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Partitions of totally positive elements in real quadratic fields

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Title: Partitions of totally positive elements in real quadratic fields

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Abstract: We consider the additive semigroup $\mathcal{O}_K^+(+)$ of totally positive integers in a real quadratic field $K = \mathbb{Q}(\sqrt{D})$. We define on $\mathcal{O}_K^+(+)$ the partition function $p_K(\alpha)$ and develop an algorithm for computing $p_K(\alpha)$ for different square-free D and different $\alpha \in \mathcal{O}_K^+$. We then investigate the behaviour of $p_K(\alpha)$, characterizing the square-free numbers D for which $p_K(\alpha)$ attains the numbers 1 through 5. Finally, we prove a sufficient condition for the number 6 to be attainable by $p_K(\alpha)$.

Keywords: real quadratic fields, totally positive elements, indecomposable elements, partitions

Contents

Introduction

The history of additive number theory goes back to Leonhard Euler (1707 – 1783), who in 1740 studied the problem of how many ways can an integer n be represented as a sum of integers, i.e. the partitions of n. This led to a number of significant results about *the partition function* $p(n)$, which represents the total number of partitions of n. One of which is the *Euler recursion formula* (see [Apo76, p. 315]):

$$p(n) = [p(n-1) + p(n-2)] - [p(n-5) + p(n-7)] + \cdots , \tag{1}$$

where $1, 2, 5, 7 \ldots$ are the pentagonal numbers. This gave a powerful way to compute the partition function, which is otherwise hard to evaluate.

Euler's discoveries initiated an extensive study of the partition function in number theory and combinatorics. For example, in 1918 the British mathematician G.H. Hardy (1877–1947) and an Indian mathematical prodigy Srinivasa Ramanujan (1887–1920) discovered its remarkable asymptotic behaviour (see [Apo76, p. 316] for reference):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$

With the development of algebraic number theory, an interesting notion that came up in the real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ was the totally positive integers. They were used in the field of universal quadratic forms and lattices and their notable additive structure was studied by T. Hejda and V. Kala [HK20]. Their work used the idea of indecomposable elements, i.e. totally positive integers, which cannot be written as a sum of two totally positive integers. These were fully characterized by A. Dress and R. Scharlau in [DS82].

Consequently, given an additive structure of totally positive integers in a real quadratic field K, a natural question is: *in how many ways can a totally positive integer be written as a sum of totally positive integers?* Building on the work mentioned above, we consider an analogous partition function $p_K(\alpha)$, where α is a totally positive integer, and study its behaviour. Our main goal in this thesis is to find an effective algorithm to compute $p_K(\alpha)$ for $K = \mathbb{Q}(\sqrt{D})$ and to study its properties for different square-free D.

In Chapter 1, we introduce the notation and notions we use throughout this thesis. This includes some elementary algebraic number theory, the definition of our partition functions $p_K(\alpha)$ for the real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ and the establishment of the notions of indecomposable and uniquely decomposable elements. In Lemma 3, Proposition 4, Proposition 6 and Proposition 9 we prove some elementary observations we use in the following two chapters.

Chapter 2 subsequently concerns the algorithm computing $p_K(\alpha)$ for different $K = Q(\sqrt{D})$ and different totally positive integers α. We describe the main idea behind our program and prove its correctness in Proposition 12. We include its proper documentation and several tables of values of $p_K(\alpha)$. These tables will later be used for reference in the last chapter.

Finally in Chapter 3, with the help of our algorithm from Chapter 2, we study the behaviour of our partition functions $p_K(\alpha)$. Namely which values it can (or cannot) attain. We do this by first analyzing the connection between $p_K(\alpha)$ and the integer partition function $p(n)$ in Theorem 13. Then, by studying the decompositions of totally positive integers into indecomposable elements in Lemma 15 through to Lemma 19 we arrive at the characterization of square-free D, for which $p_K(\alpha)$ obtains the values $1, 2, 3$ and 5, in Theorem 20. The rest of the chapter is devoted to analyzing, if values 4 and 6 may be also obtained by $p_K(\alpha)$. Theorem 23 concludes our thesis, showing the existence of particular D and particular α, for which $p_K(\alpha) = 4$ or 6.

Thus, the culmination of our inquiry into partitions of totally positive integers is represented by three main theorems: Theorem 13, Theorem 20, and Theorem 23. In particular, the proof of Theorem 20 required a long and thorough investigation into the structure of decompositions of totally positive integers into indecomposable elements. This shed light on some interesting properties of totally positive integers.

The results obtained in Chapter 2 and Chapter 3 are from the author's original research and will be submitted for publication in a reputable scientific journal in 2023.

1. Preliminaries

1.1 Algebraic number theory in quadratic fields

Throughout this thesis we will use the following notation: For a fixed square-free integer $D \geq 2$ we let K denote the real quadratic field $K = \mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$ and denote by $\mathcal{O}_K$ its ring of integers.

We know that $\{1, \omega_D\}$ forms an integral basis of $\mathcal{O}_K$, where

$$\omega_D = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

Thus, any integral element $\alpha \in \mathcal{O}_K$ can be written as $\alpha = a + b\omega_D$, $a, b \in \mathbb{Z}$. We define its *Galois conjugate* as $\alpha' := a + b\omega_D'$, where $\omega_D' = -\sqrt{D}$ if $D \equiv 2, 3 \pmod 4$ and $\omega_D' = \frac{1-\sqrt{D}}{2}$ otherwise. Hence, we can define its *trace* $\mathrm{Tr}(\alpha)$ and *norm* $N(\alpha)$ as

$$\begin{aligned} \mathrm{Tr}(\alpha) &:= \alpha + \alpha', \\ N(\alpha) &:= \alpha\alpha'. \end{aligned}$$

It can be easily verified, that $\forall \alpha, \beta \in \mathcal{O}_K$: $N(\alpha\beta) = N(\alpha)N(\beta)$ and $\mathrm{Tr}(\alpha + \beta) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\beta)$. As we will see later in this chapter, Galois conjugates are essential for the notion of totally positive elements.

An important notion from algebraic number theory are the invertible elements, i.e. $\varepsilon \in \mathcal{O}_K$ such that there exists $\varepsilon^{-1} \in \mathcal{O}_K$: $\varepsilon\varepsilon^{-1} = 1$. We call these elements *the units of* $\mathcal{O}_K$ and denote them by ε. We denote by $\mathcal{O}_K^\times$ the set of all units in $\mathcal{O}_K$. It can be observed, that all units have norm equal to ± 1. Moreover, the converse is also true, i.e. we can write $\mathcal{O}_K^\times = \{\varepsilon \in \mathcal{O}_K \mid N(\varepsilon) = \pm 1\}$.

Example. In real quadratic fields $K = \mathbb{Q}(\sqrt{D})$, where $D \equiv 2, 3 \pmod 4$, the units are exactly the solutions to the well known Pell's equation: $x^2 - Dy^2 = \pm 1$.

In the real quadratic fields $K = \mathbb{Q}(\sqrt{D})$, there exists *a fundamental unit* $\varepsilon > 1$ of $\mathcal{O}_K^+$, such that $\mathcal{O}_K^\times = \{\pm(\varepsilon)^k \mid k \in \mathbb{Z}\}$.

1.2 Totally positive integers and indecomposables

As we outlined in our introduction, we focus on the additive structure of $\mathcal{O}_K^+$ as opposed to the multiplicative structure of $\mathcal{O}_K$, which is usually studied in algebraic number theory.

Definition 1. *Let $\alpha \in \mathcal{O}_K$. We say that α is* totally positive *if $\alpha > 0$ and $\alpha' > 0$. We denote by $\alpha \succ 0$ the fact that α is totally positive.*

For $\alpha, \beta \in \mathcal{O}_K$ we denote $\alpha \succ \beta$ the fact that $\alpha - \beta \succ 0$. The relation $\succ$ on the set of totally positive integers is clearly irreflexive, asymmetric and transitive. We denote by $\mathcal{O}_K^+$ the *set of all totally positive integers* in $K = \mathbb{Q}(\sqrt{D})$. We can observe, that the set $\mathcal{O}_K^+(+)$ with the operation of addition forms a semigroup (i.e. a set with associative binary operation).

Because quadratic integers have two coordinates, we cannot order them linearly simply by the $<$-relation. Very useful to us will be the lexicographical ordering defined as follows:

Definition 2. *Let $K = \mathbb{Q}(\sqrt{D})$. Let $\alpha = a + b\omega_D$, $\beta = c + d\omega_D$ be elements in K. Given the basis $\{1, \omega_D\}$, we define* the lexicograpical ordering $>_{LEX}$ *in K as:*

$$\alpha >_{LEX} \beta \iff a > c \quad \textit{or} \quad a = c \textit{ and } b > d.$$

We denote by $\alpha \geq_{LEX} \beta$ the fact that $\alpha = \beta$ or $\alpha >_{LEX} \beta$.

It is easy to see this ordering is linear on the elements of K.

From our definition of totally positive elements above, it is not as obvious to see which integers are totally positive based on their coefficients. An important observation we will use many times throughout this thesis is that, given a coefficient $a \in \mathbb{N}$, there are only finitely many coefficients $b \in \mathbb{Z}$ such that $a + b\omega_D$ is a totally positive element.

