d}\varphi\right]^2 \\ &+\frac{\bar{H}(x,y)}{\alpha^2(x-y)^2}\left[(1-x^2)F(x)\mathrm{d}\varphi^2+\frac{\mathrm{d}x^2}{(1-x^2)F(x)}+\frac{\mathrm{d}y^2}{(y^2-1)F(y)}\right], \end{aligned} \tag{III.6}$$

where

$$F(x) = 1 + 2\alpha M\, x\,, \tag{III.7}$$

$$F(y) = 1 + 2\alpha M\, y\,, \tag{III.8}$$

$$\bar{H}(x,y) = \frac{1}{2}+\frac{\delta}{2}\left(\frac{(y^2-1)F(y)}{\alpha^2\,(x-y)^2}\right)^2 . \tag{III.9}$$

In the following Chapter 1 we will investigate this solution in full detail. More specifically, we will prove that this solution solves the vaccum Einstein field equation, and examine the algebraic type of this solution. This will clarify its existence *outside* the Plebański–Demiański class of solutions. A thorough physical and geometrical analysis of this solution will then also be provided.

New results

1. Accelerating Taub–NUT black holes

This chapter is based on the paper *Accelerating NUT black holes* [44] by Podolský and Vrátný, published in 2020 in the journal Physical Review D **102**, 084024.

1.1 Checking the vacuum equations

Finding a solution of an accelerating Taub–NUT black hole by Chng, Mann and Stelea [37] was surprising, as it was unexpected. Therefore it was desirable to prove conclusively whether it is truly a *vacuum solution* of the Einstein field equations. And if so, to clarify its relation to the Plebański–Demiański class of type D solutions.

First of all, we reparametrized the metric (III.6)–(III.9) introduced by Chng, Mann and Stelea, using $\tau = 2\lambda\,(\alpha^2 C\,\bar{t} - \varphi)$ and $\mathrm{d}\bar{s}^2 \to \mathrm{d}s^2 \equiv 2\,\mathrm{d}\bar{s}^2$, where a new real parameter $\lambda \equiv \frac{\sqrt{\delta}}{\alpha^2}$ was introduced. This approach yield a better representation for subsequent direct computations, namely

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2 H(x,y)}\left[\mathrm{d}\tau + 2\lambda\,F(x)\frac{1-2xy+y^2}{(x-y)^2}\,\mathrm{d}\varphi\right]^2 \\ &+\frac{H(x,y)}{\alpha^2(x-y)^2}\left[(1-x^2)F(x)\,\mathrm{d}\varphi^2 + \frac{\mathrm{d}x^2}{(1-x^2)F(x)} + \frac{\mathrm{d}y^2}{(y^2-1)F(y)}\right],\end{aligned} \tag{1.1}$$

where

$$H(x,y) = 1 + \lambda^2\,\frac{(y^2-1)^2F^2(y)}{(x-y)^4}\,, \tag{1.2}$$

and $F(\xi) = 1 + 2\alpha M\xi$.

For computing all the components of the Ricci tensor we developed a computer algebra method optimized for any stationary, axially symmetric solution – final equations of this method are presented in Appendix A, eq. (A11). Indeed, using this direct method the computations were lasting via Mathematica on a standard PC just around *40 seconds*, and resulted in *all zero components*. Therefore, we could confidently say that the metric (1.1)–(1.2) (or (III.6)–(III.9), respectively) *is indeed a vacuum solution of Einstein's field equations*.

In addition, we developed an equivalent and independent method utilizing the well known relations between the curvature tensors of conformally related metrics. This alternative approach (described in the Appendix B) was applied on the metric $\hat{g}_{ab} = \Omega^2\,g_{ab}$, where $\Omega^2 \equiv \alpha^2\,(1-x^2)F(x)\,(y^2-1)F(y)\,(x-y)^{10}\,H(x,y)$, verifying that all the Ricci components are identically zero. The second method took just around *15 seconds*.

1.2 Algebraic type of the spacetime

Next, it was necessary to understand the relation between this accelerating Taub–NUT metric and the Plebański–Demiański family, which seems not to involve such a solution (see Chapter III).

To this end, we had to determine the algebraic structure of the new solution. Adopting a natural null tetrad

$$
\begin{aligned}
\boldsymbol{k} &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\,\partial_t + \frac{1}{\sqrt{g_{yy}}}\,\partial_y\right),\\
\boldsymbol{l} &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\,\partial_t - \frac{1}{\sqrt{g_{yy}}}\,\partial_y\right),\\
\boldsymbol{m} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{g_{tt}}{D}}\,\partial_\varphi + \frac{g_{t\varphi}}{\sqrt{D g_{tt}}}\,\partial_t - \frac{\mathrm{i}}{\sqrt{g_{xx}}}\,\partial_x\right),
\end{aligned}
\tag{1.3}
$$

with $D = g_{tt}\,g_{\varphi\varphi} - g_{t\varphi}^2$, we were able to compute the NP scalars Ψ_A of the Weyl tensor. These were simplified to the following factorized form:

$$
\begin{aligned}
\Psi_0 = \Psi_4 &= -3\,\alpha^2\lambda\,(1-x^2)F(x)\,(y^2-1)F(y)\,\Xi(x,y)\,,\\
\Psi_1 = \Psi_3 &= -3\,\alpha^2\lambda\,\mathrm{i}\,\sqrt{(1-x^2)F(x)}\,\sqrt{(y^2-1)F(y)}\,\Sigma(x,y)\,\Xi(x,y)\,,\\
\Psi_2 &= \Big[\alpha^2\lambda\,\Pi(x,y) + \mathrm{i}\,\alpha^3 M(x-y)^5\Big]\,\Xi(x,y)\,,
\end{aligned}
\tag{1.4}
$$

where

$$
\begin{aligned}
\Xi(x,y) &= \frac{\mathrm{i}\,(x-y)^4}{\Big[(x-y)^2 - \lambda\,\mathrm{i}\,(y^2-1)(1+2\alpha M y)\Big]^3}\,,\\
\Sigma(x,y) &= xy - 1 - \alpha M x\,(1-3y^2) - \alpha M y\,(1+y^2)\,,\\
\Pi(x,y) &= 2\,\Sigma^2(x,y) - \Big[(1-x^2)F(x) - \alpha M(x-y)^3\Big]\,(y^2-1)F(y)\,.
\end{aligned}
\tag{1.5}
$$

Following the procedure summarized in Sec. I.2 (especially the scheme on Fig. I.1), we computed the scalar invariants I and J (I.6) and verified that the condition $\mathrm{I}^3 = 27\mathrm{J}^2$ *does not hold.* This clearly means that the metric (1.1) is of a general *algebraic type I*, and therefore *cannot be included in the wide Plebański–Demiański class of type D solutions.*

1.2.1 The principal null directions

Using the NP scalars (1.4)–(1.5) we were able to transform the null tetrad (1.3) via the Lorentz transformations (I.1),

$$
\boldsymbol{k}' = \boldsymbol{k} + K\,\bar{\boldsymbol{m}} + \bar{K}\,\boldsymbol{m} + K\bar{K}\,\boldsymbol{l}, \qquad \boldsymbol{l}' = \boldsymbol{l}, \qquad \boldsymbol{m}' = \boldsymbol{m} + K\,\boldsymbol{l}\,, \tag{1.6}
$$

and compute explicitly the four *distinct principal null directions.* These correspond to the null vector $\boldsymbol{k}'$ obtained by (1.6) with the following four complex coefficients:

$$
K_i = \frac{\kappa_{1,2} \pm \sqrt{\kappa_{1,2}^2 - 4}}{2}\,, \qquad \kappa_{1,2} = \frac{-2\Psi_1 \pm \sqrt{4\Psi_1^2 - 2\Psi_0(3\Psi_2 - \Psi_0)}}{\Psi_0}\,. \tag{1.7}
$$

Thus, we answered both main questions concerning this new interesting solution.

1.3 A new convenient form of the metric

Although the metric (1.1)–(1.2) was useful for the hard computations, it was no more suitable for geometrical analysis and the physical interpretation than the original metric form (III.6)–(III.9).

Therefore, we applied a specific set of transformations on the metric (III.6)–(III.9), namely

$$x = -\cos\theta\,, \quad y = -\frac{1}{\alpha\,(r - r_-)}\,, \quad \bar{t} = \frac{r_+ - r_-}{2\alpha l C}\,t\,, \tag{1.8}$$

and we introduced the *NUT parameter* l and a new real *mass parameter* m via the relations

$$l = \frac{\sqrt{\delta}}{\alpha^2}\,r_+\,, \qquad m = \sqrt{M^2 - l^2}\,. \tag{1.9}$$

Specific combinations of the mass parameter m and the NUT parameter l can be conveniently defined as

$$\begin{aligned} r_+ &\equiv m + \sqrt{m^2 + l^2}\,,\\ r_- &\equiv m - \sqrt{m^2 + l^2}\,. \end{aligned} \tag{1.10}$$

Finally, we rescaled the metric as $\mathrm{d}s^2 \equiv \frac{2\,r_+}{r_+ - r_-}\,\mathrm{d}\bar{s}^2$, and we got a *very convenient form of the metric*:

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2}\Bigg[&-\frac{\mathcal{Q}}{\mathcal{R}^2}\Big(\mathrm{d}t - 2l\big(\cos\theta - \alpha\,\mathcal{T}\sin^2\theta\big)\mathrm{d}\varphi\Big)^2 \\ &+\frac{\mathcal{R}^2}{\mathcal{Q}}\,\mathrm{d}r^2 + \mathcal{R}^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\,\mathrm{d}\varphi^2\right)\Bigg], \end{aligned} \tag{1.11}$$

where the metric functions are

$$\begin{aligned} \Omega(r,\theta) &= 1 - \alpha\,(r - r_-)\cos\theta\,,\\ P(\theta) &= 1 - \alpha\,(r_+ - r_-)\cos\theta\,,\\ \mathcal{Q}(r) &= \big(r - r_+\big)\big(r - r_-\big)\big(1 - \alpha(r - r_-)\big)\big(1 + \alpha(r - r_-)\big),\\ \mathcal{T}(r,\theta) &= \frac{(r - r_-)^2 P}{(r_+ - r_-)\,\Omega^2}\,,\\ \mathcal{R}^2(r,\theta) &= \frac{1}{r_+^2 + l^2}\left(r_+^2(r - r_-)^2 + l^2(r - r_+)^2\frac{\big[1 - \alpha^2(r - r_-)^2\big]^2}{\big[1 - \alpha\,(r - r_-)\cos\theta\big]^4}\right). \end{aligned} \tag{1.12}$$

This new metric form is described by *spherical-like coordinates*, and is more suitable for investigation of geometric properties or physical interpretation. It explicitly depends on 3 free parameters – the mass m, the NUT parameter l, and the acceleration parameter α.

Clearly, the biggest advantage of our new metric is that we can easily recover the well-known forms of the special metrics just by setting the parameters α or l to zero.

These are:

- **Taub–NUT metric ($\alpha = 0$):**

$$\mathrm{d}s^2 = -f\left(\mathrm{d}t - 2l\cos\theta\,\mathrm{d}\varphi\right)^2 + \frac{\mathrm{d}r^2}{f} + (r^2+l^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,, \qquad (1.13)$$

where

$$f \equiv \frac{\mathcal{Q}}{\mathcal{R}^2}\bigg|_{\alpha=0} = \frac{r^2 - 2mr - l^2}{r^2 + l^2}\,. \qquad (1.14)$$

- **C-metric ($l = 0$):**

$$\mathrm{d}s^2 = \frac{1}{(1-\alpha\, r\cos\theta)^2}\left[-Q\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{Q} + r^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\,\mathrm{d}\varphi^2\right)\right], \qquad (1.15)$$

where

$$\begin{aligned} P &= 1 - 2\alpha m\cos\theta\,,\\ Q \equiv \frac{\mathcal{Q}}{\mathcal{R}^2}\bigg|_{l=0} &= \left(1 - \frac{2m}{r}\right)(1-\alpha r)(1+\alpha r)\,. \end{aligned} \qquad (1.16)$$

Both of these metrics (1.13)–(1.14), or (1.15)–(1.16), respectively, are the standard forms of the well known black holes (see [1] for more details).

1.4 Physical analysis of the new metric

Then, we could take the full advantage of the new metric form (1.11)–(1.12) and perform a thorough physical and geometrical analysis (see Sec. P1.VI).

The positions of the black hole and acceleration horizons $\mathcal{H}_b^{\pm}$ and $\mathcal{H}_a^{\pm}$, respectively, located at $\mathcal{Q}(r) = 0$, were straightforward to find:

$$\begin{aligned} \mathcal{H}_b^+ :&\qquad r = r_b^+ \equiv r_+ > 0\,,\\ \mathcal{H}_b^- :&\qquad r = r_b^- \equiv r_- < 0\,,\\ \mathcal{H}_a^+ :&\qquad r = r_a^+ \equiv r_- + \alpha^{-1}\,,\\ \mathcal{H}_a^- :&\qquad r = r_a^- \equiv r_- - \alpha^{-1}\,, \end{aligned} \qquad (1.17)$$

where $r_{\pm}$ take the form (1.10).

From (1.17), it is obvious that for a sufficiently small (positive) acceleration, namely for

$$0 < \alpha < \frac{1}{2\sqrt{m^2+l^2}}\,, \qquad (1.18)$$

the ordering of its four horizons remains in the following natural order:

$$r_a^- < r_b^- < 0 < r_b^+ < r_a^+\,. \qquad (1.19)$$

1.4.1 Curvature, algebraic structure, and regularity

In Sec. P1.VI.B, we also rewrote the Newman–Penrose scalars Ψ_A (1.4)–(1.5) in terms of the new metric form (1.11)–(1.12). Adopting the natural null tetrad

$$
\begin{aligned}
\boldsymbol{k}^{(r\theta)} &= \frac{1}{\sqrt{2}}\,\Omega\left(\frac{\mathcal{R}}{\sqrt{\mathcal{Q}}}\,\partial_t + \frac{\sqrt{\mathcal{Q}}}{\mathcal{R}}\,\partial_r\right),\\
\boldsymbol{l}^{(r\theta)} &= \frac{1}{\sqrt{2}}\,\Omega\left(\frac{\mathcal{R}}{\sqrt{\mathcal{Q}}}\,\partial_t - \frac{\sqrt{\mathcal{Q}}}{\mathcal{R}}\,\partial_r\right),\\
\boldsymbol{m}^{(r\theta)} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\mathcal{R}\sqrt{P}\sin\theta}\left(\partial_\varphi + 2l\Big(\cos\theta - \alpha\,\mathcal{T}\sin^2\!\theta\Big)\,\partial_t - \mathrm{i}\,P\sin\theta\,\partial_\theta\right),
\end{aligned}
\tag{1.20}
$$

we computed the components of the Weyl tensor as

$$
\begin{aligned}
\Psi_0^{(r\theta)} &= \Psi_4^{(r\theta)} = -3\,\mathrm{i}\,\alpha^2\,l\,P\mathcal{Q}\,(r-r_-)\sin^2\theta\,X\,,\\
\Psi_1^{(r\theta)} &= \Psi_3^{(r\theta)} = 3\,\alpha\,l\,\sqrt{P\mathcal{Q}}\,\sin\theta\,S\,X\,,\\
\Psi_2^{(r\theta)} &= \Big[-r_+\sqrt{m^2+l^2}\,\Omega^5 + \mathrm{i}\,l\,W/(r-r_-)\Big]\,X\,,
\end{aligned}
\tag{1.21}
$$

where the functions X, S, W read

$$
\begin{aligned}
X(r,\theta) &= \frac{(r_+^2+l^2)(r-r_-)^3\,\Omega^4}{\Big[r_+(r-r_-)^2\,\Omega^2 - \mathrm{i}\,l\,\mathcal{Q}\Big]^3}\,,\\
S(r,\theta) &= \Big(1-\alpha^2(r-r_-)^2\Big)(r-r_+)\\
&\quad - \Big[(r-r_+) - \sqrt{m^2+l^2}\,\Big(1-\alpha^2(r-r_-)^2\Big)\Big]\,\Omega\,,\\
W(r,\theta) &= 2\,S^2 + \Big(1-\alpha^2(r-r_-)^2\Big)(r-r_+)\times\\
&\quad \Big[\sqrt{m^2+l^2}\,\Omega^3 - \alpha^2(r-r_-)^3 P\sin^2\theta\Big]\,.
\end{aligned}
\tag{1.22}
$$

Computation of these scalars took a larger amount of time then for the scalars Ψ_A from the previous metric (1.4). However, since we used the generalized metric (1.11) and chose an appropriate null tetrad (1.20), a standard form of the Newman–Penrose scalars of the Taub–NUT metric $\Psi_2^{(r\theta)} = -\frac{m+\mathrm{i}\,l}{(r+\mathrm{i}\,l)^3}$ and the accelerating C-metric $\Psi_2^{(r\theta)} = -\frac{m}{r^3}\,(1-\alpha\,r\cos\theta)^3$ could be easily obtained.

Using this new representation of the NP scalars Ψ_A, we investigated the algebraic type and regularity of the horizons and axes as well. These were thoroughly discussed in Sec. P1.VI.B. Among that, we explicitly expressed the relation $\mathrm{I}^3 - 27\mathrm{J}^2$ (see (I.6) and the scheme on Fig. I.1), namely

$$
\begin{aligned}
\mathrm{I}^3 - 27\mathrm{J}^2 = \frac{9}{4}\bigg[&\Big(r_+\sqrt{m^2+l^2}\,\Omega^5 - \mathrm{i}\,l\,\Big[W/(r-r_-) - \alpha^2 P\mathcal{Q}\,(r-r_-)\sin^2\theta\Big]\Big)^2\\
&-16\,\alpha^2 l^2 P\mathcal{Q}\sin^2\theta\,S^2\bigg]\mathcal{D}^2 X^2\,,
\end{aligned}
\tag{1.23}
$$

where $\mathcal{D}$ is defined as

$$
\mathcal{D} \equiv 4\Psi_1^2 - 2\Psi_0(3\Psi_2 - \Psi_0) = -18\,\alpha^2\,l\,\sqrt{m^2+l^2}\,P\mathcal{Q}\,\sin^2\theta\,\Omega^3 X^2\,Y\,. \tag{1.24}
$$

Clearly, the relation (1.23) is zero if the parameter $\mathcal{D}$ (1.24) vanishes. This holds for either $\alpha = 0$ or $l = 0$. Moreover, there occurs also a specific combination, namely

$$W = \alpha^2 P\mathcal{Q}\,(r - r_-)^2 \sin^2\theta \quad \text{and} \quad r_+\sqrt{m^2 + l^2}\,\Omega^5 = \pm 4\,\alpha l\sqrt{P\mathcal{Q}}\,\sin\theta\, S\,, \quad (1.25)$$

which generates a specific hypersurface of a special algebraic type. Following the scheme on Fig. I.1, we were able to assign this hypersurface the algebraic type II or type N (see the discussion in Sec. P1.VI.C).

Furthermore, studying the scalars Ψ_A (1.21), we could easily determine the location of the curvature singularity. That should occur only when the parameter X (1.22) diverges. More specifically, the parameter X diverges when

$$r_+(r - r_-)^2\,\Omega^2 - \mathrm{i}\,l\,\mathcal{Q} = 0\,. \quad (1.26)$$

Both the real and imaginary parts must be zero. The only possibility is

$$l = 0 \qquad \text{and at the same time} \qquad r = r_- = 0\,. \quad (1.27)$$

This result corresponds to the standard C-metric [1].

We also verified these results by an explicit computation (c.f. eq. (85) of Sec. P1.VI.C) and by the visualisations of the *Kretschmann scalar* (see Fig. 2 and Fig. 3).

1.4.2 Conformal infinity and the global structure

Interestingly, all the components of Ψ_A (1.21) factorized out the parameter X (1.22). In fact, the limit $X \to 0$ corresponds to *asymptotically flat regions*, and we can identify them with a *conformal infinity* $\mathcal{I}^\pm$. A similar deduction can be found also for the NP scalars (1.4) and the function Ξ.

More specifically, the conformal infinity $\mathcal{I}$ is localized at

$$\mathcal{I}: \begin{cases} x = y & \text{for the metric (1.1)}\,, \\ r = r_- + \frac{1}{\alpha\cos\theta} & \text{for the metric (1.11) and } \theta \neq \frac{\pi}{2}\,, \\ \pm\infty & \text{for the metric (1.11) and } \theta = \frac{\pi}{2}\,. \end{cases} \quad (1.28)$$

Notice again, that both relations correspond to $\Omega(r,\theta) = 0$.

Using these results, the global structure of the whole spacetime could be derived. This was visualized on Fig. 4 of an attached article, and is repeated here in Fig. 1.1.

For the metric form (III.6)–(III.9), we fixed the coordinates $\bar{t}$, φ, and plotted the diagram for a general couple (x, y). It corresponds to the coordinates (r,θ) via the relation (1.8). The coordinate θ covers just a part of the coordinate x, namely $x \in [-1, 1]$. This section is shaded in the diagram, and describes the black hole part of the solution.

A new representation (1.11)–(1.12) allows us to understand the global structure. We can notice that the distinct limit values of $y = \pm\infty$ correspond to *a single value* of r, namely to the inner horizon at r_-. These parts can thus be glued together, yielding a new view on the global structure (see the right diagram of Fig. 1.1).

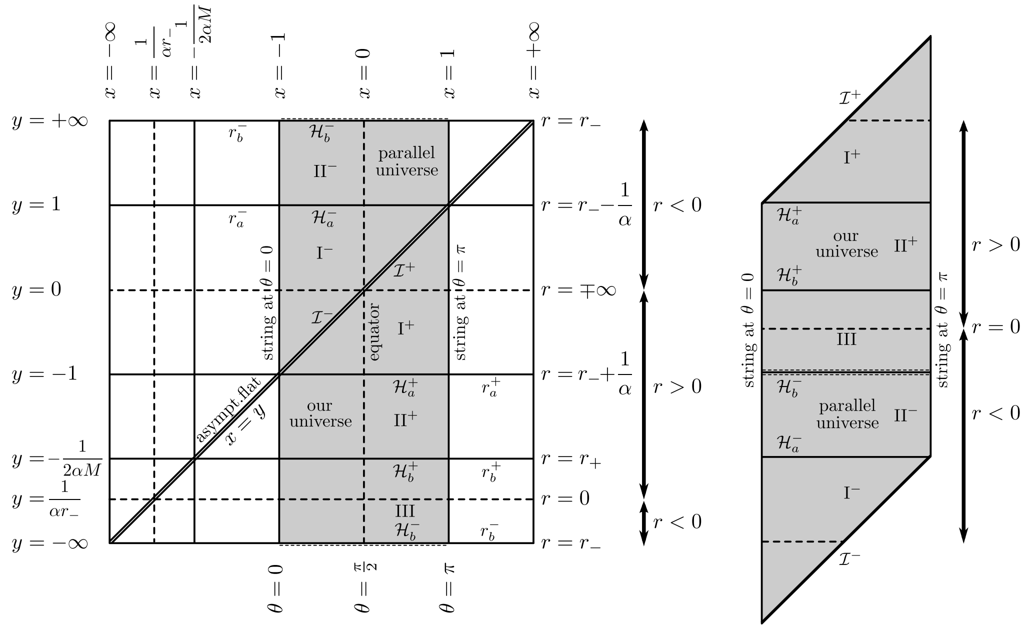

Figure 1.1: The complete global structure of accelerating NUT black holes. These sections are represented by mutually equivalent pair of coordinates x, y and θ, r. The black hole spacetime is localized in the shaded region $x \in [-1, 1]$ between the two *rotating cosmic strings*. The spacetime is separated by four Killing horizons at special values of y or r. Namely, these are the two *black-hole horizons* $\mathcal{H}_b^{\pm}$, which are located at $r_b^- = r_-$, $r_b^+ = r_+$, and the two *acceleration horizons* $\mathcal{H}_a^{\pm}$ present at $r_a^+ = r_- + \frac{1}{\alpha}$, $r_a^- = r_- - \frac{1}{\alpha}$. The lines $r = 0$ and $r = \mp\infty$ (indicated by horizontal dashed lines) are only the coordinate singularities. *Conformal infinity* $\mathcal{I}$, where the spacetime is asymptotically flat (see eq. (1.28)), is located along the diagonal line $x = y$. For more detail, see Fig. 4 of an attached article.

Let us also mention that we were able to analytically extend the solution through the acceleration horizons using the boost-rotation metric (103). Similar approach was already employed earlier for the C-metric (see [51, 52, 53]).

This reveals that there exists actually a *pair of such black holes* which are causally separated and accelerating in opposite directions. For more details, see Sec. P1.VI.D.

1.4.3 Character of the axes

Another part of our paper contains investigation of the nature of the axes $\theta = 0$ and $\theta = \pi$. Actually, in addition to the mass m, acceleration α or the NUT parameter l, there is also the fourth free parameter C hidden in the range of the coordinate $\varphi \in [0, 2\pi C)$. This parameter determines the *magnitude of the deficit (or excess) angle* around the individual axes which causes the acceleration of the black hole (for the C-metric this phenomena is discussed in detail in [1]). By an appropriate fixing of C, we can regularize this topological pathology for one of the axes.

Of course, there is also the NUT pathology around the axes, a similar as in the classic Taub–NUT solution. This can be regularized by the coordinate transformation

$$t = t_0 + 2l\varphi\,, \tag{1.29}$$

removing this pathology from the $\theta = 0$ axis. The deficit angle then vanishes for an appropriate choice

$$C = C_0 \equiv \frac{1}{1 - 2\alpha\sqrt{m^2 + l^2}}\,. \tag{1.30}$$

With the choice (1.30), the second axis $\theta = \pi$ remains with an *excess* angle

$$\delta_\pi = -\frac{8\pi\alpha\sqrt{m^2 + l^2}}{1 - 2\alpha\sqrt{m^2 + l^2}} < 0\,, \tag{1.31}$$

which we interpret as a *strut* causing the acceleration of this solution.

Analogously, we can perform the coordinate transformation

$$t = t_\pi - 2l\varphi\,, \tag{1.32}$$

and subsequently set the parameter C to

$$C = C_\pi \equiv \frac{1}{1 + 2\alpha\sqrt{m^2 + l^2}}\,. \tag{1.33}$$

This choice would regularize the $\theta = \pi$ axis, however it would introduce a deficit angle around the $\theta = 0$ axis

$$\delta_0 = \frac{8\pi\alpha\sqrt{m^2 + l^2}}{1 + 2\alpha\sqrt{m^2 + l^2}} > 0\,. \tag{1.34}$$

This would correspond to the *string* pulling the black hole. For more information about computing the deficit or excess angles see [1], or our attached article.

These strings/struts are *twisting*. This can be observed from the function $\omega \equiv \frac{g_{t\varphi}}{g_{tt}}$, evaluated on the axes $\theta = 0$ or $\theta = \pi$. This twisting parameter can be adjusted using the coordinate transformations (1.29) or (1.32). Nevertheless, its difference remains a constant $\Delta\omega = 4l$.

1.4.4 Pathological regions

The last interesting phenomena which we investigated are the pathological regions around the axes caused by the presence of the NUT parameter l. Indeed, with a non-zero l, it can be seen from metric (1.11) that there occur areas where the metric coefficient is negative,

$$g_{\varphi\varphi} < 0\,. \tag{1.35}$$

Because it determines the negative norm of the Killing vector ∂_φ, it shows the existence of regions where the *closed timelike curves* occur.

For the metric (1.11), these pathologies lie in the range such that

$$\mathcal{R}^4 P(1 - \cos^2\theta) < 4l^2 \mathcal{Q}\left(\cos\theta - \alpha\,\mathcal{T}(1 - \cos^2\theta)\right)^2. \tag{1.36}$$

The condition (1.36) is not explicitly solvable, and thus it needs to be visualized by computer. This was made in Fig. 5 of an attached article, and is recalled also here in Fig. 1.2.

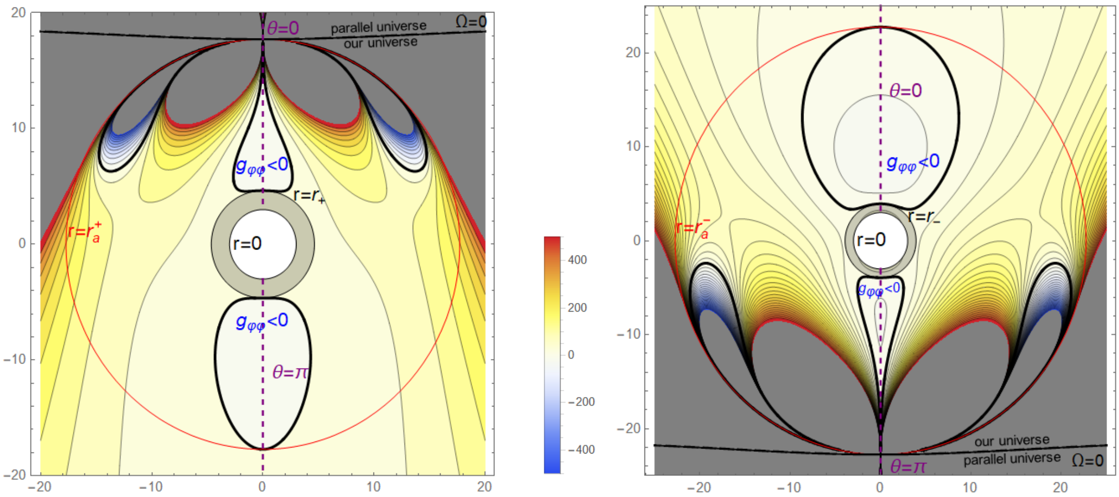

Figure 1.2: Plot of the metric function $g_{\varphi\varphi}$ and the pathological regions along the axes $\theta = 0$ and $\theta = \pi$ (1.36) in the quasi-polar coordinates $\mathrm{x} \equiv \sqrt{r^2 + l^2}\,\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + l^2}\,\cos\theta$. The visualization shows the value of $g_{\varphi\varphi}$ for $r > 0$ (left figure) and for $r < 0$ (right figure). Moreover, horizons $r_\pm$, $r_a^\pm$ and the conformal infinities are visualized. For more detail, see the Fig. 5 of an attached Paper 1.

The pathological regions modify when the coordinate transformations (1.29) or (1.32) is provided. This situation is illustrated on Fig 6 of Paper 1.

1.5 Summary

In this chapter, we have described our studies of a new interesting class of spacetimes representing accelerating black holes with a NUT parameter. Now, we summarize the most important observations. In particular:

- By developing two independent methods, we verified that the metric (III.6)–(III.9) found by Chng, Mann and Stelea in 2006 is indeed *an exact vacuum solution* to the Einstein's field equations.
- Using the metric form (1.1), we computed all the *components* Ψ_A of the Weyl tensor with respect to the null tetrad (1.3).
- From these, we calculated the corresponding curvature scalar invariants I and J (I.6). Since generically $\mathrm{I}^3 \neq 27\,\mathrm{J}^2$, the solution is of *algebraically general type I* with four distinct PNDs (given by eqs. (1.6)–(1.7)).
- This confirms the previous observations about the *deviation* of this solution from the wide Plebański–Demiański class of type D spacetimes (see Chapter III).
- In Sec. 1.3 we have summarized the *derivation* of a new metric form (1.11) of the accelerating Taub–NUT in "spherical-type" coordinates, explicitly depending on three physical parameters, namely the mass m, the acceleration α, and the NUT parameter l.

- By setting these physical parameters to zero, we recover the well-known black holes in standard coordinates, namely the *C-metric* when $l = 0$, the *Taub–NUT metric* when $\alpha = 0$, the *Schwarzschild metric*, and flat *Minkowski space.*

- Using the new metric (1.11), we provided an indepth physical and geometrical analysis of this new solution. In Sec. 1.4 we have summarized the main results, namely:

- We localized the horizons $\mathcal{H}_b^{\pm}$ and $\mathcal{H}_a^{\pm}$ at the roots of the metric function $Q(r)$ (1.17), and determined the condition (1.18) for their most natural ordering $r_a^- < r_b^- < 0 < r_b^+ < r_a^+$.

- We analyzed the curvature of this solution. More precisely, adopting the naturally chosen null tetrad (1.20), we calculated all the Ψ_A components of the Weyl tensor in terms of these new coordinates (1.21)–(1.22).

- There may be special hypersurfaces of a special algebraic type, however the overall spacetime is of a general algebraic type I.

- From the NP scalars we localized the curvature singularity at $r = 0$ while necessarily $l = 0$. It means, that the purely accelerating Taub–NUT *is non-singular.*

- We identified the asymptotically flat regions which correspond to the *conformal infinities* $\mathcal{I}^{\pm}$ given by $\Omega = 0$. This lead us to a complete understanding of the global structure of this black hole, summarized in Fig. 1.1.

- We were able to analytically extend the solution across the acceleration horizons, which revealed that there actually exists a *pair* of such (causally separated) NUT black holes uniformly accelerating in opposite directions.

- A geometrical analysis of the axes of axial symmetry at $\theta = 0$ and $\theta = \pi$ revealed that the physical source of the acceleration of this solution lies in their *topological defects*.

- We were able to fully *regularize* these defects along one of the axes of symmetry by a suitable choice of the conicity factor C.

- These cosmic *strings/struts* located along the axes of symmetry are *twisting* when $l \neq 0$. This phenomena characterizes their twist parameter ω, which is directly related to the NUT parameter l. There is always a constant difference $\Delta\omega = 4l$ between the twist parameter of each axis, and disappears only for a vanishing NUT parameter $l = 0$.

- Similarly to the case of non-accelerating Taub–NUT metric, pathological regions with *closed timelike curves* occur. These areas are visualized in Fig 1.2.

The accelerating Taub–NUT metric is an interesting solution deviating from the Plebański–Demiański class of type D black holes. We hope that the new explicit form (1.11)–(1.12) will help further investigations in the field of the black hole thermodynamics, quantum gravity, or high-energy physics (for example by extending the recent studies [54, 55]).

Furthermore, the case $m = 0$ was not explicitly studied, yet a black hole that is twisting and accelerating, even though massless, could be an interesting topic for further investigation. It can be easily computed that such a *NUT twisting string* would be of a general algebraic type I (see equation (1.23)). We thank Ibrahim Seniz for pointing this out to us.

A great success would be to find any generalization to this accelerating Taub–NUT solution, e.g. to charge the black hole ($e \neq 0 \neq g$), add the rotational parameter ($a \neq 0$), or study such black holes in (anti-)de Sitter background ($\Lambda \neq 0$). This task is however tricky due to the complexity of the metric functions $\mathcal{R}^2$ or of $\mathcal{T}$ in (1.12).

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Accelerating NUT black holes

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We present and analyze a class of exact spacetimes which describe accelerating black holes with a Newman-Unti-Tamburino (NUT) parameter. First, by two independent methods we verify that the intricate metric found by Chng, Mann, and Stelea in 2006 indeed solves Einstein's vacuum field equations of general relativity. We explicitly calculate all components of the Weyl tensor and determine its algebraic structure. As it turns out, it is actually of algebraically general type I with four distinct principal null directions. It explains why this class of solutions has not been (and could not be) found within the large Plebański–Demiański family of type D spacetimes. Then we transform the solution into a much more convenient metric form which explicitly depends on three physical parameters: mass m, acceleration α, and the NUT parameter l. These parameters can independently be set to zero, recovering thus the well-known spacetimes in standard coordinates, namely the C-metric, the Taub–NUT metric, the Schwarzschild metric, and flat Minkowski space in spherical coordinates. Using this new metric, we investigate main physical and geometrical properties of such accelerating NUT black holes. In particular, we localize and study four Killing horizons (two black-hole plus two acceleration horizons) and carefully investigate the curvature. Employing the scalar invariants we prove that there are no curvature singularities whenever the NUT parameter is nonzero. We identify asymptotically flat regions and relate them to conformal infinities. This leads to a complete understanding of the global structure of the spacetimes: each accelerating NUT black hole is a "throat" which connects "our universe" with a "parallel universe." Moreover, the analytic extension of the boost-rotation metric form reveals that there is a pair of such black holes (with four asymptotically flat regions). They uniformly accelerate in opposite directions due to the action of rotating cosmic strings or struts located along the corresponding two axes. Rotation of these sources is directly related to the NUT parameter. In their vicinity there are pathological regions with closed timelike curves.

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I. INTRODUCTION

Exact solutions of Einstein's general relativity play an important role in understanding strong gravity. Among the first and most fundamental such spacetimes, which were found, investigated and understood, were black holes. They exhibit many key features of the relativistic concept of gravity with surprising applications in modern astrophysics. It is now clear that rotating black holes reside in the hearts of almost all galaxies, and that binary black hole systems in the last stage of their evolution are the strongest sources of gravitational waves in our Universe.

In 1976, Plebański and Demiański [1] presented a nice form of a complete class of exact spacetimes of algebraic type D (including a double aligned non-null electromagnetic field and any cosmological constant), first obtained by Debever [2] in 1971. This class involves various black holes, possibly charged, rotating and accelerating. In particular, this large family of solutions contains the well-known Schwarzschild (1915), Reissner–Nordström (1916–1918), Schwarzschild–de Sitter (1918), Kerr (1963), Taub–NUT (1963) or Kerr–Newman (1965) black holes, and also the C-metric (1918, 1962) which was physically interpreted by Kinnersley–Walker (1970) as uniformly accelerating pair of black holes.

Unfortunately, these interesting types of black holes—and their combinations—had to be obtained from the general Plebański–Demiański metric by special limiting procedures (degenerate transformations), see Sec. 21.1.2 of the classic compendium [3] for more details. Moreover, it was traditionally believed that the constant coefficients of the two related Plebański–Demiański quartic metric functions directly encode the physical parameters of the spacetimes.

In 2003, Hong and Teo [4,5] came with a simple but very important idea of employing the coordinate freedom to rewrite the C-metric in a new form such that its two quartic (cubic in the uncharged case) metric functions are *factorized to simple roots*. This novel approach enormously simplified the associated calculations and—more importantly—the physical analysis of the C-metric because

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the roots themselves localize the axes of symmetry and position of horizons.

Inspired by these works of Hong and Teo, with Jerry Griffiths we applied their novel idea to the complete family of Plebański–Demiański spacetimes [1]. This "new look" enabled us to derive an alternative form of this family of type D black hole solutions, convenient for physical and geometrical interpretation, see [6–8] and Ch. 16 of [9] for summarizing review. This form of the metric reads

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left\{-\frac{\mathcal{Q}}{\varrho^2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)\mathrm{d}\varphi\right]^2 + \frac{\varrho^2}{\mathcal{Q}}\mathrm{d}r^2 + \frac{\varrho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\varrho^2}\sin^2\theta[a\mathrm{d}t - (r^2+(a+l)^2)\mathrm{d}\varphi]^2\right\}, \tag{1}$$

where $P = 1 - a_3\cos\theta - a_4\cos^2\theta$, $\mathcal{Q} = (\omega^2 k + e^2 + g^2) - 2mr + \epsilon r^2 - 2\alpha n\omega^{-1}r^3 - (\alpha^2 k + \frac{1}{3}\Lambda)r^4$, $\Omega = 1 - \alpha(l + a\cos\theta)\omega^{-1}r$, $\varrho^2 = r^2 + (l + a\cos\theta)^2$, and a_3, a_4, ϵ, n, k are uniquely determined constants. The free parameters of the solutions have a direct physical meaning, namely the mass m, electric and magnetic charges e and g, Kerr-like rotation a, Newman-Unti-Tamburino (NUT)-like parameter l, acceleration α, and the cosmological constant Λ. All the particular subclasses of the Plebański–Demiański black holes can be easily obtained from (1) by simply setting these physical parameters to zero.

At first sight, it would seem possible to obtain an exact vacuum solution for accelerating black holes with a NUT parameter simply by keeping α, m, l and setting $a = e = g = \Lambda = 0$. However, in [6] we explicitly demonstrated that in such a special case the constant α is a *redundant parameter* which can be removed by a specific coordinate transformation. In other words, the case α, m, l is just the "static" black hole with a NUT parameter l. Thus we argued convincingly in [6] that the solution which would combine the Taub–NUT metric with the C-metric is *not included* in the Plebański–Demiański family of black holes, despite the fact that a more general solution which describes accelerating *and rotating* black holes with NUT parameter is included in it (indeed, in the metric (1) it is possible to keep α, a, l, m all nonvanishing). This led us in 2005 to a "private conjecture" that the genuine accelerating Taub–NUT metric (without the Kerr-like rotation a) need not exist at all.

Quite surprisingly, such a solution was found next year in 2006 by Chng, Mann, and Stelea [10] by applying a sequence of several mathematical generating techniques. It was presented in the following form[1]

$$\mathrm{d}\bar{s}^2 = -\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2}\frac{c^2\delta}{\bar{H}(x,y)} \times\left[\mathrm{d}\bar{t} + \frac{1}{c}\left(\frac{(1-x^2)F(x)}{\alpha^2(x-y)^2} + \frac{2Mx}{\alpha}\right)\mathrm{d}\varphi\right]^2 + \frac{\bar{H}(x,y)}{\alpha^2(x-y)^2}\left[(1-x^2)F(x)\mathrm{d}\varphi^2 + \frac{\mathrm{d}x^2}{(1-x^2)F(x)} + \frac{\mathrm{d}y^2}{(y^2-1)F(y)}\right], \tag{2}$$

where

$$F(x) = 1 + 2\alpha Mx, \tag{3}$$

$$F(y) = 1 + 2\alpha My, \tag{4}$$

$$\bar{H}(x,y) = \frac{1}{2} + \frac{\delta}{2}\left(\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2}\right)^2, \tag{5}$$

see Eq. (35) in [10]. This metric explicitly contains four parameters, namely M, α, c, and δ. The authors of [10] argued that the parameter δ is related to the NUT parameter in the limiting case when the acceleration vanishes. And, complementarily, when this parameter is set to zero, the C-metric can be obtained. It is thus natural to interpret the metric (2)–(5) as an exact spacetime with uniformly accelerating black hole *and* a specific twist described by the NUT parameter. This very interesting suggestion surely deserves a deeper analysis. To our knowledge, during the last 15 years this has not yet been done, and it is the main purpose of this paper.

First, in Sec. II we will remove the redundant parameter c, simplifying the original metric of [10] to the form in which the twist can be set to zero (leading to the standard C-metric). Using it, in subsequent Sec. III we will confirm that the metric (2)–(5) is indeed a vacuum solution of Einstein's field equations (we will do this by two independent methods, based on the general results summarized in Appendices A and B). In Sec. IV we will calculate the NP scalars Ψ_A in a suitable null frame and determine the algebraic type of the Weyl tensor. Since it will turn out to be *algebraically general* with four distinct principal null directions, it *cannot belong* to the class of type D Plebański–Demiański spacetimes (1). Then, in Sec. V we will present a new metric form of the solution which is much better suited for a geometrical and physical interpretation of this class of black holes. When its three parameters l, α, and m are set to zero, standard form of the C-metric, the Taub–NUT metric, the Schwarzschild metric and eventually Minkowski space are directly obtained. Specific properties of this family of accelerating NUT black holes are investigated in Sec. VI. In particular, we study horizons, curvature singularities, asymptotically flat regions, global structure of these spacetimes, and

[1]We have only replaced the acceleration parameter A by α, and the mass parameter m by M, and the constant C by c.

specific nonregularity of the two axes of symmetry, corresponding to rotating cosmic strings or struts (surrounded by regions with closed timelike curves) which are the physical source of acceleration of the pair of black holes.

II. REMOVING THE DEGENERACY AND INITIAL COMMENTS

We immediately observe that the original metric (2) does not admit setting $c = 0$ and $\delta = 0$. The metric degenerates and its investigation is thus complicated. In fact, the constant c is redundant. To solve these problems, we found convenient to perform a transformation of the time coordinate

$$\tau = 2\lambda(\alpha^2 c\,\bar{t} - \varphi), \tag{6}$$

where the new real parameter $\lambda \geq 0$ is defined as

$$\lambda \equiv \frac{\sqrt{\delta}}{\alpha^2}. \tag{7}$$

Rescaling trivially the metric (2) by a *constant* conformal factor, $\mathrm{d}\bar{s}^2 \to \mathrm{d}s^2 \equiv 2\mathrm{d}\bar{s}^2$, we obtain a better representation of the solution

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2H(x,y)}\left[\mathrm{d}\tau + 2\lambda F(x)\frac{1-2xy+y^2}{(x-y)^2}\mathrm{d}\varphi\right]^2\\ &+\frac{H(x,y)}{\alpha^2(x-y)^2}\left[(1-x^2)F(x)\mathrm{d}\varphi^2\right.\\ &\left.+\frac{\mathrm{d}x^2}{(1-x^2)F(x)}+\frac{\mathrm{d}y^2}{(y^2-1)F(y)}\right],\end{aligned} \tag{8}$$

where the function $H \equiv 2\bar{H}$ takes the form

$$H(x,y) = 1 + \lambda^2\frac{(y^2-1)^2F^2(y)}{(x-y)^4}, \tag{9}$$

and $F(x) = 1 + 2\alpha Mx$, $F(y) = 1 + 2\alpha My$ are the linear functions (3) and (4), respectively. Without loss of generality, we may assume $\alpha \geq 0$.

It is now possible to set $\lambda = 0$, in which case $H = 1$, and the new metric reduces to a diagonal line element

$$\begin{aligned}\mathrm{d}s^2 = &\frac{1}{\alpha^2(x-y)^2}\left[-(y^2-1)F(y)\mathrm{d}\tau^2 + (1-x^2)F(x)\mathrm{d}\varphi^2\right.\\ &\left.+\frac{\mathrm{d}x^2}{(1-x^2)F(x)}+\frac{\mathrm{d}y^2}{(y^2-1)F(y)}\right].\end{aligned} \tag{10}$$

This is the *usual form of the C-metric*, see e.g. Eqs. (14.3), (14.4) in [9] with the identification $G(x) \equiv (1-x^2)F(x)$, $y \to -y$ and $m \equiv M$. In such a special case, the metric represents a spacetime with pair of Schwarzschild-like black holes of mass $M \geq 0$ and uniform acceleration α caused by cosmic strings or struts.

The full metric (8) with a *generic* λ is clearly a *one-parameter generalization* of this C-metric. Additional off-diagonal metric component $\mathrm{d}\tau\mathrm{d}\varphi$ also occurs, indicating that the parameter λ is related to an inherent twist/rotation effect in the spacetime. It will be explicitly demonstrated in Sec. V that this parameter is directly proportional to the genuine NUT parameter l.

Preliminary physical interpretation of (8) can now also be done using similar arguments as those for the C-metric, as summarized in Ch. 14 of [9]. In particular, we can comment on the character of coordinate singularities. In order to keep the *correct metric signature* of (8) and obtain the usual black-hole interpretation of the spacetime, it is necessary to require $(1-x^2)F(x) \geq 0$. In view of the roots, this restricts the range of the spatial coordinate to $x \in [-1, 1]$ and puts the constraint $0 \leq 2\alpha M < 1$. The coordinate singularities at $x = \pm 1$ are the *two poles (axes)*. On the other hand, the admitted zeros of the function $(y^2-1)F(y)$ represent the *horizons*, and $F(y)$ can be both positive and negative. More arguments on this will be given in Sec. VI, where it will also be demonstrated that the singularity of the metric (8) at $x = y$ corresponds to asymptotically flat *conformal infinity* $\mathcal{I}$.

III. CHECKING THE VACUUM EQUATIONS

Next, it is desirable to verify that the metric (8) with (3), (4), (9) is an exact solution of vacuum Einstein's field equations.

With trivial identification $\tau \equiv t$, this metric clearly belongs to the *generic class of stationary axially symmetric metrics*

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{\varphi\varphi}\mathrm{d}\varphi^2 + g_{xx}\mathrm{d}x^2 + g_{yy}\mathrm{d}y^2, \tag{11}$$

in which all the functions are independent of the temporal coordinate t and angular coordinate φ. Indeed, the explicit metric coefficients of the spacetime (8) are

$$\begin{aligned}g_{tt} &= -\frac{(y^2-1)F(y)}{\alpha^2(x-y)^2H(x,y)},\\ g_{t\varphi} &= -2\lambda\frac{(y^2-1)F(y)F(x)(1-2xy+y^2)}{\alpha^2(x-y)^4H(x,y)},\\ g_{\varphi\varphi} &= -4\lambda^2\frac{(y^2-1)F(y)F^2(x)(1-2xy+y^2)^2}{\alpha^2(x-y)^6H(x,y)}\\ &\quad+\frac{H(x,y)(1-x^2)F(x)}{\alpha^2(x-y)^2},\\ g_{xx} &= \frac{H(x,y)}{\alpha^2(x-y)^2(1-x^2)F(x)},\\ g_{yy} &= \frac{H(x,y)}{\alpha^2(x-y)^2(y^2-1)F(y)}.\end{aligned} \tag{12}$$

Interestingly, the subdeterminant

$$D \equiv g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 < 0, \tag{13}$$

turns out to be very simple, namely

$$D = -\frac{(1-x^2)F(x)(y^2-1)F(y)}{\alpha^4(x-y)^4}. \tag{14}$$

Using the expressions (11)–(14), we need to evaluate the Riemann and Ricci curvature tensors. Unfortunately, standard computer algebra systems did not provide us the results (even after several hours of calculation on a standard desktop PC) when we attempted to perform a *direct calculation* starting from (12). Therefore, we had to employ a more sophisticated approach. Actually, we developed *two independent methods*.

A. Method A

It turned out much more convenient first to analytically derive explicit expressions for the Christoffel symbols and subsequently the corresponding components of the curvature tensors of the *generic* stationary axisymmetric metric (11). These results are summarized in Appendix A.

Moreover, instead of using standard textbook definitions of the Riemann and Ricci tensors, we employed their alternative (and equivalent) versions (A8), (A10). The main advantage of this approach is that the second derivatives of the metric are all involved explicitly in the simplest possible way. It is not necessary to differentiate the Christoffel symbols which also contain the inverse metric and thus their first derivatives unnecessarily complicate the evaluation of the curvature.

In the second step, we then substituted the explicit metric functions (12), (14) into the general expressions (A5), (A9), and (A11). With a usual PC, such a symbolic-algebra computational process using MATHEMATICA lasted only around 40 seconds. The result of this computation *confirmed that all the Ricci tensor components* (A11) *are zero*. The metric (8) is thus indeed a vacuum solution in Einstein's gravity theory.

B. Method B

To verify this result (and fasten the computation), we also employed an alternative method based on the "conformal trick." Its main idea is that, by multiplying the physical metric (8) by a suitable conformal factor Ω^2, the metric components of the related unphysical metric become *polynomial expressions*. Their differentiation and combination, which are necessary to evaluate the curvature tensors, are performed much faster. Specifically, we introduced an unphysical metric $\tilde{g}_{ab}$ via the conformal relation

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{15}$$

where

$$\Omega^2 \equiv \alpha^2(1-x^2)F(x)(y^2-1)F(y)(x-y)^6\tilde{H}(x,y), \tag{16}$$

and

$$\tilde{H}(x,y) \equiv (x-y)^4 H(x,y) = (x-y)^4 + \lambda^2(y^2-1)^2F^2(y). \tag{17}$$

The metric functions $\tilde{g}_{ab}$ are then only polynomials of x and y,

$$\begin{aligned}
\tilde{g}_{tt} &= -(1-x^2)F(x)(y^2-1)^2F^2(y)(x-y)^8,\\
\tilde{g}_{t\varphi} &= -2\lambda(1-x^2)F^2(x)(y^2-1)^2F^2(y)\\
&\quad\times(1-2xy+y^2)(x-y)^6,\\
\tilde{g}_{\varphi\varphi} &= -4\lambda^2(1-x^2)F^3(x)(y^2-1)^2F^2(y)\\
&\quad\times(1-2xy+y^2)^2(x-y)^4\\
&\qquad+(1-x^2)^2F^2(x)(y^2-1)F(y)\tilde{H}^2(x,y),\\
\tilde{g}_{xx} &= (y^2-1)F(y)\tilde{H}^2(x,y),\\
\tilde{g}_{yy} &= (1-x^2)F(x)\tilde{H}^2(x,y).
\end{aligned} \tag{18}$$

Using the expressions summarized in Appendix A, we first computed the Christoffel symbols $\tilde{\Gamma}^a{}_{bc}$ and the Ricci tensor components $\tilde{R}_{ab}$ for this conformal metric $\tilde{g}_{ab}$ (it also has the stationary axisymmetric form (11), only the tilde symbol is added everywhere). Then we employed the expressions (B4)–(B6) derived in Appendix B to calculate the Ricci tensor components R_{ab} of the physical metric g_{ab}, which is (12). The computer algebra manipulation using MATHEMATICA again verified that $R_{ab} = 0$, confirming that the metric is a vacuum solution of Einstein's equations. In fact, the conformal Method B is faster than Method A: the computation took only 15 seconds.

IV. DETERMINING THE ALGEBRAIC TYPE OF THE SPACETIME

It is now necessary to determine the algebraic type of the spacetime which is given by the *algebraic structure of the Weyl tensor*. The standard procedure is to evaluate all its ten components [3,9]

$$\begin{aligned}
\Psi_0 &\equiv C_{abcd}k^a m^b k^c m^d,\\
\Psi_1 &\equiv C_{abcd}k^a l^b k^c m^d,\\
\Psi_2 &\equiv C_{abcd}k^a m^b \bar{m}^c l^d,\\
\Psi_3 &\equiv C_{abcd}l^a k^b l^c \bar{m}^d,\\
\Psi_4 &\equiv C_{abcd}l^a \bar{m}^b l^c \bar{m}^d,
\end{aligned} \tag{19}$$

in properly normalized *null tetrad* $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$. We adopt the most natural tetrad for the metric (11) in the coordinates (t, φ, x, y), namely

$$\begin{aligned} \boldsymbol{k} &\equiv \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\partial_t + \frac{1}{\sqrt{g_{yy}}}\partial_y\right), \\ \boldsymbol{l} &\equiv \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{-g_{tt}}}\partial_t - \frac{1}{\sqrt{g_{yy}}}\partial_y\right), \\ \boldsymbol{m} &\equiv \frac{1}{\sqrt{2}}\left(\sqrt{\frac{g_{tt}}{D}}\partial_\varphi + \frac{g_{t\varphi}}{\sqrt{D g_{tt}}}\partial_t - \frac{\mathrm{i}}{\sqrt{g_{xx}}}\partial_x\right), \end{aligned} \tag{20}$$

with D given by (13). All the scalar products vanish, except for

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1, \qquad \boldsymbol{m}\cdot\bar{\boldsymbol{m}} = 1. \tag{21}$$

For vacuum solutions, the Ricci tensor and Ricci scalar vanish. The Weyl tensor is thus identical to the Riemann curvature tensor, and in expressions (19) we can replace C_{abcd} by R_{abcd}. In view of the vanishing components of the null tetrad vectors (20) and the vanishing components of the Riemann tensor (A9) of the metric (11), summarized in Appendix A, the following formulas for the Weyl scalars can be derived

$$\begin{aligned} \Psi_0 &= \frac{1}{4}\left[\frac{1}{D g_{yy}}\left(\frac{g_{t\varphi}^2}{g_{tt}}R_{tyty} - 2g_{t\varphi}R_{ty\varphi y} + g_{tt}R_{\varphi y\varphi y}\right) - \frac{1}{D}R_{t\varphi t\varphi} \right.\\ &\quad \left. + \frac{1}{g_{xx}}\left(\frac{1}{g_{tt}}R_{txtx} - \frac{1}{g_{yy}}R_{xyxy}\right)\right] \\ &\quad - \frac{\mathrm{i}}{2}\frac{1}{\sqrt{-D}}\frac{1}{\sqrt{g_{xx}g_{yy}}}\left(\frac{g_{t\varphi}}{g_{tt}}R_{txty} - R_{t\varphi xy} - R_{tx\varphi y}\right), \\ \Psi_1 &= \frac{1}{2}\left[\frac{1}{\sqrt{-D}g_{yy}}\left(R_{ty\varphi y} - \frac{g_{t\varphi}}{g_{tt}}R_{tyty}\right) - \frac{\mathrm{i}}{g_{tt}\sqrt{g_{xx}g_{yy}}}R_{txty}\right], \\ \Psi_2 &= \frac{1}{4}\left[\frac{1}{D g_{yy}}\left(\frac{g_{t\varphi}^2}{g_{tt}}R_{tyty} - 2g_{t\varphi}R_{ty\varphi y} + g_{tt}R_{\varphi y\varphi y}\right) + \frac{1}{D}R_{t\varphi t\varphi} \right.\\ &\quad \left. + \frac{1}{g_{xx}}\left(\frac{1}{g_{tt}}R_{txtx} + \frac{1}{g_{yy}}R_{xyxy}\right)\right] \\ &\quad - \frac{\mathrm{i}}{2}\frac{1}{\sqrt{-D}}\frac{1}{\sqrt{g_{xx}g_{yy}}}\left(\frac{g_{t\varphi}}{g_{tt}}R_{txty} + R_{t\varphi xy} - R_{tx\varphi y}\right), \\ \Psi_3 &= \Psi_1, \\ \Psi_4 &= \Psi_0. \end{aligned} \tag{22}$$

Notice that, interestingly, the long expressions for Ψ_0 and Ψ_2 are very similar. In fact, they only differ in *signs of three terms*.

Now, by substituting the explicit components (12) of the metric and the corresponding Riemann tensor (A9) into (22), the computer algebra system MAPLE rendered the following Weyl scalars:

$$\begin{aligned} \Psi_0 &= \Psi_4 = -3\alpha^2\lambda(1-x^2)F(x)(y^2-1)F(y)\,\Xi(x,y), \\ \Psi_1 &= \Psi_3 = -3\alpha^2\lambda\mathrm{i}\sqrt{(1-x^2)F(x)}\sqrt{(y^2-1)F(y)} \\ &\quad \times \Sigma(x,y)\,\Xi(x,y), \\ \Psi_2 &= [\alpha^2\lambda\,\Pi(x,y) + \mathrm{i}\alpha^3 M(x-y)^5]\,\Xi(x,y), \end{aligned} \tag{23}$$

where the functions Ξ, Σ, and Π are defined as

$$\begin{aligned} \Xi(x,y) &= \frac{(H-4)\sqrt{H-1} + \mathrm{i}(4-3H)}{(x-y)^2 H^3}, \\ \Sigma(x,y) &= xy - 1 - \alpha M x(1-3y^2) - \alpha M y(1+y^2), \\ \Pi(x,y) &= 2\Sigma^2(x,y) - [(1-x^2)F(x) - \alpha M(x-y)^3] \\ &\quad \times (y^2-1)F(y), \end{aligned} \tag{24}$$

with $H \equiv H(x,y)$ given by (9), and $F(x)$, $F(y)$ by (3), (4). Surprisingly, the key function $\Xi(x,y)$ which *factorizes all the Weyl scalars* can be written in an explicit and compact form as

$$\Xi = \frac{\mathrm{i}(x-y)^4}{[(x-y)^2 - \lambda\mathrm{i}(y^2-1)(1+2\alpha M y)]^3}. \tag{25}$$

From these curvature scalars, we then computed the *scalar invariants* I and J, defined as

$$I \equiv \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \quad J \equiv \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}, \tag{26}$$

and using MAPLE we verified that the equality $I^3 = 27J^2$ *does not hold*. This means (see [3,9]) that the metric (8) is *algebraically general*, that is of type I.

Consequently, *the accelerating NUT metric* (8) *cannot be included in the Plebański–Demiański family* because this is of algebraic type D.

Of course, this conclusion is only valid when $\lambda \neq 0$. In the case of vanishing λ, implying $H = 1$ and thus $\Xi = \mathrm{i}/(x-y)^2$, the only nontrivial Weyl scalar remains $\Psi_2 = -M\alpha^3(x-y)^3$. Such spacetime is of algebraic type D, with double degenerate principal null directions $\boldsymbol{k}$ and $\boldsymbol{l}$. In fact, it is the C-metric (10) which belongs to the Plebański–Demiański class.

Deeper analysis of the algebraic structure will be presented in Secs. VI B and VI C.

A. The principal null directions

Actually, it is possible to determine *four principal null directions* (PNDs) of the Weyl tensor, and to prove explicitly that they are *all distinct*.

As usual [3,9], we employ the dependence of the Weyl scalars (19) on the choice of the null tetrad, namely their transformation properties under a null rotation which keeps $\boldsymbol{l}$ fixed,

$$\boldsymbol{k}' = \boldsymbol{k} + K\bar{\boldsymbol{m}} + \bar{K}\boldsymbol{m} + K\bar{K}\boldsymbol{l}, \qquad \boldsymbol{l}' = \boldsymbol{l}, \qquad \boldsymbol{m}' = \boldsymbol{m} + K\boldsymbol{l}, \tag{27}$$

where K is a complex parameter. The component Ψ_0 then transforms to

$$\Psi_0{}' = \Psi_0 + 4K\Psi_1 + 6K^2\Psi_2 + 4K^3\Psi_3 + K^4\Psi_4. \tag{28}$$

The condition for $\boldsymbol{k}'$ to be a principal null direction is $\Psi_0{}' = 0$, which is equivalent

$$\Psi_0 + 4K\Psi_1 + 6K^2\Psi_2 + 4K^3\Psi_3 + K^4\Psi_4 = 0. \tag{29}$$

Since this is a *quartic* expression in K, there are *exactly four complex roots* K_i $(i = 1, 2, 3, 4)$ to this equation. Each K_i corresponds via (27) to the principal null direction $\boldsymbol{k}_i'$.

In the case of the metric (8), the Weyl scalars with respect to the null tetrad (20) are (23). Due to the special property $\Psi_4 = \Psi_0$ and $\Psi_3 = \Psi_1$, the key algebraic equation (29) simplifies to

$$\Psi_0\left(K^2 + \frac{1}{K^2}\right) + 4\Psi_1\left(K + \frac{1}{K}\right) + 6\Psi_2 = 0, \tag{30}$$

(K must be nonvanishing in (29) because $\Psi_0 \neq 0$). It is convenient to introduce a new parameter

$$\kappa \equiv K + \frac{1}{K}, \tag{31}$$

so that (30) reduces to the quadratic equation in κ,

$$\Psi_0\kappa^2 + 4\Psi_1\kappa + 2(3\Psi_2 - \Psi_0) = 0, \tag{32}$$

with two solutions

$$\kappa_{1,2} = \frac{-2\Psi_1 \pm \sqrt{4\Psi_1^2 - 2\Psi_0(3\Psi_2 - \Psi_0)}}{\Psi_0}. \tag{33}$$

Finally, we find the roots K_i by solving (31), that is the quadratic equation $K^2 - \kappa K + 1 = 0$:

$$K_i = \frac{\kappa \pm \sqrt{\kappa^2 - 4}}{2}, \tag{34}$$

where $\kappa = \kappa_1$ and $\kappa = \kappa_2$. Indeed, we have thus obtained *four explicit complex roots* K_i corresponding to four distinct PNDs $\boldsymbol{k}_i'$, which can be expressed using (27).

V. A NEW CONVENIENT FORM OF THE METRIC

The metric (2) can be put in an alternative form which is suitable for its physical interpretation, in particular for determining the meaning of its three free parameters. This is achieved by performing the coordinate transformation

$$x = -\cos\theta, \qquad y = -\frac{1}{\alpha(r - r_-)}, \qquad \bar{t} = \frac{r_+ - r_-}{2\alpha l c}\, t. \tag{35}$$

We introduce the *NUT parameter* l as

$$l \equiv \lambda r_+ = \frac{\sqrt{\delta}}{\alpha^2} r_+, \tag{36}$$

using the definition (7), and a new real *mass parameter* m via the relation

$$m = \sqrt{M^2 - l^2}. \tag{37}$$

Specific combinations of m and l can conveniently be defined and denoted as

$$\begin{aligned} r_+ &\equiv m + \sqrt{m^2 + l^2}, \\ r_- &\equiv m - \sqrt{m^2 + l^2}, \end{aligned} \tag{38}$$

so that r_+ is *always positive* while r_- is *always negative*. Actually, it will soon be seen that these constants describe the location of two Taub–NUT horizons. From these definitions, important identities immediately follow, namely

$$\begin{aligned} r_+ + r_- &= 2m, \\ r_+ - r_- &= 2\sqrt{m^2 + l^2} = 2M \geq 0, \\ r_+ r_- &= -l^2, \\ r_+(r_+ - r_-) &= r_+^2 + l^2. \end{aligned} \tag{39}$$

The original metric (2) with (3)–(5) then becomes

$$\begin{aligned} \mathrm{d}\bar{s}^2 = \frac{1}{\Omega^2}\Bigg[&-\frac{(r_+ - r_-)^2}{2r_+^2}(1 - \alpha^2(r - r_-)^2)\frac{F(y)}{H(x,y)} \\ &\times \left(\mathrm{d}t - 2l\left(\cos\theta - \alpha\frac{(r - r_-)^2 F(x)\sin^2\theta}{(r_+ - r_-)\Omega^2}\right)\mathrm{d}\varphi\right)^2 \\ &+ \frac{1}{2}(r - r_-)^2 H(x,y) \\ &\times \left(\frac{\mathrm{d}r^2}{F(y)(r - r_-)^2(1 - \alpha^2(r - r_-)^2)} + \frac{\mathrm{d}\theta^2}{F(x)} + F(x)\sin^2\theta\,\mathrm{d}\varphi^2\right)\Bigg], \end{aligned} \tag{40}$$

where $\Omega \equiv 1 - \alpha(r - r_-)\cos\theta$. Of course, the metric functions $F(x)$, $F(y)$, and $H(x, y) \equiv 2\bar{H}$, given by (3), (4), and (9), respectively, must be expressed in terms of the new coordinates r and θ. It is useful to relabel them as

$$
\begin{aligned}
F(x) &\to P(\theta) = 1-\alpha(r_+ - r_-)\cos\theta,\\
F(y) &\to F(r) = \frac{r-r_+}{r-r_-},\\
H(x,y) &\to H(r,\theta) = 1+\frac{l^2}{r_+^2}\frac{(r-r_+)^2}{(r-r_-)^2}\\
&\qquad\times\frac{[1-\alpha^2(r-r_-)^2]^2}{[1-\alpha(r-r_-)\cos\theta]^4}. \qquad (41)
\end{aligned}
$$

Notice that H is always positive. Finally, it is natural to introduce two new functions replacing $F(r)$ and $H(r,\theta)$, namely

$$
\begin{aligned}
\mathcal{Q}(r) &\equiv F(r)(r-r_-)^2(1-\alpha^2(r-r_-)^2),\\
\mathcal{R}^2(r,\theta) &\equiv \frac{r_+}{r_+ - r_-}(r-r_-)^2 H(r,\theta), \qquad (42)
\end{aligned}
$$

and to perform a trivial *rescaling of the whole metric* by a *constant conformal factor* as

$$
\mathrm{d}s^2 \equiv \frac{2r_+}{r_+ - r_-}\mathrm{d}\bar{s}^2. \qquad (43)
$$

Thus, the exact solution found in [10] simplifies considerably to a *new convenient form of the metric*

$$
\begin{aligned}
\mathrm{d}s^2 = \frac{1}{\Omega^2}\Bigg[&-\frac{\mathcal{Q}}{\mathcal{R}^2}\left(\mathrm{d}t - 2l(\cos\theta - \alpha\mathcal{T}\sin^2\theta)\mathrm{d}\varphi\right)^2\\
&+\frac{\mathcal{R}^2}{\mathcal{Q}}\mathrm{d}r^2 + \mathcal{R}^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\mathrm{d}\varphi^2\right)\Bigg], \qquad (44)
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega(r,\theta) &= 1-\alpha(r-r_-)\cos\theta,\\
P(\theta) &= 1-\alpha(r_+ - r_-)\cos\theta,\\
\mathcal{Q}(r) &= (r-r_+)(r-r_-)(1-\alpha(r-r_-))(1+\alpha(r-r_-)),\\
\mathcal{T}(r,\theta) &= \frac{(r-r_-)^2 P}{(r_+ - r_-)\Omega^2},\\
\mathcal{R}^2(r,\theta) &= \frac{1}{r_+^2 + l^2}\Bigg(r_+^2(r-r_-)^2 + l^2(r-r_+)^2\\
&\qquad\times\frac{[1-\alpha^2(r-r_-)^2]^2}{[1-\alpha(r-r_-)\cos\theta]^4}\Bigg). \qquad (45)
\end{aligned}
$$

This new metric form can be used for investigation of geometric properties of the spacetime and for its physical interpretation. It *explicitly* contains *3 free parameters*, namely m, l and α [the first two uniquely determining the constants r_+ and r_- via the relations (38)]. They can independently be set to *any value*. In particular, it is possible to *set them to zero*, thus immediately obtaining important special subclasses of the spacetime metric (44). This is the main advantage of (44) if compared to the original form (2) in which, in particular, it is not possible to set $\alpha = 0$, and also the NUT parameter is not explicitly identified.

Let us now investigate the spacetime, based on the new form of its metric (44), (45).

