FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Marian Poljak

Off-diagonal ordered Ramsey numbers

Department of Applied Mathematics

Supervisor of the master thesis: doc. RNDr. Martin Balko, Ph.D.
Study programme: Computer Science
Study branch: Discrete Models and Algorithms

Prague 2023

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank my supervisor doc. RNDr. Martin Balko, Ph.D. for his encouragement, collaboration, proof-reading and coming up with many key ideas. I would also like to thank my family, friends and workplace for being supportive throughout this journey. Thanks to Klára Pernicová, Bára Tížková and Adéla Kostelecká for proof-reading. Lastly, thanks to Martin Mareš who helped me with making this document PDF/A-compliant.

Title: Off-diagonal ordered Ramsey numbers

Author: Marian Poljak

Department: Department of Applied Mathematics

Supervisor: doc. RNDr. Martin Balko, Ph.D., Department of Applied Mathematics

Abstract: We study ordered Ramsey numbers, an analogue of the classical Ramsey numbers for graphs with linearly ordered vertex sets. Inspired by a problem posed by Conlon, Fox, Lee and Sudakov, we focus on ordered Ramsey numbers of ordered matchings $M^<$ versus triangles. We generalize their lower bound on $r_<(M^<, K_3^<)$ for ordered matchings with any fixed interval chromatic number. We also analyze an upper bound on $r_<(M^<, K_3^<)$ for almost all ordered matchings $M^<$ with interval chromatic number 2 obtained by Rohatgi and improve it from $O(n^{24/13})$ to $O(n^{7/4})$.

Keywords: ordered graph, ordered Ramsey numbers, Ramsey theory, off-diagonal Ramsey numbers

Contents

1. Introduction

The Ramsey theory is a concept introduced in 1930 by Frank P. Ramsey and an important branch of mathematics. The focus of Ramsey theory is on the following question: "How large a system must be so that a highly organized subsystem appears within it?".

This Master thesis can be understood as a continuation of my Bachelor thesis titled *Computing and estimating ordered Ramsey numbers* [1], as we build and expand on some of its results. For that reason, some parts of this introduction might overlap with the introduction in my Bachelor thesis.

We begin by a classical problem used to introduce Ramsey numbers, the so-called *Party problem*.

Problem 1 (Party problem)**.** *Six people arrived to a party. Some pairs of people already know each other, let us call them friends. The other pairs have not met each other before, let us call them non-friends. Prove that even if we do not know anything about the friendship relations, we are guaranteed to find a group of three people such that either*

(a) all of them are friends, or

(b) all of them are non-friends.

Proof. Let us pick a person present at the party and denote him A. From his point of view, there are five other people, and he might know some of them. But we know for sure that A has at least three friends, or at least three non-friends at the party. Without loss of generality, let it be the former, as the latter can be achieved symmetrically by reversing the friendship relation. Let us denote A's friends by B, C, D.

If any two of these are friends, then they, together with A, form a triplet of people, all of which are friends. But if none of B, C, D know each other, then they form a triplet of non-friends. The statement thus holds. ☐

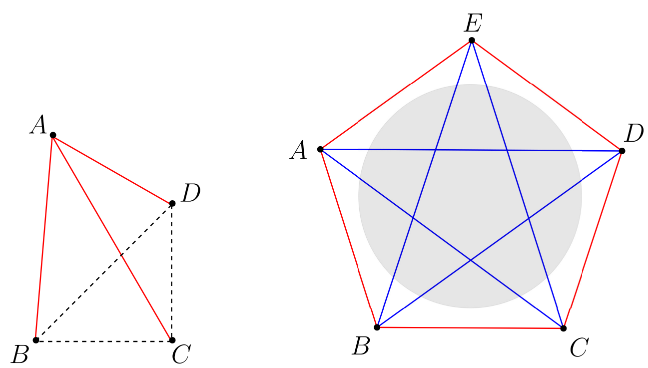

Figure 1.1: On the left, an illustration of the proof when A has three friends at the party (red color represents the friendship relation). On the right, an example of a party of five people where neither a group of three friends nor a group of three non-friends exists.

It is easy to see that if there were more than six people at the party, the statement would still hold, as we could use the same proof for six people while ignoring the extra ones. But if there were less than six people, the statement no longer holds. To see this, imagine there are five people sitting at a round table and that every person sits between two friends and does not know the two people opposite to him, see Figure 1.1. At this party, there is neither a triplet of friends, nor a triplet of non-friends. Again, with less than five people, it would be even clearer that such triplets may not exist.

It is interesting to ask how the problem changes when we look for larger groups of friends or non-friends, or for even less regular structures. How many people would then be needed to contain either the given group of friends or the given group of non-friends? Although the previous problem for groups of three friends/non-friends is easy, the situation gets out of hand quickly when increasing the sizes of the groups. In order to answer these questions, we will get a bit more formal about the problem we are facing.

1.1 Ramsey numbers

Here we state some basic definitions and give a brief introduction on Ramsey-type problems.

Definition 1. *A* ***graph*** *G is a pair (V, E) where V is a finite set of vertices and E a set of edges. An* ***edge*** *$\{u, v\}$ is an unordered pair of different vertices $u, v \in V$. We denote the* ***size*** *of the graph G by $|G| := |V|$. A* ***complete graph*** *K_N is a graph on N vertices where any two vertices are connected by an edge.*

We say that for $\{u, v\} \in E$, vertex u is a **neighbour** of vertex v and vice-versa. We will use the phrases 'forming an edge', 'being connected', 'being connected by an edge', 'being neighbours', 'being adjacent' etc. interchangeably. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively.

Definition 2. *A* ***coloring*** *of a graph $G = (V, E)$ is a mapping $f : E \to \mathcal{C}$ that maps edges into a given set of colors $\mathcal{C}$. For our use-cases in this thesis, we will always consider two-colorings with $\mathcal{C} = \{red, blue\}$, also called* red-blue colorings.

Since Ramsey theory revolves around searching for edge-colored copies of graphs in another graph, we will often work with the concept of subgraphs.

Definition 3. *We say that a graph $G' = (V', E')$ is a* ***subgraph*** *of another graph $G = (V, E)$ if V' is a subset of V and E' is a subset of E.*

We will usually work with two graphs, G (red graph) and H (blue graph). We will be interested in the smallest N such that a complete graph K_N, when colored by any red-blue coloring, always contains either a red copy of G, or a blue copy of H as a subgraph. Coming back to the introduction, red G and blue H will be our highly organized subsystems, which we will search for in a big and possibly unorganized system, the colored K_N. From now on we will not distinguish between a *coloring on N vertices* and a *coloring of K_N*.

Definition 4. *Given graphs G and H, the* ***Ramsey number*** *$r(G, H)$ is the smallest positive integer N such that any coloring of K_N contains either G as a red subgraph or H as a blue subgraph.*

Definition 5. *We say that a graph $G = (V, E)$ is **isomorphic** to another graph $G' = (V', E')$, if there exists a bijective function $f : V \to V'$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G'.*

Note that up to isomorphism, there is only one complete graph K_N for any positive integer N, validating Definition 1. Abiding by the usual convention for Ramsey numbers, we distinguish between the case when G is isomorphic to H, which is called a *diagonal* case and often abbreviated as $r(G) := r(G, G)$, and the *off-diagonal* case when G is not isomorphic to H.

Taking a look back at the Party problem, we have discovered that $r(K_3) = 6$, since we have shown a specific (so-called *avoiding*) coloring which avoids both monochromatic copies of K_3 for $N = 5$, and we have also proven that for $N = 6$, one of those copies is guaranteed. This nicely illustrates how classical Ramsey-type proofs go. We can obtain lower bounds on the Ramsey number $r(G, H)$ by showing the existence of a coloring on N vertices which avoids both graphs in their respective colors as subgraphs, implying that $r(G, H) > N$. Upper bounds are established by proving that in any coloring on N vertices, we are guaranteed to find either G or H as a subgraph in their respective color, implying $r(G, H) \leq N$.

For the Party problem, we were lucky, our estimates matched and thus we were able to determine the Ramsey number exactly. This is usually not the case, as illustrated by a folklore quote attributed to Erdős:

> *Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.*

Indeed, although it is known that $r(K_4) = 18$, for K_5 it is currently only known that $43 \leq r(K_5) \leq 48$. See the survey [2] for the most recent estimates. Ramsey numbers can grow very quickly with the size of the graphs G and H and some of the most famous and difficult open problems in Ramsey theory concern the values of Ramsey numbers for specific graphs G and H.

Ramsey [3] proved that Ramsey numbers are always finite, which is not obvious for larger graphs. This was independently rediscovered by Erdős and Szekeres [4] who also proved the following bounds for all $m, n \in \mathbb{N}$:

$$r(K_m, K_n) \leq \binom{m+n-2}{n-1} \quad \text{and} \quad [1 + o(1)]\frac{n}{e\sqrt{2}}2^{\frac{n}{2}} \leq r(K_n) \leq \frac{4^n}{\sqrt{n}}. \tag{1.1}$$

The first upper bound gives us that Ramsey numbers are finite as Corollary 2, as any two general graphs G, H can be embedded in sufficiently large complete graphs K_m, K_n. Note that the right upper bound $r(K_n) \leq \frac{4^n}{\sqrt{n}}$ follows immediately from the left one by setting $m = n$. The lower bound was obtained by Erdős in [5] by a famous probabilistic proof.

Definition 6. *In a graph $G = (V, E)$, the **degree** of a vertex $v \in V$ is the number of other vertices in V connected to v by an edge. The **maximum degree** of a graph is the maximum degree over its vertices. If we work with a coloring f of G, then the **red degree** of a vertex v means the number of red vertices adjacent to v. Analogically for **blue degree**.*

Let us prove the first upper bound from (1.1) using a well-known proof.

Proof. The statement trivially holds for m or n equal to one. We now proceed by induction. The critical insight we need for the induction step is the following simple inequality: for all integers $m, n > 1$ it holds that $r(K_m, K_n) \leq r(K_{m-1}, K_n) + r(K_m, K_{n-1})$.

To prove this, consider a coloring on $r(K_m, K_n) - 1$ vertices which does not contain a red copy of K_m nor a blue copy of K_n. Note that no vertex in this coloring can have a red degree larger than $r(K_{m-1}, K_n) - 1$. If a vertex v had a red degree $r(K_{m-1}, K_n)$, its neighbourhood would, by definition, contain either a blue K_n, or a red K_{m-1}, which would together with v form a K_m. Both of these are a contradiction. Symmetrically, the blue degrees also have to be capped at $r(K_m, K_{n-1}) - 1$.

This implies that the vertex v can have at most

$$(r(K_{m-1}, K_n) - 1) + (r(K_m, K_{n-1}) - 1)$$

neighbours. On the other hand, the degree of v is exactly $r(K_m, K_n) - 2$. Thus, $r(K_m, K_n) - 2 \leq (r(K_{m-1}, K_n) - 1) + (r(K_m, K_{n-1}) - 1)$ and the inequality follows. To obtain an explicit bound on $r(K_m, K_n)$ as stated, we extend the recurrence by induction. $\square$

Corollary 2 (Ramsey's theorem for graphs [3, 4])**.** *For any two graphs G and H, the number $r(G, H)$ is finite.*

In other words, going back to the start of the introduction, whatever graphs we choose as G and H, we can always choose a sufficiently large N such that the coloring on N vertices contains at least one of them in the appropriate color.

At first, Ramsey theory focused on estimating the Ramsey numbers of complete graphs. Despite best efforts, the original exponential estimates from (1.1) were very resilient to any improvement. At the time of writing we can state that $[1+o(1)]\frac{\sqrt{2}s}{e}2^{\frac{s}{2}} \leq r(K_n)$ and that there exists an $\epsilon > 0$ such that $r(K_n) < (4-\epsilon)^n$. The lower bound is due to Spencer [6] and improves the original bound by a factor of 2, the upper bound is a very recent breakthrough – after almost 90 years, the basis of the exponential in the upper bound has been improved from 4 by Campos et al. in a preprint [7].

Interesting results were gathered for other graphs. Due to its importance throughout this thesis, we mention the off-diagonal Ramsey number of a red K_n and a blue *triangle* K_3. A lower bound by Kim [8] combined with an upper bound by Ajtai, Komlós and Szemerédi [9] gives us the exact asymptotics for this case.

Theorem 3 ([8, 9])**.** *For any positive integer n we have that*

$$r(K_n, K_3) \in \Theta(n^2/\log n).$$

Note that both the currently best lower bound on $r(K_n)$ by Spencer [6] and a slightly weaker lower bound $r(K_n, K_3) \in \Omega(n^2/\log^2 n)$ by Spencer [10] (originally proved by Erdős in [11]) can be obtained using Lovász local lemma, a powerful tool we will also use in our proofs.

More interesting results were derived by imposing additional constraints on the original problem. For example, a notable result is that the Ramsey number of a graph with a bounded maximum degree grows at most linearly in the number of its vertices and is due to Chvátal, Rödl, Szemerédi and Trotter [12].

Theorem 4 ([12]). *For every positive integer d, there exists some $c > 0$ such that if a graph G on n vertices has maximum degree at most d, then*

$$r(G) \leq cn.$$

One of the simplest classes of graphs are *matchings*. In this thesis, we restrict ourselves to perfect matchings as per the following definition. This is mostly for notational convenience, the results do not really change for imperfect matchings.

Definition 7. *A **matching** M is a graph on even number of vertices where each vertex has degree equal to one.*

We can quickly see that for a matching M on n vertices it holds that $r(M) \leq 2n - 2$, since we can group the vertices of K_{2n-2} into $(n-1)$ pairs of two and by the pigeonhole principle, at least $n/2$ of the pairs will be connected by an edge of the same color and will thus form the required monochromatic matching.

Interest in Ramsey theory sparked many ideas, some of which modify the original version slightly. For example, we can consider more than two colors or more than two graphs. We can generalize classical graphs to k-uniform hypergraphs with $k \geq 3$, or we can reason about infinite graphs. One of these many branches, and the topic of this thesis, came to be known as *ordered Ramsey numbers*.

1.2 Ordered Ramsey numbers

What if we modified the original Party problem to pay attention to the ordering of the party guests?

Problem 5. *There are seven people at a party and all of them line up in a single line to get a photo. Some pairs of these people can be friends, which is a symmetric relation. Prove that it always holds that either*

(a) there exists a person with three friends to his right, or

(b) there exists a person with a non-friend both to his right and to his left (not necessarily right next to him).

Again, this can be easily proved, and we can also quickly verify that if there were only six people, we would be able to line them up to avoid both conditions by properly selecting the friendship edges between them, as shown in Figure 1.2 on the right.

Proof. Denote by A the leftmost person in the line. If A has three friends among the others, then we are done, since all of them are necessarily to his right. Let us assume that he has less then three friends among the others, which implies that he has at least four non-friends to his right, denoted by B, C, D, E, respectively by their ordering (see Figure 1.2).

Suppose there is a pair of non-friends among B, C, D, E, denoted by X, Y, where X is to the left of Y. Then X has a non-friend A to his left and a non-friend Y to his right and we are done. The only remaining possibility is that B, C, D, E form a friendship group, which implies that B has three friends to his right and we are again done. $\square$

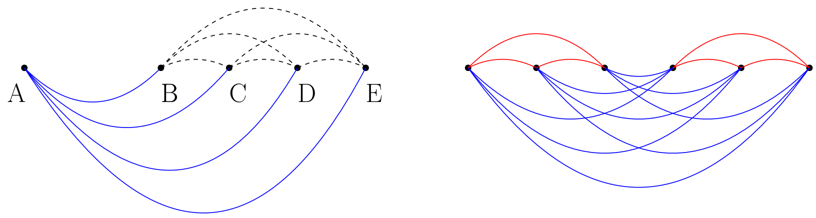

Figure 1.2: On the left, an illustration of the proof when the leftmost person has four non-friends at the party (blue color represents the non-friendship relation). On the right, an example of a party of six people where none of the given monochromatic substructures exist.

The fact that we are looking for an ordered substructure makes the original Ramsey problem somewhat different and we will soon discover where it differs from the unordered variant. To start, we define (quite analogically to the first part) the terms we work with.