Lemma 3. *Let $K = \mathbb{Q}(\sqrt{D})$.*

- *If $\alpha \succ 0$, then $N(\alpha)$, $Tr(\alpha) > 0$.*
- *Let $a \in \mathbb{N}$, $\alpha = a + b\omega_D \in \mathcal{O}_K$. Then*

$$\alpha \succ 0 \quad \textit{if and only if} \quad b \in \left(\frac{a}{-\omega_D}, \frac{a}{-\omega_D'} \right). \tag{1.1}$$

Proof. Both statements are direct consequences of our definition of totally positive integers. Specifically, for the second statement we have $\alpha \succ 0$ if and only if

$$a + b\omega_D > 0 \Leftrightarrow b > \frac{a}{-\omega_D} \quad \text{and} \quad a + b\omega_D' > 0 \Leftrightarrow b < \frac{a}{-\omega_D'} \quad \text{as } \omega_D' < 0.$$

□

Together with the notion of indecomposable elements, which we mentioned above, one can ask, if every totally positive integer can be expressed as a sum of indecomposables.

Proposition 4. *Every totally positive integer in K can be written as a finite sum of indecomposable elements.*

Proof. Let $\alpha \in \mathcal{O}_K^+$. If α is indecomposable, then we are done. Otherwise $\exists \alpha_1, \alpha_2 \in \mathcal{O}_K^+$ such that $\alpha = \alpha_1 + \beta_1$. The same process can be repeated for α_1 and α_2. This cannot be repeated indefinitely, because otherwise we would have an infinite sequence $\alpha = \alpha_0, \alpha_1, \alpha_2 \ldots$, where $\text{Tr}(\alpha_0) > \text{Tr}(\alpha_1) > \text{Tr}(\alpha_2) > \ldots > 0$ (using the Lemma 3 above), which is impossible. □

This motivates the question, if this expression is always unique. As we see in the following example, uniqueness does not always hold.

Example. Let $K = \mathbb{Q}(\sqrt{2})$. Then the integers in K are of the form: $\alpha = a + b\sqrt{2}$ and we can write the totally positive integer $4 + 2\sqrt{2}$ as

$$4 + 2\sqrt{2} = (2 + \sqrt{2}) + (2 + \sqrt{2}) = (3 + 2\sqrt{2}) + 1,$$

where 1, $2 + \sqrt{2}$, and $3 + 2\sqrt{2}$ are all indecomposable elements.

Finally it is convenient to introduce the notation of totally positive units in K. We denote $\varepsilon_+ > 1$ the smallest totally positive unit > 1 and call it *the fundamental totally positive unit*. Then the set of totally positive units, denoted by $\mathcal{O}_K^{+,\times}$, is of the form: $\mathcal{O}_K^{+,\times} = \{\varepsilon_+^n \mid n \in \mathbb{Z}\}$

1.3 Indecomposable elements

In the last section we proved in Proposition 4 the existence of a decomposition of an arbitrary totally positive integer into a sum of indecomposable elements. As we saw in the example above, the uniqueness of this sum does not always hold. This motivates the notion of *uniquely decomposable elements*.

Definition 5. *Let $\alpha \in \mathcal{O}_K^+$. We say that α is*

- indecomposable *if α cannot be written as a sum of two totally positive integers,*
- uniquely decomposable *if there is a unique way of expressing α as a sum of indecomposable elements.*

Equivalently, an element α is indecomposable if and only if $\nexists \beta \in \mathcal{O}_K^+$ such that $\alpha \succ \beta$. We now look into the notion of indecomposable elements in more detail. We start with an interesting connection between the indecomposable elements and the totally positive units in a given field.

Proposition 6. *Every totally positive unit in K is indecomposable.*

Proof. Let ε be a totally positive unit and let us assume the contrary, i.e. there exist $\alpha, \beta \in \mathcal{O}_K^+$ such that $\varepsilon = \alpha + \beta$. Then we have (using Lemma 3)

$$\begin{aligned} 1 = N(\varepsilon) = N(\alpha + \beta) &= (\alpha + \beta)(\alpha + \beta)' = (\alpha + \beta)(\alpha' + \beta') \\ &= (\alpha\alpha') + (\beta\beta') + (\alpha\beta' + \alpha'\beta) > N(\alpha) + N(\beta) \geq 2, \end{aligned}$$

as $\alpha, \beta \succ 0$. This is a contradiction, therefore ε is indecomposable. □

However, as we will see, in general the totally positive units are not the only indecomposable elements. The complete characterization of indecomposables was done by A. Dress and R. Scharlau in [DS82]. We adopt the notation from [HK20] as follows:

Let $\sigma_D = [\overline{u_0, u_1, \ldots, u_s}]$ be the periodic continued fraction of:

$$\sigma_D := \omega_D + \lfloor -\omega_D' \rfloor = \begin{cases} \sqrt{D} + \lfloor \sqrt{D} \rfloor & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2} + \lfloor \frac{1-\sqrt{D}}{2} \rfloor & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

We then have that $\omega_D = [\lceil u_0/2 \rceil, \overline{u_1, \dots, u_s}]$. This notation is helpful for defining the convergents of ω_D. As we will see, indecomposable elements correspond with the convergents and semiconvergents of ω_D.

Denote p_i/q_i its convergents. Then the sequences (p_i), (q_i) satisfy the reccurence:

$$X_{i+2} = u_{i+2} X_{i+1} + X_i \quad \text{for} \quad i \geq -1 \tag{1.2}$$

with the initial condition: $q_{-1} = 0$, $p_{-1} = q_0 = 1$, $p_0 = \lceil u_0/2 \rceil$ [Per13]. We thus have all we need to characterize indecomposable elements.

Denote $\alpha_i := p_i - q_i \omega_D'$ an $\alpha_{i,r} = \alpha_i + r\alpha_{i+1}$. Then we the following holds [DS82]:

- The sequence (α_i) satisfies the recurrence 1.2.
- We have that $\alpha_i \succ 0$ if and only if $i \geq -1$ is odd.
- The indecomposable elements in $\mathcal{O}_K^+$ are $\alpha_{i,r}$ with odd $i \geq -1$ and $0 \leq r \leq u_{i+2} - 1$, together with their conjugates.
- We have that $\alpha_{i,u_{i+2}} = \alpha_{i+2,0}$.
- The indecomposables $\alpha_{i,r}$ are increasing with increasing (i, r) (in the lexicographic sense).
- The indecomposables $\alpha_{i,r}'$ are decreasing with increasing (i, r).
- Considering the fundamental unit $\varepsilon > 1$ of $\mathcal{O}_K$, we have that $\varepsilon = \alpha_{s-1}$.

We then know that $\varepsilon_+ = \varepsilon$ if s is even and $\varepsilon_+ = \varepsilon^2 = \alpha_{2s-1}$ if s is odd. This observation corresponds to the Proposition 6 above about the relation between totally positive units and indecomposables. Lastly, we mention an important fact due to [DS82] regarding the associate elements of indecomposables:

- Denoting $s_+ := s$ if s is even and $s_+ := 2s$ if s is odd, we have $\alpha_{i+s_+} = \varepsilon_+ \alpha_i$ for all odd $i \geq -1$.

Using this notation, we can introduce an important theorem proved by T. Hejda and V. Kala in [HK20] characterizing all uniquely decomposable elements in $\mathcal{O}_K^+$:

Theorem 7 ([HK20], Theorem 4)**.** *All uniquely decomposable elements $x \in \mathcal{O}_K^+$ are the following:*

1. *$\alpha_{i,r}$ with odd $i \geq -1$ and $0 \leq r \leq u_{i+2} - 1$;*
2. *$e\alpha_{i,0}$ with odd $i \geq -1$ and with $2 \leq e \leq u_{i+1} + 1$;*
3. *$\alpha_{i,u_{i+2}-1} + f\alpha_{i+2,0}$ with odd $i \geq -1$ odd such that $u_{i+2} \geq 2$ and with $1 \leq f \leq u_{i+3}$;*
4. *$e\alpha_{i,0} + \alpha_{i,1}$ with odd $i \geq -1$ such that $u_{i+2} \geq 2$ and with $1 \leq e \leq u_{i+1}$;*
5. *$e\alpha_{i,0} + f\alpha_{i+2,0}$ with odd $i \geq -1$ such that $u_{i+2} = 1$ and with $1 \leq e \leq u_{i+1}+1$, $1 \leq f \leq u_{i+3} + 1$, $(e, f) \neq (u_{i+1} + 1, u_{i+3} + 1)$;*
6. *Galois conjugates of all of the above.*

This theorem will be especially useful in Chapter 3 for studying special properties of partition functions.

1.4 Partition functions

Before we delve into our algorithm, the last definition we need is of partition functions:

Definition 8. *Let $K = \mathbb{Q}(\sqrt{D})$, $\alpha \in \mathcal{O}_K^+$. We say that the non-increasing sequence (in the lexicographic sense): $(\lambda_1, \lambda_2, \dots, \lambda_m)$, where $\lambda_1, \lambda_2, \dots, \lambda_m \succ 0$, $m \in \mathbb{N}$; is a* partition *of α, if $\lambda_1 + \lambda_2 + \cdots + \lambda_m = \alpha$.*

For $K = \mathbb{Q}(\sqrt{D})$ we define the partition function *$p_K(\alpha)$ as a number of partitions of $\alpha \in \mathcal{O}_K^+$.*

Thus, we have $p_K(\alpha) = 1$ if $\alpha \in \mathcal{O}_K^+$ is indecomposable and $p_K(\alpha) \leq 2$ if $\alpha \in \mathcal{O}_K^+$ is uniquely decomposable. An important property of the partition functions is their invariance under conjugation and under multiplication by a unit.

