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DOCTORAL THESIS

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Nonlinear classes of mappings: properties and approximation

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I would like to express my gratitude to my supervisor Stanislav Hencl, whose guidance and expertise have been invaluable in navigating the world of Sobolev spaces. I would also like to thank all my co-authors for endeavouring on the journeys into that world with me. Special thanks go to Jani Onninen and Anastasia Molchanova who welcomed me and helped me feel at home in foreign countries.

I send my love to my parents for their unwavering support and encouragement during those journeys, even though both the mathematical and geographical lands were unknown to them.

I deeply appreciate Jenda, my friends and my family for welcoming me back whenever I returned.

I would like to commemorate all the resilience, courage, and determination of my ancestors which allowed me to live in peace and freedom.

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Abstract:

In this thesis, we explore classes of mappings suitable for models in Nonlinear Elasticity. We investigate whether, given the presence of certain desirable properties, there exists an element within the class that exhibits pathological behaviour. In the presented papers, we primarily focus on subclasses of Sobolev mappings, particularly weak closures of homeomorphisms with additional properties. These properties typically manifest themselves in the form of an additional term in the energy functional.

We show that weak limits of Sobolev homeomorphisms in $W^{1,n-1}$ satisfy the so-called (INV) condition if the integrability of the reciprocals of the Jacobians is sufficiently high. This result is sharp and we present a counterexample for cases of lower integrability. The (INV) condition is also preserved under weak limits when we add a term dependent on the cofactor matrix of the derivative, as its integrability provides some regularity for the inverses of the homeomorphisms in the sequence. Furthermore, we show that assumptions on regularity of the inverses can also ensure a.e. differentiability of the limit.

Other topics investigated in this thesis include the sizes of critical sets violating the Luzin (N) condition in the case of Sobolev homeomorphisms and the (dis)continuity of mappings of generalized distortion, where we present both positive results and counterexamples in the planar case.

Keywords: approximation of functions, Sobolev homeomorphisms, Nonlinear Elasticity

Contents

Outline

This thesis consists of five papers, which were or are to be published independently, and of introductory texts which set them into the wider context. The text is divided into four units:

- **Introduction** gives the general background for Sobolev mappings, limits of homeomorphisms and Calculus of Variations.
- **On the Luzin** (N) **condition** gives the background for
  - *Hausdorff measure of critical set for Luzin N condition*; A. Doležalová, M. Hrubešová and T. Roskovec, Journal of Mathematical Analysis and Applications, Volume 493, Issue 2, 2021.
- **On the generalized distortion** gives the background for
  - *Mappings of generalized finite distortion and continuity*; A. Doležalová, I. Kangasniemi and J. Onninen, preprint arXiv:2210.14141, 2022. Accepted to Journal of the London Mathematical Society.
- **On the** (INV) **condition** gives the background for
  - *Weak limit of homeomorphisms in* $W^{1,n-1}$ *and (INV) condition*; A. Doležalová, S. Hencl and J. Malý, preprint arXiv:2112.08041, 2021.
  - *Weak limit of homeomorphisms in* $W^{1,n-1}$*: invertibility and lower semicontinuity of energy*; A. Doležalová, S. Hencl and A. Molchanova, preprint arXiv:2212.06452, 2022.
  - *Differentiability almost everywhere of weak limits of bi-Sobolev homeomorphisms*; A. Doležalová and A. Molchanova, preprint arXiv:2302.07578, 2023.

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Introduction

The key question investigated in this thesis is the following: Given a class of mappings with nice properties, does there exist an element of the class which exhibits a pathological behaviour of some kind? Questions of this type appear naturally in Calculus of Variations and Nonlinear Elasticity, see the pioneering work of Ball [2] and Ciarlet and Nečas [7], among many others. In the research presented in this thesis, we focus on (sub)classes of Sobolev mappings.