Definition 8. *An **ordered graph** $G^<$ on N vertices is a graph G of size N whose vertices are ordered by the standard ordering of integers $<$. We can thus think of the vertex set as $[N] := \{1, 2, \ldots, N\}$ with 1 being the leftmost vertex and N being the rightmost vertex. We denote the **size** of the ordered graph $G^<$ by $|G^<| := |G|$.*

Definition 9. *We say that an ordered graph $G^<$ on vertex set V is **isomorphic** to another ordered graph $H^<$ on vertex set V' if there exists an order preserving bijective function $f : V \to V'$ such that $\{u, v\}$ is an edge of $G^<$ if and only if $\{f(u), f(v)\}$ is an edge of $H^<$.*

Note that up to isomorphism, there is only one ordered complete graph on N vertices and we denote it $K_N^<$. For other graphs though, the ordering matters. Note that in Figure 1.3, there are three ordered paths on 4 vertices, but no two of them are isomorphic.

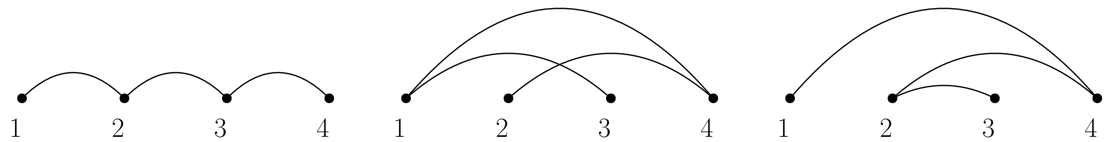

Figure 1.3: Three different non-isomorphic ordered paths on 4 vertices. The first path is a so-called *monotone path*, the third one so-called *alternating path* (see Definitions 12 and 15, respectively).

Definition 10. *An ordered graph $H^<$ on $[n]$ is an **ordered subgraph** of another ordered graph $G^<$ on $[N]$ if there exists a mapping $\phi : [n] \to [N]$ such that $\phi(i) < \phi(j)$ for $1 \leq i < j \leq n$ and also $\{\phi(i), \phi(j)\}$ is an edge of $G^<$ whenever $\{i, j\}$ is an edge of $H^<$. For an illustration, see Figure 1.4.*

Definition 11. *Given ordered graphs $G^<$ and $H^<$, the **ordered Ramsey number** $r_<(G^<, H^<)$ is defined as the smallest N such that any red-blue coloring of $K_N^<$ contains either $G^<$ as a red ordered subgraph or $H^<$ as a blue ordered subgraph.*

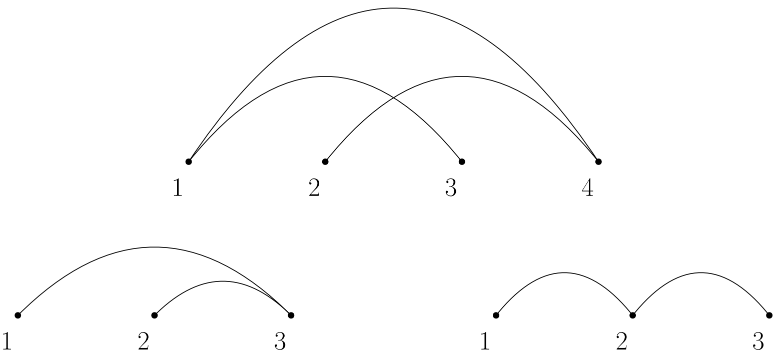

Figure 1.4: The lower left ordered graph is an ordered subgraph of the upper one, whereas the lower right ordered graph is not.

Many notions for unordered graphs have their natural counterpart for ordered graphs (like coloring, coloring on N vertices, degree of a vertex, diagonal ordered Ramsey number $r_<(G^<)$ etc.), we will thus refrain from defining them here again. Instead, with the mandatory definitions out of the way, we outline another interesting connection – the famous Erdős–Szekeres theorem on monotone subsequences.

Theorem 6 (Erdős–Szekeres theorem [4])**.** *For all positive integers m, n, every sequence of at least $(m-1)(n-1)+1$ distinct real numbers contains either an increasing subsequence of m numbers, or a decreasing subsequence of n numbers. Moreover, this is tight.*

This theorem is actually a special case of a statement about the ordered Ramsey number for monotone paths.

Definition 12. *For every positive integer n, a* ***monotone path*** *$P_n^<$ is an ordered graph on n vertices whose edges are between any two consecutive vertices. See Figure 1.5 for an example.*

Theorem 7. *For all positive integers m, n it holds that*

$$r_<(P_m^<, P_n^<) = (m-1)(n-1)+1.$$

The following proof is a variation of the proof of Theorem 6 from [13].

Proof. The lower bound $r_<(P_m^<, P_n^<) > (m-1)(n-1)$ follows from a coloring where we place $(m-1)$ blue cliques of size $(n-1)$ consecutively and color all edges between the cliques red. It is easy to verify that this coloring does not contain a red $P_m^<$ nor a blue $P_n^<$.

To prove the upper bound, let us consider a coloring on $N := (m-1)(n-1)+1$ vertices. For every vertex $i \in [N]$, we denote r_i to be the size of the longest monotone red path which makes use of vertices $\{1, \ldots, i\}$ and which ends in i. Suppose that the labels r_i are smaller than m for all $i \in [N]$, as otherwise we have a red copy of $P_m^<$ and the statement holds.

However, since the red labels are taken from the set $\{1, \ldots, m-1\}$, there is only $m-1$ of them. By the pigeonhole principle, there have to be at least $\left\lceil \frac{(m-1)(n-1)+1}{m-1} \right\rceil = n$ vertices with the same label. This, however, is a contradiction, as these n vertices sharing the same label cannot have any red edge $\{u, v\}$ (for $u < v$) between them, as then r_v would have to be at least $r_u + 1$. Thus they must form a blue copy of $K_n^<$, which trivially contains $P_n^<$. $\square$

Note that the previous proof actually established something even stronger, that is $r_<(P_m^<, K_n^<) = (m-1)(n-1)+1$. With Theorem 7, we can now derive Theorem 6 by considering the given sequence of $(m-1)(n-1)+1$ distinct real numbers and drawing colored edges between them. For two numbers u, v with u to the left of v, if $u < v$, let us draw a red edge between them, and a blue edge otherwise. This way, we get a coloring on $(m-1)(n-1)+1$ vertices. Now, Theorem 7 gives us the existence of a monochromatic monotone path of the required length, which corresponds to a monotone subsequence, as desired.

The definition of ordered Ramsey numbers is quite similar to the unordered case and it will be interesting to understand how these two relate. We can make a quick observation, which is already mentioned in [1, 14, 15]. We include its proof for completeness.

Observation 8. *For any two ordered graphs $G^<$ and $H^<$ and their unordered counterparts G and H, we have*

$$r(G, H) \leq r_<(G^<, H^<) \leq r(K_{|G|}, K_{|H|}).$$

Proof. If an ordered complete graph on N vertices contains the ordered graph $G^<$ or $H^<$ as a monochromatic ordered subgraph, it trivially also contains their unordered counterparts G or H, respectively, as a monochromatic subgraph. Omitting the ordering just makes the corresponding Ramsey number possibly lower. This proves the first inequality.

For the upper bound, the ordered graph $G^<$ is contained as an ordered subgraph in $K_{|G^<|}^<$ and the same holds for $H^<$, which implies that $r_<(G^<, H^<) \leq r_<(K_{|G^<|}^<, K_{|H^<|}^<)$. Also, we trivially have $r_<(K_{|G^<|}^<, K_{|H^<|}^<) = r(K_{|G|}, K_{|H|})$, which concludes the proof of the second inequality. $\square$

The proof implies that ordered Ramsey numbers are also finite, as any ordered graphs can be embedded into sufficiently large complete ordered graphs. It also gives us an insight that for complete ordered graphs, the behaviour of the two definitions coincides. Generally, when the edge density of ordered graphs $G^<$ and $H^<$ approaches that of a complete graph, their corresponding ordered Ramsey numbers exhibit similar behaviour as their unordered counterparts. However, sparse graphs is where a striking difference between the two notions appears.

We showed that an unordered matching M on n vertices has its Ramsey number $r(M)$ trivially linear in n. However, there are ordered matchings $M^<$ such that $r_<(M^<)$ is superpolynomial in n, as shown by Balko, Cibulka, Král and Kynčl [15].

Theorem 9 ([15])**.** *There are arbitrarily large ordered matchings $M^<$ on n vertices that satisfy*

$$r_<(M^<) \geq n^{\frac{\log n}{5 \log \log n}}.$$

We mention the independently proven version of this result from Conlon, Fox, Lee and Sudakov [14], which is slightly weaker, but applies to almost all ordered matchings.

Theorem 10 ([14]). *Let $M^<$ be a random ordered matching on n vertices. Then, asymptotically almost surely,*

$$r_<(M^<) \geq n^{\frac{\log n}{20 \log \log n}}.$$

We note that there exists an almost matching upper bound saying that for any ordered matching $M^<$ on n vertices it holds that $r_<(M^<) \leq n^{\lfloor \log n \rfloor}$, again due to Conlon et al. [14].

An important notion we will work with in this thesis is an *interval chromatic number* of an ordered graph.

Definition 13. *For a given ordered graph $G^<$, we denote its* ***interval chromatic number*** *$\chi_<(G^<)$ as the smallest number of contiguous intervals into which the vertex set of $G^<$ may be partitioned so that no two vertices from the same interval are adjacent; see Figure 1.5.*

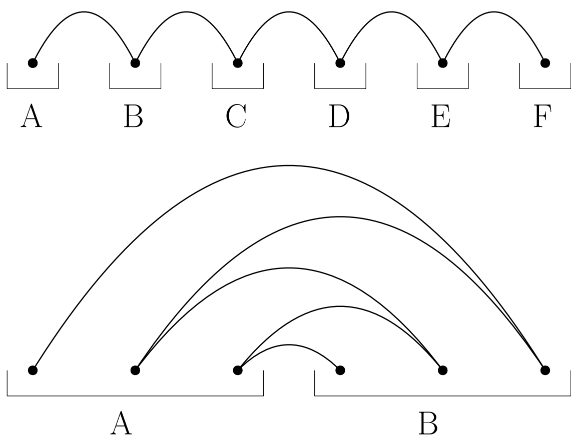

Figure 1.5: The first ordered graph, the monotone path on 6 vertices, has interval chromatic number 6. The second ordered graph, an alternating path on 6 vertices, has interval chromatic number 2. The best possible color assignments for both cases are depicted by letters.

Interval chromatic number is reminiscent of the classical definition of a chromatic number for unordered graphs, but with arbitrary vertex sets replaced by the described contiguous intervals.

Unlike the statement of Theorem 4 for unordered Ramsey numbers, Theorems 9 and 10 imply that bounding the maximum degree of an ordered graph $G^<$ is not sufficient to obtain a polynomial upper bound on $r_<(G^<)$. However, Conlon et al. [14] and Balko et al. [15] proved that if we bound the interval chromatic number as well, we do obtain a polynomial bound on $r_<(G^<)$.

Theorem 11 ([14], slightly weaker version). *There is a constant c such that for any ordered graph $G^<$ on n vertices with maximum degree d and interval chromatic number χ,*

$$r_<(G^<) \leq n^{cd \log \chi}.$$

1.3 Our contribution

In this thesis, we mostly study the off-diagonal case $r_<(M^<, K_3^<)$ where $M^<$ is an ordered matching.

In Chapter 2, we present the currently strongest known lower bound on $r_<(NM_n^<, K_3^<)$ (Theorem 15), where $NM_n^<$ is a particular class of ordered matchings (so-called *nested matchings*, see Definition 14). This is a general counterexample to a conjecture by Rohatgi [16] and allows us to improve a bound obtained by Dujmović and Wood [17] about the maximum chromatic number of k-queue graphs in Theorem 17. These results already appeared in the journal Discrete Mathematics [18] and at the conference EUROCOMB 2021 as an extended abstract [19], both of which I co-authored with my supervisor and which superseded my Bachelor thesis. In Chapter 2, we also mention some computational results, in particular, we present colorings that refute a formula suggested by Balko, Cibulka, Král and Kynčl [15] for diagonal ordered Ramsey numbers of alternating paths.

We consider general ordered matchings $M^<$ versus triangles in Chapter 3. First, in Section 3.2, we prove superlinear lower bounds on $r_<(M^<, K_3^<)$ with a fixed interval chromatic number $\chi_<(M^<)$ (Theorems 22 and 26). For $\chi_<(M^<) \geq 3$, this asymptotically matches the lower bound of Conlon, Fox, Lee and Sudakov [14] for ordered matchings with unrestricted interval chromatic number. In Section 3.3, we asymptotically improve Rohatgi's [16] upper bound on the number $r_<(M^<, K_3^<)$, where $M^<$ is a random ordered matching with interval chromatic number 2 (Theorem 29). The results of this chapter are to appear in a paper [20] co-authored by me and my supervisor at the conference EUROCOMB 2023.

Finally, in Chapter 4 we summarize our contribution and list known problems from this area together with some new open problems and possible directions for future research.

2. Nested matchings versus triangles

In this chapter we summarize the progress made on the off-diagonal ordered Ramsey numbers of specific well-behaved ordered matchings against a triangle, studied by Rohatgi [16]. Afterwards we show a general coloring which gives us a non-trivial lower bound on these ordered Ramsey numbers for ordered matchings of any size.

Definition 14. *For a positive integer n, a* ***nested matching*** $NM_n^<$ *is the ordered matching on $2n$ vertices with the edges $\{i, 2n-i+1\}$ for $1 \leq i \leq n$; see Figure 2.1.*

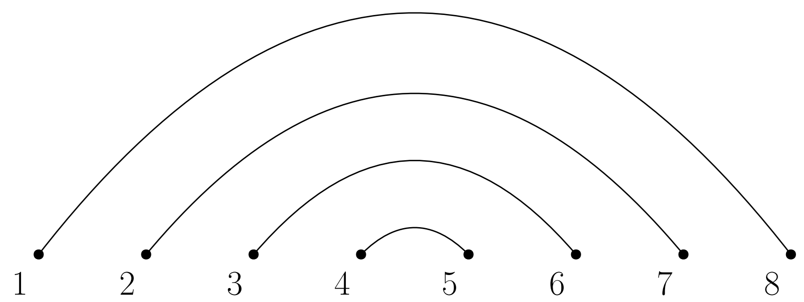

Figure 2.1: The nested matching $NM_4^<$.

Note that particularly in this chapter, it will be useful to consider the *matrix representation* of an ordered graph, which is a matrix A, where A_{ij} corresponds to an edge $\{i, j\}$. We extend this naturally to colorings of $K_N^<$, where we distinguish red and blue edges. It is enough to show the upper triangular part of the matrix since we only work with undirected graphs without self-loops. We call a coloring on N vertices *symmetric*, if it holds that the edge $\{i, j\}$ has the same color as the edge $\{N+1-j, N+1-i\}$ for all $1 \leq i < j \leq N$. See Figure 2.2 for an example.

2.1 Known results

Nested matchings were first introduced by Rohatgi [16], who proved that $4n-1 \leq r_<(NM_n^<, K_3^<) \leq 6n$ and then used this result to prove that the ordered Ramsey number of so-called *non-intersecting* ordered matchings is also nearly linear.

In my Bachelor thesis [1] we disproved a conjecture by Rohatgi stating that $r_<(NM_n^<, K_3^<) = 4n-1$ by finding an avoiding coloring (see Figure 2.2) for $n = 4, 5$ and thus showing that $r_<(NM_n^<, K_3^<) \geq 4n$ for these two particular values of n.

We also optimized the upper bound of $6n$ on the expression.