Proposition 9. *Let $K = \mathbb{Q}(\sqrt{D})$, $\alpha \in \mathcal{O}_K^+$, $\varepsilon \in \mathcal{O}_K^{+,\times}$. Then*

$$p_K(\alpha) = p_K(\alpha') = p_K(\varepsilon\alpha).$$

Proof. Observe that for conjugation and multiplying by a totally positive unit we have:

$$\begin{aligned} \text{If } \alpha = \lambda_1 + \lambda_2 + \cdots + \lambda_n \text{ then: } \alpha' &= \lambda_1' + \lambda_2' + \cdots + \lambda_n', \\ \varepsilon\alpha &= \varepsilon\lambda_1 + \varepsilon\lambda_2 + \cdots + \varepsilon\lambda_n. \end{aligned}$$

It is important to note, that the product of two totally positive integers is totally positive. This is easily observed from our definition. Then, taking into consideration, that $(\alpha')' = \alpha$ and $\varepsilon^{-1}(\varepsilon\alpha) = \alpha$, we have two well-defined bijections between the partitions of α, α' and $\varepsilon\alpha$. □

2. Algorithm for the partition function

In this chapter we use our newly built theory from Chapter 1 to construct an algorithm for computing the partition function p_K. Furthermore, we generate some tables of partition function values for a few D, which will become helpful in the following Chapter 3.

2.1 Algorithm

Our algorithm is based on a recursive property of partitions. Sadly, we do not yet have a very efficient recursion formula as the one stated in the introduction (1) for integer partitions. However, as we will see, we can make use of a simple property of partitions, that will be adequate for our purposes. We first show the idea of our algorithm on an example.

Example. Let us consider the integer partition function $p(n)$ for $n \in \mathbb{N}$. Then for $n = 4$ we have

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Similarly to our definition of partitions in K, the numbers in these sums are non-increasing. We can make use of this property. Let us adopt the notation used by G. Andrews and K. Eriksson [AE04, Chapter 6.] to denote the number of partitions n with terms $\leq m$ (i.e. the summands of the partition $\leq m$) as $p(n, m)$. Then, defining $p(0, m) := 1$ for all $m \in \mathbb{N}$, the following holds for all $n \in \mathbb{N}$:

$$p(n) = \sum_{i=1}^{n} p(n - i, i).$$

Example. We have $p(4) = p(3, 1) + p(2, 2) + p(1, 3) + p(0, 4) = 1 + 2 + 1 + 1 = 5$.

We can use the analogous property for general $p_K(\alpha)$, where, instead of $\leq$-ordering, we use the lexicographic ordering we defined in Chapter 1.

Definition 10. *Let $K = \mathbb{Q}(\sqrt{D})$, $\alpha, \beta \in \mathcal{O}_K^+$. Then we denote by $p_K(\alpha, \beta)$* the number of partitions of α with terms $\leq_{LEX}$ β. *We define $p_K(0, \beta) = 1$ for arbitrary $\beta \in \mathcal{O}_K^+$.*

Remark. It is clear from the definition, that

- if $\beta \geq_{LEX} \alpha$, then $p_K(\alpha, \beta) = p_K(\alpha, \alpha)$,
- $p_K(\alpha, \alpha) = p_K(\alpha)$.

Analogously to the lexicographical ordering, we denote by $\alpha \succeq \beta$ the fact $\alpha = \beta$ or $\alpha \succ \beta$. We analyze the relationship of these two orderings of $\mathcal{O}_K^+$.

Lemma 11. *$\alpha \succeq \beta$ implies $\alpha \geq_{LEX} \beta$.*

Proof. Let $\alpha = a + b\omega_D$, $\beta = c + d\omega_D$. By our assumption $\alpha \succeq \beta$. Thus, we have $a + b\omega_D \geq c + d\omega_D$ and $a + b\omega'_D \geq c + d\omega'_D$. We first establish that $a \geq c$.

If $a < c$, then from the first inequality we have $b > d$, as $\omega_D > 0$. But then, we get $a + b\omega'_D < c + d\omega'_D$, because $\omega'_D < 0$, which is a contradiction with the second inequality. Thus, $a \geq c$. Moreover, if $a = c$, then $b \geq d$ from the first inequality.

Furthermore, the opposite implication does not hold as we can take $\alpha := a + b\omega_D, \beta := a + (b-1)\omega_D$. Then clearly $\alpha \geq_{LEX} \beta$, but $\alpha \not\succeq \beta$, since $\alpha - \beta = \omega_D$ is not totally positive. □

As in the example above for the natural numbers, the following holds:

Proposition 12. *Let $K = \mathbb{Q}(\sqrt{D})$. Then for all $\alpha \in \mathcal{O}_K^+$*

$$p_K(\alpha) = \sum_{\beta \preceq \alpha} p_K(\alpha - \beta, \beta). \tag{2.1}$$

Proof. We prove the statement by constructing a bijection between the set of partitions of α and the set of partitions of $(\alpha - \beta)$ with terms less than or equal to β for $\beta \preceq \alpha$.

First, we consider an arbitrary partition $(\lambda_1, \lambda_2, \ldots \lambda_n)$ of α. Let $\beta \preceq \alpha$, i.e. from Lemma 11 above $\beta \leq_{LEX} \alpha \Rightarrow \beta$ can (by our definition of partition) appear in the partition of α. Hence, we set $\lambda_1 = \beta$. We thus get the partition $(\lambda_2, \lambda_3, \ldots \lambda_n)$ of $\alpha - \beta \in \mathcal{O}_K^+$, where the $\lambda_2, \lambda_3, \ldots \lambda_n$ are lexicographically less than or equal to β. Conversely, given a partition $(\mu_1, \mu_2, \ldots \mu_m)$ of $\alpha - \beta$ with terms lexicographically less than β, the sequence $(\alpha - \sum_{i=1}^m \mu_i = \beta, \mu_1, \ldots, \mu_m)$ is clearly a partition of α. Thus, we have a map

$$\begin{aligned}
\{\lambda \mid \lambda \text{ is a part. of } \alpha\} &\longrightarrow \bigcup_{\beta \preceq \alpha} \{\mu \mid \mu \text{ is a part. of } (\alpha - \beta) \text{ with } \mu_i \leq_{LEX} \beta\}, \\
(\lambda_1, \lambda_2, \ldots \lambda_n) &\longmapsto (\lambda_2, \lambda_3, \ldots \lambda_n), \\
(\alpha - \sum_{i=1}^m \mu_i, \mu_1, \ldots, \mu_m) &\longleftarrow (\mu_1, \mu_2, \ldots \mu_m),
\end{aligned}$$

which is injective and has an injective inverse, i.e. we have a bijection. □

Our recursion 2.1 above thus can be used to compute $p_K(\alpha)$ for arbitrary $\alpha \in \mathcal{O}_K^+$.

Example. Let $K = \mathbb{Q}(\sqrt{2})$ and $\alpha = 4 + 2\sqrt{2}$. Then the elements in $\mathcal{O}_K^+$ which are $\preceq \alpha$ are: $4 + 2\sqrt{2}, 3 + 2\sqrt{2}, 2 + \sqrt{2}, 1$. Thus, from 2.1 we have

$$\begin{aligned}
p_K(4 + 2\sqrt{2}) = p_K(0, 4 + 2\sqrt{2}) + p_K(1, 3 + 2\sqrt{2}) + p_K(2 + \sqrt{2}, 2 + \sqrt{2}) + \\
p_K(3 + 2\sqrt{2}, 1) = 1 + 1 + \underbrace{p_K(2 + \sqrt{2})}_{=1} + 0 = 3.
\end{aligned}$$

We thus base our algorithm on this simple recursive process. The implementation of this algorithm is the subject of the next section.

2.2 Implementation - Data structures

Before we can delve into the implementation of the algorithm itself, we need to define the data structures we use in our program. Clearly, we must find a way to describe and work with quadratic integers. It is however helpful to first introduce the class of omega.

```
class Omega:
"second element of the basis in the integer ring O_K"
INIT(self, omega)
    "the value associated to self" = omega
```

For fixed $D > 0$, `class Omega` constructs the number ω_D we defined in Chapter 1. Moreover, it defines addition, multiplication, conjugation, trace and norm of ω_D, since we can regard it as a quadratic integer. These definitions are analogous with the definitions we worked with in Chapter 1.

```
METHOD add(self, other):
    "self" = self_omega
    "other" = other_omega
    RETURN rounded(self_omega + other_omega)

METHOD multiply(self,other):
    "self" = self_omega
    "other" = other_omega
    RETURN rounded(self_omega * other_omega)

METHOD conjugate(self):
    IF D MOD 4 == 2,3:
        RETURN "new Omega" = -omega
    ELSE IF D MOD 4 == 1:
        RETURN "new Omega" = 1-omega

METHOD norm(self):
    RETURN self_omega * conjugate(self_omega)

def trace(self):
    RETURN self_omega + conjugate of self_omega
```

For the method of conjugation we use the property $\omega_D' = \mathrm{Tr}(\omega_D) - \omega_D$, where $\mathrm{Tr}(\omega_D) = 0$ if $D \equiv 2, 3 \pmod 4$ and $\mathrm{Tr}(\omega_D) = 1$ otherwise.

As we will see, `class Omega` helps us to simplify our definitions of multiplication and conjugation (as well as norm and trace) in the quadratic integers.