The motivation for such a question comes from the standard machinery of Calculus of Variations. We take an open set $\Omega \subseteq R^n$ as our elastic body in its original state and $\Omega' \subseteq R^n$ in the deformed one. We seek for a mapping $f : \Omega \to \Omega'$ which describes the deformation of Ω onto Ω'. Often we ask our mapping to satisfy some boundary condition, which might correspond to the observed behaviour of the material on the boundary of the body. The energy of the deformation f is expressed as a functional in a form

$$\mathcal{E}(f) = \int_\Omega F(Df)\, dx.$$

Such energy is independent of translations or rotations, i.e., $F(RA) = F(A)$ for every rotation $R : \mathbb{R}^n \to \mathbb{R}^n$. Note that the energy does not depend only on the norm of the derivative $|Df|$, but it can for example contains terms with the Jacobian J_f of the mapping. We usually expect that $F(Df) \to \infty$ when $|Df| \to \infty$ and also $F(Df) \to \infty$ as $J_f \to 0$. This corresponds to the intuition that both stretching and compressing the material should cost some amount of energy.

When modelling an elastic deformation, there are several restrictions we want to apply on our classes of functions so that the exhibited behaviour is reasonable from the physical point of view. Such restriction might be for example that we do not want to create any matter during elastic deformation nor lose it. That can be mathematically formulated as the Luzin (N) and (N^{-1}) condition. Similarly, we want to keep the orientation of the material. We also want to prevent the interpenetration of matter or to keep the deformation reversible in some sense, leading to different notions of invertibility. Whereas for diffeomorphisms we know that the sign of the Jacobian and of the topological degree coincide [17, Section 3.2], that does not have to be true even for Sobolev homeomorphisms [5]. Therefore it makes sense to both ask that our deformation has nonnegative Jacobian a.e. as well as investigate the topological degree.

The standard process finds a sequence of functions f_k which minimizes our energy functional. If this sequence is bounded in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ for $p \in (1, \infty)$, we can extract a weakly converging subsequence and find its weak limit f. As the approximations f_k are often nice (i.e., have the aforementioned properties), we want to investigate whether the weak limit preserves these properties in order to have a physically relevant solution. One usual choice in our work is to take $f_k \in W^{1,p}(\Omega, \mathbb{R}^n)$ to be homeomorphisms satisfying the Luzin (N) condition with $J_{f_k} > 0$ a.e. That is because such mappings should be in our class of possible solutions as those are very reasonable deformations.

Another property that we would like our function spaces or classes to have is the possibility of some discontinuities – but only of special kinds. Experiments showed that cavitations may happen in an elastic body [16]. At the same time, we do not want our body to be completely shattered. This balance lead to a widespread use of Sobolev spaces.

In Chapter 1 we investigate the Luzin (N) condition. For p small enough, even homeomorphisms from $W^{1,p}(\Omega, \mathbb{R}^n)$ can violate this condition, and thus create matter. We study in the size of the critical set from which the new material is created in Paper I.

Chapter 2 is dedicated to mappings of generalized distortion. We search for conditions under such mappings can exhibit discontinuities and when the continuity is assured. Such conditions, as well as a conjecture closing the gap between them, are presented in Paper II.

The last part of this thesis, Chapter 3, is dedicated to the (INV) condition. Pioneered by Müller and Spector [25] and later developed by Conti and De Lellis [6], this property ensures, roughly speaking, that inside of a ball stays inside the image of the sphere and, vice versa, the outside of the ball is mapped outside of the image of the sphere. Whereas for $p > n-1$ the class of $W^{1,p}$-mappings satisfying the (INV) condition is weakly closed [25], it is not true for the borderline case $p = n-1$ [6]. In Papers III and IV we present sufficient conditions under which all mappings from the weak closure of homeomorphisms satisfy (INV). Paper III also provides a new counterexample, thus showing that the assumption there is sharp. In Paper V, we investigate the differentiability of limits of homeomorphisms in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ under an assumption on the regularity of the inverse mappings.