A. The case $l = 0$: The C-metric (accelerating black holes)

For $l = 0$ the constants (38) become

$$
r_+ = 2m, \qquad r_- = 0, \qquad (46)
$$

so that the metric functions (45) reduce considerably to

$$
\begin{aligned}
\Omega(r,\theta) &= 1-\alpha r\cos\theta,\\
P(\theta) &= 1-2\alpha m\cos\theta,\\
\mathcal{Q}(r) &= r(r-2m)(1-\alpha r)(1+\alpha r),\\
\mathcal{R}^2(r,\theta) &= r^2. \qquad (47)
\end{aligned}
$$

The metric (44) thus simplifies to a diagonal line element

$$
\begin{aligned}
\mathrm{d}s^2 = &\frac{1}{(1-\alpha r\cos\theta)^2}\\
&\times\left[-Q\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{Q} + r^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\mathrm{d}\varphi^2\right)\right], \qquad (48)
\end{aligned}
$$

where

$$
\begin{aligned}
P &= 1-2\alpha m\cos\theta,\\
Q &\equiv \frac{\mathcal{Q}}{\mathcal{R}^2} = \left(1-\frac{2m}{r}\right)(1-\alpha r)(1+\alpha r). \qquad (49)
\end{aligned}
$$

This is exactly the *C-metric expressed in spherical-type coordinates*, see Eqs. (14.6) and (14.7) in [9]. As has been thoroughly described in Ch. 14 of [9], this metric represents the spacetime with a pair of Schwarzschild-like black holes of *mass m* which uniformly accelerate due to the tension of cosmic strings (or struts) located along the half-axes of symmetry at $\theta = 0$ and/or $\theta = \pi$. *Their acceleration is determined by the parameter* α. This gives the physical interpretation to the two constant parameters of the solution.

B. The case $\alpha = 0$: The Taub–NUT metric (twisting black holes)

Complementarily, it is possible to directly set $\alpha = 0$ in the metric (44). In such a case the functions (45), using the identities (39), reduce to simple quadratics

$$
\begin{aligned}
\Omega(r,\theta) &= 1,\\
P(\theta) &= 1,\\
\mathcal{Q}(r) &= (r-r_+)(r-r_-) \equiv r^2 - 2mr - l^2,\\
\mathcal{R}^2(r,\theta) &= r^2 + l^2. \qquad (50)
\end{aligned}
$$

The metric (44) remains nondiagonal, but has a compact explicit form

$$\mathrm{d}s^2 = -f(\mathrm{d}t - 2l\cos\theta\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{f} + (r^2 + l^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{51}$$

where

$$f \equiv \frac{\mathcal{Q}}{\mathcal{R}^2} = \frac{r^2 - 2mr - l^2}{r^2 + l^2}. \tag{52}$$

It is exactly the *standard Taub-NUT metric*, see Eqs. (12.1) and (12.2) in [9]. As summarized in Ch. 12 of [9], this metric is interpreted as a spacetime with black hole of *mass* m and *NUT twist parameter* l. There are horizons located at $r = r_+$ and $r = r_-$, but there is *no curvature singularity* at $r = 0$. Whenever the NUT parameter l is nonvanishing, there is an internal twist in the geometry, related to spinning cosmic strings located along the axes $\theta = 0$ and/or $\theta = \pi$. In the vicinity of these "torsion singularities" there appear closed timelike curves.

C. The case $\alpha = 0$ and $l = 0$: Schwarzschild black hole

By simultaneously setting both the acceleration α and the NUT parameter l to zero, we immediately obtain the standard spherically symmetric metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{53}$$

As is well known (see, e.g., Ch. 8 of [9]), it represents the *spherically symmetric Schwarzschild black hole of mass* m in asymptotically flat space. There is no acceleration and no twist, the axes are regular (there are no cosmic strings, struts, or torsion singularities).

D. The case $\alpha = 0$ and $l = 0$ and $m = 0$: Minkowski flat space

By setting $\alpha = 0 = m$ in (48), (49) which implies $P = 1 = \mathcal{Q}$, or by setting $l = 0 = m$ in (51), (52) which implies $f = 1$, or by setting $m = 0$ in (53), we obtain

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{54}$$

This is obviously the flat metric in spherical coordinates (Eq. (3.2) in [9]).

Since all such subcases are *directly* obtained as special cases, *it is indeed natural to interpret the general metric* (44), (45) *as* a three-parameter family of exact spacetimes with *uniformly accelerating black holes with the twist NUT parameter.*

The structure of the new family of spacetimes which represent accelerating NUT black holes is shown in Fig. 1. Previously known spacetimes are obtained in their classic form by simply setting the acceleration α, the NUT parameter l, or the mass m to zero. With these settings, algebraically general solution of Einstein's vacuum equations reduces to type D.

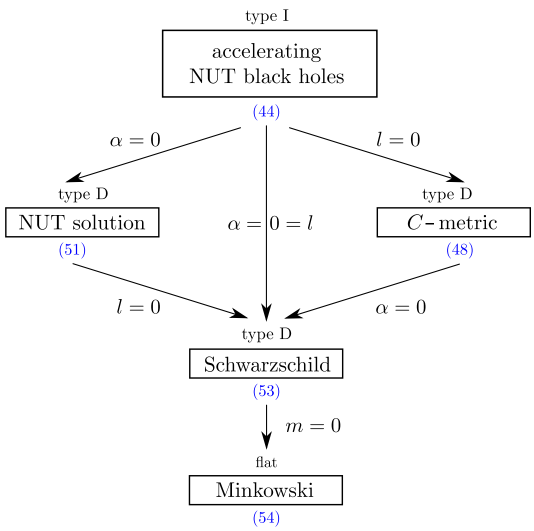

FIG. 1. Schematic structure of the complete family of accelerating black holes with a NUT parameter. This 3-parameter class of vacuum solutions to Einstein's field equations is of general algebraic type I, reducing to double degenerate type D whenever the acceleration α or the NUT parameter l (or both) vanish. By setting any of the three independent parameters α, l, m to zero, the well-known classes (namely the NUT solution, the C-metric, Schwarzschild black hole and Minkowski flat space) are obtained directly in their usual forms, whose equation numbers are also indicated in the diagram.

VI. PHYSICAL INTERPRETATION OF THE NEW METRIC FORM

A. Position of the horizons

The metric (44) is very convenient for investigation of horizons. In these coordinates, ∂_t is one of the Killing vectors (the second is ∂_φ). Its norm is $-\mathcal{Q}/(\Omega\mathcal{R})^2$, so that t is a temporal coordinate in the regions where $\mathcal{Q}(r) > 0$, while it is a spatial coordinate in the regions where $\mathcal{Q}(r) < 0$. These regions are separated by the *Killing horizons* localized at $\mathcal{Q}(r) = 0$. The form of the metric function $\mathcal{Q}$ is given by (45), which is clearly a *quartic factorized into four roots*. There are thus *four Killing horizons*, located at

$$\begin{aligned} \mathcal{H}_b^+ &: \ r = r_b^+ \equiv r_+ > 0, \\ \mathcal{H}_b^- &: \ r = r_b^- \equiv r_- < 0, \\ \mathcal{H}_a^+ &: \ r = r_a^+ \equiv r_- + \alpha^{-1}, \\ \mathcal{H}_a^- &: \ r = r_a^- \equiv r_- - \alpha^{-1}, \end{aligned} \tag{55}$$

(see Fig. 4) where $r_\pm$ are defined by (38). Recall also (39), that is $r_+ - r_- = 2\sqrt{m^2+l^2} > 0$ (unless $m = 0 = l$, in which case $r_+ = 0 = r_-$).

The horizons $\mathcal{H}_b^+, \mathcal{H}_b^-$ at r_b^+, r_b^- are *two black-hole horizons*. Interestingly, they are located *at the same values* r_+, r_- of the radial coordinate r as the two horizons in the standard (nonaccelerating) *Taub–NUT metric*, see (50).

The horizons $\mathcal{H}_a^+, \mathcal{H}_a^-$ at r_a^+, r_a^- are *two acceleration horizons*. Their presence is the consequence of the fact that the black hole accelerates whenever the parameter α is nonzero. They generalize the acceleration horizons $+\alpha^{-1}, -\alpha^{-1}$ present in the *C-metric*, see (49).

These pairs of roots are clearly ordered as $r_b^+ > r_b^-$ and $r_a^+ > r_a^-$ (naturally assuming that the acceleration parameter α is positive). Their mutual relations, however, depend on the specific values of the three physical parameters m, l, α. Concentrating on the physically most plausible case when the *acceleration is small*, the value of α^{-1} is very large, and r_a^+ becomes bigger than r_b^+. This condition $r_a^+ > r_b^+$ explicitly reads

$$\alpha < \frac{1}{2\sqrt{m^2+l^2}}. \tag{56}$$

For such a small acceleration of the black hole, the ordering of its four horizons is

$$r_a^- < r_b^- < 0 < r_b^+ < r_a^+. \tag{57}$$

The first two horizons $\mathcal{H}_a^-$ and $\mathcal{H}_b^-$ (acceleration and black-hole, respectively) are in the region $r < 0$, while the remaining two horizons $\mathcal{H}_b^+$ and $\mathcal{H}_a^+$ (black-hole and acceleration, respectively) are in the region $r > 0$. Such a situation can be naturally understood as the Taub-NUT spacetime with usual two "inner" black hole horizons $\mathcal{H}_b^\pm$, which are here surrounded by two additional "outer" acceleration horizons $\mathcal{H}_a^\pm$ (one in the region $r > 0$ and the second in the region $r < 0$).

Evaluating $\mathcal{Q}(r)$, generally given by (45), at $r = 0$ we obtain using (39)

$$\mathcal{Q}(r=0) = r_+ r_- (1 - \alpha^2 r_-^2) = -l^2(1 - \alpha^2 r_-^2). \tag{58}$$

From the condition (56) and (38) it follows that

$$1 - \alpha^2 r_-^2 > \frac{2m^2 + 3l^2 + 2m\sqrt{m^2+l^2}}{4(m^2+l^2)} > 0, \tag{59}$$

so that $\mathcal{Q}(r=0) < 0$. It implies $\mathcal{Q} < 0$ for any $r \in (r_b^-, r_b^+)$. We conclude that the coordinate t is *temporal* in the regions (r_b^+, r_a^+) and (r_a^-, r_b^-), that is *between* the black-hole and acceleration horizons, while it is spatial in the complementary three regions of the radial coordinate r.

Moreover, when the condition (56) is satisfied, the metric coefficient $P(\theta)$ in (44) is *always positive*. Indeed,

$$\begin{aligned} P_{\min} = P(\theta = 0) &= 1 - \alpha(r_+ - r_-) \\ &= 1 - 2\alpha\sqrt{m^2+l^2} > 0. \end{aligned} \tag{60}$$

Of course, for other choices of the physical parameters, *different number* and *different ordering* of the horizons can be achieved. They also may coincide, thus becoming *degenerate horizons*. In particular, in the limit of *vanishing acceleration* $\alpha \to 0$, the two outer acceleration horizons disappear (formally via the limits $r_a^+ \to +\infty$, $r_a^- \to -\infty$), and only two Taub-NUT black hole horizons $\mathcal{H}_b^+, \mathcal{H}_b^-$ remain. On the other hand, for *vanishing NUT parameter* $l \to 0$, one of the black-hole horizon disappears (formally via the limit $r_b^- \equiv r_- \to 0$), while the second becomes $r_b^+ \equiv r_+ \to 2m$. There is just one black-hole horizon at $2m$ surrounded by two acceleration horizons located at $\pm\alpha^{-1}$, which is exactly the case of the C-metric with a curvature singularity at $r = 0$.

B. Curvature of the spacetime, algebraic structure, and regularity

1. The Weyl scalars

We now employ the Weyl scalars Ψ_A given by (23), (24), (25) to discuss the algebraic properties of the spacetime, including the subcases $l = 0$ and $\alpha = 0$, the location of physical curvature singularities and its global structure.

These scalars correspond to the metric (8) with coordinates x, y, and it is thus natural to denote them as $\Psi_A^{(xy)}$. It will also be convenient to express these curvature scalars as $\Psi_A^{(r\theta)}$ for the metric form (44) with coordinates r, θ. Using the transformation (35) and definitions (41), (42) we immediately derive $\alpha^2(1-x^2)F(x)(y^2-1)F(y) = P\mathcal{Q}(r-r_-)^{-4}\sin^2\theta$, with $P = P(\theta)$ and $\mathcal{Q} = \mathcal{Q}(r)$ given by (45), and similarly we express the functions Ξ, Σ, and Π. However, it is *also necessary to properly rescale* the scalars $\Psi_A^{(xy)}$ given by (23) to get $\Psi_A^{(r\theta)}$ because the metrics (8) and (44) are *not* the same: They are related by a *constant conformal factor*,

$$g_{ab}^{(r\theta)} = \omega^2 g_{ab}^{(xy)}, \quad \text{where } \omega^2 = \frac{r_+}{r_+ - r_-}. \tag{61}$$

Indeed, $g_{ab}^{(xy)} = 2\bar{g}_{ab}$ while $g_{ab}^{(r\theta)} = 2\frac{r_+}{r_+ - r_-}\bar{g}_{ab}$, see (43). The corresponding Weyl tensor components are related as $C_{abcd}^{(r\theta)} = \omega^2 C_{abcd}^{(xy)}$, see [11]. The null tetrad (20) also needs to be rescaled in such a way that it remains properly normalized in the coordinates r, θ as (21). This requires $\boldsymbol{k}^{(r\theta)} = \omega^{-1}\boldsymbol{k}^{(xy)}$, $\boldsymbol{l}^{(r\theta)} = \omega^{-1}\boldsymbol{l}^{(xy)}$, $\boldsymbol{m}^{(r\theta)} = \omega^{-1}\boldsymbol{m}^{(xy)}$. In view of (19), we obtain the relation

$$\Psi_A^{(r\theta)} = \omega^{-2}\,\Psi_A^{(xy)}. \tag{62}$$

Using (23)–(25) and (61)–(62), we thus calculate the Weyl curvature scalars for the metric (44) with respect to the null tetrad

$$\begin{aligned}
\boldsymbol{k}^{(r\theta)} &= \frac{1}{\sqrt{2}}\Omega\left(\frac{\mathcal{R}}{\sqrt{\mathcal{Q}}}\partial_t + \frac{\sqrt{\mathcal{Q}}}{\mathcal{R}}\partial_r\right),\\
\boldsymbol{l}^{(r\theta)} &= \frac{1}{\sqrt{2}}\Omega\left(\frac{\mathcal{R}}{\sqrt{\mathcal{Q}}}\partial_t - \frac{\sqrt{\mathcal{Q}}}{\mathcal{R}}\partial_r\right),\\
\boldsymbol{m}^{(r\theta)} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\mathcal{R}\sqrt{P}\sin\theta}(\partial_\varphi + 2l(\cos\theta - \alpha\mathcal{T}\sin^2\theta)\partial_t\\
&\quad - \mathrm{i}P\sin\theta\partial_\theta).
\end{aligned} \tag{63}$$

It turns out that

$$\begin{aligned}
\Psi_0^{(r\theta)} &= \Psi_4^{(r\theta)} = -3\mathrm{i}\alpha^2 lP\mathcal{Q}(r-r_-)\sin^2\theta\, X,\\
\Psi_1^{(r\theta)} &= \Psi_3^{(r\theta)} = 3\alpha l\sqrt{P\mathcal{Q}}\sin\theta S\, X,\\
\Psi_2^{(r\theta)} &= [-r_+\sqrt{m^2+l^2}\Omega^5 + \mathrm{i}lW/(r-r_-)]\, X,
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
X(r,\theta) &= \frac{(r_+^2+l^2)(r-r_-)^3\Omega^4}{[r_+(r-r_-)^2\Omega^2 - \mathrm{i}l\mathcal{Q}]^3},\\
S(r,\theta) &= (1-\alpha^2(r-r_-)^2)(r-r_+)\\
&\quad - [(r-r_+) - \sqrt{m^2+l^2}(1-\alpha^2(r-r_-)^2)]\Omega,\\
W(r,\theta) &= 2S^2 + (1-\alpha^2(r-r_-)^2)(r-r_+)\\
&\quad \times [\sqrt{m^2+l^2}\Omega^3 - \alpha^2(r-r_-)^3P\sin^2\theta].
\end{aligned} \tag{65}$$

These functions are related to (24) via

$$\begin{aligned}
X &\equiv \frac{-\mathrm{i}(r_+-r_-)}{\alpha^2 r_+^2(r-r_-)^5}\Xi, \qquad S \equiv \alpha^2(r-r_-)^3\Sigma,\\
W &\equiv \alpha^4(r-r_-)^6\Pi,
\end{aligned} \tag{66}$$

and $\Omega = \Omega(r,\theta)$, $P = P(\theta)$, and $\mathcal{Q} = \mathcal{Q}(r)$ are given by (45).

As we have already argued in Sec. IV, this class of spacetimes with accelerating Taub–NUT black hole is generically of *type I*, i.e., it is *algebraically general*. However, it may degenerate. When *either* $\alpha = 0$ *or* $l = 0$, the only nontrivial curvature component is given by

$$\Psi_2^{(r\theta)} = [-r_+\sqrt{m^2+l^2}\Omega^5 + \mathrm{i}lW/(r-r_-)]X. \tag{67}$$

Such spacetimes are clearly of *algebraic type D*, with two double-degenerate principal null directions $\boldsymbol{k}^{(r\theta)}$ and $\boldsymbol{l}^{(r\theta)}$ of the Weyl/Riemann tensor.

This is fully consistent with the fact that the case $l = 0$ (implying $r_+ = 2m$, $r_- = 0$, see (46), and $X = (r_+ r^3\Omega^2)^{-1}$) corresponds to the type D *accelerating C-metric*, for which

$$\Psi_2^{(r\theta)} = -\frac{m}{r^3}(1-\alpha r\cos\theta)^3, \tag{68}$$

see Ch. 14 in [9].

The complementary case $\alpha = 0$, which cannot be directly obtained from $\Psi_A^{(xy)}$ given by (23), corresponds to the type D *twisting Taub–NUT metric*. It follows from (50) that in such a case $\Omega = 1$ and $\mathcal{Q}(r) = (r-r_+)(r-r_-)$. With the help of relation (39) we thus get

$$\begin{aligned}
X &= \frac{r_+^2+l^2}{[r_+(r-r_-) - \mathrm{i}l(r-r_+)]^3} = \frac{(r_+-\mathrm{i}l)(r_++\mathrm{i}l)}{(r_+-\mathrm{i}l)^3(r+\mathrm{i}l)^3},\\
S &= \sqrt{m^2+l^2}, \qquad W = \sqrt{m^2+l^2}(r-r_-),
\end{aligned} \tag{69}$$

so that

$$\begin{aligned}
\Psi_2^{(r\theta)} &= -\sqrt{m^2+l^2}(r_+-\mathrm{i}l)X = -\frac{\sqrt{m^2+l^2}}{r_+-\mathrm{i}l}\frac{r_++\mathrm{i}l}{(r+\mathrm{i}l)^3}\\
&= -\frac{\sqrt{m^2+l^2}}{r_+^2+l^2}\frac{(r_++\mathrm{i}l)^2}{(r+\mathrm{i}l)^3}.
\end{aligned} \tag{70}$$

Applying the identities

$$\begin{aligned}
r_+^2+l^2 &= r_+(r_+-r_-) = 2r_+\sqrt{m^2+l^2}, \quad \text{and}\\
(r_++\mathrm{i}l)^2 &= 2r_+(m+\mathrm{i}l),
\end{aligned} \tag{71}$$

we finally obtain

$$\Psi_2^{(r\theta)} = -\frac{m+\mathrm{i}l}{(r+\mathrm{i}l)^3}, \tag{72}$$

which is the standard form of the scalar Ψ_2 for the Taub–NUT spacetime, see Ch. 12 in [9].

2. Algebraic type and regularity of the horizons

It can be immediately observed from (64) that on the horizons (55), defined by $\mathcal{Q} = 0$, all the Weyl scalars vanish except

$$\begin{aligned}
&\Psi_2^{(r\theta)}(\text{at any horizon } r_h)\\
&\quad = -\frac{2\sqrt{m^2+l^2}}{r_+^2(r_h-r_-)^3}\left[r_+\sqrt{m^2+l^2}\Omega^3 - \mathrm{i}l\frac{W}{(r_h-r_-)\Omega^2}\right].
\end{aligned} \tag{73}$$

Therefore, *all horizons are of algebraic type D*. This is true in *a generic case with any acceleration α and any NUT parameter l*. Moreover, at these horizons the spacetime is *regular*, that is *free of curvature singularities*. This can be proved as follows:

(i) At the *acceleration horizons* r_a^+, r_a^-, the values are $r_h - r_- = \pm\alpha^{-1}$, so that $\Omega(r_h) = 1 \mp \cos\theta$ and $W(r_h) = 2\alpha^{-2}(1 \mp 2\alpha\sqrt{m^2+l^2})^2\Omega^2$, implying

$$\Psi_2^{(r\theta)}(\mathcal{H}_a^\pm) = 2\alpha^2 \frac{\sqrt{m^2+l^2}}{r_+^2} \times [\mp \alpha r_+ \sqrt{m^2+l^2}(1\mp\cos\theta)^3 + 2\mathrm{i}l(1\mp 2\alpha\sqrt{m^2+l^2})^2]. \tag{74}$$

(ii) At the *positive black hole horizon* $r_b^+ \equiv r_+ > 0$, the value of the factor is $r_h - r_- = 2\sqrt{m^2+l^2}$, so that $\Omega(r_h) = 1 - 2\alpha\sqrt{m^2+l^2}\cos\theta = P$, $W(r_h) = 2(m^2+l^2)(1-4\alpha^2(m^2+l^2))^2\Omega^2$. Thus,

$$\Psi_2^{(r\theta)}(\mathcal{H}_b^+) = -\frac{1}{4r_+^2\sqrt{m^2+l^2}} \times [r_+(1-2\alpha\sqrt{m^2+l^2}\cos\theta)^3 - \mathrm{i}l(1-4\alpha^2(m^2+l^2))^2]. \tag{75}$$

(iii) At the *negative black hole horizon* $r_b^- \equiv r_- < 0$, the expression (73) seems to diverge. However, a careful analysis of the limit $r \to r_-$ of (67) shows, using $X \to \mathrm{i}r_+(4l^3(m^2+l^2))^{-1}$, $\Omega \to 1$ and $W/(r_h - r_-) \to \sqrt{m^2+l^2}(1-6\alpha\sqrt{m^2+l^2}\cos\theta)$ that

$$\Psi_2^{(r\theta)}(\mathcal{H}_b^-) = -\frac{r_+}{4l^3\sqrt{m^2+l^2}} \times [l(1-6\alpha\sqrt{m^2+l^2}\cos\theta) + \mathrm{i}r_+]. \tag{76}$$

The expressions (74)–(76) explicitly demonstrate that *at any horizon the gravitational field is finite*, without the curvature singularities.

3. Algebraic type of the axes and principal null directions

Similarly, *along both the axes* $\theta = 0$ and $\theta = \pi$ the function $\sin\theta$ vanishes, which implies that $\Psi_0^{(r\theta)} = \Psi_1^{(r\theta)} = 0 = \Psi_3^{(r\theta)} = \Psi_4^{(r\theta)}$. This proves that the *algebraic structure of the spacetime on these axes is also of type D*, with the only curvature component (67).

Finally, let us comment on the *principal null directions* (PNDs) of the curvature tensor introduced in Sec. IV A. Using the Weyl scalars (64) we can express the key discriminant of the equation (33) as

$$\mathcal{D} \equiv 4\Psi_1^2 - 2\Psi_0(3\Psi_2 - \Psi_0) = -18\alpha^2 l\sqrt{m^2+l^2}PQ\sin^2\theta\,\Omega^3X^2Y, \tag{77}$$

where $Y(r,\theta) = l(1-\alpha^2(r-r_-)^2)(r-r_+) + \mathrm{i}r_+(r-r_-)\Omega^2$. Therefore, there are *in general two distinct roots* κ_1, κ_2 of (33), and subsequently there are *four distinct roots* K_i of (34). They correspond to *four distinct PNDs* of the Weyl tensor, confirming that the metric (44) is of algebraically general type I.

However, if (and only if) $\alpha = 0$ or $l = 0$, the discriminant (77) *everywhere vanishes* and there is *only one double root* κ of (33). In such cases, there are just two roots

$$K_{1,2} = \frac{\kappa \pm \sqrt{\kappa^2 - 4}}{2}, \tag{78}$$

corresponding to two *doubly degenerate PNDs* $\boldsymbol{k}'_{1,2}$ of type D spacetimes (the Taub–NUT metric and the *C*-metric, respectively). In particular, in this limit $K_1 \to 0$ and $K_2 \to \infty$ which effectively corresponds to PND $\boldsymbol{k}^{(r\theta)}$ and PND $\boldsymbol{l}^{(r\theta)}$ given by (63).

C. Curvature singularities and invariants

1. Investigation of possible singularities

The Weyl scalars $\Psi_A^{(xy)}$ given by (23)–(25), or their equivalent forms $\Psi_A^{(r\theta)}$ given by (64)–(65), can be used to study *curvature singularities* in the family of accelerating NUT black holes.

By inspection we observe that all functions entering these scalars are bounded[2] *except* the function $X(r,\theta)$, or equivalently $\Xi(x,y)$, whose denominator can be zero. This key function appears as a *joint factor in all the Weyl scalars* (64). Regions of spacetime where $X(r,\theta) \to \infty$ thus clearly indicate the possible presence of a physical singularity. In view of (65), such a *curvature singularity* corresponds to the vanishing denominator of X (provided its numerator remains nonzero), that is

$$r_+(r-r_-)^2\Omega^2 - \mathrm{i}lQ = 0. \tag{79}$$

Both the real and imaginary parts must vanish. Since $r_+ = m + \sqrt{m^2+l^2} > 0$, Ω is everywhere a positive conformal factor, and $Q = 0$ identifies regular horizons (as shown in previous section), the only possibility is when

$$l = 0 \text{ and at the same time } r = r_- = 0, \tag{80}$$

where in the last equality we applied the relation $r_- \equiv m - \sqrt{m^2+l^2}$ for $l = 0$. The curvature singularity thus appears *only* in the *C*-metric spacetime at the origin $r = 0$. *All other*

[2]As will be demonstrated in Sec. VI D, a possible divergence for $r \to \infty$ corresponds to asymptotically flat regions.

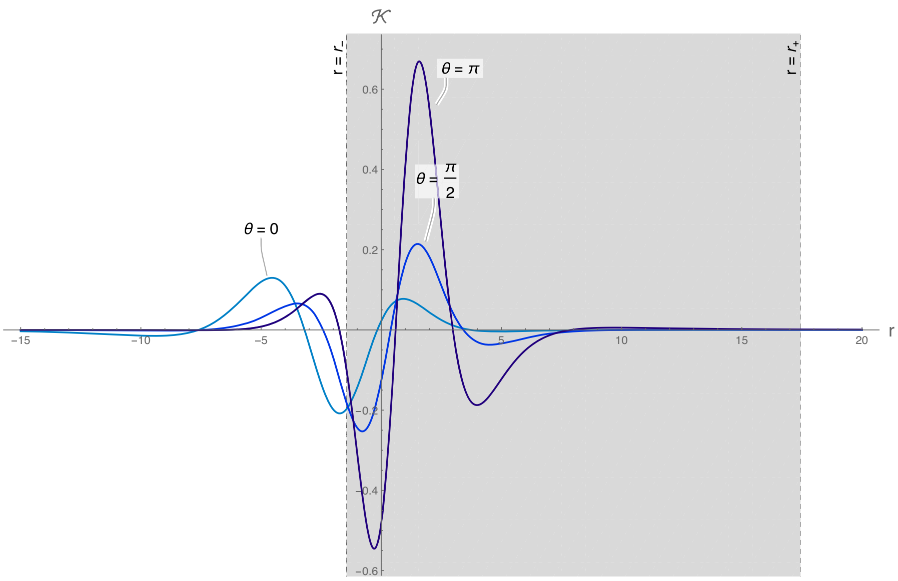

FIG. 2. The value of the Kretschmann curvature scalar (85) plotted as the function $\mathcal{K}(r)$, where r is the radial coordinate, for $\theta = 0, \frac{\pi}{2}$, and π. The black-hole parameters are $m = 8$, $l = 5$, and $\alpha = 0.025$.

spacetimes in the large class of accelerating NUT black holes are nonsingular. The presence of the NUT parameter l (even a very small one) thus makes the spacetime regular. This property is well known for classic Taub–NUT spacetime (see Ch. 12 in [9]), and the same property holds also in this new class of accelerating NUT black holes. Consequently, to describe the complete spacetime manifold, it is necessary to consider the *full range of the radial coordinate* $r \in (-\infty, +\infty)$.

To confirm these observations, we employ the *scalar curvature invariant* I defined in (26). Introducing a convenient new function Δ, defined as

$$\Delta \equiv \Psi_2 - \Psi_0, \tag{81}$$

and using the special geometrical property of the spacetime $\Psi_0 = \Psi_4$ and $\Psi_1 = \Psi_3$, this invariant is simplified to

$$I = \Psi_0^2 - 4\Psi_1^2 + 3\Psi_2^2 = 3\Delta^2 - \mathcal{D}, \tag{82}$$

where the discriminant $\mathcal{D}$ is given by (77). Explicit evaluation now leads to

$$\begin{aligned} I = 3\Big(& r_+^2(m^2+l^2)\Omega^{10} \\ & - 12\alpha^2 l^2 P\mathcal{Q}\sin^2\theta S^2 - 3\alpha^4 l^2 (r-r_-)^2 P^2 \mathcal{Q}^2 \sin^4\theta \\ & - l^2 W^2/(r-r_-)^2 - 2\mathrm{i} l r_+ \sqrt{m^2+l^2}\,\Omega^5 W/(r-r_-)\Big) X^2. \end{aligned} \tag{83}$$

Since (as already argued) even the function $W/(r-r_-)$ is finite at the black hole horizon $r_b^- \equiv r_-$, the scalar curvature invariant I becomes unbounded only if the function X diverges. This happens if, and only if, both the conditions (80) hold.

Recall also that the real part of the invariant I is proportional to the *Kretschmann scalar*,

$$\mathcal{K} \equiv R_{abcd}R^{abcd} = 16\mathcal{R}e(I), \tag{84}$$

which can thus be evaluated as

$$\begin{aligned} \mathcal{K} = 48\{ & \mathcal{R}e(\Psi_2^2) - 3\alpha^2 l^2 P\mathcal{Q}\sin^2\theta \\ & [4S^2 + \alpha^2 (r-r_-)^2 P\mathcal{Q}\sin^2\theta]\mathcal{R}e(X^2)\}. \end{aligned} \tag{85}$$

In this form it is explicitly seen that the Kretschmann scalar for the C-metric or the Taub–NUT black hole is simply obtained by setting $l = 0$ or $\alpha = 0$, respectively. In both cases, it leads to

$$\mathcal{K}_{l \text{ or } \alpha \to 0} = 48\mathcal{R}e(\Psi_2^2), \tag{86}$$

where Ψ_2 is given by (68) or (72), in full agreement with [12,13]. Interestingly, $\mathcal{K} = 48\mathcal{R}e(\Psi_2^2)$ also on the horizons (55) where $\mathcal{Q} = 0$, and on the axes $\theta = 0, \pi$ where $\sin\theta = 0$.

In the general case of accelerating NUT back holes, the Kretschmann curvature scalar $\mathcal{K}$ is given by expression (85). This explicit but somewhat complicated function of

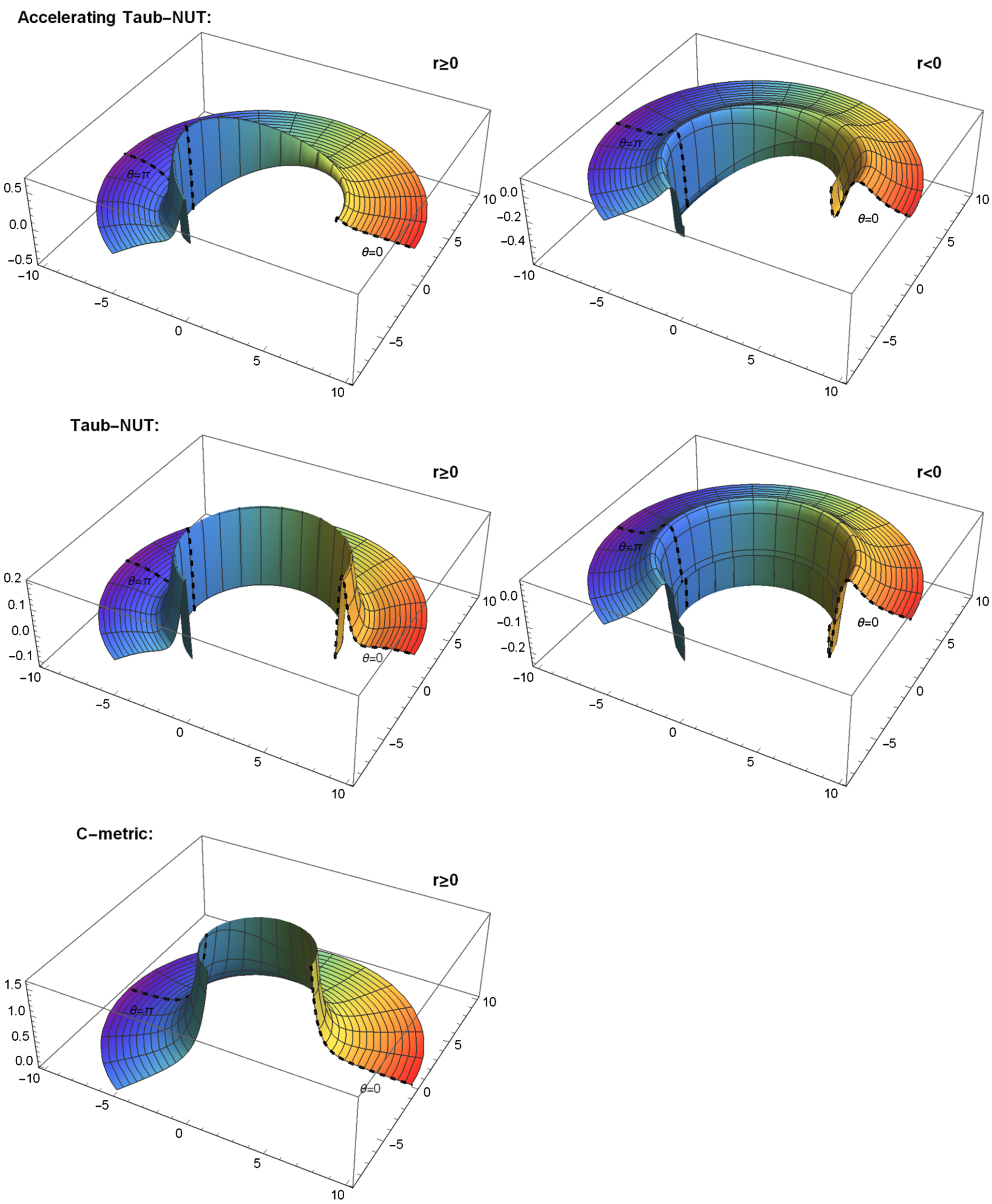

FIG. 3. The Kretschmann curvature scalar (85) visualized in quasipolar coordinates as $\mathcal{K}(\mathrm{x},\mathrm{y})$, where $\mathrm{x} \equiv \sqrt{r^2+l^2}\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2+l^2}\cos\theta$, so that $r=0$ is a circle of radius l. The left column corresponds to $r \geq 0$, while the right column represents $r<0$. The first row plots the Kretchmann scalar for the accelerating NUT black hole with $m=8$, $l=5$ and $\alpha=0.025$. It can be seen that the curvature is everywhere finite, even in the vicinity of $r=0$, and it smoothly continues across $r=0$ from $r>0$ to $r<0$. The second and third rows correspond to special cases of this metric, namely the Taub–NUT metric (with $m=8$, $l=5$, $\alpha=0$) and the C-metric (with $m=8$, $l=0$, $\alpha=0.025$). The Taub–NUT metric has no divergence of $\mathcal{K}$, which is independent of θ. On the other hand, the C-metric becomes singular as $r\to 0$, that is at $\mathrm{x}=0=\mathrm{y}$ (therefore we plot only the region $r\geq 0$). The two separate cosmic strings along the axes $\theta=0$ and $\theta=\pi$ are indicated as dashed curves.

the coordinates r and θ is visualized in the two illustrative figures.

In Fig. 2 we plot the Kretschmann scalar $\mathcal{K}(r)$ as a function of the radial coordinate r for *three fixed privileged values of* θ, namely $\theta = 0, \theta = \frac{\pi}{2}$ and $\theta = \pi$. In fact, we will argue later that the two poles/axes at $\theta = 0$ and π correspond to the position of (rotating) cosmic strings, while $\theta = \frac{\pi}{2}$ is the equatorial section "perpendicular" to them. It can be seen that for each θ there are several local maxima and local minima. Half of these extremes are in the region $r > 0$, the remaining are located in the region $r < 0$. The *curvature is everywhere finite*, and its *maximal values are localized close to the origin* $r = 0$ *inside the black hole*, that is within the shaded region $r \in (r_-, r_+) \equiv (r_b^-, r_b^+)$.

In Fig. 3 we include the angular dependence on θ. The left column corresponds to the region $r \geq 0$, while the right column represents the region $r < 0$. The first row plots the Kretchmann scalar $\mathcal{K}(r, \theta)$ for the accelerating NUT black hole (with $m = 8$, $l = 5$, $\alpha = 0.025$), the second and third rows correspond to special cases of this metric, namely the Taub–NUT metric ($m = 8$, $l = 5$, $\alpha = 0$) and the C-metric ($m = 8$, $l = 0$, $\alpha = 0.025$). From these visualizations of the Kretschmann curvature scalar it is seen that the dependence on both r and θ is smooth, and the curvature is everywhere finite, except for the C-metric at $r = 0$, in full agreement with the condition (80). The two distinct cosmic strings located on the axes $\theta = 0$ and $\theta = \pi$, respectively, are indicated as dashed curves.

2. Scalar invariants and algebraic types

Let us conclude this part by returning to the *scalar curvature invariants I and J*. We can express J, defined in (26), in terms of the discriminant $\mathcal{D}$ and the function Δ as

$$J = \frac{1}{2}\Delta(\mathcal{D} - 2\Delta^2). \tag{87}$$

Using (82), the key expression $I^3 - 27J^2$ thus takes the compact form

$$I^3 - 27J^2 = \frac{1}{4}(9\Delta^2 - 4\mathcal{D})\mathcal{D}^2, \tag{88}$$

which is explicitly

$$\begin{aligned} I^3 - 27J^2 = \frac{9}{4}\Bigg[&(r_+\sqrt{m^2 + l^2}\Omega^5 - \mathrm{i}l[W/(r - r_-) \\ &- \alpha^2 P\mathcal{Q}(r - r_-)\sin^2\theta])^2 \\ &- 16\alpha^2 l^2 P\mathcal{Q}\sin^2\theta S^2\Bigg]\mathcal{D}^2 X^2. \end{aligned} \tag{89}$$

According to standard classification scheme for determining the algebraic type (see, e.g., page 122 of [3]), the spacetime is of a *general algebraic type I* if (and only if) $I^3 \neq 27J^2$. This is clearly the generic case of (89), confirming the results of Sec. IV. Only for $\mathcal{D} = 0$ (or $X = 0$ which is, however, a subcase of $\mathcal{D} = 0$), the spacetime degenerates and *becomes algebraically special*. In particular, it follows from (77) that $\mathcal{D} = 0$ whenever $\alpha = 0$ or $l = 0$, and such spacetimes are *actually of type D everywhere*, as we have already demonstrated in previous sections.

Zeros of the big square bracket in (89) identify *algebraically more special regions in a given spacetime*. It requires

$$\begin{aligned} W &= \alpha^2 P\mathcal{Q}(r - r_-)^2\sin^2\theta \quad \text{and} \\ r_+\sqrt{m^2 + l^2}\Omega^5 &= \pm 4\alpha l\sqrt{P\mathcal{Q}}\sin\theta S. \end{aligned} \tag{90}$$

Clearly, this can happen only for the generic case of accelerating NUT black holes with $\alpha \neq 0 \neq l$. It is interesting to observe from (64) that these two conditions imply

$$\Psi_2 = -\frac{1}{3}(\Psi_0 \pm 4\Psi_1), \tag{91}$$

and thus $\mathcal{D} = 4(\Psi_0 \pm \Psi_1)^2$ and $\Delta = -\frac{4}{3}(\Psi_0 \pm \Psi_1)$, which now implies a specific relation $\mathcal{D} = \frac{9}{4}\Delta^2$. In such degenerate regions, the scalar curvature invariants take the form

$$\begin{aligned} I &= \frac{3}{4}\Delta^2, \qquad J = \frac{1}{8}\Delta^3, \quad \text{and further} \\ K &= \frac{9}{8}\Psi_1\Delta^2, \qquad L = \frac{1}{4}(\Psi_0 \pm 3\Psi_1)\Delta \\ &\Rightarrow N = \frac{9}{4}\Psi_1(3\Psi_1 \pm 2\Psi_0)\Delta^2, \end{aligned} \tag{92}$$

confirming $I^3 = 27J^2$. Therefore, using the classification scheme, as summarized in [3], for $\Delta = 0 \Leftrightarrow \Psi_0 = \mp\Psi_1$ the region is of algebraic type N (because $I = J = 0 = K = L$), while for $\Delta \neq 0$ it is of type II. It degenerates to algebraic type D if, and only if, $\Psi_1 = 0 \neq \Psi_0$ (because $I \neq 0 \neq J$ but $K = 0 = N$).

D. Description of the conformal infinity $\mathcal{I}^\pm$ and global structure

The coordinates employed in (44) are *comoving* in the sense that they are *adapted to the accelerating black holes*. This is clearly seen from the fixed position of the geometrically unique horizons which are *still at the same values* (55) of the radial coordinate r, despite the fact that the black hole moves. This has many advantages, and greatly simplifies physical and geometrical analysis of the spacetime. However, as thoroughly discussed in the simpler case of the C-metric (when $l = 0$) in [9], such accelerating comoving coordinates cannot *naturally cover the whole* conformal infinity $\mathcal{I}$ (scri).

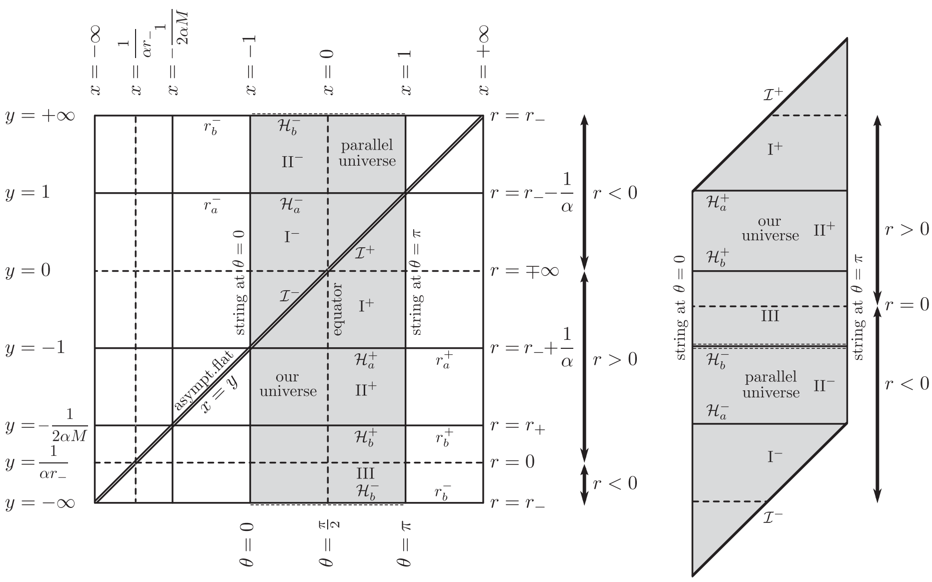

FIG. 4. The complete spacetime structure of the class of accelerating NUT black holes, suppressing the coordinates t and φ (corresponding to stationary and axial symmetry). These fundamental sections are represented by (mutually equivalent) coordinates x, y and θ, r. The black hole spacetime is localized in the shaded region $x \in [-1, 1]$ between two *rotating cosmic strings* at the two opposite poles $\theta = 0$ and $\theta = \pi$. In the complementary (vertical) direction, the spacetime is separated by four Killing horizons at special values of y and equivalently r, namely the two *black-hole horizons* $\mathcal{H}_b^\pm$ are located at $r_b^- = r_-$, $r_b^+ = r_+$ and two *acceleration horizons* $\mathcal{H}_a^\pm$ are at $r_a^+ = r_- + \frac{1}{\alpha}$, $r_a^- = r_- - \frac{1}{\alpha}$. They separate different regions of the spacetimes in which the coordinate r is spatial (regions $\mathrm{II}^\pm$) or temporal (regions $I^\pm$ and III). The values $r = 0$ and $r = \mp\infty$, indicated by horizontal dashed lines, are only coordinate singularities. *Conformal infinity* $\mathcal{I}$, where the spacetime is asymptotically flat, is located along the diagonal line $x = y$. There are thus *two asymptotically flat regions* corresponding to *our universe* where $r > 0$ and the *parallel universe* where $r < 0$, which are connected through the region III with the highest (but finite) curvature in the black hole interior $r \in (r_-, r_+)$. Notice, however, that only along the equatorial section $\theta = \frac{\pi}{2}$ the corresponding two conformal infinities $\mathcal{I}^\pm$ are represented by $r = \pm\infty$. Unlike in the C-metric or Schwarzschild black hole, with the NUT parameter l there is *no curvature singularity at* $r = 0$. It is thus obvious that there are *two complete strings* (not just semi-infinite strings) at $\theta = 0$ and $\theta = \pi$, both connecting the two distinct universes as $r \in (-\infty, +\infty)$. In fact, to obtain a geodesically complete spacetime, it is necessary to *"glue the two universes" along the regular horizon* $\mathcal{H}_b^-$ at $r = r_b^- \equiv r_-$, *both at* $y = -\infty$ and $y = +\infty$, by identifying the corresponding parts of these lower and upper boundaries of the diagram indicated by two finely dashed line segments between $x \in [-1, 1]$. Thus we obtain a *complete diagram of the spacetime* with accelerating NUT black holes, shown in the right part of this figure.

1. Asymptotically flat regions

From the Weyl scalars (64), (65) it follows that asymptotically flat regions without any curvature, locally resembling the null infinity $\mathcal{I}$ of Minkowski space, are reached for $X(r,\theta) \to 0$. It occurs in the vicinity of $\Omega \equiv 1 - \alpha(r - r_-)\cos\theta = 0$, that is for $r \to r_- + 1/(\alpha\cos\theta)$. This corresponds to *the largest possible finite positive* values of r in the angular half-range $\theta \in (0, \frac{\pi}{2})$, but to *the lowest possible finite negative* values of r for the second half-range $\theta \in (\frac{\pi}{2}, \pi)$. In the equatorial section $\theta = \frac{\pi}{2}$, such asymptotically flat region is reached *both* at $r = +\infty$ and $r = -\infty$.

It is necessary to clarify these somewhat puzzling observations. Such an understanding of the global structure of the spacetime manifold with accelerating NUT black holes will provide us with the complete picture summarized in Fig. 4.

To describe and investigate the *complete* conformal infinity $\mathcal{I}$ of spacetimes with accelerating NUT black holes, it is much more convenient to consider the metric form (8). Similarly as for the spherical-like coordinates, it directly follows from expressions (23) that the corresponding curvature scalars Ψ_A *all vanish for* $\Xi(x, y) = 0$. Such regions are thus *asymptotically flat*, representing $\mathcal{I}$. In view of the explicit form of this function (25) it is clear that this condition is equivalent to $x - y = 0$. Therefore, the *asymptotically flat infinity is located at*

$$\mathcal{I}\colon\ x = y, \tag{93}$$

see also Fig. 4. The admitted range of the coordinate x is $x \in [-1, 1]$ (see the subsequent section) and thus the range of y on $\mathcal{I}$ is also $y \in [-1, 1]$. Interestingly, it is exactly the same situation as for the C-metric (10), see [9].

It can now be understood, what are the specific drawbacks of the spherical-like coordinates r, θ of the metric (44) to represent $\mathcal{I}$. *There is no problem in the equatorial plane* $\theta = \frac{\pi}{2}$ corresponding to $x = 0$, which symmetrically divides the spacetime into two regions between the two axes (strings). Due to (93), the scri $\mathcal{I}$ in such "transverse section" is located at $y = 0$, and it follows from the transformation (35) that this occurs at *infinite values of* r,

$$\mathcal{I} \quad \text{at } \theta = \frac{\pi}{2}: \quad r = \pm\infty, \tag{94}$$

as naïvely assumed. However, at any other section $\theta =$ const., the conformal infinity $\mathcal{I}$ is located at *finite values of* r. Indeed, (93) with (35) reads $\cos\theta = 1/[\alpha(r - r_-)]$, that is

$$\mathcal{I} \text{ at any } \theta \neq \frac{\pi}{2}: \quad r = r_- + \frac{1}{\alpha\cos\theta}. \tag{95}$$

Therefore, close to the first string at $\theta = 0$ we obtain $r \to r_- + \alpha^{-1} \equiv r_a^+$, while close to the second string at $\theta = \pi$ we get $r \to r_- - \alpha^{-1} \equiv r_a^-$, see (55) and Fig. 4. Notice that this is exactly the condition for *vanishing conformal factor* in the metric (44), (45),

$$\Omega(r, \theta) = 0. \tag{96}$$

Such a behavior is analogous to the situation in the simpler C-metric [9]. However, in the present case of accelerating NUT black holes, there are *two distinct asymptotically flat regions*, namely $\mathcal{I}^+$ which is the conformal boundary of "our universe" in the region I^+, and $\mathcal{I}^-$ which is the conformal boundary of "parallel universe" in the region I^-. In order to cover the part $\theta > \frac{\pi}{2}$ of $\mathcal{I}^+$ in "our universe," it is necessary to also consider $r < 0$. And vice versa: to cover the part $\theta < \frac{\pi}{2}$ of $\mathcal{I}^-$ in parallel universe, it is necessary to also employ $r > 0$. This is surely possible, but quite cumbersome.

2. Boost-rotation metric form and its analytic extension

To further elucidate the global structure of the new solution (44) for accelerating NUT black holes, it is useful to express it in a form in which its *boost and rotation symmetries are explicitly manifested*. This will also provide a clear argument indicating that the analytically extended space-time represents *a pair of accelerated black-hole sources*. It is achieved by applying the transformation

$$\zeta = \frac{\sqrt{P}}{\alpha\Omega}\sqrt{|1 - \alpha^2(r - r_-)^2|}, \tag{97}$$

$$\rho = \frac{\sin\theta}{\Omega}\sqrt{(r - r_+)(r - r_-)}, \tag{98}$$

(so that ζ, $\rho \geq 0$) with $t' = \alpha t$ and φ unchanged. Clearly, $\zeta = 0$ at both acceleration horizons $\mathcal{H}_a^\pm$, whereas $\rho = 0$ at both black-hole horizons $\mathcal{H}_b^\pm$, and also along the two strings located at $\theta = 0$ and $\theta = \pi$. An application of the transformation (97), (98) takes the metric (44) to the form

$$\mathrm{d}s^2 = -\mathrm{e}^\mu\zeta^2(\mathrm{d}t' - A\mathrm{d}\varphi)^2 + \mathrm{e}^\lambda(\mathrm{d}\zeta^2 + \mathrm{d}\rho^2) + \mathrm{e}^{-\mu}\rho^2\mathrm{d}\varphi^2, \tag{99}$$

where the functions μ, λ, and A are

$$\begin{aligned} \mathrm{e}^\mu &= \frac{(r - r_+)(r - r_-)}{\mathcal{R}^2 P}, \\ \mathrm{e}^{-\lambda} &= \mathcal{R}^{-2}\Big((r - r_+)(r - r_-)P \\ &\quad + (m^2 + l^2)[1 - \alpha^2(r - r_-)^2]\sin^2\theta\Big), \\ A &= 2\alpha l\left(\cos\theta - \frac{\alpha}{2\sqrt{m^2 + l^2}}\frac{r - r_-}{r - r_+}P\rho^2\right). \end{aligned} \tag{100}$$

Of course, these metric functions need to be rewritten in terms of the variables ζ and ρ.

When the NUT parameter vanishes, $l = 0$, the metric becomes *static* because $A = 0$. In fact, the remaining functions e^μ and $\mathrm{e}^{-\lambda}$ then reduce exactly to expressions (14.30), (14.31) in [9] for the *C-metric*. For $m \to 0$, the metric (99) further reduces to the *uniformly accelerated flat metric*, since $\mathrm{e}^\mu \to 1$ and $\mathrm{e}^{-\lambda} \to 1$, yielding

$$\mathrm{d}s^2 = -\zeta^2\mathrm{d}t'^2 + \mathrm{d}\zeta^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2. \tag{101}$$

It is equation (14.25) in [9], equivalent to the Bondi–Rindler metric (3.14) whose coordinates are adapted to the uniform acceleration. *This weak-field limit thus provides a reasonable justification that the black-hole sources are indeed accelerating*. Moreover, in view of (97), the acceleration is given by the parameter α (see also Sec. 3.5 in [9] for more details).

Now, the metric (99) in the stationary regions II can be *analytically extended through the acceleration horizons* located at $\zeta = 0$ by transforming it to the boost-rotation symmetric form with rotating sources (see [14–16]). In particular, by performing the transformation[3]

$$T = \pm\zeta\sinh t', \qquad Z = \pm\zeta\cosh t', \tag{102}$$

the metric becomes

[3] An analogous transformation in the nonstationary regions I close to the conformal infinity $\mathcal{I}$ is $T = \pm\zeta\cosh t'$, $Z = \pm\zeta\sinh t'$.

$$\mathrm{d}s^2 = -\frac{e^{\mu}}{Z^2 - T^2}[(Z\mathrm{d}T - T\mathrm{d}Z) - A(Z^2 - T^2)\mathrm{d}\varphi]^2 + e^{\lambda}\left[\frac{(Z\mathrm{d}Z - T\mathrm{d}T)^2}{Z^2 - T^2} + \mathrm{d}\rho^2\right] + e^{-\mu}\rho^2\mathrm{d}\varphi^2. \tag{103}$$

Clearly, $\zeta^2 \equiv |Z^2 - T^2|$, so that the *acceleration horizons* $\mathcal{H}_a^{\pm}$ *are now located at* $T = \pm Z$. They separate the domains of types I and II. For the whole range of the coordinates T and Z, the boost-rotation symmetric metric (103) covers *all these regions*, with μ, λ, and A being specific functions of ρ and $Z^2 - T^2$, independent of t' and φ.

Notice, however, that the coordinates (ζ, ρ) and equivalently (r, θ) with the "+" sign in (102) each cover only *half* of the section $t' = \text{const}$ corresponding to a single domain of type II, because *necessarily* $Z > 0$. To cover also the analytically extended regions $Z < 0$, a *second copy of these coordinates is required* by choosing the "−" sign in (102). This indicates that the *complete spacetime actually contains a pair of uniformly accelerating NUT black holes*, similarly as in the case of the *C*-metric (see Ch. 14 in [9] for the details). These two black holes accelerate away from each other, and are causally separated. The analytically extended manifold thus contains *four asymptotically flat regions*, a pair of $\mathcal{I}^+$ and a pair of $\mathcal{I}^-$, each in our universe and in the parallel universe.

Let us finally remark that at large values of the radial coordinate r close to $\mathcal{I}^{\pm}$ where $\Omega = 0$, for any fixed value of θ the metric functions behave as $\mathcal{R} \sim r$, P is a constant, and $\Omega \sim r$ (the case $\theta = \frac{\pi}{2}$ must be treated separately). It thus follows from (100) that the functions $e^{\mu}, e^{-\lambda}, A$ remain *finite* in this limit, demonstrating the correct asymptotic behavior of the boost-rotation metric form (103). In fact, analogously to the procedure presented in [16], by a properly performed rescaling of the coordinates and uniquely chosen linear combination of t' and φ, for the given θ it is possible to achieve $e^{\mu}, e^{\lambda} \to 1$, and $A \to 0$ in the asymptotically flat regions of these spacetimes.

E. Character of the axes $\theta = 0$ and $\theta = \pi$: Rotating cosmic strings

We have seen in Sec. VI A that the coordinate singularities given by $\mathcal{Q}(r) = 0$ represent four horizons (55) associated with the Killing vector field ∂_t. There is also the *second Killing vector field* ∂_φ, and its degenerate points identify *the spatial axes of symmetry.*

They are located at the coordinate singularities of the function $\sin\theta$ in the new metric (44), and these appear at the poles $\theta = 0$ and $\theta = \pi$. Therefore, the range of the spatial coordinate θ must be constrained to $\theta \in [0, \pi]$. Via the simple relation $x = -\cos\theta$ this is equivalent to the range $x \in [-1, 1]$ between the two poles $x = \pm 1$ of the function $(1 - x^2)$ in the original form of the metric (8). The location of these poles is indicated in Fig. 4, defining the boundary of the physical spacetime with black holes (the shaded region). Expressed in terms of the coordinates of the boost-rotation/axially symmetric metric (103), related by (98), these poles $\theta = 0, \pi$ correspond to $\rho = 0$ which naturally identifies the corresponding two axes.

In analogy with the *C*-metric, such degenerate axes represent *cosmic strings or struts*. Their tension is the *physical source of the acceleration of the black holes.*

We have proven in Sec. VI B that the algebraic structure of (generic) type I spacetime degenerates along these axes to type D, with the only curvature component Ψ_2 given by (67). Subsequently, in Sec. VI C we have demonstrated that for $\theta = 0$ and $\theta = \pi$ the Kretschmann scalar $\mathcal{K}(r) = 48\mathcal{R}e(\Psi_2^2)$ [see the expression (85)] is everywhere finite, as is explicitly plotted in Figs. 2 and 3. There is thus *no curvature singularity along these axes*. Instead, these are basically topological defects associated with *conical singularities* given by *deficit or excess angles* around the two distinct axes. In addition, due to the nonvanishing NUT parameter l, these *cosmic strings or struts are rotating*, thus introducing an internal twist to the entire spacetime with accelerating NUT black holes. We will now analyze them in more detail.

1. Cosmic strings or struts

We have seen that there are three explicit physical parameters of the spacetime (44), namely the mass m, the acceleration α, and the NUT parameter l of the black holes [which determine the horizon parameters $r_{\pm} = m \pm \sqrt{m^2 + l^2}$, see (38) and (55)]. In fact, there is also the *fourth free parameter* C, *which is hidden in the range of the angular coordinate* $\varphi \in [0, 2\pi C)$. It has not yet been specified. We will demonstrate its physical meaning by relating it to the deficit (or excess) angles of the cosmic strings.

Let us start with investigation of the (non)regularity of the *first axis of symmetry* $\theta = 0$ in (44). Consider a small circle around it given by $\theta = \text{const.}$, with the range $\varphi \in [0, 2\pi C)$, assuming fixed t and r. The invariant length of its *circumference* is $\int_0^{2\pi C} \sqrt{g_{\varphi\varphi}}\mathrm{d}\varphi$, while its *radius* is $\int_0^{\theta} \sqrt{g_{\theta\theta}}\mathrm{d}\tilde{\theta}$. The *axis is regular* if their fraction in the limit $\theta \to 0$ is equal to 2π. In general we obtain

$$f_0 \equiv \lim_{\theta\to 0} \frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to 0} \frac{2\pi C\sqrt{g_{\varphi\varphi}}}{\theta\sqrt{g_{\theta\theta}}}. \tag{104}$$

Now, the conceptual problem is that the metric function $g_{\varphi\varphi}$ in (44), and thus the circumference, does *not* approach zero in the limit $\theta \to 0$ due to the presence of $\cos\theta$ in the first term in the metric. This problem can be resolved by the same procedure as for the classic Taub–NUT solution (see the transition between the metrics (12.1) and (12.3) in [9]): By applying the transformation of the time coordinate[4]

$$t = t_0 + 2l\varphi, \tag{105}$$

the metric (44) becomes

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left[-\frac{\mathcal{Q}}{\mathcal{R}^2}\left(\mathrm{d}t_0 + 2l\left(2\sin^2\frac{\theta}{2} + \alpha\mathcal{T}\sin^2\theta\right)\mathrm{d}\varphi\right)^2 + \frac{\mathcal{R}^2}{\mathcal{Q}}\mathrm{d}r^2 + \mathcal{R}^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\mathrm{d}\varphi^2\right)\right], \tag{106}$$

so that

$$g_{\varphi\varphi} = \frac{1}{\Omega^2}\left[\mathcal{R}^2 P\sin^2\theta - 4l^2\frac{\mathcal{Q}}{\mathcal{R}^2}\left(2\sin^2\frac{\theta}{2} + \alpha\mathcal{T}\sin^2\theta\right)^2\right],$$
$$g_{\theta\theta} = \frac{\mathcal{R}^2}{\Omega^2 P}. \tag{107}$$

For very small values of θ we obtain $g_{\varphi\varphi} \approx \mathcal{R}^2 P\theta^2/\Omega^2$ because the terms proportional to l^2 become negligible. Evaluating the limit (104) we thus obtain

$$f_0 = 2\pi C(1 - \alpha(r_+ - r_-)) \equiv 2\pi C(1 - 2\alpha\sqrt{m^2 + l^2}). \tag{108}$$

The axis $\theta = 0$ in the metric (106) can thus be made regular by the choice

$$C = C_0 \equiv \frac{1}{1 - 2\alpha\sqrt{m^2 + l^2}}. \tag{109}$$

Analogously, it is possible to regularize the *second axis of symmetry* $\theta = \pi$. Performing the complementary transformation of the time coordinate

$$t = t_\pi - 2l\varphi, \tag{110}$$

the metric (44) becomes

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left[-\frac{\mathcal{Q}}{\mathcal{R}^2}\left(\mathrm{d}t_\pi - 2l\left(2\cos^2\frac{\theta}{2} - \alpha\mathcal{T}\sin^2\theta\right)\mathrm{d}\varphi\right)^2 + \frac{\mathcal{R}^2}{\mathcal{Q}}\mathrm{d}r^2 + \mathcal{R}^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\mathrm{d}\varphi^2\right)\right], \tag{111}$$

i.e.,

$$g_{\varphi\varphi} = \frac{1}{\Omega^2}\left[\mathcal{R}^2 P\sin^2\theta - 4l^2\frac{\mathcal{Q}}{\mathcal{R}^2}\left(2\cos^2\frac{\theta}{2} - \alpha\mathcal{T}\sin^2\theta\right)^2\right],$$
$$g_{\theta\theta} = \frac{\mathcal{R}^2}{\Omega^2 P}. \tag{112}$$

[4]It leads to a closed circle instead of an open helical orbit of the axial Killing vector around $\theta = 0$. For a recent related study of geometrical and physical properties of symmetry axes of black holes with NUT parameters see [17].

For $\theta \to \pi$ we thus obtain $g_{\varphi\varphi} \approx \mathcal{R}^2 P(\pi - \theta)^2/\Omega^2$. The radius of a small circle around the axis $\theta = \pi$ is $\int_\theta^\pi \sqrt{g_{\theta\theta}}\mathrm{d}\tilde{\theta}$. Evaluating the fraction

$$f_\pi \equiv \lim_{\theta\to\pi}\frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to\pi}\frac{2\pi C\sqrt{g_{\varphi\varphi}}}{(\pi - \theta)\sqrt{g_{\theta\theta}}}, \tag{113}$$

we obtain

$$f_\pi = 2\pi C(1 + \alpha(r_+ - r_-)) \equiv 2\pi C(1 + 2\alpha\sqrt{m^2 + l^2}). \tag{114}$$

The axis $\theta = \pi$ in the metric (111) is thus regular for the unique choice

$$C = C_\pi \equiv \frac{1}{1 + 2\alpha\sqrt{m^2 + l^2}}. \tag{115}$$

It is now explicitly seen that it is not possible to regularize simultaneously both the axes because $C_0 \neq C_\pi$ and $t_0 \neq t_\pi = t_0 + 4l\varphi$ (unless $\alpha = 0 = l$ which is just the Schwarzschild solution, regular for the standard choice $C = 1$).

When the second axis of symmetry $\theta = \pi$ is made regular by the choice (115), there is necessarily a *deficit angle* δ_0 (conical singularity) along the first axis $\theta = 0$, namely

$$\delta_0 \equiv 2\pi - f_0 = \frac{8\pi\alpha\sqrt{m^2 + l^2}}{1 + 2\alpha\sqrt{m^2 + l^2}} > 0. \tag{116}$$

The corresponding tension in this *cosmic string located along* $\theta = 0$ *pulls the black hole, causing its uniform acceleration*. Such string *extends to the full range of the radial coordinate* $r \in (-\infty, +\infty)$, connecting thus our universe with the parallel universe through the nonsingular NUT black-hole interior, see Fig. 4. Moreover, as argued in Sec. VI D, there is a *pair* of causally separated NUT black holes accelerating away from each other by the action of two such cosmic strings, one string in each copy $Z > 0$ and $Z < 0$.

Complementarily, when the first axis of symmetry $\theta = 0$ is made regular by the choice (109), there is necessarily an *excess angle* δ_π along the second axis $\theta = \pi$, namely

$$\delta_\pi \equiv 2\pi - f_\pi = -\frac{8\pi\alpha\sqrt{m^2 + l^2}}{1 - 2\alpha\sqrt{m^2 + l^2}} < 0. \tag{117}$$

This represents the *cosmic strut located along* $\theta = \pi$ *between the two black holes, pushing them away from each other* in opposite spatial directions $\pm Z$.

In particular, for black holes with vanishing NUT parameter $l = 0$, the general results (116) and (117) reduce to

$$\delta_0 = \frac{8\pi\alpha m}{1+2\alpha m} \quad \text{and} \quad \delta_\pi = -\frac{8\pi\alpha m}{1-2\alpha m}, \tag{118}$$

which fully agree with the known expressions for the C-metric, see Eqs. (14.15)–(14.17) in [9].

2. Rotation of these cosmic strings or struts

With a generic NUT parameter l, these *cosmic strings/struts are rotating*. This can be seen by calculating the *angular velocity* parameter ω of the metric along the two different axes [10],

$$\omega \equiv \frac{g_{t\varphi}}{g_{tt}}. \tag{119}$$

For the general form of the new metric (44) we obtain $\omega = -2l(\cos\theta - \alpha\mathcal{T}\sin^2\theta)$. Evaluating it on the axis $\theta = 0$ and the axis $\theta = \pi$, we immediately get

$$\omega_0 = -2l \quad \text{and} \quad \omega_\pi = 2l, \tag{120}$$

respectively. Both cosmic strings/struts thus rotate. In fact, they are *contrarotating* with exactly opposite angular velocities $\pm 2l$ *determined solely by the NUT parameter.*

If the first axis of symmetry $\theta = 0$ is made regular by considering the metric (106) with the time t_0, then $\omega = 2l(2\sin^2\frac{\theta}{2} + \alpha\mathcal{T}\sin^2\theta)$ and the corresponding angular velocities of the axes are

$$\omega_0 = 0 \quad \text{and} \quad \omega_\pi = 4l, \tag{121}$$

On the other hand, when the second axis $\theta = \pi$ is regularized by switching to the metric (111) with t_π, then $\omega = -2l(2\cos^2\frac{\theta}{2} - \alpha\mathcal{T}\sin^2\theta)$ and the angular velocities of the axes are

$$\omega_0 = -4l \quad \text{and} \quad \omega_\pi = 0. \tag{122}$$

Clearly, there is always a *constant difference* $\Delta\omega \equiv \omega_\pi - \omega_0 = 4l$ between the angular velocities of the two rotating cosmic strings or struts, directly given by the NUT parameter l.

F. Regions with closed timelike curves around the rotating strings

In the vicinity of the rotating cosmic strings or struts, which are located along $\theta = 0$ and $\theta = \pi$, the spacetime with accelerating NUT black holes can serve as a specific time machine. Indeed, similarly as in the classic Taub–NUT solution, there are *closed timelike curves*.