We can thus proceed to define the `class Quadint`.

```
class Quadint:
"integers of the form a + b*omega_D"
INIT(self,a,b):
    "first coordinate of self" = a
    "second coordinate of self" = b
```

In `Quadint`, we identify the quadratic integer $\alpha = a + b\omega_D$ as an ordered pair (a, b). We define the methods of addition, multiplication and conjugation in the following way:

```
METHOD add(self, other):
    "self" = (a,b)
    "other" = (c,d)
    RETURN "new Quadint" = (a+b, c+d)

METHOD multiply(self, other):
    "self" = (a,b)
    "other" = (c,d)
    "e" = a*c + -"norm of omega"*b*d
    "f" = a*d + b*c + "trace of omega"*b*d
    RETURN "new Quadint" = (e,f)

METHOD conjugate(self):
    "self" = (a,b)
    RETURN "new Quadint" = (a + "trace of omega"*b,-b)
```

Here we use the `"norm of omega"` and `"trace of omega"` defined in `class Omega`. Furthermore, the definitions of multiplication and conjugation use properties of quadratic integers. Specifically

- $(a + b\omega_D)(c + d\omega_D) = ac + (ad + bc)\omega_D + bd(\omega_D)^2$, where $(\omega_D)^2 = -N(\omega_D) + \mathrm{Tr}(\omega_D)\omega_D$.
- $a + b\omega'_D = a + b(\mathrm{Tr}(\omega_D) - \omega_D)$.

Hence, we easily define trace and norm in the `class Quadint`, as in Chapter 1, using multiplication and addition of two quadratic integers defined above.

```
 METHOD norm(self):
    "new Quadint = (a,b)" = self * conjugate(self)
    RETURN round(a)

METHOD trace(self):
    "new Quadint = (a,b)" = self + conjugate(self)
    RETURN round(a)
```

Lastly we define the methods for lexicographical ordering and the $\prec$-ordering for totally positive integers.

```
METHOD lex(self, other):
    "self" = (a,b)
    "other" = (c,d)
    IF a > c:
        RETURN TRUE
    ELSE IF a = c AND b > d:
        RETURN TRUE
```

```
        ELSE:
            RETURN FALSE

    METHOD totally_positive(self):
        "self" = (a,b)
        IF a + b*"omega" > 0 AND a + b*"conjugate of omega" > 0:
            RETURN TRUE
        ELSE:
            RETURN FALSE
```

Apart from the method of trace and norm, we use every method in `class Omega` defined above in our implementation of the partition function.

2.3 Implementation - Partition function

We can proceed to develop our function for computing partition numbers. Firstly we construct functions `max(a)` and `min(a)`, which for $a \in \mathbb{N}$ output the greatest and the lowest integer b such that $a + b\omega_D \in \mathcal{O}_K^+$, respectively. For ,this we use Lemma 3, setting $b = \lfloor \frac{a}{-\omega_D'} \rfloor$ as the maximum and $b = \lceil \frac{a}{-\omega_D'} \rceil$ as the minimum.

```
    FUNCTION max(a):
        RETURN floor(a/-"conjugate of omega")

    FUNCTION min(a):
        RETURN ceil(a/-"omega")
```

Hence, we define the augmented partition function $p_K(\alpha, \beta)$:

```
    FUNCTION partition(self, other):
        "returns the number of partitions of the integer 'self'
        with the greatest part 'other'"
        SET p = 0

        IF NOT totally_positive(self) OR NOT totally_positive(other):
            RETURN p = 0

        IF lex(other,self):
            SET other = self

        FOR s -> "first coordinate of other" to 1:
            FOR t -> "min(s)" to "max(s)":
                SET beta = ("new Quadint" = (s,t))
                IF "totally_positive(self - beta)":
                    INCREMENT p by "partition(self-beta, beta)"
                ELSE IF self - beta == 0:
                    INCREMENT p by 1
        RETURN p
```

In our implementation of partition function we use the recursion (2.1) we proved in Proposition 12. It is also evident from Proposition 12, that this implementation always arrives at the correct value. Moreover, our algorithm always ends, because for any $\alpha \in \mathcal{O}_K^+$, there are only finitely many $\beta \in \mathcal{O}_K^+$ such that $\beta \leq_{LEX} \alpha$.

2.4 Tables

Hence, with our algorithm we can generate some tables of partition function values for different D. In every table we label the rows and columns with integers representing the first and the second coordinates of quadratic integers, respectively. Denoting a as the a-th row and b as the b-th column, the number in the corresponding cell (a,b) is $p_K(a + b\omega_D)$. As we know from Chapter 1, the conjugates of quadratic integers have the same number of partitions. Hence, it suffices to include quadratic integers with the second coordinate ≥ 0.

The implementation of our algorithm available at:

`https://github.com/davidstern2001/partitions_of_tot_pos_int.git`

was used to generate the following tables. They consist of partition values for $K = \mathbb{Q}(\sqrt{D})$, where $D = 2, 3$ and 5.

	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1												
2	2	1											
3	3	2	1										
4	6	4	3										
5	10	8	6	2									
6	19	16	12	6	2								
7	34	29	23	13	6								
8	62	54	44	28	16	4							
9	108	98	81	56	33	13	3						
10	190	175	149	107	69	33	12	1					
11	329	308	267	199	134	73	33	8					
12	570	538	472	365	257	153	79	28	6				
13	973	926	820	652	475	301	169	73	23	2			
14	1658	1583	1415	1151	866	577	346	172	69	16			
15	2789	2678	2412	2000	1541	1071	676	368	169	56	10		
16	4667	4497	4082	3436	2707	1945	1285	748	383	153	44	4	
17	7740	7488	6842	5838	4679	3462	2371	1458	801	368	134	29	1

Table 2.1: Partition values for $K = \mathbb{Q}(\sqrt{2})$

	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1												
2	2	1											
3	3	2											
4	6	4	2										
5	10	7	4										
6	18	14	9	3									
7	29	25	16	7	1								
8	52	45	32	16	6								
9	87	76	57	32	14	2							
10	149	133	103	64	32	10							
11	244	224	177	118	64	25	4						
12	410	378	309	215	128	57	18						
13	669	624	521	376	237	118	45	7					
14	1098	1034	878	656	432	237	103	29	2				
15	1776	1684	1448	1115	760	447	215	76	14				
16	2876	2737	2388	1876	1328	819	432	177	52	4			
17	4601	4400	3876	3105	2262	1456	819	376	133	25			
18	7349	7053	6267	5110	3811	2549	1512	760	309	87	9		
19	11633	11203	10029	8298	6317	4365	2701	1456	656	224	45		
20	18365	17722	15986	13385	10390	7373	4747	2701	1328	521	149	16	
21	28779	27839	25263	21394	16872	12263	8156	4860	2549	1115	378	76	3
22	44929	43558	39743	33992	27200	20191	13814	8570	4747	2262	878	244	32

Table 2.2: Partition values for $K = \mathbb{Q}(\sqrt{3})$

	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	2	2	2	1									
3	4	4	4	4	2								
4	8	9	10	9	8	4	2						
5	14	18	21	21	18	14	9	4	1				
6	29	36	43	46	43	36	29	18	10	4			
7	54	71	84	92	92	84	71	54	36	21	9	2	
8	106	136	166	183	191	183	166	136	106	71	43	21	8

Table 2.3: Partition values for $K = \mathbb{Q}(\sqrt{5})$

3. Specific values of the partition function

With the tables in Chapter 2, we can observe some interesting properties of the partition functions. For example, in all the tables above, we can find quadratic integers with 1, 2 and 4 partitions. Is this a special fact in $\mathbb{Q}(\sqrt{D})$ for $D = 2, 3, 5$ or does this property hold in general? In this final chapter, we will answer this particular question together with proving a number of other interesting properties and facts.

3.1 The connection to $p(n)$

We begin with the study of partitions of natural numbers in different $K = \mathbb{Q}(\sqrt{D})$, i.e. for fixed $n \in \mathbb{N}$, we look at $p_{\mathbb{Q}(\sqrt{D})}(n)$ for different D. We base our analysis on the comparison between our partition functions p_K and the integer partition function p.

Observe, that $p_K(1) = 1 = p(1)$ for all D, because 1 is always a unit in $\mathcal{O}_K^+$ and all totally positive units are indecomposable (as we proved in Proposition 6). From our tables in Chapter 2, we may think that $p_K(2) = 2 = p(2)$ for all D. It is however not as trivial. For instance, there may exist D, where $(1+\omega_D)+(1-\omega_D) = 2$. Our strategy in this case will be showing $(1+\omega_D)$ or $(1-\omega_D)$ is not totally positive for an arbitrary D. We can generalize this idea:

Theorem 13. *Let $n \in \mathbb{N}$. Then there exists $D_n > 0$ such that $\forall D \geq D_n$: $p_{\mathbb{Q}(\sqrt{D})}(n) = p(n)$. Furthermore,*

- *for $D \equiv 2, 3 \pmod 4$: $D_n = (\lfloor \frac{n}{2} \rfloor)^2$,*
- *for $D \equiv 1 \pmod 4$: $D_n = \begin{cases} (n-1)^2 & \text{if } n \text{ is even}, \\ n^2 & \text{if } n \text{ is odd}. \end{cases}$*

Proof. We first prove the existence of D_n. Let $n \in \mathbb{N}$ be fixed and let $(\lambda_1, \ldots, \lambda_m)$ be an arbitrary partition of n in K. Write $\lambda_i := a_i + b_i\omega_D$, for all $1 \leq i \leq m$. As λ_i are totally positive, we have $a_i > 0$, i.e. $a_i \in \{1, 2, \ldots n\}$ since we consider the partition of n. Let us fix the sequence of integers $(a_1, \ldots, a_m)$. We want to show that for a large enough D, it must hold that $b_i = 0 \quad \forall i \in \{1, \ldots, m\}$.
Observe that from (1.1) $b_i \in \left(\frac{a_i}{-\omega_D}, \frac{a_i}{-\omega_D'}\right)$ for all i, so we have

$$\frac{n}{-\omega_D} \leq \frac{a_i}{-\omega_D} < b_i < \frac{a_i}{-\omega_D'} \leq \frac{n}{-\omega_D'},$$

as $\omega_D' < 0$. Because $-\omega_D' \xrightarrow{D\to\infty} +\infty$ and $-\omega_D \xrightarrow{D\to\infty} -\infty$, there exists $D_n > 0$ such that $b_i \in (-1, 1)$. Thus, since b_i is an integer for all i, we have $b_i = 0$.

