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Report on the doctoral thesis "Nonlinear classes of mappings: properties and approximation" by Anna Doležalová

In summary:

I think this is a very nice thesis significantly exceeding average expectations in breadth, depth, originality and potential impact of results. While all associated papers were written in cooperation with co-authors, I am convinced that the personal contributions of Mrs. Doležalová were substantial.

Detailed report:

This thesis covers selected topics in what could be called genuinely nonlinear analysis: properties of functions in classes that, while typically non-compact subsets of well-understood Banach spaces, are neither linear nor convex. This means that very many classical techniques developed to obtain existence and properties of solutions of PDE's or variational problems fail if the problem is constrained to such a function class: In one way or another, important standard methods, for instance the use of cut-off functions for localization, mollification by convolution and many other ways of manipulating admissible functions, require a set up in a linear space or a convex subset thereof, without additional constraints further restricting the admissible class. In particular, there is no really general approach to such problems; instead, properties of the concrete function class have to be studied and exploited in detail.

The predominant examples studied in the thesis are vector valued "deformation" maps $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, weakly differentiable in the Sobolev space $W^{1,p}$, that also satisfy certain local or global (almost-everywhere) injectivity properties. These play a crucial role in continuum mechanics, in models of deformable solids where states are encoded by the map describing the deformation of the solid from its original "reference" shape Ω to its actual state. There, injectivity corresponds to the natural physical requirement that matter should not interpenetrate. In the models for nonlinear elastic solids mathematically pioneered by John Ball starting from the late 1970's and subsequently studied by many other authors, the appropriate ambient space is the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^3)$, with some $p \in (1,\infty)$: weakly differentiable functions with gradient in L^p. To see some of the difficulties encountered, consider for instance the set I of almost everywhere injective deformations $y \in W^{1,p}(\Omega;\mathbb{R}^3)$ which are also orientation preserving in the sense that $\det \nabla f > 0$ a.e.: $I \subset W^{1,p}$ is not open, not dense, not convex, and not smooth. However, it does have non-empty interior in $C^1 \cap W^{1,p}$ if we use the stronger topology of C^1. That already illustrates how complicated

I is from an abstract topological point of view, and how it and similiar classes can depend in a very sensitive way on the regularity of their ambient space.

If $p \geq 3$ (the dimension), I is weakly sequentially closed as was show by Ciarlet and Nečas (for $p > 3$) in 1987, a crucial ingredient to be able to find minimizers of variational problems in this class by the direct method. While the restriction $p \geq 3$ is sharp, similar useful results are available in the literature also in the range $2 < p < 3$ for various related but slightly more restricted function classes, for instance functions satisfying the Müller-Spector condition (INV). Still, we do not yet understand this subject comprehensively, especially for the lower critical case $p = 2$ (or $p = n - 1$ in dimension $n \neq 3$).

This case has become a focus of recent active research with positive results so far mostly limited to dimension $n = 2$ for classes of Sobolev homeomorphism and their weak closure, for instance by Iwaniec an Onninen (2017). As one of the main underlying themes of her thesis, Mrs. Doležalová has presented a significant contribution to the ongoing research effort in this topic for dimension $n \geq 3$, providing both counterexamples and positive results. Here, it should be pointed out that the case $p = 2$ is of more than just academic interest. It is the suitable choice for variational problem with functionals exhibiting quadratic growth, and this includes widely used models of linear elasticity. Mathematically, these have been justified only asymptotically in the small strain regime (deformations converging to the identity). Practically, however, they are used for simulation for moderately large strains, where injectivity constraints are practically relevant but so far either ignored, treated in purely heuristic fashion or under unrealistically strong extra regularity assumptions. In this context, the related results of the thesis (in Paper III and IV) are a major step ahead.

To my knowledge, all main theorems in each of the five paper comprising the thesis are new, and auxiliary results drawn from the existing literature are properly quoted.

Paper I is a refinement of an earlier paper by Kauhanen (2002) and presents detailed constructions of surprisingly badly behaved Sobolev homeomorphisms just below the first critical threshold $p = n$ (in dimension n): they can map sets of measure zero to sets of positive measure (i.e., violate Lusin's property (N)), and it is not possible to bound the dimension of such a bad set in any way.