1. On the Luzin (N) condition

1.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}^n$ be a mapping. We say that f satisfies the Luzin (N) condition if

$$\text{for every } E \subseteq \Omega \text{ with } |E| = 0 \text{ we have } |f(E)| = 0$$

and that f satisfies the Luzin (N^{-1}) condition if

$$\text{for every } E \subseteq \Omega \text{ with } |f(E)| = 0 \text{ we have } |E| = 0.$$

These two conditions are important in models of Nonlinear Elasticity: The first one prevents creation of matter, whereas the second one prevents its loss. (Note that the Luzin (N^{-1}) condition is often replaced by the assumption $J_f > 0$ a.e., which is more suitable for the Calculus of Variations approach.) They also play a crucial role in the abstract setting of Sobolev spaces, as they are tightly tied to the validity of the area and co-area formula and the change-of-variables formula (for details see [17, Section A.8]).

Similarly to the case of the continuity, if $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ for $p > n$, we know that f satisfies the (N) condition (see Marcus and Mizel [24]). If we restrict ourselves to homeomorphisms, this holds true even for $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ (see Reshetnyak [28]). Both of these results are sharp, as was shown in [27] and [23]. Moreover, Malý and Martio [23] and Kauhanen [22] studied the size of the critical set of measure zero which is enlarged to a set of positive measure.

To compare sizes of sets of measure zero, we use general Hausdorff measures determined by gauge functions. A continuous function $h : [0, \infty) \to [0, \infty)$ is called a gauge function if it is nondecreasing and $h(0) = 0$. Such function then defines the corresponding Hausdorff measure $\mathcal{H}^h$ in the following way: Let $E \subseteq \mathbb{R}^n$, then

$$\mathcal{H}^h(E) := \lim_{\delta \to 0+} \left(\inf \left\{ \sum_{i=1}^{\infty} h(\operatorname{diam} U_i) : E \subseteq \bigcup_i U_i, \operatorname{diam}(U_i) < \delta \right\} \right).$$

This allows us to compare sizes of sets on a finer scale than with the classical Hausdorff measures. Note that we obtain the classical Hausdorff measure when choosing $h(t) = t^\alpha$, so we indeed refine the classical scale of measures. In literature there might be small differences in the definition, however, the difference lies only in a multiplicative constant. Since we want to distinguish whether the set has zero measure or finite positive measure, the precise choice plays no role.

Beside refining the scale for measuring sizes of sets, we can also get closer to $W^{1,n}(\Omega, \mathbb{R}^n)$ than by just taking $W^{1,p}(\Omega, \mathbb{R}^n)$, $p < n$. One can for example consider only functions $f \in \bigcap_{p<n} W^{1,p}$. We go a step further and use the grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$ (see [15] and [20] for details). Where the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ uses the Lebesgue norm to have $||f||_{1,p} = ||f||_p + ||Df||_p$, the Grand Lebesgue space $W^{1,p)}(\Omega, \mathbb{R}^n)$ uses the norm $|| \cdot ||_{p)}$ which is defined as

$$||f||_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} \right)^{\frac{1}{p-\varepsilon}}.$$

We have $W^{1,n} \subsetneq W^{1,n)}(\Omega, \mathbb{R}^n) \subsetneq \bigcap_{p<n} W^{1,p}$, so this space indeed enables us to get closer to the borderline case.

1.2 Results of Paper I

In Paper I, we studied the following questions: Given a homeomorphism $f \in W^{1,p}$, $p < n$, we know that it can violate the (N) condition and that the Hausdorff dimension of the critical set might be any number $d \in [0, n)$. If we refine the scales both on the side of function spaces (and use the grand Sobolev spaces) and of gauge functions, can we get similar results or is there a fundamental bound? This is motivated by the fact that $W^{1,n}$-mappings satisfy the (N) condition outside of a set of measure zero [23], so there is a kind of jump in the size of the critical set when we take $p = n$.

We can look at the problem also in this way: We take a class of nice mappings (homeomorphisms from the grand Sobolev space) and we want to know whether we can find functions which behave badly. One type of bad behaviour is mapping a very small set (i.e., not only of Lebesgue n-dimensional measure zero, but small in the language of Hausdorff measures) onto a big set. Another type is that the function is nice, but to obtain the (N) condition we need to omit a big set (again, on the scale of Hausdorff measures), therefore there is no universal bound for the size of omitted set as in the $W^{1,n}$ case.