Now we prove the second part of the statement, i.e. determining D_n. For the sake of clarity, we denote $E_n := D_n$ for $D \equiv 2, 3 \pmod 4$ and $F_n := D_n$ for $D \equiv 1 \pmod 4$. Let us denote by $\mathcal{B}_i$ the set of values b_i can attain, if, for a fixed integer a_i, the quadratic integer $\lambda_i = a_i + b_i\omega_D$ is totally positive. Clearly $\mathcal{B}_i = \left(\frac{a_i}{-\omega_D}, \frac{a_i}{-\omega'_D}\right) \cap \mathbb{Z}$. We want to find E_n, F_n such that for an arbitrary partition $(\lambda_1, \lambda_2, \dots, \lambda_m)$ we have $\mathcal{B}_i = \{0\}$ for all $i \in \{1, \dots, m\}$, i.e. all the terms in the partition cannot have nonzero second coordinate.

Firstly, we show that, if $\mathcal{B}_i = \{0\}$ for some $1 \leq i \leq m$, then the same holds for $\mathcal{B}_j$, where $i \leq j \leq m$. That is $\{0\} = \mathcal{B}_i = \mathcal{B}_{i+1} = \cdots = \mathcal{B}_m$.

Since $\lambda_1, \dots \lambda_m$ are lexicographicaly non-increasing, we have $a_i \geq a_{i+1} \geq \cdots \geq a_m$. Thus $\{0\} = \mathcal{B}_i \supseteq \mathcal{B}_{i+1} \supseteq \cdots \supseteq \mathcal{B}_m$, which implies $\mathcal{B}_j = \{0\} \quad \forall i \leq j \leq m$.

Furthermore, if $b_2, b_3, \dots, b_m = 0$, then surely $b_1 = 0 - \sum_{i=2}^m b_i = 0$, as we partition $n = n + 0\omega_D$. Combining these two observations, we only need to find E_n, F_n, such that $\mathcal{B}_2 = \{0\}$ for all possible a_2. We can do so by finding the largest possible a_2, which can appear in the partition of n, and then computing E_n, F_n, such that $\mathcal{B}_2 = \{0\}$, respectively. The same will then hold for arbitrary a_2.

Clearly, the case where a_2 is the largest, is the case

$$n = \lambda_1 + \lambda_2 = \begin{cases} (k + b_1\omega_D) + (k + b_2\omega_D) & \text{if } n = 2k \text{ is even,} \\ ((k+1) + b_1\omega_D) + (k + b_2\omega_D) & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Otherwise, if $a_2 > k$, then the sequence (λ_1, λ_2) would not be a non-increasing sequence, and thus could not be a partition of n.

Furthermore, if n is even, then $b_1 \geq 0$ and $b_2 = -b_1 \leq 0$ (otherwise we would also not have a non-increasing sequence). Hence, in this case $\mathcal{B}_2 = \left\{\lceil \frac{a_2}{-\omega_D} \rceil, \dots, 0\right\}$. Otherwise $\mathcal{B}_2 = \left\{\lceil \frac{a_2}{-\omega_D} \rceil, \dots, \lfloor \frac{a_2}{-\omega'_D} \rfloor\right\}$ as we established above. It remains to find E_n, F_n such that $\mathcal{B}_2 = \{0\}$. We consider the cases:

- If $D \equiv 2, 3 \pmod 4$, then $\omega_D = \sqrt{D}$, thus $\frac{a_2}{\omega_D} = \frac{a_2}{\sqrt{D}} = \frac{a_2}{-\omega'_D}$. Therefore, we only have to solve the inequality $\lfloor \frac{a_2}{\sqrt{D}} \rfloor \leq \frac{a_2}{\sqrt{D}} < 1$. We have $a_2 < \sqrt{D} \Rightarrow E_n = (a_2)^2 = k^2 = (\lfloor \frac{n}{2} \rfloor)^2$.
- If $D \equiv 1 \pmod 4$, then $\omega_D = \frac{1+\sqrt{D}}{2}$. As in this case $\omega_D = \frac{1+\sqrt{D}}{2} \neq \frac{\sqrt{D}-1}{2} = -\omega'_D$, we need to consider the following possibilities:
  - if n is even, then we are solving the inequality $\lceil \frac{a_2}{-\omega_D} \rceil \geq \frac{a_2}{-\omega_D} > -1$, which is equivalent to $a_2 < \frac{1+\sqrt{D}}{2} \quad \Rightarrow F_n = (2a_2 - 1)^2 = (2k-1)^2 = (n-1)^2$.
  - if n is odd, then we have the inequality $\frac{a_2}{\omega_D} < \frac{a_2}{-\omega'_D} < 1$, that is, we only need to solve $a_2 < \frac{\sqrt{D}-1}{2} \quad \Rightarrow F_n = (2a_2+1)^2 = (2k+1)^2 = n^2$.

□

Remark. From the proof above, we can observe, that our estimates of the values E_n and F_n are optimal, i.e. they are the smallest possible values satisfying Theorem 13.

3.2 Values 2, 3 and 5

In this section we study, for which D does the partition function p_K attain the values $2, 3$ and 5. We first observe, that $p(2) = 2, p(3) = 3$ and $p(4) = 5$. Thus we can use Theorem 13 and arrive at:

Proposition 14. *Let $K = \mathbb{Q}(\sqrt{D})$. Then p_K attains the value:*

- 1 *for arbitrary D*
- 2 *for arbitrary D*
- 3 *for $D = 2, 3$ and $D > 5$*
- 5 *for $D > 5$*

Proof. The case where $p_k(\alpha) = 1$, for $\alpha \in \mathcal{O}_K^+$, we already analyzed above. For other values, we use Theorem 13. Thus

- $E_2 = 1,\ F_2 = 1 \Rightarrow p_K(2) = p(2) = 2$ for $D > 1$,
- $E_3 = 1,\ F_3 = 9 \Rightarrow p_K(3) = p(3) = 3$ for all $D > 1$ except $D = 5$,
- $E_4 = 4,\ F_4 = 9 \Rightarrow p_K(4) = p(4) = 5$ for $D > 5$.

□

Notice $p_K(3) \neq 3$ for $D = 5$ and $p_K(4) \neq 5$ for $D = 2, 3, 5$, since we see (using our algorithm) that

- $3 = 2 + 1 = (2 - \omega_5) + (1 + \omega_5) = 1 + 1 + 1 \Rightarrow p_K(3) = 4$ in $\mathbb{Q}(\sqrt{5})$.
- $4 = 3+1 = (2+\sqrt{D})+(2-\sqrt{D}) = 2+2 = 2+1+1 = 1+1+1+1 \Rightarrow p_K(4) = 6$ in $\mathbb{Q}(\sqrt{D})$, where $D = 2$ and 3.
- $4 = 3 + 1 = (3 - \omega_5) + (1 + \omega_5) = (2 + \omega_5) + (2 - \omega_5) = 2 + 2 = (2 - 2\omega_5) + (1 + \omega_5) + (1 + \omega_5) = 2 + 1 + 1 = 1 + 1 + 1 + 1 \Rightarrow p_K(4) = 9$ in $K = \mathbb{Q}(\sqrt{5})$.

Looking at the tables in Chapter 2, one can ask, if p_K attains these values (in the respective fields) at all. The remainder of this section is devoted to answering this question. Specifically, our goal is to prove the non-existence of $\alpha \in \mathcal{O}_K^+$ in a given field K, such that $p_K(\alpha) = 3, 5$.

Recall from Proposition 4 that every totally positive integer can be written as a sum of indecomposable elements. Hence, for $\alpha \in \mathcal{O}_K^+$, we can write:

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i,$$

where α_i are pairwise distinct indecomposable elements and $k_i \in \mathbb{N}$, for all $i \in \{1, \dots, r\}$. Without loss of generality, we consider the sequence $(k_1, \dots, k_r)$ to be non-increasing.

From the structure of the sum above, we can estimate the number of partitions of α. If, for instance, $\alpha = \alpha_1$, then α is indecomposable and thus $p_K(\alpha) = 1$. Furthermore, if $\alpha = \alpha_1 + \alpha_2$, we can see, that $p_K(\alpha) \geq 2$. Continuing in this sense, we classify the number of partitions of totally positive integers depending on their indecomposable sums, i.e. the sums of the form described above.

Lemma 15. *Consider the sum of indecomposables above.*

- *Let $r = 1$, then*
  - *if $\alpha = \alpha_1$, then $p_K(\alpha) = 1$,*
  - *if $\alpha = 2\alpha_1$, then $p_K(\alpha) \geq 2$,*
  - *if $\alpha = 3\alpha_1$, then $p_K(\alpha) \geq 3$,*
  - *if $\alpha = 4\alpha_1$, then $p_K(\alpha) \geq 5$,*
  - *if $\alpha = 5\alpha_1$, then $p_K(\alpha) \geq 7$,*

    $\vdots$
- *Let $r = 2$, then*
  - *if $\alpha = \alpha_1 + \alpha_2$, then $p_K(\alpha) \geq 2$,*
  - *if $\alpha = 2\alpha_1 + \alpha_2$, then $p_K(\alpha) \geq 4$,*
  - *if $\alpha = 3\alpha_1 + \alpha_2$, then $p_K(\alpha) \geq 7$,*

    $\vdots$
  - *if $\alpha = 2\alpha_1 + 2\alpha_2$, then $p_K(\alpha) \geq 9$,*

    $\vdots$
- *Let $r = 3$, then*
  - *if $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, then $p_K(\alpha) \geq 5$,*
  - *if $\alpha = 2\alpha_1 + \alpha_2 + \alpha_3$, then $p_K(\alpha) \geq 11$,*

    $\vdots$
- *For $r \geq 4$ we have $p_K(\alpha) \geq 15$.*

Proof. We will prove for example the case $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. We have

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 = (\alpha_1 + \alpha_2) + \alpha_3 = \alpha_1 + (\alpha_2 + \alpha_3) = \alpha_2 + (\alpha_1 + \alpha_3),$$

which are all distinct partitions of α, thus $p_K(\alpha) \geq 5$. The rest of the statements we get by straightforward checking. □

We use this observation to arrive at a characterization of elements with a specific number of partitions.