Paper II is the only one among the five which studies a function class not given by Sobolev homeomorphisms or closures thereof, instead looking a generalized notion of functions of finite distortion in $W^{1,n}$ (in dimension n), given by a pointwise constraint for ∇f: $|\nabla f(x)|^n \leq K(x) J_f + \Sigma(x)$ with given functions K, Σ. Here, $J_f := \det \nabla f$ is allowed to change sign, and injectivity of f is not expected. Possible applications include the regularity of solutions of mean-curvature type PDEs. It aims at a fusion of known results on continuity of functions of finite distortion on the one hand ($\Sigma = 0$), and on the other hand, functions with gradient in suitable Orlicz spaces just slightly smaller than the critical space L^n, just enough for embedding into the continuous functions ($K = 0$ and Σ in such an Orlicz space). As I am not an expert on the geometric equations presented as a motivation, it is hard for me to tell to what the degree the results of this paper are applicable in a natural broader context. Still, at the very least, this paper is an interesting theoretical contribution a bit in the flavor of classical functional analysis for critical embeddings, mostly relying on careful analytical estimates in its proofs.

The remaining three papers all study classes of weak closures of Sobolev homeomorphisms in $W^{1,n-1}$ in dimension $n \geq 2$, with $n = 3$ being the most relevant practical example. These classes are very natural choices for admissible sets of deformations of solids. From the point of view of my personal research interests, this part presented in **Paper III-V** contains the most important results of the thesis. While these papers are not yet published, I have no doubt that they will find their place in the literature.

Paper III shows that condition (INV) is weakly closed and thus can indeed be used as a viable injectivity constraint for variational problems in $W^{1,n-1}$, as long as there is sufficient control of the distortion of f (III, Thm. 3.1). Here, this control is meant to be provided by a bound on the functional to be minimized, and many hyperelastic energies do have this property. In particular, the phenomenon of the counter-example in the seminal paper of Conti and De Lellis (2003) (or the new example in III, Thm. 1.2) can be avoided by enforcing a sufficiently strong bound on the distortion. For nonlinear elastic functionals coercive in $W^{1,2}$ in dimension $n = 3$, this is the first directly applicable result that is one the one hand compatible with the proof of existence of minimizers via a direct method, and on the other hand provides the strong local and global invertibility property (INV) – non-interpenetration of mass – of all admissible deformations. In particular, it can be seen as a significant addition to the results of Müller, Tang and Yan (1994), the most general somewhat comparable paper that was available so far. The results and the new counter-example are also used to show that weak and strong closures of the set of Sobolev homeomorphisms in $W^{1,n-1}$ do not have to coincide in dimension $n = 3$ (III, Thm. 1.3), a fundamental difference to dimension $n = 2$.

Paper IV contains a related result for yet another case, where distortion control is replaced by a control of J_f (local changes of volume) and $\operatorname{cof} \nabla f$ (the cofactor matrix of $(n-1)$-dimensional subdeterminants of ∇f that in particular controls local changes of surface area), combined with Lusin's condition (N) *imposed* on the admissible class of Sobolev homeomorphisms. Again, using the weak closure of this class as an admissible set of deformations in $W^{1,n-1}$ is shown to be a viable way of enforcing an injectivity constraint which implies condition (INV) and also preserves Lusin's (N) (IV, Thm 1.1). In addition, for a suitable, quite general class of functionals on nonlinear elastic type, weak lower semi-continuity on this admissible set and the one of Paper III is shown, and the existence of minimizers is proved in detail. The sharpness of the results is again demonstrated by new counter-examples. For me, the observation that one can directly include Lusin's (N) in the admissible class for better invertibility properties without loosing the well-posedness of associated variational problems is a particularly interesting and original contribution of this paper.