Proposition 12 ([1, 18]). *For every positive integer n,*

$$r_<(NM_n^<, K_3^<) \leq \left(3+\sqrt{5}\right)n + 1 < 5.3n + 1.$$

(a) (b)

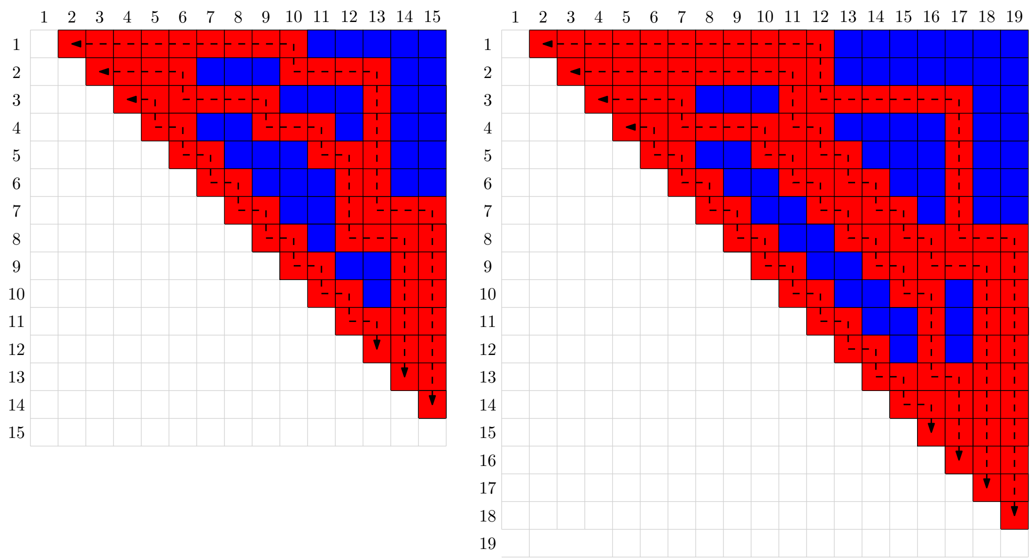

Figure 2.2: (a) The matrix representation of a coloring of the edges of $K_{15}^<$ vertices without a red $NM_4^<$ and a blue $K_3^<$. (b) The matrix representation of a symmetric coloring of the edges of $K_{19}^<$ without a red $NM_5^<$ and a blue $K_3^<$.

We continued this line of research to learn when does an ordered graph contain a copy of a nested matching as an ordered subgraph. This aligns closely with the notion of so-called k-*queue* graphs studied by Dujmović and Wood [17], as an ordered graph $G^<$ without a copy of $NM_{k+1}^<$ as an ordered subgraph directly corresponds to an ordered k-queue layout of G.

Lemma 13 ([1, 17])**.** *For every positive integer n, if an ordered graph $G^<$ on N vertices with $N \geq 2n$ does not contain $NM_n^<$ as an ordered subgraph, then the number of edges in $G^<$ is at most $(n-1)(2N-2n+1)$. Moreover, this upper bound is tight.*

The tightness of this bound can be achieved by a general construction. In an ordered graph $G^<$, we avoid $NM_n^<$ as a copy if we lead $(n-1)$ pairwise disjoint "routes" in the matrix A and set all entries of A with positions in these routes as edges and all other entries of A as non-edges. For $k \in [n-1]$, the kth *route* in the matrix A is the set of positions $\{(i_\ell, j_\ell) : \ell \in [2N-4k+3]\}$ such that the following four conditions hold:

- $(i_1, j_1) = (k, k)$, being the route starting cell,
- $(i_{2N-4k+3}, j_{2N-4k+3}) = (N-k+1, N-k+1)$, being the route ending cell,
- $(i_{\ell+1} = i_\ell + 1 \;\&\; j_{\ell+1} = j_\ell)$ for every $\ell \in [2N-4k+2]$, specifying that a route can go down in A and
- $(i_{\ell+1} = i_\ell \;\&\; j_{\ell+1} = j_\ell + 1)$ for every $\ell \in [2N-4k+2]$, specifying that a route can go to the right in A.

Note that a route can contain even those entries of A that lie on or below the main diagonal of A. We say that an ordered graph $G^<$ is *covered* by a set R of routes if every position in the matrix representation of $G^<$ that corresponds to an edge of $G^<$ is contained in some route from R.

Lemma 14 ([1, 18]). *For every positive integer n, every ordered graph $G^<$ that is covered by $n-1$ pairwise disjoint routes does not contain $NM_n^<$ as an ordered subgraph.*

2.2 General lower bound on nested matchings versus triangles

The avoiding colorings for $n = 4, 5$ we mentioned above are quite rare and ad-hoc, especially for $n = 4$. For $n \geq 6$, we found a slightly stronger and general coloring.

Theorem 15 ([18]). *For every $n \geq 6$, we have $r_<(NM_n^<, K_3^<) \geq 4n + 1$.*

Proof. Let $n \geq 6$ be an integer. The matrix representation A of the coloring χ is illustrated in Figures 2.3 and 2.4. We now describe the construction of χ formally by listing all its blue edges. Note that χ is symmetric.

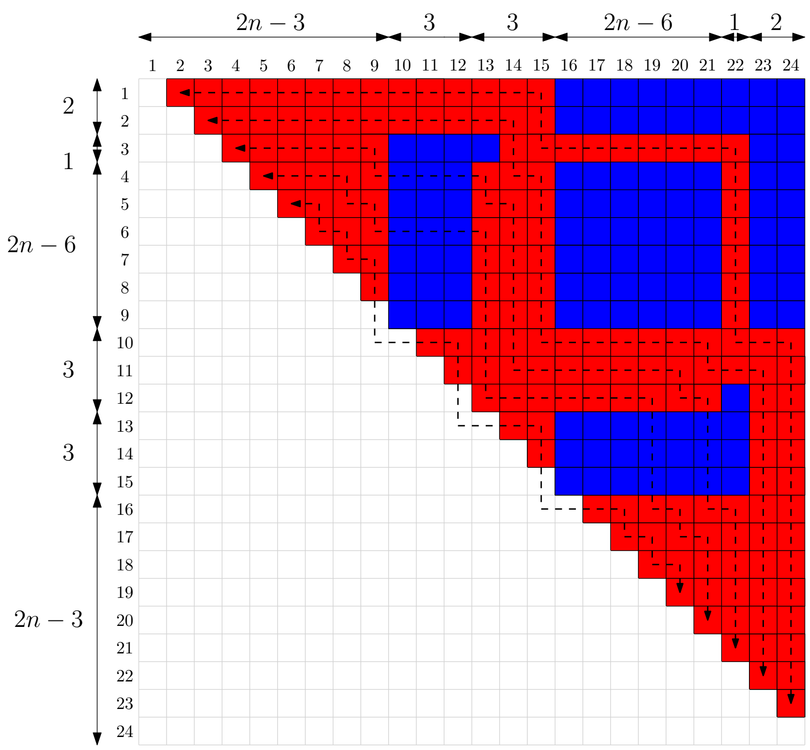

Figure 2.3: The matrix representation of the coloring χ of the edges of $K_{4n}^<$ for $n = 6$.

The blue edges in χ are decomposed into the following sets: the set

$$S = \{\{i, j\} : 4 \leq i \leq 2n-3, 2n+4 \leq j \leq 4n-3\}$$

which forms a $(2n-6) \times (2n-6)$ square in A, the set

$$L = \{\{i, j\} : i \in \{1, 2\}, 2n+4 \leq j \leq 4n\} \cup \{\{i, j\} : 1 \leq i \leq 2n-3, j \in \{4n, 4n-1\}\}$$

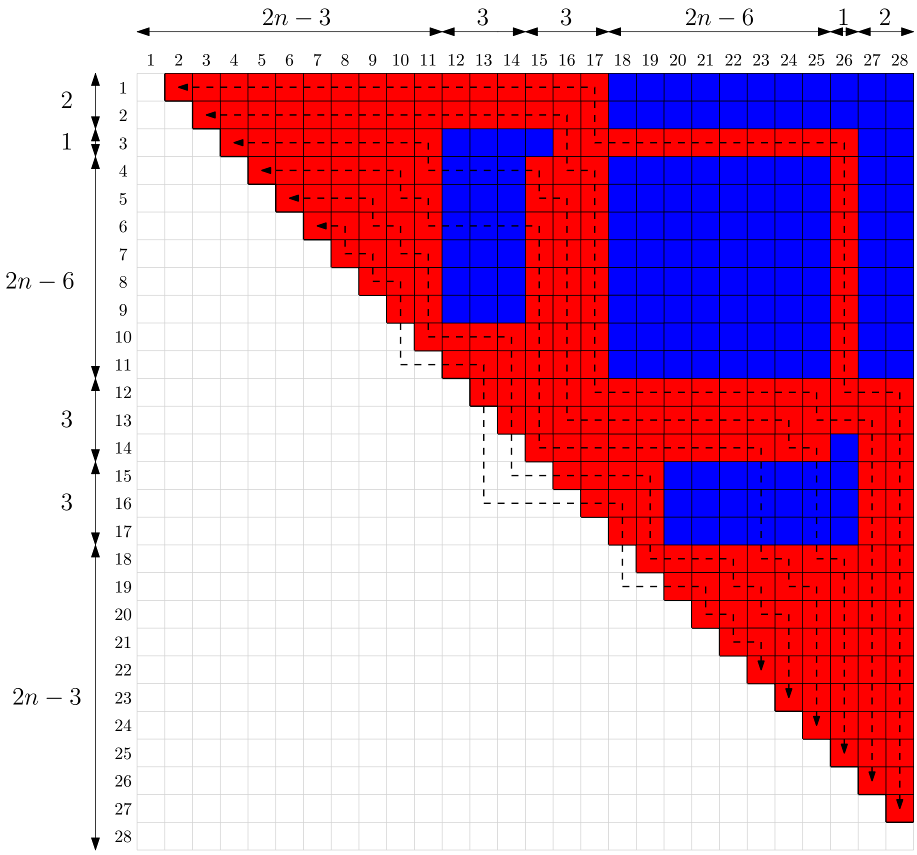

Figure 2.4: The matrix representation of the coloring χ of the edges of $K^<_{4n}$ for $n = 7$.

which corresponds to the L-shaped upper right corner of A of width and height $2n - 3$, and two sets

$$R_1 = \{\{i, j\} : 3 \leq i \leq 9, 2n - 2 \leq j \leq 2n\}$$

and

$$R_2 = \{\{i, j\} : 2n + 1 \leq i \leq 2n + 3, 4n - 8 \leq j \leq 4n - 2\}$$

which form two 3×7 rectangles in A. Finally, there are two single blue edges $e_1 = \{3, 2n + 1\}$ and $e_2 = \{2n, 4n - 2\}$. All the remaining edges of $K^<_{4n}$ are red in χ.

We show that the red edges of χ can be covered by $n - 1$ pairwise disjoint routes. Then it will follow from Lemma 14 that there is no red copy of the nested matching $NM^<_n$ in χ. The set of routes covering the ordered graph formed by red edges in χ is constructed inductively with respect to n. As the basis of the induction, we use the set of routes for $n = 6$ that is illustrated in Figure 2.3. For $n \geq 7$, we use essentially the same $n - 2$ routes we had for $n - 1$, we only elongate them. However, we additionally have to cover two new diagonals formed by entries on positions (i, j) with $j - i \in \{1, 2\}$ and $n - 1 \leq i \leq 3n + 1$; see Figure 2.4. Covering these two new diagonals by an $(n - 1)$st route is clearly possible and thus we can cover the whole ordered graph by $n - 1$ pairwise disjoint routes. Note that some entries of the two new diagonals might be covered by the first $n - 2$ routes, but this makes covering their entries by the $(n - 1)$st route only simpler.

To prove that χ does not contain a blue triangle for any $n \geq 6$, we consider the ordered graph formed by edges that are blue in χ. First, there is no blue triangle containing the edge $e_1 = \{3, 2n+1\}$, as in any such blue triangle there is another blue edge incident to vertex 3. However, all other edges containing vertex 3 are of the form $\{3, i\}$ for $i \in \{2n-2, 2n-1, 2n\} \cup \{4n-1, 4n\}$ and there is no blue edge of the form $\{2n+1, i\}$ for these i. By symmetry, there is no blue triangle containing the edge $e_2 = \{2n, 4n-2\}$.

The edges from $S \cup L$ form a bipartite graph and thus there is no blue triangle with vertices in $S \cup L$ and any blue triangle in χ has to have an edge in $R_1 \cup R_2$. Since both sets R_1 and R_2 induce a bipartite graph, any blue triangle in χ contains at most one edge in R_1 and at most one edge in R_2.

Consider a blue triangle T with an edge from R_1. By the definition of R_1, this edge contains a vertex $i \in \{2n-2, 2n-1, 2n\}$. Since there is at most one edge of T in R_1, there is an edge $\{i, j\}$ of T that is not contained in R_1. The vertex j satisfies $j > i$, as all blue edges $\{i, k\}$ with $k \leq i$ lie in R_1. However, the only blue edge of this form is for $i = 2n$ and $j = 4n-2$, which gives the edge e_2 and we already know that e_2 is not contained in a blue triangle. Thus there is no blue triangle with an edge in R_1. By symmetry, there is also no blue triangle with an edge from R_2 and, altogether, χ contains no blue triangle. $\square$

It is likely that our construction can be modified to obtain stronger lower bounds on $r_<(NM_n^<, K_3^<)$. However, the coloring χ is easy to describe for any $n \geq 6$ and one can show that it does not contain the forbidden monochromatic ordered subgraphs without employing too complicated case analysis. We also note that some of the blue edges might be colored red without introducing a red copy of $NM_n^<$ in the resulting coloring.

Improving the trivial lower bound observed by Rohatgi [16] for general n has an interesting implication because of the relation nested matchings have with the k-queue graphs. In particular, we mention the following problem of Dujmović and Wood [17] about the chromatic number of k-queue graphs, where *chromatic number* of a graph G is the minimum number of colors we can assign to vertices of G so that there is no edge connecting two vertices of the same color.

Problem 16 ([17])**.** *What is the maximum chromatic number χ_k of a k-queue graph?*

Dujmović and Wood [17] note that $\chi_k \in \{2k+1, \ldots, 4k\}$ and they prove that the lower bound is attainable for $k = 1$. With the stronger lower bound from Theorem 15, we are ready to improve the lower bound on χ_k for $k \geq 3$.

Corollary 17. *For every $k \geq 3$, the maximum chromatic number of k-queue graphs is at least $2k+2$.*

Proof. For $k \in \mathbb{N}$, let N be a positive integer such that $r_<(NM_{k+1}^<, K_3^<) > N$. We will show that $\chi_k \geq \lceil N/2 \rceil$. Since $r_<(NM_{k+1}^<, K_3^<) > N$, there is a coloring of $K_N^<$ without a red copy of $NM_{k+1}^<$ and a blue copy of $K_3^<$. Let $R^<$ be the ordered subgraph of $K_N^<$ formed by red edges. Suppose for contradiction that the chromatic number $\chi(R^<)$ of $R^<$ is less than $\lceil N/2 \rceil$. Then, by the pigeonhole principle, there is an independent set in $R^<$ of size $s \geq 3$. However, since there is no blue copy of $K_3^<$, we have $s < 3$, a contradiction.

By Theorem 15 and the two counterexamples for Rohatgi's conjecture in Figure 2.2, we have $r_<(NM^<_{k+1}K^<_3) > 4(k+1)-1$ for every $k \geq 3$. Applying this estimate to the previous observation, we obtain $\chi_k \geq \left\lceil \frac{4(k+1)-1}{2} \right\rceil = 2k+2$. $\square$

2.3 Computational proofs and alternating paths

Rohatgi's conjecture [16] about $r_<(NM^<_n, K^<_3)$ was disproved in [1] with the help of a SAT solver based GUI utility written by me [21]. The computer generated output was useful in more ways throughout our research and deserves at least a short description. I would like to refer an interested reader to my Bachelor thesis [1], where I devoted a whole chapter to the approach on computing ordered Ramsey numbers.

With ordered graphs $G^<$ and $H^<$ and a positive integer N as an input, we construct a SAT formula whose satisfiability is equivalent to the existence of a coloring on N vertices avoiding both $G^<$ and $H^<$ as ordered subgraphs in their respective colors. This SAT solver backend is combined with a graphical interface to build the ordered graphs $G^<$, $H^<$, set N and possibly display graphical output, like the matrix representations of a coloring. The tool is free to use on GitHub [21] and might be useful to anyone researching ordered Ramsey numbers.