To identify these pathological causality-violating regions, let us again consider simple curves in the spacetime which are *circles around the axes of symmetry* $\theta = 0$ and $\theta = \pi$ such that only the periodic angular coordinate $\varphi \in [0, 2\pi C)$ changes, while the remaining three coordinates t, r and θ *are kept fixed*. The corresponding tangent (velocity) vectors are thus proportional to the *Killing vector field* ∂_φ. Its norm is determined just by the metric coefficient $g_{\varphi\varphi}$, which for the general metric (44) reads

$$g_{\varphi\varphi} = \frac{1}{\Omega^2}\left[\mathcal{R}^2 P\sin^2\theta - 4l^2\frac{\mathcal{Q}}{\mathcal{R}^2}(\cos\theta - \alpha\mathcal{T}\sin^2\theta)^2\right]. \tag{123}$$

When $l = 0$, i.e., for nonrotating cosmic strings, this metric coefficient is always positive, so that the circles are *spacelike curves*. However, with the NUT parameter l, there are regions where $g_{\varphi\varphi} < 0$ in which the circles (orbits of the axial symmetry) are *closed timelike curves*. These pathological regions are explicitly given by the condition

$$\mathcal{R}^4 P(1-\cos^2\theta) < 4l^2\mathcal{Q}(\cos\theta - \alpha\mathcal{T}(1-\cos^2\theta))^2, \tag{124}$$

where the functions P, $\mathcal{Q}$, $\mathcal{T}$, $\mathcal{R}$ have been defined in (45). Although this condition is quite difficult to be solved analytically, some general observations can easily be made.

In particular, the condition can not be satisfied in the regions where $\mathcal{Q}(r) < 0$. Assuming that the acceleration α is not too large, satisfying (56) which implies (57), the closed timelike curves can thus only appear *between* the black hole horizon $\mathcal{H}_b$ and the acceleration horizon $\mathcal{H}_a$, that is only in the region II^+ given by $r \in (r_b^+, r_a^+)$ or in the region II^- given by $r \in (r_a^-, r_b^-)$. On the contrary, the pathological domain can not occur in the region III inside the black hole or close to the conformal infinities $\mathcal{I}^\pm$ which are the boundaries of the dynamical regions $I^\pm$ where r is temporal because $\mathcal{Q} < 0$, see Fig. 4.

These observations are nicely confirmed by plotting the values of the relevant function $g_{\varphi\varphi}(r,\theta)$ given by (123), obtained numerically for various choices of the black-hole parameters. A typical example $m = 0.5$, $l = 3$, $\alpha = 0.05$ is presented in Fig. 5, for $r > 0$ (left) and $r < 0$ (right). The grey curves are contour lines (isolines) of a constant value of $g_{\varphi\varphi}(r,\theta)$, red color depicts large positive values, while blue color depicts negative values (dark gray domains indicate extremely large values, both positive and negative). Zeros of $g_{\varphi\varphi}$ in light yellow, determining the boundary of the pathological regions given by the condition (124), are exactly indicated by the thick black curves. As expected, these regions with closed timelike curves occur close to the both axes $\theta = 0$ and $\theta = \pi$, were the rotating cosmic strings at located. Such regions are indeed restricted to the concentric domains (two annuli) between the black hole

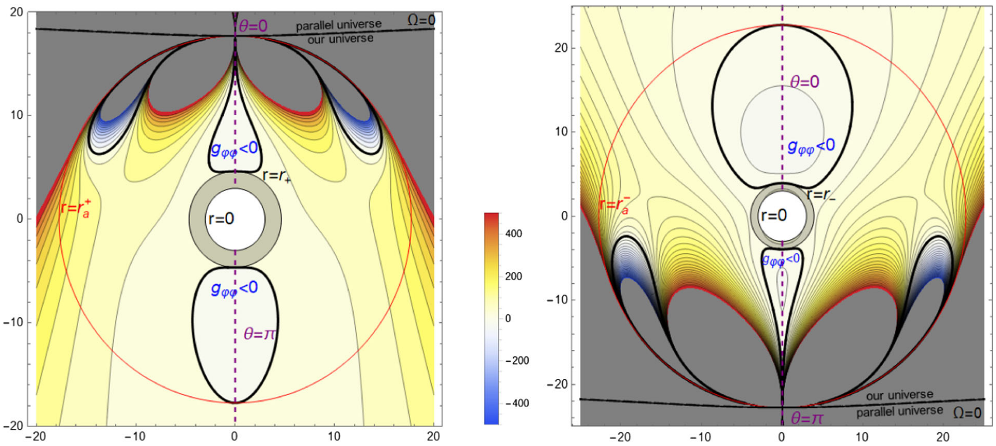

FIG. 5. Plot of the metric function $g_{\varphi\varphi}$ (123) for the general accelerating NUT black hole (44) with rotating cosmic strings on both axes $\theta = 0$ and $\theta = \pi$. Its values are visualized in quasipolar coordinates $\mathrm{x} \equiv \sqrt{r^2 + l^2} \sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + l^2} \cos\theta$ for $r \geq 0$ (left) and $r \leq 0$ (right). The gray annulus in the center of each figure localizes the black hole bordered by its horizons $\mathcal{H}_b^\pm$ at $r_+ > 0$ and $r_- < 0$. The acceleration horizons $\mathcal{H}_a^\pm$ at r_a^+ and r_a^- (big red circles) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. The grey curves are contour lines $g_{\varphi\varphi}(r, \theta) = \mathrm{const}$, and the values are color-coded from red (positive values) to blue (negative values). Extremely large/low values are cut and depicted in dark gray. The thick black curves in the light yellow domain are the isolines $g_{\varphi\varphi} = 0$ determining the boundary of the pathological regions (124) with closed timelike curves. They occur close to both the axes $\theta = 0$ and $\theta = \pi$ (purple dashed lines), but also near the acceleration horizons, forming an additional symmetric pair of "lobes" around $\theta = 0$ just below $\mathcal{H}_a^+$ and around $\theta = \pi$ just above $\mathcal{H}_a^-$. This plot for the choice $m = 0.5$, $l = 3$, $\alpha = 0.05$ is typical.

horizons $\mathcal{H}_b^\pm$ at $r_b^\pm = r_\pm$ and the acceleration horizons $\mathcal{H}_a^\pm$ at $r_a^\pm = r_- \pm \alpha^{-1}$.

Interestingly, for $r > 0$ there is another pair of symmetric "lobes" around $\theta = 0$ near the acceleration horizon $\mathcal{H}_a^+$ (big red circle). At a given r close to r_a^+, these lobes extend to surprisingly large values of θ. Similarly, there is a "mirror" pair of such pathological regions near $\mathcal{H}_a^-$ and $\theta = \pi$ for $r < 0$. In both cases, the lobes are localized around such axis, along which the acceleration horizon $\mathcal{H}_a$ closely approaches the conformal infinity $\mathcal{I}$ at $\Omega = 0$.

In Fig. 5 we visualized the regions containing the closed timelike curves for the accelerating black hole with a big value of the NUT parameter $l = 6m = 3$. However, our investigation of a large set of the parameters m, l, and α shows that the overall picture displayed here is quite generic.

Similarly, it is possible to investigate the regions with closed timelike curves in the special cases when one of the axes is regular. The *case with regular axis* $\theta = 0$ is described by the metric (106), and the corresponding metric function (107) gives for fixed t_0 the condition

$$\mathcal{R}^4 P(1 + \cos\theta) < 4l^2 \mathcal{Q}(1 - \cos\theta)(1 + \alpha\mathcal{T}(1 + \cos\theta))^2, \tag{125}$$

while the complementary *case with regular axis* $\theta = \pi$ is described by the metric (111), and the corresponding metric function (112) yields for fixed t_π

$$\mathcal{R}^4 P(1 - \cos\theta) < 4l^2 \mathcal{Q}(1 + \cos\theta)(1 - \alpha\mathcal{T}(1 - \cos\theta))^2. \tag{126}$$

For a direct comparison with Fig. 5, analogous visualizations of the pathological regions in such special cases are shown in Fig. 6 for the same choice of the black-hole parameters.

Finally, we can observe that the conditions (124)–(126) for the pathological regions simplify considerably in the *absence of acceleration*. Indeed, for $\alpha = 0$ the key functions reduce to $P = 1$, $\mathcal{Q} = (r - r_+)(r - r_-) \equiv r^2 - 2mr - l^2$ and $\mathcal{R}^2 = r^2 + l^2$, see (50), so that the above three conditions (124)–(126) for the regions with closed timelike curves become, respectively,

$$\cos^2\theta > \frac{r^2 + l^2}{r^2 + l^2 + 4l^2 f},$$
$$\cos\theta < -\frac{r^2 + l^2 - 4l^2 f}{r^2 + l^2 + 4l^2 f},$$
$$\cos\theta > \frac{r^2 + l^2 - 4l^2 f}{r^2 + l^2 + 4l^2 f}, \tag{127}$$

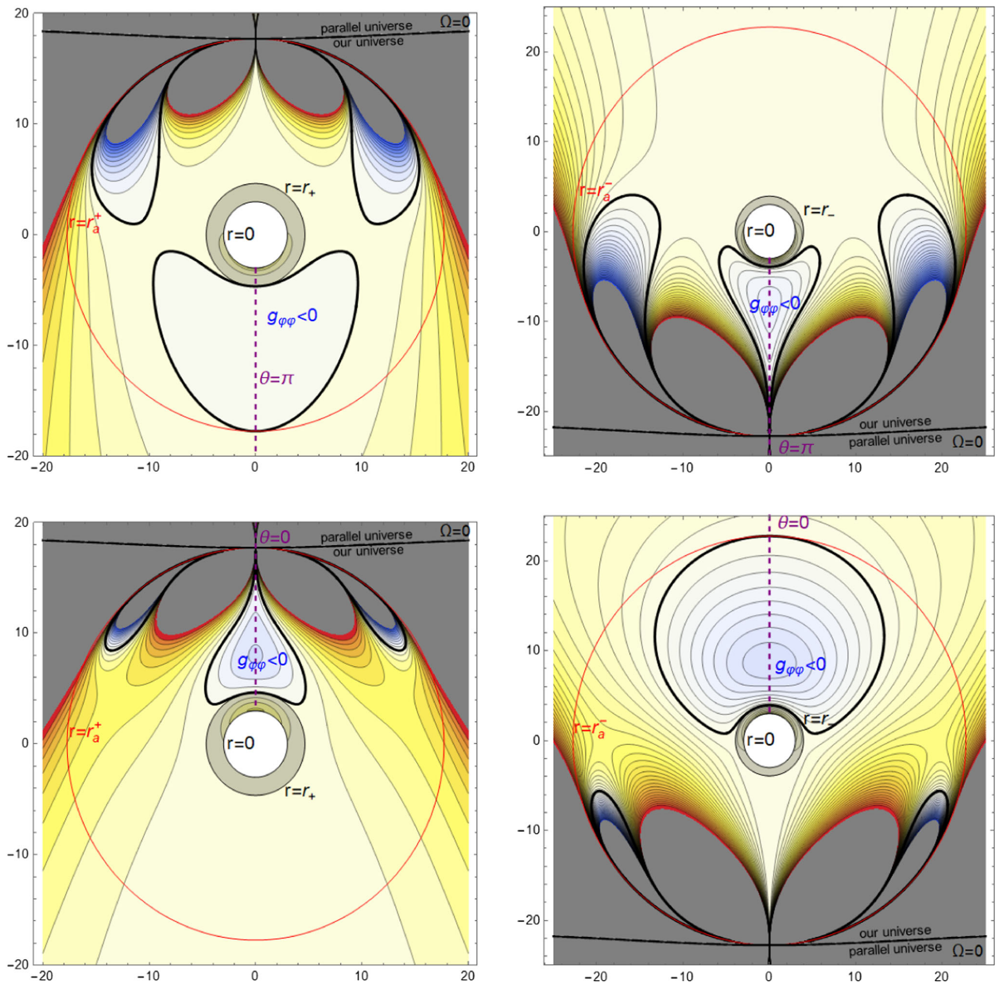

FIG. 6. The functions $g_{\varphi\varphi}$ given by (107) and (112) for the accelerating NUT black hole metric (106) with the regular axis $\theta = 0$ (top row) and for the metric (111) with the regular axis $\theta = \pi$ (bottom row). The regions with closed timelike curves surround the remaining rotating cosmic string, and there is always an additional symmetric pair of such pathological regions near the acceleration horizons.

where $f(r) \equiv \mathcal{Q}/\mathcal{R}^2$, see (52). The result (127) fully agrees with the equation for the Taub–NUT spacetime presented in Sec. 12.1.4 of the monograph [9].

VII. CONCLUDING SUMMARY

We presented and carefully investigated a remarkable class of spacetimes which represent accelerating black holes with a NUT parameter. In particular:

(i) By two independent methods we verified in Sec. III that the metric (2) found by Chng, Mann and Stelea in 2006 is indeed an exact solution to Einstein's vacuum field equations.

(ii) To achieve this, we employed a modified version (8) of the solution in which one redundant parameter was removed and the original metric simplified, so that the standard C-metric (10) is immediately obtained by setting the NUT-like twist parameter λ to zero.

(iii) Using the metric form (8), in Sec. IV we calculated all components of the Weyl tensor in the natural null tetrad (20), namely the NP scalars Ψ_A (23), and the corresponding curvature scalar invariants I and J (26).

(iv) Since generically $I^3 \neq 27J^2$, the Weyl tensor is of algebraically general type I with four distinct principal null directions, explicitly given by expressions (27) with (34), (33).

(v) It explains why this class of solutions with accelerating NUT black holes has not been previously found within the large Plebański–Demiański family of type D spacetimes.

(vi) In Sec. V we derived and introduced a new metric form (44) of these solutions in "spherical-type" coordinates which is much more convenient for understanding of this class of black holes.

(vii) In particular, its metric functions (45), with $r_\pm \equiv m \pm \sqrt{m^2 + l^2}$ given by (38), explicitly depend on three physical parameters, namely the mass m, the acceleration α, and the NUT parameter l.

(viii) These black-hole parameters can be separately set to zero, recovering the well-known spacetimes in standard coordinates, namely the C-metric (48) when $l = 0$, the Taub–NUT metric (51) when $\alpha = 0$, the Schwarzschild metric (53), and flat Minkowski space (54).

(ix) The structure of this complete family of accelerating NUT black holes is shown in Fig. 1. By setting $\alpha = 0$ or $l = 0$, algebraically general spacetime reduces to the type D.

(x) Using the new metric (44), in Sec. VI we investigated main physical and geometrical properties of this family of accelerating NUT black holes. In particular:

(xi) In Sec. VI A we localized the position of the horizons associated with the Killing vector field ∂_t. There are two black-hole horizons $\mathcal{H}_b^\pm$ located at $r_b^- \equiv r_-$ and $r_b^+ \equiv r_+$ plus two acceleration horizons $\mathcal{H}_a^\pm$ at $r_a^+ \equiv r_- + \frac{1}{\alpha}$ and $r_a^- \equiv r_- - \frac{1}{\alpha}$. For small acceleration $\alpha < \frac{1}{2\sqrt{m^2+l^2}}$ they are ordered as $r_a^- < r_b^- < 0 < r_b^+ < r_a^+$, see (57).

(xii) We carefully analyzed the curvature of the spacetime in Sec. VI B. We expressed the Weyl scalars (64) in the new coordinates and frames. For $l = 0$ and $\alpha = 0$, only the Newtonian component $\Psi_2^{(r\theta)}$ remains, and its special subcases (68) and (72) fully agree with standard expressions for the C-metric and the Taub–NUT metric, which are both of algebraic type D.

(xiii) Evaluating these Weyl scalars on the horizons, we proved that they are all regular (that is free of curvature singularities), and of a double degenerate algebraic type D.

(xiv) Using the curvature invariants, including the Kretschmann scalar, we proved in Sec. VI C that there are no curvature singularities whenever the NUT parameter l is nonzero. This is visualized in Figs. 2 and 3. Maximal (finite) values of the curvature are inside the black hole.

(xv) Curvature singularity appears only in the C-metric case $l = 0$ at $r = 0$. All other spacetimes in the class of accelerating NUT black holes are nonsingular, and to describe their complete manifold it is thus necessary to consider the full range of the coordinate $r \in (-\infty, +\infty)$.

(xvi) There may occur special regions in a given spacetime which are of algebraic type D, II or N, according to the values of the scalar curvature invariants (92).

(xvii) In Sec. VI D we identified asymptotically flat regions which correspond to the conformal infinities $\mathcal{I}^\pm$ given by $\Omega = 0$. These are simply given by the condition $x = y$ in the coordinates of the metric form (8).

(xviii) Using the spherical-like coordinates of (44), the position of $\mathcal{I}^\pm$ is given by the conditions (94) and (95), which look less intuitive.

(xix) All these investigations lead us to a complete understanding of the global structure of this class of spacetimes, summarized in Fig. 4. The accelerating NUT black hole can be understood as a "throat" of maximal curvature which connects our universe located in the region $r > 0$ with the second (also asymptotically flat) parallel universe in the region $r < 0$.

(xx) Analytic extension across the acceleration horizons, using the boost-rotation symmetric form of the metric (103), revealed that there is actually a pair of such (causally separated) NUT black holes, which together involve four asymptotically flat regions. The two black holes uniformly accelerate in opposite directions, as in the case of the C-metric with $l = 0$.

(xxi) We clarified in Sec. VI E that the physical source of the acceleration of this pair of black holes is the tension (or compression) in the rotating cosmic strings (or struts) located along the corresponding two axes of axial symmetry at $\theta = 0$ and $\theta = \pi$.

(xxii) These strings or struts are related to the deficit or excess angles which introduce topological defects along the axes. However, their curvature remains finite, and of algebraic type D.

(xxiii) In general, there are strings/struts along both the axes, but one of the axis can be made fully regular by a suitable choice of the constant C in the range $\varphi \in [0, 2\pi C)$. The first axis $\theta = 0$ is regular in the metric form (106) with the choice (109), whereas the second axis $\theta = \pi$ is regular in the form (111) with the choice (115). In the first case, there is a cosmic strut along $\theta = \pi$ with the excess angle (117), while in the second case there is a cosmic string along $\theta = 0$ with the deficit angle (116).

(xxiv) In addition to the deficit/excess angles, these cosmic strings/struts located along the axes of symmetry are characterized by their rotation parameter ω (angular velocity). Their values are directly related to the NUT parameter l, see expressions (120)–(122).

(xxv) There is always a constant difference $\Delta\omega = 4l$ between the angular velocities of the two rotating cosmic strings or struts. If, and only if $l = 0$, both the axes are nontwisting.

(xxvi) In the neighborhood of these rotating strings/struts there occur pathological regions with closed timelike curves. They are given by the conditions (124)–(126) and visualized in Figs. 5 and 6.

We hope that, with these geometrical and physical insights, the new explicit form (44) of the class of accelerating NUT black holes can be used as an interesting example for various types of investigations in Einstein's general relativity, black hole thermodynamics, quantum gravity, or high-energy physics, for example by extending the recent studies [18,19].

ACKNOWLEDGMENTS

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APPENDIX A: CURVATURE OF GENERAL STATIONARY AXISYMMETRIC SPACETIMES

Let us assume a general form of stationary axisymmetric metric in coordinates (t, φ, x, y) given by (11), that is

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{t\varphi} & 0 & 0 \\ g_{t\varphi} & g_{\varphi\varphi} & 0 & 0 \\ 0 & 0 & g_{xx} & 0 \\ 0 & 0 & 0 & g_{yy} \end{pmatrix}, \tag{A1}$$

in which all the metric functions *can only depend* on x and y. The inverse matrix is

$$g^{\mu\nu} = \begin{pmatrix} g_{\varphi\varphi}/D & -g_{t\varphi}/D & 0 & 0 \\ -g_{t\varphi}/D & g_{tt}/D & 0 & 0 \\ 0 & 0 & 1/g_{xx} & 0 \\ 0 & 0 & 0 & 1/g_{yy} \end{pmatrix}, \tag{A2}$$

where

$$D \equiv g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2. \tag{A3}$$

The corresponding *Christoffel symbols of the first kind* $\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} - g_{\beta\gamma,\alpha})$ are

$$\begin{array}{llll}
\Gamma_{ttt} = 0, & \Gamma_{\varphi tt} = 0, & \Gamma_{xtt} = -\frac{1}{2} g_{tt,x}, & \Gamma_{ytt} = -\frac{1}{2} g_{tt,y}, \\
\Gamma_{tt\varphi} = 0, & \Gamma_{\varphi t\varphi} = 0, & \Gamma_{xt\varphi} = -\frac{1}{2} g_{t\varphi,x}, & \Gamma_{yt\varphi} = -\frac{1}{2} g_{t\varphi,y}, \\
\Gamma_{ttx} = \frac{1}{2} g_{tt,x}, & \Gamma_{\varphi tx} = \frac{1}{2} g_{t\varphi,x}, & \Gamma_{xtx} = 0, & \Gamma_{ytx} = 0, \\
\Gamma_{tty} = \frac{1}{2} g_{tt,y}, & \Gamma_{\varphi ty} = \frac{1}{2} g_{t\varphi,y}, & \Gamma_{xty} = 0, & \Gamma_{yty} = 0, \\
\Gamma_{t\varphi\varphi} = 0, & \Gamma_{\varphi\varphi\varphi} = 0, & \Gamma_{x\varphi\varphi} = -\frac{1}{2} g_{\varphi\varphi,x}, & \Gamma_{y\varphi\varphi} = -\frac{1}{2} g_{\varphi\varphi,y}, \\
\Gamma_{t\varphi x} = \frac{1}{2} g_{t\varphi,x}, & \Gamma_{\varphi\varphi x} = \frac{1}{2} g_{\varphi\varphi,x}, & \Gamma_{x\varphi x} = 0, & \Gamma_{y\varphi x} = 0, \\
\Gamma_{t\varphi y} = \frac{1}{2} g_{t\varphi,y}, & \Gamma_{\varphi\varphi y} = \frac{1}{2} g_{\varphi\varphi,y}, & \Gamma_{x\varphi y} = 0, & \Gamma_{y\varphi y} = 0, \\
\Gamma_{txx} = 0, & \Gamma_{\varphi xx} = 0, & \Gamma_{xxx} = \frac{1}{2} g_{xx,x}, & \Gamma_{yxx} = -\frac{1}{2} g_{xx,y}, \\
\Gamma_{txy} = 0, & \Gamma_{\varphi xy} = 0, & \Gamma_{xxy} = \frac{1}{2} g_{xx,y}, & \Gamma_{yxy} = \frac{1}{2} g_{yy,x}, \\
\Gamma_{tyy} = 0, & \Gamma_{\varphi yy} = 0, & \Gamma_{xyy} = -\frac{1}{2} g_{yy,x}, & \Gamma_{yyy} = \frac{1}{2} g_{yy,y},
\end{array} \tag{A4}$$

and usual *Christoffel symbols of the second kind* $\Gamma^{\alpha}{}_{\beta\gamma} \equiv g^{\alpha\sigma}\Gamma_{\sigma\beta\gamma}$ are thus

$$\begin{aligned}
&{\Gamma^t}_{tt} = 0, && {\Gamma^\varphi}_{tt} = 0,\\
&{\Gamma^t}_{t\varphi} = 0, && {\Gamma^\varphi}_{t\varphi} = 0,\\
&{\Gamma^t}_{tx} = \tfrac{1}{2}(g_{\varphi\varphi}g_{tt,x} - g_{t\varphi}g_{t\varphi,x})/D, && {\Gamma^\varphi}_{tx} = \tfrac{1}{2}(g_{tt}g_{t\varphi,x} - g_{t\varphi}g_{tt,x})/D,\\
&{\Gamma^t}_{ty} = \tfrac{1}{2}(g_{\varphi\varphi}g_{tt,y} - g_{t\varphi}g_{t\varphi,y})/D, && {\Gamma^\varphi}_{ty} = \tfrac{1}{2}(g_{tt}g_{t\varphi,y} - g_{t\varphi}g_{tt,y})/D,\\
&{\Gamma^t}_{\varphi\varphi} = 0, && {\Gamma^\varphi}_{\varphi\varphi} = 0,\\
&{\Gamma^t}_{\varphi x} = \tfrac{1}{2}(g_{\varphi\varphi}g_{t\varphi,x} - g_{t\varphi}g_{\varphi\varphi,x})/D, && {\Gamma^\varphi}_{\varphi x} = \tfrac{1}{2}(g_{tt}g_{\varphi\varphi,x} - g_{t\varphi}g_{t\varphi,x})/D,\\
&{\Gamma^t}_{\varphi y} = \tfrac{1}{2}(g_{\varphi\varphi}g_{t\varphi,y} - g_{t\varphi}g_{\varphi\varphi,y})/D, && {\Gamma^\varphi}_{\varphi y} = \tfrac{1}{2}(g_{tt}g_{\varphi\varphi,y} - g_{t\varphi}g_{t\varphi,y})/D,\\
&{\Gamma^t}_{xx} = 0, && {\Gamma^\varphi}_{xx} = 0,\\
&{\Gamma^t}_{xy} = 0, && {\Gamma^\varphi}_{xy} = 0,\\
&{\Gamma^t}_{yy} = 0, && {\Gamma^\varphi}_{yy} = 0,
\end{aligned} \tag{A5}$$

$$\begin{aligned}
&{\Gamma^x}_{tt} = -\tfrac{1}{2}g_{tt,x}/g_{xx}, && {\Gamma^y}_{tt} = -\tfrac{1}{2}g_{tt,y}/g_{yy},\\
&{\Gamma^x}_{t\varphi} = -\tfrac{1}{2}g_{t\varphi,x}/g_{xx}, && {\Gamma^y}_{t\varphi} = -\tfrac{1}{2}g_{t\varphi,y}/g_{yy},\\
&{\Gamma^x}_{tx} = 0, && {\Gamma^y}_{tx} = 0,\\
&{\Gamma^x}_{ty} = 0, && {\Gamma^y}_{ty} = 0,\\
&{\Gamma^x}_{\varphi\varphi} = -\tfrac{1}{2}g_{\varphi\varphi,x}/g_{xx}, && {\Gamma^y}_{\varphi\varphi} = -\tfrac{1}{2}g_{\varphi\varphi,y}/g_{yy},\\
&{\Gamma^x}_{\varphi x} = 0, && {\Gamma^y}_{\varphi x} = 0,\\
&{\Gamma^x}_{\varphi y} = 0, && {\Gamma^y}_{\varphi y} = 0,\\
&{\Gamma^x}_{xx} = \tfrac{1}{2}g_{xx,x}/g_{xx}, && {\Gamma^y}_{xx} = -\tfrac{1}{2}g_{xx,y}/g_{yy},\\
&{\Gamma^x}_{xy} = \tfrac{1}{2}g_{xx,y}/g_{xx}, && {\Gamma^y}_{xy} = \tfrac{1}{2}g_{yy,x}/g_{yy},\\
&{\Gamma^x}_{yy} = -\tfrac{1}{2}g_{yy,x}/g_{xx}, && {\Gamma^y}_{yy} = \tfrac{1}{2}g_{yy,y}/g_{yy}
\end{aligned} \quad . \tag{A6}$$

Now, we compute the *Riemann curvature tensor*. However, instead of using the usual definition

$$R^\mu{}_{\nu\kappa\lambda} \equiv \Gamma^\mu{}_{\nu\lambda,\kappa} - \Gamma^\mu{}_{\nu\kappa,\lambda} + \Gamma^\mu{}_{\rho\kappa}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\mu{}_{\rho\lambda}\Gamma^\rho{}_{\nu\kappa}, \tag{A7}$$

for our purposes we found that it is much more convenient to employ the equivalent expression

$$R_{\mu\nu\kappa\lambda} = \frac{1}{2}(g_{\mu\lambda,\kappa\nu} + g_{\kappa\nu,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) + \Gamma_{\sigma\mu\lambda}\Gamma^\sigma{}_{\nu\kappa} - \Gamma_{\sigma\mu\kappa}\Gamma^\sigma{}_{\nu\lambda}. \tag{A8}$$

Its advantage is that there is no need to differentiate the complicated Christoffel symbols of the second kind. This greatly simplifies subsequent computer algebra manipulations. Direct evaluation using (A4) and (A5) leads to

$$
\begin{aligned}
R_{t\varphi t\varphi} &= \frac{1}{4}\left(\frac{g_{t\varphi,x}^2 - g_{tt,x}g_{\varphi\varphi,x}}{g_{xx}} + \frac{g_{t\varphi,y}^2 - g_{tt,y}g_{\varphi\varphi,y}}{g_{yy}}\right),\\
R_{t\varphi tx} &= 0,\\
R_{t\varphi ty} &= 0,\\
R_{t\varphi\varphi x} &= 0,\\
R_{t\varphi\varphi y} &= 0,\\
R_{t\varphi xy} &= \frac{1}{4(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2)}\Big(g_{tt}(g_{t\varphi,y}g_{\varphi\varphi,x} - g_{t\varphi,x}g_{\varphi\varphi,y}) - g_{t\varphi}(g_{tt,y}g_{\varphi\varphi,x} - g_{tt,x}g_{\varphi\varphi,y}) + g_{\varphi\varphi}(g_{tt,y}g_{t\varphi,x} - g_{tt,x}g_{t\varphi,y})\Big),\\
R_{txtx} &= -\frac{1}{2}g_{tt,xx} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,x}^2 - 2g_{t\varphi}g_{tt,x}g_{t\varphi,x} + g_{\varphi\varphi}g_{tt,x}^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{tt,x}g_{xx,x}}{g_{xx}} - \frac{g_{tt,y}g_{xx,y}}{g_{yy}}\right),\\
R_{txty} &= -\frac{1}{2}g_{tt,xy} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,x}g_{t\varphi,y} - g_{t\varphi}(g_{tt,x}g_{t\varphi,y} + g_{t\varphi,x}g_{tt,y}) + g_{\varphi\varphi}g_{tt,x}g_{tt,y}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{tt,x}g_{xx,y}}{g_{xx}} + \frac{g_{tt,y}g_{yy,x}}{g_{yy}}\right),\\
R_{tx\varphi x} &= -\frac{1}{2}g_{t\varphi,xx} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,x}g_{\varphi\varphi,x} - g_{t\varphi}(g_{t\varphi,x}^2 + g_{tt,x}g_{\varphi\varphi,x}) + g_{\varphi\varphi}g_{tt,x}g_{t\varphi,x}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{t\varphi,x}g_{xx,x}}{g_{xx}} - \frac{g_{t\varphi,y}g_{xx,y}}{g_{yy}}\right),\\
R_{tx\varphi y} &= -\frac{1}{2}g_{t\varphi,xy} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,y}g_{\varphi\varphi,x} - g_{t\varphi}(g_{t\varphi,x}g_{t\varphi,y} + g_{tt,y}g_{\varphi\varphi,x}) + g_{\varphi\varphi}g_{tt,y}g_{t\varphi,x}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{t\varphi,x}g_{xx,y}}{g_{xx}} + \frac{g_{t\varphi,y}g_{yy,x}}{g_{yy}}\right),\\
R_{txxy} &= 0,\\
R_{tyty} &= -\frac{1}{2}g_{tt,yy} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,y}^2 - 2g_{t\varphi}g_{tt,y}g_{t\varphi,y} + g_{\varphi\varphi}g_{tt,y}^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} - \frac{g_{tt,x}g_{yy,x}}{g_{xx}} + \frac{g_{tt,y}g_{yy,y}}{g_{yy}}\right),\\
R_{ty\varphi x} &= R_{tx\varphi y} - R_{t\varphi xy},\\
R_{ty\varphi y} &= -\frac{1}{2}g_{t\varphi,yy} + \frac{1}{4}\left(\frac{g_{tt}g_{t\varphi,y}g_{\varphi\varphi,y} - g_{t\varphi}(g_{t\varphi,y}^2 + g_{tt,y}g_{\varphi\varphi,y}) + g_{\varphi\varphi}g_{tt,y}g_{t\varphi,y}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} - \frac{g_{t\varphi,x}g_{yy,x}}{g_{xx}} + \frac{g_{t\varphi,y}g_{yy,y}}{g_{yy}}\right),\\
R_{tyxy} &= 0,\\
R_{\varphi x\varphi x} &= -\frac{1}{2}g_{\varphi\varphi,xx} + \frac{1}{4}\left(\frac{g_{tt}g_{\varphi\varphi,x}^2 - 2g_{t\varphi}g_{t\varphi,x}g_{\varphi\varphi,x} + g_{\varphi\varphi}g_{t\varphi,x}^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{\varphi\varphi,x}g_{xx,x}}{g_{xx}} - \frac{g_{\varphi\varphi,y}g_{xx,y}}{g_{yy}}\right),\\
R_{\varphi x\varphi y} &= -\frac{1}{2}g_{\varphi\varphi,xy} + \frac{1}{4}\left(\frac{g_{tt}g_{\varphi\varphi,x}g_{\varphi\varphi,y} - g_{t\varphi}(g_{t\varphi,x}g_{\varphi\varphi,y} + g_{\varphi\varphi,x}g_{t\varphi,y}) + g_{\varphi\varphi}g_{t\varphi,x}g_{t\varphi,y}}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} + \frac{g_{\varphi\varphi,x}g_{xx,y}}{g_{xx}} + \frac{g_{\varphi\varphi,y}g_{yy,x}}{g_{yy}}\right),\\
R_{\varphi xxy} &= 0,\\
R_{\varphi y\varphi y} &= -\frac{1}{2}g_{\varphi\varphi,yy} + \frac{1}{4}\left(\frac{g_{tt}g_{\varphi\varphi,y}^2 - 2g_{t\varphi}g_{t\varphi,y}g_{\varphi\varphi,y} + g_{\varphi\varphi}g_{t\varphi,y}^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2} - \frac{g_{\varphi\varphi,x}g_{yy,x}}{g_{xx}} + \frac{g_{\varphi\varphi,y}g_{yy,y}}{g_{yy}}\right),\\
R_{\varphi yxy} &= 0,\\
R_{xyxy} &= -\frac{1}{2}(g_{xx,yy} + g_{yy,xx}) + \frac{1}{4}\left(\frac{g_{xx,y}^2 + g_{xx,x}g_{yy,x}}{g_{xx}} + \frac{g_{yy,x}^2 + g_{xx,y}g_{yy,y}}{g_{yy}}\right).
\end{aligned}
\tag{A9}
$$

Finally, we employ a general expression for the *Ricci tensor*,

$$
R_{\nu\lambda} \equiv g^{\mu\kappa}R_{\mu\nu\kappa\lambda} = \frac{1}{2}g^{\mu\kappa}(g_{\mu\lambda,\kappa\nu} + g_{\kappa\nu,\mu\lambda} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}) + g^{\mu\kappa}\Gamma_{\sigma\mu\lambda}\Gamma^{\sigma}{}_{\nu\kappa} - g^{\mu\kappa}\Gamma_{\sigma\mu\kappa}\Gamma^{\sigma}{}_{\nu\lambda}, \tag{A10}
$$

which yields the following nontrivial components for the Ricci tensor of the metric (A1):

$$
\begin{aligned}
R_{tt} &= -\frac{1}{2}\left(\frac{g_{tt,xx}}{g_{xx}}+\frac{g_{tt,yy}}{g_{yy}}\right) - 2(\Gamma^x{}_{t\varphi}\Gamma^\varphi{}_{tx}+\Gamma^y{}_{t\varphi}\Gamma^\varphi{}_{ty})\\
&\quad - \Gamma^x{}_{tt}(\Gamma^t{}_{tx}-\Gamma^\varphi{}_{\varphi x}+\Gamma^x{}_{xx}-\Gamma^y{}_{xy}) - \Gamma^y{}_{tt}(\Gamma^t{}_{ty}-\Gamma^\varphi{}_{\varphi y}-\Gamma^x{}_{xy}+\Gamma^y{}_{yy}),\\
R_{t\varphi} &= -\frac{1}{2}\left(\frac{g_{t\varphi,xx}}{g_{xx}}+\frac{g_{t\varphi,yy}}{g_{yy}}\right) - \Gamma^x{}_{tt}\Gamma^t{}_{\varphi x} - \Gamma^y{}_{tt}\Gamma^t{}_{\varphi y} - \Gamma^\varphi{}_{tx}\Gamma^x{}_{\varphi\varphi} - \Gamma^\varphi{}_{ty}\Gamma^y{}_{\varphi\varphi} - \Gamma^x{}_{t\varphi}(\Gamma^x{}_{xx}-\Gamma^y{}_{xy}) + \Gamma^y{}_{t\varphi}(\Gamma^x{}_{xy}-\Gamma^y{}_{yy}),\\
R_{\varphi\varphi} &= -\frac{1}{2}\left(\frac{g_{\varphi\varphi,xx}}{g_{xx}}+\frac{g_{\varphi\varphi,yy}}{g_{yy}}\right) - 2(\Gamma^x{}_{t\varphi}\Gamma^t{}_{\varphi x}+\Gamma^y{}_{t\varphi}\Gamma^t{}_{\varphi y})\\
&\quad - \Gamma^x{}_{\varphi\varphi}(-\Gamma^t{}_{tx}+\Gamma^\varphi{}_{\varphi x}+\Gamma^x{}_{xx}-\Gamma^y{}_{xy}) - \Gamma^y{}_{\varphi\varphi}(-\Gamma^t{}_{ty}+\Gamma^\varphi{}_{\varphi y}-\Gamma^x{}_{xy}+\Gamma^y{}_{yy}),\\
R_{xx} &= -\frac{1}{2}\left(\frac{g_{\varphi\varphi}g_{tt,xx}-2g_{t\varphi}g_{t\varphi,xx}+g_{tt}g_{\varphi\varphi,xx}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}+\frac{g_{yy,xx}+g_{xx,yy}}{g_{yy}}\right) + (\Gamma^t{}_{tx})^2 + 2\Gamma^\varphi{}_{tx}\Gamma^t{}_{\varphi x} + (\Gamma^\varphi{}_{\varphi x})^2\\
&\quad + \Gamma^x{}_{xx}(\Gamma^t{}_{tx}+\Gamma^\varphi{}_{\varphi x}) + \Gamma^y{}_{xy}(\Gamma^x{}_{xx}+\Gamma^y{}_{xy}) + \Gamma^y{}_{xx}(\Gamma^t{}_{ty}+\Gamma^\varphi{}_{\varphi y}-\Gamma^x{}_{xy}-\Gamma^y{}_{yy}),\\
R_{xy} &= -\frac{1}{2}\left(\frac{g_{\varphi\varphi}g_{tt,xy}-2g_{t\varphi}g_{t\varphi,xy}+g_{tt}g_{\varphi\varphi,xy}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}\right) + \Gamma^t{}_{tx}\Gamma^t{}_{ty} + \Gamma^\varphi{}_{tx}\Gamma^t{}_{\varphi y} + \Gamma^\varphi{}_{ty}\Gamma^t{}_{\varphi x} + \Gamma^\varphi{}_{\varphi x}\Gamma^\varphi{}_{\varphi y}\\
&\quad + \Gamma^x{}_{xy}(\Gamma^t{}_{tx}+\Gamma^\varphi{}_{\varphi x}) + \Gamma^y{}_{xy}(\Gamma^t{}_{ty}+\Gamma^\varphi{}_{\varphi y}) + \Gamma^x{}_{xy}\Gamma^y{}_{xy} - \Gamma^x{}_{yy}\Gamma^y{}_{xx},\\
R_{yy} &= -\frac{1}{2}\left(\frac{g_{\varphi\varphi}g_{tt,yy}-2g_{t\varphi}g_{t\varphi,yy}+g_{tt}g_{\varphi\varphi,yy}}{g_{tt}g_{\varphi\varphi}-g_{t\varphi}^2}+\frac{g_{xx,yy}+g_{yy,xx}}{g_{xx}}\right) + (\Gamma^t{}_{ty})^2 + 2\Gamma^\varphi{}_{ty}\Gamma^t{}_{\varphi y} + (\Gamma^\varphi{}_{\varphi y})^2\\
&\quad + \Gamma^y{}_{yy}(\Gamma^t{}_{ty}+\Gamma^\varphi{}_{\varphi y}) + \Gamma^x{}_{xy}(\Gamma^x{}_{xy}+\Gamma^y{}_{yy}) + \Gamma^x{}_{yy}(\Gamma^t{}_{tx}+\Gamma^\varphi{}_{\varphi x}-\Gamma^x{}_{xx}-\Gamma^y{}_{xy}).
\end{aligned} \tag{A11}
$$

APPENDIX B: RICCI TENSORS OF CONFORMALLY RELATED METRICS

For the conformally related metrics (15),

$$
\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{B1}
$$

the corresponding Ricci tensors are connected as (see, e.g., [11])

$$
\begin{aligned}
\tilde{R}_{ab} &= R_{ab} - 2\Omega^{-1}\nabla_a\nabla_b\Omega - \Omega^{-1}g_{ab}g^{cd}\nabla_c\nabla_d\Omega\\
&\quad + 4\Omega^{-2}\nabla_a\Omega\nabla_b\Omega - \Omega^{-2}g_{ab}g^{cd}\nabla_c\Omega\nabla_d\Omega.
\end{aligned} \tag{B2}
$$

This implies relation between the physical and unphysical Ricci tensors R_{ab} and $\tilde{R}_{ab}$, respectively,

$$
\begin{aligned}
R_{ab} &= \tilde{R}_{ab} + \frac{1}{\Omega^2}[(\tilde{g}_{ab}\tilde{g}^{cd} + 2\delta^c_a\delta^d_b)(\Omega_{,cd} - \tilde{\Gamma}^e{}_{cd}\Omega_{,e})\Omega\\
&\quad - 3\tilde{g}_{ab}\tilde{g}^{cd}\Omega_{,c}\Omega_{,d}].
\end{aligned} \tag{B3}
$$

For the metric (15), (18), the conformal factor (16) is independent of φ and t, so that the resulting metric is again stationary and axisymmetric, in which case the relations (B3) simplify to

$$
\begin{aligned}
R_{tt} &= \tilde{R}_{tt} + \frac{\Phi}{\Omega}\tilde{g}_{tt} - \frac{2}{\Omega}(\tilde{\Gamma}^x{}_{tt}\Omega_{,x} + \tilde{\Gamma}^y{}_{tt}\Omega_{,y}),\\
R_{t\varphi} &= \tilde{R}_{t\varphi} + \frac{\Phi}{\Omega}\tilde{g}_{t\varphi} - \frac{2}{\Omega}(\tilde{\Gamma}^x{}_{t\varphi}\Omega_{,x} + \tilde{\Gamma}^y{}_{t\varphi}\Omega_{,y}),\\
R_{\varphi\varphi} &= \tilde{R}_{\varphi\varphi} + \frac{\Phi}{\Omega}\tilde{g}_{\varphi\varphi} - \frac{2}{\Omega}(\tilde{\Gamma}^x{}_{\varphi\varphi}\Omega_{,x} + \tilde{\Gamma}^y{}_{\varphi\varphi}\Omega_{,y}),\\
R_{xx} &= \tilde{R}_{xx} + \frac{\Phi}{\Omega}\tilde{g}_{xx} + \frac{2}{\Omega}(\Omega_{,xx} - \tilde{\Gamma}^x{}_{xx}\Omega_{,x} - \tilde{\Gamma}^y{}_{xx}\Omega_{,y}),\\
R_{xy} &= \tilde{R}_{xy} + \frac{\Phi}{\Omega}\tilde{g}_{xy} + \frac{2}{\Omega}(\Omega_{,xy} - \tilde{\Gamma}^x{}_{xy}\Omega_{,x} - \tilde{\Gamma}^y{}_{xy}\Omega_{,y}),\\
R_{yy} &= \tilde{R}_{yy} + \frac{\Phi}{\Omega}\tilde{g}_{yy} + \frac{2}{\Omega}(\Omega_{,yy} - \tilde{\Gamma}^x{}_{yy}\Omega_{,x} - \tilde{\Gamma}^y{}_{yy}\Omega_{,y}),
\end{aligned} \tag{B4}
$$

where

$$
\begin{aligned}
\Phi &\equiv -\frac{1}{\tilde{D}}[(\tilde{g}_{\varphi\varphi}\tilde{\Gamma}^x{}_{tt} - 2\tilde{g}_{t\varphi}\tilde{\Gamma}^x{}_{t\varphi} + \tilde{g}_{tt}\tilde{\Gamma}^x{}_{\varphi\varphi})\Omega_{,x}\\
&\quad + (\tilde{g}_{\varphi\varphi}\tilde{\Gamma}^y{}_{tt} - 2\tilde{g}_{t\varphi}\tilde{\Gamma}^y{}_{t\varphi} + \tilde{g}_{tt}\tilde{\Gamma}^y{}_{\varphi\varphi})\Omega_{,y}]\\
&\quad + \frac{1}{\tilde{g}_{xx}}(\Omega_{xx} - \tilde{\Gamma}^x{}_{xx}\Omega_{,x} - \tilde{\Gamma}^y_{,xx}\Omega_{,y})\\
&\quad + \frac{1}{\tilde{g}_{yy}}(\Omega_{yy} - \tilde{\Gamma}^x_{,yy}\Omega_{,x} - \tilde{\Gamma}^y{}_{yy}\Omega_{,y}) - \frac{3}{\Omega}\left(\frac{\Omega^2_{,x}}{\tilde{g}_{xx}} + \frac{\Omega^2_{,y}}{\tilde{g}_{yy}}\right),
\end{aligned} \tag{B5}
$$

and the determinant for the metric (18) reads

$$\tilde{D} \equiv \tilde{g}_{tt}\tilde{g}_{\varphi\varphi} - \tilde{g}_{t\varphi}^2 = \Omega^4 D$$
$$= -(x-y)^8(1-x^2)^3F^3(x)(y^2-1)^3F^3(y)\tilde{H}^2(x,y). \tag{B6}$$

[1] J. F. Plebański and M. Demiański, Rotating, charged and uniformly accelerating mass in general relativity, Ann. Phys. (N.Y.) **98**, 98 (1976).
[2] R. Debever, On type D expanding solutions of Einstein–Maxwell equations, Bull. Soc. Math. Belg. **23**, 360 (1971).
[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[4] K. Hong and E. Teo, A new form of the *C*-metric, Classical Quantum Gravity **20**, 3269 (2003).
[5] K. Hong and E. Teo, A new form of the rotating *C*-metric, Classical Quantum Gravity **22**, 109 (2005).
[6] J. B. Griffiths and J. Podolský, Accelerating and rotating black holes, Classical Quantum Gravity **22**, 3467 (2005).
[7] J. Podolský and J. B. Griffiths, Accelerating Kerr–Newman black holes in (anti-)de Sitter space-time, Phys. Rev. D **73**, 044018 (2006).
[8] J. B. Griffiths and J. Podolský, A new look at the Plebański–Demiański family of solutions, Int. J. Mod. Phys. D **15**, 335 (2006).
[9] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).
[10] B. Chng, R. Mann, and C. Stelea, Accelerating Taub-NUT and Eguchi–Hanson solitons in four dimensions, Phys. Rev. D **74**, 084031 (2006).
[11] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, London, 1984).
[12] S. Guha and S. Chakraborty, On the gravitational entropy of accelerating black holes, Int. J. Mod. Phys. D **29**, 2050034 (2020).
[13] G. Clément, D. Gal̕tsov, and M. Guenouche, NUT wormholes, Phys. Rev. D **93**, 024048 (2016).
[14] J. Bičák and B. Schmidt, Asymptotically flat radiative space-times with boost-rotation symmetry, Phys. Rev. D **40**, 1827 (1989).
[15] J. Bičák and V. Pravda, Spinning *C* metric: Radiative spacetime with accelerating, rotating black holes, Phys. Rev. D **60**, 044004 (1999).
[16] J. B. Griffiths and J. Podolský, Global aspects of accelerating and rotating black hole space-times, Classical Quantum Gravity **23**, 555 (2006).
[17] I. Kolář and P. Krtouš, Symmetry axes of Kerr-NUT-(A)dS spacetimes, Phys. Rev. D **100**, 064014 (2019).
[18] M. Appels, R. Gregory, and D. Kubizňák, Thermodynamics of Accelerating Black Holes, Phys. Rev. Lett. **117**, 131303 (2016).
[19] A. Anabalón, M. Appels, R. Gregory, D. Kubizňák, R. B. Mann, and A. Övgün, Holographic thermodynamics of accelerating black holes, Phys. Rev. D **98**, 104038 (2018).

2. New improved form of black holes of type D

This chapter is based on the paper *New improved form of black holes of type D* [45] by Podolský and Vrátný, published in 2021 in the journal Physical Review D **104**, 084078.

2.1 Derivation of the new metric form

In this contribution, we studied the whole Plebański–Demiański class of black hole solutions in asymptotically flat universe (that is for $\Lambda = 0$). More specifically, we derived a new metric form based on the well-known representation of this large family originally found by Griffiths and Podolský in 2006 (II.28)–(II.32). We also provided a thorough physical and geometrical analysis of this solution.

The main reason, why we decided to (re)analyze this solution, was a general simplicity and the ability to factorize the metric function $Q(r)$ given by (II.32) in the case $\Lambda = 0$. This factorization was already introduced by Griffiths and Podolský in 2005 (see the equation (15) in [38]). The question was whether also the *second* metric function $P(\theta)$ could be factorized. We achieved this.

First of all, we introduced a set of new parameters, namely

$$\begin{aligned} m &\equiv \frac{a^2 - l^2}{\omega^2 k}\,\tilde{m} - \alpha\frac{l}{\omega}(a^2 - l^2 + e^2 + g^2)\,,\\ e^2 &\equiv \frac{a^2 - l^2}{\omega^2 k}\,\tilde{e}^2\,,\\ g^2 &\equiv \frac{a^2 - l^2}{\omega^2 k}\,\tilde{g}^2\,. \end{aligned} \tag{2.1}$$

Using it, we were able to pull a common constant factor S^{-1} out of the metric functions $\mathcal{P}(\theta)$ (II.31) and $\mathcal{Q}(r)$ (II.32), that is

$$\mathcal{Q}(r) = S^{-1}\,Q(r)\,, \qquad \mathcal{P}(\theta) = S^{-1}\,P(\theta)\,, \tag{2.2}$$

where the prefactor is simply

$$S^{-1} \equiv \frac{\omega^2 k}{a^2 - l^2}\,. \tag{2.3}$$

Then, we performed the transformations $t \to S\,t$ and $\varphi \to S\,\varphi$ which effectively pulled out the constant S from the complete metric. This could be removed by a simple conformal rescaling of the whole metric, $\mathrm{d}\tilde{s}^2 = S\,\mathrm{d}s^2$.

The last step was to appropriately fix the twist parameter ω. Recall that from (II.11) it is clear that ω represents a *twist behavior*. By studying the choices made for ω in [38], [40] for gaining the standard forms of the well-known metrics, we finally decided to fix the parameter as

$$\omega \equiv \frac{a^2 + l^2}{a}\,. \tag{2.4}$$

With such a unique choice, we derived a *completely new form of the Plebański–Demiański metric* describing all the type D black hole solutions with zero cosmological constant, namely

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg(-\frac{Q}{\rho^2}\Big[\mathrm{d}t - \Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)\,\mathrm{d}\varphi\Big]^2 + \frac{\rho^2}{Q}\,\mathrm{d}r^2 + \frac{\rho^2}{P}\,\mathrm{d}\theta^2 + \frac{P}{\rho^2}\,\sin^2\theta\,\Big[a\,\mathrm{d}t - \Big(r^2 + (a+l)^2\Big)\,\mathrm{d}\varphi\Big]^2\bigg)\,, \tag{2.5}$$

where

$$\Omega = 1 - \frac{\alpha\,a}{a^2+l^2}\,r\,(l + a\cos\theta)\,, \tag{2.6}$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2\,, \tag{2.7}$$

$$P(\theta) = \Big(1 - \frac{\alpha\,a}{a^2+l^2}\,r_+(l + a\cos\theta)\Big)\Big(1 - \frac{\alpha\,a}{a^2+l^2}\,r_-(l + a\cos\theta)\Big), \tag{2.8}$$

$$Q(r) = \Big(r - r_+\Big)\Big(r - r_-\Big)\Big(1 + \alpha\,a\,\frac{a-l}{a^2+l^2}\,r\Big)\Big(1 - \alpha\,a\,\frac{a+l}{a^2+l^2}\,r\Big)\,, \tag{2.9}$$

and the convenient parameters $r_\pm$ determining the two main roots of $Q(r)$ are

$$r_\pm \equiv m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}\,. \tag{2.10}$$

The whole metric is now described by the following physical parameters:

m	.....	mass ,
a	.....	Kerr-like rotation ,
l	.....	NUT parameter ,
e	.....	electric charge ,
g	.....	magnetic charge ,
α	.....	acceleration .

The great advantage of this new metric form is that we can now easily obtain the standard forms of the most important black holes by simply setting the corresponding physical parameters to zero, namely

- $\alpha = 0$ **: Kerr–Newman–NUT black holes** (P2.III.A) ,
- $l = 0$ **: Accelerating Kerr–Newman black holes** (P2.III.B) ,
- $a = 0$ **: Charged Taub–NUT black holes** (P2.III.C) ,
- $e = 0 = g$ **: Uncharged accelerating Kerr–NUT black holes** (P2.III.D) .

The new metric (2.5) describes black hole solutions with *distinct* horizons only when the condition $m^2 + l^2 > a^2 + e^2 + g^2$ holds. This representation, however, also admits *extreme* and *hyperextreme* cases for which $m^2 + l^2 \le a^2 + e^2 + g^2$. For these cases we use a slightly modified metric functions (2.8), (2.9), namely

$$P(\theta) = 1 - 2\,\alpha\,a\,\frac{l + a\,\cos\theta}{a^2+l^2}\,m + \alpha^2 a^2\frac{(l + a\,\cos\theta)^2}{(a^2+l^2)^2}(a^2 - l^2 + e^2 + g^2)\,, \tag{2.11}$$

$$Q(r) = \Big(r^2 - 2m\,r + (a^2 - l^2 + e^2 + g^2)\Big)\Big(1 + \alpha\,a\,\frac{a-l}{a^2+l^2}\,r\Big)\Big(1 - \alpha\,a\,\frac{a+l}{a^2+l^2}\,r\Big).$$

Extreme and hyperextreme solutions are described in section (P2.IV).

2.2 Physical analysis of the new metric form

Another benefit of this new metric form is its suitability for physical and geometrical analysis. The fact that the metric is a direct generalization of the standard forms of well-known black hole metrics means that physically relevant variables or invariants are easily comparable with the ones from these simpler solutions. Similarly, other well studied phenomena such as the *ergoregions*, *pathological regions*, *cosmological strings/struts*, *singularities*, etc., are also expected to appear in the general case.

First, we introduced a natural null tetrad inspired by the null tetrad (II.9):

$$\begin{aligned}
\mathbf{k} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}\Big(\big(r^2+(a+l)^2\big)\,\partial_t + a\,\partial_\varphi\Big) + \sqrt{Q}\,\partial_r\right],\\
\mathbf{l} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}\Big(\big(r^2+(a+l)^2\big)\,\partial_t + a\,\partial_\varphi\Big) - \sqrt{Q}\,\partial_r\right],\\
\mathbf{m} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{P}\sin\theta}\Big(\partial_\varphi + \big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\big)\,\partial_t\Big) + \mathrm{i}\,\sqrt{P}\,\partial_\theta\right].
\end{aligned} \tag{2.12}$$

Adopting this null tetrad the only non-zero NP scalars corresponding to the Weyl and Ricci tensors are

$$\begin{aligned}
\Psi_2 &= \frac{\Omega^3}{\big[r+\mathrm{i}\,(l+a\cos\theta)\big]^3}\left[-\,(m+\mathrm{i}\,l)\Big(1-\mathrm{i}\,\alpha\,a\,\frac{a^2-l^2}{a^2+l^2}\Big)\right.\\
&\qquad\left.+\frac{(e^2+g^2)}{r-\mathrm{i}\,(l+a\cos\theta)}\Big(1+\frac{\alpha\,a}{a^2+l^2}\big[a\,r\cos\theta+\mathrm{i}\,l\,(l+a\cos\theta)\big]\Big)\right], \qquad (2.13)\\
\Phi_{11} &= \tfrac{1}{2}(e^2+g^2)\,\frac{\Omega^4}{\rho^4}, \qquad (2.14)
\end{aligned}$$

while the Ricci scalar R vanishes (c.f. the NP scalars of the previous form of the metric (II.15), (II.16)).

The spin coefficients are given by

$$\begin{aligned}
\kappa &= \nu = 0\,, \qquad \sigma = \lambda = 0\,,\\
\varrho &= \mu = -\frac{\sqrt{Q}}{\sqrt{2}\,\rho^3}\Big(1+\mathrm{i}\,\frac{\alpha\,a}{a^2+l^2}\,(l+a\cos\theta)^2\Big)\Big(r-\mathrm{i}\,(l+a\cos\theta)\Big),\\
\tau &= \pi = -\frac{a\sqrt{P}\,\sin\theta}{\sqrt{2}\,\rho^3}\Big(1-\mathrm{i}\,\frac{\alpha\,a}{a^2+l^2}\,r^2\Big)\Big(r-\mathrm{i}\,(l+a\cos\theta)\Big),
\end{aligned} \tag{2.15}$$

which correspond to the expressions (II.10)–(II.12). Also the remaining coefficients $\alpha=\beta$ and $\epsilon=\gamma$ are non-zero, but we do not explicitly write them here due to their complexity.

Therefore, both PNDs $\mathbf{k}$ and $\mathbf{l}$ are *geodetic* and *shear-free*, but with *expansion* and a generally non-zero *twist* given by

$$\Theta \equiv -\mathcal{R}e\,(\rho) = -\mathcal{R}e\,(\mu) = \frac{\sqrt{Q}}{\sqrt{2}\,\rho^3}\Big(r+\frac{\alpha\,a}{a^2+l^2}\,(l+a\cos\theta)^3\Big), \tag{2.16}$$

$$\omega \equiv -\mathcal{I}m\,(\rho) = -\mathcal{I}m\,(\mu) = -\frac{\Omega\sqrt{Q}}{\sqrt{2}\,\rho^3}\,(l+a\cos\theta)\,. \tag{2.17}$$

The Newman–Penrose scalars of the *Maxwell 2-form* $\mathbf{F}$ for the charged solution ($e \neq 0 \neq g$) was also evaluated. Its *4-potential* is

$$\mathbf{A} = -\sqrt{e^2+g^2}\,\frac{r}{\rho^2}\left[\mathrm{d}t - (a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta)\,\mathrm{d}\varphi\right], \tag{2.18}$$

and the corresponding NP scalars gives only one non-zero component

$$\Phi_1 \equiv \tfrac{1}{2}F_{ab}(k^a l^b + \bar{m}^a m^b) = \frac{\sqrt{e^2+g^2}\;\Omega^2}{\Big(r + \mathrm{i}\,(l + a\cos\theta)\Big)^2}\,. \tag{2.19}$$

Furthermore, we were able to explicitly expressed the corresponding Weyl scalar $\mathcal{C} \equiv C_{abcd}C^{abcd}$ (see the equations (65)–(66)). However, we mistakenly called it the Kretschmann scalar $\mathcal{K}$ which is true only for the uncharged solution (see Sec. I.4 for a full explanation, in particular equation (I.20)).

2.2.1 Horizons

The black hole horizons $\mathcal{H}_b^\pm$ and the acceleration horizons $\mathcal{H}_a^\pm$ given by the equation $Q(r) = 0$ are immediately apparent from the factorized form of the metric function $Q(r)$ given by (2.9), that is

$$\mathcal{H}_b^+ \quad \text{at} \quad r_b^+ \equiv r_+ = m + \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}\,, \tag{2.20}$$

$$\mathcal{H}_b^- \quad \text{at} \quad r_b^- \equiv r_- = m - \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}\,, \tag{2.21}$$

$$\mathcal{H}_a^+ \quad \text{at} \quad r_a^+ \equiv +\frac{1}{\alpha}\,\frac{a^2+l^2}{a^2+a\,l}\,, \tag{2.22}$$

$$\mathcal{H}_a^- \quad \text{at} \quad r_a^- \equiv -\frac{1}{\alpha}\,\frac{a^2+l^2}{a^2-a\,l}\,, \tag{2.23}$$

where $r_\pm$ were already introduced in (2.10).

Clearly r_+ is positive for any choice of parameters (assuming $m > 0$), but r_- can have any sign:

$$r_- > 0 \quad \Leftrightarrow \quad l^2 < a^2 + e^2 + g^2\,, \tag{2.24}$$

$$r_- < 0 \quad \Leftrightarrow \quad l^2 > a^2 + e^2 + g^2\,, \tag{2.25}$$

$$r_- = 0 \quad \Leftrightarrow \quad l^2 = a^2 + e^2 + g^2\,. \tag{2.26}$$

Moreover, for a sufficiently small (positive) acceleration

$$\alpha < \frac{1}{r_+}\,\frac{a^2+l^2}{a^2+a\,l}\,, \tag{2.27}$$

the 4 horizons of the Plebański–Demiański black hole takes the most natural ordering

$$r_a^- < r_b^- < r_b^+ < r_a^+\,, \tag{2.28}$$

where the two black hole horizons $\mathcal{H}_b^\pm$ are surrounded by two "outer" acceleration horizons $\mathcal{H}_a^\pm$.

2.2.2 Ergoregions

Due to the presence of the rotation parameter a, the existence of ergoregions, commonly known for the Kerr black hole and other related metrics, can be expected. The ergoregions occur in the situation when

$$g_{tt} = \frac{1}{\Omega^2 \rho^2} \left(P\, a^2 \sin^2\theta - Q\right) > 0\,. \tag{2.29}$$

This equation is not directly solvable by analytic means, but we provided some visualization by computer plotting, see the Fig. 2.1.

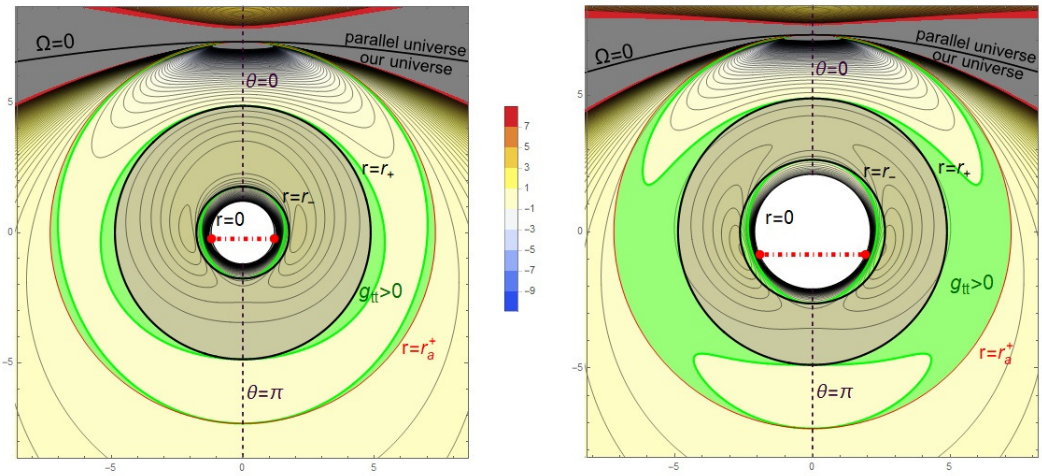

Figure 2.1: Plot of the metric function g_{tt} given by (2.29) in quasi-polar coordinates $\mathrm{x} \equiv \sqrt{r^2 + (a+l)^2}\,\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + (a+l)^2}\,\cos\theta$ for $r \geq 0$. The green regions localize the ergoregions. The gray annulus in the center of each figure localizes the black hole within its horizons $\mathcal{H}_b^{\pm}$ at r_+ and r_-. The acceleration horizon $\mathcal{H}_a^{+}$ at r_a^{+} (big red circle) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. For more details, see the Fig. 1 of the attached Paper 2.

2.2.3 Curvature singularities

Curvature singularities of a generic type D black hole (2.5) were discussed in Sec. P2.V.C. They correspond to the case when the only non-zero component of the Weyl tensor Ψ_2 (2.13) *diverges*. This happens only for $\rho^2 = r + \mathrm{i}\,(l + a\cos\theta) = 0$, that is when both the real and imaginary parts vanish:

$$r = 0 \qquad \text{and at the same time} \qquad l + a\cos\theta = 0\,. \tag{2.30}$$

This condition can be also seen from the Ricci scalar (2.14), or the Weyl scalar $\mathcal{C}$ (equations (65)–(66) of the attached paper).

From the conditions (2.30), we can discuss all possible cases of the mutual relation between the Kerr-like rotation a and the NUT parameter l. More specif-

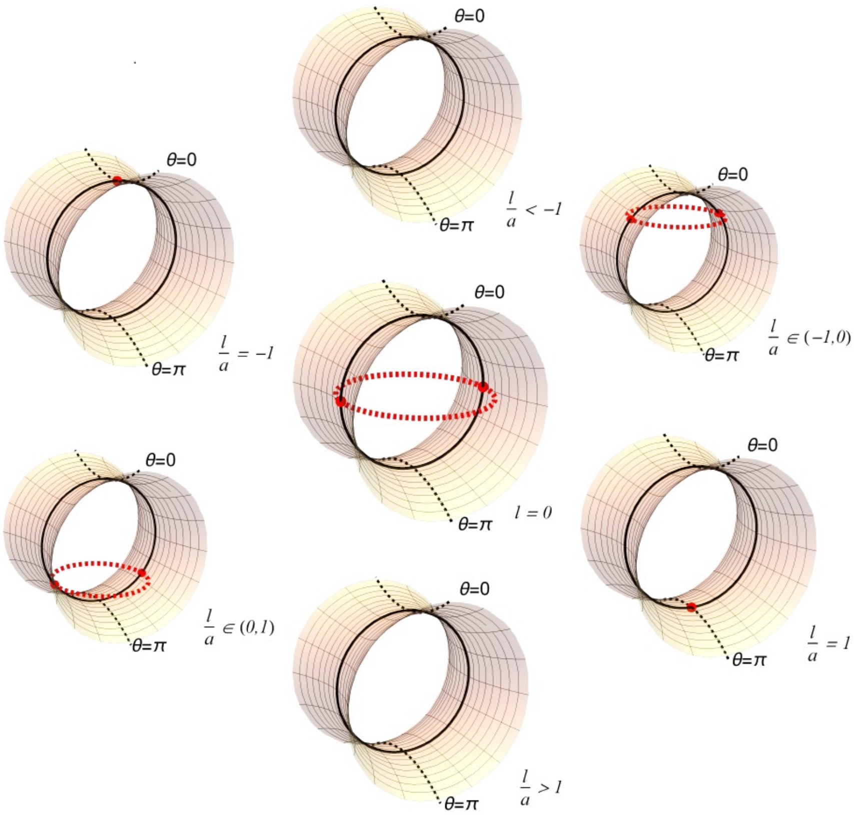

Figure 2.2: A schematic visualization of the ring singularity of the generic PD black hole (2.5) for fixed coordinates t and φ. Visualization depicts the closest neighborhood of the origin $r = 0$ (black circle), the ring singularity (red circle) for various values of the Kerr-like rotation a or the NUT parameter l. For more details, see the Fig. 3 of an attached article.

ically, the character of the curvature singularity is:

$$
\begin{array}{rccl}
l = 0\,,\ a = 0: & \text{singularity at } r = 0 & \text{for any } \theta\,, \\
l = 0\,,\ a \neq 0: & \text{ring singularity at } r = 0 & \text{for } \theta = \pi/2\,, \\
0 < |l| < |a|: & \text{ring singularity at } r = 0 & \text{for } \cos\theta = -l/a\,, \\
l = +a: & \text{singularity at } r = 0 & \text{for } \theta = \pi\,, \\
l = -a: & \text{singularity at } r = 0 & \text{for } \theta = 0\,, \\
|l| > |a| > 0: & \text{no singularity}\,, & \\
l \neq 0\,,\ a = 0: & \text{no singularity}\,. &
\end{array}
\qquad (2.31)
$$

We illustrate all these situations in a schematic visualization in Fig. 2.2, showing the closest neighborhood of the coordinate origin $r = 0$ (denoted by a black vertical circle). The coordinates t and φ are fixed, and we plot (a part of) the radial coordinate r and the angular coordinate $\theta \in [0, \pi]$.

2.2.4 Global structure and the conformal diagrams

We carefully derived the coordinate transformations to compactified coordinates $\{\tilde{T}_h^{\pm}, \tilde{R}_h^{\pm}\}$ (eq. (118) and (119) of the attached paper) and the appropriate angular coordinate ϕ_h (Ibid., eq. (104)). This allowed us to rigorously construct the *conformal Penrose diagrams*. It generalized the previous works on special cases of non-accelerating black holes [56]–[63], or black holes with non-zero acceleration [53, 64].

Each couple of compactified coordinates $\{\tilde{T}_h^{\pm},\ \tilde{R}_h^{\pm}\}$ covers the corresponding horizon $\mathcal{H}_h^{\pm}$. The distinct regions of the manifold between the horizons are characterized by two integers (i, j). There are 5 types of regions, namely:

Region	**Description**	**Specification of** (i,j)
I:	asymptotic time-dependent domain between $\mathcal{H}_a^+$ and $\mathcal{I}$	$(n-2m+1, n+2m-1)$
II:	stationary region between $\mathcal{H}_b^+$ and $\mathcal{H}_a^+$	$(2n-m, 2n+m-1)$
III:	time-dependent domain between the black-hole horizons	$(n-2m, n+2m)$
IV:	stationary region between $\mathcal{H}_a^-$ and $\mathcal{H}_b^-$	$(2n-m+1, 2n+m)$
V:	asymptotic time-dependent domain between $\mathcal{I}$ and $\mathcal{H}_a^-$	$(n-2m+1, n+2m-1)$

The complete global structure was visualized for two cases: one for a special θ on which the ring curvature singularity at $r = 0$ occurs, the second without any curvature singularity (see Fig. 5 and Fig. 4 of the attached paper, respectively). We present here only the case with the singularity, however in a slightly modified form then Fig. 5 of the publication. Fig. 2.3 here takes into account the fact, recently pointed out by MacCallum [65], that although the geodesics end for a certain θ in a singularity, the diagram can still be extended for other types of curves bypassing the curvature singularity at $r = 0$ via different values of θ.