Example.
- If $\alpha \in \mathcal{O}_K^+$ has exactly one indecomposable sum (i.e. sum of indecomposables): $\alpha = \alpha_1 + \alpha_2$, then $p_K(\alpha) = 2$.
- If $\alpha = 3\alpha_1 = \beta_1 + \beta_2$ are the only indecomposable sums of α, where $\alpha_1, \beta_1, \beta_2$ are distinct, then $\alpha = 2\alpha_1 + \alpha_1 = \alpha_1 + \alpha_1 + \alpha_1 = \beta_1 + \beta_2$ are all of the partitions of α. Hence $p_K(\alpha) = 4$.

The main idea of our proof is to characterize the totally positive elements α, for which $p_K(\alpha) = 3, 5$; by their sums into indecomposables. This will help us to prove their non-existence in the fields $\mathbb{Q}(\sqrt{D})$, where $D = 2, 3, 5$.

Firstly, to simplify our characterization, we need to make an observation:

Lemma 16. *If $\alpha = k\alpha_1 = l\beta_1$, then $k = l$ and $\alpha_1 = \beta_1$*

Proof. Without loss of generality let $k \geq l$. Clearly $l \mid k$, otherwise $\beta_1 = \frac{k}{l}\alpha_1 \notin \mathcal{O}_K$. Thus we have $\beta_1 = m\alpha_1$ for $m \in \mathbb{N}$. If $m = 1$, we are done. Otherwise we get $\beta_1 = \alpha_1 + (m-1)\alpha_1$. However, we assumed β_1 is indecomposable, so this cannot occur. □

Thus we can finally characterize the decompositions.

Lemma 17. *Let $K = \mathbb{Q}(\sqrt{D})$, $\alpha \in \mathcal{O}_K^+$.*

1. *$p_K(\alpha) = 3$ if and only if it has one of the following forms:*
   - $\alpha = 2\alpha_1 = \beta_1 + \beta_2$,
   - $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2$,
   - $\alpha = 3\alpha_1$.
2. *$p_K(\alpha) = 5$ if and only if it has one of the following forms:*
   - $\alpha = 2\alpha_1 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2 = \delta_1 + \delta_2$,
   - $\alpha = 2\alpha_1 = 2\beta_1 + \beta_2$,
   - $\alpha = 3\alpha_1 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2$,
   - $\alpha = 4\alpha_1$,
   - $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2 = \delta_1 + \delta_2$,
   - $\alpha = \alpha_1 + \alpha_2 = 2\beta_1 + \beta_2$,
   - $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

We assume all the indecomposables in each expression of α are distinct and each α has no other decompostition.

Proof. We see from Lemma 15 that the only combinations of sums into indecomposables of α can be only of these forms (given $p_K(\alpha)$). We excluded all the forms $\alpha = k\alpha_1 = l\beta_1$, since from Lemma 16 they are not distinct. □

We proceed with a powerful lemma, which eliminates the majority of expressions of α we need to take into consideration.

Lemma 18. *Let $K = \mathbb{Q}(\sqrt{D})$, $\alpha = \alpha_1 + \alpha_2 \in \mathcal{O}_K^+$ be the sum into indecomposables.*

- *If $D = 5$, then $p_K(\alpha) \neq 3, 5$.*
- *If $D = 2, 3, 5$, then $p_K(\alpha) \neq 5$.*

Proof. Our proof is based on the observation that in each K there are only finitely many fundamental indecomposable elements. By "fundamental" we mean, there is a finite set $\Phi_K = \{\phi_1, \phi_2, \ldots, \phi_m\}$ such that for every $\alpha \in \mathcal{O}_K^+$ indecomposable, there exists a unit $\varepsilon_+^n \in \mathcal{O}_K^+$, $n \in \mathbb{Z}$, such that $\alpha = \varepsilon_+^n \phi_i$, for some $i \in \{1, \ldots, m\}$. This fact is due to A. Dress and R. Scharlau [DS82], who proved there are only finitely many unique indecomposables up to conjugation and multiplication by ε_+. For reference, see [BK18, Section 2.]

Thus we have for $\alpha_1 = \varepsilon_+^k \phi_i$, $\alpha_2 = \varepsilon_+^l \phi_j$, where $k, l \in \mathbb{Z}$ and $i, j \in \{1, \ldots, m\}$. Therefore, we can get the associated form:

$$\varepsilon_+^{-k}\alpha = \varepsilon_+^{-k}(\alpha_1 + \alpha_2) = \varepsilon_+^{k-k}\phi_i + \varepsilon_+^{l-k}\phi_j = \phi_i + \varepsilon_+^n \phi_j, \quad \text{where } n := l - k.$$

Therefore, using Proposition 9 we arrive at the fact, that $p_K(\alpha) = p_K(\phi_i + \varepsilon_+^n \phi_j)$ for some $n \in \mathbb{Z}; i, j \in \{1, \ldots, m\}$. Hence, it suffices to check, that every totally positive element of the form $\phi_i + \varepsilon_+^n \phi_j$ cannot have 3 or 5 partitions in the respective fields above.

We regrettably cannot use an inductive argument, as for all $i, j \in \{1, \ldots, m\}$ we have $\phi_i + \varepsilon_+^k \phi_j \not\prec \phi_i + \varepsilon_+^l \phi_j$ for $k \leq l$, because $\varepsilon_+^l - \varepsilon_+^k$ cannot be totally positive, as otherwise ε_+^l would not be indecomposable (this is impossible, as all totally positive units are indecomposable from Proposition 6). Therefore, in the rest of the proof, we proceed by checking each possibility for $\phi_i + \varepsilon_+^n \phi_j$.

We begin with the case, where $D = 5$. By analyzing the continuous fraction of $\omega_5 = \frac{1+\sqrt{5}}{2} = [1, \bar{1}]$, we deduce that the only fundamental indecomposible element in $K = \mathbb{Q}(\sqrt{5})$ is 1, i.e. $\Phi_K = \{1\}$. Hence, we have exactly one form for $\phi_i + \varepsilon_+^n \phi_j$: $1 + \varepsilon_+^n$, where $n \in \mathbb{Z}$ and $\varepsilon_+ = 1 + \omega_5 = \frac{3+\sqrt{5}}{2}$. Observe, that from Proposition 9 we have $p_K(1 + \varepsilon_+^n) = p_K((1 + \varepsilon_+^n)') = p_K(1 + \varepsilon_+^{-n})$. Thus, it suffices to check only the partitions for $n \geq 0$.

- If $n \in \{0, 1, 2, 3, 4, 5\}$, we can check by our algorithm, that

$$p_K(\alpha) \in \{2, 4, 10, 54, 753\}, \text{ i.e. } p_K(\alpha) \neq 3, 5.$$

- If $n > 5$, then we claim, that $p_K(\alpha) > 5$.

Furthermore, we observe, that $\forall 0 < k \leq n$ we have $\varepsilon_+^k \prec 1 + \varepsilon_+^n$.

Clearly $\varepsilon_+^k < 1 + \varepsilon_+^n$. Also $(\varepsilon_+^k)' = \varepsilon_+^{-k} < 1 < 1 + \varepsilon_+^{-n} = (1 + \varepsilon_+^n)'$, as $0 < \varepsilon_+^{-k} < 1$ for each $k \geq 0$. Moreover, we claim, that each ε_+^k gives a unique partition of $1 + \varepsilon_+^n$. We cannot have the same partition for different $k, l < n$, as we have

$$\varepsilon_+^k + \varepsilon_+^l \leq 2\varepsilon_+^{n-1} < \varepsilon_+^n < 1 + \varepsilon_+^n,$$

where we used the estimate: $\varepsilon_+ = \frac{3+\sqrt{5}}{2} > 2$. Consequently $\varepsilon_+^k + \varepsilon_+^l = 1 + \varepsilon_+^n$ if and only if $k = n$ and $l = 0$.

Thus every integer of the form $1 + \varepsilon_+^n$ has at least n partitions. Specifically $p_K(\alpha) > 5$ for $n > 5$. This establishes the proof, that there does not exist a totally positive integer $\alpha = \alpha_1 + \alpha_2$ in $\mathbb{Q}(\sqrt{5})$, such that $p_K(\alpha) = 3$ or 5.

We continue with the case: $D = 2$. From the continuous fraction of $\sqrt{2} = [1, \bar{2}]$, the fundamental indecomposable elements are $\{1, (2 + \sqrt{2})\}$, where the fundamental totally positive unit is $\varepsilon_+ = 3 + 2\sqrt{2}$. This gives us the following four possibilities:

$$\phi_i + \varepsilon_+^n \phi_j \in \{1 + \varepsilon_+^n, (2 + \sqrt{2}) + \varepsilon_+^n, 1 + (2 + \sqrt{2})\varepsilon_+^n, (2 + \sqrt{2}) + (2 + \sqrt{2})\varepsilon_+^n\},$$

where $n \in \mathbb{Z}$. We can further reduce these possibilities using Proposition 9.