We add one result proved by this utility. Balko, Cibulka, Král and Kynčl [15] researched diagonal ordered Ramsey numbers for alternating paths, which were briefly mentioned in Figures 1.3 and 1.5.

Definition 15. *An* ***alternating path*** $Alt^<_n$ *is a specific ordered path on* n *vertices. If we denote the vertices* $1, 2, \ldots, n$ *as per their ordering* $<$*, then* $Alt^<_n$ *has edges between 1 and* n*,* n *and 2, 2 and* $(n-1)$ *and so on (until there is no isolated vertex left).*

Balko et al. [15] proved that the ordered Ramsey number $r_<(Alt^<_n)$ is linear in n, unlike the case with most other ordered paths, as also demonstrated by Theorem 7, which says that $r_<(P^<_n)$ is quadratic in n. Due to their computed results, they suggested that $r_<(Alt^<_n)$ might follow the formula $\lfloor (n-2) \cdot \frac{1+\sqrt{5}}{2} \rfloor + n$. This formula was proved by a computer to hold up to $n = 9$ and verified again by computational approach. However, for $n = 10$, our long running SAT solver experiments showed that $r_<(Alt^<_{10}) = 23$, invalidating the formula. See Figure 2.5 for the relevant coloring on 22 vertices which does not contain a red $Alt^<_{10}$ nor a blue $Alt^<_{10}$. We also found a coloring proving $r_<(Alt^<_{13}) > 30$, which further invalidates the formula. Some relevant colorings found by the SAT solver algorithm are attached to this thesis, see A.1. After our computer experiments, we can update the table for small values of $r_<(Alt^<_n)$.

n	2	3	4	5	6	7	8	9	10	11	12	13
$r_<(Alt^<_n)$	2	4	7	9	12	15	17	20	23	≥ 25	≥ 28	≥ 31

We suspect that for $n = 11, 12$, the given lower bound is the desired ordered Ramsey number.

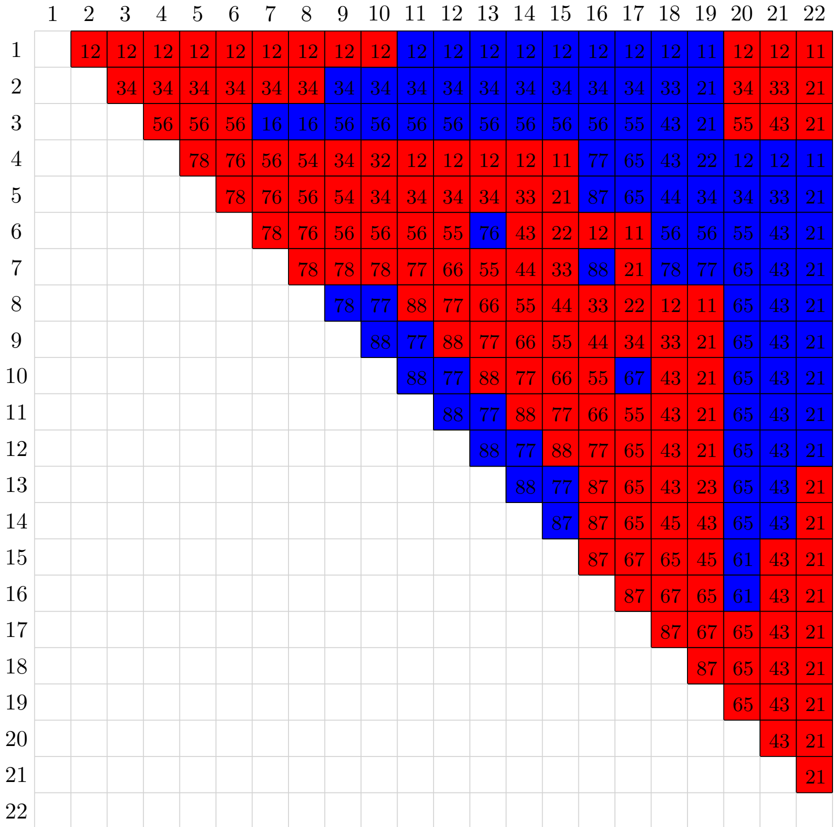

Figure 2.5: The matrix representation of a symmetric coloring of the edges of $K_{22}^{<}$ avoiding $Alt_{10}^{<}$ in both colors. For every edge $\{u, v\}$ where $u < v$ colored by c, there are two digits expressing the maximum length of alternating paths with $\{u, v\}$ as the innermost edge. The left digit expresses the longest alternating path of color c, which ends with vertex u as its 'last' (or innermost) vertex, e.g. its last edge going 'left' from v to u. Symmetrically for the right digit, it expresses the longest alternating path of color c, which ends with vertex v as its 'last' (or innermost) vertex, e.g. its last edge going 'right' from u to v.

3. General ordered matchings versus triangles

In this chapter, we will consider the ordered Ramsey numbers $r_<(M^<, K_3^<)$ for general ordered matchings $M^<$.

3.1 Known results

Finding the growth rate of the ordered Ramsey number $r_<(M^<, K_3^<)$ for an ordered matching $M^<$ has been of significant interest and it is one of the first non-trivial cases where the exact asymptotics is not known.

We can combine Theorem 3 and Observation 8 to get an upper bound:

$$r_<(M^<, K_3^<) \leq r_<(K_n^<, K_3^<) = r(K_n, K_3) \in O(n^2/\log n).$$

On the other hand, Conlon, Fox, Lee and Sudakov [14] showed the following.

Theorem 18 ([14])**.** *There exists a positive constant c such that, for all even positive integers n, there is an ordered matching $M^<$ on n vertices with*

$$r_<(M^<, K_3^<) \geq c \left(\frac{n}{\log n}\right)^{4/3}.$$

Conlon et al. [14] expect that the upper bound $r_<(M^<, K_3^<) \in O(n^2/\log n)$ is far from optimal and posed the following open problem, which is also mentioned in a survey on recent developments in graph Ramsey theory by Conlon, Fox and Sudakov [22].

Problem 19 ([14])**.** *Does there exist an $\epsilon > 0$ such that for any ordered matching $M^<$ on n vertices $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$?*

This problem is unexpectedly quite hard. It seems we should only account for the obvious sparsity of $M^<$ compared to $K_n^<$ in order to get a better bound. However, there exist ordered matchings, which, although sparse, can be as hard to find as an ordered subgraph as an ordered complete graph. This is implied by a theorem proved by Conlon et al. [14].

Theorem 20 ([14])**.** *There is an ordered matching $M^<$ on $2n$ vertices with interval chromatic number 2 and an ordered graph $G^<$ on $2^{\Omega(n^{1/12})}$ vertices with edge density $1 - O(n^{-1/6})$ which does not contain $M^<$ as an ordered subgraph.*

We will get more in-depth look into why is this problem hard and which ordered matchings pose the problem when proving our results in Section 3.3. A partial progress on Problem 19 was done by Rohatgi [16], who studied the behaviour of these ordered Ramsey numbers for random ordered matchings and proved a subquadratic bound for almost all ordered matchings with interval chromatic number 2.

Theorem 21 ([16])**.** *There is a constant c such that for every positive integer n, if an ordered matching $M^<$ on $2n$ vertices with $\chi_<(M^<) = 2$ is picked uniformly at random, then with high probability*

$$r_<(M^<, K_3^<) \leq cn^{24/13}.$$

3.2 Lower bound on $r_<(M^<, K_3^<)$ for fixed $\chi_<(M^<)$

The ordered matching $M^<$ on n vertices constructed for the proof of Theorem 18 (so-called *jumbled matching*) has the property that every two disjoint intervals of length $\Theta(\sqrt{n})$ has at least one and at most constant number of edges between them, its interval chromatic number is also $\Theta(\sqrt{n})$. In our proofs, we will construct ordered matchings of similar flavour, taking into account a fixed interval chromatic number as well.

3.2.1 Case $\chi_<(M^<) = 2$

We will start by proving the following theorem restricted on ordered matchings $M^<$ with interval chromatic number 2, which yields a slightly weaker bound than the one from Theorem 18.

Theorem 22. *There exists a positive constant c such that, for all positive integers n, if an ordered matching $M^<$ on $2n$ vertices with $\chi_<(M^<) = 2$ is picked uniformly at random, then with high probability*

$$r_<(M^<, K_3^<) \geq c\left(\frac{n}{\log n}\right)^{5/4}.$$

The proof is carried out using a similar probabilistic argument used by Conlon, Fox, Lee, and Sudakov [14] and is based on the Lovász local lemma.

We use the following result about the edge-density between disjoint intervals in a random ordered matching with interval chromatic number 2.

Lemma 23. *Let $M^<$ be a uniform random ordered matching on $[2n]$ satisfying $\chi_<(M^<) = 2$. Then, asymptotically almost surely, $M^<$ contains an edge between any two intervals $I \subseteq [n]$ and $J \subseteq \{n+1, \dots, 2n\}$, each of length at least $2\sqrt{n\log n}$, and at most $12s\sqrt{\log n/n}$ edges between any two disjoint intervals of lengths at most $2\sqrt{n \log n}$ and $s \geq 2\sqrt{n\log n}$, respectively.*

Proof. For sets $A \subseteq [n]$ and $B \subseteq \{n+1, \dots, 2n\}$ with $|A| = t = |B|$, the probability that $M^<$ has no edge between A and B is at most

$$\left(\frac{n-t}{n}\right)\left(\frac{n-t-1}{n-1}\right)\cdots\left(\frac{n-2t+1}{n-t+1}\right) \leq \left(\frac{n-t}{n}\right)^t \leq e^{-t^2/n}$$

where we used the inequalities $\frac{n-t-i}{n-i} < \frac{n-t}{n}$ for every $i > 0$ and $1 - x \leq e^{-x}$ for every $x \in \mathbb{R}$. There are at most n^2 intervals $I \subseteq [n]$ and $J \subseteq \{n+1, \dots, 2n\}$, each of size t, and thus the probability that there there are two such intervals with no edge of $M^<$ between them is at most $n^2e^{-t^2/n}$. This probability goes to 0 with increasing n for $t \geq \frac{3}{2}\sqrt{n \log n}$. Thus, it suffices to take t as an integer between $\frac{3}{2}\sqrt{n\log n}$ and $2\sqrt{n \log n}$.

Now, consider disjoint subsets C and D of $[2n]$ with $|C| = 2\sqrt{n\log n}$ and $|D| = s \geq 2\sqrt{n \log n}$. Set $r = 12s\sqrt{\log n/n}$. We show that asymptotically almost surely there are at most r edges of $M^<$ between C and D. This is trivial for $r > 2\sqrt{n\log n}$ as there are always at most $|C| = 2\sqrt{n \log n}$ edges of $M^<$ between

C and D. Thus, we assume $r \leq 2\sqrt{n \log n}$. Then, the probability that there are r edges of $M^<$ between C and D is at most

$$\binom{2\sqrt{n\log n}}{r}\left(\frac{s}{n}\right)\left(\frac{s-1}{n-1}\right)\cdots\left(\frac{s-r+1}{n-r+1}\right) \leq \left(\frac{2es\sqrt{n\log n}}{rn}\right)^r$$
$$\leq \left(\frac{6s\sqrt{\log n}}{r\sqrt{n}}\right)^r$$

as the ith edge of such r edges has the other vertex in D with probability $\left(\frac{s-i+1}{n-i+1}\right)$. The remaining edges can be assigned arbitrarily. There are at most n^2 pairs of disjoint intervals I and J with $|I| = 2\sqrt{n \log n}$ and $|J| = s$ and thus the probability that there there are two such intervals with at least r edges of $M^<$ between them is at most $n^2\left(\frac{6s\sqrt{\log n}}{r\sqrt{n}}\right)^r$. Since $s \geq 2\sqrt{n\log n}$ we then have $r \geq 24 \log n$. Thus, since also $\frac{6s\sqrt{\log n}}{r\sqrt{n}} \leq 1/2$ by the choice of r, the upper bound goes to zero with increasing n. □

We note that there is an explicit construction of an ordered matching $M_t^<$ on $2t^2$ vertices that satisfies a similar statement with intervals of size only t; see Subsection 3.2.2.

The key ingredient in our probabilistic argument is the famous Lovász local lemma, see [23] for example. We now recall its statement.

Lemma 24 (The Lovász local lemma)**.** *Let $\{A_1, \ldots, A_n\}$ be a finite set of events in a probability space. A directed graph $D = (V, E)$ is the* dependency graph *of $A_1, \ldots, A_n$ if each event A_i is mutually independent of all the events from $\{A_j \colon (i, j) \notin E\}$. Let $x_1, \ldots, x_n$ be real numbers such that $0 \leq x_i < 1$ and $\Pr[A_i] \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ for every $i \in [n]$. Then,*

$$\Pr\left[\overline{A_1} \cap \cdots \cap \overline{A_n}\right] \geq \prod_{i=1}^{n}(1 - x_i).$$

In particular, the probability that none of the events $A_1, \ldots, A_n$ occur is positive.

We apply the Lovász local lemma to prove the following auxiliary result, which is also used in the proof of Theorem 26. For positive integers r, s, we use $K^<_{r,s}$ to denote the ordered complete bipartite graph where the color classes of sizes r and s form consecutive intervals in this order.

Lemma 25. *Let $\alpha, \beta, \gamma > 0$ and $\delta \geq 0$ be real numbers satisfying the following three inequalities: $\alpha + \beta + \gamma - \delta \leq \frac{3}{2}$, $\beta \leq 2\gamma$ and $\alpha + \gamma \leq 1$. For a sufficiently large integer n, let $\mathcal{G}$ be a family of ordered graphs, each on n^β vertices and with $40n^{\frac{3}{2}-\alpha+\delta} \log n$ edges, and assume that $|\mathcal{G}| \leq e^{n^\beta \log n}$. Then, there is a red-blue coloring χ of the edges of $K^<_{n^\beta}$ such that the following three conditions are satisfied:*

(a) there is no blue triangle in χ,

(b) there is no red copy of any ordered graph from $\mathcal{G}$ in χ, and

(c) there is no red copy of $K^<_{10n^{1-\alpha}\log n, 10n^{1-\alpha}\log n}$ in χ.

Proof. We color each edge of $K^{<}_{n^\beta}$ independently at random red with probability $\frac{1}{2n^\gamma}$ and blue with probability $1-\frac{1}{2n^\gamma}$. Let P_i be the events corresponding to blue triangles in our random coloring, let Q_i be the events corresponding to red ordered graphs from $\mathcal{G}$, and R_i the events corresponding to the red ordered complete bipartite graphs as in the statement of the lemma. We denote the index sets of the events P_i, Q_i, and R_i by I_P, I_Q, and I_R, respectively. Clearly, we have $|I_P| \leq \binom{n^\beta}{3} \leq n^{3\beta}$ and $|I_Q| = |\mathcal{G}| \leq e^{n^\beta \log n}$. Since n is sufficiently large, we obtain

$$\begin{aligned}|I_R| &\leq \binom{n^\beta}{20n^{1-\alpha}\log n} \leq \left(\frac{en^\beta}{20n^{1-\alpha}\log n}\right)^{20n^{1-\alpha}\log n} \leq (n^{\alpha+\beta-1})^{20n^{1-\alpha}\log n}\\ &\leq e^{20n^{1-\alpha}\log^2 n}.\end{aligned}$$

We apply the Lovász local lemma (Lemma 24) with the events P_i, Q_i, and R_i. It suffices to verify the conditions of the lemma as then it follows that the probability that none of these events hold is positive. That is, there is a coloring χ satisfying the statement of Lemma 25.

We choose $x = \frac{1}{4n^{3\gamma}}$, $y = e^{-2n^\beta \log n}$, and $z = e^{-21n^{1-\alpha}\log^2 n}$. It follows from the choice of y and z and our estimates on I_Q and I_R that $y|I_Q| \in o(1)$ and $z|I_R| \in o(1)$. We now verify the conditions of Lemma 24.