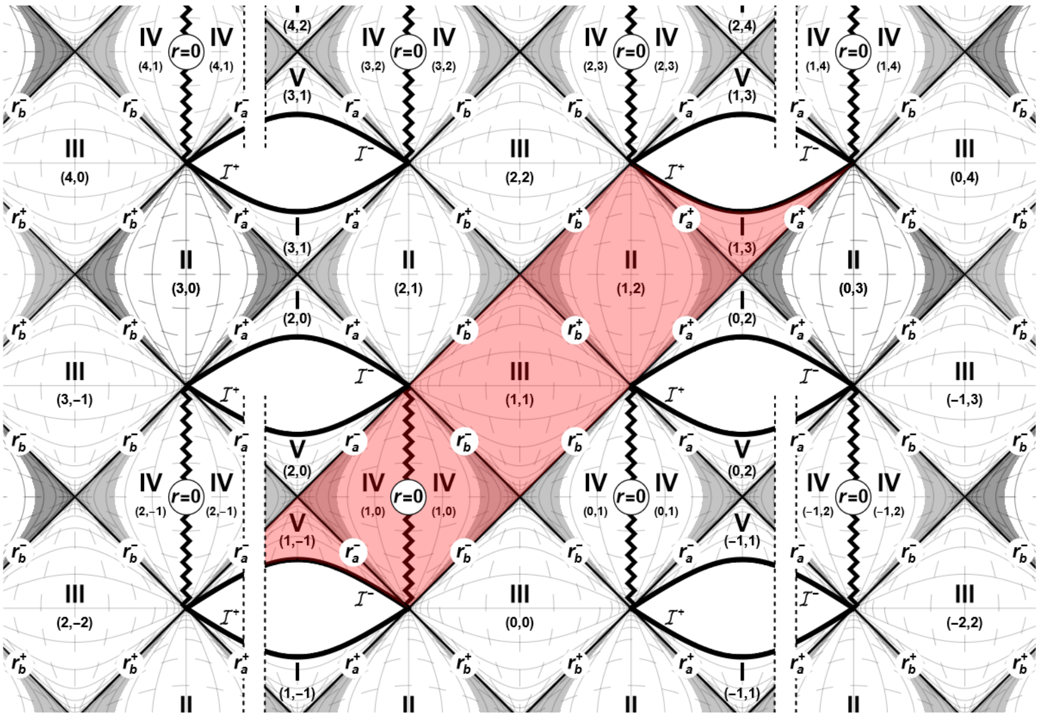

Figure 2.3: Penrose conformal diagram of the completely extended spacetime (2.5) for the section $\theta, \phi_h =$ const. containing the curvature singularity at $r = 0$. In this case, the regions IV are “cut in half” by this singularity, but it can be extended to a negative r with curves having a *different* value of θ at $r = 0$.

2.2.5 Regularity of the axes

Similarly as in Chapter 1, we investigated the nature of the axes $\theta = 0$ and $\theta = \pi$. Also in this case the angular coordinate $\varphi \in [0, 2\pi C)$ has the range depending on the parameter C. This *conicity factor* represents a possible topological incompleteness around the axes which physically causes the acceleration of this black hole.

We started from the metric (2.5) with a vanishing NUT pathology around the $\theta = 0$ axis, and we just removed its conicity. This was easily achieved by the convenient choice

$$\begin{aligned} C = C_0 \equiv & \left[\left(1 - \alpha\,\frac{a^2 + al}{a^2 + l^2}\,r_+\right)\left(1 - \alpha\,\frac{a^2 + al}{a^2 + l^2}\,r_-\right)\right]^{-1} \\ = & \left[1 - 2\alpha m\,\frac{a^2 + al}{a^2 + l^2} + \alpha^2\left(\frac{a^2 + al}{a^2 + l^2}\right)^2 (a^2 - l^2 + e^2 + g^2)\right]^{-1}, \end{aligned} \tag{2.32}$$

which necessarily left an excess angle around the second axis $\theta = \pi$, namely

$$\delta_\pi = -8\pi\,\alpha\,\frac{a^2\left[m(a^2 + l^2) - \alpha a l(a^2 - l^2 + e^2 + g^2)\right]}{(a^2 + l^2)^2 - 2\alpha m(a^2 + al)(a^2 + l^2) + \alpha^2(a^2 + al)^2(a^2 - l^2 + e^2 + g^2)}\,.$$

Notice that it vanishes whenever $\alpha a = 0$, that is for non-accelerating black holes.

On the other hand, performing the coordinate transformation

$$t = t_\pi - 4l\varphi\,, \tag{2.33}$$

which effectively removes the NUT pathology around the complementary $\theta = \pi$ axis, and assuming an appropriate choice of the conicity parameter

$$\begin{aligned} C = C_\pi \equiv & \left[\left(1 + \alpha\,\frac{a^2 - al}{a^2 + l^2}\,r_+\right)\left(1 + \alpha\,\frac{a^2 - al}{a^2 + l^2}\,r_-\right)\right]^{-1} \\ = & \left[1 + 2\alpha m\,\frac{a^2 - al}{a^2 + l^2} + \alpha^2\left(\frac{a^2 - al}{a^2 + l^2}\right)^2 (a^2 - l^2 + e^2 + g^2)\right]^{-1}, \end{aligned} \tag{2.34}$$

the axis $\theta = \pi$ becomes completely regular, with a deficit angle

$$\delta_0 = 8\pi\,\alpha\,\frac{a^2\left[m(a^2 + l^2) - \alpha a l(a^2 - l^2 + e^2 + g^2)\right]}{(a^2 + l^2)^2 + 2\alpha m(a^2 - al)(a^2 + l^2) + \alpha^2(a^2 - al)^2(a^2 - l^2 + e^2 + g^2)}$$

around the axis $\theta = 0$. Again, it vanishes for $\alpha a = 0$.

Interestingly, there exists a specific combination of physical parameters

$$m(a^2 + l^2) = \alpha a l\,(a^2 - l^2 + e^2 + g^2)\,, \tag{2.35}$$

which *regularizes both axes*. Nevertheless, such a combination does not satisfy the natural restrictions on the acceleration (2.27). For more information see the book by Griffiths and Podolský [1], or our attached Paper 2.

From the function $\omega \equiv \frac{g_{t\varphi}}{g_{tt}}$ on the axes $\theta = 0$ and $\theta = \pi$, it is also clear that these strings/struts are *twisting*. It is possible to modify the twist of the individual axes by the coordinate transformation (2.33), but the *difference* always remains the same, namely $\Delta\omega = 4l$.

Another interesting phenomena which we have investigated is the occurrence of the pathological regions around the rotating strings/struts along the axes, caused by the presence of the NUT parameter l. Indeed, with a non-zero l there are areas where $g_{\varphi\varphi} < 0$. This causes the existence of closed timelike curves. For the form of the metric (2.5), these pathologies lie in the range of the coordinates for which

$$\mathcal{R}^4 P(1-\cos^2\theta) < 4l^2 \mathcal{Q}\Big(\cos\theta - \alpha\,\mathcal{T}(1-\cos^2\theta)\Big)^2 . \tag{2.36}$$

These regions has been visualized in Fig. 6 of the attached Paper 2.

2.2.6 Thermodynamic properties

Finally, we computed the basic thermodynamic properties such as the *entropy* S and the *temperature* T. They are directly connected to the area of the horizons $\mathcal{A}$ and the surface gravity κ via the standard relations

$$S \equiv \frac{1}{4}\mathcal{A}\,, \qquad T \equiv \frac{1}{2\pi}\,\kappa\,, \tag{2.37}$$

see [66] for more details.

The area of all four horizons is given by the following expressions

$$\text{area of } \mathcal{H}_b^+ \text{ is} \qquad \mathcal{A}_b^+ = \frac{4\pi C\left(r_+^2 + (a+l)^2\right)}{\Big(1-\alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_+\Big)\Big(1+\alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_+\Big)}\,, \tag{2.38}$$

$$\text{area of } \mathcal{H}_b^- \text{ is} \qquad \mathcal{A}_b^- = \frac{4\pi C\left(r_-^2 + (a+l)^2\right)}{\Big(1-\alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_-\Big)\Big(1+\alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_-\Big)}\,, \tag{2.39}$$

$$\text{area of } \mathcal{H}_a^+ \text{ is} \qquad \text{infinite}\,, \tag{2.40}$$

$$\text{area of } \mathcal{H}_a^- \text{ is} \qquad \text{infinite}\,, \tag{2.41}$$

from which we can easily compute the entropy S via (2.37).

The surface gravity of the horizons is:

$$\text{surface gravity of } \mathcal{H}_b^+ \text{ is} \quad \kappa_b^+ = \frac{\frac{1}{2}\Big(r_+ - r_-\Big)\Big(1+\alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_+\Big)\Big(1-\alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_+\Big)}{r_+^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_b^- \text{ is} \quad \kappa_b^- = -\frac{\frac{1}{2}\Big(r_+ - r_-\Big)\Big(1+\alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_-\Big)\Big(1-\alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_-\Big)}{r_-^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_a^+ \text{ is} \quad \kappa_a^+ = -\alpha\,\frac{a^2}{a^2+l^2}\,\frac{\Big(r_a^+ - r_+\Big)\Big(r_a^+ - r_-\Big)}{(r_a^+)^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_a^- \text{ is} \quad \kappa_a^- = \alpha\,\frac{a^2}{a^2+l^2}\,\frac{\Big(r_a^- - r_+\Big)\Big(r_a^- - r_-\Big)}{(r_a^-)^2 + (a+l)^2}\,,$$

which determines the temperature T, via (2.37).

It can be easily seen that *extreme* black holes, for which $r_+ = r_-$, have zero temperature, because $\kappa_b^- = 0 = \kappa_b^+$.

2.3 Summary

In this second Chapter, we have summarized the new metric form of the large Plebański–Demiański class of type D black holes (2.5)–(2.9). This form is much more convenient for geometrical or physical analysis. In particular:

- In Sec. 2.1, we outlined the derivation of a new form of the metric. By introducing a reparametrization (2.1), applying the special conformal rescaling S (2.3), and fixing a useful gauge of the twist parameter ω (2.4), we were able to significantly simplify and neatly factorize the metric functions (2.6)–(2.9).

- The metric depends on six physical parameters, namely on the mass m, the rotational parameter a, the NUT twist parameter l, the acceleration parameter α, and the charges e and g, respectively.

- Putting these parameters to zero, we recover the standard forms of the well-known simpler metrics, such as the Kerr–Newman–NUT black hole (for $\alpha = 0$), accelerating Kerr–Newman black hole (for $l = 0$), charged Taub–NUT black hole (for $a = 0$), and uncharged accelerating Kerr–NUT black holes (for $e = 0 = g$).

- By setting the Kerr-like rotation a to zero, the new metric (2.5) becomes completely independent of the acceleration α, and simplifies directly to the charged Taub–NUT black hole. This confirms the previous observation that there is no accelerating NUT black hole in the Plebański–Demiański class of type D spacetimes.

- We evaluated the NP scalars of the Weyl and Ricci tensors in the natural tetrad (2.12). The only non-zero components are the Ψ_2 and Φ_{11}, see (2.13) and (2.14), confirming the type D algebraic structure.

- We calculated the spin coefficients. Both the double-degenerate PNDs are expanding and (generally) twisting.

- There are four distinct horizons localized as the roots of the metric function $Q(r)$. These are a pair of black-hole horizons $\mathcal{H}_b^{\pm}$ at $r_b^{\pm}$ and a pair of acceleration horizons $\mathcal{H}_a^{\pm}$ at $r_a^{\pm}$. The roots $r_b^{\pm}$ and $r_a^{\pm}$ are explicitly expressed and simple, see (2.20)–(2.23).

- For a sufficiently small α (2.27), the four horizons follows the natural ordering $r_a^- < r_b^- < r_b^+ < r_a^+$.

- Similarly to the Kerr black hole, there are ergoregions due to the non-zero rotational parameter a. These were visualized in Fig. 2.1.

- Using the curvature scalars, we clarified the presence of a ring curvature singularity. It can occur if and only if $r = 0$ and at the same time $l + a\cos\theta = 0$. Various possibilities were summarized in (2.31) and illustrated in Fig. 2.2.

- We constructed the corresponding Kruskal–Szekeres-type coordinates, and generated the corresponding *Penrose conformal diagrams.* In Fig. 2.3, we presented the version of the Penrose diagram for an appropriate $\theta = const.$ including the singularity. This figure is, however, modified compared to Fig. 5 of the attached Paper 2 due to the recent findings of MacCallum.

- The physical source of the acceleration comes from the topological defects along the two axes of axial symmetry at $\theta = 0$ and $\theta = \pi$. By an appropriate choice of the conicity parameter C, we managed to regularize one of the axes. For a vanishing acceleration α, both the axes are regular.

- These cosmic strings/struts are twisting. This is characterized by the twist parameter ω, which is always directly related to the NUT parameter l. The difference between the twist parameters of each axis is always the constant $\Delta\omega = 4l$.

- The NUT-like pathology in the neighborhood of these rotating strings or struts was studied. These regions with closed timelike curves are generally given by the condition (2.36).

- The metric form (2.5) is also suitable for an easy analysis of the black hole thermodynamics. We have explicitly evaluated the area of the four horizons, their surface gravity, and thus their related temperature T and entropy S (2.37).

To conclude, all this demonstrates the usefulness of the new improved metric form of the family of type D black holes. We hope that various other investigations of this interesting class of accelerating and rotating black holes with charges and a NUT parameter can now be performed.

Although the results, published in Paper 2, cover all the main aspects of this family of black holes, the cosmological constant Λ was missing.

As the next step we generalized our results to *any value of* Λ, completing thus the derivation of the new better form of *full family of type D black holes.* This is the contents of our Paper 3, summarized in the following Chapter 3.

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New improved form of black holes of type D

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We derive a new metric form of the complete family of black hole spacetimes (without a cosmological constant) presented by Plebański and Demiański in 1976. It further improves the convenient representation of this large family of exact black holes found in 2005 by Griffiths and Podolský. The main advantage of the new metric is that the key functions are considerably simplified, fully explicit, and factorized. All four horizons are thus clearly identified, and degenerate cases with extreme horizons can easily be discussed. Moreover, the new metric depends only on six parameters with direct geometrical and physical meaning, namely m, a, l, α, e, g which characterize mass, Kerr-like rotation, Newman-Unti-Tamburino (NUT) parameter, acceleration, electric and magnetic charges of the black hole, respectively. This general metric reduces directly to the familiar forms of either (possibly accelerating) Kerr–Newman, charged Taub–NUT solution, or (possibly rotating and charged) *C*-metric by simply setting the corresponding parameters to zero, without the need of any further transformations. In addition, it shows that the Plebański–Demiański family does not involve accelerating black holes with just the NUT parameter, which were discovered by Chng, Mann and Stelea in 2006. It also enables us to investigate various physical properties, such as the character of singularities, horizons, ergoregions, global conformal structure including the Penrose diagrams, cosmic strings causing the acceleration of the black holes, their rotation, pathological regions with closed timelike curves, or explicit thermodynamic properties. It thus seems that our new metric is a useful representation of this important family of black hole spacetimes of algebraic type D in the asymptotically flat settings.

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I. INTRODUCTION

In this contribution, we derive and analyze a new coordinate representation of the Plebański–Demiański spacetimes [1] describing a large class of black holes (identified also by Debever [2]). It contains, as special cases, all the well-known simpler black holes, namely the Schwarzschild (1915), Reissner–Nordström (1916–1918), Kerr (1963), Taub–NUT (1963) or Kerr–Newman (1965) black holes, and also the *C*-metric (1918, 1962), physically interpreted by Kinnersley–Walker (1970) as uniformly accelerating pair of black holes, see e.g., [3,4]. These accelerating black holes can also be charged, rotating, and can admit the NUT twist parameter.

The class of Plebański–Demiański spacetimes, which includes all these famous black holes, is a family of exact solutions to Einstein–Maxwell equations of algebraic type D with double-aligned non-null electromagnetic field (in the present paper we restrict ourselves only to the case of vanishing cosmological constant)—see Chapter 16 of the monograph [4] for the recent review and number of related references.

Our new form of the metric, which further improves the convenient representation of the class of Plebański–Demiański black holes found by Griffiths and Podolský [5–7], reads

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left(-\frac{Q}{\rho^2}\left[\mathrm{d}t-\left(a\sin^2\theta+4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2+\frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2+\frac{P}{\rho^2}\sin^2\theta\left[a\mathrm{d}t-(r^2+(a+l)^2)\mathrm{d}\varphi\right]^2\right), \quad (1)$$

where

$$\Omega = 1-\frac{\alpha a}{a^2+l^2}r(l+a\cos\theta), \quad (2)$$

$$\rho^2 = r^2+(l+a\cos\theta)^2, \quad (3)$$

$$P(\theta) = \left(1-\frac{\alpha a}{a^2+l^2}r_+(l+a\cos\theta)\right)\times\left(1-\frac{\alpha a}{a^2+l^2}r_-(l+a\cos\theta)\right), \quad (4)$$

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$$Q(r) = (r - r_+)(r - r_-) \times \left(1 + \alpha a \frac{a - l}{a^2 + l^2} r\right)\left(1 - \alpha a \frac{a + l}{a^2 + l^2} r\right). \quad (5)$$

The main roots of $Q(r)$, which identify the two black-hole horizons, are (independently of α) located at

$$r_+ \equiv m + \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}, \quad (6)$$

$$r_- \equiv m - \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}, \quad (7)$$

with the (naturally positive) physical parameters

m ..... mass,
a ..... Kerr-like rotation,
l ..... NUT parameter,
e ..... electric charge,
g ..... magnetic charge,
α ..... acceleration.

This is a further simplification of the previous Griffiths–Podolský form of the metric. The generic structure of the metric has remained basically the same (compare (1) with Eq. (16.18) in [4], renaming $\mathcal{P} \to P$, $\mathcal{Q} \to Q$ and $\varrho \to \rho$), but the new metric functions $P(\theta)$ and $Q(r)$ are now much more compact and explicit than previous $\mathcal{P}(\theta)$ and $\mathcal{Q}(r)$. They are nicely factorized, with P determining the deficit angles corresponding to the cosmic strings along the axes $\theta = 0, \pi$ of the black holes (causing the acceleration), while the roots of Q clearly determine the four horizons. Moreover, the ambiguous twist parameter ω has been removed by its most convenient fixing.

To see these improvements explicitly, let us recall the original Griffiths–Podolský form [5] of the metric functions, namely

$$\Omega = 1 - \alpha\left(\frac{l}{\omega} + \frac{a}{\omega}\cos\theta\right) r,$$
$$\rho^2 = r^2 + (l + a\cos\theta)^2, \quad (8)$$

and

$$\mathcal{P}(\theta) = 1 - a_3 \cos\theta - a_4 \cos^2\theta, \quad (9)$$

$$\mathcal{Q}(r) = \left[(\omega^2 k + \tilde{e}^2 + \tilde{g}^2)\left(1 + 2\alpha \frac{l}{\omega} r\right) - 2\tilde{m} r + \frac{\omega^2 k}{a^2 - l^2} r^2\right] \times \left[1 + \alpha \frac{a - l}{\omega} r\right]\left[1 - \alpha \frac{a + l}{\omega} r\right], \quad (10)$$

where the constants are

$$a_3 = 2\alpha \frac{a}{\omega}\tilde{m} - 4\alpha^2 \frac{al}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2),$$
$$a_4 = -\alpha^2 \frac{a^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2), \quad (11)$$

and $\omega^2 k$ is given by

$$\frac{\omega^2 k}{a^2 - l^2} = \frac{1 + 2\alpha \frac{l}{\omega}\tilde{m} - 3\alpha^2 \frac{l^2}{\omega^2}(\tilde{e}^2 + \tilde{g}^2)}{1 + 3\alpha^2 \frac{l^2}{\omega^2}(a^2 - l^2)}, \quad (12)$$

which implies the expression

$$\omega^2 k + \tilde{e}^2 + \tilde{g}^2 = \frac{(a^2 - l^2 + \tilde{e}^2 + \tilde{g}^2) + 2\alpha \frac{l}{\omega}(a^2 - l^2)\tilde{m}}{1 + 3\alpha^2 \frac{l^2}{\omega^2}(a^2 - l^2)}. \quad (13)$$

Substituting (11)–(13) into (9) and (10) gives explicit but cumbersome expressions for the key metric functions $\mathcal{P}(\theta)$ and $\mathcal{Q}(r)$. This is now simplified in the new compact form of the metric (1)–(5).

II. DERIVATION OF THE NEW METRIC

The first step in improving the form of the spacetime is to concentrate on the first factor of the metric function $\mathcal{Q}(r)$ given by (10), which is quadratic in r. It can be rewritten as

$$\left[(\omega^2 k + \tilde{e}^2 + \tilde{g}^2)\left(1 + 2\alpha \frac{l}{\omega} r\right) - 2\tilde{m} r + \frac{\omega^2 k}{a^2 - l^2} r^2\right] = \frac{\omega^2 k}{a^2 - l^2}\left[r^2 - 2\tilde{m}\frac{a^2 - l^2}{\omega^2 k} r + \left(1 + 2\alpha \frac{l}{\omega} r\right)\left(a^2 - l^2 + \frac{a^2 - l^2}{\omega^2 k}(\tilde{e}^2 + \tilde{g}^2)\right)\right]. \quad (14)$$

It can now be observed that this rather complicated expression nicely simplifies if we introduce a *new set of the mass and charge parameters* m, e, g in such a way that

$$\begin{aligned} m &\equiv \frac{a^2-l^2}{\omega^2 k}\tilde{m} - \alpha\frac{l}{\omega}(a^2-l^2+e^2+g^2), \\ e^2 &\equiv \frac{a^2-l^2}{\omega^2 k}\tilde{e}^2, \\ g^2 &\equiv \frac{a^2-l^2}{\omega^2 k}\tilde{g}^2. \end{aligned} \tag{15}$$

Indeed, the factor (14) then takes the explicit form

$$\frac{\omega^2 k}{a^2-l^2}[r^2-2mr+(a^2-l^2+e^2+g^2)]. \tag{16}$$

Provided $m^2+l^2>a^2+e^2+g^2$, it *has two explicit roots* r_+ and r_- given by (6) and (7), respectively. The metric function (10) can thus be factorized to

$$\begin{aligned} \mathcal{Q}(r) &= S^{-1}(r-r_+)(r-r_-) \\ &\quad\times\left(1+\alpha\frac{a-l}{\omega}r\right)\left(1-\alpha\frac{a+l}{\omega}r\right), \end{aligned} \tag{17}$$

where the *constant* S is a shorthand for the inverse of (12), namely

$$S^{-1} \equiv \frac{\omega^2 k}{a^2-l^2}. \tag{18}$$

Substitution from (15) into (12), rewritten as

$$\begin{aligned} &\frac{\omega^2 k}{a^2-l^2}\left[1+3\alpha^2\frac{l^2}{\omega^2}(a^2-l^2)\right] \\ &\quad = 1+2\alpha\frac{l}{\omega}\tilde{m}-3\alpha^2\frac{l^2}{\omega^2}(\tilde{e}^2+\tilde{g}^2), \end{aligned} \tag{19}$$

yields the explicit expression for S in terms of the new physical parameters

$$S = 1-2\alpha\frac{l}{\omega}m+\alpha^2\frac{l^2}{\omega^2}(a^2-l^2+e^2+g^2). \tag{20}$$

Notice that it can also be expressed in terms of the roots r_+ and r_- as

$$\begin{aligned} S &= 1-\alpha\frac{l}{\omega}(r_++r_-)+\alpha^2\frac{l^2}{\omega^2}r_+r_- \\ &= \left(1-\alpha\frac{l}{\omega}r_+\right)\left(1-\alpha\frac{l}{\omega}r_-\right). \end{aligned} \tag{21}$$

One may be worried about the change of the "main physical parameters" introduced by (15). However, by inspecting the expressions (19), (20) it is immediately seen that

$$\begin{aligned} &\alpha\frac{l}{\omega}=0 \quad \text{implies} \quad S=\frac{a^2-l^2}{\omega^2 k}=1, \\ &\text{and consequently } m=\tilde{m},\, e=\tilde{e},\, g=\tilde{g}. \end{aligned} \tag{22}$$

It means, that *in all the subcases* $\alpha=0$ or $l=0$ (namely for Schwarzschild, Reissner–Nordström, Kerr, Taub–NUT or Kerr–Newman black holes, and also for their accelerating generalizations with vanishing NUT parameter l) the mass parameter m and the charges e, g actually *remain the same*. And since there are no accelerating *purely* NUT black holes in the Plebański–Demiański class of type D solutions, see [8], the difference between m, e, g and $\tilde{m}, \tilde{e}, \tilde{g}$ occurs *only if* $\alpha a l \neq 0$, cf. (30). That is the most general case of *accelerating* black holes *with both the rotation a and the NUT parameter l*, whose geometric and physical properties have not yet been studied.

After factorizing the function $\mathcal{Q}(r)$, as the second step we now turn to the metric function $\mathcal{P}(\theta)$ determined by the constants a_3 and a_4. It is known that these two Plebański–Demiański metric functions are related, and for vanishing cosmological constant they share the root structure. It can thus be expected that also the function $\mathcal{P}(\theta)$ could be factorized by the suitable reparametrization (15). This is indeed the case. Expressing (11) in terms of the new parameters m, e, g we get

$$\begin{aligned} a_3 &= 2\alpha\frac{a}{\omega}\frac{\omega^2 k}{a^2-l^2}\left[m-\alpha\frac{l}{\omega}(a^2-l^2+e^2+g^2)\right], \\ a_4 &= -\alpha^2\frac{a^2}{\omega^2}\frac{\omega^2 k}{a^2-l^2}(a^2-l^2+e^2+g^2). \end{aligned} \tag{23}$$

Using (18), (20) and substituting (23) into (9) we obtain

$$\begin{aligned} \mathcal{P}(\theta) &= S\frac{\omega^2 k}{a^2-l^2} - a_3\cos\theta - a_4\cos^2\theta \\ &= \frac{\omega^2 k}{a^2-l^2}\left[1-2\alpha\frac{l+a\cos\theta}{\omega}m+\alpha^2\frac{(l+a\cos\theta)^2}{\omega^2}(a^2-l^2+e^2+g^2)\right] \\ &= S^{-1}\left(1-\alpha r_+\frac{l+a\cos\theta}{\omega}\right)\left(1-\alpha r_-\frac{l+a\cos\theta}{\omega}\right). \end{aligned} \tag{24}$$

The metric function $\mathcal{P}(\theta)$ is thus also factorized when $m^2+l^2>a^2+e^2+g^2$, i.e., when the roots r_+ and r_- exist.

To summarize, we have obtained the key expressions (17) and (24), which can be written as

$$\mathcal{Q}(r)=S^{-1}Q(r), \qquad \mathcal{P}(\theta)=S^{-1}P(\theta), \tag{25}$$

where

$$Q(r)=(r-r_+)(r-r_-)\left(1+\alpha\frac{a-l}{\omega}r\right)\left(1-\alpha\frac{a+l}{\omega}r\right), \tag{26}$$

$$P(\theta)=\left(1-\alpha r_+\frac{l+a\cos\theta}{\omega}\right)\left(1-\alpha r_-\frac{l+a\cos\theta}{\omega}\right). \tag{27}$$

Putting these into the original metric [5–7] (which has the same form as (1) with Q, P replaced by $\mathcal{Q}$, $\mathcal{P}$, respectively) we get

$$\begin{aligned}\mathrm{d}s^2=&\frac{S}{\Omega^2}\bigg(-\frac{Q}{\rho^2}S^{-2}\Big[\mathrm{d}t-\Big(a\sin^2\theta+4l\sin^2\tfrac{1}{2}\theta\Big)\mathrm{d}\varphi\Big]^2+\frac{\rho^2}{Q}\mathrm{d}r^2\\&+\frac{\rho^2}{P}\mathrm{d}\theta^2+\frac{P}{\rho^2}\sin^2\theta\,S^{-2}\Big[a\mathrm{d}t-(r^2+(a+l)^2)\mathrm{d}\varphi\Big]^2\bigg).\end{aligned} \tag{28}$$

The third step in deriving the new metric is now based on an observation (first made in [9]) that it is possible to *rescale the coordinates t and φ* by a constant scaling factor $S\neq 0$ (because their range has not yet been specified). In other words, we perform the transformation $t\to St$ and $\varphi\to S\varphi$ which effectively removes the constants S from the conformal metric $\mathrm{d}\hat{s}^2\equiv S^{-1}\mathrm{d}s^2$. Moreover, a *constant conformal factor* S^{-1} does not change the geometry of the spacetime (recall also (22), according to which $S=1$ whenever $\alpha a l=0$). Therefore, the Plebański–Demiański black-hole solutions can equivalently be represented by the metric $\mathrm{d}\hat{s}^2$. Dropping the hat, we arrive at the metric (1).

In fact, this specific rescaling procedure removes the two coordinate singularities hidden in the expression (21) for S at $\alpha l r_\pm=\omega$, making our new metric form (1)–(5) somewhat richer.

To complete the derivation, it only remains to *fix the remaining twist parameter ω*. In the original Griffiths–Podolský form of the metric [5], this was left as a free parameter which could be set to *any* value (if at least one of the parameters a or l are nonzero, otherwise $\omega\equiv 0$—see the discussion in [5,7]) using the remaining coordinate freedom. This ambiguity is unfortunate since the metric explicitly contains nonunique ω coupled *both* to the Kerr-like rotation a *and* the NUT parameter l. We can now improve this drawback. It was found in [9], and conveniently employed in [10], that the most suitable gauge choice of the twist parameter is

$$\omega\equiv\frac{a^2+l^2}{a}, \tag{29}$$

so that

$$\frac{a}{\omega}=\frac{a^2}{a^2+l^2}, \qquad \frac{l}{\omega}=\frac{al}{a^2+l^2}. \tag{30}$$

Substituting this gauge into the expressions (8), (27) and (26), we obtain the explicit metric functions Ω, P and Q presented in (2), (4) and (5), respectively. The new form of the metric (1)–(5), which nicely represents the large family of type D black holes, is thus completely derived.

III. MAIN SUBCLASSES OF TYPE D BLACK HOLES

When $m^2+l^2>a^2+e^2+g^2$, the new metric (1)–(5) naturally generalizes the standard forms of the most important black hole solutions. These are now easily obtained by setting the corresponding physical parameters to zero.

A. Kerr–Newman–NUT black holes ($\alpha=0$: no acceleration)

By setting the acceleration parameter α to zero, the functions (2), (4) reduce to $\Omega=1$, $P=1$, so that the generic metric (1) simplifies as

$$\begin{aligned}\mathrm{d}s^2=&-\frac{Q}{\rho^2}\Big[\mathrm{d}t-\Big(a\sin^2\theta+4l\sin^2\tfrac{1}{2}\theta\Big)\mathrm{d}\varphi\Big]^2+\frac{\rho^2}{Q}\mathrm{d}r^2\\&+\rho^2\mathrm{d}\theta^2+\frac{\sin^2\theta}{\rho^2}\Big[a\mathrm{d}t-(r^2+(a+l)^2)\mathrm{d}\varphi\Big]^2,\end{aligned} \tag{31}$$

where

$$Q(r)=(r-r_+)(r-r_-), \tag{32}$$

$$\rho^2=r^2+(l+a\cos\theta)^2. \tag{33}$$

The two roots of $Q(r)$ identify the two black-hole horizons located at

$$r_\pm\equiv m\pm\sqrt{m^2+l^2-a^2-e^2-g^2}. \tag{34}$$

Famous subcases are readily obtained, namely the black holes solution of Kerr–Newman ($l=0$), charged Taub–NUT ($a=0$), Kerr ($l=0$, $e=0=g$), Reissner–Nordström ($a=0$, $l=0$), and Schwarzschild ($a=0$, $l=0$, $e=0=g$).

B. Accelerating Kerr–Newman black holes ($l = 0$: no NUT)

Without the NUT parameter l, the new metric (1) simplifies to

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg(-\frac{Q}{\rho^2}\bigg[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\bigg]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\bigg[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi\bigg]^2\bigg), \tag{35}$$

where

$$\Omega = 1 - \alpha r\cos\theta, \tag{36}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{37}$$

$$P(\theta) = (1-\alpha r_+\cos\theta)(1-\alpha r_-\cos\theta), \tag{38}$$

$$Q(r) = (r-r_+)(r-r_-)(1+\alpha r)(1-\alpha r). \tag{39}$$

This is a compact factorized form of the class of accelerating, rotating, and charged black holes. The spacetime admits 4 horizons, namely two black hole horizons at $r_\pm = m \pm \sqrt{m^2 - a^2 - e^2 - g^2}$ and two acceleration horizons at $\pm\alpha^{-1}$. For vanishing charges ($e = 0 = g$), it is equivalent to the rotating C-metric identified by Hong and Teo [11]. For vanishing acceleration ($\alpha = 0$), the standard form of Kerr–Newman solution in Boyer–Lindquist coordinates is recovered.

C. Charged Taub–NUT black holes ($a = 0$: no rotation)

By setting the Kerr-like rotation parameter a to zero, the new metric (1) considerably simplifies and *becomes independent of the acceleration* α (because the metric functions (2)–(5) depend on α only via the product αa). Indeed, $\Omega = 1$, $P = 1$, so that

$$\mathrm{d}s^2 = -\frac{Q}{\rho^2}\Big(\mathrm{d}t - 4l\sin^2\tfrac{1}{2}\theta\,\mathrm{d}\varphi\Big)^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + (r^2+l^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{40}$$

where

$$Q(r) = (r-r_+)(r-r_-), \tag{41}$$

$$\rho^2 = r^2 + l^2. \tag{42}$$

This explicitly demonstrates that *there is no accelerating NUT black hole in the Plebański–Demiański family* of spacetimes. This observation was made already in the original works [5–7], and recently clarified. It was proven in [8] that the metric for accelerating (nonrotating) black holes with purely NUT parameter—which was found in 2006 by Chng, Mann and Stelea [12] and analyzed in detail in [8]—is of algebraic type I. Therefore, it *cannot* be contained in the Plebański–Demiański class which is of type D.

The charged Taub–NUT spacetime (40) is nonsingular (its curvature does not diverge at $r = 0$), away from the axis $\theta = \pi$ (where the rotating cosmic string is located) it is asymptotically flat as $r \to \pm\infty$, and the interior of the black hole is located between the two horizons $r_+ > 0$ and $r_- > 0$, where $r_\pm = m \pm \sqrt{m^2 + l^2 - e^2 - g^2}$.

D. Uncharged accelerating Kerr–NUT black holes ($e = 0 = g$: vacuum)

Another nice feature of our new metric (1)–(5) is that it *has the same form for vacuum spacetimes* without the electromagnetic field. Indeed, the electric and magnetic charges e and g, which generate the electromagnetic field, *enter only the expressions for* $r_\pm$ introduced in (6), (7). In other words, e and g just change the positions of the two black hole horizons. In the vacuum case, these constant parameters simplify to

$$r_\pm \equiv m \pm \sqrt{m^2 + l^2 - a^2}. \tag{43}$$

The metric (1)–(5) with (43) represents the full class of accelerating Kerr–NUT black holes. It reduces to accelerating Kerr black hole when $l = 0$, and nonaccelerating Kerr–NUT black hole when $\alpha = 0$. For $a = 0$ it simplifies *directly* to the Taub–NUT black hole (40) without acceleration.

IV. EXTREME BLACK HOLES AND HYPEREXTREME CASES

The new form of the generic black hole (1)—and also all its subclasses—naturally admits a special case with a *degenerate* horizon, which is the situation when the *two horizons coincide*, $r_+ = r_-$. In view of (6), (7), this occurs if and only if the *extremality condition*

$$m^2 + l^2 = a^2 + e^2 + g^2 \tag{44}$$

is satisfied, and in such a case the *extremal horizon is located at*

$$r = m. \tag{45}$$

Consequently, the metric functions take the form

$$P(\theta) = \bigg(1 - \frac{\alpha a m}{a^2+l^2}(l + a\cos\theta)\bigg)^2, \tag{46}$$

$$Q(r) = (r-m)^2\bigg(1 + \alpha a\frac{a-l}{a^2+l^2}r\bigg)\bigg(1 - \alpha a\frac{a+l}{a^2+l^2}r\bigg), \tag{47}$$

while all the remaining expressions in the metric (1) remain the same. Apart from the degenerate black hole horizon at $r = m$ with zero surface gravity (and thus zero temperature), there are two acceleration horizons.

This large family of *extremal accelerating Kerr–Newman–NUT black holes* admits various natural subclasses which are easily obtained by setting the corresponding physical parameters α, l, a, e, g to zero. In particular, Kerr–Newman–NUT black holes *without acceleration* ($\alpha = 0$) take the form

$$\mathrm{d}s^2 = -\frac{Q}{\rho^2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}[a\mathrm{d}t - (r^2 + (a+l)^2)\mathrm{d}\varphi]^2, \tag{48}$$

where

$$\frac{Q}{\rho^2} = \frac{(r-m)^2}{r^2 + (l + a\cos\theta)^2}. \tag{49}$$

The subcases are Kerr–Newman ($l = 0$), charged Taub–NUT ($a = 0$), Kerr ($l = 0$, $e = 0 = g$), and Reissner–Nordström ($a = 0$, $l = 0$) extremal black holes, satisfying the extremality condition (44).

Interestingly, in our recent work [10] we proved the equivalence of degenerate horizons in this family (48), (49) of type D black holes to a complete class of extremal isolated horizons with axial symmetry.

Finally, if the physical parameters satisfy the relation

$$m^2 + l^2 < a^2 + e^2 + g^2, \tag{50}$$

the *black hole horizons are absent*. This case represents *hyperextreme* spacetimes with very large rotation a and/or charges e, g. The metric function $Q(r)$ does not admit the real roots r_+, r_-. Instead, it involves a nonfactorizable quadratic term of the form (16). In such a case, the metric (1) remains valid, but its metric functions P and Q are

$$P(\theta) = 1 - 2\alpha a\frac{l + a\cos\theta}{a^2 + l^2}m + \alpha^2 a^2\frac{(l + a\cos\theta)^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2), \tag{51}$$

$$Q(r) = (r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)) \times \left(1 + \alpha a\frac{a - l}{a^2 + l^2}r\right)\left(1 - \alpha a\frac{a + l}{a^2 + l^2}r\right). \tag{52}$$

This exact spacetime represents a *naked singularity* of mass m with rotation a, NUT parameter l, electromagnetic charges e, g, and acceleration α caused by the tension of rotating cosmic strings attached to it along the axes. There are only two acceleration horizons. For $\alpha = 0$, the metric simplifies considerably to the form (48) with

$$\frac{Q}{\rho^2} = \frac{r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)}{r^2 + (l + a\cos\theta)^2}. \tag{53}$$

The new metric form (1)–(5) thus nicely describes the complete family of black holes of type D, as well as their extreme cases and hyperextreme spacetimes with naked singularities.

V. PHYSICAL DISCUSSION OF THE NEW METRIC

To study the global structure of the spacetime and to analyze its physical properties, it is first necessary to determine the gravitational field, in particular the specific curvature of the geometry, and the electromagnetic field. These are encoded in the Newman–Penrose scalars—the components of the Riemann and Maxwell tensors with respect to the null tetrad. Its most natural choice is

$$\begin{aligned}
\mathbf{k} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}((r^2 + (a+l)^2)\partial_t + a\partial_\varphi) + \sqrt{Q}\partial_r\right],\\
\mathbf{l} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}((r^2 + (a+l)^2)\partial_t + a\partial_\varphi) - \sqrt{Q}\partial_r\right],\\
\mathbf{m} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{P}\sin\theta}\left(\partial_\varphi + \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\partial_t\right) + \mathrm{i}\sqrt{P}\partial_\theta\right].
\end{aligned} \tag{54}$$

A direct calculation reveals that the only nontrivial Newman–Penrose scalars corresponding to the Weyl and Ricci tensors are

$$\Psi_2 = \frac{\Omega^3}{[r + \mathrm{i}(l + a\cos\theta)]^3}\left[-(m + \mathrm{i}l)\left(1 - \mathrm{i}\alpha a\frac{a^2 - l^2}{a^2 + l^2}\right) + \frac{(e^2 + g^2)}{r - \mathrm{i}(l + a\cos\theta)}\left(1 + \frac{\alpha a}{a^2 + l^2}[ar\cos\theta + \mathrm{i}l(l + a\cos\theta)]\right)\right], \tag{55}$$

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{\Omega^4}{\rho^4}, \tag{56}$$

while the Ricci scalar vanishes (indeed, $R = 0$ for electrovacuum solutions with $\Lambda = 0$). Recall also (2), (3), i.e.,

$$\Omega = 1 - \frac{\alpha a}{a^2+l^2} r(l + a\cos\theta), \qquad \rho^2 = r^2 + (l + a\cos\theta)^2. \tag{57}$$

The curvature for the main subclasses of type D black holes, summarized in Sec. III, are now easily obtained by setting up the corresponding physical parameters to zero:

(i) Kerr–Newman–NUT ($\alpha = 0$: no acceleration)

$$\Psi_2 = \frac{1}{[r + \mathrm{i}(l + a\cos\theta)]^3} \times \left[-(m+\mathrm{i}l) + \frac{e^2+g^2}{r - \mathrm{i}(l + a\cos\theta)}\right], \tag{58}$$

(ii) Accelerating Kerr–Newman ($l = 0$: no NUT)

$$\Psi_2 = \frac{(1-\alpha r\cos\theta)^3}{(r+\mathrm{i}a\cos\theta)^3} \times \left[-m(1-\mathrm{i}\alpha a) + (e^2+g^2)\frac{1+\alpha r\cos\theta}{r-\mathrm{i}a\cos\theta}\right], \tag{59}$$

(iii) Charged Taub–NUT ($a = 0$: no rotation)

$$\Psi_2 = -\frac{m+\mathrm{i}l}{(r+\mathrm{i}l)^3} + \frac{e^2+g^2}{(r^2+l^2)(r+\mathrm{i}l)^2}. \tag{60}$$

Of course, these expressions further simplify if (some of) the remaining parameters are zero. In particular, the *Kerr–Newman* black hole is recovered from (58) if $l = 0$. The *C-metric* (accelerating charged black holes without rotation) are obtained from (59) when $a = 0$. The *Reissner–Nordström* black hole follows from (60) when $l = 0$. The *uncharged* (vacuum) black holes are obtained for $e = 0 = g$. Moreover, all these particular expressions for Ψ_2 agree with those presented in the corresponding chapters of the monograph [4].

It is also useful to calculate the *spin coefficients* for the null tetrad (54). It turns out that

$$\begin{aligned} &\kappa = \nu = 0, \quad \sigma = \lambda = 0, \\ &\varrho = \mu = -\frac{\sqrt{Q}}{\sqrt{2}\rho^3}\left(1 + \mathrm{i}\frac{\alpha a}{a^2+l^2}(l + a\cos\theta)^2\right)(r - \mathrm{i}(l + a\cos\theta)) \\ &\tau = \pi = -\frac{a\sqrt{P}\sin\theta}{\sqrt{2}\rho^3}\left(1 - \mathrm{i}\frac{\alpha a}{a^2+l^2}r^2\right)(r - \mathrm{i}(l + a\cos\theta)). \end{aligned} \tag{61}$$

Also the coefficients $\alpha = \beta$ and $\epsilon = \gamma$ are nonzero (we do not write them because they are not simple). Both double-degenerate principal null directions generated by $\mathbf{k}$ and $\mathbf{l}$ are thus geodetic and shear-free. However, they have *expansion* and *twist* given by $\varrho = \mu \equiv -(\Theta + \mathrm{i}\omega)$, that is

$$\Theta = \frac{\sqrt{Q}}{\sqrt{2}\rho^3}\left(r + \frac{\alpha a}{a^2+l^2}(l + a\cos\theta)^3\right), \tag{62}$$

$$\omega = -\frac{\Omega\sqrt{Q}}{\sqrt{2}\rho^3}(l + a\cos\theta). \tag{63}$$

It is now explicitly seen that these *black-hole spacetimes of algebraic type D are nontwisting* (for a general r, θ) *if and only if* $a = 0 = l$. Moreover, *on the horizons* identified by $Q(r) = 0$, both the expansion and the twist *vanish* ($\Theta = 0 = \omega$).

For investigation of the curvature singularities and asymptotically flat regions, it is also useful to evaluate the *Kretschmann scalar*

$$\mathcal{K} \equiv R_{abcd}R^{abcd} = 48\mathcal{R}e(\Psi_2^2), \tag{64}$$

for type D spacetimes. Interestingly, it takes the factorized form

$$\mathcal{K} = 48\frac{\Omega^6}{\rho^{12}}K_+K_-, \tag{65}$$

where

$$\begin{aligned} K_\pm &= m\left(F_\pm \pm \alpha a\frac{a^2-l^2}{a^2+l^2}F_\mp\right) \mp l\left(F_\mp \mp \alpha a\frac{a^2-l^2+e^2+g^2}{a^2+l^2}F_\pm\right) - (e^2+g^2)\left(1 + \frac{\alpha a}{a^2+l^2}rL\right)T_\pm, \\ F_\pm &= (r \mp L)(r^2 \pm 4rL + L^2), \qquad T_\pm = (r^2 \pm 2rL - L^2), \qquad L = l + a\cos\theta. \end{aligned} \tag{66}$$

These expressions characterize the gravitational field.

When e, g are not zero, the black-hole spacetime also contains a specific *electromagnetic field* represented by the Maxwell 2-form $\mathbf{F} = \frac{1}{2} F_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b = \mathrm{d}\mathbf{A}$. Its 1-form potential $\mathbf{A} = A_a \mathrm{d}x^a$ is

$$\mathbf{A} = -\sqrt{e^2+g^2}\,\frac{r}{\rho^2}\Big[\mathrm{d}t - \Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)\mathrm{d}\varphi\Big]. \quad (67)$$

Therefore, the nonzero components of $F_{ab} = A_{b,a} - A_{a,b}$ are

$$\begin{aligned}
F_{tr} &= -\sqrt{e^2+g^2}\rho^{-4}(r^2 - (l + a\cos\theta)^2),\\
F_{\varphi r} &= -F_{tr}\Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big),\\
F_{t\theta} &= 2a\sqrt{e^2+g^2}\rho^{-4} r\sin\theta(l + a\cos\theta),\\
F_{\varphi\theta} &= -2\sqrt{e^2+g^2}\rho^{-4} r\sin\theta(l + a\cos\theta)(r^2 + (a+l)^2).
\end{aligned} \quad (68)$$

The corresponding Newman–Penrose scalars are $\Phi_0 \equiv F_{ab} k^a m^b = 0$, $\Phi_2 \equiv F_{ab} \bar{m}^a l^b = 0$, and

$$\Phi_1 \equiv \frac{1}{2} F_{ab}(k^a l^b + \bar{m}^a m^b) = \frac{\sqrt{e^2+g^2}\,\Omega^2}{(r + \mathrm{i}(l + a\cos\theta))^2}. \quad (69)$$

It follows that $\Phi_1 \bar{\Phi}_1 = 2\Phi_{11}$, in fully agreement with (56).

A. Position of the horizons

The new metric form (1) is very convenient for investigation of horizons. Clearly, the "radial" coordinate r is spatial in the regions where $Q(r) > 0$, while it is a temporal coordinate where $Q(r) < 0$. These regions are separated by horizons localized at $Q(r) = 0$. In the case when $m^2 + l^2 > a^2 + e^2 + g^2$, the metric function Q is given by (5),

$$Q(r) = (r - r_+)(r - r_-) \times \Big(1 + \alpha a \frac{a-l}{a^2+l^2} r\Big)\Big(1 - \alpha a \frac{a+l}{a^2+l^2} r\Big). \quad (70)$$

It is a *quartic expression explicitly factorized into four real roots*, so that there are *four horizons*, namely

$$\mathcal{H}_b^+ \quad \text{at} \quad r_b^+ \equiv r_+ = m + \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}, \quad (71)$$

$$\mathcal{H}_b^- \quad \text{at} \quad r_b^- \equiv r_- = m - \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}, \quad (72)$$

$$\mathcal{H}_a^+ \quad \text{at} \quad r_a^+ \equiv +\frac{1}{\alpha}\frac{a^2+l^2}{a^2+al}, \quad (73)$$

$$\mathcal{H}_a^- \quad \text{at} \quad r_a^- \equiv -\frac{1}{\alpha}\frac{a^2+l^2}{a^2-al}, \quad (74)$$

see the definitions of $r_\pm$ introduced in (6), (7). It is clear that $r_+ > 0$ *for an arbitrary choice* of the physical parameters (assuming $m > 0$), but r_- *can take any sign*. In particular,

$$r_- > 0 \Leftrightarrow l^2 < a^2 + e^2 + g^2, \quad (75)$$

$$r_- < 0 \Leftrightarrow l^2 > a^2 + e^2 + g^2, \quad (76)$$

$$r_- = 0 \Leftrightarrow l^2 = a^2 + e^2 + g^2. \quad (77)$$

The horizons $\mathcal{H}_b^\pm$ at $r_b^\pm$ are *two black-hole horizons*. Interestingly, in our new metric form these are *independent of the acceleration parameter* α. In fact, they are located *at the same values* r_+, r_- as the two horizons in the class of standard (nonaccelerating) *Kerr–Newman–NUT black holes* given by $\alpha = 0$, see [4].

The horizons $\mathcal{H}_a^\pm$ at $r_a^\pm$ are *two acceleration horizons*. Their presence is the consequence of the fact that the black holes accelerate whenever the parameter α is nonzero. It is interesting that their location is now *independent of mass* m *and charges* e, g of the black holes. The values of $r_a^\pm$ depend only on the acceleration α and the specific combination of the twist parameters a, l. Moreover, when $l = 0$ these are simply given just by the acceleration parameter as $r_a^\pm = \pm\alpha^{-1}$. They retain *the same values* as in the *C-metric* [4] even if it is generalized to include the charges and rotation, that is for *accelerating Kerr–Newman black holes*.

Of course, there may be *less than 4 horizons*. As already discussed in Sec. IV, when the physical parameters satisfy the extremality relation $m^2 + l^2 = a^2 + e^2 + g^2$ the two *black-hole horizons* $\mathcal{H}_b^+, \mathcal{H}_b^-$ *coincide* because $r_+ = r_-$. In such a degenerate case the *extremal horizon is located at*

$$r_b^+ = r_b^- = m, \quad (78)$$

see (44) and (45), while the two distinct acceleration horizons $\mathcal{H}_a^\pm$ given by (73) and (74) remain the same. This is the horizon structure for the family of extremal accelerating Kerr–Newman–NUT black holes, recently studied in [10]. If the parameters satisfy $m^2 + l^2 < a^2 + e^2 + g^2$ the *black-hole horizons* $\mathcal{H}_b^+, \mathcal{H}_b^-$ *are absent*. Such hyperextreme spacetimes involve accelerating naked singularities with just two acceleration horizons $\mathcal{H}_a^\pm$.

In the limit $\alpha \to 0$ of vanishing acceleration, from (73), (74) we formally obtain $r_a^\pm \to \pm\infty$ which is consistent with the fact that the two horizons $\mathcal{H}_a^\pm$ *disappear* for nonaccelerating Kerr–Newman–NUT black holes. In the complementary limit $a \to 0$ of vanishing Kerr-like rotation, we *also* obtain $r_a^\pm \to \infty$. This explicitly confirms that there are no accelerating *purely* NUT black holes in the Plebański–Demiański family of type D spacetimes. Indeed, by setting $a = 0$ the metric (1) becomes independent of α, and the metric reduces to (40) representing charged Taub–NUT black holes without acceleration. Nevertheless, accelerating black holes with purely NUT parameter exist *outside* the Plebański–Demiański family [12]—they are of algebraic type I, and have been recently analyzed in detail in [8].

Returning now to the *generic case* with four distinct horizons, it immediately follows from (71)–(74) that (assuming non-negative parameters α, a, and l)

$$r_b^- < r_b^+ \quad \text{always,} \quad \text{while} \quad r_a^- < r_a^+ \quad \text{for} \quad 0 \le l < a. \tag{79}$$

In the limiting case $l \to a$ we obtain $r_a^+ = \alpha^{-1}$, $r_a^- \to -\infty$, while for $l > a$ there is $0 < r_a^+ < r_a^-$.

The physically most natural ordering of the horizons

$$r_a^- < r_b^- < r_b^+ < r_a^+, \tag{80}$$

in which the two black hole horizons $\mathcal{H}_b^\pm$ are surrounded by two "outer" acceleration horizons $\mathcal{H}_a^\pm$, requires a *sufficiently small acceleration*. The condition $r_b^+ < r_a^+$ explicitly reads

$$\alpha < \frac{1}{r_+} \frac{a^2 + l^2}{a^2 + al}, \tag{81}$$

while $r_a^- < r_b^-$ for any $0 \le l < a$ because in such a case $r_a^- < 0$ but $0 < r_b^-$.

By evaluating $\mathcal{Q}$ given by (70) at $r = 0$ we obtain

$$\mathcal{Q}(r = 0) = r_+ r_- = a^2 - l^2 + e^2 + g^2 > 0 \quad \text{for} \quad l < a. \tag{82}$$

Consequently, $\mathcal{Q} > 0$ for any (r_a^-, r_b^-). It follows that the coordinate r is *spatial* in the regions (r_a^-, r_b^-) and (r_b^+, r_a^+), that is *between* the black-hole and acceleration horizons, while it is temporal in the complementary three regions.

Moreover, using the condition (81) we infer that

$$\frac{\alpha a}{a^2 + l^2} r_-(l + a\cos\theta) < \frac{\alpha a}{a^2 + l^2} r_+(l + a\cos\theta) < \alpha r_+ \frac{a^2 + al}{a^2 + l^2} < 1. \tag{83}$$

It means that both brackets in the metric coefficient $P(\theta)$ given by (4) are positive, and thus the function P in (1) is *always positive*, retaining the correct signature of the spacetime.

B. Ergoregions

With the rotation parameter a, the family of black holes (1) contains *ergoregions* similar to those known from the famous Kerr solution.

The boundary of the ergoregion is defined by the condition $g_{tt} = 0$, where the corresponding metric coefficient reads

$$g_{tt} = \frac{1}{\Omega^2 \rho^2} (P a^2 \sin^2\theta - Q). \tag{84}$$

The corresponding condition is thus

$$Q(r_e) = a^2 \sin^2\theta \; P(\theta), \tag{85}$$

where the metric functions $P(\theta)$ and $Q(r)$ are given by (4) and (5), respectively. For a fixed value of the angular coordinate θ, the right-hand side of (85) is some constant. And since the function $Q(r)$ is of the fourth order, it follows that there are (at most) *four distinct boundaries r_e of the ergoregions* in the direction of θ. These are associated with the corresponding four horizons $\mathcal{H}_b^\pm$ and $\mathcal{H}_a^\pm$, defining the surfaces of infinite redshift, and also the stationary limit at which observers on fixed r and θ cannot "stand still".

Solving the Eq. (85) explicitly is generally complicated but can be plotted using computer, see Fig. 1. It is also obvious that *the ergoregion boundary "touches" the corresponding horizon at the poles $\theta = 0$ and $\theta = \pi$* because there the condition (85) reduces to $Q(r_e) = 0$.

In the case of *vanishing acceleration* $\alpha = 0$, the metric functions (4) and (5) simplify to $P = 1$ and $Q = (r - r_+)(r - r_-)$. Equation (85) reduces to $r_e^2 - 2mr_e + (a^2\cos^2\theta - l^2 + e^2 + g^2) = 0$ which has two roots

$$r_{e\pm}(\theta) = m \pm \sqrt{m^2 + l^2 - e^2 - g^2 - a^2\cos^2\theta}. \tag{86}$$

This explicitly localizes the two ergoregions for the Kerr–Newman–NUT black holes. As for the standard Kerr black hole, it extends most from the corresponding horizon in the equatorial plane $\theta = \pi/2$, in which case $r_{e\pm} = m \pm \sqrt{m^2 + l^2 - e^2 - g^2}$.

On the other hand, *for $a = 0$ there are no ergoregions* because the condition (85) reduces to $Q(r_e) = 0$, i.e., the boundaries coincide with the black hole horizons $\mathcal{H}_b^\pm$ at $r_\pm$ of the Taub–NUT spacetime (possibly charged). In fact, such horizons become the *Killing horizons* associated with the Killing vector field ∂_t, located at $m \pm \sqrt{m^2 + l^2 - e^2 - g^2}$. To summarize, the ergoregions are related only to the Kerr-like rotation represented by the parameter a, not to the NUT parameter l. There are no ergoregions in the purely NUT spacetimes.

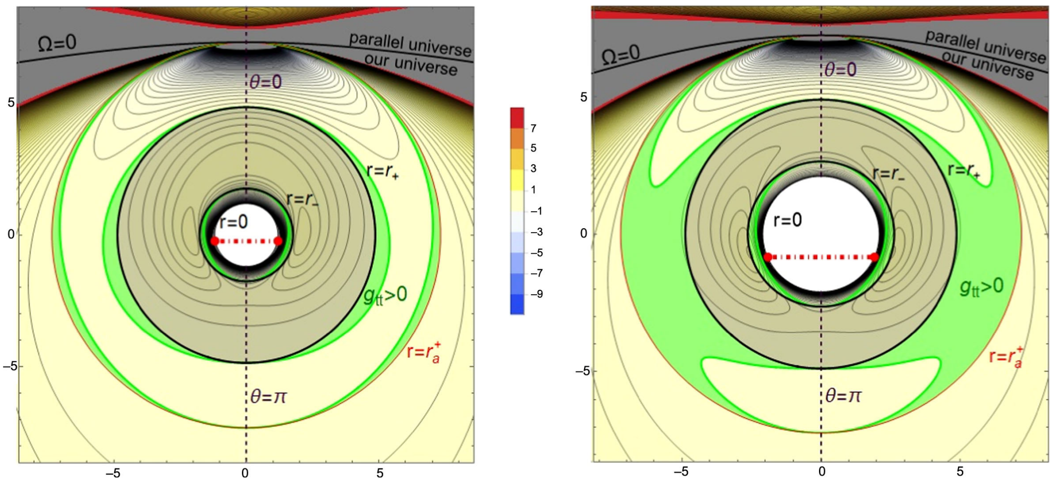

FIG. 1. Plot of the metric function g_{tt} (84) for the accelerating black hole (1) with axes $\theta = 0$ and $\theta = \pi$. The values of g_{tt} are visualized in quasipolar coordinates $\mathrm{x} \equiv \sqrt{r^2 + (a+l)^2}\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + (a+l)^2}\cos\theta$ for $r \geq 0$. The grey annulus in the center of each figure localizes the black hole bordered by its horizons $\mathcal{H}_b^{\pm}$ at r_+ and r_- $(0 < r_- < r_+)$. The acceleration horizon $\mathcal{H}_a^+$ at r_a^+ (big red circle) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. The grey curves are contour lines $g_{tt}(r,\theta) = \text{const.}$, and the values are color-coded from red (positive values) to blue (negative values). The green curves are the isolines $g_{tt} = 0$ determining the boundary of the ergoregions (85) in which $g_{tt} > 0$ (green regions). They occur close to the horizons near the equatorial plane $\theta = \pi/2$. The plot is made for the choice $m = 3$, $a = 1$, $l = 0.2$, $e = g = 1.6$, $\alpha = 0.12$ (left) and $m = 3$, $a = 1.5$, $l = 0.6$, $e = g = 1.6$, $\alpha = 0.12$ (right). For larger values of a and l the ergoregions are bigger and shifted toward $\theta = \pi$. In fact, it can be seen that the ergoregion above the black hole horizon at r_+ *is merged* with the ergoregion below the acceleration horizon at r_a^+ in the equatorial part.

C. Curvature singularities

By inspecting the Weyl NP scalar Ψ_2 given explicitly by the expression (55) we conclude that the curvature singularities occur if and only if $r + \mathrm{i}(l + a\cos\theta) = 0$ (or its complex conjugate). Notice that this complex equation *implies* also $\rho^2 = r^2 + (l + a\cos\theta)^2 = 0$ which represents the curvature singularity in the Ricci scalar Φ_{11} given by (56) when the electric and magnetic charges e, g are nonzero. Both the real and imaginary parts must vanish, so that the *curvature singularity condition* reads

$$r = 0 \quad \text{and at the same time} \quad l + a\cos\theta = 0. \tag{87}$$

The presence of the curvature singularity is confirmed by the behavior of the Kretschmann scalar $\mathcal{K} \equiv R_{abcd}R^{abcd}$ given by (65). The second condition (87), that is $L = 0$, implies $\Omega = 1$, $\rho^2 = r^2$, $F_\pm = r^3$, $T_\pm = r^2$, and

$$K_\pm = m\left(1 \pm \alpha a \frac{a^2 - l^2}{a^2 + l^2}\right) r^3 \mp l\left(1 \mp \alpha a \frac{a^2 - l^2 + e^2 + g^2}{a^2 + l^2}\right) r^3 - (e^2 + g^2) r^2.$$

In the limit $r \to 0$ the Kretschmann scalar thus diverges,

$$\mathcal{K} = 48 \frac{K_+ K_-}{r^{12}} \to \infty, \tag{88}$$

because $K_+ K_- \sim r^6$ in the vacuum case, and $K_+ K_- \sim r^4$ in the electrovacuum case.

Now, the important observation is that the necessary (but not sufficient) singularity condition $l + a\cos\theta = 0$ *can only be satisfied if* $|l| \leq |a|$. Otherwise, the expression $l + a\cos\theta$ remains nonzero because $\cos\theta$ is bounded to the range $[-1, 1]$.

We thus conclude that in the whole family of type D spacetimes (1) the *curvature singularity structure depends on the relative values of the two twist parameters*, i.e., the Kerr-like rotation a versus the NUT parameter l, as follows:

$$\begin{aligned}
&l = 0, \quad a = 0: && \text{singularity at } r = 0 && \text{for any } \theta,\\
&l = 0, \quad a \neq 0: && \text{singularity at } r = 0 && \text{for } \theta = \pi/2,\\
&0 < |l| < |a|: && \text{singularity at } r = 0 && \text{for } \cos\theta = -l/a,\\
&l = +a: && \text{singularity at } r = 0 && \text{for } \theta = \pi,\\
&l = -a: && \text{singularity at } r = 0 && \text{for } \theta = 0,\\
&|l| > |a| > 0: && \text{no singularity}, &&\\
&l \neq 0, a = 0: && \text{no singularity}. &&
\end{aligned} \tag{89}$$

Recall that throughout this paper we naturally assume that all physical parameters m, e, g, α, a, l are non-negative. However, for the sake of completeness, in the above table we have admitted the situation in which a and l can be *any* *real numbers*. In fact, the reflection symmetry $\varphi \to -\varphi$ of the metric (1), or equivalently $t \to -t$, can be used to change $a \to -a$ or $l \to -l$ when $l = 0$ or $a = 0$, respectively. However, in the generic case when *both* a and l are nontrivial, their relative sign plays the role.

Of course, these results agree with the standard character of the singularity $r = 0$ of the Schwarzschild, Reissner–Nordström and (possibly charged) C-metric spacetimes ($l = 0$, $a = 0$), the ring singularity structure of the Kerr and Kerr–Newman black holes ($l = 0$, $\alpha = 0$), and the absence of curvature singularities in (possibly charged) Taub–NUT spacetime ($a = 0$, $\alpha = 0$).

Finally, it may be useful to graphically represent the global curvature and horizon structure of these black hole spacetimes. On a schematic picture in Fig. 2 we depict the section $t = \text{const.}$, $\varphi = \text{const.}$, taking the full range of $\theta \in [0, \pi]$ *distinct* from the specific value $\cos\theta = -l/a$. Therefore, the curvature singularity located at $r = 0$ is *not* encountered, and it is possible to consider the *full range of the coordinate* $r \in (-\infty, +\infty)$. In the vicinity of $r = 0$ the curvature of the spacetime is maximal, in the region $r > 0$ (the right part of the surface) it decreases to zero, and similarly in the region $r < 0$ (the left part of the surface)—far away from the origin the spacetime becomes asymptotically flat. The angular coordinate $\theta \in [0, \pi]$ is plotted perpendicularly, completing the full circles $r = \text{const.}$ (considering also the antipodal section $\varphi + \pi$ in the second half of the circle). The resulting "neck" or "wormhole" connects two distinct universes. Positions of the two black hole horizons $\mathcal{H}_b^+$ at $r_b^+ \equiv r_+$ and $\mathcal{H}_b^-$ at $r_b^- \equiv r_-$ are indicated by red and green circles, respectively. Here we assume $0 < l < a \leq \sqrt{a^2 + e^2 + g^2}$, so that $0 < r_- < r_+$. In this plot we also show the position of the two distinct full axes $\theta = 0$ and $\theta = \pi$. These are indicated by dashed lines on top and bottom of the surface.

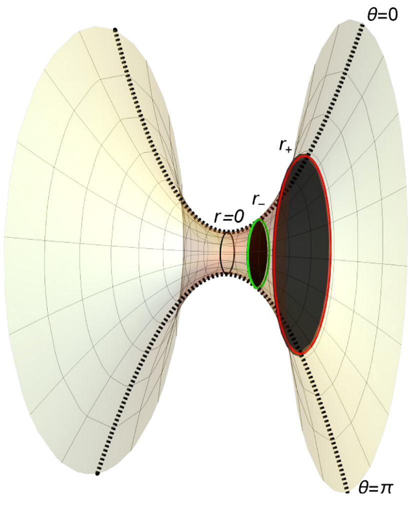

FIG. 2. A schematic visualization of the curvature structure of the generic black hole spacetime (1) using a section with fixed coordinates t and φ. Away from the singularity located at $\cos\theta = -l/a$, $r = 0$ it is possible to cross $r = 0$ from the asymptotically flat universe in the region $r > 0$ (right part) to another universe in the region $r < 0$ (left part). In this diagram we also plot the positions of the two black hole horizons $\mathcal{H}_b^+$ and $\mathcal{H}_b^-$ at r_+ and r_- (red and green circles, respectively), and the two distinct infinite axes $\theta = 0$ and $\theta = \pi$ (dashed lines).

It should be emphasized that this is only a *schematic picture*, not an embedding and rigorous construction (it cannot be done because the r-coordinate is *temporal* between the horizons $\mathcal{H}_b^-$ and $\mathcal{H}_b^+$, and also because the "point" $\cos\theta = -l/a$, $r = 0$ is actually the *curvature singularity*.

Using the same schematic plot of the central domain of the black hole spacetime, we can also indicate the location of the curvature singularity at $r = 0$, $\cos\theta = -l/a$ for various values of the NUT parameter l (assuming the same a and other physical parameters). As in Fig. 2, the origin $r = 0$ is plotted in Fig. 3 as a black circle around the "neck," and the two axes located at $\theta = 0$ and $\theta = \pi$ are indicated by dashed lines on top and bottom of the surface.

There are 7 such plots in Fig. 3 corresponding to 7 specific values of l/a. When the NUT parameter vanishes, $l = 0$, the curvature singularity is located at $r = 0$ for $\theta = \pi/2$. In the middle plot in Fig. 2 such a singularity is indicated by *red dots*. In fact, considering also the additional angular coordinate $\varphi \in [0, 2\pi)$, this forms a *ring singularity* of the Kerr–Newman black hole, shown here as the red dashed circle in extra dimension. In the case when $l = a$ the curvature singularity is located at the pole $\theta = \pi$ (the bottom right plot), while for $l = -a$ it is located at the opposite pole $\theta = 0$ (the top left plot). In the generic case $|l| < |a|$, the ring curvature singularity is located at specific θ between these extremes, such that $\cos\theta = -l/a$ (the bottom left and the top right plots). Finally, when $|l| > |a|$, there is no curvature singularity (the top and the bottom plots).

In a similar way, by the red dots and the red dashed line we have indicated the position of the ring-like curvature singularity at $r = 0$ in Fig. 1.

D. Conformal diagrams: Global structure and infinities

In Sec. V A we have already clarified that the coordinate singularities of the metric located at $r_b^\pm$ and $r_a^\pm$ correspond to *four* distinct horizons $\mathcal{H}_b^\pm$ and $\mathcal{H}_a^\pm$ (provided $m^2 + l^2 \geq a^2 + e^2 + g^2$). We will now explicitly construct coordinates which cover the whole spacetime, including these horizons given by the roots $Q(r) = 0$ of the quartic function (70). They will enable us to subsequently derive the corresponding Penrose conformal diagrams showing the global structure of this family of type D black holes represented by the metric (1).