- $p_K(1 + \varepsilon_+^n) = p_K((1 + \varepsilon_+^n)') = p_K(1 + \varepsilon_+^{-n})$,

- $p_K((2+\sqrt{2})+\varepsilon_+^n) = p_K(((2+\sqrt{2})+\varepsilon_+^n)') = p_K((2-\sqrt{2})+\varepsilon_+^{-n}) = p_K(\varepsilon_+^{-1}(2+\sqrt{2}) + \varepsilon_+^{-n}) = p_K((2+\sqrt{2}) + \varepsilon_+^{-n+1})$,
- $p_K(1 + (2+\sqrt{2})\varepsilon_+^n) = p_K((2+\sqrt{2}) + \varepsilon_+^{-n})$,
- $p_K((2+\sqrt{2}) + (2+\sqrt{2})\varepsilon_+^n) = p_K(((2+\sqrt{2}) + (2+\sqrt{2})\varepsilon_+^n)') = p_K((2-\sqrt{2}) + (2-\sqrt{2})\varepsilon_+^{-n}) = p_K((2+\sqrt{2}) + (2+\sqrt{2})\varepsilon_+^{-n})$.

Here we used $(2-\sqrt{2}) = \varepsilon_+^{-1}(2+\sqrt{2})$. In all the items above, we showed, that it suffices to check the number of partitions only for $n \geq 0$. Also, we see, that the second and the third possibility are equivalent. Observe, that we can use the fact $\forall 1 \leq k \leq n:\ \varepsilon_+^k \prec 1 + \varepsilon_+^n$ we proved above for the case $D = 5$, since we used a general argument. Furthermore, it holds that $\forall 1 \leq k < n:\ (2+\sqrt{2})\varepsilon_+^k \prec 1 + \varepsilon_+^n$, as

$$(2+\sqrt{2})\varepsilon_+^k < 1 + (3+2\sqrt{2})\varepsilon_+^k \leq 1 + \varepsilon_+^n,$$
$$(2-\sqrt{2})\varepsilon_+^{-k} < 1 < 1 + \varepsilon_+^{-n},$$

since $0 < \varepsilon_+^{-k} < 1$ for all $k \geq 1$. We already proved above, that all ε_+^k give unique partitions of $1 + \varepsilon_+^n$. The same can be said about the terms $(2+\sqrt{2})\varepsilon_+^k$, since for $k, l < n$:

$$\varepsilon_+^k + (2+\sqrt{2})\varepsilon_+^l < (3+2\sqrt{2})\varepsilon_+^{n-1} = \varepsilon_+^n < 1 + \varepsilon_+^n,$$
$$(2+\sqrt{2})\varepsilon_+^k + (2+\sqrt{2})\varepsilon_+^l < (12+8\sqrt{2})\varepsilon_+^{n-2} < \varepsilon_+^n < 1 + \varepsilon_+^n, \quad \text{if } l \leq n-2.$$

Hence, the totally positive integer $(2+\sqrt{2})\varepsilon_+^k$ gives a unique partition of $1 + \varepsilon_+^n$ for all $1 \leq k < n$. The same argument can be similarly applied to the other two cases: $(2+\sqrt{2}) + \varepsilon_+^n$ and $(2+\sqrt{2}) + (2+\sqrt{2})\varepsilon_+^n$.

We thus proved that for all $n \geq 0:\ p_K(\phi_1 + \varepsilon_+^n\phi_2) \geq 2n - 1$. Hence, we have to only check by our algorithm $p_K(\phi_1 + \varepsilon_+^n\phi_2)$ for $n \in \{0, 1, 2, 3\}$. We get:

a) $p_K(1 + \varepsilon_+^n) \in \{2, 3, 19, 1732\}$,

b) $p_K((2+\sqrt{2}) + \varepsilon_+^n) \in \{2, 8, 301\}$,

c) $p_K((2+\sqrt{2}) + (2+\sqrt{2})\varepsilon_+^n) \in \{3, 6, 34067\}$.

This completes the proof that $\alpha = \alpha_1 + \alpha_2$ cannot have 5 partitions in $K = \mathbb{Q}(\sqrt{2})$.

In the last case: $D = 3$, we have $\omega_3 = \sqrt{3} = [1, \overline{1, 2}]$, thus the fundamental indecomposable elements in $\mathbb{Q}(\sqrt{3})$ are $\Phi_K = \{1\}$ and the fundamental unit is $\varepsilon_+ = 2 + \sqrt{3}$. Similarly to the case $D = 5$, we only check the possibilities for $n \in \{0, 1, \ldots, 5\}$:

$$p_K(1 + \varepsilon_+^n) \in \{2, 6, 32, 1098, 377024\}, \text{ i.e. } p_K(\alpha) \neq 3, 5.$$

This finishes our proof. $\square$

Lemma 19. *1. Let $K = \mathbb{Q}(\sqrt{5})$. Then $\nexists \alpha \in \mathcal{O}_K^+$ such that $p_K(\alpha) = 3$.*

2. Let $K = \mathbb{Q}(\sqrt{D})$, where $D = 2, 3, 5$. Then $\nexists \alpha \in \mathcal{O}_K^+$ such that $p_K(\alpha) = 5$.

Proof. The proof uses Lemma 16 above. We first characterize the elements α, for which $p_K(\alpha) = 3, 5$; by their sum into indecomposables. We then proceed by showing that such totally positive elements cannot exist in a given field.

Let us prove the first statement. In Lemma 16 above, we characterized totally positive elements α, where $p_K(\alpha) = 3$. We omit all cases, where $\alpha = \alpha_1 + \alpha_2$, as we have studied them in the previous Lemma 18. Therefore, the only remaining instance is $\alpha = 3\alpha_1$. We know, that the continued fraction for $\frac{1+\sqrt{5}}{2}$ is $[1, \overline{1}]$. Hence, using Theorem 7, we have $\alpha = 3\alpha_1$ is not uniquely decomposable, i.e. such α with only one decomposition $3\alpha_1$ cannot exist.

The second statement we prove analogously. Firstly, we have the characterization of α, such that $p_K(\alpha) = 5$. Of course, we omit the cases discussed in Lemma 18). This leaves us with the following possibilities:

- $\alpha = 4\alpha_1$,
- $\alpha = 2\alpha_1 = 2\beta_1 + \beta_2$,
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

Using Theorem 7, we see, that the first and third cases cannot occur, since elements of these forms cannot be uniquely decomposable in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ or in $\mathbb{Q}(\sqrt{5})$ (we look at the continuous fraction of $\sqrt{2}, \sqrt{3}$ and $\frac{1+\sqrt{5}}{2}$).

Let us look at the remaining case, i.e. where $\alpha = 2\alpha_1$. From Proposition 9 we can multiply α by a unit and the number of partitions will stay the same. Thus we find a totally positive unit ε such that $\varepsilon\alpha_1$ is a fundamental indecomposable element (see the proof of Lemma 18 for reference). We consider the cases:

- If $K = \mathbb{Q}(\sqrt{2})$, then we can find a unit ε such that $\varepsilon\alpha_1 = 1$ or $\varepsilon\alpha_1 = 2+\sqrt{2}$. Hence

$$p_K(\alpha) = \begin{cases} p_K(2) = 2 \\ p_K(4 + 2\sqrt{2}) = 3 \end{cases}$$

- If $K = \mathbb{Q}(\sqrt{3})$, then we can find a unit ε such that $\varepsilon\alpha_1 = 1$. Hence

$$p_K(\alpha) = p_K(2) = 2$$

- If $K = \mathbb{Q}(\sqrt{5})$, then we can similarly find a unit ε such that $\varepsilon\alpha_1 = 1$. Then

$$p_K(\alpha) = p_K(2) = 2.$$

In summary, the only remaining case above also cannot occur in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ or in $\mathbb{Q}(\sqrt{5})$. This finishes the proof. □

Theorem 20. *Let $K = \mathbb{Q}(\sqrt{D})$. Then p_K attains the value:*

- 1 *for all D,*
- 2 *for all D,*
- 3 *if and only if $D \neq 5$,*
- 5 *if and only if $D \neq 2, 3, 5$.*

Proof. This result is a direct consequence of Proposition 14 and Lemma 19. □

3.3 Values 4 and 6

After we analyzed values 1,2,3 and 5, we may ask, whether there is a characterization of D, for which $p_{\mathbb{Q}(\sqrt{D})}$ attains the values 4 and 6. This question is very interesting, as these values do not appear as the number of partitions of natural numbers. This section looks into this problem in more detail.

Proposition 21. *Let* $K = \mathbb{Q}(\sqrt{D})$, $\alpha := \sigma_D + 3$, *then* $p_K(\alpha) = 4$.

Proof. Let D be arbitrary. Recall, that $\sigma_D := \omega_D + \lfloor -\omega'_D \rfloor$, thus we have $\alpha = (\lfloor -\omega'_D \rfloor + 3) + \omega_D$. For abbreviation, we denote $a := (\lfloor -\omega'_D \rfloor + 3)$. We then see $p_K(\alpha) \geq 4$, since

$$\alpha = (a-1) + \omega_D = ((a-2) + \omega_D) + 2 = ((a-2) + \omega_D) + 1 + 1,$$

where all the elements (including α) are totally positive. We prove this by showing for $k \geq 0$: $((a-k) + \omega_D) \succ 0$ if and only if $0 \leq k \leq 2$.