1. Events P_i:

   Each event P_i depends on exactly $3n^\beta$ events P_j and on at most $|I_Q|$ events Q_j and on at most $|I_R|$ events R_j. Thus,

$$\begin{aligned}&x\prod_{j\in I_P, j\sim i}(1-x)\prod_{j\in I_Q, j\sim i}(1-y)\prod_{j\in I_R, j\sim i}(1-z)\\ &= (1-o(1))\cdot xe^{-3xn^\beta}e^{-y|I_Q|}e^{-z|I_R|} = (1-o(1))\cdot\frac{1}{4n^{3\gamma}}e^{-\frac{3}{4}n^{\beta-3\gamma}}\\ &\geq \frac{1}{8n^{3\gamma}} = \Pr[P_i].\end{aligned}$$

   The last inequality holds for a sufficiently large n if and only if $\beta < 3\gamma$, which follows from our stronger assumption $\beta \leq 2\gamma$.

2. Events Q_i:

   Every ordered graph corresponding to the event Q_i contains $40n^{\frac{3}{2}-\alpha+\delta}\log n$ edges and thus Q_i depends on at most $40n^{\frac{3}{2}-\alpha+\beta+\delta}\log n$ events P_j. It then follows that

$$\begin{aligned}&y\prod_{j\in I_P, j\sim i}(1-x)\prod_{j\in I_Q, j\sim i}(1-y)\prod_{j\in I_R, j\sim i}(1-z)\\ &= (1-o(1))\cdot ye^{-x(40n^{\frac{3}{2}-\alpha+\beta+\delta}\log n)}e^{-y|I_Q|}e^{-z|I_R|}\\ &= (1-o(1))\cdot e^{-2n^\beta\log n}e^{-10n^{\frac{3}{2}-\alpha+\beta-3\gamma+\delta}\log n}.\end{aligned}$$

   We want the last expression to be at least $\Pr[Q_i] = (1-\frac{1}{2n^\gamma})^{40n^{\frac{3}{2}-\alpha+\delta}\log n}$ which is at most $e^{-20n^{\frac{3}{2}-\alpha-\gamma+\delta}\log n}$. Therefore, it suffices to show that

$$(1-o(1))\cdot e^{-2n^\beta\log n}e^{-10n^{\frac{3}{2}-\alpha+\beta-3\gamma+\delta}\log n} \geq e^{-20n^{\frac{3}{2}-\alpha-\gamma+\delta}\log n}.$$

This is true if
$$2n^{\beta} + 10n^{\frac{3}{2}-\alpha+\beta-3\gamma+\delta} \le 20n^{\frac{3}{2}-\alpha-\gamma+\delta}$$
and the right hand side grows faster than the left one to beat the $(1-o(1))$-term above. This follows from our assumptions $\alpha+\beta+\gamma-\delta \le \frac{3}{2}$ and $\beta \le 2\gamma$ as then $n^{\beta} \le n^{\frac{3}{2}-\alpha-\gamma+\delta}$ and also $n^{\frac{3}{2}-\alpha+\beta-3\gamma+\delta} \le n^{\frac{3}{2}-\alpha-\gamma+\delta}$.

3. Events R_i:

Every ordered complete bipartite graph corresponding to the event R_i contains $(10n^{1-\alpha}\log n)^2 = 100n^{2-2\alpha}\log^2 n$ edges and thus R_i depends on at most $100n^{2-2\alpha+\beta}\log^2 n$ events P_i. It follows that
$$\begin{aligned} &z \prod_{j\in I_P, j\sim i}(1-x) \prod_{j\in I_Q, j\sim i}(1-y) \prod_{j\in I_R, j\sim i}(1-z) \\ &= (1-o(1)) \cdot z e^{-x(100n^{2-2\alpha+\beta}\log^2 n)} e^{-y|I_Q|} e^{-z|I_R|} \\ &= (1-o(1)) \cdot e^{-21n^{1-\alpha}\log^2 n} e^{-25n^{2-2\alpha+\beta-3\gamma}\log^2 n}. \end{aligned}$$

We want the last expression to be at least $\Pr[R_i] = (1-\frac{1}{2n^{\gamma}})^{100n^{2-2\alpha}\log^2 n}$, which is at most $e^{-50n^{2-2\alpha-\gamma}\log^2 n}$. That is, it suffices to check that
$$(1-o(1)) \cdot e^{-21n^{1-\alpha}\log^2 n} e^{-25n^{2-2\alpha+\beta-3\gamma}\log^2 n} \ge e^{-50n^{2-2\alpha-\gamma}\log^2 n}.$$
This is true if
$$21n^{1-\alpha} + 25n^{2-2\alpha+\beta-3\gamma} \le 50n^{2-2\alpha-\gamma}$$
and the right hand side grows faster than the left one to beat the $(1-o(1))$-term above. This follows from our assumptions $\alpha+\gamma \le 1$ and $\beta \le 2\gamma$ as then $n^{1-\alpha} \le n^{2-2\alpha-\gamma}$ and $n^{2-2\alpha+\beta-3\gamma} \le n^{2-2\alpha-\gamma}$.

Altogether, all conditions of Lemma 24 are satisfied and we obtain the desired coloring χ. □

We now proceed with the proof of Theorem 22. The approach is similar to the one used by Conlon, Fox, Lee, and Sudakov [14].

Proof of Theorem 22. Let $M^<$ be a random ordered matching with $\chi_<(M^<) = 2$ on $m = 800n\log n$ vertices. We will prove that $n^{5/4} \le r_<(M^<, K_3^<)$ for n sufficiently large. We choose $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$, and $\gamma = \frac{1}{4}$. Note that this choice of parameters satisfies the conditions in the statement of Lemma 25 with $\delta = 0$.

We set $N = n^{\alpha+\beta} = n^{5/4}$ and we partition $[N]$ into consecutive intervals $V_1, \dots, V_{n^{\beta}}$, each of length n^{α}. Let Φ be the set of injective embeddings of $M^<$ into $[N]$ that respect the vertex ordering of $M^<$. For $\phi \in \Phi$, let $G^<(\phi)$ be the ordered graph on the vertex set $[n^{\beta}]$ with an edge between i and j if and only if there is an edge of $M^<$ between $\phi^{-1}(V_i)$ and $\phi^{-1}(V_j)$.

Let
$$\mathcal{H} = \{G^<(\phi) : \phi \in \Phi, |E(G^<(\phi))| \ge 40n^{\frac{3}{2}-\alpha}\log n\}.$$

Since the mappings in Φ respect the order of the vertices of $M^<$, any ordered graph $G^<(\phi)$ is determined by the last vertex in $\phi^{-1}(V_i)$ for every $i \in [n^{\beta}]$. Therefore, for n sufficiently large, we obtain
$$|\mathcal{H}| \le \binom{m+n^{\beta}}{n^{\beta}} \le \left(\frac{e(m+n^{\beta})}{n^{\beta}}\right)^{n^{\beta}} = \left(e(800n^{1-\beta}\log n + 1)\right)^{n^{\beta}} \le e^{n^{\beta}\log n}.$$

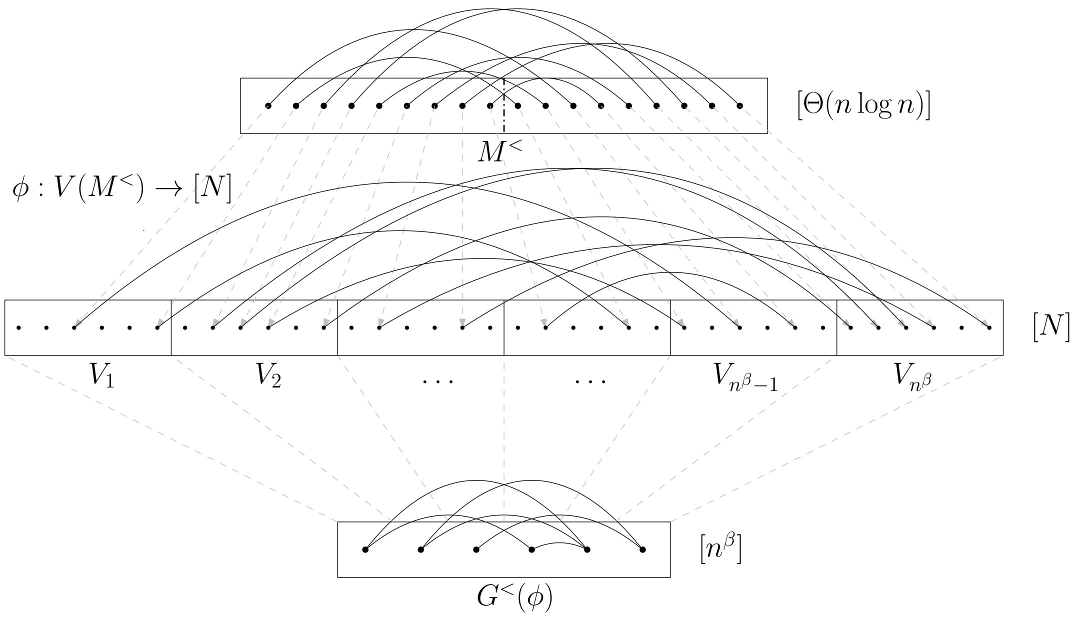

Figure 3.1: An illustration of the mapping ϕ used in the proof of Theorem 22 and Theorem 26.

Let $\mathcal{G}$ be the class of ordered graphs such that for every $H^< \in \mathcal{H}$ there is $G^< \in \mathcal{G}$ with exactly $40n^{\frac{3}{2}-\alpha} \log n$ edges such that $G^<$ is an ordered subgraph of $H^<$. Note that we can choose $\mathcal{G}$ so that $|\mathcal{G}| \leq |\mathcal{H}|$.

Applying Lemma 25 to $\mathcal{G}$ with our choice of α, β, γ, and δ, we obtain a red-blue coloring χ' of the edges of $K^<_{n^\beta}$ that avoids a blue triangle, a red copy of any ordered graph from $\mathcal{G}$, and a red copy of $K^<_{10n^{1-\alpha}\log n, 10n^{1-\alpha}\log n}$.

Let χ be the red-blue coloring of the edges of the ordered complete graph on $[N]$ where we color all edges between V_i and V_j with color $\chi'(i,j)$ for all $i, j \in [n^\beta]$. We color all edges within the sets V_i red. Note that χ contains no blue triangle, since χ' does not contain a blue triangle.

Suppose for contradiction that for some $\phi \in \Phi$, the ordered matching $\phi(M^<)$ is a red copy of $M^<$ in χ. We use P_1 and P_2 to denote the left and the right color class of $\phi(M^<)$, respectively, each of size $m/2 = 400n \log n$. Let $W_i = V(\phi(M^<)) \cap V_i$ for each i and let $S \subset [n^\beta]$ be the set of indices i for which $|W_i| \leq 2\sqrt{m \log m}$. We set $L = [n^\beta] \setminus S$.

By Lemma 23, for any pair of indices $i, j \in L$ with $W_i \subseteq P_1$ and $W_j \subseteq P_2$, there is an edge of $\phi(M^<)$ between W_i and W_j since $|W_i|, |W_j| > 2\sqrt{m \log m}$ and $M^<$ is a random ordered matching with $\chi_<(M^<) = 2$. Then, $\chi'(i,j)$ is red as all edges of $\phi(M^<)$ are red in χ. Thus, if there are at least $10n^{1-\alpha} \log n$ sets W_i with $i \in L$ inside each of the two color classes P_1 and P_2 of $\phi(M^<)$, then we have a red copy of $K^<_{10n^{1-\alpha}\log n, 10n^{1-\alpha}\log n}$ in χ'. This is impossible by the choice of χ'.

Hence, one of the color classes of $\phi(M^<)$ contains less than $10n^{1-\alpha} \log n$ sets W_i with $i \in L$. By symmetry, we can assume that it is the color class P_1. Since the size of any set W_i is at most $|V_i| = n^\alpha$, each set W_i is incident to at most n^α edges of the ordered matching $\phi(M^<)$. Overall, all sets $W_i \subseteq P_1$ with $i \in L$ are incident to at most $10n \log n$ edges of $\phi(M^<)$. Therefore, there are at least $390n \log n$ edges of $\phi(M^<)$ incident to sets $W_i \subseteq P_1$ with $i \in S$.

Consider $W_i \subseteq P_1$ and $W_j \subseteq P_2$ such that $i \in S$. We recall that $|W_i| \leq$

$2\sqrt{m \log m}$ and $|W_j| \leq n^\alpha$. By Lemma 23, there are at most $12n^\alpha\sqrt{\log m/m} \leq n^{\alpha-1/2}$ edges of $\phi(M^<)$ between W_i and W_j for n sufficiently large. Since there are at least $390n \log n$ edges of $\phi(M^<)$ incident to sets $W_i \subseteq P_1$ with $i \in S$, there are at least

$$\frac{390n \log n}{n^{\alpha-1/2}} = 390n^{\frac{3}{2}-\alpha} \log n > 40n^{\frac{3}{2}-\alpha} \log n$$

red edges in the coloring χ'. This implies that $G^<(\phi) \in \mathcal{H}$. However, $G^<(\phi)$ has all edges red in the coloring χ' which contradicts the choice of χ'.

Altogether, there is no red copy of $M^<$ and no blue copy of $K_3^<$ in χ and thus $r_<(M^<, K_3^<) > N = n^{5/4}$. □

3.2.2 Case $\chi_<(M^<) \geq 3$

We will now show that the obtained lower bound can be improved for ordered matchings $M^<$ with $\chi_<(M^<) \geq 3$ and prove the following theorem.

Theorem 26. *For every integer $k \geq 3$, there exists a positive constant $c = c(k)$ such that, for all positive integers n, there exists an ordered matching $M^<$ on $2n$ vertices with $\chi_<(M^<) = k$ satisfying*

$$r_<(M^<, K_3^<) \geq c\left(\frac{n}{\log n}\right)^{4/3}.$$

Note that the lower bound from Theorem 26 already asymptotically matches the bound from Theorem 18 by Conlon, Fox, Lee, and Sudakov [14]. Thus, the best known lower bound on $r_<(M^<, K_3^<)$ for general ordered matchings $M^<$ can be obtained also for ordered matchings with any bounded interval chromatic number as long as this number is at least 3. The proof of Theorem 26 is again probabilistic and based on ideas used by Conlon, Fox, Lee, and Sudakov [14].

First, we construct the following auxiliary ordered matching $M_t^<$ with interval chromatic number 2. For a positive integer t, let $[2t^2]$ be the vertex set of $M_t^<$. We partition the set $[t^2]$ into t consecutive intervals $I_1, \ldots, I_t$, each of size t and, similarly, let $J_1, \ldots, J_t$ be the partition of the set $\{t^2+1, \ldots, 2t^2\}$ into t consecutive intervals, each of size t. See Figure 3.2 for an illustration. Then, for all distinct integers i and j with $1 \leq i, j \leq t$ we put an edge between the jth vertex of I_i and the ith vertex of J_j. Note that there is exactly one edge between each I_i and J_j.

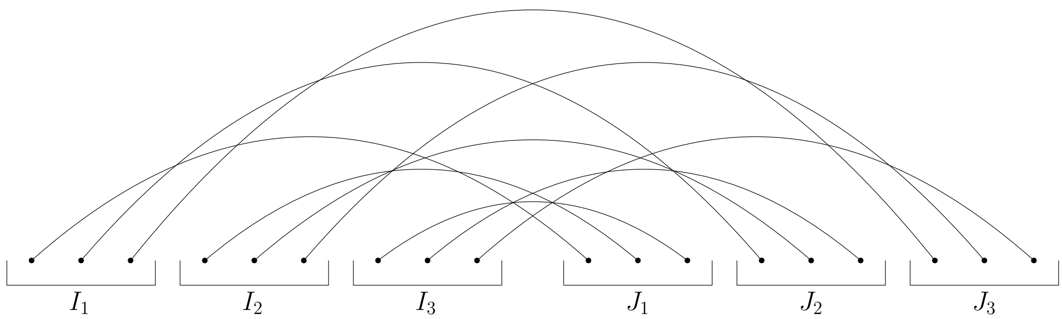

Figure 3.2: An illustration of the ordered matching $M_t^<$ for $t = 3$.

The ordered matching $M_t^<$ then satisfies the following property.