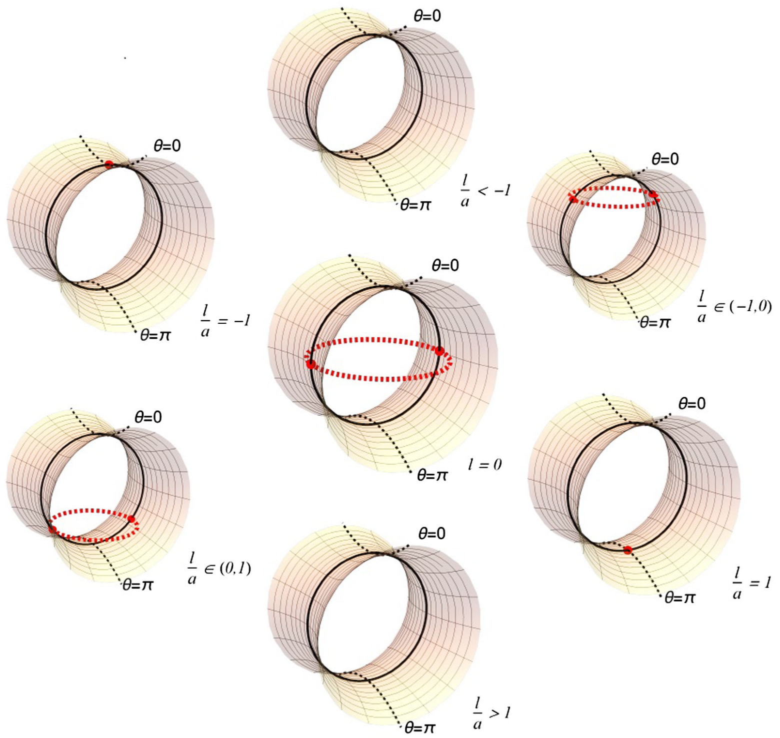

FIG. 3. Schematic visualization of the curvature singularity located at $r = 0$, $\cos\theta = -l/a$ in the black hole spacetime (1) for 7 distinct choices of the NUT parameter l. For $|l| \geq |a|$ such singularity is absent and it is possible to regularly cross $r = 0$ at *any* θ, entering another asymptotically flat universe.

To this end, we first introduce the *retarded* and *advanced* *null* coordinates

$$u = t - r_* \quad \text{and} \quad v = t + r_*, \tag{90}$$

with the *tortoise* coordinate

$$r_* \equiv \int \frac{r^2 + (a+l)^2}{Q(r)} \mathrm{d}r, \tag{91}$$

and also the corresponding *untwisted angular* coordinates

$$\phi_u \equiv \varphi - a \int \frac{\mathrm{d}r}{Q(r)} \quad \text{and} \quad \phi_v \equiv \varphi + a \int \frac{\mathrm{d}r}{Q(r)}. \tag{92}$$

Using the *advanced* pair of coordinates $\{v, \phi_v\}$, the metric (1) takes the form

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg[&\frac{a^2 P \sin^2\theta - Q}{\rho^2}(\mathrm{d}v - \mathcal{T}\mathrm{d}\phi_v)^2 \\ &+ 2(\mathrm{d}v - \mathcal{T}\mathrm{d}\phi_v)(\mathrm{d}r - aP\sin^2\theta\,\mathrm{d}\phi_v) \\ &+ \rho^2\left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\,\mathrm{d}\phi_v^2\right)\bigg]. \end{aligned} \tag{93}$$

The function

$$\mathcal{T}(\theta) \equiv a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta \tag{94}$$

was introduced to abbreviate the expression. It also enters a useful identity

$$r^2 + (a+l)^2 - a\mathcal{T} = r^2 + (l + a\cos\theta)^2 \equiv \rho^2. \tag{95}$$

Obviously, the metric (93) is regular at $Q(r) = 0$, so that *the coordinate singularity at the horizons has been removed.*

By employing the complementary *retarded* pair of coordinates $\{u, \phi_u\}$, the metric (1) reads

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg[&\frac{a^2 P \sin^2\theta - Q}{\rho^2}(\mathrm{d}u - \mathcal{T}\mathrm{d}\phi_u)^2 \\ &- 2(\mathrm{d}u - \mathcal{T}\mathrm{d}\phi_u)(\mathrm{d}r + aP\sin^2\theta\, \mathrm{d}\phi_u) \\ &+ \rho^2\bigg(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta\, \mathrm{d}\phi_u^2\bigg)\bigg], \end{aligned} \tag{96}$$

which is also regular at $Q(r) = 0$.

Actually, these metrics are a considerable generalization of the original coordinate forms of the rotating Kerr–Newman black hole solutions, see Eq. (1) in [13], Eq. (5.31) in [14], or Eq. (11.4) in [4]. Now it includes not only the usual physical parameters m, a, e (and/or g), but also the NUT parameter l and the acceleration parameter α.

As usual, the next step in construction of the maximal analytic extension of the manifold is to introduce *both the null coordinates u and v simultaneously* (dropping r as a coordinate). Clearly, for fixed values of ϕ_v and θ the *radial null geodesics* are simply given by $v = \text{const.}$, while for fixed values of ϕ_u and θ the complementary radial null geodesics are given by $u = \text{const.}$ Therefore, by employing both the coordinates u and v, the causal structure of the spacetime is naturally revealed. Using the relation (90) we immediately obtain

$$v - u = 2r_*(r), \tag{97}$$

so that

$$2\mathrm{d}r = \frac{Q}{r^2 + (a+l)^2}(\mathrm{d}v - \mathrm{d}u). \tag{98}$$

This relation can be used to eliminate the dr-term either from the metric (93) or (96).

Moreover, due to the simple *factorized* form (70) of the metric function $Q(r)$, the integral (91) defining the function $r_*(r)$ in (97) can be calculated explicitly as

$$\begin{aligned} r_*(r) = k_b^+ \log\left|1 - \frac{r}{r_b^+}\right| + k_b^- \log\left|1 - \frac{r}{r_b^-}\right| \\ + k_a^+ \log\left|1 - \frac{r}{r_a^+}\right| + k_a^- \log\left|1 - \frac{r}{r_a^-}\right|, \end{aligned} \tag{99}$$

where the auxiliary constant coefficients are

$$\begin{aligned} k_b^+ &= \frac{(a^2+l^2)^2[r_+^2 + (a+l)^2]}{2m(a^2+l^2+\alpha a(a-l)r_+)(a^2+l^2-\alpha a(a+l)r_+)}, \\ k_b^- &= -\frac{(a^2+l^2)^2[r_-^2 + (a+l)^2]}{2m(a^2+l^2+\alpha a(a-l)r_-)(a^2+l^2-\alpha a(a+l)r_-)}, \\ k_a^+ &= -\frac{(a^2+l^2)[(a^2+l^2)^2 + \alpha^2 a^2 (a+l)^4]}{2\alpha a^2(a^2+l^2-\alpha a(a+l)r_+)(a^2+l^2-\alpha a(a+l)r_-)}, \\ k_a^- &= \frac{(a^2+l^2)[(a^2+l^2)^2 + \alpha^2 a^2 (a^2-l^2)^2]}{2\alpha a^2(a^2+l^2+\alpha a(a-l)r_+)(a^2+l^2+\alpha a(a-l)r_-)}, \end{aligned} \tag{100}$$

each associated with the corresponding horizon $\mathcal{H}_h^\pm$ located at $r = r_h^\pm$, where $h = b$ (for the black-hole horizons) or $h = a$ (for the acceleration horizons). Inverting the function (99), we can express the metric functions Q, ρ^2 and Ω^2 in terms of the null coordinates $v - u$ instead of r by using the relation (97).

To obtain the maximal extension of the black-hole manifold represented by (1), we now "glue together" different "coordinate patches" (charts of the complete atlas) *crossing all the horizons*, until a curvature singularity or conformal infinity (the scri $\mathcal{I}$) is reached. In order to derive the correct causal structure, it is essential to employ the null coordinates u and v. Therefore, we apply the coordinate patches of the metric form (93) for extending the spacetime across the horizons in the null direction given by the *advanced* coordinate v, while we apply the coordinate patches of the metric form (96) for extending the spacetime across the horizons in the complementary null direction given by the *retarded* coordinate u. Since both these metrics are regular for $Q = 0$, the coordinate singularities at *all the horizons* $\mathcal{H}_h^\pm$ are removed, step-by-step.

However, to perform this procedure exactly and correctly, two complicated issues must also be clarified. The first problem is the fact, that the distinct coordinate patches (93) and (96) employ *distinct angular coordinates* ϕ_v *and* ϕ_u, respectively. The second problem is to prove that thus obtained maximal extension of the manifold is *analytic*.

To resolve the first problem associated with distinct angular coordinates ϕ_v and ϕ_u, we can employ the general strategy suggested by Boyer and Lindquist [15] for the Kerr spacetime and subsequently used also for the charged Kerr–Newman spacetime by Carter [13]. The trick is based on using the specific *Killing vector fields* which are *the null generators of the horizons*. In terms of the two coordinate patches (93) and (96), such special vector fields read

$$\xi^a \equiv \partial_u + \Omega_h \partial_{\phi_u}, \quad \text{and also} \quad \xi^a \equiv \partial_v + \Omega_h \partial_{\phi_v}, \tag{101}$$

where the angular velocity of the given horizon $\mathcal{H}$ is

$$\Omega_h = \frac{a}{r_h^2 + (a+l)^2}. \tag{102}$$

Indeed, using the corresponding metric coefficients of (93) and (96), evaluated at $Q = 0$, it is straightforward to show that $\xi^a \xi_a(\mathcal{H}) = 0$ whenever $\Omega_h = a/(\rho_h^2 + a\mathcal{T})$. Applying the identity (95), we obtain the expression (102) for *both* the Killing vector fields (101).

Now, following [13,15] we introduce a special angular coordinate ϕ_h *which is constant along the trajectories of both the Killing vector fields* (101). Being the generators of the specific *bifurcate Killing horizon* (a 2-dimensional spatial intersection of the "advanced" and the "retarded" null horizons), via such new angular coordinate ϕ_h a suitable transition between the corresponding patches is achieved. Technically, it is introduced by the 1-form condition

$$2\mathrm{d}\phi_h \equiv \mathrm{d}\phi_u + \mathrm{d}\phi_v - \Omega_h(\mathrm{d}u + \mathrm{d}v), \tag{103}$$

because $\mathrm{d}\phi_h(\xi^a) = 0$ for *both* the Killing vector fields (101). Using (90) and (92), this condition can be integrated to

$$\phi_h = \varphi - \Omega_h t. \tag{104}$$

Unfortunately, the specific choice of the angular coordinate ϕ_h *depends on the given horizon* via its value r_h and thus Ω_h. For this reason, it is not possible to find a single and simple global coordinate ϕ which would conveniently "cover" all the four horizons. This drawback was met many years ago already in the Kerr spacetime, so it is not surprising that it reappears in the current context of the complete family of type D black holes.

An explicit *general* metric form constructed in this way reads

$$\mathrm{d}s^2 = \frac{1}{4\Omega^2}\bigg[-\frac{Q}{\rho^2}((1-\mathcal{T}\Omega_h)(\mathrm{d}u+\mathrm{d}v) - 2\mathcal{T}\mathrm{d}\phi_h)^2 + Q\rho^2\frac{(\mathrm{d}u-\mathrm{d}v)^2}{[r^2+(a+l)^2]^2} + 4\frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P\sin^2\theta}{\rho^2}((a-[r^2+(a+l)^2]\Omega_h)(\mathrm{d}u+\mathrm{d}v) - 2[r^2+(a+l)^2]\mathrm{d}\phi_h)^2\bigg]. \tag{105}$$

For *nontwisting* black holes without the Kerr-like rotation ($a = 0$) and the NUT parameter ($l = 0$), the metric functions simplify to $\Omega = 1$, $P = 1$, $\rho^2 = r^2$, $\mathcal{T} = 0$, $\Omega_h = 0$, so that

$$\mathrm{d}s^2 = -\frac{Q}{r^2}\mathrm{d}u\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi_h^2), \tag{106}$$

which is the usual form of the spherically symmetric black holes in the double-null coordinates [4].

On any 2-dimensional section $\theta =$ const. and $\phi_h =$ const., using (102), the general metric (105) reduces to

$$\mathrm{d}\sigma^2 = \frac{1}{4\Omega^2}\bigg[-\frac{(1-\mathcal{T}\Omega_h)^2}{\rho^2}Q(\mathrm{d}u+\mathrm{d}v)^2 + \frac{\rho^2}{[r^2+(a+l)^2]^2}Q(\mathrm{d}u-\mathrm{d}v)^2 + a^2\frac{P\sin^2\theta}{\rho^2}\frac{(r+r_h)^2(r-r_h)^2}{[r_h^2+(a+l)^2]^2}(\mathrm{d}u+\mathrm{d}v)^2\bigg], \tag{107}$$

which is indeed null at any horizon r_h because $Q(r_h) = 0$.

Let us now move to the second problem, which is the global extension and investigation of the degree of smoothness (analyticity) of the horizons $\mathcal{H}_h^{\pm}$. Restricting ourselves to the sections given by constant values of the angular coordinates θ and ϕ_h, we introduce *the couples of new null coordinates* $U_h^{\pm}$ and $V_h^{\pm}$, defined as

$$U_h^{\pm} = (-1)^i \mathrm{sign}(k_h^{\pm}) \exp\left(-\frac{u}{2k_h^{\pm}}\right), \tag{108}$$

$$V_h^{\pm} = (-1)^j \mathrm{sign}(k_h^{\pm}) \exp\left(+\frac{v}{2k_h^{\pm}}\right). \tag{109}$$

Each couple covers the corresponding horizon $\mathcal{H}_h^{\pm}$. Moreover, it is characterized by a *particular choice of two integers* (i, j) which specify a certain region in the manifold. Generally, there are 5 types of regions which are separated by the four types of horizons $\mathcal{H}_h^{\pm}$, namely

Region	Description	Specification of (i,j)
I:	asymptotic time-dependent domain between $\mathcal{H}_a^+$ and $\mathcal{I}$	$(n-2m+1, n+2m-1)$
II:	stationary region between $\mathcal{H}_b^+$ and $\mathcal{H}_a^+$	$(2n-m, 2n+m-1)$
III:	time-dependent domain between the black-hole horizons	$(n-2m, n+2m)$
IV:	stationary region between $\mathcal{H}_a^-$ and $\mathcal{H}_b^-$	$(2n-m+1, 2n+m)$
V:	asymptotic time-dependent domain between $\mathcal{I}$ and $\mathcal{H}_a^-$	$(n-2m+1, n+2m-1)$

where m, n are arbitrary integers. The corresponding Kruskal–Szekeres-type dimensionless coordinates for every distinct region are

$$T_h^{\pm}=\frac{1}{2}(V_h^{\pm}+U_h^{\pm}),\qquad R_h^{\pm}=\frac{1}{2}(V_h^{\pm}-U_h^{\pm}). \tag{110}$$

Of course, the presence of the *curvature singularity* at $r=0$ (implying $r_*=0$) for certain values of θ restricts the range of the corresponding coordinates U_b^- and V_b^- in the region IV to the domain outside $U_b^-V_b^-=\pm 1$.

In terms of these coordinates, the extension across the horizon is regular (in fact, analytic). Indeed, by multiplying and dividing the null coordinates (108) and (109) we obtain

$$U_h^{\pm}V_h^{\pm}=\left(1-\frac{r}{r_b^+}\right)^{\frac{k_b^+}{k_h^{\pm}}}\left(1-\frac{r}{r_b^-}\right)^{\frac{k_b^-}{k_h^{\pm}}}\left(1-\frac{r}{r_a^+}\right)^{\frac{k_a^+}{k_h^{\pm}}}\left(1-\frac{r}{r_a^-}\right)^{\frac{k_a^-}{k_h^{\pm}}}, \tag{111}$$

$$\frac{U_h^{\pm}}{V_h^{\pm}}=(-1)^{i+j}\exp\left(-\frac{t}{k_h^{\pm}}\right). \tag{112}$$

The terms $(\mathrm{d}u\pm\mathrm{d}v)^2$ in the metric (107) become

$$(\mathrm{d}u\pm\mathrm{d}v)^2=\frac{4(k_h^{\pm})^2}{U_h^{\pm}V_h^{\pm}}\left(\frac{V_h^{\pm}}{U_h^{\pm}}(\mathrm{d}U_h^{\pm})^2\mp 2\mathrm{d}U_h^{\pm}\mathrm{d}V_h^{\pm}+\frac{U_h^{\pm}}{V_h^{\pm}}(\mathrm{d}V_h^{\pm})^2\right). \tag{113}$$

A nonanalytic behavior across the horizon r_h may thus occur only at zeros of the product $U_h^{\pm}V_h^{\pm}$. However, they exactly cancel the zeros of the functions $Q(r)$ in the metric (107). For example, by choosing the black hole horizon $r_h=r_b^+\equiv r_+$, be get $U_b^+V_b^+\propto(r-r_+)$ which clearly compensates the corresponding root $Q\propto(r-r_+)$ in (5). Notice also that the last term in (107) actually vanishes. Therefore, the metric (107) remains finite at r_+. Of course, the same argument applies to the remaining three horizons.

Now we can construct the *Penrose conformal diagrams* which visualize the global structure of the extended manifold. This is achieved by a suitable conformal rescaling of $U_h^{\pm}$ and $V_h^{\pm}$ to the corresponding compactified null coordinates $\tilde{u}_h^{\pm}$ and $\tilde{v}_h^{\pm}$ defined as

$$\tan\frac{\tilde{u}_h^{\pm}}{2}\equiv-\mathrm{sign}(k_h^{\pm})(U_h^{\pm})^{-\mathrm{sign}(k_h^{\pm})}=(-1)^{i+1}\exp\left(+\frac{u}{2|k_h^{\pm}|}\right), \tag{114}$$

$$\tan\frac{\tilde{v}_h^{\pm}}{2}\equiv-\mathrm{sign}(k_h^{\pm})(V_h^{\pm})^{-\mathrm{sign}(k_h^{\pm})}=(-1)^{j+1}\exp\left(-\frac{v}{2|k_h^{\pm}|}\right). \tag{115}$$

Applying the identity $\arctan x+\arctan y=\arctan(\frac{x+y}{1-xy})$ $(mod\ \pi)$ we get

$$\tilde{T}_h^{\pm}\equiv\frac{1}{2}(\tilde{v}_h^{\pm}+\tilde{u}_h^{\pm})=-\arctan\left[\frac{(-1)^j\exp(-\frac{t+r_*}{2|k_h^{\pm}|})+(-1)^i\exp(\frac{t-r_*}{2|k_h^{\pm}|})}{1-(-1)^{i+j}\exp(-\frac{r_*}{|k_h^{\pm}|})}\right], \tag{116}$$

$$\tilde{R}_h^{\pm}\equiv\frac{1}{2}(\tilde{v}_h^{\pm}-\tilde{u}_h^{\pm})=-\arctan\left[\frac{(-1)^j\exp(-\frac{t+r_*}{2|k_h^{\pm}|})-(-1)^i\exp(\frac{t-r_*}{2|k_h^{\pm}|})}{1+(-1)^{i+j}\exp(-\frac{r_*}{|k_h^{\pm}|})}\right]. \tag{117}$$

From these general relations it follows that

$$\tilde{T}_h^{\pm}=\begin{cases}(-1)^{j+1}\arctan\frac{\cosh\frac{t}{2|k_h^{\pm}|}}{\sinh\frac{r_*}{2|k_h^{\pm}|}} & \text{for } i+j \text{ even},\\ (-1)^{j}\arctan\frac{\sinh\frac{t}{2|k_h^{\pm}|}}{\cosh\frac{r_*}{2|k_h^{\pm}|}} & \text{for } i+j \text{ odd}, r_*<0,\\ (-1)^{j}\arctan\frac{\sinh\frac{t}{2|k_h^{\pm}|}}{\cosh\frac{r_*}{2|k_h^{\pm}|}}+\pi & \text{for } i+j \text{ odd}, r_*\geq 0,\end{cases} \tag{118}$$

and

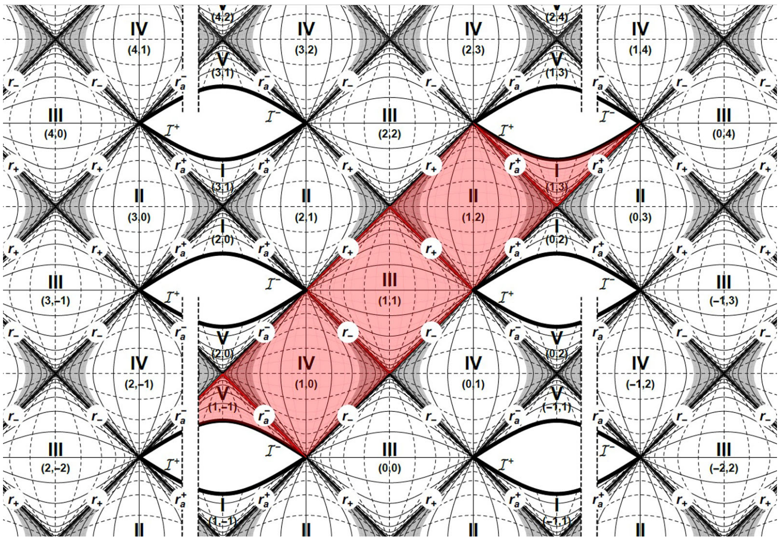

FIG. 4. Penrose conformal diagram of the completely extended spacetime (1) showing the global structure of this family of accelerating and rotating charged black holes. We assume the ordering of the four horizons as $r_a^- < r_- < r_+ < r_a^+$, see (80), which occurs for reasonably small acceleration parameter α, restricted by (81), and small values of the NUT parameter l such that $|l| < |a|$. Here we show a typical 2-dimensional section $\theta, \phi_h = \text{const}$ without the curvature singularity at $r = 0$, i.e., for any $\theta = \text{const}$ such that $\cos\theta \neq -l/a$. The double dashed vertical parallel lines indicate a separation of distinct asymptotically flat regions close to $\mathcal{I}^\pm$ (different "parallel universes" that are not necessarily identified). Grey areas in regions II and IV close to the horizons denote the ergoregions.

$$\tilde{R}_h^\pm = \begin{cases} (-1)^j \arctan \dfrac{\sinh\frac{t}{2|k_h^\pm|}}{\cosh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ even}, \\ (-1)^{j+1} \arctan \dfrac{\cosh\frac{t}{2|k_h^\pm|}}{\sinh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ odd}, r_* < 0, \\ (-1)^{j+1} \arctan \dfrac{\cosh\frac{t}{2|k_h^\pm|}}{\sinh\frac{r_*}{2|k_h^\pm|}} + \pi & \text{for } i+j \text{ odd}, r_* \geq 0. \end{cases} \tag{119}$$

Recall that the function $r_*(r)$ is given by (99) and the coefficients $k_h^\pm$ by (100). In particular, the lines of constant r thus coincide with the lines of constant r_*. Moreover, the condition (81) for a reasonably small values of the acceleration parameter α guarantees that $k_a^+, k_b^- < 0$ while $k_a^-, k_b^+ > 0$. Therefore, for every single region the coordinate r_* spans the whole range $(-\infty, +\infty)$, and similarly the coordinate t.

The explicit relations (118), (119) between the compactified coordinates $\{\tilde{T}_h^\pm, \tilde{R}_h^\pm\}$ and the original coordinates $\{t, r\}$ of the metric (1) for all (i, j) can be used for graphical construction of the Penrose diagram which represents the global structure of the extended black-hole manifold, composed of various "diamond" regions. The resulting picture is shown in Figs. 4 and 5. Fig. 4 is the Penrose diagram of a *generic* 2-dimensional section through the whole spacetime for any $\theta = \text{const}$ such that $\cos\theta \neq -l/a$. It *does not contain* the curvature singularity at $r = 0$. Fig. 5 is the complementary Penrose diagram for *the special* value of θ such that $\cos\theta = -l/a$ which *contains* the curvature singularity at $r = 0$ in all its regions IV (see Sec. V C and Fig. 3).

It can be seen that the complete manifold consists of an *infinite number of the regions* I, II, III, IV and V, each identified by the specific pair of integers (i, j). These regions are *separated by the corresponding horizons*. Namely, the regions I and II are separated by the acceleration horizon $\mathcal{H}_a^+$ at r_a^+, with the asymptotic region I also bounded by the conformal infinity $\mathcal{I}$ (the scri) for very large values of r. The regions II and III are separated by the black-hole horizon $\mathcal{H}_b^+$ at $r_b^+ \equiv r_+$, while the regions III and IV are separated by the inner black-hole horizon $\mathcal{H}_b^-$ at $r_b^- \equiv r_-$. Finally, the regions IV and V (if present) are separated by the acceleration horizon $\mathcal{H}_a^-$ at r_a^-, with the asymptotic region V bounded by the conformal infinity $\mathcal{I}$ with negative values of r. The curves in each region represent the lines of constant t and r (dashed or solid, respectively).

In the diagonal *null directions* of these Penrose diagrams we can identify the particular coordinate patches covered by the "advanced" metric form (93), extending from the

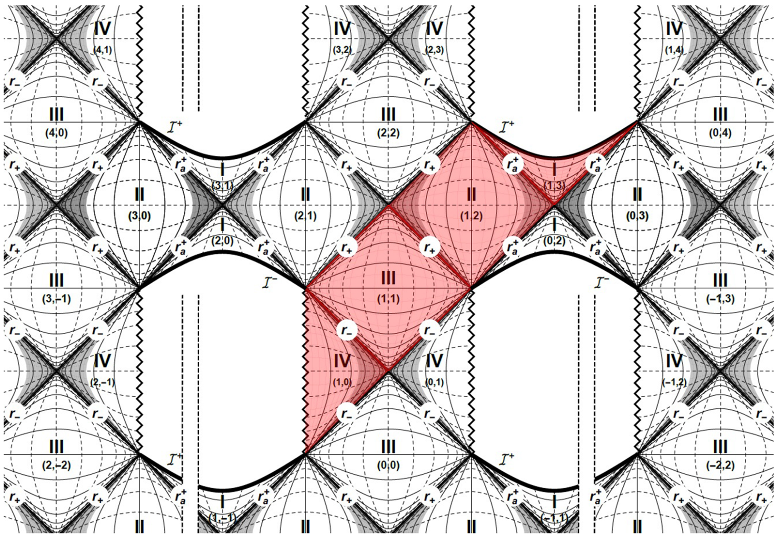

FIG. 5. Penrose conformal diagram of the spacetime (1) representing the same black hole as in Fig. 4 but for the section $\theta, \phi_h = \text{const}$ containing the curvature singularity at $r = 0$, i.e., for the special value of θ such that $\cos\theta = -l/a$. In this section, the regions IV are "cut in half" by this singularity at $r = 0$, so that the acceleration horizon at $r_a^- < 0$ can not be reached, and the region V is thus absent.

bottom left $\mathcal{I}^-$ to the top right $\mathcal{I}^+$ [for example the pink regions I–V between $(1,-1)$ and $(1,\ 3)$], and also the complementary "retarded" metric form (96), extending from the bottom right $\mathcal{I}^-$ to the top left $\mathcal{I}^+$ [these are not colored but also contain the regions I–V, for example between $(-1,1)$ and $(3,\ 1)$]. These patches "share" the "central regions" III [for example $(1,\ 1)$]. Each of such central region III is bounded by the inner and outer black-hole horizons at r_- and r_+, localizing thus the interior of the corresponding black hole. In the whole extended universe, there are thus *infinitely many black holes*—they are identified by the regions III, and labeled by the corresponding specification (i,j), for example $(0, 0)$, $(1, 1)$, $(2, 2)$, $(-2,2)$, $(-1,3)$, $(0,\ 4)$, etc.

Recall that all these black holes are *rotating, NUTed, charged, and accelerating*. Due to their rotation, there are *ergoregions* associated with *all the horizons*, see Sec. V B and Fig. 1. They are represented by the grey areas in the regions II and IV close to the horizons.

As shown in Sec. V C and schematically depicted in Fig. 2, there are two *distinct asymptotically flat universes* associated with each original coordinate patch given by the metric (1), one for $r \to +\infty$ and the other for $r \to -\infty$. These can now be identified in the Penrose diagram in Fig. 4 as the regions I and V beyond the acceleration horizons close to $\mathcal{I}$, respectively. However, the maximal extension has now revealed that *each black hole*, identified by the specific region III, *is in fact associated with four asymptotically flat regions*, namely the pair of the regions I and a pair of the regions V. Two such regions are in the *causal future*, while the remaining two are in the *past*. Moreover, each asymptotically flat region bounded by $\mathcal{I}$ is *"shared" by two distinct black holes*.

For example, the *"infinite chain" of black holes* (regions III) given by $\ldots, (3,-1), (1,1), (-1,3), \ldots$ are located in the "future universes" (regions I) $\ldots, (5,-1), (3,1), (1,3), (-1,5), \ldots$, while their "past universes" (regions V) are $\ldots, (3,-3), (1,-1), (-1,1), (-3,3), \ldots$, respectively. However, these "past universes" *need not be the same* asymptotically flat regions. Therefore, we inserted the double dashed vertical parallel lines in them to indicate their separation: in general the two regions such as $(1,-1)$ are *different* "causal-past parallel universes" *with respect to the distinct causal-future universes of the chain of the black holes*. Of course, it is possible to "artificially" identify (some of) them—both the black-hole regions III and/or their asymptotically flat regions I and V. Since there are *infinitely many possibilities of such identifications*, a plethora of various topologically extremely complicated manifolds can be constructed.

Finally, let us remark that the conformal infinities $\mathcal{I}$ plotted in Figs. 4, 5 *does not look null*. This may be surprising because in all the regions I and V the spacetime is *asymptotically flat* (excluding the cosmic strings along the axes $\theta = 0$ and $\theta = \pi$, arising as specific topological defects which we will investigate in the next three sections of this paper). Being Minkowski-like, the scri $\mathcal{I}$ is indeed null. However, it should be emphasized that the Penrose diagrams in Fig. 4 and Fig. 5 are *just 2-dimensional sections* through the global conformal structure of the four-dimensional Lorentzian manifold which is not spherically symmetric. In particular, it turns out that in the presence of acceleration, the *null* conformal infinity $\mathcal{I}$ of the asymptotically flat regions is indeed represented as the *non-null curve* in the given *section*. This has been thoroughly discussed and analyzed in our previous work on the C-metric [16], see also Chapter 14 in [4].

The global extension of the type D black-hole family of spacetimes obtained in this section seems to be more elegant and also more complete than the preliminary investigation [17] which employed rather complicated transformations to the Weyl–Lewis–Papapetrou form and subsequently to the boost-rotation-symmetric form of the metric. Moreover, here it is explicitly compactified.

E. Cosmic strings (or struts) and deficit angles at $\theta = 0$ and $\theta = \pi$

As shown already in previous works [5,7], the metric form (1) is convenient for explicit analysis of the *regularity of the poles/axes* located at $\theta = 0$ and $\theta = \pi$, respectively. This is now further improved with the new metric functions (2)–(5).

The spatial axes of symmetry are associated with the Killing vector field ∂_φ, identified as its degenerate points. These are located at the coordinate singularities of the function $\sin\theta$ in the metric (1) which appear at $\theta = 0$ and $\theta = \pi$. Therefore, the range of the spatial coordinate θ must be constrained to $\theta \in [0, \pi]$.

Recall that there are six physical parameters in the new metric (1), namely m, a, l, α, e, g, which represent mass, Kerr-like rotation, NUT parameter, acceleration, electric and magnetic charges of the black hole, respectively. However, there is also the *seventh free parameter—the conicity C hidden in the range of the angular coordinate*

$$\varphi \in [0, 2\pi C), \tag{120}$$

which has not yet been specified. We will demonstrate its physical meaning by relating it to the *deficit (or excess) angles* of the *cosmic strings (or struts)*. Their tension is the *physical source of the acceleration of the black holes*. These are basically topological defects associated with *conical singularities* around the two distinct axes. In addition, for nonvanishing NUT parameter l these cosmic strings or struts are *rotating*, thus introducing specific internal twist to the entire spacetime. We will now analyze them in more detail.

Let us start with investigation of the (non)regularity of the *first axis of symmetry* $\theta = 0$ in the metric (1). Consider a small circle around it given by $\theta = \text{const.}$, with the range of φ given by (120), assuming fixed t and r. The invariant length of its *circumference* is $\int_0^{2\pi C}\sqrt{g_{\varphi\varphi}}\,\mathrm{d}\varphi$, while its *radius* is $\int_0^{\theta}\sqrt{g_{\theta\theta}}\,\mathrm{d}\theta$. The axis is regular if their fraction in the limit $\theta \to 0$ is equal to 2π. However, in general we obtain

$$f_0 \equiv \lim_{\theta\to 0}\frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to 0}\frac{2\pi C\sqrt{g_{\varphi\varphi}}}{\theta\sqrt{g_{\theta\theta}}}. \tag{121}$$

For the metric (1), the relevant metric functions are

$$g_{\varphi\varphi} = \frac{1}{\Omega^2\rho^2}\Big[P(r^2+(a+l)^2)^2\sin^2\theta - Q\Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)^2\Big], \qquad g_{\theta\theta} = \frac{\rho^2}{\Omega^2 P}. \tag{122}$$

For very small values of θ, the second term in $g_{\varphi\varphi}$ proportional to Q becomes negligible with respect to the first term proportional to P, so that we obtain $g_{\varphi\varphi} \approx P(r^2+(a+l)^2)^2\theta^2/\Omega^2\rho^2$. Straightforward evaluation of the limit (121) gives

$$f_0 = 2\pi C P(0) = 2\pi C\Big(1-\alpha\frac{a^2+al}{a^2+l^2}r_+\Big)\Big(1-\alpha\frac{a^2+al}{a^2+l^2}r_-\Big). \tag{123}$$

The axis $\theta = 0$ in the metric (1) can thus be made regular by the unique choice

$$C = C_0 \equiv \Big[\Big(1-\alpha\frac{a^2+al}{a^2+l^2}r_+\Big)\Big(1-\alpha\frac{a^2+al}{a^2+l^2}r_-\Big)\Big]^{-1} = \Big[1-2\alpha m\frac{a^2+al}{a^2+l^2}+\alpha^2\Big(\frac{a^2+al}{a^2+l^2}\Big)^2(a^2-l^2+e^2+g^2)\Big]^{-1}, \tag{124}$$

where we have employed the relations (6), (7). Notice that for vanishing acceleration α, this regularization condition is simply $C_0 = 1$.

Analogously, it is possible to regularize the *second axis of symmetry* $\theta = \pi$. Now, the conceptual problem is that the metric function $g_{\varphi\varphi}$ in (122), and thus the circumference, does *not* approach zero in the limit $\theta \to \pi$ due to the presence of the term $4l\sin^2\frac{1}{2}\theta$. This problem can be resolved by the same procedure as for the classic Taub–NUT solution (see the transition between the metrics (12.1) and (12.3) in [4]), namely by applying the transformation of the time coordinate

$$t_\pi \equiv t - 4l\varphi. \tag{125}$$

The metric (1) then becomes

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg(-\frac{Q}{\rho^2}\Big[\mathrm{d}t_\pi - \Big(a\sin^2\theta - 4l\cos^2\tfrac{1}{2}\theta\Big)\mathrm{d}\varphi\Big]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta[a\mathrm{d}t_\pi - (r^2+(a-l)^2)\mathrm{d}\varphi]^2\bigg), \tag{126}$$

i.e.,

$$g_{\varphi\varphi} = \frac{1}{\Omega^2\rho^2}\bigg[P(r^2+(a-l)^2)^2\sin^2\theta - Q\Big(a\sin^2\theta - 4l\cos^2\tfrac{1}{2}\theta\Big)^2\bigg], \quad g_{\theta\theta} = \tfrac{\rho^2}{\Omega^2 P}. \tag{127}$$

Thus, for $\theta \to \pi$ we get $g_{\varphi\varphi} \approx P(r^2 + (a-l)^2)^2 (\pi-\theta)^2/\Omega^2\rho^2$. The radius of a small circle around the axis $\theta = \pi$ is $\int_\theta^\pi \sqrt{g_{\theta\theta}}\mathrm{d}\theta$, so that the fraction

$$f_\pi \equiv \lim_{\theta\to\pi} \frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to\pi} \frac{2\pi C\sqrt{g_{\varphi\varphi}}}{(\pi-\theta)\sqrt{g_{\theta\theta}}}, \tag{128}$$

is

$$f_\pi = 2\pi C P(\pi) = 2\pi C\left(1+\alpha\frac{a^2-al}{a^2+l^2}r_+\right)\left(1+\alpha\frac{a^2-al}{a^2+l^2}r_-\right). \tag{129}$$

The axis $\theta = \pi$ in the metric (126) can thus be made regular by the unique choice

$$C = C_\pi \equiv \left[\left(1+\alpha\frac{a^2-al}{a^2+l^2}r_+\right)\left(1+\alpha\frac{a^2-al}{a^2+l^2}r_-\right)\right]^{-1} = \left[1 + 2\alpha m\frac{a^2-al}{a^2+l^2} + \alpha^2\left(\frac{a^2-al}{a^2+l^2}\right)^2(a^2-l^2+e^2+g^2)\right]^{-1}. \tag{130}$$

With such a choice, there is a *deficit angle* δ_0 (conical singularity) *along the first axis* $\theta = 0$, namely

$$\delta_0 \equiv 2\pi - f_0 = 8\pi\alpha\frac{a^2[m(a^2+l^2) - \alpha al(a^2-l^2+e^2+g^2)]}{(a^2+l^2)^2 + 2\alpha m(a^2-al)(a^2+l^2) + \alpha^2(a^2-al)^2(a^2-l^2+e^2+g^2)}. \tag{131}$$

For black holes without the NUT parameter ($l = 0$) this expression simplifies to

$$\delta_0 = \frac{8\pi\alpha m}{1 + 2\alpha m + \alpha^2(a^2+e^2+g^2)}, \tag{132}$$

recovering the previous results for rotating charged C-metric, see Chapter 14 in [4]. The tension in the *cosmic string along $\theta = 0$ pulls the black hole, causing its uniform acceleration*. Such a string *extends to the full range of the radial coordinate* $r \in (-\infty, +\infty)$, connecting "our universe" with the "parallel universe" through the nonsingular black-hole interior close to $r = 0$.

Complementarily, when the first axis of symmetry $\theta = 0$ is made regular by the choice (124), there is necessarily an *excess angle* δ_π along the second axis $\theta = \pi$, namely

$$\delta_\pi \equiv 2\pi - f_\pi = -8\pi\alpha\frac{a^2[m(a^2+l^2) - \alpha al(a^2-l^2+e^2+g^2)]}{(a^2+l^2)^2 - 2\alpha m(a^2+al)(a^2+l^2) + \alpha^2(a^2+al)^2(a^2-l^2+e^2+g^2)}, \tag{133}$$

which simplifies to

$$\delta_\pi = -\frac{8\pi\alpha m}{1 - 2\alpha m + \alpha^2(a^2+e^2+g^2)}, \tag{134}$$

for $l = 0$. As in the C-metric, this represents the *cosmic strut located along $\theta = \pi$ between the pair of black holes, pushing them away from each other* in opposite spatial directions.

We observe that $\delta_0 = 0 = \delta_\pi$ whenever $\alpha = 0$. In such a case *both the axes are regular*, there is no physical cause of the acceleration and the Kerr–Newman–NUT black holes do not move.

Interestingly, both the axes $\theta = 0$ and $\theta = \pi$ can be *simultaneously regular even for nonvanishing acceleration* α when all six physical parameters satisfy the special constraint

$$m(a^2 + l^2) = \alpha a l(a^2 - l^2 + e^2 + g^2). \qquad (135)$$

The nontrivial constraint requires *both* $a \neq 0$ and $l \neq 0$. Actually, this is a nice compact form of the condition given on page 313 of [4], when the relations (15) for the physical parameters and also the convenient gauge choice (30) are employed. This again demonstrates the advantages of the new form of the metric (1).

However, the condition (135) is *not satisfied for small values of the acceleration* α obeying the inequality (81) which guarantees the natural ordering of the four horizons (80). Indeed, (135) can be rewritten as $m(a^2 + l^2) = \alpha a l r_+ r_-$. Now applying (81), and assuming m, a, l *all positive*, we get the relation

$$m < \frac{l}{a + l} r_- < r_-. \qquad (136)$$

It is in clear contradiction with (7) which implies $m > r_-$.

F. Rotation of the cosmic strings (or struts)

With a generic NUT parameter l, the *cosmic strings (or struts) are rotating*. This can be seen by calculating the *angular velocity* parameter ω_θ of the metric, see [12], along the two different axes $\theta = 0$ and $\theta = \pi$, namely

$$\omega_\theta \equiv \frac{g_{t\varphi}}{g_{tt}}. \qquad (137)$$

For the general form of the new metric (1), where

$$g_{t\varphi} = \frac{1}{\Omega^2\rho^2}\Big[Q\Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big) - a(r^2 + (a + l)^2)P\sin^2\theta\Big],$$
$$g_{tt} = \frac{-1}{\Omega^2\rho^2}[Q - a^2 P\sin^2\theta], \qquad (138)$$

we obtain

$$\omega_\theta = -\frac{Q(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta) - a(r^2 + (a + l)^2)P\sin^2\theta}{Q - a^2 P\sin^2\theta}. \qquad (139)$$

Now we take any fixed value of r away from the horizons, so that $Q \neq 0$ is a nonvanishing constant. Then the limits $\theta \to 0$ and $\theta \to \pi$ are

$$\omega_0 = 0 \quad \text{and} \quad \omega_\pi = -4l, \qquad (140)$$

respectively. The first axis $\theta = 0$ is thus *nonrotating*, while the second axis $\theta = \pi$ rotates and its *angular velocity is directly and solely determined by the NUT parameter l*. Notice that ω_π is independent of the Kerr-like parameter a, and it also does not depend on the conicity parameter C. The rotational character of the axis is thus a specific feature determined by the NUT parameter l, which is clearly *independent* of the possible deficit angles defining the cosmic string/strut along the same axis.

By changing the time coordinate as (125), we obtain the alternative metric (126) for which

$$g_{t_\pi\varphi} = \frac{1}{\Omega^2\rho^2}\Big[Q\Big(a\sin^2\theta - 4l\cos^2\tfrac{1}{2}\theta\Big) - a(r^2 + (a - l)^2)P\sin^2\theta\Big],$$
$$g_{t_\pi t_\pi} = \frac{-1}{\Omega^2\rho^2}[Q - a^2 P\sin^2\theta], \qquad (141)$$

so that

$$\omega_\theta = -\frac{Q(a\sin^2\theta - 4l\cos^2\frac{1}{2}\theta) - a(r^2 + (a - l)^2)P\sin^2\theta}{Q - a^2 P\sin^2\theta}. \qquad (142)$$

The corresponding angular velocities of the two axes are thus

$$\omega_0 = 4l \quad \text{and} \quad \omega_\pi = 0. \qquad (143)$$

In this case, the situation is complementary to (140): the axis $\theta = 0$ rotates, while the axis $\theta = \pi$ is nonrotating.

It is interesting to observe that there is a *constant difference* $\Delta\omega \equiv \omega_0 - \omega_\pi = 4l$ between the angular velocities of the two rotating cosmic strings or struts, directly given by the NUT parameter l (irrespective of the value of a or the choice of C). The NUT parameter is thus responsible for the *difference* between the magnitude of rotation of the two axes $\theta = 0$ and $\theta = \pi$.

G. Closed timelike curves around the rotating strings (or struts)

In the vicinity of the rotating cosmic strings or struts located along $\theta = 0$ or $\theta = \pi$, the black-hole spacetime with twist can serve as a specific time machine because (as in the classic Taub–NUT solution) there are *closed timelike curves*.

To identify these pathological causality-violating regions we will consider simple curves in the spacetime, namely *circles around the axes of symmetry* $\theta = 0$ or $\theta = \pi$ such that only the *periodic* angular coordinate $\varphi \in [0, 2\pi C)$ changes, while the remaining coordinates t, r and θ are kept fixed. The corresponding tangent (velocity) vectors are thus proportional to the *Killing vector field* ∂_φ. Its norm is determined just by the metric coefficient $g_{\varphi\varphi}$, which for the general metric (1) has the form (122). There exist regions such that $g_{\varphi\varphi} < 0$, where the circles (orbits of the axial symmetry) are *closed timelike curves*. These pathological regions are explicitly given by the condition

$$P(\theta)(r^2 + (a+l)^2)^2 \sin^2\theta < Q(r)\Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)^2, \tag{144}$$

where the functions $P(\theta)$, $Q(r)$ are given by (4), (5). In particular, for $l = 0$, $g = 0$, $\alpha = 0$ this reduces to $r^2 + a^2 + \rho^{-2}(2mr - e^2)a^2 \sin^2\theta < 0$ which is exactly the condition (27) derived in [13] for the Kerr–Newman family of black holes.

Although this condition is difficult to be solved analytically, some general observations can be made. Clearly, the condition cannot be satisfied in the regions where $Q(r) < 0$. Naturally assuming a sufficiently small acceleration α satisfying the inequality (81), the function $P(\theta)$ is positive, while the four distinct horizons are ordered as $r_a^- < r_b^- < r_b^+ < r_a^+$, see (80). For $l < a$, the metric function Q satisfies $Q(r) > 0$ only in the regions (r_a^-, r_b^-) and (r_b^+, r_a^+), in which r is a *spatial* coordinate. The closed timelike curves can thus *only appear between the black hole horizon* $\mathcal{H}_b^\pm$ *and the corresponding acceleration horizon* $\mathcal{H}_a^\pm$, that is only in the region IV given by $r \in (r_a^-, r_b^-)$ or in the region II given by $r \in (r_b^+, r_a^+)$. On the contrary, the pathological domain can not occur in the region III inside the black hole or close to the conformal infinities $\mathcal{I}^\pm$ which are the boundaries of the dynamical regions I and V where r is temporal because $Q < 0$. This fact is explicitly seen in the exact plots shown in Fig. 6.

Moreover, it can be proven analytically that *these pathological regions with closed timelike curves do not overlap with the ergoregions* (shown in Fig. 1), although they are both in the same domains II and IV. Recall that the ergoregions are identified by the condition $g_{tt} > 0$ (together with $g_{rr} > 0$), that is

$$Q < Pa^2 \sin^2\theta, \tag{145}$$

see Eq. (84). By substituting this inequality into (144), which is the condition $g_{\varphi\varphi} < 0$ for the pathological regions, we obtain the relation

$$r^2 + (a+l)^2 < a^2 \sin^2\theta + 4al \sin^2\tfrac{1}{2}\theta, \tag{146}$$

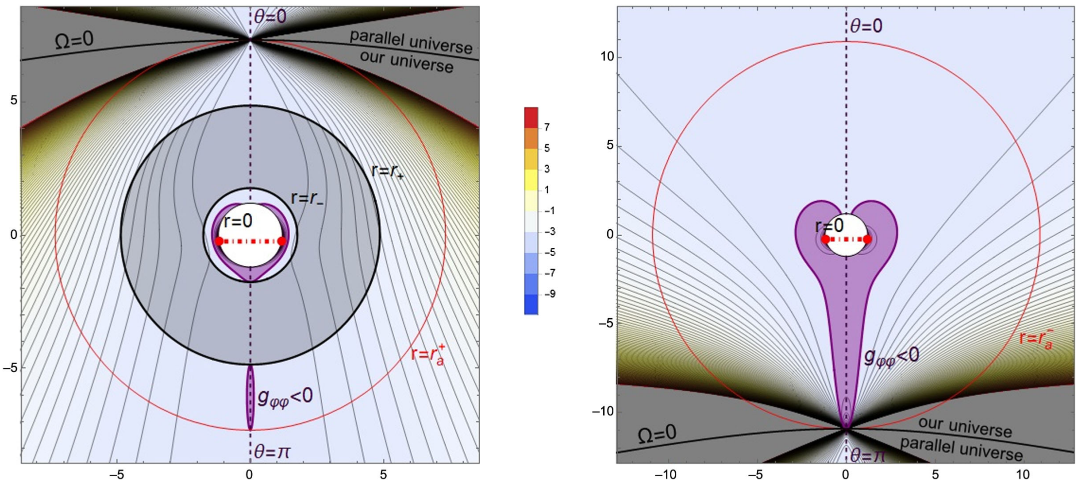

FIG. 6. Plot of the metric function $g_{\varphi\varphi}$ (122) for the accelerating black hole (1) with a regular axis $\theta = 0$ and rotating cosmic string along $\theta = \pi$. The values of $g_{\varphi\varphi}$ are visualized in quasipolar coordinates $x \equiv \sqrt{r^2 + (a+l)^2}\sin\theta$, $y \equiv \sqrt{r^2 + (a+l)^2}\cos\theta$ for $r \geq 0$ (left) and $r \leq 0$ (right). The grey annulus in the center of the left figure localizes the black hole bordered by its horizons $\mathcal{H}_b^\pm$ at r_+ and r_- ($0 < r_- < r_+$). The acceleration horizons $\mathcal{H}_a^\pm$ at r_a^+ and r_a^- (big red circles) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. The grey curves are contour lines $g_{\varphi\varphi}(r,\theta) = \text{const}$, and the values are color-coded from red (positive values) to blue (negative values); extremely large values are cut. The purple curves are the isolines $g_{\varphi\varphi} = 0$ determining the boundary of the pathological regions (144) with closed timelike curves. They occur close to the axis $\theta = \pi$ (purple regions where $g_{\varphi\varphi} < 0$). This plot is for the choice $m = 3$, $a = 1$, $l = 0.2$, $e = g = 1.6$, and $\alpha = 0.12$.

that is the same as $r^2 + a^2\cos^2\theta + 2al\cos\theta + l^2 < 0$. In view of (3), we have thus obtained

$$\rho^2 \equiv r^2 + (l + a\cos\theta)^2 < 0, \tag{147}$$

which is a contradiction.

Interestingly, there is thus no intersection of the pathological regions with the ergoregions. This is in accord with a physical intuition: the pathological regions with closed timelike curves are located here in the vicinity of the twisting axis $\theta = \pi$, while the ergoregions are concentrated mostly near the equatorial plane $\theta = \frac{\pi}{2}$ of the rotating black hole horizons.

H. Thermodynamic properties

Finally, we evaluate basic thermodynamic quantities of this class of black holes, namely the *entropy*

$$S \equiv \frac{1}{4}\mathcal{A}, \tag{148}$$

given by the horizon area $\mathcal{A}$, and the *temperature*

$$T \equiv \frac{1}{2\pi}\kappa, \tag{149}$$

given by the corresponding horizon surface gravity κ, see [18].

We obtain the *horizon area* by integrating both angular coordinates of the metric (1) for *fixed values of* t *and* $r = r_h$,

$$\mathcal{A}(r_h) = \int_0^{2\pi C}\int_0^{\pi}\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\,\mathrm{d}\theta\mathrm{d}\varphi, \tag{150}$$

where the metric functions are given by (122). Using the fact that $Q(r_h) = 0$ on any horizon, this expression simplifies to

$$\mathcal{A} = 2\pi C(r_h^2 + (a+l)^2)\int_0^{\pi}\frac{\sin\theta}{\Omega^2(r_h)}\mathrm{d}\theta. \tag{151}$$

Applying the explicit form of the conformal factor (2), an integration immediately leads to

$$\mathcal{A} = \frac{4\pi C(r_h^2 + (a+l)^2)}{(1 - \alpha\frac{a^2+al}{a^2+l^2}r_h)(1 + \alpha\frac{a^2-al}{a^2+l^2}r_h)}. \tag{152}$$

With the gauge (30), this is the same expression as Eq. (51) in [10]. In particular, for the *four distinct horizons* $\mathcal{H}$ introduced in (71)–(74) we thus obtain that

$$\text{area of } \mathcal{H}_b^+ \text{ is } \mathcal{A}_b^+ = \frac{4\pi C(r_+^2 + (a+l)^2)}{(1 - \alpha\frac{a^2+al}{a^2+l^2}r_+)(1 + \alpha\frac{a^2-al}{a^2+l^2}r_+)}, \tag{153}$$

$$\text{area of } \mathcal{H}_b^- \text{ is } \mathcal{A}_b^- = \frac{4\pi C(r_-^2 + (a+l)^2)}{(1 - \alpha\frac{a^2+al}{a^2+l^2}r_-)(1 + \alpha\frac{a^2-al}{a^2+l^2}r_-)}, \tag{154}$$

$$\text{area of } \mathcal{H}_a^+ \text{ is infinite}, \tag{155}$$

$$\text{area of } \mathcal{H}_a^- \text{ is infinite}. \tag{156}$$

The area of the acceleration horizons $\mathcal{H}_a^\pm$ is thus *unbounded*, while the black-hole horizons $\mathcal{H}_b^\pm$ have *finite values* given by (153), (154).

Interestingly, there exists a relation between these horizon areas and the conicities, namely

$$\mathcal{A}_b^+\mathcal{A}_b^- = 16\pi^2C^2C_0C_\pi(r_+^2 + (a+l)^2)(r_-^2 + (a+l)^2), \tag{157}$$

where C_0 and C_π, given by (124) and (130), are the specific conicities which regularize either the $\theta = 0$ or the $\theta = \pi$ axis, respectively. For *vanishing acceleration* α the conicities are $C = C_0 = C_\pi = 1$, so that the two horizons of the complete family of Kerr–Newman–NUT black holes (31)–(33) located at $r_\pm = m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}$ have the corresponding areas

$$\mathcal{A}_b^\pm = 4\pi(r_\pm^2 + (a+l)^2). \tag{158}$$

This simple expression reduces to the well-known formulas for Kerr–Newman black holes ($l = 0$), charged Taub–NUT ($a = 0$), Kerr ($l = 0,\ e = 0 = g$), Reissner–Nordström ($a = 0,\ l = 0$), and Schwarzschild ($a = 0,\ l = 0,\ e = 0 = g$) with a single horizon of the area $\mathcal{A}_b = 4\pi r_h^2 = 16\pi m^2$.

The *surface gravity* κ is defined as the "acceleration" of the null normal ξ^a generating the horizon at r_h via the relation $\xi_{a;b}\xi^b = \kappa\xi_a$ (so that $\kappa^2 = -\frac{1}{2}\xi_{a;b}\xi^{a;b}$). Previously in [10] we showed that for the general metric form (1) this can be expressed as

$$\kappa = \frac{1}{2}\frac{Q'(r_h)}{r_h^2 + (a+l)^2}, \tag{159}$$

where the prime denotes the derivative with respect to the coordinate r. With the new factorized form (5) of the metric function $Q(r)$ this can now be easily evaluated, yielding

$$\text{surface gravity of } \mathcal{H}_b^+ \text{ is } \kappa_b^+ = \frac{\frac{1}{2}(r_+ - r_-)(1 + \alpha \frac{a^2-al}{a^2+l^2} r_+)(1 - \alpha \frac{a^2+al}{a^2+l^2} r_+)}{r_+^2 + (a+l)^2}, \tag{160}$$

$$\text{surface gravity of } \mathcal{H}_b^- \text{ is } \kappa_b^- = -\frac{\frac{1}{2}(r_+ - r_-)(1 + \alpha \frac{a^2-al}{a^2+l^2} r_-)(1 - \alpha \frac{a^2+al}{a^2+l^2} r_-)}{r_-^2 + (a+l)^2}, \tag{161}$$

$$\text{surface gravity of } \mathcal{H}_a^+ \text{ is } \kappa_a^+ = -\alpha \frac{a^2}{a^2+l^2} \frac{(r_a^+ - r_+)(r_a^+ - r_-)}{(r_a^+)^2 + (a+l)^2}, \tag{162}$$

$$\text{surface gravity of } \mathcal{H}_a^- \text{ is } \kappa_a^- = \alpha \frac{a^2}{a^2+l^2} \frac{(r_a^- - r_+)(r_a^- - r_-)}{(r_a^-)^2 + (a+l)^2}. \tag{163}$$

Recall that the specific values r_+, r_-, r_a^+, r_a^- of the horizons position are explicitly given by (71)–(74). In particular,

$$\frac{1}{2}(r_+ - r_-) = \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}. \tag{164}$$

Notice that the surface gravities κ (and thus the corresponding temperatures T) of the black-hole horizon $\mathcal{H}_b^+$ and the acceleration horizon $\mathcal{H}_a^-$ are *positive*, while they are *negative* for the complementary horizons $\mathcal{H}_b^-$ and $\mathcal{H}_a^+$.

It is also very interesting that even in the most general case the *product of the area and the surface gravity of the black-hole horizons are the same*, and expressed simply as

$$\mathcal{A}_b^+ \kappa_b^+ = -\mathcal{A}_b^- \kappa_b^- = 2\pi C (r_+ - r_-). \tag{165}$$

Consequently, the *product of the temperature and the entropy* of the black-hole horizons $\mathcal{H}_b^\pm$ is

$$(TS)^+ = -(TS)^- = \frac{1}{2} C \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}. \tag{166}$$

Moreover, it is seen from (160) and (161) that

$$\kappa_b^+ = 0 = \kappa_b^- \quad \text{if and only if} \quad r_+ = r_- \tag{167}$$

(assuming a reasonably small acceleration α). This fully confirms that an *extremal horizon has vanishing surface gravity.* As described in Sec. IV, if the extremality condition (44) is satisfied the double-degenerate extremal horizon is located at

$$r_h = m, \tag{168}$$

and the metric function $Q(r)$ takes the form (47),

$$Q(r) = (r - m)^2 \left(1 + \alpha a \frac{a - l}{a^2 + l^2} r\right) \left(1 - \alpha a \frac{a + l}{a^2 + l^2} r\right). \tag{169}$$

Clearly, $Q(r_h) = 0$ and also $Q'(r_h) = 0$, so that $\kappa = 0$ due to (159). Such a degenerate black-hole horizon at $r = m$ in the family of accelerating extremal Kerr–Newman–NUT spacetimes has zero surface gravity, and thus zero thermodynamic temperature T.

Let us consider the special case with *vanishing acceleration* ($\alpha = 0$). In such a situation, the expressions (160)–(163) simplify:

$$\text{surface gravity of } \mathcal{H}_b^+ \text{ is } \kappa_b^+ = \frac{\sqrt{m^2 + l^2 - a^2 - e^2 - g^2}}{r_+^2 + (a+l)^2}, \tag{170}$$

$$\text{surface gravity of } \mathcal{H}_b^- \text{ is } \kappa_b^- = -\frac{\sqrt{m^2 + l^2 - a^2 - e^2 - g^2}}{r_-^2 + (a+l)^2}, \tag{171}$$

$$\text{surface gravity of } \mathcal{H}_a^\pm \text{ is } \kappa_a^\pm = 0. \tag{172}$$

(Actually, both the acceleration horizons $\mathcal{H}_a^\pm$ disappear in this limit.) Writing (170) fully explicitly, we obtain the surface gravity of the black-hole horizon $\mathcal{H}_b^+$

$$\kappa_b^+ = \frac{\sqrt{m^2 + l^2 - a^2 - e^2 - g^2}}{(m + \sqrt{m^2 + l^2 - a^2 - e^2 - g^2})^2 + (a+l)^2}. \tag{173}$$

This generalizes for the case $l \neq 0$ and $g \neq 0$ the expression

$$\kappa = \frac{\sqrt{m^2 - a^2 - e^2}}{2m(m + \sqrt{m^2 - a^2 - e^2}) - e^2}, \tag{174}$$

which is the usual surface-gravity formula for the Kerr–Newman black hole, see Eq. (12.5.4) in [18]. For the Schwarzschild black hole it reads $\kappa = 1/(4m)$.

Finally, let us remark that our explicit and fully general expressions (160)–(163) for the surface gravity κ of each of the 4 horizons at r_h agree with the results obtained *directly* from the definition $\xi_{a;b}\xi^b = \kappa\xi_a$ if the appropriate null normal generator ξ^a of the horizon is employed. In particular, the corresponding Killing vector field is

$$\xi^a \equiv \partial_t + \Omega_h \partial_\varphi, \tag{175}$$

where the constant Ω_h is the *angular velocity of the given horizon* $\mathcal{H}$. Using (138) and (122), the norm $\xi^a\xi_a$ of the Killing vector ξ^a at the horizon (where $Q = 0$) vanishes if and only if

$$\Omega_h = \frac{a}{r_h^2 + (a+l)^2}. \tag{176}$$

For the particular horizons $r_b^\pm \equiv r_\pm$ and $r_a^\pm$ given by (71)–(74) this gives the constants

$$\Omega_b^\pm = \frac{a}{r_\pm^2 + (a+l)^2}, \tag{177}$$

$$\Omega_a^\pm = \frac{\alpha^2 a^3 (a \pm l)^2}{(a^2+l^2)^2 + \alpha^2 a^2 (a+l)^2 (a \pm l)^2}. \tag{178}$$

It can be seen that for vanishing Kerr-like rotation ($a = 0$) the angular velocities of all four horizons become zero, whereas for vanishing NUT parameter ($l = 0$) they all remain nonzero,

$$\Omega_b^\pm = \frac{a}{r_\pm^2 + a^2}, \qquad \Omega_a^\pm = \frac{\alpha^2 a}{1 + \alpha^2 a^2}. \tag{179}$$

I. Concluding summary

In this work we presented a new metric form (1)–(7) of the remarkable family of exact black holes of algebraic type D, initially found by Debever (1971) and by Plebański and Demiański (1976). Moreover, we demonstrated that this improved metric representation has many advantages which simplify the investigation of its geometrical and physical properties. In particular:

(i) In Sec. II we started with a convenient Griffiths–Podolský (2005, 2006) form of this class of spacetimes, but we further improved it. By introducing a modified set of the mass and charge parameters m, e, g, applying a special conformal rescaling S, and choosing a useful gauge of the twist parameter ω, we obtained an explicit compact form of the metric.

(ii) The metric functions (2)–(5) are very simple, depending only on the radial coordinate r and the angular coordinate θ. Moreover, the key functions $P(\theta)$ and $Q(r)$ are factorized. They explicitly localize the axes of symmetry and the horizons, respectively.

(iii) The metric depends on six parameters m, a, l, α, e, g with direct physical meaning, namely they represent the mass, Kerr-like rotation, NUT parameter, acceleration, electric, and magnetic charges of the black hole, respectively.

(iv) Interestingly, the new metric (1) depends on the parameters a, l, α directly, while the dependence on the remaining three parameters m, e, g is encoded in the two constants r_+ and r_- defined by (6) and (7). In fact, these expressions localize the two black-hole horizons, and they only appear in the factorized metric functions P and Q.

(v) Very nice feature of the new metric form (1)–(5) is that any of its six physical parameters can be independently set to zero, and this can be done in any order. In this way, specific subclasses of type D black holes are easily obtained.

(vi) This property is demonstrated in Sec. III where the general family of accelerating, charged, rotating and NUTed black holes naturally reduce to its large subclasses with five physical parameters. These are the Kerr–Newman–NUT black holes without acceleration ($\alpha = 0$), accelerating Kerr–Newman black holes without NUT ($l = 0$), charged Taub–NUT black holes without rotation ($a = 0$), and accelerating Kerr–NUT black holes without electric or magnetic charges ($e = 0$ or $g = 0$).

(vii) All the metric functions (2)–(5) depend on the acceleration α only via the product αa. Therefore, by setting the Kerr-like rotation a to zero, the new metric (1) becomes independent of α, and simplifies directly to charged Taub–NUT black holes. This explicitly confirms the previous observation made by Griffiths and Podolský that there is no accelerating NUT black hole in the Plebański–Demiański family of type D spacetimes. Quite surprisingly, such a solution for accelerating nonrotating black hole with purely NUT parameter exists [8,12], but it is of distinct algebraic type I.

(viii) The simplest subcases of our general metric (1) with just the mass m and one additional physical parameter reveal the famous black holes, namely the Schwarzschild, Reissner–Nordström, Kerr, Taub–NUT or the C-metric solutions, all in their standard coordinate forms.

(ix) As shown in Sec. IV, the improved metric (1) naturally contains also extreme black holes with double-degenerate horizons ($r_+ = r_-$) located at $r = m$, whenever $m^2 + l^2 = a^2 + e^2 + g^2$. Such a family of extremal accelerating Kerr–Newman–NUT black holes also admits various subclasses, obtained by setting any of the parameters α, l, a, e, g to zero. In fact, they represent the complete class of extremal isolated horizons with axial symmetry [10].

(x) The hyperextreme cases, when the parameters satisfy the relation $m^2 + l^2 < a^2 + e^2 + g^2$, represent exact spacetimes with an accelerated naked singularity. The metric functions P, Q are not (fully) factorizable, and take the form (51), (52). There are thus only two acceleration horizons, which are absent when $\alpha a = 0$.

The new convenient metric (1) considerably simplifies the investigation of various properties of this large family of black holes, as demonstrated in the subsequent sections of our work, namely:

(i) First, in Sec. V we evaluated the Weyl and Ricci tensors of (1), expressed as the Newman–Penrose scalars in the natural tetrad (54) adapted to the double-degenerate principal null directions. The only such scalars are Ψ_2 and Φ_{11}, confirming the type D algebraic structure of the gravitational field, aligned with the non-null electromagnetic field (67)–(69).

(ii) Their explicit form (55) and (56) reveals that generic black-hole spacetimes are asymptotically flat at $\Omega = 0$. For vanishing acceleration α, the spacetimes (1) become asymptotically flat for large values of the radial coordinate $|r|$ (except along the axes of symmetry $\theta = 0$ and $\theta = \pi$ if the cosmic strings or struts are present).

(iii) Both the double-degenerate principal null directions are expanding. They are twisting if and only if $a = 0 = l$. On the horizons, the expansion and twist always vanish.

(iv) In general, there are four distinct horizons identified in Sec. V A as the roots of the metric function $Q(r)$. Since its form (70) is fully factorized, the corresponding positions are simply expressed in terms of the physical parameters as (71)–(74). There is a pair of black-hole horizons $\mathcal{H}_b^\pm$ at $r_b^\pm \equiv r_\pm = m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}$, and a pair of acceleration horizons $\mathcal{H}_a^\pm$ at $r_a^\pm \equiv \pm\alpha^{-1}(a^2 + l^2)/(a^2 \pm al)$, which simplifies to $r_a^\pm \equiv \pm\alpha^{-1}$ when $l = 0$.

(v) Interestingly, these positions of the black-hole horizons are independent of the acceleration α, while the acceleration horizons do not depend on the mass m and the charges e, g.

(vi) For sufficiently small acceleration α such that $\alpha r_+ < (a^2 + l^2)/(a^2 + al)$, with $0 \le l < a$, the four horizons are ordered as $r_a^- < r_b^- < r_b^+ < r_a^+$, see (81).

(vii) Whenever the Kerr-like rotation parameter a is nonzero, each of these four horizons is accompanied by the corresponding ergoregion, see Sec. V B. It "touches' the horizon at its poles, extending from the horizon near the equatorial region. This is shown in Fig. 1. For the Kerr–Newman–NUT black holes without acceleration, the ergoregions are bounded by the surface $r_{e\pm}(\theta) = m \pm \sqrt{m^2 + l^2 - e^2 - g^2 - a^2\cos^2\theta}$.

(viii) Using the Weyl scalar Ψ_2 and also the Kretschmann scalar $\mathcal{K} \equiv R_{abcd}R^{abcd}$, in Subsec. V C we clarified the presence and the structure of the curvature singularities. Such a singularity is present at $r = 0$, but only if $l + a\cos\theta = 0$ which requires $|l| \le |a|$. There is thus no curvature singularity in the black-hole spacetimes with large NUT parameter $|l| > |a| \ge 0$.

(ix) For $0 \le |l| \le |a|$ the curvature singularity is present at $r = 0$, but only in the section with special value of the angular coordinate θ such that $\cos\theta = -l/a$. Various possibilities are summarized in (89).

(x) This singularity has a ring structure which can be crossed from the asymptotically flat region $r > 0$ to the distinct asymptotically flat region $r < 0$, as schematically shown in Fig. 2 and Fig. 3. Only in the section $\cos\theta = -l/a$ (or for any value of θ if $l = 0 = a$) we have to restrict the range of r to two separate domains $r > 0$ and $r < 0$.

(xi) To complete our understanding of the global causal structure of the entire family of black-hole spacetimes (1), in Sec. V D we introduced the retarded and advanced null coordinates in which the corresponding metric forms (93) and (96) have no coordinate singularities at the horizons.

(xii) Then we explicitly constructed the corresponding Kruskal–Szekeres-type coordinates which enabled us to perform the maximal analytic extension across all the horizons. It revealed an infinite number of time-dependent regions (of type I, III, V) and stationary regions (of type II, IV) which are separated by the black hole and acceleration horizons $\mathcal{H}_b^\pm$ and $\mathcal{H}_a^\pm$.