A crucial novel part at the heart of the proofs in **Paper III and IV** is the precise treatment of possible "bubbles" of mass forming on interior surfaces that in principle could develop along a weakly converging sequence of homeomorphisms. These are the main obstacle that could cause a loss of condition (INV) in the limit and need to be defined and estimated precisely enough to be ruled out. To mathematically describe such bubbles, the proofs use a comparison between the given function and smoother extensions (obtained by minimizing a surface Dirichlet energy subject to given boundary conditions) from their values on a well-chosen surface grid, where in particular continuity of the map can be assured. The core idea then is the observation that by the assumed energy bound, tiny local surface pieces cannot stretch to arbitrarily large area after deformation: the deformed local surface area cannot concentrate along the sequence as it is controlled by the equi-integrable cofactors of the sequence. This ultimately prevents the formation of bubbles from arbitrarily small surface pieces. The necessary estimates to implement this idea rigorously employ a whole arsenal of refined topological and analytic tools in concert, including a generalized notion of topological degree suitable for maps in $W^{1,n-1} \cap L^\infty$, optimal versions of change-of-variables and area formulas for maps of low regularity and the isoperimetric inequality for sets of finite perimeter. Apart from being remarkable results in themselves, I think there is a realistic chance that the theorems of Paper III and IV and the ideas developed for their proofs will contribute to a better understanding of the role of linearized elasticity in presence of injectivity constraints in future research.

Finally, **Paper V** proves a fine property, namely differentiability a.e., for weak limits of homeomorphisms similar to the classes studied in Paper III and IV. (Unlike before, a control of the inverse gradient is now assumed along the sequences, but this class is comparable to the previous ones.) For the considered Sobolev homeomorphisms themselves, this property is known, but extending the results to functions in the weak closure is important because this is where minimizers of variational problems can naturally be found. Again, condition (INV) and Lusin's (N) play a major role, as well as a weak form of a non-smooth inverse mapping theorem developed for the proof (Prop. 4.3). Among other things, results like this are potential ingredients for the approximation of admissible deformations by more regular maps, in particular for numerical purposes. So far, this question (sometimes referred to as the Ball-Evans problem) is wide open as the usual mollification techniques are not compatible with injectivity constraints. While it is known that this not possible in full generality – a Lavrentiev phenomenon can sometimes occur – even restricted scenarios avoiding this issue would be of tremendous interest.

All papers of the thesis were written in collaboration with co-authors, and I have no direct way of determining the extent of Mrs. Doležalová's personal contributions in each paper. However, in the thesis' preface, she has presented a concise and convincing overview of the topics, results and crucial examples. In addition, her ability to successfully cooperate with quite many different people – no co-author appears more than twice – strongly speaks in her favor. Finally, while strictly speaking not part of the thesis itself, I also had the opportunity to attend two seminars where she presented some of the thesis' results (on Paper II and Paper III/IV, respectively), explaining a few technical details of the proofs and demonstrating insight in depth. For these reasons, I am convinced that her personal contributions to the obtained results were indeed substantial.

The diverse cooperation and also the broad assembly of individually deep results obtained in the thesis in my opinion exceeds the average by a fair margin. The question investigated are natural, and the results obtained represent significant progress in the field. The papers are well written, and while there are of course still a handful of typos and minor oversights, I have not found any bigger issue. Existing literature is adequately quoted, and interconnections are outlined well. All in all, this is a very nice thesis with a lot of content and many new interesting, subtle and strong results that require comprehensive knowledge of a broad array of mathematical techniques.

(Stefan Krömer)

Attachment: A few questions, remarks and minor issues

I include the list below mostly because Paper III-V are apparently still in the preprint phase and their final version could profit a little from a few minor corrections or additions. Questions mainly come simply from personal curiosity.

1. P. I: Is there a special reason for considering the grand Sobolev space $W^{1,n)}$? Clearly, you want to look at something bigger than $W^{1,n}$ but as close as possible. While definitely closer than $W^{1,p}$ for any $p < n$, the particular choice of $W^{1,n)}$ still seems to be a bit arbitrary to me.

2. P. II: I'd be interested in more detailed explanation of the link between your results and possible applications to the Gauss map of solutions to mean curvature equations. Is there particular application matching the assumptions on K and Σ of one of your theorems?

3. P. III, p.7, Lemma 2.9: I think Φ can be chosen monotone, and this should be stated in the lemma. (Otherwise, it is possible to have $\inf \Phi = 0$ on arbitrary subintervals of $(0, \infty)$, which might be problematic for applications.)