We approach this problem with the help of Ponomarev's construction from [27]. There are two main results, summarized in Theorems 1.1 and 1.2 in Paper II. The first one constructs a mapping $f \in W^{1,n)}$ which violates the (N) condition on a set of big measure. In that case, our assumptions on the gauge function h allow it to be e.g. $t^n \log\log\log\ldots(1/t)$, and therefore in some sense we can get as close to the Lebesgue measure as we wish. This shows that there is no universal bound such that after omitting a suitable set of finite $\mathcal{H}^h$ measure the (N) condition would be satisfied.

Theorem A. *Let $Q_0 = [-1, 1]^n$, $\tau : (0, \infty) \to [1, \infty)$ be a continuous decreasing function such that $\lim_{t\to 0+} \tau(t) = \infty$ and for all $p \in (0, 1]$ there exists $x_p \in (0, 1)$ such that for all $t \in (0, x_p)$ we have*

$$\frac{1}{\tau(pt)} > t^n.$$

Let $h : [0, \infty) \to [0, \infty)$ be a gauge function, i.e., a continuous non-decreasing function such that $h(0) = 0$, satisfying $h(t) = t^n \tau(t)$ on $(0, \infty)$. Then there exists a homeomorphism $f : Q_0 \to Q_0$ such that

1. *f is the identity on the boundary of Q_0,*
2. *$f \in W^{1,n)}(Q_0, Q_0)$,*
3. *$Jf > 0$ a.e. in Q_0,*
4. *if $E \subseteq Q_0$ with $\mathcal{H}^h(E) = 0$, then $|f(E)| = 0$,*
5. *there exists a set C_A such that $\mathcal{H}^h(C_A) \in (0, \infty)$, $|C_A| = 0$ and $|f(C_A)| > 0$.*

The second result holds for any gauge function h and shows that indeed we can find $f \in W^{1,n)}$ such that the Luzin (N) condition is violated on a set of zero $\mathcal{H}^h$ measure, and therefore f, despite being from a function space very close to $W^{1,n}$, stretches a very small set onto a set of positive Lebesgue measure.

Theorem B. *Let $Q_0 = [-1, 1]^n$ and let $h : [0, \infty) \to [0, \infty)$ be a gauge function, i.e., a continuous non-decreasing function such that $h(0) = 0$. Then there exists a homeomorphism $f : Q_0 \to Q_0$ such that*

1. *f is the identity on the boundary of Q_0,*
2. *$f \in W^{1,n)}(Q_0, Q_0)$,*
3. *$Jf > 0$ a.e. in Q_0,*
4. *there exists a set $C_A \subseteq Q_0$ such that $\mathcal{H}^h(C_A) = 0$, $|C_A| = 0$ and $|f(C_A)| > 0$.*

2. On the generalized distortion

2.1 Introduction

In this chapter, we consider $f : \mathbb{R}^n \to \mathbb{R}^n$ to be a mapping from the space $W^{1,n}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ which satisfies

$$|Df(x)|^n \leq K(x)J_f(x) + \Sigma(x) \tag{2.1}$$

for a.e. $x \in \mathbb{R}^n$, where the assumptions on $K : \mathbb{R}^n \to [1, \infty]$ and $\Sigma : \mathbb{R}^n \to [0, \infty]$ are to be specified. We call f a mapping of generalized distortion.

This topic was already investigated by Simon [29], Astala, Iwaniec and Martin [1] and Kangasniemi and Onninen [21]. The motivation for this property stems from the well known results regarding each of the terms in the inequality: If we consider

$$|Df(x)|^n \leq K(x)J_f(x)$$

and assume K to be finite a.e., we obtain mappings of finite distortion and therefore we have the continuity of mapping f (see [17, Section 2.1]). Similarly, for

$$|Df(x)|^n \leq \Sigma(x)$$

and $\Sigma \in L^p_{loc}(\mathbb{R}^n)$, $p > n$, we also have the continuity of f as in that case $f \in W^{1,p}_{loc}(\mathbb{R}^n)$. However, a combination of these two terms is not enough to give us the continuity in the general case. This can be easily seen by taking $f : B(0, 1/2) \to \mathbb{R}^n$ as

$$f(x) = f(x_1, \ldots, x_n) := \left(\log\log\log\left(\frac{1}{|x|}\right), 0, \ldots, 0 \right).$$