(xiii) The complicated global structure of this large family of spacetimes is visualized in the Penrose diagrams obtained by a suitable conformal compactification, drawn in Fig. 4 and Fig. 5. The complete manifold contains an infinite number of black holes in various asymptotically flat universes identified by distinct (future and past) conformal infinities $\mathcal{I}$—unless a special topological identification is made.

(xiv) In Sec. V E we clarified that the physical source of acceleration of the black holes is the tension (or compression) in the rotating cosmic strings (or struts) located along the two axes of axial symmetry at $\theta = 0$ and $\theta = \pi$. Such strings or struts are related to the deficit or excess angles which introduce topological defects along these axes (while the curvature remains finite).

(xv) In general, there are strings/struts along both the axes, but one of the axis can be made fully regular by a suitable choice of the conicity parameter C in the range $\varphi \in [0, 2\pi C)$. The first axis $\theta = 0$ is regular in the metric form (1) with the choice (124), whereas the second axis $\theta = \pi$ is regular in the form (126) with the choice (130). In the first case, there is a cosmic strut along $\theta = \pi$ with the excess angle (133), while in the second case there is a cosmic string along $\theta = 0$ with the deficit angle (131). For vanishing acceleration, both the axes can be made regular simultaneously (except for a possible NUT-like pathology).

(xvi) In addition to the deficit/excess angles, these cosmic strings/struts located along the axes of symmetry are characterized by their rotation parameter ω (angular velocity). We demonstrated in Sec. V F that their values are directly related to the NUT parameter l, see expressions (140) and (143).

(xvii) There is always a constant difference $\Delta\omega = 4l$ between the angular velocities of the two rotating cosmic strings or struts. If and only if $l = 0$, both the axes are nontwisting.

(xviii) In the neighborhood of these rotating strings/struts there occur pathological regions with closed timelike curves. As shown in Sec. V G, these regions are generally given by the condition (144). They appear close to the rotating strings/struts, but only between the black hole horizon $\mathcal{H}_b^\pm$ and the corresponding acceleration horizon $\mathcal{H}_a^\pm$ (that is in the domains of type II and IV), see Fig. 6.

(xix) Although the pathological regions with closed time-like curves are located in the same domains as the ergoregions, they do not overlap with each other.

(xx) The convenient metric form (1) with straightforward identification of the horizons is also suitable for an easy investigation of the black hole thermodynamics. Indeed, in Sec. V H we explicitly evaluated the area of the four horizons (153)–(156), their surface gravity (160)–(163), and their angular velocity (177)–(178).

(xxi) These expressions generalize the usual formulas for the Kerr–Newman family to black holes with acceleration α and NUT parameter l. They reveal interesting relations for the horizons temperature and entropy, for example $(TS)^+ = -(TS)^- = \frac{1}{2}C\sqrt{m^2 + l^2 - a^2 - e^2 - g^2}$.

To conclude, the simple new metric form (1)–(7) has clear advantages. We hope that it will be employed for various studies and applications of this interesting class of accelerating and rotating black holes which charges and the NUT parameter.

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[1] J. F. Plebański and M. Demiański, Rotating, charged and uniformly accelerating mass in general relativity, Ann. Phys. (N.Y.) **98**, 98 (1976).
[2] R. Debever, On type D expanding solutions of Einstein–Maxwell equations, Bull. Soc. Math. Belg. **23**, 360 (1971).
[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[4] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).
[5] J. B. Griffiths and J. Podolský, Accelerating and rotating black holes, Classical Quantum Gravity **22**, 3467 (2005).
[6] J. Podolský and J. B. Griffiths, Accelerating Kerr–Newman black holes in (anti-)de Sitter space-time, Phys. Rev. D **73**, 044018 (2006).
[7] J. B. Griffiths and J. Podolský, A new look at the Plebański–Demiański family of solutions, Int. J. Mod. Phys. D **15**, 335 (2006).
[8] J. Podolský and A. Vrátný, Accelerating NUT black holes, Phys. Rev. D **102**, 084024 (2020).
[9] A. Vrátný, Spacetimes with accelerating sources, Master thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2018.
[10] D. Matejov and J. Podolský, Uniqueness of extremal isolated horizons and their identification with horizons of all type D black holes, Classical Quantum Gravity **38**, 135032 (2021).
[11] K. Hong and E. Teo, A new form of the rotating *C*-metric, Classical Quantum Gravity **22**, 109 (2005).
[12] B. Chng, R. Mann, and C. Stelea, Accelerating Taub-NUT and Eguchi–Hanson solitons in four dimensions, Phys. Rev. D **74**, 084031 (2006).
[13] B. Carter, Global structure of the Kerr family of gravitational fields, Phys. Rev. **174**, 1559 (1968).
[14] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[15] R. H. Boyer and R. W. Lindquist, Maximal analytic extension of the Kerr metric, J. Math. Phys. (N.Y.) **8**, 265 (1967).
[16] J. B. Griffiths, P. Krtouš, and J. Podolský, Interpreting the *C*-metric, Classical Quantum Gravity **23**, 6745 (2006).
[17] J. B. Griffiths and J. Podolský, Global aspects of accelerating and rotating black hole space-times, Classical Quantum Gravity **23**, 555 (2006).
[18] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago and London, 1984).

3. New improved form of black holes of type D with Λ

This last chapter is based on the paper *New form of all black holes of type D with a cosmological constant* [46] by Podolský and Vrátný, published as a preprint in December 2022, and accepted to the journal Physical Review D on March 9, 2023. The version shown here is the proofreading of the accepted manuscript.

In this paper, we generalized our previous work [45], in which we studied the Plebański–Demiański class of black hole solutions for zero cosmological constant (see the previous Chapter 2). Recently, we succeeded in generalizing our new metric form (2.5)–(2.9) by including also a cosmological constant $\Lambda \neq 0$, still preserving the (partial) factorization of the metric functions.

We also provided a thorough analysis of this solution. Among other things we computed and studied its special and extreme cases, localized the horizons, evaluated the curvature tensors and scalars, visualized ergoregions and pathological regions around the axes, performed the Kruskalization followed by the construction of the conformal diagram, and started to analyze the thermodynamics of these black holes.

3.1 Derivation of the new metric form

As in the previous Chapter 2, we started with a set of changes on the metric form (II.28) and its metric functions $\mathcal{P}(\theta)$ (II.31) and $\mathcal{Q}(r)$ (II.32). We applied the reparametrization

$$\begin{aligned} m &\equiv S\,\tilde{m} - \alpha\frac{l}{\omega}(a^2 - l^2 + e^2 + g^2)\,, \\ e^2 &\equiv S\,\tilde{e}^2\,, \\ g^2 &\equiv S\,\tilde{g}^2\,, \\ \Lambda &\equiv S\,\tilde{\Lambda}\,, \end{aligned} \tag{3.1}$$

where the prefactor S is defined as

$$S \equiv \frac{a^2 - l^2}{\omega^2 k}\,. \tag{3.2}$$

Notice, that now we also rescale the cosmological constant $\tilde{\Lambda}$ (compare the equations (2.1)). This can raise some questions since the cosmological constant enters the Einstein field equations (EFE). However, at the end we will rescale the whole metric $\mathrm{d}s^2 \to S\,\mathrm{d}s^2$, and the field equations *require* the corresponding rescaling of $\tilde{\Lambda}$ to Λ.

Indeed, using (3.1), the metric functions $\mathcal{P}(\theta)$ and $\mathcal{Q}(r)$, changes as

$$\mathcal{Q}(r) = S^{-1}\,Q(r)\,, \qquad \mathcal{P}(\theta) = S^{-1}\,P(\theta)\,, \tag{3.3}$$

and the coordinate transformations $t \to S\,t$ and $\varphi \to S\,\varphi$, enabled us to pull out S completely from the whole metric as a specific constant conformal factor (see Sec. P3.II for the full derivation).

We have also set the twist parameter to

$$\omega \equiv \frac{a^2+l^2}{a}\,. \tag{3.4}$$

Thus we derived *a new representation for the complete family of type D black holes, including any cosmological constant*:

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg(&-\frac{Q}{\rho^2}\Big[\mathrm{d}t-\big(a\sin^2\theta+4l\sin^2\tfrac{1}{2}\theta\big)\mathrm{d}\varphi\Big]^2+\frac{\rho^2}{Q}\,\mathrm{d}r^2 \\ &+\frac{\rho^2}{P}\,\mathrm{d}\theta^2+\frac{P}{\rho^2}\,\sin^2\theta\,\Big[a\,\mathrm{d}t-\big(r^2+(a+l)^2\big)\,\mathrm{d}\varphi\Big]^2\bigg), \end{aligned} \tag{3.5}$$

where

$$\Omega = 1-\frac{\alpha\,a}{a^2+l^2}\,r\,(l+a\cos\theta)\,, \tag{3.6}$$

$$\rho^2 = r^2+(l+a\cos\theta)^2\,, \tag{3.7}$$

$$\begin{aligned} P(\theta) = 1 &- 2\Big(\frac{\alpha\,a}{a^2+l^2}\,m-\frac{\Lambda}{3}\,l\Big)(l+a\,\cos\theta) \\ &+\Big(\frac{\alpha^2a^2}{(a^2+l^2)^2}(a^2-l^2+e^2+g^2)+\frac{\Lambda}{3}\Big)(l+a\,\cos\theta)^2\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} Q(r) = &\Big[r^2-2m\,r+(a^2-l^2+e^2+g^2)\Big]\Big(1+\alpha\,a\,\frac{a-l}{a^2+l^2}\,r\Big)\Big(1-\alpha\,a\,\frac{a+l}{a^2+l^2}\,r\Big) \\ &-\frac{\Lambda}{3}\,r^2\Big[r^2+2\alpha\,a\,l\,\frac{a^2-l^2}{a^2+l^2}\,r+(a^2+3l^2)\Big]\,, \end{aligned} \tag{3.9}$$

with the physical parameters

m	.....	mass parameter ,
a	.....	Kerr-like rotation ,
l	.....	NUT parameter ,
e	.....	electric charge ,
g	.....	magnetic charge ,
α	.....	acceleration ,
Λ	.....	cosmological constant .

For $\Lambda = 0$ and $m^2+a^2 \geq a^2-l^2+e^2+g^2$ *both* the metric functions $P(\theta)$ and $Q(r)$ can be fully and nicely factorized (see equations (2.8) and (2.9)). For $\Lambda \neq 0$ this is not generally possible. Nevertheless, we were able to factorize (at least) the metric function $P(\theta)$, and appropriately simplify the function $Q(r)$. By defining the convenient parameters

$$r_{\Lambda+} \equiv \mu+\sqrt{\mu^2+l^2-a^2-e^2-g^2-\lambda}\,, \tag{3.10}$$

$$r_{\Lambda-} \equiv \mu-\sqrt{\mu^2+l^2-a^2-e^2-g^2-\lambda}\,, \tag{3.11}$$

where

$$\mu \equiv m-\frac{\Lambda}{3}\,l\,\frac{a^2+l^2}{\alpha\,a}\,, \qquad \lambda \equiv \frac{\Lambda}{3}\,\frac{(a^2+l^2)^2}{\alpha^2a^2}\,, \tag{3.12}$$

the metric functions are simplified to

$$P(\theta) = \Big(1 - \frac{\alpha\, a}{a^2+l^2}\, r_{\Lambda+}\,(l + a\,\cos\theta)\Big)\Big(1 - \frac{\alpha\, a}{a^2+l^2}\, r_{\Lambda-}\,(l + a\,\cos\theta)\Big), \tag{3.13}$$

$$Q(r) = \Big(r - r_{\Lambda+}\Big)\Big(r - r_{\Lambda-}\Big)\Big(1 + \alpha\, a\, \frac{a-l}{a^2+l^2}\, r\Big)\Big(1 - \alpha\, a\, \frac{a+l}{a^2+l^2}\, r\Big) \tag{3.14}$$
$$\qquad - \frac{\Lambda}{3}\left[r^4 + \frac{(a^2+l^2)^2}{\alpha^2 a^2}\right].$$

This is possible only for $\mu^2 + l^2 \geq a^2 + e^2 + g^2 + \lambda$, in which case the expressions (3.10), (3.11) yield real constants. Notice also the similarity between these two metric functions and the relation (II.8) already pointed out by Griffiths and Podolský in 2006 [40].

Similarly as in Chapter 2, the advantage of this new metric representation is that we can easily gain the standard forms of the most important black holes by simply switching off the appropriate parameters, namely

- $\Lambda = 0$ **: Black holes in flat universe** (P3.IV.A) ,
- $\alpha = 0$ **: Kerr–Newman–NUT–(anti-)de Sitter black holes** (P3.IV.B) ,
- $l = 0$ **: Accelerating Kerr–Newman–(anti-)de Sitter black holes** (P3.IV.C) ,
- $a = 0$ **: Charged Taub–NUT–(anti-)de Sitter black holes** (P3.IV.D) ,
- $e = 0 = g$ **: Uncharged accelerating Kerr–NUT–(anti-)de Sitter black holes** (P3.IV.E) .

Notice again, how easily can we now perform the transition to the *charged Taub–NUT–(anti-)de Sitter black hole* for a vanishing Kerr-like rotation ($a = 0$). This confirms that no solution representing (possibly charged) accelerating Taub–NUT with a (possibly non-zero) cosmological constant exists in the complete family of type D black holes.

3.2 Physical analysis of the new metric form

The new representation of the full family of exact type D black holes (3.5)–(3.9) can be used for a thorough physical analysis (see section P3.V). The procedure we have chosen is essentially similar to the one we used in Chapter 2. We defined a natural null tetrad (2.12):

$$\begin{aligned}
\mathbf{k} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}\Big(\Big(r^2 + (a+l)^2\Big)\,\partial_t + a\,\partial_\varphi\Big) + \sqrt{Q}\,\partial_r\right],\\
\mathbf{l} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}\Big(\Big(r^2 + (a+l)^2\Big)\,\partial_t + a\,\partial_\varphi\Big) - \sqrt{Q}\,\partial_r\right],\\
\mathbf{m} &= \frac{1}{\sqrt{2}}\,\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{P}\sin\theta}\Big(\partial_\varphi + \Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)\,\partial_t\Big) + \mathrm{i}\,\sqrt{P}\,\partial_\theta\right].
\end{aligned} \tag{3.15}$$

The only nontrivial NP scalars corresponding to the Weyl tensor and the Ricci tensor are

$$\Psi_2 = \frac{\Omega^3}{\big[r+\mathrm{i}\,(l+a\cos\theta)\big]^3}\bigg[-(m+\mathrm{i}\,l)\Big(1-\mathrm{i}\,\alpha\,a\,\frac{a^2-l^2}{a^2+l^2}\Big)-\mathrm{i}\,\frac{\Lambda}{3}\,l\,(a^2-l^2)$$
$$+\frac{(e^2+g^2)}{r-\mathrm{i}\,(l+a\cos\theta)}\Big(1+\frac{\alpha\,a}{a^2+l^2}\big[a\,r\cos\theta+\mathrm{i}\,l\,(l+a\cos\theta)\big]\Big)\bigg]\,, \qquad (3.16)$$

$$\Phi_{11} = \tfrac{1}{2}(e^2+g^2)\,\frac{\Omega^4}{\rho^4}\,, \qquad (3.17)$$

while the Ricci scalar now equals to

$$R = 4\,\Lambda\,. \qquad (3.18)$$

For $\Lambda = 0$, we simply recover equations (2.13)–(2.14) and the vanishing Ricci scalar.

Let us also recall the relation between the Kretschmann scalar $\mathcal{K} \equiv R_{abcd}\,R^{abcd}$ and its related Weyl scalar $\mathcal{C} \equiv C_{abcd}\,C^{abcd}$ derived in Sec. I.4. Using it, we computed and explicitly expressed these scalars for the metric (3.5)–(3.9), see equations (98), (99).

The *4-potential* of the charged solution is

$$\mathbf{A} = -\sqrt{e^2+g^2}\,\frac{r}{\rho^2}\,\Big[\mathrm{d}t-(a\sin^2\theta+4l\sin^2\tfrac{1}{2}\theta)\,\mathrm{d}\varphi\Big]\,, \qquad (3.19)$$

and the corresponding Newman–Penrose scalars have only the component

$$\Phi_1 \equiv \tfrac{1}{2}F_{ab}(k^a l^b+\bar{m}^a m^b) = \frac{\sqrt{e^2+g^2}\;\Omega^2}{\Big(r+\mathrm{i}\,(l+a\cos\theta)\Big)^2}\,. \qquad (3.20)$$

Since the only nontrivial NP Weyl scalar is Ψ_2, *both vectors* $\mathbf{k}$ *and* $\mathbf{l}$ *are the principal null directions.* Both are double-degenerate, yielding that the *metric is of algebraic type D.* The electromagnetic field for $e \neq 0 \neq g$ is non-null and double-aligned.

The *spin coefficients* for the null tetrad (3.15) are the same as (2.15). Both $\mathbf{k}$, $\mathbf{l}$ (3.15) are *geodetic* ($\kappa = 0 = \nu$) and *shear-free* ($\sigma = 0 = \lambda$), with *expansion* Θ and *twist* ω, namely

$$\Theta = \frac{\sqrt{Q}}{\sqrt{2}\,\rho^3}\,\Big(r+\frac{\alpha\,a}{a^2+l^2}\,(l+a\cos\theta)^3\Big)\,, \qquad (3.21)$$

$$\omega = -\frac{\Omega\sqrt{Q}}{\sqrt{2}\,\rho^3}\,(l+a\cos\theta)\,. \qquad (3.22)$$

Notice that all the equations (3.19)–(3.22) depend on the cosmological constant Λ only implicitly via the metric functions $Q(r)$, $P(\theta)$.

From (3.22) we see directly that the black hole is everywhere non-twisting if (and only if) $a = 0 = l$. The *conformally flat regions*, where Ψ_2 (3.16) vanishes, correspond to $\Omega = 0$. This is *the conformal infinity.* The curvature singularity localized at the region where Ψ_2 (3.16) diverges occurs if and only if $\rho^2 = 0$. This can happen only when

$$r = 0 \qquad \text{and at the same time} \qquad l + a\cos\theta = 0\,. \qquad (3.23)$$

It means that we obtained the very same result as for the asymptotically flat black holes (2.31).

3.2.1 Horizons

One of the key topics in our latest publication [46] was the calculation and general classification of possible horizons. They are fully determined by the quartic equation $Q(r) = 0$. Their explicit calculation is however quite cumbersome and does not provide simple answers on a desired analysis. We had to proceed in a systematic way.

First of all, we defined the polynomial coefficients of $Q(r)$ in the following way:

$$Q(r) = q_4\, r^4 + q_3\, r^3 + q_2\, r^2 + q_1\, r + q_0\,, \tag{3.24}$$

where

$$\begin{aligned}
q_4 =& -\alpha^2 a^2\, \frac{a^2 - l^2}{(a^2 + l^2)^2} - \frac{\Lambda}{3}\,,\\
q_3 =& \, 2\,\alpha\, a \left[\alpha\, a\, m\, \frac{a^2 - l^2}{(a^2 + l^2)^2} - \frac{l}{a^2 + l^2} - l\, \frac{a^2 - l^2}{a^2 + l^2}\, \frac{\Lambda}{3} \right],\\
q_2 =& \, 1 + 4\,\alpha\, a\, m\, \frac{l}{a^2 + l^2} - \alpha^2 a^2\, \frac{a^2 - l^2}{(a^2 + l^2)^2}\, (a^2 - l^2 + e^2 + g^2) - (a^2 + 3l^2)\, \frac{\Lambda}{3}\,,\\
q_1 =& -2m - 2\,\alpha\, a\, \frac{l}{a^2 + l^2}\, (a^2 - l^2 + e^2 + g^2)\,,\\
q_0 =& \, a^2 - l^2 + e^2 + g^2\,.
\end{aligned} \tag{3.25}$$

Then we employed the analysis presented in [67]. More specifically, with the parameters

$$N \equiv 8q_4 q_2 - 3q_3^2\,, \tag{3.26}$$

$$R \equiv 8q_4^2 q_1 - 4q_4 q_3 q_2 + q_3^3\,, \tag{3.27}$$

$$S \equiv 256 q_4^3 q_0 - 64 q_4^2 q_3 q_1 + 16 q_4 q_3^2 q_2 - 3q_3^4\,, \tag{3.28}$$

$$\begin{aligned}
\Delta \equiv\, & 256 q_4^3 q_0^3 - 192 q_4^2 q_3 q_1 q_0^2 - 128 q_4^2 q_2^2 q_0^2 + 144 q_4^2 q_2 q_1^2 q_0\\
& -27 q_4^2 q_1^4 + 144 q_4 q_3^2 q_2 q_0^2 - 6 q_4 q_3^2 q_1^2 q_0 - 80 q_4 q_3 q_2^2 q_1 q_0\\
& +18 q_4 q_3 q_2 q_1^3 + 16 q_4 q_2^4 q_0 - 4 q_4 q_2^3 q_1^2 - 27 q_3^4 q_0^2\\
& +18 q_3^3 q_2 q_1 q_0 - 4 q_3^3 q_1^3 - 4 q_3^2 q_2^3 q_0 + q_3^2 q_2^2 q_1^2\,.
\end{aligned} \tag{3.29}$$

We concluded that the following possibilities arise:

For $\Delta > 0$:
The metric function $Q(r)$ has either *none*, or *four distinct real roots.* That depends on:

- **If** $N < 0$ **and** $N^2 > S$: *all four roots are real and distinct.*
- **If** $N < 0$ **and** $N^2 < S$: there exist *two pairs of complex conjugate non-real roots.*
- **If** $N \geq 0$**:** there are also *two pairs of complex conjugate non-real roots.*

For $\Delta < 0$:
The metric function $Q(r)$ has *two distinct real roots* and *two complex conjugate non-real roots.*

For $\Delta = 0$:
The only case when the metric function $Q(r)$ has at least one *multiple root*. Here are the different cases that can occur:

• **If $N < 0$ together with**:

  • $N^2 < S$: there exists *one real double root* and *two complex conjugate roots*.

  • $N^2 = S$: there are *two distinct real double roots*.

  • $N^2 > S$ **and** $N^2 > -3S$: there occurs *one real double root* and *two distinct simple real roots*.

  • $N^2 = -3S$: there is *one real triple root* and *one distinct simple real root*.

• **If $N > 0$ together with**:

  • $S = 0$: there exists *one real double root* and *two complex conjugate roots*.

  • $S > 0$ **and** $R \neq 0$: there is also *one real double root* and *two complex conjugate roots*.

  • $S = N^2$ **and** $R = 0$: there are only *two complex conjugate double roots*.

• **If $N = 0$ together with**:

  • $S > 0$: there is *one real double root* and *two complex conjugate roots*.

  • $S = 0$ (implying $R = 0$): there is *one real quadruple root at* $r_h = -\frac{q_3}{4\,q_4}$.

Our main interest lies in the most physically relevant case of *four distinct roots*, that is when

$$\Delta > 0 \qquad \text{and} \qquad N < 0 \qquad \text{and} \qquad N^2 > S\,. \tag{3.30}$$

Under these special conditions, the metric function $Q(r)$ is fully factorized with four distinct horizons r_b^+, r_b^-, r_c^+, r_c^-. More specifically, the metric function reads

$$Q(r) = -\mathcal{N}\left(r - r_b^+\right)\left(r - r_b^-\right)\left(r - r_c^+\right)\left(r - r_c^-\right), \tag{3.31}$$

where

$$\mathcal{N} = \alpha^2\, a^2\, \frac{a^2 - l^2}{(a^2 + l^2)^2} + \frac{\Lambda}{3}\,, \tag{3.32}$$

while the roots localize the horizons $\mathcal{H}_h^\pm$, namely

$$\mathcal{H}_b^+ \text{ at } r_b^+ \quad \text{is the } \textbf{outer black-hole horizon}, \tag{3.33}$$

$$\mathcal{H}_b^- \text{ at } r_b^- \quad \text{is the } \textbf{inner black-hole horizon}, \tag{3.34}$$

$$\mathcal{H}_c^+ \text{ at } r_c^+ \quad \text{is the } \textbf{outer cosmo-acceleration horizon}, \tag{3.35}$$

$$\mathcal{H}_c^- \text{ at } r_c^- \quad \text{is the } \textbf{inner cosmo-acceleration horizon}. \tag{3.36}$$

We presume a natural ordering of these horizons as

$$r_c^- < r_b^- < r_b^+ < r_c^+\,. \tag{3.37}$$

We were able even to explicitly find these roots. Their complexity is however big (see equations (140)–(148) of the attached Paper 3).

3.2.2 Ergoregions

Similarly as in the case $\Lambda = 0$ with $a \neq 0$, the occurrence of *ergoregions* can be expected in the vicinity of the horizons. More precisely, their presence will appear for

$$g_{tt} = \frac{1}{\Omega^2 \rho^2} \left(P\, a^2 \sin^2\theta - Q \right) > 0 \, . \tag{3.38}$$

The condition (3.38) depends on the cosmological constant via the metric functions P and Q. We presented the visualization of these areas for different values of a and Λ, see Fig. 3.1.

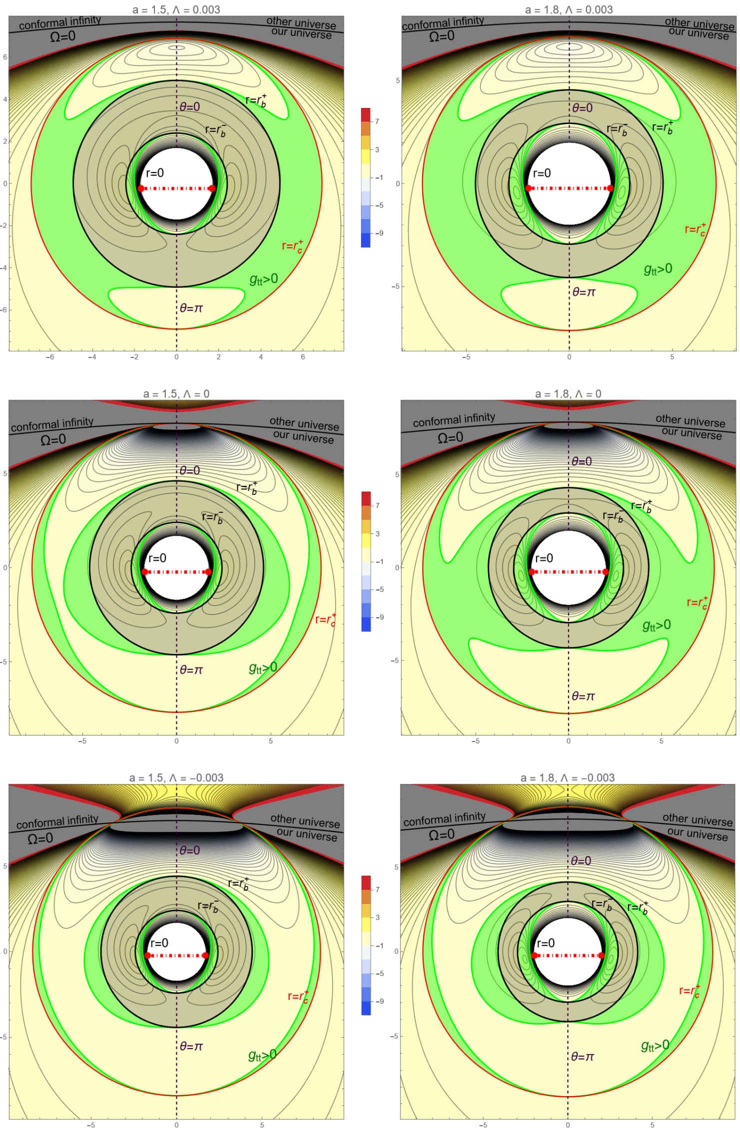

Figure 3.1: Plot of the metric function g_{tt} given by (3.38) in quasi-polar coordinates $\mathrm{x} \equiv \sqrt{r^2 + (a+l)^2}\,\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + (a+l)^2}\,\cos\theta$ for $r \geq 0$. Ergoregions are localized within the green areas between the gray annulus in the center which localizes the black hole horizons $\mathcal{H}_b^{\pm}$ at r_b^+ and r_b^-. The cosmo-acceleration horizon $\mathcal{H}_c^+$ at r_c^+ (big red circle) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also illustrated. For more details, see the attached Paper 3.

Notice that the ergoregions occur not only in the vicinity of the black hole horizons but also near the cosmo-acceleration horizons as well. Not only does the area of ergoregions increase for a larger Kerr-like rotation a, as it was in the asymptotically flat universe (see Fig. 2.1), but the cosmological constant also affects the size of these regions. Moreover, for a sufficiently large rotation a or Λ, these ergoregions around the different horizons can even *merge near the equatorial plane*.

3.2.3 Global structure and the conformal diagrams

Similarly, as in the case of $\Lambda = 0$ summarized in the previous Chapter 2 (see Sec. 2.2.4), we were able to explicitly construct the compactified coordinates $\{\tilde{T}_h^{\pm}, \tilde{R}_h^{\pm}\}$ (eq. (175) and (176) of the attached paper).

There are 5 types of regions bounded by the black hole horizons $\mathcal{H}_b^{\pm}$ and the cosmo-acceleration horizons $\mathcal{H}_c^{\pm}$. They are characterized by two integers (i, j), namely

Region	**Description**	**Specification of** (i, j)
I:	asymptotic time-dependent domain between $\mathcal{H}_c^+$ and $\mathcal{I}^+$	$(n - 2m + 1, n + 2m - 1)$
II:	stationary region between $\mathcal{H}_b^+$ and $\mathcal{H}_c^+$	$(2n - m, 2n + m - 1)$
III:	time-dependent domain between the black-hole horizons	$(n - 2m, n + 2m)$
IV:	stationary region between $\mathcal{H}_c^-$ and $\mathcal{H}_b^-$	$(2n - m + 1, 2n + m)$
V:	asymptotic time-dependent domain between $\mathcal{I}^-$ and $\mathcal{H}_c^-$	$(n - 2m + 1, n + 2m - 1)$

They form the conformal Penrose diagram, see the Fig. 3.2 (or the original Fig. 2 of the Paper 3 for more details).

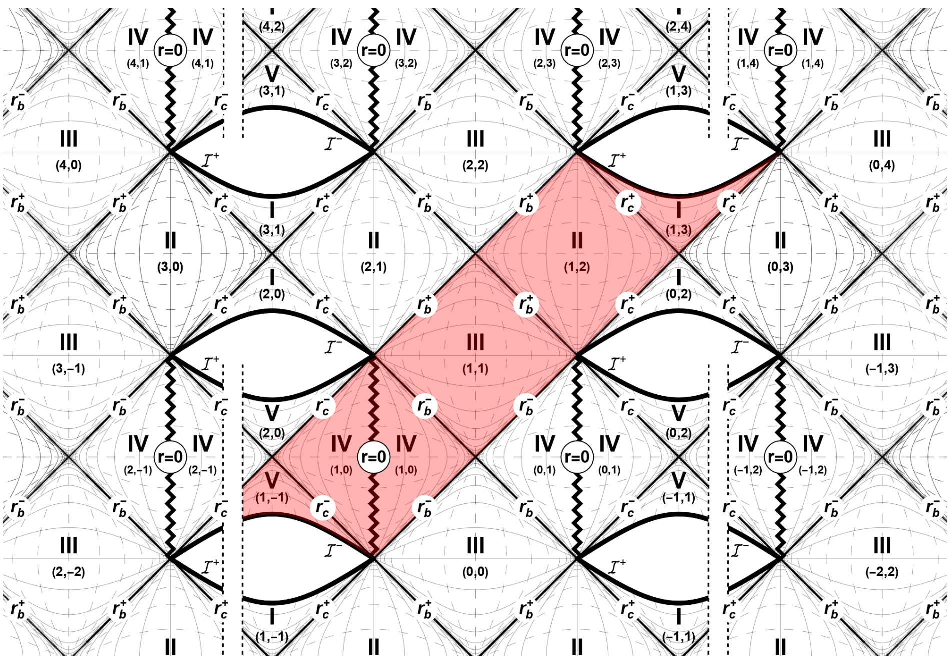

Figure 3.2: Penrose conformal diagram of the completely extended spacetime (3.5) for the section $\phi_h = \text{const.}$ and such θ that the spacetime contains the curvature singularity at $r = 0$. In this case, the regions IV are "cut in half" by this ring singularity at $r = 0$, but can be non-geodetically extended.

3.2.4 Regularity of the axes

Recall that there are *seven physical parameters* in the metric (3.5): mass m, acceleration α, Kerr-like rotation and NUT twist parameters a and l, electric and magnetic charges e and g, and the cosmological constant Λ.

But there is also an additional free parameter – the conicity parameter C hidden in the range of the angular coordinate $\varphi \in [0, 2\pi C)$ corresponding to the magnitude of the deficit/excess angle of the cosmic string/strut.

The conical degeneracy of the $\theta = 0$ axis can be removed by a suitable choice of C, namely

$$C_0 \equiv \Big[1 - 2\Big(\frac{\alpha\, a\, m}{a^2 + l^2} - \frac{\Lambda}{3}\, l\Big)(a + l) + \Big(\frac{\alpha^2 a^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\Big)(a + l)^2\Big]^{-1}.$$

This leaves a deficit/excess angle on the other axis:

$$\delta_\pi = \frac{-8\pi a\,\Big[\alpha a\,[\,m(a^2 + l^2) - \alpha a l(a^2 - l^2 + e^2 + g^2)] - \frac{2}{3}\Lambda l(a^2 + l^2)^2\Big]}{\Big[1 + \frac{1}{3}\Lambda(a + l)(a + 3l)\Big](a^2 + l^2)^2 - 2\alpha a m(a + l)(a^2 + l^2) + \alpha^2 a^2 (a + l)^2 (a^2 - l^2 + e^2 + g^2)}.$$

To regularize the second axis, we perform the coordinate transformation

$$t_\pi \equiv t - 4l\,\varphi\,, \tag{3.39}$$

and chose

$$C_\pi \equiv \Big[1 + 2\Big(\frac{\alpha\, a\, m}{a^2 + l^2} - \frac{\Lambda}{3}\, l\Big)(a - l) + \Big(\frac{\alpha^2 a^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\Big)(a - l)^2\Big]^{-1}.$$

This regularizes the axis $\theta = \pi$, with a deficit/excess angle on the $\theta = 0$ axis

$$\delta_0 = \frac{8\pi a\,\Big[\alpha a\,[\,m(a^2 + l^2) - \alpha a l(a^2 - l^2 + e^2 + g^2)] - \frac{2}{3}\Lambda l(a^2 + l^2)^2\Big]}{\Big[1 + \frac{1}{3}\Lambda(a - l)(a - 3l)\Big](a^2 + l^2)^2 + 2\alpha a m(a - l)(a^2 + l^2) + \alpha^2 a^2 (a - l)^2 (a^2 - l^2 + e^2 + g^2)}.$$

Most interestingly, a coincidence of the physical parameters

$$\tfrac{2}{3}\Lambda l(a^2 + l^2)^2 = \alpha a\Big[m(a^2 + l^2) - \alpha a l(a^2 - l^2 + e^2 + g^2)\Big] \tag{3.40}$$

fully regularizes both axes.

The strings/struts are *twisting*. This can be seen from the function $\omega \equiv \frac{g_{t\varphi}}{g_{tt}}$, and its evaluation on the axes $\theta = 0$ or $\theta = \pi$. The twisting parameters ω on each axes can be adjusted using (3.39), but its difference remains always constant $\Delta\omega = 4l$.

As we have already mentioned, in the $\Lambda = 0$ case, we can expect the pathology around the axis $\theta = \pi$ caused by the presence of the NUT parameter. Such a region with closed timelike curves is defined by the condition

$$P(\theta)\Big(r^2 + (a + l)^2\Big)^2 \sin^2\theta < Q(r)\Big(a \sin^2\theta + 4l \sin^2\tfrac{1}{2}\theta\Big)^2, \tag{3.41}$$

where the metric functions $P(\theta)$, $Q(r)$ are explicitly given by (3.8), (3.9). This is plotted on Fig. 3 of Paper 3.

3.2.5 Thermodynamic properties

We also evaluated the basic thermodynamic quantities of this class of black holes, namely the entropy and the temperature T, generalizing the $\Lambda = 0$ case (2.37).

The area of both *black hole horizons* is now given as:

$$\text{area of } \mathcal{H}_b^+ \text{ is } \quad \mathcal{A}_b^+ = \frac{4\pi C\left[(r_b^+)^2 + (a+l)^2\right]}{\Big(1 - \alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_b^+\Big)\Big(1 + \alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_b^+\Big)}\,, \tag{3.42}$$

$$\text{area of } \mathcal{H}_b^- \text{ is } \quad \mathcal{A}_b^- = \frac{4\pi C\left[(r_b^-)^2 + (a+l)^2\right]}{\Big(1 - \alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_b^-\Big)\Big(1 + \alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_b^-\Big)}\,, \tag{3.43}$$

whereas the area of the *cosmo-acceleration horizons* depends on Λ. If $\Lambda \leq 0$ then both are infinite. On the other hand, if $\Lambda > 0$ they are finite and equal to

$$\text{area of } \mathcal{H}_c^+ \text{ is } \quad \mathcal{A}_c^+ = \frac{4\pi C\left[(r_c^+)^2 + (a+l)^2\right]}{\Big(1 - \alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_c^+\Big)\Big(1 + \alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_c^+\Big)}\,, \tag{3.44}$$

$$\text{area of } \mathcal{H}_c^- \text{ is } \quad \mathcal{A}_c^- = \frac{4\pi C\left[(r_c^-)^2 + (a+l)^2\right]}{\Big(1 - \alpha\,\dfrac{a^2+al}{a^2+l^2}\,r_c^-\Big)\Big(1 + \alpha\,\dfrac{a^2-al}{a^2+l^2}\,r_c^-\Big)}\,. \tag{3.45}$$

The surface gravity from which we can evaluate the temperature of the horizons via (2.37) is:

$$\text{surface gravity of } \mathcal{H}_b^+ \text{ is } \quad \kappa_b^+ = \frac{1}{2k_b^+} = -\frac{\mathcal{N}}{2}\,\frac{(r_b^+ - r_b^-)(r_b^+ - r_c^+)(r_b^+ - r_c^-)}{(r_b^+)^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_b^- \text{ is } \quad \kappa_b^- = \frac{1}{2k_b^-} = -\frac{\mathcal{N}}{2}\,\frac{(r_b^- - r_b^+)(r_b^- - r_c^+)(r_b^- - r_c^-)}{(r_b^-)^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_c^+ \text{ is } \quad \kappa_c^+ = \frac{1}{2k_c^+} = -\frac{\mathcal{N}}{2}\,\frac{(r_c^+ - r_b^+)(r_c^+ - r_b^-)(r_c^+ - r_c^-)}{(r_c^+)^2 + (a+l)^2}\,,$$

$$\text{surface gravity of } \mathcal{H}_c^- \text{ is } \quad \kappa_c^- = \frac{1}{2k_c^-} = -\frac{\mathcal{N}}{2}\,\frac{(r_c^- - r_b^+)(r_c^- - r_b^-)(r_c^- - r_c^+)}{(r_c^-)^2 + (a+l)^2}\,.$$

From these expressions, it immediately follows that any *extremal* horizon has a vanishing surface gravity, and thus zero temperature $T = 0$.

3.3 Summary

In this final Chapter 3, we have built on the results of Paper 2, that is [45], and we further generalized the new metric form of the Plebański–Demiański metric by including a non-zero *cosmological constant* Λ. This solution was then physically and geometrically investigated. In particular:

- In Sec. 3.1, we summarized a new metric form (3.5)–(3.9) of the general accelerating, rotating and charged black hole with a NUT parameter and

$\Lambda \neq 0$. This was achieved by further improving the Griffiths–Podolský form of this class of spacetimes using a new set of the physical parameters m, e, g, by applying a unique conformal rescaling S, and suitably fixing the twist parameter ω.

- This large family of black hole solution depends generally on 7 arbitrary parameters, namely the mass m, the rotation and NUT parameters a and l, the electric and magnetic charges e and g, the acceleration parameter α, and the cosmological constant Λ.

- The clear advantage of this new metric is its ability to transit to the simpler black hole solutions in their standard formats. These are the black holes in asymptotically flat universe (for $\Lambda = 0$), discussed in detail in the previous Chapter 2, Kerr–Newman–NUT–(anti-)de Sitter black holes (for $\alpha = 0$), accelerating Kerr–Newman–(anti-)de Sitter black holes (for $l = 0$), charged Taub–NUT–(anti-)de Sitter black holes (with $a = 0$), and uncharged accelerating Kerr–NUT–(anti-)de Sitter black holes (when $e = 0$ or $g = 0$).

- The new metric form (3.5) becomes completely independent of the acceleration α when the Kerr-like rotation a is set to zero: The solution then simplifies directly to the charged Taub–NUT–(anti-)de Sitter black holes. This further confirms the previous observation that there is no accelerating NUT black hole present in the Plebański–Demiański family of type D spacetimes (see Chapter III for more details).

- Applying the null tetrad (3.15), we calculated all the NP scalars. The only non-zero components are Ψ_2 (3.16) and Φ_{11} (3.17). The Ricci scalar is simply $R = 4\Lambda$.

- The spin coefficients indicate that both principal null directions are *geodesic*, *shear-free*, expanding and generally *twisting*.

- From the curvature tensors we localized the presence of the *ring curvature singularity*. It is located at $\rho^2 = 0$, i.e. $r = 0$, and at the same time $l + a\cos\theta = 0$. This requires $|l| \leq |a|$. Otherwise, no curvature singularity is present (see the classification (2.31)).

- In Sec. 3.2.1, we identified the *four distinct horizons* corresponding to the roots of the metric function $Q(r)$. We also provided a *general* classification based on the number and multiplicity of its roots.

- For non-zero Kerr-like rotation a, each of these four horizons is accompanied by the corresponding ergoregion. This was visualized in Sec. 3.2.2 on Fig. 3.1.

- In Sec. 3.2.3, the global structure was visualized by the rigorously constructed Penrose conformal diagram, see Fig. 3.2.

- The regularization of the axisymmetric axes $\theta = 0$ and $\theta = \pi$, which we interpret as the physical source of acceleration of the black holes, was considered in Sec. 3.2.4. By an appropriate fixing of the conicity parameter C, we were able to regularize one of the axes.

- There exists a unique choice of the physical parameters (3.40) which regularizes both the axes simultaneously.
- These cosmic strings/struts are twisting when $l \neq 0$. In their vicinity there are pathological regions with closed timelike curves. These regions are given by the condition (3.41).
- The new metric (3.5) is also convenient for the investigation of thermodynamic quantities, such as the temperature T or the entropy S, see Sec. 3.2.5.

This demonstrates that the new metric form (3.5)–(3.9) has considerable advantages.

Among further investigations which should be done, let us mention the in-depth analysis of the various other cases given by the classification diagram of possible horizons in Sec. 3.2.1. In fact, we are already preparing a publication which is concerned with these cases — the four-horizons cases given by a different horizon ordering, the reduced solutions when one or more of the horizons disappear "at infinity", the discussion of its exact roots, and their simplification for a vanishing cosmological constant.

Also the multiple horizons should be analyzed. This topic has been recently studied, for example in the works [68]–[74]. We hope that the new form of the metric may simplify these investigations.

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New form of all black holes of type D with a cosmological constant

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We present an improved metric form of the complete family of exact black hole spacetimes of algebraic type D, including any cosmological constant. This class was found by Debever in 1971, Plebański and Demiański in 1976, and conveniently reformulated by Griffiths and Podolský in 2005. In our new form of this metric the key functions are simplified, partially factorized, and fully explicit. They depend on seven parameters with direct physical meanings, namely $m, a, l, \alpha, e, g, \Lambda$ which characterize mass, Kerr-like rotation, NUT parameter, acceleration, electric and magnetic charges of the black hole, and the cosmological constant, respectively. Moreover, this general metric reduces directly to the familiar forms of (possibly accelerating) Kerr-Newman–(anti–)de Sitter spacetime, charged Taub-NUT–(anti–)de Sitter solution, or (possibly rotating and charged) *C*-metric with a cosmological constant by simply setting the corresponding parameters to zero. In addition, it shows that the Plebański-Demiański family does not involve accelerating NUT black holes without the Kerr-like rotation. The new improved metric also enables us to study various physical and geometrical properties, namely the character of singularities, two black hole and two cosmo-acceleration horizons (in a generic situation), the related ergoregions, global structure including the Penrose conformal diagrams, parameters of cosmic strings causing the acceleration of the black holes, their rotation, pathological regions with closed timelike curves, or thermodynamic quantities.

DOI:

I. INTRODUCTION

Black holes belong to the most remarkable predictions of Einstein's general relativity. Although their existence had been doubted for many decades, it is now widely accepted that such *totaly* gravitationally collapsed "objects" indeed exist in our Universe. Recent (and spectacular) observational proofs of this fact are the detections of gravitational waves emitted from binary black hole coalescences, achieved by the LIGO Scientific Collaboration-Virgo Collaboration [1,2], and also the first direct image of a shadow of a supermassive black hole in M87* and in Sgr A*, obtained by the Event Horizon Telescope Collaboration [3,4].

First *exact spacetimes* representing black holes were found very soon after the final formulation of Einstein's field equations of general relativity in November 1915. Namely, it is the important solution of Schwarzschild (1916), Reissner-Nordström solution with an electric charge (1916–1918), and Kottler-Weyl-Trefftz solution with a cosmological constant Λ (1918–1922). These were followed in 1960s by rotating Kerr (1963), twisting Taub-NUT (1963) or Kerr-Newman charged black holes (1965), and also the so called *C*-metric (1918, 1962), physically interpreted by Kinnersley-Walker (1970) as uniformly accelerating pair of black holes.

All these fundamental exact solutions are spherically/axially symmetric, and are of algebraic type D. In fact, they belong to a general family of type D spacetimes with any Λ and an aligned electromagnetic field. Nonaccelerating solutions of this family were obtained in 1968 by Carter [5]. In the vacuum $\Lambda = 0$ case, they include all the particular subclasses identified by Kinnersley [6]. Debever [7] in 1971 found a wider class of such black holes which also admit acceleration. In 1976 a better metric representation of this complete class of type D exact solutions to Einstein-Maxwell equations with double-aligned non-null electromagnetic field and Λ was found in a seminal work [8] by Plebański and Demiański (for more details and further references see [9,10], in particular Chap. 16).

Unfortunately, the familiar forms of the well-known black holes *were not included explicitly* in the original Plebański-Demiański metric (specific degenerate transformations had to be applied), and the *physical interpretation* of its seven free parameters was not clear. Both these drawbacks were overcome in 2006 in the works of Griffiths and Podolský [11–13], see also [10], enabling easier analysis of physical and geometrical properties of these exact black holes.

In our recent paper [14] we demonstrated that this Griffiths-Podolský form of the generic black hole metric

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of type D can be *further improved*. This was achieved by introducing a modified set of the mass and charge parameters, an appropriate conformal rescaling, and a useful gauge choice of the twist parameter. The new improved form of the metric is simple, fully explicit, and with factorized metric functions. It is thus possible to investigate and evaluate various properties of this large family of rotating, charged, and accelerating black holes, namely their singularities, horizons, ergoregions, infinities, cosmic strings, or thermodynamics [14].

In such studies we restricted ourselves only to the case $\Lambda = 0$. It is the purpose of the present paper to extend the new improved coordinate representation found in [14] to *any value of the cosmological constant*, thus completing our program to improve the metric description of the *full class* of Plebański-Demiański black holes of algebraic type D.

In Sec. II we systematically derive the new form of the metric, with the results summarized in Sec. III. In subsequent Sec. IV all the main subclasses of this large family of type D black holes are discussed—these are obtained by simply setting the corresponding physical parameters $\Lambda, \alpha, l, a, e, g$ to zero. The second part of our paper, which is contained in the long Sec. V, is devoted to the physical and geometrical analysis of this class of black holes which can be done fully explicitly using our improved form of the generic metric. Such a study includes determining the curvature of the gravitational field, evaluation of the electromagnetic field, the structure and location of horizons, finding the related ergoregions, analytic extension and global structure, regularization of the symmetry axes, properties of the possible cosmic strings or struts, their rotation related to the NUT parameter, regions with closed timelike curves in their vicinity, and calculation of the entropy and temperature of the black hole and cosmo-acceleration horizons. Final summary with further remarks is contained in Sec. VI.

II. DERIVATION OF THE NEW FORM OF THE METRIC

First, let us recall the convenient representation of the complete class of Plebański-Demiański black holes of algebraic type D found by Griffiths and Podolský in 2005 [11–13]. It is summarized in Eq. (16.18) of [10] as

$$d\tilde{s}^2 = \frac{1}{\Omega^2}\left(-\frac{\mathcal{Q}}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)d\varphi\right]^2 + \frac{\rho^2}{\mathcal{Q}}dr^2 + \frac{\rho^2}{\mathcal{P}}d\theta^2 + \frac{\mathcal{P}}{\rho^2}\sin^2\theta[a\,dt - (r^2+(a+l)^2)d\varphi]^2\right), \quad (1)$$

where the metric functions are

$$\Omega = 1 - \frac{\alpha}{\omega}(l + a\cos\theta)r, \quad (2)$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2, \quad (3)$$

$$\mathcal{P}(\theta) = 1 - a_3\cos\theta - a_4\cos^2\theta, \quad (4)$$

$$\mathcal{Q}(r) = (\omega^2 k + \tilde{e}^2 + \tilde{g}^2) - 2\tilde{m}r + \epsilon r^2 - 2\alpha\frac{n}{\omega}r^3 - \left(\alpha^2 k + \frac{\tilde{\Lambda}}{3}\right)r^4. \quad (5)$$

The constants a_3 and a_4 in (4) are

$$a_3 = 2\alpha\frac{a}{\omega}\tilde{m} - 4\alpha^2\frac{al}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) - 4\frac{\tilde{\Lambda}}{3}al, \quad (6)$$

$$a_4 = -\alpha^2\frac{a^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) - \frac{\tilde{\Lambda}}{3}a^2, \quad (7)$$

while the coefficients ϵ, n, and k in (5)–(7) are determined by the relations,

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}\tilde{m} - (a^2 + 3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right], \quad (8)$$

$$n = \frac{\omega^2 k}{a^2 - l^2}l - \alpha\frac{a^2 - l^2}{\omega}\tilde{m} + (a^2 - l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) + \frac{\tilde{\Lambda}}{3}\right], \quad (9)$$

and

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right)k = 1 + 2\alpha\frac{l}{\omega}\tilde{m} - 3\alpha^2\frac{l^2}{\omega^2}(\tilde{e}^2 + \tilde{g}^2) - \tilde{\Lambda}l^2, \quad (10)$$

which implies

$$\frac{\omega^2 k}{a^2 - l^2} = \frac{1 - \tilde{\Lambda}l^2 + 2\alpha\frac{l}{\omega}\tilde{m} - 3\alpha^2\frac{l^2}{\omega^2}(\tilde{e}^2 + \tilde{g}^2)}{1 + 3\alpha^2\frac{l^2}{\omega^2}(a^2 - l^2)}, \quad (11)$$

$$(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) = \frac{(1 - \tilde{\Lambda}l^2)(a^2 - l^2) + (\tilde{e}^2 + \tilde{g}^2) + 2\alpha\frac{l}{\omega}(a^2 - l^2)\tilde{m}}{1 + 3\alpha^2\frac{l^2}{\omega^2}(a^2 - l^2)}. \quad (12)$$

The fully explicit form of the metric (1) is thus quite complicated because substituting (6)–(12) into (4) and (5) gives cumbersome expressions. Another fundamental problem is the actual physical meaning of the seven parameters

$\tilde{m}, a, l, \tilde{e}, \tilde{g}, \alpha, \tilde{\Lambda}$. These have been clearly interpreted only in special subcases when some of the other parameters were set to zero. In such subcases, they represent *mass, Kerr-like rotation, NUT parameter, electric charge, magnetic charge, acceleration*, and *cosmological constant*, respectively. Their meaning in a *completely general situation* is still an open problem. Moreover, there is an additional (auxiliary) *twist parameter* ω. In previous works [11–13] it was argued that ω is related *both* to a and l, and in some cases can be scaled appropriately using the remaining coordinate freedom. A satisfactory insight into all these problems is still missing. It is the aim of the present work to clarify such issues. We achieve this by presenting a new compact, explicit and considerably simplified form of the Plebański-Demiański metric, namely (47)–(51), for a complete family of black holes.

The first step in improving the form of the spacetime is to introduce a *new set of the mass and charge parameters* m, e, g. Following our previous paper [14], we define them as

$$\begin{aligned} m &\equiv S\tilde{m} - \alpha\frac{l}{\omega}(a^2 - l^2 + e^2 + g^2), \\ e^2 &\equiv S\tilde{e}^2, \\ g^2 &\equiv S\tilde{g}^2, \end{aligned} \tag{13}$$

where S is a *specific scaling constant*

$$S \equiv \frac{a^2 - l^2}{\omega^2 k}. \tag{14}$$

Notice that

$$(\omega^2 k + \tilde{e}^2 + \tilde{g}^2) = S^{-1}(a^2 - l^2 + e^2 + g^2), \tag{15}$$

which is a much simpler expression than (12).

In terms of these new parameters m, e, g, the coefficients (6)–(9) take the form,

$$a_3 = S^{-1}\frac{a}{\omega}\left[2\alpha m - 2\alpha^2\frac{l}{\omega}(a^2 - l^2 + e^2 + g^2) - \frac{4}{3}\tilde{\Lambda}Sl\omega\right], \tag{16}$$

$$a_4 = -S^{-1}\frac{a^2}{\omega^2}\left[\alpha^2(a^2 - l^2 + e^2 + g^2) + \frac{1}{3}\tilde{\Lambda}S\omega^2\right], \tag{17}$$

$$\begin{aligned} \epsilon = S^{-1}\bigg[1 + 4\alpha\frac{l}{\omega}m - \alpha^2\frac{a^2 - l^2}{\omega^2}(a^2 - l^2 + e^2 + g^2) \\ -\frac{1}{3}\tilde{\Lambda}S(a^2 + 3l^2)\bigg], \end{aligned} \tag{18}$$

$$n = S^{-1}\left[l - \alpha\frac{a^2 - l^2}{\omega}m + \frac{1}{3}\tilde{\Lambda}S(a^2 - l^2)l\right]. \tag{19}$$

The key metric functions (4), (5) thus nicely simplify to

$$\mathcal{P}(\theta) = S^{-1}P(\theta), \qquad \mathcal{Q}(r) = S^{-1}Q(r), \tag{20}$$

where

$$P(\theta) = 1 - 2\left(\frac{\alpha}{\omega}m - \frac{1}{3}\tilde{\Lambda}Sl\right)(l + a\cos\theta) + \left(\frac{\alpha^2}{\omega^2}(a^2 - l^2 + e^2 + g^2) + \frac{1}{3}\tilde{\Lambda}S\right)(l + a\cos\theta)^2, \tag{21}$$

$$Q(r) = [r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)]\left(1 + \alpha\frac{a - l}{\omega}r\right)\left(1 - \alpha\frac{a + l}{\omega}r\right) - \frac{1}{3}\tilde{\Lambda}Sr^2\left[r^2 + 2\alpha\frac{l}{\omega}(a^2 - l^2)r + (a^2 + 3l^2)\right]. \tag{22}$$

With (20), the metric (1) now reads

$$\begin{aligned} \mathrm{d}\tilde{s}^2 = \frac{S}{\Omega^2}\bigg(-\frac{Q}{\rho^2}S^{-2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2 \\ + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 \\ + \frac{P}{\rho^2}\sin^2\theta S^{-2}[a\mathrm{d}t - (r^2 + (a + l)^2)\mathrm{d}\varphi]^2\bigg). \end{aligned} \tag{23}$$

Recall that it is a solution to the Einstein-Maxwell field equations with a cosmological constant $\tilde{\Lambda}$.

As the second step, we now *rescale the coordinates* t *and* φ by a *constant scaling factor* $S \neq 0$. (This is possible because their ranges have not yet been specified.) In other words, we perform the transformation,

$$t \to St, \qquad \varphi \to S\varphi, \tag{24}$$

which completely removes all the constants S from the conformally related metric,

$$\mathrm{d}s^2 \equiv S^{-1}\mathrm{d}\tilde{s}^2, \tag{25}$$

that is

$$\begin{aligned}ds^2 = \frac{1}{\Omega^2}\bigg(&-\frac{Q}{\rho^2}\Big[dt - \Big(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\Big)d\varphi\Big]^2\\ &+\frac{\rho^2}{Q}dr^2 + \frac{\rho^2}{P}d\theta^2\\ &+\frac{P}{\rho^2}\sin^2\theta[a dt - (r^2+(a+l)^2)d\varphi]^2\bigg). \end{aligned}\tag{26}$$

Since the energy-momentum tensor of the Maxwell field $4\pi T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}$ in four dimensions is trace-free, Einstein's equations read $R_{ab} = \Lambda g_{ab} + 8\pi T_{ab}$, and the Ricci scalar is $R = 4\Lambda$. Under the constant conformal rescaling (25) of the metric, the Ricci tensor is invariant: $g_{ab} = S^{-1}\tilde{g}_{ab}$ implies $R_{ab} = \tilde{R}_{ab}$ and $R = \tilde{R}S$. Consequently, the new metric (26) is a solution to the Einstein-Maxwell field equations with a cosmological constant Λ, provided

$$\Lambda \equiv \tilde{\Lambda}S, \qquad F_{ab} \equiv \tilde{F}_{ab}\sqrt{S}. \tag{27}$$

The corresponding metric functions (21), (22) are thus

$$P(\theta) = 1 - 2\Big(\frac{\alpha}{\omega}m - \frac{\Lambda}{3}l\Big)(l + a\cos\theta) + \Big(\frac{\alpha^2}{\omega^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\Big)(l + a\cos\theta)^2, \tag{28}$$

$$Q(r) = [r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)]\Big(1 + \alpha\frac{a-l}{\omega}r\Big)\Big(1 - \alpha\frac{a+l}{\omega}r\Big) - \frac{\Lambda}{3}r^2\Big[r^2 + 2\alpha\frac{l}{\omega}(a^2 - l^2)r + (a^2 + 3l^2)\Big]. \tag{29}$$

As the third step, it remains to *fix the auxiliary twist parameter* ω, coupled with *both* the Kerr-like rotation a *and* the NUT parameter l. It was found in [15] and conveniently employed in [14,16,17] that the most suitable gauge choice of this twist parameter is

$$\omega \equiv \frac{a^2 + l^2}{a}, \tag{30}$$

so that

$$\frac{a}{\omega} = \frac{a^2}{a^2 + l^2}, \qquad \frac{l}{\omega} = \frac{al}{a^2 + l^2}. \tag{31}$$

Substituting these expressions into (2), (28) and (29), we obtain the explicit functions Ω, P and Q, namely

$$\Omega = 1 - \frac{\alpha a}{a^2 + l^2}r(l + a\cos\theta), \tag{32}$$

$$P(\theta) = 1 - 2\Big(\frac{\alpha a}{a^2 + l^2}m - \frac{\Lambda}{3}l\Big)(l + a\cos\theta) + \Big(\frac{\alpha^2 a^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\Big)(l + a\cos\theta)^2, \tag{33}$$

$$Q(r) = [r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)]\Big(1 + \alpha a\frac{a-l}{a^2+l^2}r\Big)\Big(1 - \alpha a\frac{a+l}{a^2+l^2}r\Big) - \frac{\Lambda}{3}r^2\Big[r^2 + 2\alpha a l\frac{a^2 - l^2}{a^2+l^2}r + (a^2 + 3l^2)\Big]. \tag{34}$$

In fact, for a generic class of black holes the metric functions P and Q can be further simplified. To this end, let us define convenient parameters μ, λ, and $\mathcal{A}$ (representing the "modified" mass, cosmological constant, and acceleration, respectively) as

$$\mu \equiv m - \lambda\mathcal{A} \qquad = m - \frac{\Lambda}{3}l\frac{a^2 + l^2}{\alpha a}, \tag{35}$$

$$\lambda \equiv \frac{\Lambda}{3}\frac{(a^2 + l^2)^2}{\alpha^2 a^2}, \tag{36}$$

$$\mathcal{A} \equiv \frac{\alpha l}{a^2 + l^2}. \tag{37}$$

Moreover, we introduce a *pair of special constants* $r_{\Lambda+}$ and $r_{\Lambda-}$ by

$$r_{\Lambda\pm} \equiv \mu \pm \sqrt{\mu^2 + l^2 - a^2 - e^2 - g^2 - \lambda}. \tag{38}$$

From these definitions it immediately follows that

$$\begin{aligned}\frac{\alpha a}{a^2 + l^2}(r_{\Lambda+} + r_{\Lambda-}) &= 2\Big(\frac{\alpha a}{a^2 + l^2}m - \frac{\Lambda}{3}l\Big),\\ \frac{\alpha^2 a^2}{(a^2 + l^2)^2}r_{\Lambda+}r_{\Lambda-} &= \frac{\alpha^2 a^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3},\end{aligned} \tag{39}$$

so that (33) can be reexpressed as

$$P(\theta) = \left(1 - \frac{\alpha a}{a^2+l^2} r_{\Lambda+}(l + a\cos\theta)\right)\left(1 - \frac{\alpha a}{a^2+l^2} r_{\Lambda-}(l + a\cos\theta)\right). \tag{40}$$

The metric function $P(\theta)$ *is thus nicely factorized*.

Using (35)–(38), the expression (34) for the metric function $Q(r)$ is also simplified to

$$Q(r) = [r^2 - 2\mu r + (a^2 - l^2 + e^2 + g^2 + \lambda)]\left(1 + \alpha a \frac{a-l}{a^2+l^2} r\right)\left(1 - \alpha a \frac{a+l}{a^2+l^2} r\right) - \lambda\left[1 + \frac{\alpha^2 a^2}{(a^2+l^2)^2} r^4\right]. \tag{41}$$

In the cases when $\mu^2 + l^2 > a^2 + e^2 + g^2 + \lambda$, the definition (38) yields *two real distinct constants* $r_{\Lambda+}$ and $r_{\Lambda-}$, and (41) takes the form,

$$Q(r) = (r - r_{\Lambda+})(r - r_{\Lambda-})\left(1 + \alpha a \frac{a-l}{a^2+l^2} r\right)\left(1 - \alpha a \frac{a+l}{a^2+l^2} r\right) - \frac{\Lambda}{3}\left[r^4 + \frac{(a^2+l^2)^2}{\alpha^2 a^2}\right]. \tag{42}$$

Interestingly, *when* $\Lambda = 0$, the constants $r_{\Lambda\pm}$ defined by (38) reduce to

$$r_\pm \equiv m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}. \tag{43}$$

These parameters then identify (independently of the acceleration α) the *two black hole horizons* because they are also the roots of the metric functions $Q(r)$ given by (42), cf. [14].

Finally, although the unique scaling constant S defined by (14) *does not enter the final form* of the metric (26) with (32)–(34), it may be useful to present its explicit form in terms of the new parameters. Substitution from (13) into (11) with $\Lambda = \tilde{\Lambda} S$ yields the relation,

$$S = 1 - 2\alpha \frac{l}{\omega} m + \alpha^2 \frac{l^2}{\omega^2}(a^2 - l^2 + e^2 + g^2) + \Lambda l^2, \tag{44}$$

that is, using (30), (37)–(39),

$$S = (1 - \mathcal{A} r_{\Lambda+})(1 - \mathcal{A} r_{\Lambda-}). \tag{45}$$

The rescaling transformation (25) thus actually removes two coordinate singularities hidden in the expression (45) at $\mathcal{A} r_{\Lambda\pm} = 1$. This fact was already observed for the $\Lambda = 0$ case in our previous article [14].

Moreover, it can be seen that $S = 1$ whenever $\mathcal{A} r_{\Lambda+} = 0 = \mathcal{A} r_{\Lambda-}$. For $\Lambda = 0$, this happens if $l = 0$ or $\alpha = 0$ or $a = 0$, in which cases $m = \tilde{m}$, $e = \tilde{e}$, $g = \tilde{g}$.

For $\Lambda \neq 0$, the value of the scaling factor is generically $S \neq 1$. In the case $l = 0$ it follows from (44) that $S = 1$, but in the case $l \neq 0$ we get $S = 1 + \Lambda l^2$ even if $\alpha = 0$ or $a = 0$. Generally, $S = 1$ *only for a special value* of the cosmological constant,

$$\Lambda = \frac{\alpha a}{a^2+l^2}\left[2\frac{m}{l} - \frac{\alpha a}{a^2+l^2}(a^2 - l^2 + e^2 + g^2)\right]. \tag{46}$$

III. SUMMARY OF THE NEW FORM OF A GENERIC BLACK HOLE

It is now useful to summarize our new metric representation of the *complete family of black holes* contained in the class of Plebański-Demiański spacetimes [8]. Recall that such spacetimes are the most general exact solutions to Einstein-Maxwell equations of algebraic type D with double-aligned non-null electromagnetic field (see Chap. 16 of the monograph [10] for the recent review and number of related references).

The new metric form, which improves the previous representation found by Griffiths and Podolský [11–13], reads

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left(-\frac{Q}{\rho^2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta[a\mathrm{d}t - (r^2 + (a+l)^2)\mathrm{d}\varphi]^2\right), \tag{47}$$

where

$$\Omega = 1 - \frac{\alpha a}{a^2+l^2} r(l + a\cos\theta), \tag{48}$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2, \tag{49}$$

$$P(\theta) = 1 - 2\left(\frac{\alpha a}{a^2+l^2}m - \frac{\Lambda}{3}l\right)(l + a\cos\theta) + \left(\frac{\alpha^2 a^2}{(a^2+l^2)^2}(a^2-l^2+e^2+g^2) + \frac{\Lambda}{3}\right)(l+a\cos\theta)^2, \tag{50}$$

$$Q(r) = [r^2 - 2mr + (a^2-l^2+e^2+g^2)]\left(1+\alpha a\frac{a-l}{a^2+l^2}r\right)\left(1-\alpha a\frac{a+l}{a^2+l^2}r\right) - \frac{\Lambda}{3}r^2\left[r^2 + 2\alpha a l\frac{a^2-l^2}{a^2+l^2}r + (a^2+3l^2)\right]. \tag{51}$$

The spacetime depends on *seven physical parameters*, namely

- m ….. mass parameter,
- a ….. Kerr-like rotation,
- l ….. NUT parameter,
- e ….. electric charge,
- g ….. magnetic charge,
- α ….. acceleration,
- Λ ….. cosmological constant.

This metric is compact and fully explicit, and the ambiguous twist parameter ω has been removed by its most convenient choice. Moreover, the standard forms of famous black hole spacetimes—namely Kerr-Newman–(A)dS, charged Taub-NUT–(A)dS, their accelerated versions, and others—can easily be obtained as direct subcases of (47)–(51) by setting the corresponding physical parameters to zero.

When $\Lambda = 0$, both metric functions P and Q are factorized, see [14] for more details. With $\Lambda \neq 0$ this cannot be in general achieved. However, it is possible to explicitly factorize the function P and compactify the function Q as

$$P(\theta) = \left(1 - \frac{\alpha a}{a^2+l^2}r_{\Lambda+}(l+a\cos\theta)\right)\left(1 - \frac{\alpha a}{a^2+l^2}r_{\Lambda-}(l+a\cos\theta)\right), \tag{52}$$

$$Q(r) = (r-r_{\Lambda+})(r-r_{\Lambda-})\left(1+\alpha a\frac{a-l}{a^2+l^2}r\right)\left(1-\alpha a\frac{a+l}{a^2+l^2}r\right) - \frac{\Lambda}{3}\left[r^4 + \frac{(a^2+l^2)^2}{\alpha^2 a^2}\right], \tag{53}$$

using the two specific constants,

$$r_{\Lambda\pm} \equiv \mu \pm \sqrt{\mu^2 + l^2 - a^2 - e^2 - g^2 - \lambda}, \tag{54}$$

where

$$\mu \equiv m - \frac{\Lambda}{3}l\frac{a^2+l^2}{\alpha a}, \qquad \lambda \equiv \frac{\Lambda}{3}\frac{(a^2+l^2)^2}{\alpha^2 a^2}. \tag{55}$$

This is possible provided $\mu^2 + l^2 > a^2 + e^2 + g^2 + \lambda$, in which case the expressions (54) yield two distinct real constants (or a double root of P given by $r_{\Lambda+} = r_{\Lambda-} = \mu$ in the specific situation when $\mu^2 + l^2 = a^2 + e^2 + g^2 + \lambda$).

The new form of the metric (47)–(51) nicely represents the *complete family of type D black holes*. Moreover, it naturally generalizes the standard forms of the most important black hole solutions, with two black hole horizons (outer and inner) and two cosmological/acceleration horizons.