Observation 27. *There is at least one edge of $M_t^<$ between any two intervals $I \subseteq \cup_{i=1}^t I_i$ and $J \subseteq \cup_{j=1}^t J_j$, each of length at least $2t$.*

Proof. The interval I contains some interval I_i and J contains some interval J_j, so there is an edge of $M^<$ between I and J. $\square$

For positive integers $k \geq 3$ and t, we now construct the ordered matching $M_{k,t}^<$ on $m = k(k-1)t^2$ vertices that is used in the proof of Theorem 26. The main idea is to define $M_{k,t}^<$ as an intertwined union of the ordered matchings $M_t^<$; see Figure 3.3 for an illustration with $k = 3$ and $t = 3$.

The vertex set $[m]$ of $M_{k,t}^<$ is partitioned into consecutive intervals $P_1, \ldots, P_k$, each of size $m/k = (k-1)t^2$. For every $i \in [k]$, the interval P_i is partitioned into consecutive intervals $B_{i,1}, \ldots, B_{i,(k-1)t}$, each of size t. We call each interval $B_{i,j}$ a *block* of $M_{k,t}^<$. For every $i \in [k]$ and $j \in \{0, 1, \ldots, k-2\}$, let a_j be the $(j+1)$st smallest element of $[k] \setminus \{i\}$ and let C_{i,a_j} be the set of vertices that is the union of the blocks $B_{i,\ell}$ where ℓ is congruent to j modulo $k-1$; see Figure 3.3. We call each set C_{i,a_j} a *superblock* of $M_{k,t}^<$. Note that the size of each superblock is t^2. We now place the edges so that any pair $C_{i,j}$ and $C_{j,i}$ of superblocks induces a copy $M^<(i,j)$ of the ordered matching $M_t^<$.

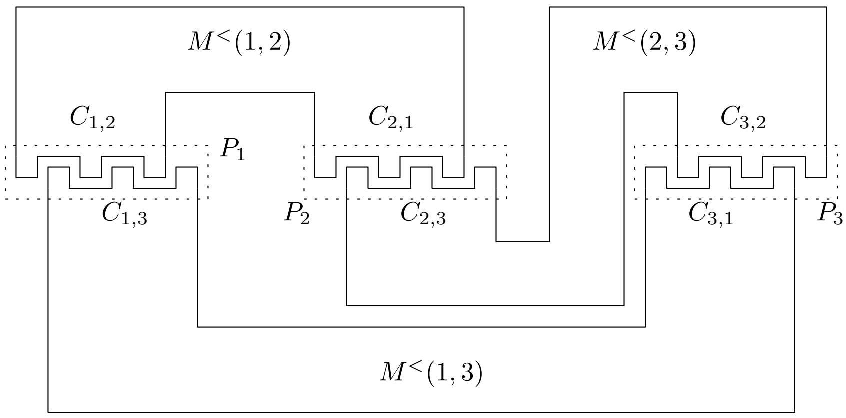

Figure 3.3: An illustration of the ordered matching $M_{k,t}^<$ for $k = 3$ and $t = 3$.

Observe that $\chi_<(M_{k,t}^<) = k$ as the sets $P_1, \ldots, P_k$ form the color classes of $M_{k,t}^<$. We now state the key properties of the ordered matching $M_{k,t}^<$.

Lemma 28. *Let $i, j \in [k]$ be two distinct integers. For any pair of intervals $I \subseteq P_i$ and $J \subseteq P_j$, each of length at least $2kt$, there is an edge of $M_{k,t}^<$ between I and J. Moreover, there are at most $(2k+1)^2$ edges between any two disjoint intervals $I' \subseteq P_i$ and $J' \subseteq P_j$, each of size at most $2kt$.*

Proof. First, let I and J be the two intervals from the first part of the statement. Since $|I| \geq 2kt$, the interval I intersects each superblock $C_{i,j'}$, $j' \in [k] \setminus \{i\}$, in an interval of length at least $2t$. Analogously, J intersects each superblock $C_{j,i'}$, $i' \in [k] \setminus \{j\}$, in an interval of length at least $2t$. Then, by Observation 27, there is an edge of $M^<(i,j) \subseteq M_{k,t}^<$ between the sets $I \cap C_{i,j}$ and $J \cap C_{j,i}$.

Let I' and J' be the intervals from the second part of the statement. Since $|I'| \leq 2kt$ and since the size of each block of $M_{k,t}^<$ is t, the interval I' can intersect at most $2k+1$ blocks. An analogous claim is true for the interval J'. Since there is at most one edge between any pair of blocks, it follows that there can be at most $(2k+1)^2$ edges of $M_{k,t}^<$ between I' and J'. $\square$

We now proceed with the proof of Theorem 26. The proof is similar to the proof of Theorem 22.

Proof of Theorem 26. For a given integer $k \geq 3$, we choose t sufficiently large and express the number $m = k(k-1)t^2$ of vertices of $M^<_{k,t}$ as $m = 500k^3 n \log n$ for some positive integer n. We will prove that $n^{4/3} \leq r_<(M^<_{k,t}, K^<_3)$. We set $\alpha = \frac{2}{3}$, $\beta = \frac{2}{3}$, and $\gamma = \frac{1}{3}$. Note that this choice of parameters satisfies the conditions in the statement of Lemma 25 with $\delta = \alpha - 1/2 = 1/6$.

We set $N = n^{\alpha+\beta} = n^{4/3}$ and we partition $[N]$ into consecutive intervals $V_1, \dots, V_{n^\beta}$, each of length n^α. Similarly as before, we let Φ be the set of injective embeddings of $M^<_{k,t}$ into $[N]$ that respect the vertex ordering of $M^<_{k,t}$. For $\phi \in \Phi$, let $G^<(\phi)$ be the ordered graph on the vertex set $[n^\beta]$ with an edge between i and j if and only if there is an edge of $M^<_{k,t}$ between $\phi^{-1}(V_i)$ and $\phi^{-1}(V_j)$. We also set

$$\mathcal{H} = \{G^<(\phi) : \phi \in \Phi, |E(G^<(\phi))| \geq 40n \log n\}.$$

Analogously as in the proof of Theorem 22, we obtain $|\mathcal{H}| \leq e^{n^\beta \log n}$. Let $\mathcal{G}$ be the class of ordered graphs such that for every $H^< \in \mathcal{H}$ there is $G^< \in \mathcal{G}$ with exactly $40n \log n$ edges such that $G^<$ is an ordered subgraph of $H^<$. Note that we can choose $\mathcal{G}$ so that $|\mathcal{G}| \leq |\mathcal{H}|$.

Applying Lemma 25 to $\mathcal{G}$ with our choice of α, β, γ, and δ, we get a red-blue coloring χ' of the edges of $K^<_{n^\beta}$ that avoids a blue triangle, a red copy of any ordered graph from $\mathcal{G}$, and a red copy of $K^<_{10n^{1-\alpha}\log n, 10n^{1-\alpha}\log n}$.

Let χ be the red-blue coloring of the edges of the ordered complete graph on $[N]$ where we color all edges between V_i and V_j with color $\chi'(i,j)$ for all $i, j \in [n^\beta]$. We color all edges within the sets V_i red. Note that χ contains no blue triangle, since χ' does not contain a blue triangle.

Suppose for contradiction that for some $\phi \in \Phi$, the ordered matching $\phi(M^<_{k,t})$ is a red copy of $M^<$ in χ. We use $\phi(P_1), \dots, \phi(P_k)$ to denote the color classes of $\phi(M^<_{k,t})$. Let $W_i = V(\phi(M^<)) \cap V_i$ for each i and let $S \subset [n^\beta]$ be the set of indices i for which $|W_i| \leq 2kt$. We set $L = [n^\beta] \setminus S$.

By Lemma 28, for any pair of indices $i, j \in L$ with W_i and W_j that are contained in different color classes of $\phi(M^<_{k,t})$, there is an edge of $\phi(M^<_{k,t})$ between W_i and W_j since $|W_i|, |W_j| > 2kt$. Then, $\chi'(i,j)$ is red as all edges of $\phi(M^<)$ are red in χ. Thus, if there are two color classes $\phi(P_a)$ and $\phi(P_b)$, each with at least $10n^{1-\alpha}\log n$ sets W_i with $i \in L$, then we have a red copy of $K^<_{10n^{1-\alpha}\log n, 10n^{1-\alpha}\log n}$ in χ'. This is impossible by the choice of χ'.

Thus, at most one color class of $\phi(M^<_{k,t})$ contains at least $10n^{1-\alpha}\log n$ sets W_i with $i \in L$. Since $k \geq 3$, there are two color classes of $\phi(M^<_{k,t})$, without loss of generality $\phi(P_1)$ and $\phi(P_2)$, such that each one of them contains less than $10n^{1-\alpha}\log n$ sets W_i with $i \in L$. Note that $|\phi(P_1)| = |\phi(P_2)| = m/k = 500k^2 n \log n$. Since the size of any set W_i is at most $|V_i| = n^\alpha$, each set W_i is incident to at most n^α edges of the ordered matching $\phi(M^<_{k,t})$. Overall, all sets $W_i \subseteq \phi(P_1) \cup \phi(P_2)$ with $i \in L$ are incident to at most $20n \log n$ edges of $\phi(M^<)$. Therefore, there are at least $(500k^2 - 20)n \log n \geq 480k^2 n \log n$ edges of $\phi(M^<)$ incident to sets $W_i \subseteq \phi(P_1) \cup \phi(P_2)$ with $i \in S$.

Consider $W_i \subseteq \phi(P_1)$ and $W_j \subseteq \phi(P_2)$ such that $i, j \in S$. We recall that $|W_i|, |W_j| \leq 2kt$. By Lemma 28, there are at most $(2k+1)^2 < 10k^2$ edges of $\phi(M^<_{k,t})$ between W_i and W_j. Since there are at least $480k^2 n \log n$ edges of $\phi(M^{<}_{k,t})$ incident to sets $W_i \subseteq \phi(P_1)$ with $i \in S$, there are at least

$$\frac{480k^2 n \log n}{10k^2} > 40n \log n$$

red edges in the coloring χ'. This implies that $G^{<}(\phi) \in \mathcal{H}$. However, $G^{<}(\phi)$ has all edges red in the coloring χ' which contradicts the choice of χ'.

Altogether, there is no red copy of $M^{<}$ nor a blue copy of $K_3^{<}$ in χ and thus $r_{<}(M^{<}, K_3^{<}) > N = n^{4/3}$. □

3.3 Upper bound on $r_{<}(M^{<}, K_3^{<})$

Here, we improve the exponent from Theorem 21 from 24/13 to 7/4.

Theorem 29. *There is a constant c such that for every positive integer n, if an ordered matching $M^{<}$ on $2n$ vertices with $\chi_{<}(M^{<}) = 2$ is picked uniformly at random, then with high probability*

$$r_{<}(M^{<}, K_3^{<}) \leq cn^{7/4}.$$

The proof is carried out using a multi-thread scanning procedure whose variants were recently used by Cibulka and Kynčl [24], He and Kwan [25], and Rohatgi [16].

First, note that the set of ordered matchings on $2n$ vertices with interval chromatic number 2 is in one-to-one correspondence with the set of permutations on $[n]$. Since it is often convenient to work with the permutation corresponding to a given ordered matching $M^{<}$ on $[2n]$ with $\chi_{<}(M^{<}) = 2$, we define the permutation $\pi_{M^{<}}$ as the permutation on $[n]$ that maps i to $j - n$ for every edge $\{i, j\}$ of $M^{<}$. A uniform random ordered matching on $[2n]$ then corresponds to a uniform permutation on $[n]$ selected uniformly at random.

Let χ be a red-blue coloring of the edges of $K_{2N}^{<}$ for some positive integer N. Let A be an $N \times N$ matrix where an entry on position $(i, j) \in [N] \times [N]$ contains the color of the edge $\{i, N + j\}$ in χ. Note that a red copy of $M^{<}$ with one color class in $[N]$ and the other one in $\{N + 1, \ldots, 2N\}$ corresponds to an $n \times n$ submatrix of A with red entries on positions $(i, \pi_{M^{<}}(i))$ for $i = 1, \ldots, n$.

We now describe a procedure that we use to find a red copy of $M^{<}$ in χ; see Figure 3.4 for an illustration. Let T be a positive integer. We try to find a red copy of $M^{<}$ in rows $t+1, \ldots, t+n$ for every $t \in \{0, 1, \ldots, T-1\}$. First, we scan through the row $\pi_{M^{<}}(1) + t$ of A from left to right until we find a red entry in some position $(\pi_{M^{<}}(1) + t, j_1)$. For every $i \in \{2, \ldots, n\}$, after we have finished scanning through rows $\pi_{M^{<}}(1) + t, \ldots, \pi_{M^{<}}(i-1) + t$, we scan through the row $\pi_{M^{<}}(i) + t$ of A, starting from column $j_{i-1} + 1$, until we find a red entry in some position $(\pi_{M^{<}}(i) + t, j_i)$.

We call this *multi-thread scanning* for $M^{<}$ and we call the set $Th(t)$ of entries of A that are revealed in step t a *thread*. Note that a thread $Th(t)$ successfully finds a red copy of $M^{<}$ if and only if some red copy of $M^{<}$ lies in the rows $t+1, \ldots, t+n$ of A. Moreover, if the thread $Th(t)$ does not find a red copy of $M_{<}$, then it reveals at least $N - n$ blue entries of A.

For a permutation π on $[n]$, we say that a subset $C \subseteq [n]$ with $|C| = k$ is a *shift* of another subset $D \subseteq [n]$ in π if there is a positive integer Δ such that

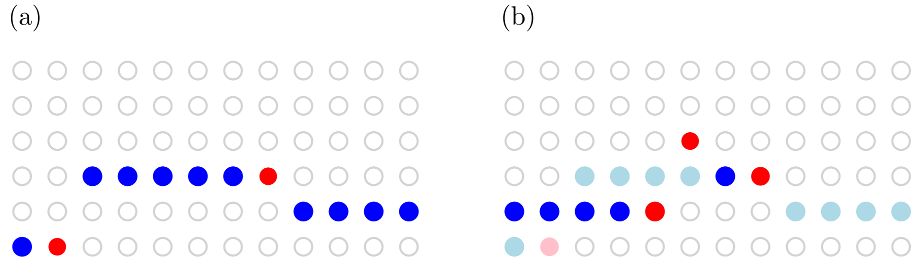

Figure 3.4: An illustration of the multi-thread scanning procedure for the ordered matching $M^<$ with the corresponding permutation $\pi(M^<) = 132$. (a) Thread $Th(0)$ did not find a red copy of $M^<$. (b) Thread $Th(1)$ successfully found a red copy of $M^<$. The entries whose color was previously revealed by thread $Th(0)$ are denoted by light blue and light red.

$\pi(c_i) = \pi(d_i) + \Delta$ for each $i \in [k]$ where $c_1 < \cdots < c_k$ and $d_1 < \cdots < d_k$ are the elements of C and D, respectively. Let $L(\pi)$ be the largest positive integer k for which there are sets $C, D \subseteq [n]$, each of size k, such that C is a shift of D. This notion captures the maximum size of a pattern that a permutation can share with its translation.

We now state the following upper bound on ordered Ramsey numbers of ordered matchings $M^<$ with restricted $L(\pi_{M^<})$ versus triangles, which is used later to derive Theorem 29. A similar result was proved by Rohatgi [16], but it yields asymptotically weaker bounds.

Theorem 30. *For a positive integer n, let $M^<$ be an ordered matching on $2n$ vertices with $\chi_<(M^<) = 2$ and $L(\pi_{M^<}) \leq \ell$. If $N \geq 4n(\sqrt{n\ell}+1)$, then every red-blue coloring χ of the edges of $K^<_{2N}$ on $[2N]$ satisfies at least one of the following three claims:*

1. *χ contains a blue copy of $K^<_3$,*
2. *χ contains a red copy of $K^<_{2n}$, or*
3. *χ contains a red copy of $M^<$ between $[N]$ and $\{N+1,\dots,2N\}$.*

Proof. Let χ be a red blue coloring of the edges of $K^<_{2N}$ on $[2N]$. Suppose for contradiction that χ satisfies none of the three claims from the statement of the theorem.