4. P. III, p.14: **The construction of the good grid in the proof of Lemma 3.5 could be a bit more precise.** What exactly do you mean by "in such a way that the Sobolev regularity on the intersections [...] is controlled"? (Also, typo: "controled".) Of course you can choose many $(n-2)$-dimensional lines (or circles, or similar) where the Sobolev function has a well defined continuous representative. As far as I can see, however, this by itself does not guarantee that these representatives coincide on intersection points of such lines.

5. **P. IV, assumptions of Thm. 1.1: (1.3) is not enough.** You clearly intend to use that the intergrand of the functional is polyconvex, with $A(|\cdot|)$ convex (cf. proof of Lemma 4.3). But the convexity of A assumed in (1.3) does not suffice for that, a composition of convex functions is not necessarily convex. For instance, additionally assume in (1.3) that A is nondecreasing.

6. P. IV, proof strategy to obtain condition (INV) in comparison to the one of P. III: P. III uses the notion of shapes as defined there, which includes hollowed cubes. As far as I can see, this is essentially for Lemma 3.6 in P. III and its applications. P. IV avoids this to some extent. Also, the choice of a good grid on the surface of a good shape in P. III has been replaced by choice derived from a Vitali covering in P. IV. It might be useful to comment on these differences a bit. Are there essential advantages of one over the other?

7. P. IV, p. 15, Step 3: Similar to 4. above: It is not entirely obvious to me that the boundary values of $g_{j,m}$ (or, equivalently, f_m) will converge uniformly on $T_j \setminus S_j$. This boundary $T_j \setminus S_j$ again has intersection points of the original spheres from the Vitali covering where the uniform convergence is achieved by construction. However, the continuous representatives of f_j and f on the surface of the Vitali balls in ∂B can in principle depend on these balls and may or may not coincide on intersection points of Vitali spheres. **This should be explained more carefully.**

8. P. IV, literature: I think you should also quote [Müller, Tang and Yan (1994)] ([33] of P.III). Discuss relations to your results, in particular their Thm. 4.2 (lower semicontinuity/existence) and Thm. 5.1 (which implies the Ciarlet-Nečas condition given good boundary values with degree 1).

9. P. IV, literature: Maybe also mention [Henao, Mora-Corral and Oliva, Global invertibility... (2021)], as another reference for the case $p > n - 1$.

10. P. IV, p. 4, (2.3): Typos: I think it should be $|J_{n-1}h(x)|$ and $N(h, A, y)$ under the integrals. (Also, two lines below: delete "of sizes".)

11. P. IV, p. 5, proof of Lemma 2.3, "Moreover, we assume": Moreover, we may assume

12. P. IV, Lem 4.1: What happens if the assumption that f satisfies (INV) is dropped? Can we then still get that f satisfies Lusin's (N)? Do you know of a counterexample? The example of Ponomarev does not immediately apply, because the existence of an appropriate generating sequence f_m is not clear (cannot set $f_m = f$, because f_m has to satisfy (N)). Nor does the example of Lemma 4.5, apparently.

13. P. IV, p. 10, Def. 2.11, "satisfies condition (INV) in a ball": Here and throughout the paper, I'd prefer "for the ball" instead of "in a ball". For me, "in" suggest a domain and this could lead to misunderstanding by readers jumping to later parts of the paper, mistakenly thinking that this means that (INV) holds for all balls within.

14. P. V, top of p.2: You mean $\operatorname{osc}_B f := \sup_{x_1,x_2 \in B} |f(x_1) - f(x_2)|$, I suppose. Or $\operatorname{ess\,sup}$? Recall the definiton of osc. I doubt it is fully consistent in the literature and could lead to needless misunderstandings.

15. P. V, Thm 1.1: Differtiability a.e. is clearly a local property. However, the assumptions of the theorem also have a global aspect: the f_m are onto with fixed image. In particular, it is not directly possible to apply the theorem to restrictions of the given sequence to a smaller domain. Do you know of a counterexample if this assumption of a fixed image is dropped?