IV. THE MAIN SUBCLASSES OF TYPE D BLACK HOLES

These are easily obtained by setting the appropriate physical parameters to zero, as follows.

A. Black holes in flat universe ($\Lambda = 0$: no cosmological constant)

In the case $\Lambda = 0$, we get $\mu = m$ and $\lambda = 0$. When $m^2 + l^2 > a^2 + e^2 + g^2$ (which guarantees that two distinct roots r_+ and r_- exist) the metric functions (52), (53) thus take the form,

$$P(\theta) = \left(1 - \frac{\alpha a}{a^2+l^2}r_+(l+a\cos\theta)\right)\left(1 - \frac{\alpha a}{a^2+l^2}r_-(l+a\cos\theta)\right), \tag{56}$$

$$Q(r) = (r-r_+)(r-r_-)\left(1+\alpha a\frac{a-l}{a^2+l^2}r\right)\left(1-\alpha a\frac{a+l}{a^2+l^2}r\right), \tag{57}$$

where

$$r_{\pm} \equiv m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2}, \tag{58}$$

cf. (43). The constants r_+ and r_- now directly identify (independently of the acceleration α) the *two black hole horizons* because they are also the *roots of the metric functions* $Q(r)$ given by (57). This large family of black holes was thoroughly analyzed in our previous work [14], and it is not necessary to repeat all the arguments and results here.

B. Kerr-Newman-NUT–(anti-)de Sitter black holes ($\alpha = 0$: No acceleration)

By setting the acceleration parameter α to zero, the metric function (48) reduces to $\Omega = 1$, while (49) remains the same. Concerning the functions P and Q given by (52) and (53), respectively, one has to be more careful in evaluating the limits of the terms $\alpha a r_{\Lambda\pm}$ because the acceleration $\alpha \to 0$ appears also in the denominator of the parameters μ and λ, defined by (55), which enter $r_{\Lambda\pm}$. In this case it is more convenient to directly set $\alpha = 0$ in the most general forms of these metric functions (50) and (51). In any case, we obtain the metric,

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{Q}{\rho^2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 \\ &+ \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta[a\mathrm{d}t - (r^2 + (a+l)^2)\mathrm{d}\varphi]^2, \end{aligned} \tag{59}$$

where

$$\rho^2 = r^2 + (l + a\cos\theta)^2. \tag{60}$$

$$P(\theta) = 1 + 2\frac{\Lambda}{3}l(l + a\cos\theta) + \frac{\Lambda}{3}(l + a\cos\theta)^2, \tag{61}$$

$$\begin{aligned} Q(r) = &\, r^2 - 2mr + (a^2 - l^2 + e^2 + g^2) \\ &-\frac{\Lambda}{3}r^2(r^2 + a^2 + 3l^2). \end{aligned} \tag{62}$$

This result is the same as the limit $\alpha \to 0$ of the metric functions (52) and (53). Indeed,

$$\lim_{\alpha\to 0}\frac{\alpha a}{a^2 + l^2}r_{\Lambda\pm} = -\frac{\Lambda}{3}l \pm \sqrt{\left(\frac{\Lambda}{3}l\right)^2 - \frac{\Lambda}{3}} \equiv L_{\pm}, \tag{63}$$

$$\lim_{\alpha\to 0}\mathcal{A}r_{\Lambda\pm} = lL_{\pm}, \tag{64}$$

so that

$$L_+ + L_- = -2\frac{\Lambda}{3}l, \qquad L_+L_- = \frac{\Lambda}{3}. \tag{65}$$

Thus $\lim_{\alpha\to 0}P(\theta) = (1 - L_+(l + a\cos\theta))(1 - L_-(l + a\cos\theta))$ gives (61), which can be rewritten as

$$P(\theta) = (1 + \Lambda l^2) + \frac{4}{3}\Lambda a\, l\cos\theta + \frac{1}{3}\Lambda a^2\cos^2\theta. \tag{66}$$

In a similar way, the limit of (53) using (39) yields (62). Moreover, in the limit of vanishing acceleration the scaling factor (45), using (64) and (65), becomes

$$\lim_{\alpha\to 0}S = 1 + \Lambda l^2. \tag{67}$$

We must emphasize that the forms (66) and (62) of the metric functions $P(\theta)$ and $Q(r)$ *are different* from the analogous metric functions for the Kerr-Newman-NUT–(anti-)de Sitter black holes as given by Eq. (16.23) in [10]. In fact, they are *equivalent reparametrization* of this solution. Indeed, we have to take into account the nontrivial scaling (20), that is

$$\mathcal{P}(\theta) = S^{-1}P(\theta), \qquad \mathcal{Q}(r) = S^{-1}Q(r), \tag{68}$$

where S is the constant (67). Straightforward calculation using the relations (13), (27) between the physical parameters then yields

$$\mathcal{P}(\theta) = 1 + \frac{4}{3}\tilde{\Lambda}\, a\, l\cos\theta + \frac{1}{3}\tilde{\Lambda}a^2\cos^2\theta, \tag{69}$$

$$\begin{aligned} \mathcal{Q}(r) = &\,(a^2 - l^2 + \tilde{e}^2 + \tilde{g}^2) - 2\tilde{m}r + r^2 \\ &- \tilde{\Lambda}\left[(a^2 - l^2)l^2 + \left(\frac{1}{3}a^2 + 2l^2\right)r^2 + \frac{1}{3}r^4\right], \end{aligned} \tag{70}$$

which is exactly the form of the metric functions given by Eq. (16.23) in [10].

All famous subcases of this general family of (nonaccelerating) Kerr-Newman-NUT-(anti–)de Sitter black holes, expressed now in a compact way by the metric (59) with (60)–(62) [or (66), equivalent to (61)], are readily obtained. These are the black hole solutions of Kerr-Newman-(anti–)de Sitter ($l = 0$), charged Taub-NUT-(anti–)de Sitter ($a = 0$), Kerr-(anti–)de Sitter ($l = 0$, $e = 0 = g$), Reissner-Nordström-(anti–)de Sitter ($a = 0$, $l = 0$), and Schwarzschild-(anti–)de Sitter ($a = 0$, $l = 0$, and $e = 0 = g$). Of course, by setting $\Lambda = 0$, the corresponding black holes in asymptotically flat universe are obtained (the same as in Sec. IV A).

C. Accelerating Kerr-Newman-(anti–)de Sitter black holes ($l = 0$: no NUT)

Without the NUT parameter l, the new metric (47) reduces to

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\bigg(-\frac{Q}{\rho^2}[\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi]^2\bigg), \tag{71}$$

where

$$\Omega = 1 - \alpha r\cos\theta, \tag{72}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{73}$$

$$P(\theta) = (1 - \alpha r_{\Lambda+}\cos\theta)(1 - \alpha r_{\Lambda-}\cos\theta), \tag{74}$$

$$Q(r) = (r - r_{\Lambda+})(r - r_{\Lambda-})(1+\alpha r)(1-\alpha r) - \frac{\Lambda}{3}\left(r^4 + \frac{a^2}{\alpha^2}\right), \tag{75}$$

where the specific constants $r_{\Lambda\pm}$ are now simplified to

$$r_{\Lambda\pm} \equiv m \pm \sqrt{m^2 - a^2 - e^2 - g^2 - \frac{\Lambda}{3}\frac{a^2}{\alpha^2}}. \tag{76}$$

The metric functions $P(\theta)$ and $Q(r)$ can be equivalently rewritten as

$$P(\theta) = 1 - 2\alpha m\cos\theta + \left[\alpha^2(a^2+e^2+g^2) + \frac{\Lambda}{3}a^2\right]\cos^2\theta, \tag{77}$$

$$Q(r) = [r^2 - 2mr + (a^2+e^2+g^2)](1+\alpha r)(1-\alpha r) - \frac{\Lambda}{3}r^2[r^2+a^2]. \tag{78}$$

In this explicit form we easily obtain all possible subcases by simply setting the corresponding physical parameters to zero. For vanishing acceleration ($\alpha = 0$), the metric of the Kerr-Newman-(anti–)de Sitter black hole solution is recovered, which then yields the standard form of the Kerr-Newman solution in the Boyer-Lindquist coordinates in the case of vanishing cosmological constant ($\Lambda = 0$). Contrarily, by setting $\Lambda = 0$ first, we obtain the general metric of accelerating Kerr-Newman black holes. For vanishing charges ($e = 0 = g$), it is equivalent to the rotating C-metric, first identified by Hong and Teo [18].

D. Charged Taub-NUT-(anti–)de Sitter black holes ($a = 0$: No rotation)

By setting the Kerr-like rotation parameter a to zero, the new metric (47) considerably simplifies and *becomes independent of the acceleration* α [because the metric functions (48)–(53) depend on α only via the product αa]. Indeed, $\Omega = 1$ and $P = 1 + \Lambda l^2$, so that

$$\mathrm{d}s^2 = -\frac{Q}{\rho^2}(\mathrm{d}t - 4l\sin^2\tfrac{1}{2}\theta\mathrm{d}\varphi)^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \rho^2\left(\frac{\mathrm{d}\theta^2}{1+\Lambda l^2} + (1+\Lambda l^2)\sin^2\theta\mathrm{d}\varphi^2\right), \tag{79}$$

where

$$Q(r) = (1 - \Lambda l^2)r^2 - 2mr + (e^2+g^2-l^2) - \frac{\Lambda}{3}r^4, \tag{80}$$

$$\rho^2 = r^2 + l^2. \tag{81}$$

This explicitly demonstrates that *there is no accelerating "purely" NUT-(anti–)de Sitter black hole in the Plebański-Demiański family* of spacetimes.

For $\Lambda = 0$, this observation was made already in the original works [11–13], and recently clarified in [19]. It was proven that the metric for accelerating (nonrotating) black holes with purely NUT parameter—which was found by Chng *et al.* [20] in 2006 and analyzed in detail in [19]—is of algebraic type I. Therefore, it *cannot* be contained in the Plebański-Demiański class which is of type D. We have just shown that the same is true also in the case of a nonvanishing cosmological constant Λ.

It should again be emphasized that the metric function (80) for $Q(r)$ *is different* from the analogous metric function for the charged Taub-NUT-(anti–)de Sitter black hole as given by Eq. (12.19) in [10]. Actually, it is simpler. Such a difference is caused by the nontrivial rescaling $S = 1 + \Lambda l^2$; see (67), (68). Considering the relations (13), (20) and (27), we get

$$\mathcal{P}(\theta) = 1, \tag{82}$$

$$\mathcal{Q}(r) = r^2 - l^2 - 2\tilde{m}r + \tilde{e}^2 + \tilde{g}^2 - \tilde{\Lambda}\left(\frac{1}{3}r^4 + 2l^2r^2 - l^4\right), \tag{83}$$

which is the expression (70) for $a = 0$, exactly the same as the metric function presented in Eq. (12.19) of [10] for the case $\epsilon = +1$ (with $\tilde{g} = 0$).

It will be shown below that the charged Taub-NUT-(anti–)de Sitter spacetime (79) is *nonsingular* (its curvature does not diverge at $r = 0$), away from the axis $\theta = \pi$ (where the rotating cosmic string is located) it is asymptotically (anti–)de Sitter, and the interior of the black hole is located between its two horizons, that can be surrounded by two "outer" cosmological horizons.

E. Uncharged accelerating Kerr-NUT-(anti–)de Sitter black holes ($e = 0 = g$: Vacuum with Λ)

Another nice feature of our new metric (47)–(53) is that it *has the same form for vacuum spacetimes* without the electromagnetic field. Indeed, the electric and magnetic charges e and g, which generate the electromagnetic field, *enter only the expressions for* $r_{\Lambda\pm}$ introduced in (54). In other words, e and g just change the values of these two constant parameters. In such a vacuum case, they simplify to

$$r_{\Lambda\pm} \equiv \mu \pm \sqrt{\mu^2 + l^2 - a^2 - \lambda}. \tag{84}$$

The metric (47)–(53) with (84) represents the full class of accelerating Kerr-NUT-(anti–)de Sitter black holes. It reduces to accelerating Kerr-(anti–)de Sitter black hole when $l = 0$, and nonaccelerating Kerr-NUT-(anti–)de Sitter black hole when $\alpha = 0$. For $a = 0$ it simplifies *directly* to the Taub-NUT-(anti–)de Sitter black hole (79) without acceleration and charges.

V. PHYSICAL ANALYSIS OF THE NEW METRIC

The explicit new metric form (47)–(53) [or, more generally, (50)–(51)] of the complete class of accelerating Kerr-Newman-NUT-(anti–)de Sitter black holes is very convenient for investigation of geometric and physical properties of this large family of black holes. This will now be demonstrated by deriving and presenting some of the key quantities and facts, namely those concerning the global structure of the spacetime, the stringy sources of the acceleration, and thermodynamic properties.

A. Curvature of the gravitational field and the electromagnetic field

First, it is necessary to determine the *gravitational field*, namely the specific curvature of the geometry. It is encoded in the corresponding Newman-Penrose (NP) scalars, that is, components of the curvature tensors with respect to the null tetrad. Its most natural choice is

$$\begin{aligned}
\mathbf{k} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}((r^2+(a+l)^2)\partial_t + a\partial_\varphi) + \sqrt{Q}\partial_r\right],\\
\mathbf{l} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{Q}}((r^2+(a+l)^2)\partial_t + a\partial_\varphi) - \sqrt{Q}\partial_r\right],\\
\mathbf{m} &= \frac{1}{\sqrt{2}}\frac{\Omega}{\rho}\left[\frac{1}{\sqrt{P}\sin\theta}\left(\partial_\varphi + \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\partial_t\right) + \mathrm{i}\sqrt{P}\partial_\theta\right].
\end{aligned} \tag{85}$$

A direct calculation shows that the only nontrivial Newman-Penrose scalars corresponding to the Weyl tensor and the Ricci tensor are

$$\begin{aligned}
\Psi_2 = \frac{\Omega^3}{[r+\mathrm{i}(l+a\cos\theta)]^3}\Bigg[&-(m+\mathrm{i}l)\left(1-\mathrm{i}\alpha a\frac{a^2-l^2}{a^2+l^2}\right) - \mathrm{i}\frac{\Lambda}{3}l(a^2-l^2)\\
&+\frac{(e^2+g^2)}{r-\mathrm{i}(l+a\cos\theta)}\left(1+\frac{\alpha a}{a^2+l^2}[ar\cos\theta + \mathrm{i}l(l+a\cos\theta)]\right)\Bigg],
\end{aligned} \tag{86}$$

$$\Phi_{11} = \frac{1}{2}(e^2+g^2)\frac{\Omega^4}{\rho^4}, \tag{87}$$

respectively, where

$$\begin{aligned}
\Omega &= 1 - \frac{\alpha a}{a^2+l^2}r(l+a\cos\theta),\\
\rho^2 &= r^2 + (l+a\cos\theta)^2,
\end{aligned} \tag{88}$$

cf. (48), (49). The Ricci scalar is simply

$$R = 4\Lambda, \tag{89}$$

which is the usual relation valid for any solution of Einstein-Maxwell equations with a cosmological constant Λ. While Φ_{11} is independent of Λ, the Weyl curvature component Ψ_2 contains the term proportional to $\Lambda l(a^2 - l^2)$. The dependence of Ψ_2 on the cosmological constant thus disappears if (and only if) $l = 0$ or $l = \pm a$.

For an invariant identification of curvature singularities and regions which asymptotically become conformally flat, it is necessary to evaluate the key (second-order) *scalar invariants*, namely the *Kretschmann invariant* $\mathcal{K}$ and the *Weyl invariant* $\mathcal{C}$,

$$\mathcal{K} \equiv R_{abcd}R^{abcd}, \tag{90}$$

$$\mathcal{C} \equiv C_{abcd}C^{abcd}. \tag{91}$$

This can be conveniently achieved in the NP formalism. Indeed, it is well known that

$$C^*_{abcd}C^{*abcd} = 32(\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2), \quad (92)$$

in which $C^*_{abcd} \equiv C_{abcd} + \mathrm{i}C^\sim_{abcd}$, where $C^\sim_{abcd}$ is the dual tensor to Weyl, $C^\sim_{abcd} \equiv \frac{1}{2}\epsilon_{cdef}C^{ef}_{ab}$. Since $C^\sim_{abcd}C^{\sim abcd} = -C_{abcd}C^{abcd}$, we get $C_{abcd}C^{abcd} + \mathrm{i}C^\sim_{abcd}C^{abcd} = \frac{1}{2}C^*_{abcd}C^{*abcd}$; see e.g. [9], or Eq. (17) in [21]. Therefore, the Weyl invariant is

$$\mathcal{C} = 16\,\mathcal{R}e(\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2). \quad (93)$$

From the definition of the Weyl tensor it follows that the Kretschmann invariant reads

$$\mathcal{K} = \mathcal{C} + 2R_{ab}R^{ab} - \frac{1}{3}R^2, \quad (94)$$

where $R = 4\Lambda$, while $R_{ab}R^{ab}$ can be expressed as[1]

$$\frac{1}{8}R_{ab}R^{ab} = \Phi_{00}\Phi_{22} + \Phi_{02}\bar{\Phi}_{02} - 2(\Phi_{01}\bar{\Phi}_{12} + \bar{\Phi}_{01}\Phi_{12}) + 2\Phi_{11}^2 + \frac{1}{32}R^2. \quad (95)$$

For the *black hole spacetimes* (47)–(53), which are of algebraic type D, the only nontrivial NP scalars are Ψ_2 and Φ_{11}, as given by (86) and (87), respectively. Therefore, the corresponding scalar curvature invariants are

$$\mathcal{C} = 48\,\mathcal{R}e(\Psi_2^2), \quad (96)$$

$$\mathcal{K} = \mathcal{C} + 32\Phi_{11}^2 + \frac{8}{3}\Lambda^2. \quad (97)$$

Interestingly the Weyl invariant takes the explicit factorized form,

$$\mathcal{C} = 48\frac{\Omega^6}{\rho^{12}}C_+C_-, \quad (98)$$

where

$$C_\pm = m\left(F_\pm \pm \alpha a\frac{a^2-l^2}{a^2+l^2}F_\mp\right) \mp l\left(\left[1+\frac{1}{3}\Lambda(a^2-l^2)\right]F_\mp \mp \alpha a\frac{a^2-l^2+e^2+g^2}{a^2+l^2}F_\pm\right) - (e^2+g^2)\left(1+\frac{\alpha a}{a^2+l^2}rL\right)T_\pm, \quad (99)$$

in which $F_\pm = (r\mp L)(r^2 \pm 4rL + L^2)$, $T_\pm = (r^2 \pm 2rL - L^2)$, and $L = l + a\cos\theta$.

[1]There are nine independent (real) quantities encoded in the complex NP scalars $\Phi_{AB} = \bar{\Phi}_{BA}$. Due to their usual definition, the projections on the null tetrad (85) of the Ricci tensor R_{ab} and of the related traceless Ricci tensor $S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$ give the same results. The additional tenth independent component of R_{ab} is given by $\frac{1}{4}Rg_{ab}$ containing the Ricci scalar R, so that $R_{ab}R^{ab}$ also involves the term $\frac{1}{16}R^2g_{ab}g^{ab} = \frac{1}{4}R^2$.

This is a generalization of the previously known expressions for the Kerr-Newman geometry; see [21,22] and elsewhere, in which case $\Lambda, l, g, \alpha = 0$ so that $\Omega = 1$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $C_\pm = m(r\mp a\cos\theta)(r^2 \pm 4ar\cos\theta + a^2\cos^2\theta) - e^2(r^2 \pm 2ar\cos\theta - a^2\cos^2\theta)$.

The spacetime also contains *electromagnetic field* represented by the Maxwell tensor F_{ab}, forming a 2-form $\mathbf{F} = \frac{1}{2}F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b = \mathrm{d}\mathbf{A}$. Its 1-form potential $\mathbf{A} = A_a\mathrm{d}x^a$ is

$$\mathbf{A} = -\sqrt{e^2+g^2}\frac{r}{\rho^2}\left[\mathrm{d}t - \left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]. \quad (100)$$

Therefore, the nonzero components of $F_{ab} = A_{b,a} - A_{a,b}$ are

$$F_{\mathrm{tr}} = -\sqrt{e^2+g^2}\rho^{-4}(r^2-(l+a\cos\theta)^2),$$
$$F_{\varphi r} = -F_{\mathrm{tr}}\left(a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta\right),$$
$$F_{t\theta} = 2a\sqrt{e^2+g^2}\rho^{-4}r\sin\theta(l+a\cos\theta),$$
$$F_{\varphi\theta} = -2\sqrt{e^2+g^2}\rho^{-4}r\sin\theta(l+a\cos\theta)(r^2+(a+l)^2). \quad (101)$$

The corresponding Newman-Penrose scalars are $\Phi_0 \equiv F_{ab}k^am^b = 0$, $\Phi_2 \equiv F_{ab}\bar{m}^al^b = 0$, and

$$\Phi_1 \equiv \frac{1}{2}F_{ab}(k^al^b + \bar{m}^am^b) = \frac{\sqrt{e^2+g^2}\Omega^2}{(r+\mathrm{i}(l+a\cos\theta))^2}. \quad (102)$$

It follows that $\Phi_1\bar{\Phi}_1 = 2\Phi_{11}$, in fully agreement with (87). The electromagnetic field thus vanishes if (and only if) $e = 0 = g$.

Since the only nontrivial NP Weyl scalar is Ψ_2, *both vectors* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are principal null directions* (PNDs). In fact, both are double-degenerate, demonstrating that the *gravitational field is of algebraic type D*. The electromagnetic field is non-null, and double-aligned with these PNDs because the only nonzero NP Maxwell scalar is Φ_1.

Moreover, by evaluating the *spin coefficients* for the null tetrad (85) one obtains

$$\kappa = \nu = 0, \qquad \sigma = \lambda = 0,$$
$$\varrho = \mu = -\frac{\sqrt{Q}}{\sqrt{2}\rho^3}\left(1+\mathrm{i}\frac{\alpha a}{a^2+l^2}(l+a\cos\theta)^2\right)\times(r-\mathrm{i}(l+a\cos\theta)),$$
$$\tau = \pi = -\frac{a\sqrt{P}\sin\theta}{\sqrt{2}\rho^3}\left(1-\mathrm{i}\frac{\alpha a}{a^2+l^2}r^2\right)(r-\mathrm{i}(l+a\cos\theta)). \quad (103)$$

Also $\alpha = \beta$ and $\epsilon = \gamma$ are nonzero, but we do not write them here due to their complexity.

Both double-degenerate PNDs generated by **k** and **l** (85) are thus *geodetic* ($\kappa = 0 = \nu$) and *shear-free* ($\sigma = 0 = \lambda$). However, they have *expansion* Θ and *twist* ω defined, respectively, by the real and imaginary parts of $\varrho \equiv -(\Theta + \mathrm{i}\omega) \equiv \mu$, namely

$$\Theta = \frac{\sqrt{Q}}{\sqrt{2}\rho^3}\left(r + \frac{\alpha a}{a^2+l^2}(l + a\cos\theta)^3\right), \tag{104}$$

$$\omega = -\frac{\Omega\sqrt{Q}}{\sqrt{2}\rho^3}(l + a\cos\theta). \tag{105}$$

It is now immediately seen from (105) that

(i) The black-hole spacetime is everywhere nontwisting if (and only if)

$$a = 0 = l. \tag{106}$$

In addition, on the horizons identified by $Q(r) = 0$ (see below) both the expansion and the twist always vanish ($\Theta = 0 = \omega$).

By inspecting the NP scalars (86)–(89) and (102), it is also obvious that

(ii) The curvature singularities occur if (and only if)

$$r = 0 \quad \text{and at the same time} \quad l + a\cos\theta = 0. \tag{107}$$

Indeed, *both* these conditions must be satisfied to have $r + \mathrm{i}(l + a\cos\theta) = 0$. With its complex conjugate, this implies

$$\rho^2 \equiv r^2 + (l + a\cos\theta)^2 = 0. \tag{108}$$

This agrees with the Weyl scalar (98).

(iii) The region of a generic spacetime is conformally flat if (and only if)

$$\Omega = 0. \tag{109}$$

With this condition, the spacetime is also locally vacuum, cf. (87), with a cosmological constant Λ. The condition $\Omega = 0$ thus localizes the asymptotic *(anti–)de Sitter/Minkowski conformal infinity.*

(iv) In the case when $m = 0 = l$ and also $e = 0 = g$ then $\Psi_2 = 0 = \Phi_{11}$, so that

$$\text{the space time is everywhere conformally flat and vacuum.} \tag{110}$$

The metric (47)–(55) then represents de Sitter spacetime (for $\Lambda > 0$), anti–de Sitter spacetime (for $\Lambda < 0$), and Minkowski spacetime (for $\Lambda = 0$).

Curvature of the subclasses of type D black holes, summarized in Sec. IV, are easily obtained from the general expression (86) by setting up the corresponding physical parameters to zero:

(i) Kerr-Newman-NUT-(anti–)de Sitter ($\alpha = 0$: No acceleration)

$$\Psi_2 = \frac{1}{[r + \mathrm{i}(l + a\cos\theta)]^3}\left[-m - \mathrm{i}l\left[1 + \frac{1}{3}\Lambda(a^2 - l^2)\right] + \frac{e^2+g^2}{r - \mathrm{i}(l + a\cos\theta)}\right], \tag{111}$$

(ii) Accelerating Kerr-Newman-(anti–)de Sitter ($l = 0$: No NUT)

$$\Psi_2 = \frac{(1 - \alpha r\cos\theta)^3}{(r + \mathrm{i}a\cos\theta)^3}\left[-m(1 - \mathrm{i}\alpha a) + (e^2+g^2)\frac{1 + \alpha r\cos\theta}{r - \mathrm{i}a\cos\theta}\right], \tag{112}$$

(iii) Charged Taub-NUT-(anti–)de Sitter ($a = 0$: No rotation)

$$\Psi_2 = -\frac{m + \mathrm{i}l(1 - \frac{1}{3}\Lambda l^2)}{(r + \mathrm{i}l)^3} + \frac{e^2+g^2}{(r^2+l^2)(r+\mathrm{i}l)^2}. \tag{113}$$

Observe that the cosmological constant Λ appears in the Weyl curvature scalar Ψ_2 only if the NUT parameter l is also present.

These expressions further simplify if some of the remaining parameters are zero. In particular, the curvature of *Kerr-Newman-(anti–)de Sitter* black hole is obtained from (111) if $l = 0$. The curvature for *generalized C-metric with* Λ (accelerating charged black holes without rotation) are obtained from (112) when $a = 0$. The curvature of *Reissner-Nordström-(anti–)de Sitter* black hole follows from (113) when $l = 0$. The *uncharged* (vacuum) black holes are obtained for $e = 0 = g$.

B. Horizons

Next step is the investigation of *horizons of the black hole metric* (47), namely their *number*, possible *degeneration*, and *location*. It is immediately seen that the "radial" coordinate r is spatial in the regions where $Q(r) > 0$, while it is a temporal coordinate where $Q(r) < 0$. These regions are separated by horizons $\mathcal{H}$ located at r_h such that

$$Q(r_h) = 0, \tag{114}$$

where the key metric function $Q(r)$ is explicitly given by expression (51). In the particular "under-extreme" case

$\mu^2 + l^2 > a^2 + e^2 + g^2 + \lambda$, the alternative form of this function (53) with $r_{\Lambda+} \neq r_{\Lambda-}$ can be used.

These observations are in accordance with the behaviour of the *determinant of the metric* (47) constrained on a constant r which, due to the identity $\rho^2 = r^2 + (a+l)^2 - a(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta)$, is simply

$$\det(g_{\mu\nu}|_{r=\text{const}}) = -\frac{\rho^2}{\Omega^6}\mathcal{Q}\sin^2\theta. \tag{115}$$

Such a 3-surface is thus timelike when $\mathcal{Q} > 0$, while it is spacelike when $\mathcal{Q} < 0$. On any horizon the determinant vanishes (degenerates) due to (114).

Moreover, the determinant of the *complete* metric (47) reads $\det g_{\mu\nu} = -\Omega^{-8}\rho^4\sin^2\theta$. This indicates nonregularity only at $\Omega = 0$ (conformal infinity), $\rho = 0$ (curvature singularity), $\mathcal{Q} = 0$ (horizons), and $\theta = 0$ or $\theta = \pi$ (poles/axes with possible cosmic strings).

Since the function $\mathcal{Q}(r)$ does not directly enter the Weyl scalar (86) or the Ricci scalar (87)—and thus the invariants $\mathcal{C}$ and $\mathcal{K}$ given by (96) and (97)—there is *no curvature/physical obstacle* located at any of the horizons r_h. Explicit extension of the coordinate system across the horizons $\mathcal{H}$ will be presented in Sec. V F.

To analyze the number, possible degeneration, and location of the horizons, it is thus necessary to find all root of the equation (114). Because the function (51) is a *polynomial of the fourth order*, it admits *up to four real roots*. In the *generic black hole spacetime* (47) there is thus *four possible horizons* $\mathcal{H}$. We can call and denote them as follows:

(i) two black hole horizons $\mathcal{H}_b^\pm$ located at $r_b^\pm$,
(ii) two cosmo-acceleration horizons $\mathcal{H}_c^\pm$ located at $r_c^\pm$.

While the terminology *black hole horizon* is common and standard, we hereby introduce a new name *cosmo-acceleration horizon* which combines the usual names for *cosmological* and for *acceleration* horizons. These are mutually combined in this family of spacetimes due to the presence of *both* the acceleration α *and* the cosmological constant Λ.

Let us now analyze these horizons explicitly. The generic key metric function $\mathcal{Q}(r)$ is the *quartic polynomial* of r, namely

$$\mathcal{Q}(r) = q_4 r^4 + q_3 r^3 + q_2 r^2 + q_1 r + q_0, \tag{116}$$

where the coefficients are

$$\begin{aligned}
q_4 &\equiv -\alpha^2 a^2 \frac{a^2 - l^2}{(a^2+l^2)^2} - \frac{\Lambda}{3},\\
q_3 &\equiv 2\alpha a\left[\alpha a m \frac{a^2-l^2}{(a^2+l^2)^2} - \frac{l}{a^2+l^2} - l\frac{a^2-l^2}{a^2+l^2}\frac{\Lambda}{3}\right],\\
q_2 &\equiv 1 + 4\alpha a m \frac{l}{a^2+l^2} - \alpha^2 a^2 \frac{a^2-l^2}{(a^2+l^2)^2}(a^2 - l^2 + e^2 + g^2) - (a^2+3l^2)\frac{\Lambda}{3},\\
q_1 &\equiv -2m - 2\alpha a \frac{l}{a^2+l^2}(a^2 - l^2 + e^2 + g^2),\\
q_0 &\equiv a^2 - l^2 + e^2 + g^2.
\end{aligned} \tag{117}$$

The *quartic equation* $\mathcal{Q}(r) = 0$ can have from zero to maximally four explicit real roots r_h corresponding to the horizons. In particular, we may observe that

(i) Maximally four horizons is the *general case* which will be discussed in detail in subsequent Sec. V C. Some of the roots of (114) may coincide, resulting in *degenerate* horizons (doubly, triply, or even quadruply).
(ii) Maximally three horizons occur in spacetimes with the physical parameters related in such a way that $q_4 = 0$, that is for

$$\frac{\Lambda}{3} = -\alpha^2 a^2 \frac{a^2-l^2}{(a^2+l^2)^2}. \tag{118}$$

For these black hole spacetimes the metric function $\mathcal{Q}(r)$ reduces to a *cubic function*. Notice that in the case $l = 0$, this condition is simply $\alpha^2 = -\Lambda/3$, i.e., a specific relation between the acceleration of the (rotating and charged) black hole and the *negative* cosmological constant (while the complementary case $a = 0$ requires $\Lambda = 0$). Further analysis of this case will be presented in our subsequent paper.
(iii) Maximally two horizons occur in spacetimes with such parameters that—in addition to the condition (118)—also the second coefficient in (116) vanishes, $q_3 = 0$, that is for $\alpha a = 0 \Rightarrow \Lambda = 0$, or for

$$\alpha a m = l\left(\frac{a^2+l^2}{a^2-l^2} - \alpha^2 a^2 \frac{a^2-l^2}{a^2+l^2}\right). \tag{119}$$

Equation (114) is then a quadratic equation $q_2 r^2 + q_1 r + q_0 = 0$, from which both horizons

r_h can be easily calculated. If $q_1^2 - 4q_2q_0 = 0$, these two horizons coincide (it is double degenerate), and for $q_1^2 - 4q_2q_0 < 0$ there is no horizon.

(iv) Maximally one horizon occurs when both the constraints (118) and (119) are satisfied, and moreover $q_2 = 0$, that is

$$\alpha^2 a^2 (e^2 + g^2) = (a^2 + 3l^2)\left(\frac{a^2 + l^2}{a^2 - l^2}\right)^2. \quad (120)$$

The single horizon is then located at

$$r_h = -\frac{q_0}{q_1} = \frac{1}{4\alpha a l}\left(a^2 + 3l^2 + \alpha^2 a^2 \frac{(a^2 - l^2)^3}{(a^2 + l^2)^2}\right). \quad (121)$$

For $q_1 = 0$ there is no horizon.

These three conditions (118), (119), and (120) characterize very special black hole spacetimes in which the physical parameters Λ, m, and $e^2 + g^2$ have particular values in terms of the Kerr-like rotational parameter a, NUT parameter l, and acceleration α.

It is a usual procedure that *the general quartic equation* (114), (116) can be solved by first dividing it by a nonzero prefactor q_4 and then performing the substitution,

$$r \equiv x - \frac{q_3}{4q_4}, \quad (122)$$

leading to the *depressed (reduced) quartic equation* without the cubic term,

$$\frac{1}{q_4} Q(x) = x^4 + \frac{N}{8\mathcal{N}^2}x^2 - \frac{R}{8\mathcal{N}^3}x + \frac{S}{256\mathcal{N}^4} = 0, \quad (123)$$

where $\mathcal{N} \equiv -q_4$, the coefficients are

$$N \equiv 8q_4q_2 - 3q_3^2, \quad (124)$$

$$R \equiv 8q_4^2q_1 - 4q_4q_3q_2 + q_3^3, \quad (125)$$

$$S \equiv 256q_4^3q_0 - 64q_4^2q_3q_1 + 16q_4q_3^2q_2 - 3q_3^4, \quad (126)$$

and the constants q_i are explicitly defined by (117).

Moreover, the *discriminant* Δ of the general quartic polynomial (116) is

$$\begin{aligned}\Delta \equiv\, & 256q_4^3q_0^3 - 192q_4^2q_3q_1q_0^2 - 128q_4^2q_2^2q_0^2 + 144q_4^2q_2q_1^2q_0 \\ & - 27q_4^2q_1^4 + 144q_4q_3^2q_2q_0^2 - 6q_4q_3^2q_1^2q_0 \\ & - 80q_4q_3q_2^2q_1q_0 + 18q_4q_3q_2q_1^3 + 16q_4q_2^4q_0 \\ & - 4q_4q_2^3q_1^2 - 27q_3^4q_0^2 + 18q_3^3q_2q_1q_0 - 4q_3^3q_1^3 \\ & - 4q_3^2q_2^3q_0 + q_3^2q_2^2q_1^2. \end{aligned} \quad (127)$$

This is simply related to the discriminant of the depressed quartic function (123) via

$$\Delta = \mathcal{N}^6 \Delta_{\text{depressed}},$$

so that *the signs* of Δ and $\Delta_{\text{depressed}}$ *are the same*.

In terms of these key quantities Δ, N, S and R, a *complete* analysis and a full description of the number and the possible multiplicity of roots can now be performed. Following [23], we can summarize that

For $\Delta > 0$:

The metric function $Q(r)$ has either *four distinct real roots*, or none, and that depends on:

(i) If $N < 0$ and $N^2 > S$ then *all four roots are real and distinct*.
(ii) If $N < 0$ and $N^2 < S$ then there are *two pairs of complex conjugate nonreal roots*.
(iii) If $N \geq 0$ then there are also *two pairs of complex conjugate nonreal roots*.

For $\Delta < 0$:

The function $Q(r)$ has *two distinct real roots* and *two complex conjugate nonreal roots*.

For $\Delta = 0$:

This is the only case when the metric function $Q(r)$ has at least one *multiple root*.

The different cases that can occur are

(1) If $N < 0$ together with
  (a) $N^2 < S$: there is *one real double root* and *two complex conjugate roots*.
  (b) $N^2 = S$: there are *two distinct real double roots*.
  (c) $N^2 > S$ and $N^2 > -3S$: there is *one real double root* and *two distinct simple real roots*.
  (d) $N^2 = -3S$: there is *one real triple root* and *one distinct simple real root*.
(2) If $N > 0$ together with
  (a) $S = 0$: there is *one real double root* and *two complex conjugate roots*.
  (b) $S > 0$ and $R \neq 0$: there is also *one real double root* and *two complex conjugate roots*.
  (c) $S = N^2$ and $R = 0$: there are only *two complex conjugate double roots*.
(3) If $N = 0$ together with
  (a) $S > 0$: there is *one real double root* and *two complex conjugate roots*.
  (b) $S = 0$ (implying $R = 0$): there is *one real quadruple root* $x = 0$, that is $r_h = -\frac{q_3}{4q_4}$.

This exhausts all the possibilities.

C. The case with two black hole and two cosmo-acceleration horizons

We will now concentrate on physically most interesting case in which there are *four distinct real roots*. This may appear only in the case when $q_4 \neq 0$ (otherwise there are maximally three horizons), i.e., when the cosmological constant Λ is *not* "finely tuned" to acceleration α and the two twist parameters a and l, that is for

$$\frac{\Lambda}{3} \neq -\alpha^2 a^2 \frac{a^2 - l^2}{(a^2 + l^2)^2}. \tag{128}$$

In particular, we can observe that for $\Lambda = 0$ there are no nonaccelerating or nonrotating black holes ($\alpha a = 0$) with four horizons.

In such generic black hole spacetimes there are *two black hole horizons* $\mathcal{H}_b^+$ and $\mathcal{H}_b^-$ and also *two cosmo-acceleration horizons* $\mathcal{H}_c^+$ and $\mathcal{H}_c^-$. With the assumption that they are generically distinct, we can rewrite the key metric function $Q(r)$ given by (116), (117) in a factorized form as

$$Q(r) = -\mathcal{N}(r - r_b^+)(r - r_b^-)(r - r_c^+)(r - r_c^-), \tag{129}$$

where $\mathcal{N} \equiv -q_4$ reads

$$\mathcal{N} = \alpha^2 a^2 \frac{a^2 - l^2}{(a^2 + l^2)^2} + \frac{\Lambda}{3}, \tag{130}$$

while the four roots $r_b^+, r_b^-, r_c^+, r_c^-$ localize the four distinct horizons, namely

$$\mathcal{H}_b^+ \text{ at } r_b^+ \quad \text{is the outer black hole horizon}, \tag{131}$$

$$\mathcal{H}_b^- \text{ at } r_b^- \quad \text{is the inner black hole horizon}, \tag{132}$$

$$\mathcal{H}_c^+ \text{ at } r_c^+ \quad \text{is the outer cosmo-acceleration horizon}, \tag{133}$$

$$\mathcal{H}_c^- \text{ at } r_c^- \quad \text{is the inner cosmo-acceleration horizon}. \tag{134}$$

In view of the classification scheme summarized above, this occurs if (and only if)

$$\Delta > 0 \quad \text{and} \quad N < 0 \quad \text{and} \quad N^2 > S. \tag{135}$$

Moreover, we can assume a natural ordering of these horizons as

$$r_c^- < r_b^- < r_b^+ < r_c^+, \tag{136}$$

so that the cosmological horizons are located "outside" the black hole horizons. Because $Q(r) < 0$ for all $r > r_c^+$ when $\mathcal{N} > 0$, such an ordering guarantees that these four horizons separate the corresponding five regions of the spacetime in such a way that they are, symbolically expressed,

$$\text{time-dependent} < \text{stationary} < \text{time-dependent} < \text{stationary} < \text{time-dependent}. \tag{137}$$

It means, for example, that in the whole range $r \in (r_b^+, r_c^+)$, the coordinate r is spatial. Therefore, the region between the outer black hole horizon $\mathcal{H}_b^+$ and the outer cosmo-acceleration horizon $\mathcal{H}_c^+$ is stationary.

The natural ordering (136) implying (137) is present for a large range of values of the cosmological constant Λ, including $\Lambda = 0$. In fact, it is a straightforward generalization of the ordering of two black hole horizons and two acceleration horizons in the family of type D black holes spacetimes without the cosmological constant; see Eq. (80) in our previous paper [14]. The ordering (137) depends on the constraint $\mathcal{N} > 0$ which, using (130), reads

$$\alpha^2 a^2 \frac{a^2 - l^2}{(a^2 + l^2)^2} + \frac{\Lambda}{3} > 0. \tag{138}$$

In the $\Lambda = 0$ case, this condition reduces simply to $|l| < |a|$, while in the case $l = 0$ it is

$$\frac{\Lambda}{3} > -\alpha^2. \tag{139}$$

Notice also that for $|l| \geq |a|$ only (a sufficiently large) $\Lambda > 0$ is admitted.

An *explicit* evaluation of the four distinct roots of the metric function $Q(r)$ in the factorized form (129) in terms of the seven physical parameters $m, a, l, e, g, \alpha, \Lambda$ is quite cumbersome, leading to rather complicated expressions. Nevertheless, it may be useful to present them here. Using a standard procedure of *Wolfram Mathematica 13* one obtains

$$r_b^\pm = \frac{1}{2}\left(\sqrt{V} - H \pm \sqrt{G - 2F/\sqrt{V}}\right), \tag{140}$$

$$r_c^\pm = \frac{1}{2}\left(-\sqrt{V} - H \pm \sqrt{G + 2F/\sqrt{V}}\right), \tag{141}$$

where

$$V = H^2 + \frac{1}{3\mathcal{N}}\left[2X - \left(Z + \mathrm{i}\sqrt{Y^3 - Z^2}\right)^{\frac{1}{3}} - \left(Z - \mathrm{i}\sqrt{Y^3 - Z^2}\right)^{\frac{1}{3}}\right], \tag{142}$$

$$H = -\frac{K}{\mathcal{N}}, \quad G = 3H^2 + \frac{2X}{\mathcal{N}} - V, \quad F = H^3 + \frac{2L}{\mathcal{N}} - \frac{KX}{\mathcal{N}^2}, \tag{143}$$

and

$$K = \frac{\alpha a}{a^2 + l^2}\left[\left(\frac{\alpha a}{a^2 + l^2} m - \frac{\Lambda}{3} l\right)(a^2 - l^2) - l\right], \tag{144}$$

$$L = m + \frac{\alpha a l}{a^2 + l^2}(a^2 - l^2 + e^2 + g^2), \tag{145}$$

$$X = 1 + 4\frac{\alpha a l}{a^2 + l^2}m - \alpha^2 a^2 \frac{a^2 - l^2}{(a^2 + l^2)^2}(a^2 - l^2 + e^2 + g^2) - (a^2 + 3l^2)\frac{\Lambda}{3}, \tag{146}$$

$$Y = X^2 + 12KL - 12(a^2 - l^2 + e^2 + g^2)\mathcal{N}, \tag{147}$$

$$Z = X^3 + 18KLX - 54L^2\mathcal{N} + 18(a^2 - l^2 + e^2 + g^2) \times (3K^2 + 2\mathcal{N}X). \tag{148}$$

Although these expression are fully explicit, they are not telling much, and so we prefer to postpone their discussion to our subsequent paper. For example, it is possible to show that the complicated discriminant (127) can be nicely expressed as

$$\Delta = \frac{4}{27}(Y^3 - Z^2). \tag{149}$$

The condition $\Delta = 0$ for the existence of multiple roots thus simplifies to $Y^3 = Z^2$.

D. Ergoregions

For the generic black hole metric (47) the condition,

$$g_{tt} \equiv \frac{1}{\Omega^2\rho^2}(Pa^2\sin^2\theta - Q) = 0, \tag{150}$$

defines the *boundary of the ergoregions*, that are the surface of infinite redshift and also the stationary limit at which observers on fixed r and θ cannot “stand still”. It can be seen that for a vanishing Kerr-like rotation parameter a such a boundary coincides with a horizon determined by $Q = 0$, but *for any* $a \neq 0$ there exists a nontrivial ergoregion between the $g_{tt} = 0$ boundary and the horizon. Moreover, the existence of ergoregions is related only to the Kerr-like rotation parameter a, not to the twist NUT parameter l.

There is an ergoregion associated with *any of the four horizons* $\mathcal{H}_b^{\pm}$ and $\mathcal{H}_c^{\pm}$. Indeed, the ergoregion boundary (150) is located at

$$Q(r_e) = a^2\sin^2\theta P(\theta), \tag{151}$$

where the metric functions $P(\theta)$ and $Q(r)$ are given by (50) and (51), or (52) and (53), respectively. For a fixed value of the angular coordinate θ, the right-hand side of (151) is a specific constant. Because the function $Q(r)$ is of the fourth order, it follows that there are (at most) *four* boundaries r_e of the ergoregions in the direction of θ.

From (151) it is also obvious that *the ergoregion boundary “touches” the corresponding horizon at the poles* because for $\theta = 0$ and $\theta = \pi$ the condition (151) reduces to $Q(r_e) = 0$.

It is generally complicated to explicitly solve Eq. (151), but it can be plotted using a computer. Typical results are shown and discussed in Fig. 1.

E. Curvature singularities

By inspecting the Newman-Penrose scalars Ψ_2 and Φ_{11} given explicitly as (86) and (87), we have already concluded that the *curvature singularities occur if and only if* $\rho^2 = 0$, that is when

$$r = 0 \quad \text{and at the same time} \quad l + a\cos\theta = 0; \tag{152}$$

see (107). The presence of these curvature singularities has also been confirmed by the behavior of the Weyl invariant $\mathcal{C} \equiv C_{abcd}C^{abcd}$ and the Kretschmann invariant $\mathcal{K} \equiv R_{abcd}R^{abcd}$, evaluated in (96) and (97).

Now, the condition $l + a\cos\theta = 0$ *can only be satisfied if* $|a| \geq |l|$. Otherwise, $l + a\cos\theta$ remains nonzero because $\cos\theta$ is bounded to the range $[-1, 1]$. Therefore, the curvature singularity structure of the complete family of type D spacetimes (47) depends on *relative values of the two twist parameters*, that is the Kerr-like rotation parameter a and the NUT parameter l, as follows:

$$\begin{aligned}
l = 0, a = 0&: \text{ singularity at } r = 0 \text{ for any } \theta,\\
l = 0, a \neq 0&: \text{ singularity at } r = 0 \text{ for } \theta = \pi/2,\\
l \neq 0, a = 0&: \text{ no singularity},\\
|l| > |a| > 0&: \text{ no singularity},\\
l = +a&: \text{ singularity at } r = 0 \text{ for } \theta = \pi,\\
l = -a&: \text{ singularity at } r = 0 \text{ for } \theta = 0,\\
|a| > |l| > 0&: \text{ singularity at } r = 0 \text{ for } \cos\theta = -l/a.
\end{aligned} \tag{153}$$

These results agree with the well-known character of the $r = 0$ singularity of the Schwarzschild-(anti–)de Sitter, Reissner-Nordström-(anti–)de Sitter and (possibly charged) C-metric spacetimes ($l = 0$, $a = 0$, in this order), the ring singularity structure of the Kerr-Newman-(anti–)de Sitter black holes ($l = 0$, $\alpha = 0$), and the absence of curvature singularities in the Taub-NUT-(anti–)de Sitter spacetime ($a = 0$, $\alpha = 0$). For a recent detailed analysis of the singular ring structure in these Kerr-like metrics see [24].

Moreover, from the generic form (51) of the metric function $Q(r)$, or equivalently (116), evaluated at $r = 0$ we obtain

$$Q(0) = q_0 \equiv a^2 - l^2 + e^2 + g^2. \tag{154}$$

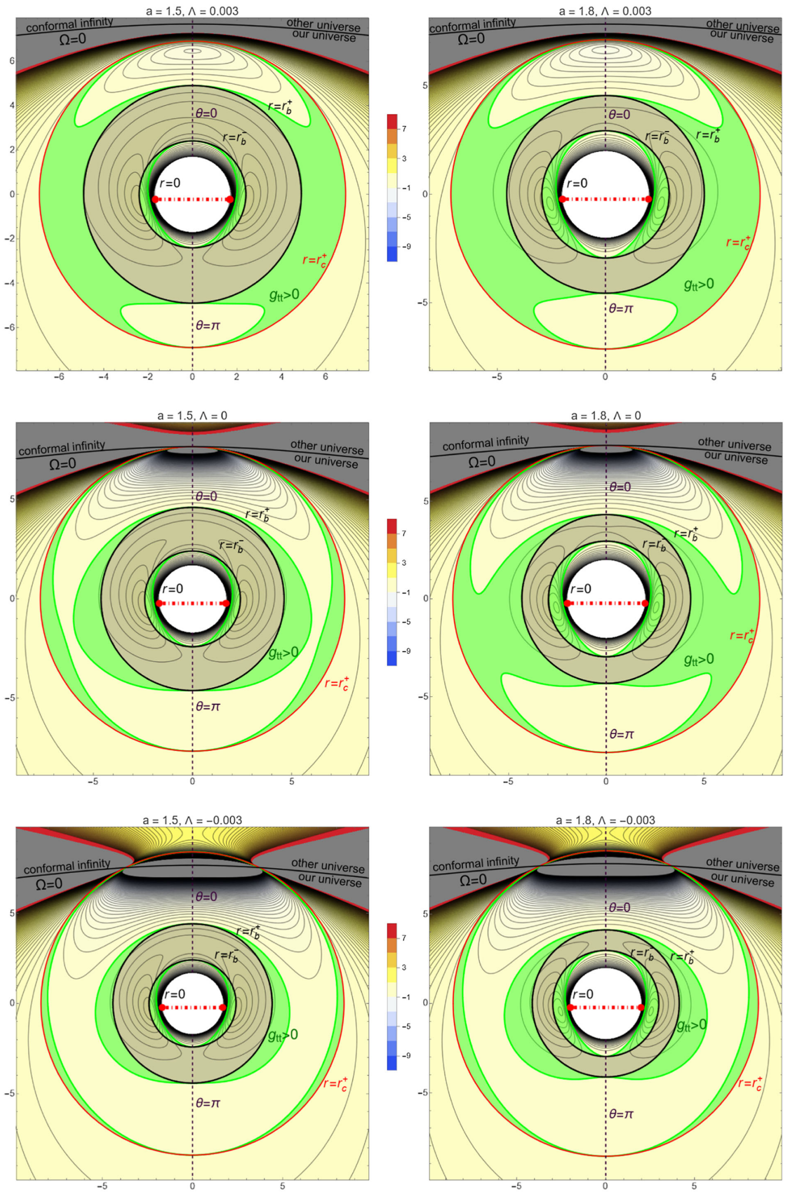

FIG. 1. Plot of the metric function g_{tt} (150) for the generic spacetime (47). The values of g_{tt} are visualized in quasipolar coordinates $\mathrm{x} \equiv \sqrt{r^2 + (a+l)^2}\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + (a+l)^2}\cos\theta$ for $r \geq 0$. The gray annulus around the center of each figure localizes the black hole bordered by its horizons $\mathcal{H}_b^{\pm}$ at r_b^+ and r_b^- $(0 < r_b^- < r_b^+)$. The cosmo-acceleration horizon $\mathcal{H}_c^+$ at r_c^+ (red circle) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. The gray curves are contour lines $g_{tt}(r,\theta) = \text{const}$, and the values are color-coded from red (positive values) to blue (negative values). The green curves are the isolines $g_{tt} = 0$ determining the boundary of the ergoregions (151) in which $g_{tt} > 0$ (green regions). All six plots are made for the same choice $m = 3$, $l = 0.2$, $e = 1.6 = g$, $\alpha = 0.12$. There are two distinct choices of the Kerr-like rotation parameter, namely $a = 1.5$ (left) and $a = 1.8$ (right). The rows visualize three different signs of the cosmological constant, namely $\Lambda = 0.003$ (top), $\Lambda = 0$ (middle) and $\Lambda = -0.003$ (bottom). For larger values of a and Λ the ergoregions are bigger. In fact, the ergoregion above the black hole horizon $\mathcal{H}_b^+$ *is merged* with the ergoregion below the cosmo-acceleration horizon $\mathcal{H}_c^+$ in the equatorial part near $\theta = \frac{\pi}{2}$.

The singularity at $r = 0$ occurs only if $a^2 \geq l^2$, see (153), so that it is located only in the *stationary* region where $Q > 0$. In fact, in view of the natural ordering (136) and the scheme (137), the ring singularity must be contained in the region $r \in (r_c^-, r_b^-)$ *between the horizons* $\mathcal{H}_c^-$ *and* $\mathcal{H}_b^-$. The alternative possibility $r \in (r_b^+, r_c^+)$ would correspond to a *naked singularity* in the stationary region located *outside* the horizon $\mathcal{H}_b^+$.

F. Global structure and conformal diagrams

Now we analyze the global structure and the maximal extension of the spacetime. As in the previous parts, we will assume the generic case with *four distinct horizons* $\mathcal{H}_b^\pm$ and $\mathcal{H}_c^\pm$ located at $r_b^\pm$ and $r_c^\pm$, that are ordered as $r_c^- < r_b^- < r_b^+ < r_c^+$; see (136).

The procedure is basically the same as in Sec. V.D of our previous paper [14], and extends special cases of non-accelerating black holes, see e.g. [22,25–31], or black holes with acceleration [32,33]. First, the *retarded* and *advanced null* coordinates are defined,

$$u = t - r_* \quad \text{and} \quad v = t + r_*, \tag{155}$$

where the *tortoise* coordinate is

$$r_* = \int \frac{r^2 + (a+l)^2}{Q(r)} \mathrm{d}r, \tag{156}$$

and also the corresponding *untwisted angular* coordinates are introduced by

$$\phi_u = \varphi - a \int \frac{\mathrm{d}r}{Q(r)} \quad \text{and} \quad \phi_v = \varphi + a \int \frac{\mathrm{d}r}{Q(r)}. \tag{157}$$

Using the *advanced* pair of coordinates $\{v, \phi_v\}$, the metric (47) takes the form,

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2} \bigg[& \frac{a^2 P \sin^2\theta - Q}{\rho^2} (\mathrm{d}v - \mathcal{T}\mathrm{d}\phi_v)^2 \\ & + 2(\mathrm{d}v - \mathcal{T}\mathrm{d}\phi_v)(\mathrm{d}r - aP\sin^2\theta \mathrm{d}\phi_v) \\ & + \rho^2 \left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta \mathrm{d}\phi_v^2 \right) \bigg], \end{aligned} \tag{158}$$

where $\mathcal{T}(\theta) \equiv a \sin^2\theta + 4l \sin^2\frac{1}{2}\theta$, while using the *retarded* pair of coordinates $\{u, \phi_u\}$ it reads

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2} \bigg[& \frac{a^2 P \sin^2\theta - Q}{\rho^2} (\mathrm{d}u - \mathcal{T}\mathrm{d}\phi_u)^2 \\ & - 2(\mathrm{d}u - \mathcal{T}\mathrm{d}\phi_u)(\mathrm{d}r + aP\sin^2\theta \mathrm{d}\phi_u) \\ & + \rho^2 \left(\frac{\mathrm{d}\theta^2}{P} + P\sin^2\theta \mathrm{d}\phi_u^2 \right) \bigg]. \end{aligned} \tag{159}$$

Both these metrics are *regular* at $Q(r) = 0$, so that *the coordinate singularities at the horizons has been removed*.

The next step in construction of the maximal (analytic) extension of the manifold is to introduce *both the null coordinates u and v simultaneously*, revealing thus the causal structure. The coordinate r is eliminated using the relation (155) which implies

$$2\,\mathrm{d}r = \frac{Q}{r^2 + (a+l)^2} (\mathrm{d}v - \mathrm{d}u). \tag{160}$$

In addition, it is necessary to construct a *unique angular coordinate* ϕ_h across the horizon ar r_h using the specific relation,

$$\phi_h = \varphi - \Omega_h t, \quad \text{where} \quad \Omega_h = \frac{a}{r_h^2 + (a+l)^2}. \tag{161}$$

The constant Ω_h is the *angular velocity of the horizon*. Actually, $2\mathrm{d}\phi_h = \mathrm{d}\phi_u + \mathrm{d}\phi_v - \Omega_h(\mathrm{d}u + \mathrm{d}v)$. This it the unique way how to properly combine the distinct angular coordinates ϕ_v and ϕ_u (for more details see [14]).

Unfortunately, the specific choice of the angular coordinate ϕ_h *depends on the given horizon* via its value r_h and thus Ω_h. For this reason, it is not possible to find a single and simple global coordinate ϕ which would conveniently "cover" *all the four horizons*. This drawback was met many years ago already in the Kerr spacetime, so it is not surprising that it reappears in the current context of the complete family of type D black holes with seven physical parameters.

An explicit *general* metric form of this family constructed in this way reads

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{4\Omega^2} \bigg[& -\frac{Q}{\rho^2} ((1 - \mathcal{T}\Omega_h)(\mathrm{d}u + \mathrm{d}v) - 2\mathcal{T}\mathrm{d}\phi_h)^2 + Q\rho^2 \frac{(\mathrm{d}u - \mathrm{d}v)^2}{[r^2 + (a+l)^2]^2} + 4\frac{\rho^2}{P}\mathrm{d}\theta^2 \\ & + \frac{P\sin^2\theta}{\rho^2} ((a - [r^2 + (a+l)^2]\Omega_h)(\mathrm{d}u + \mathrm{d}v) - 2[r^2 + (a+l)^2]\mathrm{d}\phi_h)^2 \bigg]. \end{aligned} \tag{162}$$

For *nontwisting* black holes without the Kerr-like rotation ($a = 0$) and the NUT parameter ($l = 0$), the metric functions simplify to $\Omega = 1$, $P = 1$, $\rho^2 = r^2$, $\mathcal{T} = 0$, $\Omega_h = 0$, so that

$$
\mathrm{d}s^2 = -\frac{Q}{r^2}\mathrm{d}u\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi_h^2), \qquad (163)
$$

which is the usual form of the spherically symmetric black holes in the double-null coordinates [10].

It remains to analyze the *global extension of* (162) and to study the degree of smoothness (analyticity) of the four distinct horizons $\mathcal{H}_b^{\pm}$ and $\mathcal{H}_c^{\pm}$ where $Q(r_h) = 0$. Restricting to any two-dimensional section $\theta = \text{const}$ and $\phi_h = \text{const}$ the general metric (162) reduces to

$$
\begin{aligned}
\mathrm{d}\sigma^2 = \frac{1}{4\Omega^2}\Big[&-\frac{(1-\mathcal{T}\Omega_h)^2}{\rho^2}Q(\mathrm{d}u+\mathrm{d}v)^2 \\
&+\frac{\rho^2}{[r^2+(a+l)^2]^2}Q(\mathrm{d}u-\mathrm{d}v)^2 \\
&+a^2\frac{P\sin^2\theta}{\rho^2}\frac{(r+r_h)^2(r-r_h)^2}{[r_h^2+(a+l)^2]^2}(\mathrm{d}u+\mathrm{d}v)^2\Big], \qquad (164)
\end{aligned}
$$

which is null at any horizon r_h where $Q(r_h) = 0$. Due to the simple *factorized* form (129) of the metric function $Q(r)$, the integral (156) defining the function $r_*(r)$ can be calculated explicitly as

$$
\begin{aligned}
r_*(r) = k_b^+ \log\left|1-\frac{r}{r_b^+}\right| + k_b^- \log\left|1-\frac{r}{r_b^-}\right| \\
+ k_c^+ \log\left|1-\frac{r}{r_c^+}\right| + k_c^- \log\left|1-\frac{r}{r_c^-}\right|, \qquad (165)
\end{aligned}
$$

where the auxiliary coefficients are

$$
\begin{aligned}
k_b^+ &= -\frac{(r_b^+)^2+(a+l)^2}{\mathcal{N}(r_b^+-r_b^-)(r_b^+-r_c^+)(r_b^+-r_c^-)}, \\
k_b^- &= -\frac{(r_b^-)^2+(a+l)^2}{\mathcal{N}(r_b^--r_b^+)(r_b^--r_c^+)(r_b^--r_c^-)}, \\
k_c^+ &= -\frac{(r_c^+)^2+(a+l)^2}{\mathcal{N}(r_c^+-r_b^+)(r_c^+-r_b^-)(r_c^+-r_c^-)}, \\
k_c^- &= -\frac{(r_c^-)^2+(a+l)^2}{\mathcal{N}(r_c^--r_b^+)(r_c^--r_b^-)(r_c^--r_c^+)}. \qquad (166)
\end{aligned}
$$

Each of these constants is associated with the corresponding horizon $\mathcal{H}_h^{\pm}$ located at $r = r_h^{\pm}$, where $h = b$ (for the black hole horizons) or $h = c$ (for the cosmo-acceleration horizons).

We can express the metric functions $Q(r)$, $\rho^2(r)$ and $\Omega^2(r)$ entering (164) in terms of the null coordinates $v - u$ instead of r by using the inversion of the relation $2r_*(r) = v - u$. Finally, we introduce *the couples of new null coordinates* $U_h^{\pm}$ and $V_h^{\pm}$, defined as

$$
U_h^{\pm} = (-1)^i \operatorname{sign}(k_h^{\pm}) \exp\left(-\frac{u}{2k_h^{\pm}}\right), \qquad (167)
$$

$$
V_h^{\pm} = (-1)^j \operatorname{sign}(k_h^{\pm}) \exp\left(+\frac{v}{2k_h^{\pm}}\right). \qquad (168)
$$

Each couple covers the corresponding horizon $\mathcal{H}_h^{\pm}$. Moreover, it is characterized by a *particular choice of two integers* (i, j) which specify a certain region in the manifold. Generally, there are *five types of regions* which are separated by the four types of horizons $\mathcal{H}_h^{\pm}$, namely

Region	**Description**	**Specification of** (i, j)
I:	asymptotic time-dependent domain between $\mathcal{H}_c^+$ and $\mathcal{I}^+$	$(n-2m+1, n+2m-1)$
II:	stationary region between $\mathcal{H}_b^+$ and $\mathcal{H}_c^+$	$(2n-m, 2n+m-1)$
III:	time-dependent domain between the black-hole horizons	$(n-2m, n+2m)$
IV:	stationary region between $\mathcal{H}_c^-$ and $\mathcal{H}_b^-$	$(2n-m+1, 2n+m)$
V:	asymptotic time-dependent domain between $\mathcal{I}^-$ and $\mathcal{H}_c^-$	$(n-2m+1, n+2m-1)$

where m, n are arbitrary integers. The corresponding *Kruskal–Szekeres-type* dimensionless coordinates for every distinct region are

$$
T_h^{\pm} = \frac{1}{2}(V_h^{\pm} + U_h^{\pm}), \qquad R_h^{\pm} = \frac{1}{2}(V_h^{\pm} - U_h^{\pm}). \qquad (169)
$$

[The presence of the *curvature singularity* at $r = 0$ (implying $r_* = 0$) for certain values of θ restricts the range of the coordinates U_b^- and V_b^- in the region IV to the domain outside $U_b^- V_b^- = \pm 1$.]

In terms of these coordinates, the *extension across the horizon is regular* (in fact, analytic). Indeed, by multiplying and dividing the null coordinates (167) and (168) we obtain the relations,

$$
U_h^{\pm}V_h^{\pm} = \left(1-\frac{r}{r_b^+}\right)^{\frac{k_b^+}{k_h^{\pm}}}\left(1-\frac{r}{r_b^-}\right)^{\frac{k_b^-}{k_h^{\pm}}}\left(1-\frac{r}{r_c^+}\right)^{\frac{k_c^+}{k_h^{\pm}}}\left(1-\frac{r}{r_c^-}\right)^{\frac{k_c^-}{k_h^{\pm}}}, \qquad (170)
$$

$$\frac{U_h^\pm}{V_h^\pm} = (-1)^{i+j}\exp\left(-\frac{t}{k_h^\pm}\right), \tag{171}$$

while the terms $(\mathrm{d}u \pm \mathrm{d}v)^2$ in the metric (164) become

$$(\mathrm{d}u \pm \mathrm{d}v)^2 = \frac{4(k_h^\pm)^2}{U_h^\pm V_h^\pm}\left(\frac{V_h^\pm}{U_h^\pm}(\mathrm{d}U_h^\pm)^2 \mp 2\mathrm{d}U_h^\pm \mathrm{d}V_h^\pm + \frac{U_h^\pm}{V_h^\pm}(\mathrm{d}V_h^\pm)^2\right). \tag{172}$$

A nonanalytic behavior across the horizon r_h may thus occur only at zeros of the product $U_h^\pm V_h^\pm$. However, they exactly cancel the zeros of the functions $Q(r)$ in the metric (164). For example, by choosing the black hole horizon $r_h = r_b^+$, we get $U_b^+ V_b^+ \propto (r - r_b^+)$ which obviously compensates the corresponding root $Q \propto (r - r_b^+)$ in (129). Notice also that the last term in (164) actually vanishes. Therefore, the metric (164) remains finite at r_b^+. Of course, the same argument applies to the remaining three horizons.

Maximal extension (the complete atlas) of the black hole manifold represented by (47) is obtained by ‘gluing together” the different “coordinate patches” *crossing all the horizons*, until a curvature singularity or conformal infinity (the scri $\mathcal{I}$) is reached. Such an extension has to be performed both along the *advanced* null coordinate v and the *retarded* null coordinate u, using the corresponding coordinates $U_h^\pm$ and $V_h^\pm$. By this step-by-step procedure, the coordinate singularities at *all the horizons* $\mathcal{H}_h^\pm$ are removed.

Finally, we construct the *Penrose conformal diagrams* visualizing the global structure of this extended manifold. This is achieved by a suitable conformal rescaling of $U_h^\pm$ and $V_h^\pm$ to the *compactified* null coordinates $\tilde{u}_h^\pm$ and $\tilde{v}_h^\pm$ defined as

$$\tan\frac{\tilde{u}_h^\pm}{2} = -\mathrm{sign}(k_h^\pm)(U_h^\pm)^{-\mathrm{sign}(k_h^\pm)}, \tag{173}$$

$$\tan\frac{\tilde{v}_h^\pm}{2} = -\,\mathrm{sign}(k_h^\pm)(V_h^\pm)^{-\mathrm{sign}(k_h^\pm)}. \tag{174}$$

Consequently, for $\tilde{T}_h^\pm = \frac{1}{2}(\tilde{v}_h^\pm + \tilde{u}_h^\pm)$ and $\tilde{R}_h^\pm = \frac{1}{2}(\tilde{v}_h^\pm - \tilde{u}_h^\pm)$ we obtain the following explicit expressions in terms of the original coordinates t, r of the metric (47):

$$\tilde{T}_h^\pm = \begin{cases} (-1)^{j+1}\arctan\dfrac{\cosh\frac{t}{2|k_h^\pm|}}{\sinh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ even},\\[2ex] (-1)^{j}\arctan\dfrac{\sinh\frac{t}{2|k_h^\pm|}}{\cosh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ odd}, r_*<0,\\[2ex] (-1)^{j}\arctan\dfrac{\sinh\frac{t}{2|k_h^\pm|}}{\cosh\frac{r_*}{2|k_h^\pm|}}+\pi & \text{for } i+j \text{ odd}, r_*\geq 0, \end{cases} \tag{175}$$

and

$$\tilde{R}_h^\pm = \begin{cases} (-1)^{j}\arctan\dfrac{\sinh\frac{t}{2|k_h^\pm|}}{\cosh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ even},\\[2ex] (-1)^{j+1}\arctan\dfrac{\cosh\frac{t}{2|k_h^\pm|}}{\sinh\frac{r_*}{2|k_h^\pm|}} & \text{for } i+j \text{ odd}, r_*<0,\\[2ex] (-1)^{j+1}\arctan\dfrac{\cosh\frac{t}{2|k_h^\pm|}}{\sinh\frac{r_*}{2|k_h^\pm|}}+\pi & \text{for } i+j \text{ odd}, r_*\geq 0. \end{cases} \tag{176}$$

Recall that the function $r_*(r)$ is given by (165) and the coefficients $k_h^\pm$ by (166). In particular, the lines of constant r thus coincide with the lines of constant r_*. For every single region the coordinate r_* spans the whole range $(-\infty, +\infty)$, and similarly the coordinate t.

These explicit relations between the compactified coordinates $\{\tilde{T}_h^\pm, \tilde{R}_h^\pm\}$ and the original coordinates $\{t, r\}$ of the metric (47) for all (i, j) can be used for graphical construction of the Penrose diagram, composed of various “diamond” regions. The resulting picture is shown in Fig. 2 for the *special value of* θ such that $\cos\theta = -l/a$ which *contains the curvature singularity at* $r = 0$ in all its regions IV (see Sec. V E). In particular, for vanishing NUT parameter $l = 0$ this is the equatorial plane $\theta = \frac{\pi}{2}$.