We clearly see from the definition of the indecomposable elements, that for $k \geq 0$: $((a-k) + \omega_D) \succ 0$ if and only if $a - k > -\omega'_D$, which is equivalent to $3 + \lfloor -\omega'_D \rfloor + \omega'_D > k$, where $\lfloor -\omega'_D \rfloor + \omega'_D \in (-1, 0)$. Thus $k \in \{0, 1, 2\}$.

Hence, we showed that if $\beta = k \in \mathcal{O}_K^+$ is in the partition of α, then $\beta \in \{1, 2\}$. For the proposition to hold, it suffices to show there does not exist a term $\beta = b + c\omega_D$ in the partition of α, where $c > 1$ or $c < 0$. We use the fact, that $\beta \in \mathcal{O}_K$ is in the partition of α, if $\beta \in \mathcal{O}_K^+$ and $\alpha - \beta \in \mathcal{O}_K^+$.

We first use that $\beta \in \mathcal{O}_K^+$. Let us first consider the case, where $c > 1$. We want $\beta \in \mathcal{O}_K^+$. Thus, from the estimation (1.1) we proved in Lemma 3, we have:

$$1 < c < \frac{b}{-\omega'_D} \leq \frac{a-1}{-\omega'_D} = \frac{\lfloor -\omega'_D \rfloor + 2}{-\omega'_D},$$

where the left hand side is < 2 for all D, except $D = 2, 5, 13$. Thus, since c is an integer, there cannot be β of this form. We will analyze the cases for $D = 2, 5, 13$ separately.

Analogously for $c < 0$, we have:

$$\frac{\lfloor -\omega'_D \rfloor + 2}{-\omega_D} = \frac{a-1}{-\omega_D} \leq \frac{b}{-\omega_D} < c < 0,$$

where the right hand side is > -2 for all D, except $D = 2$. Thus, (for $D \neq 2$) the only possibility for $c < 0$ is $c = -1$. Then, however, in order that β is in the partition of α, $\alpha - \beta$ must also be totally positive. Hence, we have $\alpha - \beta = (a-b) + 2\omega_D \succ 0$, which holds, if $a - b > -2\omega'_D$. This is equivalent to $2 + \omega'_D > b$. Thus, we have the inequality $0 < b < 2 + \omega'_D$, which does not have a solution for all D, except $D = 5$. Hence, we proved that, for all $D \neq 5$, there cannot be such β in the partition of α.

By our implementation of the algorithm described in Chapter 2 we can easily verify the individual cases, where $D = 2, 5, 13$; and thus completing our proof. ☐

Thus, we can always find an element α in K such that $p_K(\alpha) = 4$. For the number 6, we employ a similar strategy. We, however, find, that it is more intricate than the previous case.

For simplicity of notation, we denote $\xi_D := -\omega'_D = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{\sqrt{D}-1}{2} & \text{if } D \equiv 1 \pmod 4. \end{cases}$

Proposition 22. *Let $K = \mathbb{Q}(\sqrt{D})$, $D \neq 5$.*

1. *If $\xi_D - \lfloor \xi_D \rfloor > \frac{1}{2}$, then for $\alpha := 2\sigma_D + 4$: $p_K(\alpha) = 9$*
2. *If $\xi_D - \lfloor \xi_D \rfloor < \frac{1}{2}$, then for $\alpha := 2\sigma_D + 3$: $p_K(\alpha) = 6$*

Proof. We observe, that α in both statements is indeed totally positive. This is easily seen from the definition of σ_D. Also observe, we can unify both cases into one, as we have

$$\alpha = (\lfloor 2\xi_D \rfloor + 3) + 2\omega_D = \begin{cases} (2\lfloor \xi_D \rfloor + 4) + 2\omega_D = 2\sigma_D + 4 & \text{if } \xi_D - \lfloor \xi_D \rfloor > \frac{1}{2}, \\ (2\lfloor \xi_D \rfloor + 3) + 2\omega_D = 2\sigma_D + 3 & \text{if } \xi_D - \lfloor \xi_D \rfloor < \frac{1}{2}, \end{cases}$$

because $\lfloor 2\xi_D \rfloor = 2\lfloor \xi_D \rfloor + 1$ if $\xi_D - \lfloor \xi_D \rfloor > \frac{1}{2}$, and $\lfloor 2\xi_D \rfloor = 2\lfloor \xi_D \rfloor$ otherwise. Thus, for a few of the following steps of the proof, we let $D > 0$ be arbitrary.

Now we prove, that (for almost all D) if $\beta = a + b\omega_D$ is in a partition of α, then $b \in \{0, 1, 2\}$. Our proof of this fact is analogous to the proof in Proposition 21. Clearly, if $b > 2$, then from Lemma 3

$$2 < b < \frac{a}{\xi_D} \leq \frac{\lfloor 2\xi_D \rfloor + 2}{\xi_D},$$

where the right hand side is < 3 for all $D \neq 5, 13, 17$. Thus, (for $D \neq 5, 13, 17$) no such b or, more precisely, no such β cannot exist, since we require β to be totally positive. Analogously, if $b < 0$, then $\alpha - \beta = (\lfloor 2\xi_D \rfloor + 4 - a) + (2 - b)\omega_D$. We want $\alpha - \beta$ to be totally positive. This holds, if $(\lfloor 2\xi_D \rfloor + 4 - a) + (2 - b)\omega_D' > 0$, which can be rewritten to

$$3 \leq 2 - b < \frac{(\lfloor 2\xi_D \rfloor + 3 - a)}{\xi_D} \leq \frac{\lfloor 2\xi_D \rfloor + 2}{\xi_D}.$$

Here, similarly to the argument above, the right hand side < 3 for all $D \neq 5, 13, 17$. This proves our observation. We will now assume, that $D \neq 5, 13, 17$. We will come back to these cases later.

We proved, that the only terms in the partition of α are of the form a, $a + \omega_D$, $a + 2\omega_D$, where $a > 0$. We can write $a = k$, $a + \omega_D = \sigma_D + k$, $a + 2\omega_D = 2\sigma_D + k$ for an appropriate choice of $k \in \mathbb{N}$ in each case. We first note, which terms of the form above are totally positive, and then we exhaust all the options of the partitions of α. Observe, that

- $k \succ 0$ for all $k > 0$.
- $\sigma_D + k \succ 0$ if and only if $k \geq 1 > \xi_D - \lfloor \xi_D \rfloor$
- $2\sigma_D + k \succ 0$ if and only if $k > 2(\xi_D - \lfloor \xi_D \rfloor)$, that is
  - if $\xi_D - \lfloor \xi_D \rfloor > \frac{1}{2}$, then the condition holds if and only if $k \geq 2$,
  - if $\xi_D - \lfloor \xi_D \rfloor < \frac{1}{2}$, then the condition holds if and only if $k \geq 1$.

Hence, if $\xi_D - \lfloor \xi_D \rfloor > \frac{1}{2}$, then we have

$$\begin{aligned}\alpha &= (2\sigma_D + 3) + 1\\ &= (2\sigma_D + 2) + 2\\ &= (2\sigma_D + 2) + 1 + 1\\ &= (\sigma_D + 3) + (\sigma_D + 1)\\ &= (\sigma_D + 2) + (\sigma_D + 2)\\ &= (\sigma_D + 2) + (\sigma_D + 1) + 1\\ &= (\sigma_D + 1) + (\sigma_D + 1) + 2\\ &= (\sigma_D + 1) + (\sigma_D + 1) + 1 + 1.\end{aligned}$$

These are (as we analyzed above) all the partitions of α. Thus $p_K(\alpha) = 9$.

Analogously, if $\xi_D - \lfloor \xi_D \rfloor < \frac{1}{2}$, then we get

$$\begin{aligned}\alpha &= (2\sigma_D + 2) + 1\\ &= (2\sigma_D + 1) + 2\\ &= (\sigma_D + 2) + (\sigma_D + 1)\\ &= (2\sigma_D + 1) + 1 + 1\\ &= (\sigma_D + 1) + (\sigma_D + 1) + 1.\end{aligned}$$

Hence, we have $p_K(\alpha) = 6$.

For $D = 13, 17$ we can check by our algorithm that:

- $p_K(\alpha) = p_K(2\sigma_{13} + 3) = p_K(5 + 2\omega_{13}) = 6$ and $\xi_{13} - \lfloor \xi_{13} \rfloor < \frac{1}{2}$,
- $p_K(\alpha) = p_K(2\sigma_{17} + 4) = p_K(6 + 2\omega_{17}) = 9$ and $\xi_{17} - \lfloor \xi_{17} \rfloor > \frac{1}{2}$.

This finishes our proof. □

Remark. For $D = 5$, we have (by the computation of our algorithm):

$$p_K(\alpha) = p_K(2\sigma_5 + 4) = p_K(4 + 2\omega_5) = 10, \quad \text{since } \xi_5 - \lfloor \xi_5 \rfloor > \frac{1}{2}.$$

This is because $(2 + \omega_5) + (2 - \omega_5)$ is also a valid partition of α.

We can thus conclude with:

Theorem 23. *Let $K = \mathbb{Q}(\sqrt{D})$. Then $p_K(\alpha)$ attains the value:*

- *4 for all D,*
- *6 for all D, such that $\xi_D - \lfloor \xi_D \rfloor < \frac{1}{2}$.*

Proof. Both statements are direct consequences of Proposition 21 and Proposition 22. □

Observe, that for the attainability of the value 6 we do not have a complete characterization, but only a sufficient condition. It, however implies, that $p_K(\alpha)$ attains 6 for infinitely many D. The first values of D satisfying this condition are: $2, 6, 10, 11, 13, 15, 19, 26, 29, \ldots$.

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