Let A be the matrix $N \times N$ matrix where an entry on position $(i,j) \in [N]\times[N]$ contains the color of the edge $\{i, N+j\}$ in χ. We set $T = \sqrt{n/\ell}$ and we run the multi-thread scanning for $M^<$ in A with T threads. For every $t \in \{0,1,\dots,T-1\}$, the blue entries from the thread $Th(t)$ intersect each row of A in a set that we call *segment*. Let $S(t)$ be the set of segments obtained from $Th(t)$.

Observe that each segment forms an interval of blue entries in a row of A. Moreover, each segment has length less than $2n$ as otherwise there is a vertex of $K^<_{2N}$ incident to at least $2n$ blue edges in χ and, since there is no blue triangle in χ, the neighborhood of such a vertex induces a red copy of $K^<_{2n}$. This is impossible by our assumptions on χ.

Claim 31. *Fix t and t' with $0 \leq t' < t \leq T$. Assume that k segments in $S(t)$ intersect with some segments from $S(t')$. Then, $L(\pi_{M^<}) \geq k$.*

Each segment from $S(t)$ intersects at most one segment from $S(t')$ as no two segments from $S(t)$ lie in the same row of A and the same claim is true for segments from $S(t')$. Moreover, if two segments intersect, then they are contained in the same row of A. Let s_1^t and s_2^t be two segments from $S(t)$ and let $s_1^{t'}$ and $s_2^{t'}$ be two segments from $S(t')$ and assume $s_1^t \cap s_1^{t'} \neq \emptyset$ and $s_2^t \cap s_2^{t'} \neq \emptyset$. Then, the columns of A intersected by s_1^t are to the left of the columns intersected by s_2^t if and only if the columns of A intersected by $s_1^{t'}$ are to the left of the columns intersected by $s_2^{t'}$. This is because no two segments from $S(t)$ intersect the same column from A and the same claim is true for segments from $S(t')$. Altogether, the k segments from $S(t)$ intersect exactly k segments in $S(t')$ and indices of their rows decreased by t and $t-t'$, respectively, form a shift in $\pi_{M^<}$ of size k. The claim follows.

Consider some $t \in \{0,1,\dots,T-1\}$. Since no thread succeeds in χ, the thread $Th(t)$ reveals at least $N-n$ blue entries of A. The claim and our assumption $L(\pi_{M^<}) \leq \ell$ imply that the segments from $S(t)$ intersect at most ℓ segments from $S(t')$ for every $t' < t$. Since each segment has length at most $2n$, the thread $Th(t)$ reveals at least $N - 2n - 2tn\ell$ new blue entries of A. This is at least $N/2$ by our assumption $N \geq 4n(\sqrt{n\ell}+1)$ and by the choice of T since

$$N - 2n - 2tn\ell \geq N - 2n(T\ell+1) = N - 2n(\sqrt{n\ell}+1) \geq N/2.$$

Thus, the total number of blue entries in A is at least $TN/2$. Since the multi-thread scanning visited $n+T$ rows of A, there is a vertex v of $K_{2N}^<$ incident to at least $\frac{TN}{2(T+n)}$ blue edges in χ. Now, our assumption $N \geq 4n(\sqrt{n\ell}+1)$ and the choice of T implies

$$\frac{TN}{2(T+n)} \geq \frac{\sqrt{\frac{n}{\ell}}4n(\sqrt{n\ell}+1)}{2\left(\sqrt{\frac{n}{\ell}}+n\right)} = 2n.$$

Thus, the blue neighborhood of the vertex v contains either a blue triangle or a red copy of $K_{2n}^<$. This contradicts our assumptions on χ. $\square$

For every $\varepsilon > 0$, Theorem 30 immediately implies that $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$ for every ordered matching with $\chi_<(M^<) = 2$ and $L(\pi_{M^<}) \leq n^{1-2\varepsilon}$. We show that this is the case for uniform random ordered matchings with interval chromatic number 2 by using the following result by He and Kwan [25] about the maximum length of a shift in a uniform random permutation on $[2n]$.

Lemma 32 ([25])**.** *A uniform random permutation π on $[n]$ satisfies $L(\pi) \leq 3\sqrt{n}$ with high probability.*

Now, it suffices to show that Theorem 30 together with Lemma 32 implies Theorem 29.

Proof of Theorem 29. Let $M^<$ be the uniform random ordered matching on $[2n]$. By Lemma 32, we have $L(\pi) \leq 3\sqrt{n}$ with high probability. Thus, applying Theorem 30 with $\ell = 3\sqrt{n}$, we obtain

$$r_<(M^<, K_3^<) \leq 4n(\sqrt{3n^{3/2}}+1) \in O(n^{7/4})$$

with high probability. $\square$

4. Conclusion

In the thesis, we mainly focused on studying off-diagonal ordered Ramsey numbers $r_<(M^<, K_3^<)$ for ordered matchings $M^<$. We have improved some known lower and upper bounds for $M^<$ with fixed interval chromatic number. Now, we list some interesting known open problems about these ordered Ramsey numbers.

We recall the main problem in this area posed by Conlon, Fox, Lee and Sudakov [14] (Problem 19), which asks whether there exists an $\epsilon > 0$ such that for any ordered matching $M^<$ on n vertices $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$. This problem is still open and seems to be difficult, but there are many interesting intermediate questions that one could try to tackle.

The following variant of this problem for random ordered matchings with interval chromatic number was conjectured by Rohatgi [16].

Conjecture 33 ([16])**.** *For every integer $k \geq 2$, there is a constant $\varepsilon = \varepsilon(k) > 0$ such that*

$$r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$$

for almost every ordered matching $M^<$ on n vertices with $\chi_<(M^<) = k$.

It follows from Theorem 29 that $\varepsilon(2) \geq 1/4$. The conjecture is open for all cases with $k \geq 3$. Our results suggest that $\varepsilon(2) > \varepsilon(3)$ might hold.

Concerning the ordered matchings $M^<$ with interval chromatic number 2, even in this case the growth rate of $r_<(M^<, K_3^<)$ is not understood. However, we believe that in this case the ordered Ramsey number is subquadratic.

Conjecture 34. *There exists an $\epsilon > 0$ such that for any ordered matching $M^<$ on n vertices with $\chi_<(M^<) = 2$ we have $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$.*

In this thesis, we considered the variant of this problem for random ordered matchings with interval chromatic number 2, but there is still a gap between our bounds. It would be very interesting to close it.

Problem 35. *What is the growth rate of $r_<(M^<, K_3^<)$ for uniform random ordered matchings $M^<$ on n vertices with $\chi_<(M^<) = 2$?*

It follows from our results that the answer to Problem 35 lies somewhere between $\Omega((n/\log n)^{5/4})$ and $O(n^{7/4})$. We do not know which of these bounds is closer to the truth.

As a short addendum to Section 2.3, we also restate the briefly discussed question of Balko, Cibulka, Král and Kynčl [15], which we were able to prove computationally for $n \leq 7$.

Problem 36. *Does $r_<(Alt_n^<) \leq r_<(P^<)$ hold for all ordered paths $P^<$ on $n \in \mathbb{N}$ vertices?*

Bibliography

[1] Marian Poljak. Computing and estimating ordered Ramsey numbers, 2020. Bachelor thesis.

[2] Stanisław P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30, 1994.

[3] Frank P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc. (2)*, 30(4):264–286, 1929.

[4] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[5] Paul Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[6] Joel H. Spencer. Ramsey's theorem - a new lower bound. *J. Comb. Theory, Ser. A*, 18:108–115, 1975.

[7] Marcelo Campos, Simon Griffiths, Robert Morris, and Julian Sahasrabudhe. An exponential improvement for diagonal Ramsey, 2023.

[8] Jeong Han Kim. The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms*, 7(3):173–207, 1995.

[9] Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980.

[10] Joel Spencer. Asymptotic lower bounds for Ramsey functions. *Discrete Mathematics*, 20:69–76, 1977.

[11] Paul Erdős. Graph Theory and Probability II. *Canadian Journal of Mathematics*, 13:346–352, 1961.

[12] Václav Chvátal, Vojtěch Rödl, Endre Szemerédi, and William T. Trotter, Jr. The Ramsey number of a graph with bounded maximum degree. *J. Combin. Theory Ser. B*, 34(3):239–243, 1983.

[13] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fourth edition, 2010.

[14] David Conlon, Jacob Fox, Choongbum Lee, and Benny Sudakov. Ordered Ramsey numbers. *J. Combin. Theory Ser. B*, 122:353–383, 2017.

[15] Martin Balko, Josef Cibulka, Karel Král, and Jan Kynčl. Ramsey numbers of ordered graphs. *Electron. J. Combin.*, 27(1), Jan 2020.

[16] Dhruv Rohatgi. Off-diagonal ordered Ramsey numbers of matchings. *Electron. J. Combin.*, 26(2):Paper No. 2.21, 18, 2019.

[17] Vida Dujmović and David R. Wood. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–357, 2004.

[18] Martin Balko and Marian Poljak. On off-diagonal ordered Ramsey numbers of nested matchings. *Discrete Mathematics*, 346(2):113223, 2023.

[19] Martin Balko and Marian Poljak. On off-diagonal ordered Ramsey numbers of nested matchings. In Jaroslav Nešetřil, Guillem Perarnau, Juanjo Rué, and Oriol Serra, editors, *Extended Abstracts EuroComb 2021*, pages 241–247, Cham, 2021. Springer International Publishing.

[20] Martin Balko and Marian Poljak. On ordered Ramsey numbers of matchings versus triangles, 2023. Accepted to the European Conference on Combinatorics, Graph Theory and Applications 2023 (EUROCOMB'23).

[21] Marian Poljak. Ordered Ramsey numbers utility. https://github.com /marismmm/Ordered-Ramsey-numbers-utility.

[22] David Conlon, Jacob Fox, and Benny Sudakov. Recent developments in graph Ramsey theory. In *Surveys in combinatorics 2015*, volume 424 of *London Math. Soc. Lecture Note Ser.*, pages 49–118. Cambridge Univ. Press, Cambridge, 2015.

[23] Noga Alon and Joel H. Spencer. *The probabilistic method.* Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, fourth edition, 2016.

[24] Josef Cibulka and Jan Kynčl. Better upper bounds on the Füredi-Hajnal limits of permutations. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2280–2293. SIAM, Philadelphia, PA, 2017.

[25] Xiaoyu He and Matthew Kwan. Universality of random permutations. *Bull. Lond. Math. Soc.*, 52(3):515–529, 2020.

A. Attachments

A.1 Colorings lower-bounding $r_<(Alt_n^<)$

As an addendum to the coloring proving that $r_<(Alt_{10}^<) > 22$, we include several more colorings for alternating paths of higher order.

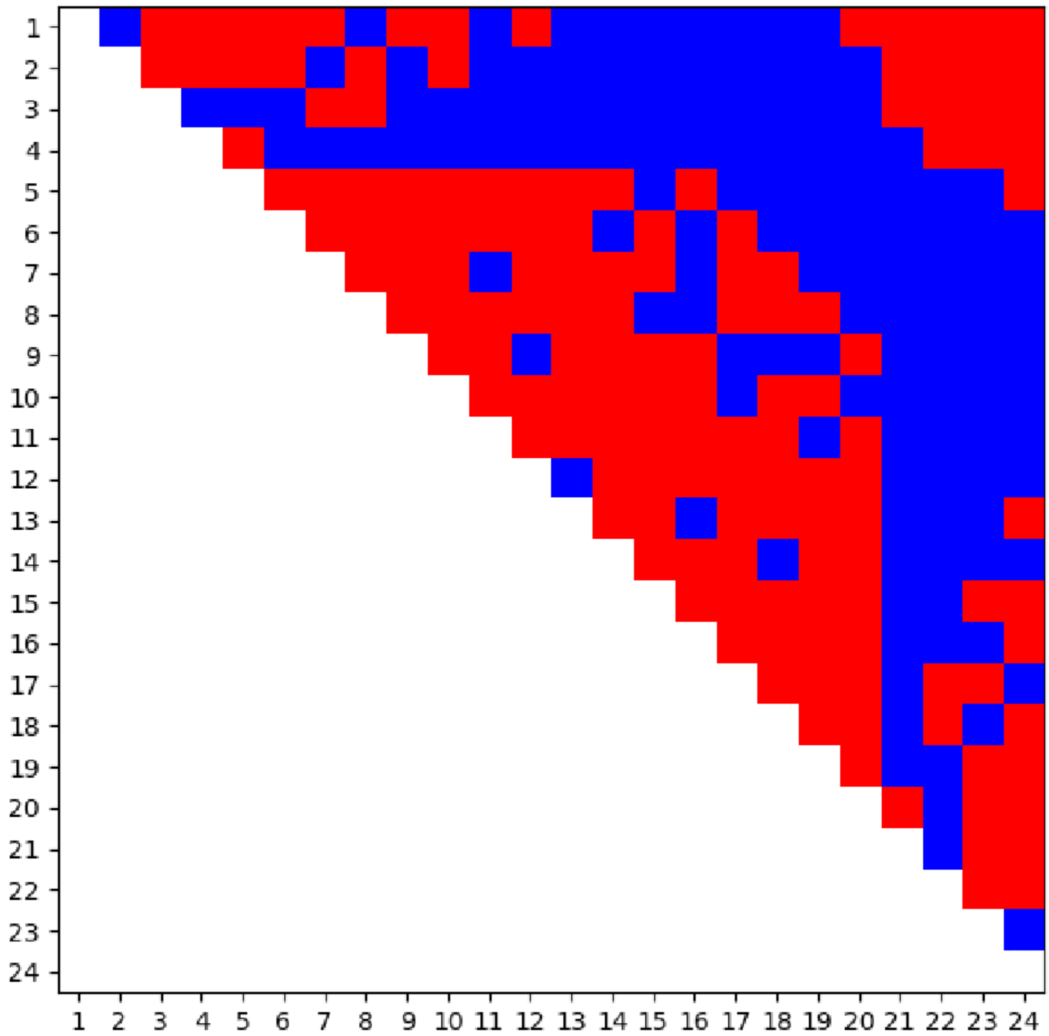

Figure A.1: The matrix representation of a symmetric coloring of $K_{24}^<$ avoiding $Alt_{11}^<$ in both colors.

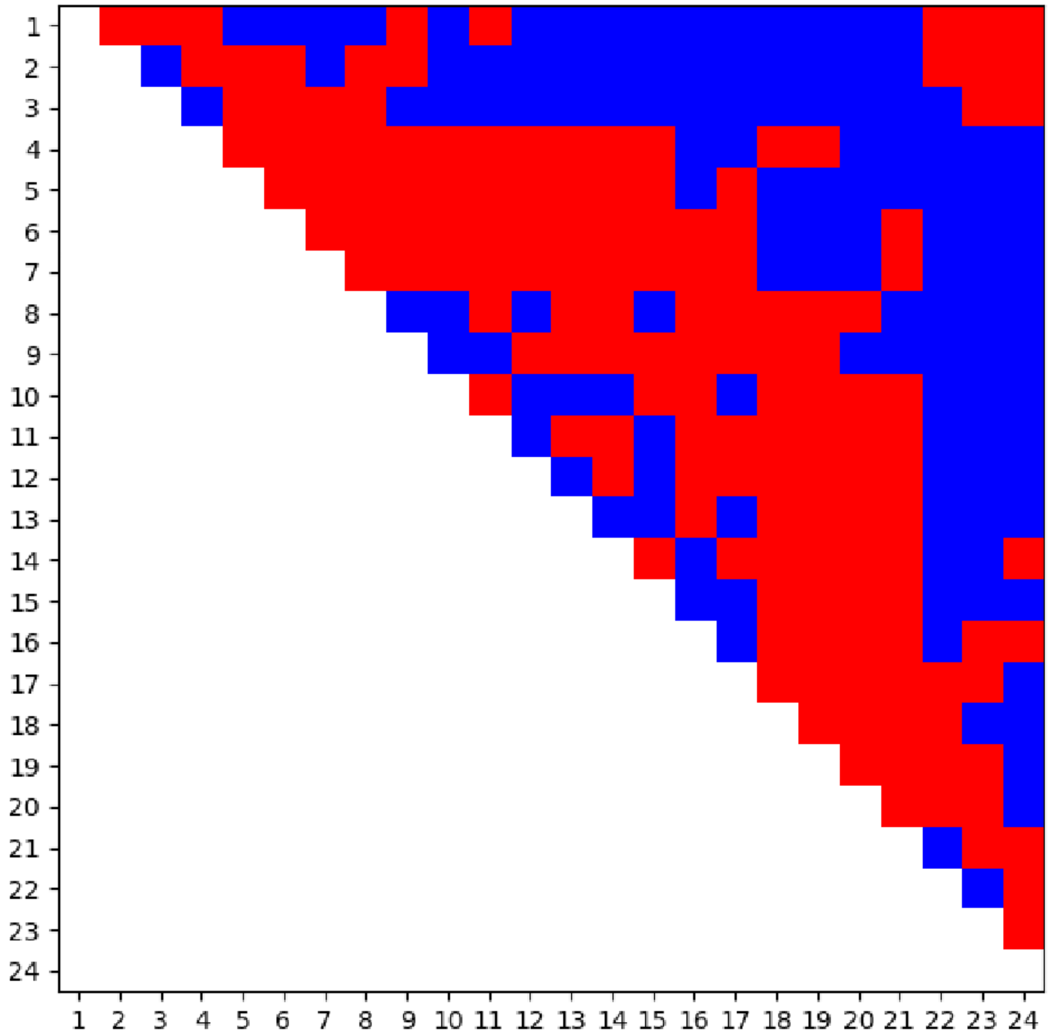

Figure A.2: The matrix representation of another symmetric coloring of $K_{24}^{<}$ avoiding $Alt_{11}^{<}$ in both colors.