The complete manifold consists of an *infinite number of the regions* I, II, III, IV and V, each identified by the specific pair of integers (i, j). These regions are *separated by the corresponding horizons*. Namely, the regions I and II are separated by the cosmo-acceleration horizon $\mathcal{H}_c^+$ at r_c^+, with the asymptotic region I also bounded by the conformal infinity $\mathcal{I}$ (the scri) for very large values of r. The regions II and III are separated by the black hole horizon $\mathcal{H}_b^+$ at r_b^+, while the regions III and IV are separated by the inner black hole horizon $\mathcal{H}_b^-$ at r_b^-. Finally, the regions IV and V are separated by the cosmo-acceleration horizon $\mathcal{H}_c^-$ at r_c^-, with the asymptotic region V bounded by the conformal infinity $\mathcal{I}$ with negative values of r. The curves in each region represent the lines of constant t and r (dashed or solid, respectively).

In the “diagonal” null directions of these Penrose diagrams we can identify the particular coordinate patches covered by the “advanced” metric form (158), extending from the bottom left $\mathcal{I}^-$ to the top right $\mathcal{I}^+$ [for example the pink regions I–V between $(1, -1)$ and (1,3)], and also the complementary “retarded” metric form (159), extending from the bottom right $\mathcal{I}^-$ to the top left $\mathcal{I}^+$ [these are not colored but also contain the regions I–V, for example between $(-1, 1)$ and (3,1)]. These patches “share” the “central regions” III [for example (1,1)]. Each of such central region III is bounded by the inner and outer black hole horizons at r_b^- and r_b^+, localizing thus the interior of the corresponding black hole. In the whole extended

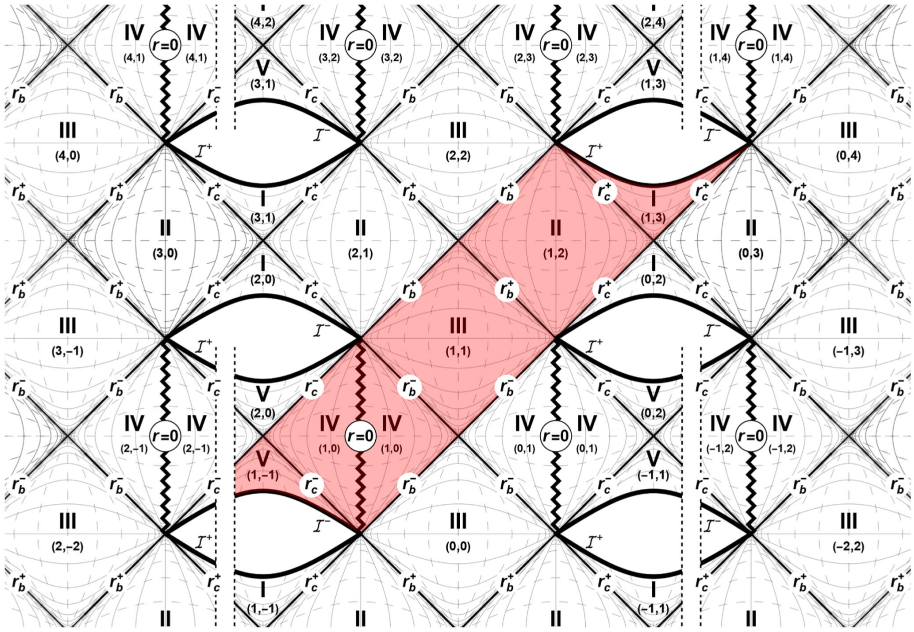

FIG. 2. Penrose conformal diagram of the completely extended spacetime (47) showing the global structure of this family of accelerating and rotating charged NUT black holes with a cosmological constant. We assume the ordering of the four distinct horizons as $r_c^- < r_b^- < r_b^+ < r_c^+$; see (136). Here we show a two-dimensional section $\theta, \phi_h =$ const with the curvature singularity at $r = 0$, i.e., for $\theta =$ const such that $\cos\theta = -l/a$. In such a section, the corresponding regions IV are "cut in half" by this curvature singularity at $r = 0$, indicated by the vertical zigzag lines. The double dashed vertical parallel lines indicate a separation of distinct asymptotically flat regions close to $\mathcal{I}^\pm$ (different "parallel universes" that are not necessarily identified).

universe, there are thus *infinitely many black holes*—they are identified by the different regions III.

Provided $|l| \le |a|$, such black hole has the *curvature singularity* at $r = 0$ in the region IV bounded by the inner black hole horizon $\mathcal{H}_b^-$ at r_b^- (and also the inner cosmo-acceleration horizon $\mathcal{H}_c^-$ at r_c^-). In the section given by the special value of θ such that $\cos\theta = -l/a$ it is *not possible to cross* from the values $r > 0$ to $r < 0$. This is indicated by the vertical zigzag lines in the regions IV. However, as recently pointed out by MacCallum [34] in his interesting revisit of the maximal extension of the Kerr black hole spacetime, there is a "missing triangle" in usual plots (such as in [10]). Although it is not possible to cross the curvature singularity $r = 0$ on this specific section, due to its *ring* structure there exist curves that decrease from $r > 0$ to $r = 0$ *and continue to* $r < 0$, provided their value of θ is *different* form $\cos\theta = -l/a$. On such a section there is no curvature singularity, so that the coordinate boundary $r = 0$ is no obstacle for continuation of the curve. The same argument is valid not only for the Kerr black hole but also for the whole family of rotating black hole spacetimes (such that $|l| \le |a|$) investigated here. Therefore, in Fig. 2 we represent the curvature singularity in (any) region IV simply by a vertical zigzag line. The "missing triangle" on the left of $r = 0$ is the extension of the "present triangle" on the right, continuing from positive to negative values of the coordinate r, and vice versa, because the curvature singularity can be "bypassed" on any section such that $\cos\theta \ne -l/a$.

Each of these black holes, identified by the specific region III, *is associated with four asymptotic regions*, namely the pair of the regions I with *future conformal infinity* $\mathcal{I}^+$ and a pair of the regions V with *past conformal infinity* $\mathcal{I}^-$. Moreover, each asymptotically conformally flat region bounded by $\mathcal{I}$ is *"shared" by two distinct black holes*. For example, the conformal infinities $\mathcal{I}^+$ of the *"infinite horizontal chain" of black holes* (regions III) given by …, $(3,-1)$, $(1,1)$, $(-1,3)$, … are located in the "future universes" (regions I) …, $(5,-1)$, $(3,1)$, $(1,3)$, $(-1,5)$, …, while their "past universes" (regions V) are …, $(3,-3)$, $(1,-1)$, $(-1,1)$, $(-3,3)$, …, respectively. However, these "past universes" *need not be the same*. Therefore, we inserted the double dashed vertical parallel lines in them to indicate their separation. Of course, it is possible to "artificially" identify (some of) them—both the black hole regions III and/or their asymptotic regions

I and V. An infinite plethora of various topologically complicated manifolds can thus be constructed.

Let us emphasize that the Penrose conformal diagram shown in Fig. 2 represents the global structure of a *generic* black hole spacetime of type D (47) with *four distinct horizons*. It remains to investigate a great number of other special situations for particular choices of the physical parameters with degenerate (multiple) horizons or with a reduced number of horizons, as identified in Sec. V B and Sec. V C. Other specific situations also occur, for example $|a| = |l|$. In all these cases the Penrose diagram will have different forms.

G. Regularization of the axes of symmetry $\theta = 0$ and $\theta = \pi$

As shown in previous works [11,13,14], the metric (47) is convenient for explicit analysis of the *regularity of the poles/axes* located at $\theta = 0$ and $\theta = \pi$, respectively, which are the boundaries of the range $\theta \in [0, \pi]$.[2] This is now further improved with the new metric functions (48)–(53).

Recall that there are seven physical parameters in the metric (47), namely $m, a, l, e, g, \alpha, \Lambda$, which represent mass, Kerr-like rotation, NUT parameter, electric and magnetic charges, acceleration, and cosmological constant of the black hole, respectively. But it should be emphasized that, in fact, there is also the *eighth free parameter—the conicity C hidden in the range of the angular coordinate,*

$$\varphi \in [0, 2\pi C), \tag{177}$$

which has not yet been specified. It is directly related to the *deficit (or excess) angles* of the *cosmic strings (or struts)* located along the axes. The tension associated with these topological defects is the *physical source of the acceleration* of the black holes.

First, let us consider a small circle around the *first axis of symmetry* $\theta = 0$ in the metric (47) given by $\theta =$ const, with the range of φ given by (177), assuming fixed t and r. The invariant length of its circumference is $\int_0^{2\pi C} \sqrt{g_{\varphi\varphi}}\mathrm{d}\varphi$, while its radius is $\int_0^{\theta} \sqrt{g_{\theta\theta}}\mathrm{d}\theta$, so that

$$f_0 \equiv \lim_{\theta\to 0} \frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to 0} \frac{2\pi C\sqrt{g_{\varphi\varphi}}}{\theta\sqrt{g_{\theta\theta}}}. \tag{178}$$

For the metric (47) near the axis $\theta = 0$ we get

$$g_{\varphi\varphi} \approx \frac{P}{\Omega^2\rho^2}(r^2 + (a+l)^2)^2\theta^2, \qquad g_{\theta\theta} = \frac{\rho^2}{\Omega^2 P}, \tag{179}$$

and thus, using (50),

$$\begin{aligned} f_0 &= 2\pi C P(0) \\ &= 2\pi C\left[1 - 2\left(\frac{\alpha a m}{a^2+l^2} - \frac{\Lambda}{3}l\right)(a+l) + \left(\frac{\alpha^2 a^2}{(a^2+l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\right)(a+l)^2\right]. \end{aligned} \tag{180}$$

Therefore, the axis $\theta = 0$ in the metric (47) *can always be made regular* by the unique choice of $C = C_0$ such that

$$C_0 \equiv \left[1 - 2\left(\frac{\alpha a m}{a^2+l^2} - \frac{\Lambda}{3}l\right)(a+l) + \left(\frac{\alpha^2 a^2}{(a^2+l^2)^2}(a^2 - l^2 + e^2 + g^2) + \frac{\Lambda}{3}\right)(a+l)^2\right]^{-1}. \tag{181}$$

Notice that for $l = -a$, this is simply $C_0 = 1$.

Analogously, we can regularize the *second axis of symmetry* $\theta = \pi$. By applying the transformation of the time coordinate,

$$t_\pi \equiv t - 4l\varphi, \tag{182}$$

the metric (47) becomes

$$\mathrm{d}s^2 = \frac{1}{\Omega^2}\left(-\frac{Q}{\rho^2}\left[\mathrm{d}t_\pi - \left(a\sin^2\theta - 4l\cos^2\tfrac{1}{2}\theta\right)\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta[a\mathrm{d}t_\pi - (r^2 + (a-l)^2)\mathrm{d}\varphi]^2\right), \tag{183}$$

[2]Usually, $\theta = 0$ and $\theta = \pi$ are considered as two *semiaxes* of the *same axis of rotation* (a single symmetry axis). This is natural in the simplest spacetimes for which the coordinates (r, θ, φ) represent spherical(like) symmetry with $r > 0$ only. However, in the present context of generic black hole spacetimes with the Kerr parameter a and the NUT parameter l, the range of the "radial coordinate" is $r \in (-\infty, +\infty)$. In such a case, *both* the axes given by $\theta = 0$ and $\theta = \pi$ have this full range of r, and thus they are not the same (unless they are "artificially" identified, which would lead to nontrivial topologies). Therefore, they form *two distinct infinite axes* connecting two different asymptotically flat regions in the whole spacetime. This fact is explained in more detail in our previous papers, in particular see Fig. 4 of [19] and Fig. 2 of [14].

Now, for $\theta \to \pi$ the radius of a small circle around the axis $\theta = \pi$ is $\int_\theta^\pi \sqrt{g_{\theta\theta}}\mathrm{d}\theta$, so that

$$f_\pi \equiv \lim_{\theta\to\pi} \frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to\pi} \frac{2\pi C\sqrt{g_{\varphi\varphi}}}{(\pi-\theta)\sqrt{g_{\theta\theta}}}, \qquad (184)$$

where for the metric (183) now

$$g_{\varphi\varphi} \approx \frac{P}{\Omega^2\rho^2}(r^2+(a-l)^2)^2(\pi-\theta)^2, \quad g_{\theta\theta} = \frac{\rho^2}{\Omega^2 P}. \qquad (185)$$

Using (50) we obtain

$$\begin{aligned} f_\pi &= 2\pi C P(\pi) \\ &= 2\pi C\left[1 + 2\left(\frac{\alpha a m}{a^2+l^2} - \frac{\Lambda}{3}l\right)(a-l) + \left(\frac{\alpha^2 a^2}{(a^2+l^2)^2}(a^2-l^2+e^2+g^2) + \frac{\Lambda}{3}\right)(a-l)^2\right]. \end{aligned} \qquad (186)$$

The axis $\theta = \pi$ in the metric (183) *can always be made regular* by the unique choice $C = C_\pi$ where

$$C_\pi \equiv \left[1 + 2\left(\frac{\alpha a m}{a^2+l^2} - \frac{\Lambda}{3}l\right)(a-l) + \left(\frac{\alpha^2 a^2}{(a^2+l^2)^2}(a^2-l^2+e^2+g^2) + \frac{\Lambda}{3}\right)(a-l)^2\right]^{-1}. \qquad (187)$$

Notice that for $l = a$, this is simply $C_\pi = 1$.

H. Cosmic strings (or struts) and deficit (or excess) angles

Regularizing the second axis $\theta = \pi$ by the choice (187) there remains a *deficit/excess angle* $\delta_0 \equiv 2\pi - f_0$ (conical singularity representing a cosmic string/strut) *along the first axis* $\theta = 0$, namely

$$\delta_0 = \frac{8\pi a[\alpha a[m(a^2+l^2) - \alpha a l(a^2-l^2+e^2+g^2)] - \frac{2}{3}\Lambda l(a^2+l^2)^2]}{[1+\frac{1}{3}\Lambda(a-l)(a-3l)](a^2+l^2)^2 + 2\alpha a m(a-l)(a^2+l^2) + \alpha^2 a^2(a-l)^2(a^2-l^2+e^2+g^2)}.$$

For *nonrotating black holes* ($a = 0$) we immediately obtain $\delta_0 = 0$ which means that *both axes* $\theta = 0$ *and* $\theta = \pi$ *are regular.* In such a case, the possible cosmic strings are absent, so that there is *no source of acceleration*. This is fully consistent with our previous observation made in Sec. IV D that there is no accelerating "purely" NUT-(anti–)de Sitter black hole in the Plebański-Demiański family of spacetimes. Indeed, by setting the Kerr-like rotation parameter a to zero, the metric (47) becomes independent of the acceleration α, and simplifies directly to (79).

For black holes *without the NUT parameter* ($l = 0$) this expression simplifies to

$$\delta_0 = \frac{8\pi\alpha m}{1 + 2\alpha m + \alpha^2(a^2+e^2+g^2) + \frac{1}{3}\Lambda a^2}, \qquad (188)$$

recovering the previous results for rotating charged C-metric with a cosmological constant; see Chap. 14 in [10] [and generalizing Eq. (132) of [14] to any Λ]. The tension in the cosmic string along $\theta = 0$ characterized by $\delta_0 > 0$ pulls the black hole, *causing its uniform acceleration*. Such a string extends to the *full range* of the radial coordinate $r \in (-\infty, +\infty)$, connecting "our Universe" with the "parallel universe" through the nonsingular black hole interior close to $r = 0$.

Complementarily, when the first axis of symmetry $\theta = 0$ is made regular by the choice (181), there is necessarily an *excess/deficit angle* $\delta_\pi \equiv 2\pi - f_\pi$ along the second axis $\theta = \pi$, namely

$$\delta_\pi = \frac{-8\pi a[\alpha a[m(a^2+l^2) - \alpha a l(a^2-l^2+e^2+g^2)] - \frac{2}{3}\Lambda l(a^2+l^2)^2]}{[1+\frac{1}{3}\Lambda(a+l)(a+3l)](a^2+l^2)^2 - 2\alpha a m(a+l)(a^2+l^2) + \alpha^2 a^2(a+l)^2(a^2-l^2+e^2+g^2)}.$$

For $a = 0$ it gives $\delta_\pi = 0$, while for $l = 0$ it simplifies to

$$\delta_\pi = \frac{-8\pi\alpha m}{1 - 2\alpha m + \alpha^2(a^2+e^2+g^2) + \frac{1}{3}\Lambda a^2}, \qquad (189)$$

[generalizing Eq. (134) of [14] to any Λ]. This represents the *cosmic strut* characterized by $\delta_\pi < 0$ located along $\theta = \pi$ between the pair of black holes, pushing them away from each other in opposite spatial directions.

Interestingly, *both axes* $\theta = 0$ and $\theta = \pi$ can be made *simultaneously regular* ($\delta_0 = 0 = \delta_\pi$) if (and only if) seven physical parameters of the black hole spacetime satisfy the special constraint,

$$\frac{2}{3}\Lambda l(a^2+l^2)^2 = \alpha a[m(a^2+l^2) - \alpha a l(a^2-l^2+e^2+g^2)]. \tag{190}$$

For such a special value of the cosmological constant Λ, the rotating charged black holes with the NUT parameter $l \neq 0$ accelerate without the presence of the cosmic strings or struts. In the $\Lambda = 0$ case the simpler condition given by Eq. (135) of [14] is recovered. The condition (190) also corrects the wrong sign of the Λ-term in the corresponding unnumbered equation on p. 313 of [10].

I. Rotation of the cosmic strings (or struts)

With a NUT parameter $l \neq 0$ these *cosmic strings (or struts) are rotating*. The *angular velocity* parameter ω_θ of the metric (47) is

$$\omega_\theta \equiv \frac{g_{t\varphi}}{g_{tt}} = -\frac{Q(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta) - a(r^2+(a+l)^2)P\sin^2\theta}{Q - a^2P\sin^2\theta}. \tag{191}$$

Now we consider any fixed value of r away from the horizons (so that $Q \neq 0$ is a constant). Then the limits $\theta \to 0$ and $\theta \to \pi$ near the two different axes $\theta = 0$ and $\theta = \pi$ give

$$\omega_0 = 0 \quad \text{and} \quad \omega_\pi = -4l, \tag{192}$$

respectively. The first axis $\theta = 0$ is thus *nonrotating*, while the second axis $\theta = \pi$ rotates, and its *angular velocity is directly (and solely) determined by the NUT parameter l*. Indeed, ω_π does not depend on the Kerr-like parameter a, nor the conicity parameter C. The rotational character of the axis is thus a specific feature related to the NUT parameter l, which is independent of the possible deficit angles defining the cosmic string/strut along the same axis.

By changing the time coordinate as in (182), we obtain the alternative metric (183) for which

$$\omega_\theta \equiv \frac{g_{t_\pi\varphi}}{g_{t_\pi t_\pi}} = -\frac{Q(a\sin^2\theta - 4l\cos^2\frac{1}{2}\theta) - a(r^2+(a-l)^2)P\sin^2\theta}{Q - a^2P\sin^2\theta}. \tag{193}$$

The corresponding angular velocities of the two axes are thus

$$\omega_0 = 4l \quad \text{and} \quad \omega_\pi = 0. \tag{194}$$

In this case, the situation is complementary to (192): the axis $\theta = 0$ rotates, while the axis $\theta = \pi$ does not rotate.

Interestingly, there is a *constant difference,*

$$\Delta\omega \equiv \omega_0 - \omega_\pi = 4l, \tag{195}$$

between the angular velocities of the two cosmic strings or struts given by l (irrespective of the value of a or the choice of C). The NUT parameter l is thus responsible for the *difference* between the magnitude of rotation of the two axes $\theta = 0$ and $\theta = \pi$.

J. Pathological regions with closed timelike curves near the rotating strings (or struts)

In the close vicinity of the rotating cosmic strings or struts located along $\theta = 0$ or $\theta = \pi$, the black hole spacetime can serve as a time machine because there are closed timelike curves. To identify such "pathological" causality-violating regions, let us consider *circles* around the axes of symmetry $\theta = 0$ or $\theta = \pi$ such that only the *periodic* angular coordinate $\varphi \in [0, 2\pi C)$ changes, while the remaining coordinates t, r and θ are constant. The corresponding velocity vectors are thus proportional to the *Killing vector field* ∂_φ whose norm is determined just by the metric coefficient $g_{\varphi\varphi}$ of the general metric (47). There exist regions with

$$g_{\varphi\varphi} < 0, \tag{196}$$

in which the circles (orbits of the axial symmetry) are *closed timelike curves*. Such pathological regions are given by the condition,

$$P(\theta)(r^2+(a+l)^2)^2\sin^2\theta < Q(r)(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta)^2, \tag{197}$$

where the functions $P(\theta)$, $Q(r)$ are explicitly given by (50), (51).

Since $P(\theta) > 0$, this condition can only be satisfied in the regions where $Q(r) > 0$. In the generic case admitting four distinct horizons (129), with $\mathcal{N} > 0$, ordered as $r_c^- < r_b^- < r_b^+ < r_c^+$, the pathological regions with closed timelike curves can only appear in the stationary region $r \in (r_b^+, r_c^+)$ *between the outer black hole horizon* $\mathcal{H}_b^+$ *and the outer cosmo-acceleration horizon* $\mathcal{H}_c^+$, or in the stationary region $r \in (r_c^-, r_b^-)$ *between the inner cosmo-acceleration horizon* $\mathcal{H}_c^-$ *and the inner black hole horizon* $\mathcal{H}_b^-$ containing the curvature singularity at $r = 0$; see the scheme (137). These are, respectively, the regions II and the regions IV in the Penrose conformal diagram shown in Fig. 2.

Moreover, it can be proven analytically that these *pathological regions with closed timelike curves do not intersect with the ergoregions* (shown in Fig. 1), although they are both in the same domains II and IV. Indeed, the ergoregions are identified by the condition $g_{tt} > 0$ (together with $g_{rr} > 0$), that is

$$Q < Pa^2 \sin^2\theta; \tag{198}$$

see Eq. (150). Substituting this inequality into (197) we obtain

$$r^2 + (a+l)^2 < a^2 \sin^2\theta + 4al \sin^2 \tfrac{1}{2}\theta. \tag{199}$$

This is the same relation as $r^2 + a^2\cos^2\theta + 2al\cos\theta + l^2 < 0$, and in view of (49) it reads

$$\rho^2 \equiv r^2 + (l + a\cos\theta)^2 < 0, \tag{200}$$

which is a contradiction.

The pathological regions with closed timelike curves are indicated in Fig. 3 for several choices of the cosmological constant. They are the purple regions near the rotating cosmic string (strut) at $\theta = \pi$.

K. Thermodynamic quantities

In this final section we evaluate some basic thermodynamic quantities of the large class of black holes (47), namely the *entropy,*

$$S \equiv \frac{1}{4}\mathcal{A}, \tag{201}$$

given by the horizon area $\mathcal{A}$, and the *temperature,*

$$T \equiv \frac{1}{2\pi}\kappa, \tag{202}$$

given by the corresponding horizon surface gravity κ; see [35].

The *horizon area* is obtained easily by integrating both angular coordinates of the metric (47) for *fixed values of* t *and* $r = r_h$,

$$\mathcal{A}(r_h) = \int_0^{2\pi C}\int_{\theta_{\min}}^{\theta_{\max}} \sqrt{g_{\theta\theta}g_{\varphi\varphi}}\,d\theta d\varphi. \tag{203}$$

Because $Q(r_h) = 0$ on any horizon, this expression simplifies to

$$\mathcal{A} = 2\pi C(r_h^2 + (a+l)^2)\int_{\theta_{\min}}^{\theta_{\max}} \frac{\sin\theta}{\Omega^2(r_h)}d\theta. \tag{204}$$

Applying the explicit form of the conformal factor (48), that is

$$\Omega(r_h) = 1 - \frac{\alpha a r_h}{a^2 + l^2}(l + a\cos\theta), \tag{205}$$

a simple integration leads to

$$\mathcal{A} = 2\pi C(r_h^2 + (a+l)^2)\frac{a^2 + l^2}{\alpha a^2 r_h}\left[\frac{-1}{\Omega(r_h)}\right]_{\theta_{\min}}^{\theta_{\max}}. \tag{206}$$

Let us now assume the generic case of *four distinct horizons* $\mathcal{H}$ introduced in (131)–(134). For the black hole horizons $\mathcal{H}_b^\pm$ the integration range is a full spherical angle, $[\theta_{\min}, \theta_{\max}] = [0, \pi]$, and this leads to the following result:

$$\text{area of } \mathcal{H}_b^\pm \text{ is} \quad \mathcal{A}_b{}^\pm = \frac{4\pi C[(r_b^\pm)^2 + (a+l)^2]}{(1 - \alpha\frac{a^2+al}{a^2+l^2}r_b^\pm)(1 + \alpha\frac{a^2-al}{a^2+l^2}r_b^\pm)}. \tag{207}$$

For *vanishing acceleration* α the area of the black hole horizons is simply

$$\mathcal{A}_b^\pm = 4\pi C((r_b^\pm)^2 + (a+l)^2). \tag{208}$$

This reduces to the well-known expressions for Kerr–Newman-NUT-(anti–)de Sitter black holes, and in particular the Schwarzschild solution with a single horizon of the area $\mathcal{A}_b = 4\pi r_b^2$.

Concerning the cosmo-acceleration horizons $\mathcal{H}_c^\pm$, it is necessary to discuss three cases depending on the sign of the cosmological constant. In our previous work [14] we demonstrated that for $\Lambda = 0$ the area of both $\mathcal{H}_a^+ \equiv \mathcal{H}_c^+$ and $\mathcal{H}_a^- \equiv \mathcal{H}_c^-$ is *infinite*. The same is true for $\Lambda < 0$. In this case the reason is that *the cosmo-acceleration horizons extend up to conformal infinity given by* $\Omega = 0$. This can be seen, e.g., from the corresponding pictures in the bottom row of Fig. 1 and Fig. 3 in which $\mathcal{H}_c^\pm$ are indicated by big red circles. Consequently, $\Omega(r_c^+, \theta_{\min}) = 0$ and $\Omega(r_c^-, \theta_{\max}) = 0$. In both cases, the expression (206) for $\mathcal{A}_c^\pm$ diverges.

For a *positive* cosmological constant $\Lambda > 0$ the integration (206) over the full admitted range $[\theta_{\min}, \theta_{\max}] = [0, \pi]$ implies that

$$\text{area of } \mathcal{H}_c^\pm \text{ is} \quad \mathcal{A}_c{}^\pm = \frac{4\pi C[(r_c^\pm)^2 + (a+l)^2]}{\left(1 - \alpha\frac{a^2+al}{a^2+l^2}r_c^\pm\right)\left(1 + \alpha\frac{a^2-al}{a^2+l^2}r_c^\pm\right)}. \tag{209}$$

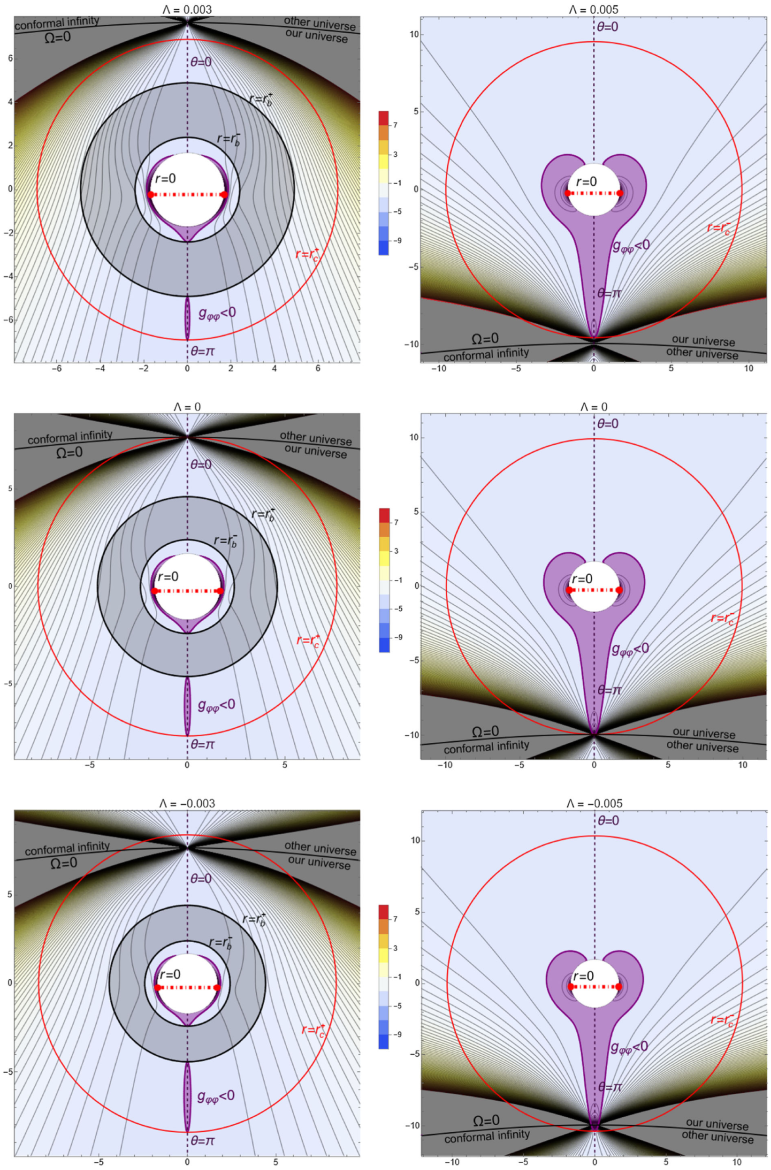

FIG. 3. Plot of the metric function $g_{\varphi\varphi}$ for the accelerating black hole (47) with a regular axis $\theta = 0$ and rotating cosmic string (strut) along the axis $\theta = \pi$. The values of $g_{\varphi\varphi}$ are visualized in quasipolar coordinates $\mathrm{x} \equiv \sqrt{r^2 + (a+l)^2}\sin\theta$, $\mathrm{y} \equiv \sqrt{r^2 + (a+l)^2}\cos\theta$ for $r \geq 0$ (left) and $r \leq 0$ (right). The gray annulus in the center of the left figure localizes the black hole bordered by its horizons $\mathcal{H}_b^+$ at r_b^+ and $\mathcal{H}_b^-$ at r_b^- ($0 < r_b^- < r_b^+$). The cosmo-acceleration horizons $\mathcal{H}_c^\pm$ at r_c^+ and r_c^- (big red circles) and the conformal infinity $\mathcal{I}$ at $\Omega = 0$ are also shown. The gray curves are contour lines $g_{\varphi\varphi}(r, \theta) = \text{const}$, and the values are color-coded from red (positive values) to blue (negative values); extremely large values are cut. The purple curves are the isolines $g_{\varphi\varphi} = 0$ determining the boundary of the pathological regions (197) with closed timelike curves. They occur close to the axis $\theta = \pi$ (purple regions where $g_{\varphi\varphi} < 0$). This plot is for the choice $m = 3$, $a = 1.5$, $l = 0.2$, $e = 1.6 = g$, and $\alpha = 0.12$. The top row is plotted for *positive* values of the cosmological constant ($\Lambda = 0.003$ on the left for $r \geq 0$, $\Lambda = 0.005$ on the right for $r \leq 0$), the middle row is for $\Lambda = 0$, while the bottom row is plotted for *negative* values of the cosmological constant ($\Lambda = -0.003$ on the left for $r \geq 0$, $\Lambda = -0.005$ on the right for $r \leq 0$).

Interestingly, *these areas of cosmo-acceleration horizons $\mathcal{H}_c^\pm$ are finite*.

Indeed, from the general form (51) of the metric function $Q(r)$, namely

$$Q(r) = [r^2 - 2mr + (a^2 - l^2 + e^2 + g^2)]\left(1 + \alpha a \frac{a-l}{a^2+l^2} r\right)\left(1 - \alpha a \frac{a+l}{a^2+l^2} r\right) - \frac{\Lambda}{3} r^2 \left[r^2 + 2\alpha a l \frac{a^2-l^2}{a^2+l^2} r + (a^2 + 3l^2)\right], \tag{210}$$

evaluated at the horizons $r_c^\pm$ [which are defined as the two roots of $Q(r_c) = 0$], it follows that

$$\left(1 - \alpha \frac{a^2+al}{a^2+l^2} r_c\right)\left(1 + \alpha \frac{a^2-al}{a^2+l^2} r_c\right) = \frac{\Lambda}{3} r_c^2 \frac{r_c^2 + 2\alpha a l \frac{a^2-l^2}{a^2+l^2} r_c + (a^2+3l^2)}{r_c^2 - 2mr_c + (a^2 - l^2 + e^2 + g^2)}, \tag{211}$$

An infinite value of $\mathcal{A}_c^\pm$ given by (209) would require the left-hand side of (211) to be *zero*, implying its roots $r_c = \pm \frac{1}{\alpha} \frac{a^2+l^2}{a^2 \pm al}$. By substituting such values into the numerator of the right-hand side of (211) we get $r_c^2 + 2\alpha a l \frac{a^2-l^2}{a^2+l^2} r_c + (a^2+3l^2) = \frac{(a^2+l^2)^2}{\alpha^2 a^2 (a \pm l)^2} + (a \pm l)^2$ which is *strictly positive*. For $\Lambda > 0$ we thus get a contradiction, so that $\mathcal{A}_c^\pm$ must be finite.

For $m = a = l = e = g = \alpha = 0$ (so that $C = 1$) the function reduces to $Q(r) = r^2(1 - \frac{\Lambda}{3} r^2)$. The cosmological horizons are thus located at $r_c^2 = \frac{3}{\Lambda}$, and their areas given by (209) are $\mathcal{A}_c = 4\pi r_c^2 = 12\pi/\Lambda$ which is the well-known result for the de Sitter space.

The *temperature* of the horizon is determined by its *surface gravity* κ. In [14,16] we showed that for the general metric form (47) this can be expressed as

$$\kappa = \frac{1}{2} \frac{Q'(r_h)}{r_h^2 + (a+l)^2}, \tag{212}$$

where the prime denotes the derivative with respect to r. With the factorized form (129) of the metric function $Q(r)$, using the constant parameters (166), this can be easily evaluated as

$$\text{surface gravity of } \mathcal{H}_b^+ \text{ is} \quad \kappa_b^+ = \frac{1}{2k_b^+} = -\frac{\mathcal{N}}{2} \frac{(r_b^+ - r_b^-)(r_b^+ - r_c^+)(r_b^+ - r_c^-)}{(r_b^+)^2 + (a+l)^2}, \tag{213}$$

$$\text{surface gravity of } \mathcal{H}_b^- \text{ is} \quad \kappa_b^- = \frac{1}{2k_b^-} = -\frac{\mathcal{N}}{2} \frac{(r_b^- - r_b^+)(r_b^- - r_c^+)(r_b^- - r_c^-)}{(r_b^-)^2 + (a+l)^2}, \tag{214}$$

$$\text{surface gravity of } \mathcal{H}_c^+ \text{ is} \quad \kappa_c^+ = \frac{1}{2k_c^+} = -\frac{\mathcal{N}}{2} \frac{(r_c^+ - r_b^+)(r_c^+ - r_b^-)(r_c^+ - r_c^-)}{(r_c^+)^2 + (a+l)^2}, \tag{215}$$

$$\text{surface gravity of } \mathcal{H}_c^- \text{ is} \quad \kappa_c^- = \frac{1}{2k_c^-} = -\frac{\mathcal{N}}{2} \frac{(r_c^- - r_b^+)(r_c^- - r_b^-)(r_c^- - r_c^+)}{(r_c^-)^2 + (a+l)^2}. \tag{216}$$

It can now be seen from (213) and (214) that

$$\kappa_b^+ = 0 = \kappa_b^- \quad \text{if} \quad r_b^+ = r_b^-, \tag{217}$$

and from (213) and (215) that

$$\kappa_b^+ = 0 = \kappa_c^+ \quad \text{if} \quad r_b^+ = r_c^+. \tag{218}$$

This confirms that *extremal horizons have vanishing surface gravity*, and thus zero thermodynamic temperature $T = \frac{1}{2\pi}\kappa$.

VI. SUMMARY

We presented a new metric form (47)–(51) of the large family of exact black holes of algebraic type D, initially found by Debever (1971) and by Plebański and Demiański (1976). It generalizes our previous paper on this topic [14] to *any value of the cosmological constant* Λ. We also demonstrated that this improved metric representation simplify the investigation of various geometrical and physical properties. In particular:

(i) In Sec. II we recalled the Griffiths–Podolský (2005, 2006) form of this class of spacetimes, and we further improved it by introducing a modified set of the mass and charge parameters m, e, g, applying a conformal rescaling S, and choosing a gauge of the twist parameter ω.

(ii) As summarized in Sec. III, the metric (47) and its functions (48)–(51) are simple, depending only on the radial coordinate r and the angular coordinate θ. Moreover, the key functions $P(\theta)$ and $Q(r)$ can be further compactified to (52)–(53). In particular, $P(\theta)$ is factorized.

(iii) The metric depends on seven parameters $m, a, l, e, g, \alpha, \Lambda$ with direct physical meaning. They represent the mass parameter, Kerr-like rotation, NUT parameter, electric and magnetic charges, acceleration of the black hole, and the cosmological constant, respectively.

(iv) Another nice feature of the new metric form (47)–(51) is that any of its seven physical parameters can be independently set to zero (and this can be done in any order). As shown in Sec. IV, specific subclasses of type D black holes are thus easily obtained. These are the black holes with $\Lambda = 0$, obtained and analyzed previously in [14], Kerr-Newman-NUT-(anti–)de Sitter black holes without acceleration ($\alpha = 0$), accelerating Kerr-Newman-(anti–)de Sitter black holes without NUT ($l = 0$), charged Taub-NUT-(anti–)de Sitter black holes without rotation ($a = 0$), and accelerating Kerr-NUT-(anti–)de Sitter black holes without electric or magnetic charges ($e = 0$ or $g = 0$).

(v) All the metric functions (48)–(51) depend on the acceleration α only via the product αa. Consequently, by setting the Kerr-like rotation a to zero, the new metric (47) always becomes independent of α, and simplifies directly to the charged Taub-NUT-(anti–)de Sitter black holes. This explicitly confirms the previous observation made by Griffiths and Podolský that there is no accelerating purely NUT black hole in the Plebański–Demiański family of type D spacetimes. Quite surprisingly, such a solution for accelerating nonrotating black hole with just the NUT parameter and $\Lambda = 0$ exists [19,20], but it is of distinct algebraic type I. Its possible generalization to any cosmological constant Λ remains an open problem.

(vi) The simplest subcases of the metric (47) with just the mass parameter m and a cosmological constant Λ, plus one additional physical parameter, give famous black holes, namely the Schwarzschild-(anti–)de Sitter, Reissner-Nordström-(anti–)de Sitter, Kerr-(anti–)de Sitter, Taub-NUT-(anti–)de Sitter black holes, or black holes accelerating in de Sitter or anti–de Sitter universes—all in their usual coordinate forms.

(vii) As shown in Sec. V, our convenient metric (47)–(51) considerably simplifies the study of physical and geometrical properties of this large family of black holes. First of all, the Weyl and Ricci curvature tensors, expressed as the Newman-Penrose scalars Ψ_2 and Φ_{11} [with respect to the natural tetrad (85) adapted to the double-degenerate principal null directions] can be evaluated, confirming the type D algebraic structure of the gravitational field, aligned with the non-null electromagnetic field (100)–(102).

(viii) Their form (86) and (87), together with the explicit expressions (96) and (97) for the Kretschmann scalar $\mathcal{K} \equiv R_{abcd}R^{abcd}$ and the Weyl scalar $\mathcal{C} \equiv C_{abcd}C^{abcd}$, clarifies the presence and the structure of the curvature singularity. It is located at $\rho^2 = 0$, i.e., at $r = 0$, but only if also $l + a\cos\theta = 0$, which requires $|l| \leq |a|$. There is no curvature singularity in the black hole spacetimes with large NUT parameter $|l| > |a| \geq 0$.

(ix) Both the double-degenerate principal null directions $\mathbf{k}$ and $\mathbf{l}$ given by (85) are geodetic, shear-free, and expanding. They are twisting if and only if $a = 0 = l$.

(x) The generic black hole spacetime becomes asymptotically conformally flat at the conformal infinity localized by the condition $\Omega = 0$.

(xi) In general, there are four distinct horizons identified by the roots $Q(r_h) = 0$ of the metric function $Q(r)$—which is explicitly given by (51)—a pair of black hole horizons $\mathcal{H}_b^{\pm}$ at $r_b^{\pm}$, and a pair of cosmo-acceleration horizons $\mathcal{H}_c^{\pm}$ at $r_c^{\pm}$. The positions of these four horizons are explicitly given by expressions (140) and (141), respectively. Their natural ordering is $r_c^- < r_b^- < r_b^+ < r_c^+$.

(xii) Of course, there may be less then four horizons, and they can be degenerate (corresponding to multiple roots of $Q(r_h) = 0$), as explicitly listed in Sec. V B.

(xiii) Whenever the Kerr-like rotation parameter a is nonzero, each of these four horizons is accompanied by the corresponding ergoregion; see Sec. V D and Fig. 1.

(xiv) The ringlike curvature singularity at $r = 0$ such that $\cos\theta = -l/a$ (requiring $a^2 \geq l^2$) is, for the black hole solution, located in the stationary region IV between the inner cosmo-acceleration horizon $\mathcal{H}_c^-$ and the inner black hole horizon $\mathcal{H}_b^-$ (assuming the natural ordering $r_c^- < r_b^- < r_b^+ < r_c^+$).

(xv) in Sec. V F we analyzed the global causal structure of the generic family of black hole spacetimes (47) by constructing the Kruskal–Szekeres-type coordinates which enabled us to perform the maximal analytic extension across all the horizons. It revealed an infinite number of time-dependent regions (of type I, III, V) and stationary regions (of type II, IV) which are separated by the black hole and cosmo-acceleration horizons $\mathcal{H}_b^{\pm}$ and $\mathcal{H}_c^{\pm}$.

(xvi) This global structure is visualized in the Penrose diagrams obtained by a suitable conformal compactification, drawn in Fig. 2. The complete manifold contains an infinite number of black holes in various universes identified by distinct (future and past) conformal infinities $\mathcal{I}$.

(xvii) In Sec. V G we investigated the regularization of the two axes of axial symmetry $\theta = 0$ and $\theta = \pi$ by an appropriate setting of the conicity parameter C in the range $\varphi \in [0, 2\pi C)$. The first axis $\theta = 0$ is regular in the metric form (47) with the choice (181), while the second axis $\theta = \pi$ is regular in the metric form (183) with the choice (187).

(xviii) Both these choices lead to the existence of a cosmic string or a strut identified by the deficit or excess angle on the complementary axis, see the expressions for δ_0 and δ_π in Sec. V H. Such topological defects are the physical source of acceleration of the black holes.

(xix) Interestingly, both the axes of symmetry can be made regular simultaneously for the particular choice (190) of the physical parameters.

(xx) In addition to such deficit/excess angles, the cosmic strings/struts are characterized by their rotation ω (angular velocity). In Sec. V I we demonstrated that their values are directly related to the NUT parameter l, see the expressions (192) and (194). There is always a constant difference $\Delta\omega = 4l$ between the angular velocities of the two rotating cosmic strings or struts.

(xxi) In the vicinity of these rotating strings/struts there are pathological regions with closed timelike curves; see Sec. V J and Fig. 3.

(xxii) Although the pathological regions with closed timelike curves are located in the same domains as the ergoregions, they do not overlap with each other, see the end of Sec. V J.

(xxiii) The new metric form (47) is also convenient for the investigation of thermodynamic quantities. In Sec. V K we evaluated the area and the surface gravity of the black hole and cosmo-acceleration horizons, simply related to their entropy and temperature.

All this demonstrates the usefulness of the new improved metric of the complete family of type D accelerating and rotating black holes with charges and the NUT parameter in (anti–)de Sitter universe. Various other investigations can now be performed. Among them is a systematic analysis of the degenerate cases with smaller number of horizons, and with multiple horizons. Recently, such extremal isolated horizons have been studied, for example in the works [16,17,36–40]. Also, extension of the Plebański-Demiański solutions (including a cosmological constant) to the framework of the metric-affine gravity (MAG) theory was constructed in [41]. It would be nice to see if the new and more explicit metric (47)–(51) simplifies such investigations.

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[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).
[2] R. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, Phys. Rev. X **11**, 021053 (2021).
[3] The Event Horizon Telescope Collaboration, First M87 event horizon Telescope results. I. The shadow of the supermassive black hole, Astrophys. J. Lett. **875**, L1 (2019).
[4] The Event Horizon Telescope Collaboration, First Sagittarius A* Event Horizon Telescope results. I. The shadow of the supermassive black hole in the center of the Milky Way, Astrophys. J. Lett. **930**, L12 (2022).
[5] B. Carter, Hamilton–Jacobi and Schrödinger separable solutions of Einstein's equations, Commun. Math. Phys. **10**, 280 (1968).
[6] W. Kinnersley, Type D vacuum metrics, J. Math. Phys. (N.Y.) **10**, 1195 (1969).
[7] R. Debever, On type D expanding solutions of Einstein–Maxwell equations, Bull. Soc. Math. Belg. **23**, 360 (1971).
[8] J. F. Plebański and M. Demiański, Rotating, charged and uniformly accelerating mass in general relativity, Ann. Phys. (N.Y.) **98**, 98 (1976).
[9] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
[10] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).
[11] J. B. Griffiths and J. Podolský, Accelerating and rotating black holes, Classical Quantum Gravity **22**, 3467 (2005).
[12] J. Podolský and J. B. Griffiths, Accelerating Kerr–Newman black holes in (anti-)de Sitter space-time, Phys. Rev. D **73**, 044018 (2006).
[13] J. B. Griffiths and J. Podolský, A new look at the Plebański–Demiański family of solutions, Int. J. Mod. Phys. D **15**, 335 (2006).
[14] J. Podolský and A. Vrátný, New improved form of black holes of type D, Phys. Rev. D **104**, 084078 (2021).

[15] A. Vrátný, Spacetimes with accelerating sources, Master thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2018.

[16] D. Matejov and J. Podolský, Uniqueness of extremal isolated horizons and their identification with horizons of all type D black holes, Classical Quantum Gravity **38**, 135032 (2021).

[17] D. Matejov and J. Podolský, Extremal isolated horizons with a cosmological constant and the related unique type D black holes, Phys. Rev. D **105**, 064016 (2022).

[18] K. Hong and E. Teo, A new form of the rotating *C*-metric, Classical Quantum Gravity **22**, 109 (2005).

[19] J. Podolský and A. Vrátný, Accelerating NUT black holes, Phys. Rev. D **102**, 084024 (2020).

[20] B. Chng, R. Mann, and C. Stelea, Accelerating Taub-NUT and Eguchi–Hanson solitons in four dimensions, Phys. Rev. D **74**, 084031 (2006).

[21] C. Cherubini, D. Bini, S. Capozziello, and R. Ruffini, Second order scalar invariants of the Riemann tensor: Applications to black hole spacetimes, Int. J. Mod. Phys. D **11**, 827 (2002).

[22] K. Lake and T. Zannias, Global structure of Kerr–de Sitter spacetimes, Phys. Rev. D **92**, 084003 (2015).

[23] E. L. Rees, Graphical discussion of the roots of a quartic equation, Am. Math. Mon. **29**, 51 (1922).

[24] P. T. Chruściel, M. Maliborski, and N. Yunes, Structure of the singular ring in Kerr-like metrics, Phys. Rev. D **101**, 104048 (2020).

[25] R. H. Boyer and R. W. Lindquist, Maximal analytic extension of the Kerr metric, J. Math. Phys. (N.Y.) **8**, 265 (1967).

[26] B. Carter, Global structure of the Kerr family of gravitational fields, Phys. Rev. **174**, 1559 (1968).

[27] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).

[28] G. W. Gibbons and S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, Phys. Rev. D **15**, 2538 (1977).

[29] S. Akcay and R. A. Matzner, The Kerr–de Sitter universe, Classical Quantum Gravity **28**, 085012 (2011).

[30] P. T. Chruściel, C. R. Ölz, and S. J. Szybka, Space-time diagrammatics, Phys. Rev. D **86**, 124041 (2012).

[31] J. Borthwick, Maximal Kerr–de Sitter spacetimes, Classical Quantum Gravity **35**, 215006 (2018); **39**, 219501(C) (2022).

[32] J. B. Griffiths and J. Podolský, Global aspects of accelerating and rotating black hole space-times, Classical Quantum Gravity **23**, 555 (2006).

[33] J. B. Griffiths, P. Krtouš, and J. Podolský, Interpreting the *C*-metric, Classical Quantum Gravity **23**, 6745 (2006).

[34] M. A. H. MacCallum, Two discs and a missing triangle: The maximally extended Kerr black hole revisited, *Proceedings of the Carter Fest Meeting Black Holes and Other Cosmic Systems* in Honour of Brandon Carter's 80th birthday, 2022 (IAP, Paris & Observatoire de Paris, Meudon, 2022) and subsequent private communication.

[35] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, London, 1984).

[36] J. Lewandowski and T. Pawlowski, Extremal isolated horizons: A local uniqueness theorem, Classical Quantum Gravity **20**, 587 (2003).

[37] H. K. Kunduri and J. Lucietti, A classification of near-horizon geometries of extremal vacuum black holes, J. Math. Phys. (N.Y.) **50**, 082502 (2009).

[38] H. K. Kunduri and J. Lucietti, Uniqueness of near-horizon geometries of rotating extremal AdS_4 black holes, Classical Quantum Gravity **26**, 055019 (2009).

[39] H. K. Kunduri and J. Lucietti, Classification of near-horizon geometries of extremal black holes, Living Rev. Relativity **16**, 8 (2013).

[40] E. Buk and J. Lewandowski, Axisymmetric, extremal horizons at the presence of a cosmological constant, Phys. Rev. D **103**, 104004 (2021).

[41] S. Bahamonde, J. G. Valcarcel, and L. Järv, Plebański-Demiański solutions with dynamical torsion and nonmetricity fields, J. Cosmol. Astropart. Phys. 04 (2022) 011.

Conclusions

In this thesis, we have studied exact black hole solutions of Einstein's field equations that belong to the large Plebański–Demiański family of all type D spacetimes with a cosmological constant and double aligned Maxwell field.

In the first part, I got acquainted the reader with my personal journey throughout the study of theoretical physics. I formulated the problems I have been working on, and briefly presented their current state of knowledge.

One particular section deserves to be mentioned explicitly, namely that containing the relations between the *Kretschmann scalar*, the analogous *Weyl scalar*, and the scalar invariant I. Direct computer algebra calculation of the Kretschmann scalar is often very complicated, and we offer an approach how to retrieve the expression using the NP quantities. We believe that this could be helpful even for other unrelated problems.

The second part of this thesis contains the new original results.

In the **first chapter**, we analyzed a solution, originally found by Chng, Mann and Stelea in 2006 [37], which describes an accelerating black hole with a twist parameter NUT. Finding such a solution was surprising since it was shown by Griffiths and Podolský [38] that no such solution is included in the large Plebański–Demiański family of type D black holes.

We proved that *this solution is a vacuum solution* of the Einstein field equations, that means that its Ricci tensor $R_{\mu\nu}$ vanishes identically. This was verified by two independent methods. We computed all the Weyl tensor components in the null tetrad, and determined the algebraic type. It turned out that the solution is of the *general algebraic type I* with four distinct principal null directions. This answered the main question, why this solution was not found in the Plebański–Demiański class of type D solutions.

We then introduced a new metric representation of this spacetime described by three physical parameters — the mass m, acceleration α, and NUT parameter l. It enabled an easy transitions to the standard forms of C-metric or the Taub–NUT metric, just by putting an appropriate physical parameter to zero.

Using this new convenient metric representation, we were able to compute and analyze the main physical and geometrical properties, such as the location and the nature of the Killing horizons, the curvature of the black hole, asymptotically flat regions and the global structure of this metric. Furthermore, we analyzed the axes which turned out to have the conicity, causing an acceleration of the black hole, and also the twist, which is a clear contribution of the NUT parameter l. Along these axes, pathological regions with closed timelike curves occur.

These results were published in Physical Review D in 2020 [44].

In the **second chapter**, we presented a new representation of the whole Plebański–Demiański class of black holes without a cosmological constant, $\Lambda = 0$. This metric further improves the convenient representation found by Griffiths and Podolský in 2005.

This new form of the solution explicitly depends on *6 physical parameters*, namely the mass m, acceleration α, Kerr-like rotation a, NUT parameter l, and on electric and magnetic charges e and g. The great advantage of this new metric is that we obtain the well-known black holes just by setting the corresponding parameters to zero. These are, for example, (possibly accelerating) Kerr–Newman black hole, (possibly charged) Taub–NUT solution, or (possibly rotating and charged) C-metric. No (possibly charged) accelerating Taub–NUT exists in this form of the general and large family, which further confirms our conclusions from Chapter 1.

Very useful in the subsequent analysis of the solution proved to be the simple, explicit and factorized form of the key metric function $P(\theta)$ and $Q(r)$. We were thus able to easily localize the horizons, and study their properties such as their degeneration. Moreover, it enabled us to investigate various physical and geometrical phenomena, such as the character of the singularities, visualization of the ergoregions, nature of the axes (their conicity, rotational behavior or the pathological regions with closed timelike curves caused by the presence of the parameter NUT). Additionally, we provided the Kruskalization and generated the corresponding Penrose conformal diagrams. We also expressed the area and the surface gravity of the black hole horizons and the acceleration horizons, from which we were able to calculate the basic thermodynamic quantities.

In 2021, we published all these results in the exhaustive paper in Physical Review D [45].

In the **third chapter**, we built on the results of paper [45], and we further *generalized* the new metric form of the Plebański–Demiański solution by admitting *any value of the cosmological constant*, $\Lambda \neq 0$. This was achieved by generalizing the key metric functions $P(\theta)$ and $Q(r)$.

Thus we derived a new representation of *all* black holes of algebraic type D, determined by *7 physical parameters*, namely the mass m, acceleration α, Kerr-like rotation a, NUT parameter l, the electric and magnetic charges e and g, and the cosmological constant Λ.

Our new metric form simplifies to the standard metrics of the well-known black holes, namely to the Kerr–Newman–NUT–(anti-)de Sitter black hole (for $\alpha = 0$), accelerating Kerr–Newman–(anti-)de Sitter black hole (for $l = 0$), charged Taub–NUT–(anti-) de Sitter black hole (for $a = 0$), accelerating Kerr–NUT–(anti-)de Sitter black hole (for $e = g = 0$) and their analogies in asymptotically flat universe (when $\Lambda = 0$), just by setting the appropriate physical parameters to zero.

Even for the $\Lambda \neq 0$ case we explicitly proved that *no accelerating Taub–NUT–(anti-)de Sitter solution exists in this large class of spacetimes.*

Using this convenient representation, we were able to analyze various physical and geometrical properties of this class of black holes. We localized the horizons and generally classified their multiplicity. We visualized the ergoregions, clarified the character of singularities, and described the global structure, providing the Penrose conformal diagrams. The nature of the cosmic strings or struts along the axes $\theta = 0$, or $\theta = \pi$, respectively, was also elucidated. We calculated their conicity, which causes the acceleration of the black holes, and we managed to regularize it for a balanced values of the physical parameters. Both axes are twisting, and are surrounded by a pathological regions caused by the presence of the NUT parameter l. We also evaluated the main thermodynamic properties, namely the entropy or the temperature of the horizons.

All these results have been recently summarized in a comprehensive publication *New form of all black holes of type D with a cosmological constant*, accepted to Physical Review D [46] in March 2023.

Bibliography

[1] Griffiths, J. B. and Podolský, J.: *Exact Space-Times in Einstein's General Relativity*, Cambridge University Press, Cambridge (2009)

[2] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E.: *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge (2003)

[3] Einstein, A.: *Die Feldgleichungen der Gravitation*, Preuss. Akad. Wiss. Sitz. **48** 844–847 (1915)

[4] Einstein, A.: *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, Preuss. Akad. Wiss. Sitz. **47** 831–839 (1915)

[5] Schwarzschild, K.: *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **7** 189–196 (1916)

[6] Isaacson, W.: *Einstein: Jeho život a vesmír*, Ladislav Horáček - Paseka, Praha a Litomyšl (2010)

[7] Einstein, A.: *Näherungsweise Integration der Feldgleichungen der Gravitation*, Preuss. Akad. Wiss. Sitz. 688–696 (1916)

[8] Einstein, A.: *Über Gravitationswellen*, Preuss. Akad. Wiss. Sitz. 154–167 (1918)

[9] Einstein, A.: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie*, Preuss. Akad. Wiss. Sitz. **6** 142–152 (1917)

[10] de Sitter, W.: *Over de relativiteit der traagheid: Beschouingen naar aanleiding van Einstein's hypothese*, Koninklijke Akademie van Wetenschappen te Amstedam **25** 1268–1278 (1917a), Proc. Akad. Amsterdam **19** 1217–1225 (1918)

[11] Einstein, A.: *Die Grundlage der allgemeinen Relativitätstheorie*, Ann. Phys., Leipzig, **49** 769 (1916)

[12] Thorne, K.: *Warping spacetime* in *The future of theoretical physics and cosmology: celebrating Stephen Hawking's 60th birthday*, Cambridge University Press, Cambridge, page 74 (2003)

[13] Pais, A.: *Subtle Is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press., Oxford (1982)

[14] Cervantes-Cota, J. L., Galindo-Uribarri, S. and Smoot, G. F.: *A Brief History of Gravitational Waves*, Universe **2** 22 (2016)

[15] Weber, J.: *Gravitational radiation*, Phys. Rev. Lett. **18** 498–501 (1967)

[16] Weber, J.: *Gravitational-wave-detector events*, Phys. Rev. Lett. **20** 1307–1308 (1968)

[17] Weber, J.: *Evidence for discovery of gravitational radiation*, Phys. Rev. Lett. **22** 1320–1324 (1969)

[18] Taylor, J. H., Fowler, L. A. and McCulloch, P. M.: *Overall measurements of relativistic effects in the binary pulsar PSR 1913 + 16*, Nature **277** 437–440 (1979)

[19] Taylor, J. H. and Weisberg, J. M.: *A New Test of General Relativity: Gravitational Radiation and the Binary Pulsar PSR 1913+16*, Astrophysical Journal **253** 908–920 (1979)

[20] Caltech/MIT/LIGO Lab: *Gravitational Waves, As Einstein Predicted*, Available online: https://www.ligo.caltech.edu/image/ligo20160211a, ligo20160211a (2016)

[21] Taub, A. H.: *Empty space-times admitting a three parameter group of motions*, Ann. Math. (The Annals of Mathematics) **53** 472–490 (1951)

[22] Newman, E., Tamburino, L. and Unti, T.: *Empty-space generalization of the Schwarzschild metric*, Journal of Mathematical Physics **4** 915–923 (1963)

[23] Ehlers, J. and Kundt, W.: *Exact solutions of the gravitational field equations* in *Gravitation: an introduction to current research*, (ed. L. Witten), Wiley, New York 49–101 (1962)

[24] Kinnersley, W. and Walker, M.: *Uniformly accelerating charges mass in general relativity*, Phys. Rev. D **2** 1359–1370 (1970)

[25] Hong, K. and Teo, E.: *A new form of the C-metric*, Class. Quant. Grav. **20** 3269–3277 (2003)

[26] Hong, K. and Teo, E.: *A new form of the rotating C-metric*, Class. Quant. Grav. **22** 109–117 (2005)

[27] Kerr, R. P.: *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*, Phys. Rev. Lett. **11** 237–238 (1963)

[28] Debever, R.: *On type D expanding solutions of Einstein–Maxwell equations*, Bull. Soc. Math. Belg. **23** 360–376 (1971)

[29] Plebański, J. F. and Demiański, M.: *Rotating, charged and uniformly accelerating mass in general relativity*, Ann. Phys. NY, **98** 98–127 (1976)

[30] Abbott, Benjamin P., et al. (LIGO Scientific Collaboration and Virgo Collaboration): *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116** 061102 (2016)

[31] Arun, K. G., et al. (LISA Collaboration): *New horizons for fundamental physics with LISA*, Living Rev. Rel. **25** 1 (2022)

[32] Doeleman, S.: *Imaging an Event Horizon: submm-VLBI of a Super Massive Black Hole*, Astro2010: The Astronomy and Astrophysics Decadal Survey, Science White Papers 68 (2010)

[33] Kazunori, A. et al.: *First M87 Event Horizon Telescope results*, The shadow of the supermassive black hole, Astrophys. J. Lett. **875** 1 L6 (2019)

[34] Kazunori, A. et al.: *First M87 Event Horizon Telescope Results. VII. Polarization of the Ring. The shadow of the supermassive black hole*, Astrophys. J. Lett. **910** 1 L12 (2021)

[35] Kazunori, A. et al.: *First Sagittarius A* Event Horizon Telescope Results. III. Imaging of the Galactic Center Supermassive Black Hole*, Astrophys. J. Lett. **930** 2 L14 (2022)

[36] Brill, D. R.: *Electromagnetic fields in a homogeneous, nonisotropic universe*, Phys. Rev. B **133** 845–848 (1964)

[37] Chng, B., Mann, R. and Stelea, C.: *Accelerating Taub–NUT and Eguchi–Hanson solitons in four dimensions*, Phys. Rev. D **74** 084031 (2006)

[38] Griffiths, J. B. and Podolský, J.: *Accelerating and rotating black holes*, Class. Quantum Grav. **22** 3467–3479 (2005)

[39] Podolský, J. and Griffiths, J. B.: *Accelerating Kerr–Newman black holes in (anti-)de Sitter space-time*, Phys. Rev. D **73** 044018 (2006)

[40] Griffiths, J. B. and Podolský, J.: *A new look at the Plebański–Demiański family of solutions*, Int. J. Mod. Phys. D **15** 335–369 (2006)

[41] Vrátný, A.: *Studium prostoročasů typu Taub–NUT se zrychlením*, Bachelor Thesis, Faculty of Mathematics and Physics, Charles University Prague (2016)

[42] Vrátný, A.: *Spacetimes with accelerating sources*, Master Thesis, Faculty of Mathematics and Physics, Charles University Prague (2018)

[43] Vrátný, A. and Podolský, J.: *Accelerating Taub-NUT Black Hole*, WDS'19 Proceedings of Contributed Papers, Matfyzpress, Prague, 151–157 (2019)

[44] Podolský, J. and Vrátný, A.: *Accelerating NUT black holes*, Phys. Rev. D **102** 084024 (2020)

[45] Podolský, J. and Vrátný, A.: *New improved form of black holes of type D*, Phys. Rev. D **104** 084078 (2021)

[46] Podolský, J. and Vrátný, A.: *New form of all black holes of type D with a cosmological constant*, arXiv:2212.08865 (2022), accepted to Phys. Rev. D (2023)

[47] Penrose, R.: *A spinor approach to general relativity*, Ann. Phys. (USA) **10** 171–201 (1960)

[48] d'Inverno, R. A. and Russell-Clark, R. A.: *Classification of the Harrison metrics*, Journal of Mathematical Physics **12** 1258–1263 (1971)

[49] Cherubini, C., Bini, D., Capozziello, S. and Ruffini, R.: *Second order scalar invariants of the Riemann tensor: applications to black hole spacetimes*, Int. J. Mod. Phys. D **11** 827–841 (2002)

[50] Teo, E.: *Accelerating black diholes and static black di-rings*, Phys. Rev. D **73** 024016 (2006)

[51] Bičák, J. and Schmidt, B.: *Asymptotically flat radiative space-times with boost-rotation symmetry*, Phys. Rev. D **40** 1827–1853 (1989)

[52] Bičák, J. and Pravda, V.: *Spinning C metric: Radiative spacetime with accelerating, rotating black holes*, Phys. Rev. D **60** 044004 (1999)

[53] Griffiths, J. B. and Podolský, J.: *Global aspects of accelerating and rotating black hole space-times*, Class. Quantum Grav. **23** 555–568 (2006)

[54] Appels, M., Gregory, R. and Kubizňák, D.: *Thermodynamics of accelerating black holes*, Phys. Rev. Lett. **117** 131303 (2016)

[55] Anabalón, A., Appels, M., Gregory, R., Kubizňák, D., Mann, R. B. and Övgün, A.: *Holographic thermodynamics of accelerating black holes*, Phys. Rev. D **98** 104038 (2018)

[56] Carter, B.: *Global structure of the Kerr family of gravitational fields*, Phys. Rev. **174** 1559–1571 (1968)

[57] Hawking, S. W. and Ellis, G. F. R.: *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge (1973)

[58] Boyer, R. H. and Lindquist, R. W.: *Maximal analytic extension of the Kerr metric*, J. Math. Phys. **8** 265–281 (1967)

[59] Gibbons, G. W. and Hawking, S. W.: *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D **15** 2538–2551 (1977)

[60] Akcay, S. and Matzner, R. A.: *The Kerr–de Sitter universe*, Class. Quantum Grav. **28** 085012 (2011)

[61] Chruściel, P. T., Ölz, C. R. and Szybka, S. J.: *Space-time diagrammatics*, Phys. Rev. D **86** 124041 (2012)

[62] Borthwick, J.: *Maximal Kerr–de Sitter spacetimes*, Class. Quantum Grav. **35** 215006 (2018), Corrigendum: **39** 219501 (2022)

[63] Lake, K. and Zannias, T.: *Global structure of Kerr–de Sitter spacetimes*, Phys. Rev. D **92** 084003 (2015)

[64] Griffiths, J. B., Krtouš, P. and Podolský, J.: *Interpreting the C-metric*, Class. Quantum Grav. **23** 6745–6766 (2006)

[65] MacCallum, M. A. H.: *Two discs and a missing triangle: the maximally extended Kerr black hole revisited*, Talk at the Carter Fest meeting *Black holes and other cosmic systems* in honour of Brandon Carter's 80th birthday, IAP, Paris, & Observatoire de Paris, Meudon, and subsequent private communication, July 4–6 (2022)

[66] Wald, R. M.: *General Relativity*, University of Chicago Press, Chicago and London (1984)

[67] Rees, E. L.: *Graphical discussion of the roots of a quartic equation*, American Mathematical Monthly **29** 51–55 (1922)

[68] Lewandowski, J. and Pawlowski, T.: *Extremal isolated horizons: a local uniqueness theorem*, Class. Quantum Grav. **20** 587–606 (2003)

[69] Kunduri, H. K. and Lucietti, J.: *A classification of near-horizon geometries of extremal vacuum black holes*, J. Math. Phys. **50** 082502 (2009)

[70] Kunduri, H. K. and Lucietti, J.: *Uniqueness of near-horizon geometries of rotating extremal AdS_4 black holes*, Class. Quantum Grav. **26** 055019 (2009)

[71] Kunduri, H. K. and Lucietti, J.: *Classification of near-horizon geometries of extremal black holes*, Living Rev. Relativ. **16** 8 (2013)

[72] Buk, E. and Lewandowski, J.: *Axisymmetric, extremal horizons at the presence of a cosmological constant*, Phys. Rev. D **103** 104004 (2021)

[73] Matejov, D. and Podolský, J.: *Uniqueness of extremal isolated horizons and their identification with horizons of all type D black holes*, Class. Quantum Grav. **38** 135032 (2021)

[74] Matejov, D. and Podolský, J.: *Extremal isolated horizons with a cosmological constant and the related unique type D black holes*, Phys. Rev. D **105** 064016 (2022)

List of publications

Vrátný A. and Podolský J.: *Accelerating Taub-NUT Black Hole*, in WDS'19 Proceedings of Contributed Papers (Prague: Matfyzpress, Charles University) 151–157 (2019)

Podolský J. and Vrátný A.: *Accelerating NUT black holes*, Phys. Rev. D **102** 084024, 27pp (2020)

Podolský J. and Vrátný A.: *New improved form of black holes of type D*, Phys. Rev. D **104** 084078, 26pp (2021)

Podolský J. and Vrátný A.: *New form of all black holes of type D with a cosmological constant*, arXiv:2212.08865, 38pp (2022), accepted to Phys. Rev. D (2023)

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XI Black Holes Workshop, Lisboa,
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Short-term visit to Fakultät für Mathematik, Universität Wien,
2019

Week of Doctoral Students 2019, Prague,
oral presentation *Accelerating Taub–NUT black hole*,
2019

XII Black Holes Workshop, Guimaraes,
poster *Conformal tools for stationary axisymmetric metrics*,
2019

Relativistic Seminar, Prague,
oral presentation *Accelerating Taub–NUT black hole*,
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Black hole spacetimes in a general dimension, their properties and interpretation
2022–2024

GAČR 23-05914S
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Advanced techniques applied to black-hole and gravitational-wave exact spacetimes
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