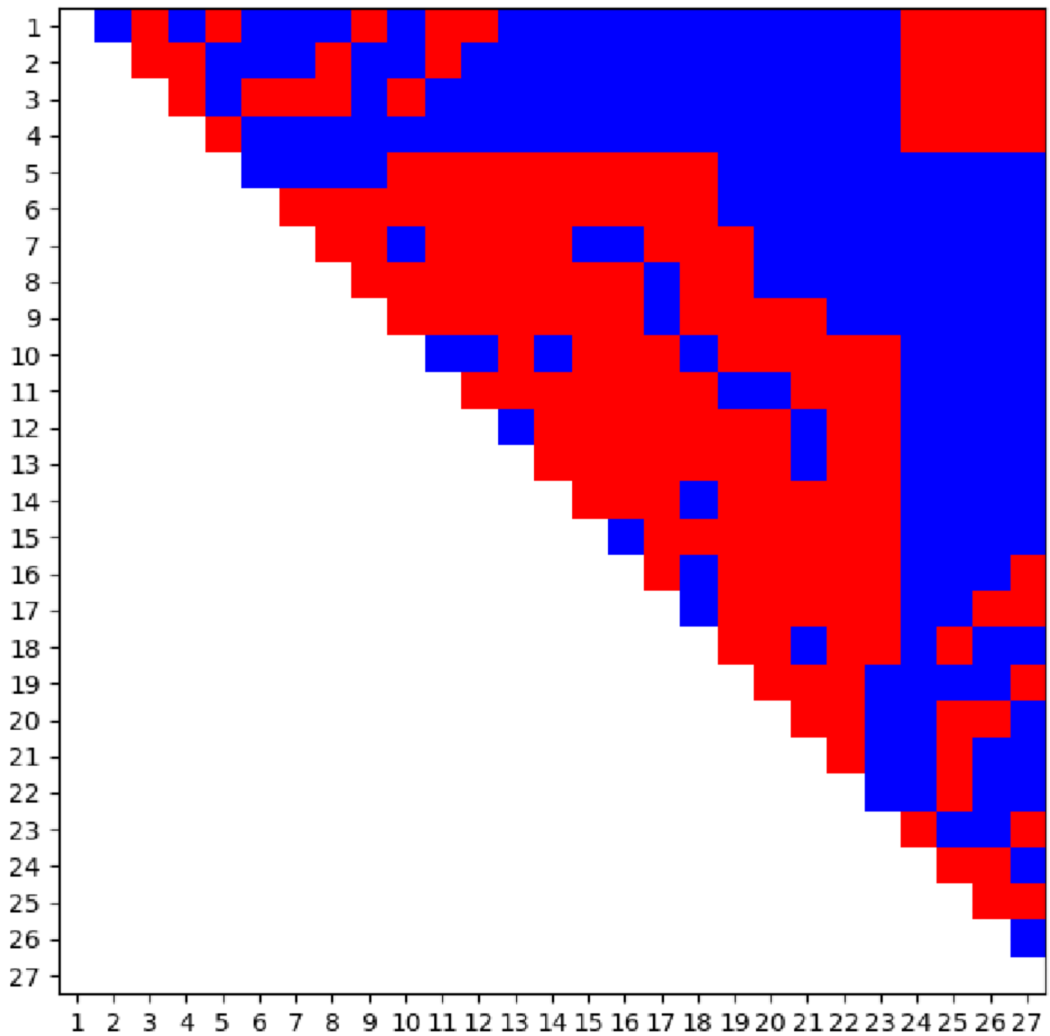

Figure A.3: The matrix representation of a symmetric coloring of $K_{27}^{<}$ avoiding $Alt_{12}^{<}$ in both colors.

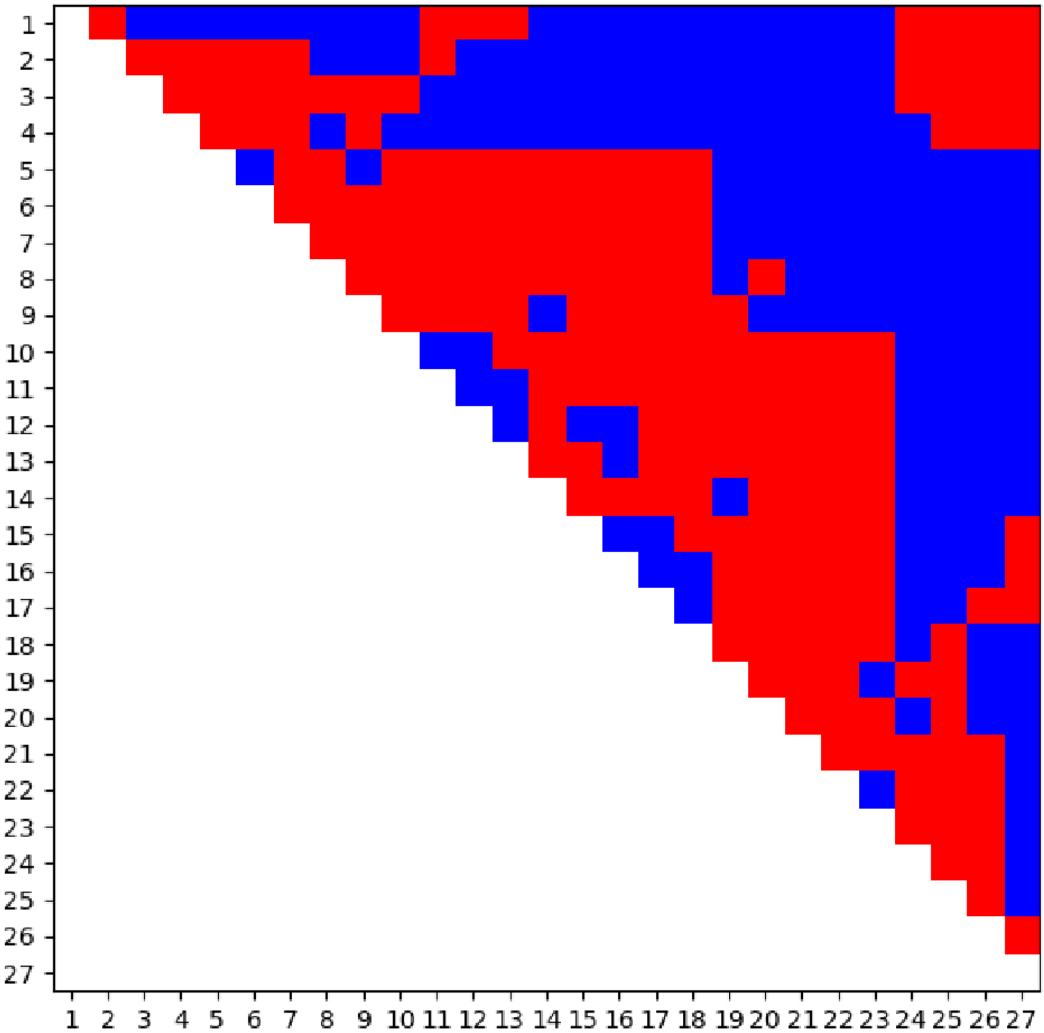

Figure A.4: The matrix representation of another symmetric coloring of $K_{27}^{<}$ avoiding $Alt_{12}^{<}$ in both colors.

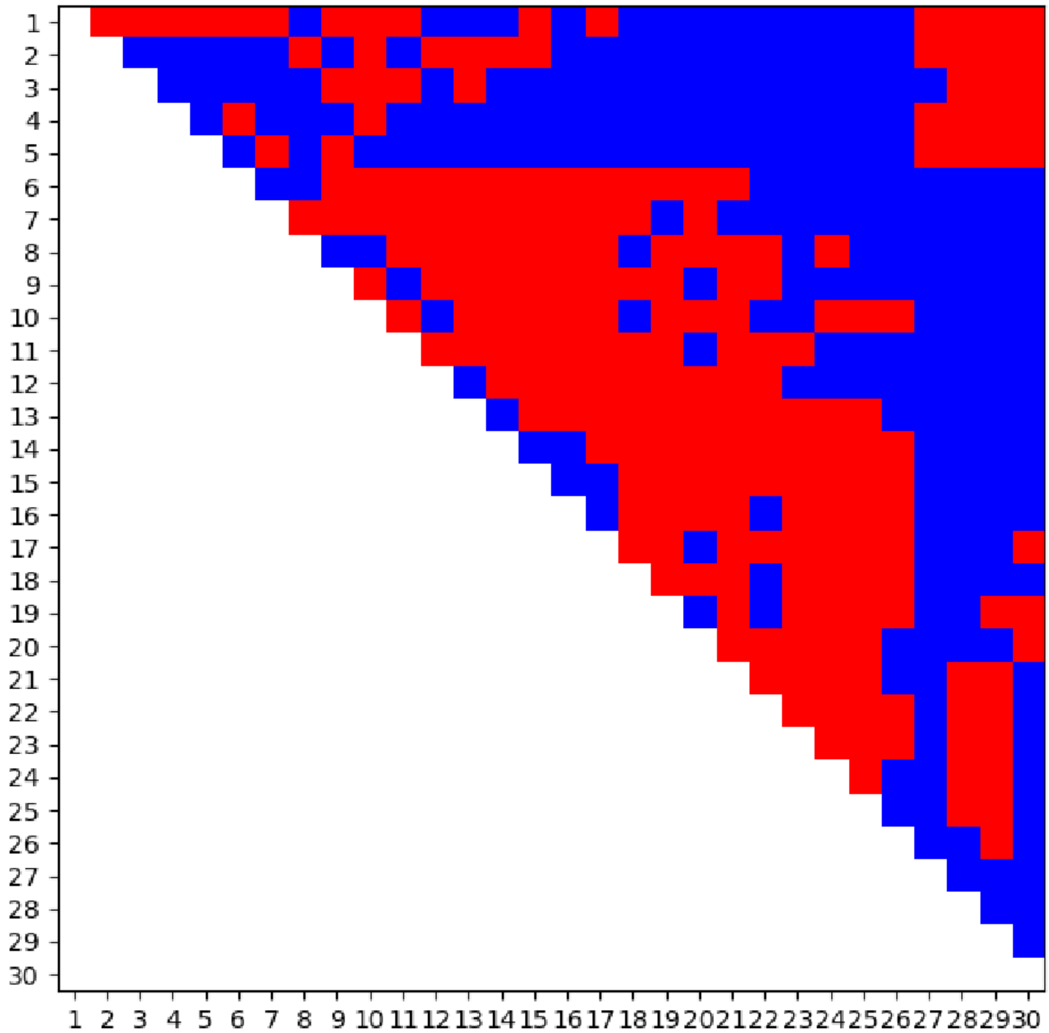

Figure A.5: The matrix representation of a coloring of $K_{30}^{<}$ avoiding $Alt_{13}^{<}$ in both colors.

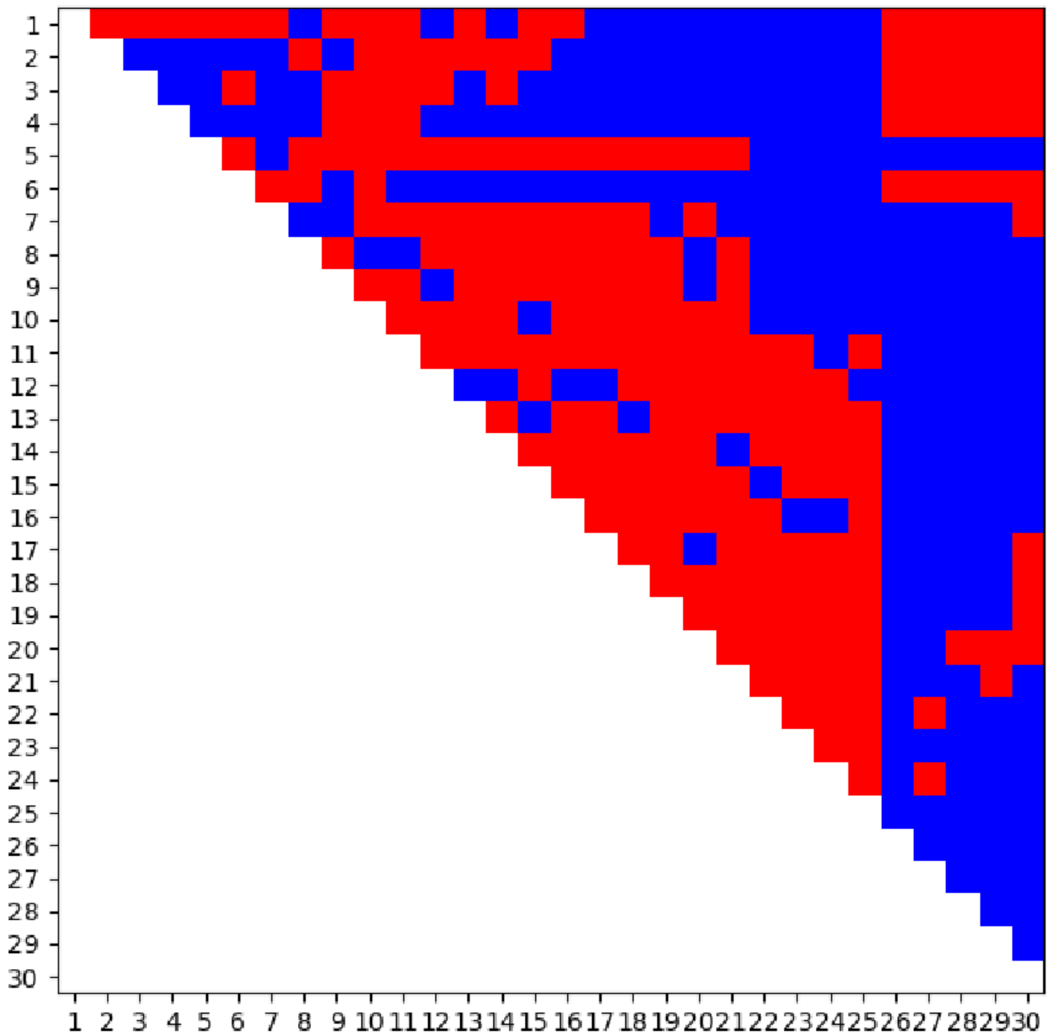

Figure A.6: The matrix representation of another coloring of $K_{30}^{<}$ avoiding $Alt_{13}^{<}$ in both colors.