We set $K := 1$ and $\Sigma(x) := |Df(x)|^n = \left[\log\left(\frac{1}{|x|}\right)\log\log\left(\frac{1}{|x|}\right)\right]^{-n}$. This example shows that we need to make stronger assumptions on K and Σ in order to have continuous functions.

2.2 Results of Paper II

In Paper II, we follow up on the work of Kangasniemi and Onninen [21] and investigate the assumptions on K and Σ under which is f continuous.

The case with $K \in \mathcal{L}^\infty_{loc}(\mathbb{R}^n)$ and $\Sigma \log^\mu(e + \Sigma) \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ is completely solved as there is a sharp assumption on μ under which f is continuous, see Theorem C and Figure A. The sharpness can be proven by taking $f : B(0, 1/2) \to \mathbb{R}^n$ as the triple logarithm mentioned above.

Theorem C. *Suppose that $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies (2.1) in Ω, with*

$$K \in L^\infty_{loc}(\Omega) \qquad \textit{and} \qquad \Sigma \log^\mu(e + \Sigma) \in L^1_{loc}(\Omega),$$

for some $\mu > n - 1$. Then f has a continuous representative.

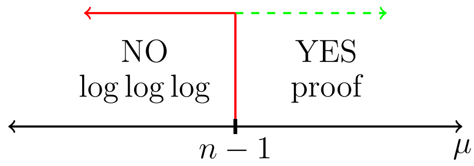

Figure A: Continuity in the case $K \in \mathcal{L}^{\infty}_{loc}(\mathbb{R}^n)$ and $\Sigma \log^{\mu}(e+\Sigma) \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$.

When slightly relaxing the assumption on K, the situation becomes more complicated. By having counterexamples for the planar case, we are able to approach from both sides, however, there is still a gap remaining, see Section 1.2 in Paper II and Figure C.

Theorem D. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfy* (2.1) *with*

$$\exp(\lambda K) \in L^1_{loc}(\Omega) \qquad \text{and} \qquad \Sigma \log^{\mu}(e+\Sigma) \in L^1_{loc}(\Omega),$$

for some $\mu > \lambda > n+1$. Then f has a continuous representative.

Theorem E. *For every $\mu \in (0,2)$, there exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$ satisfying* (2.1) *with*

$$\exp(\lambda K) \in L^1(\Omega) \qquad \text{and} \qquad \Sigma \log^{\mu}(e+\Sigma) \in L^1(\Omega)$$

for every $\lambda > 0$.

This counterexample is based on dividing the circle into a cusp and the remaining part, see Figure B. On the cusp, we can ask for Σ being the leading term and its higher integrability is ensured by the shape of the cusp (given by γ). On the remaining part, KJ_f is the leading term. One can think of the mapping as pinching the center of the disc and dragging it to infinity, while the rest of the circle is stretched along accordingly.

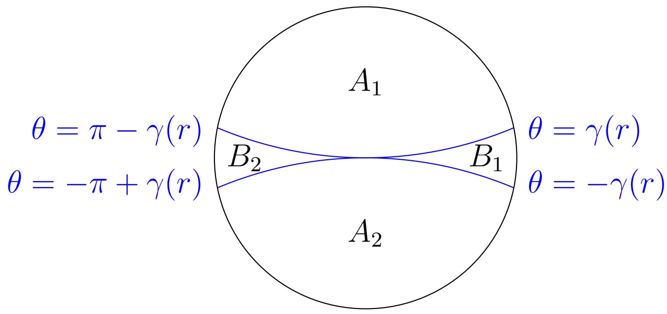

Figure B: The regions in the cusp counterexample. The Jacobian is positive on A_1 and A_2 and negative on B_1 and B_2.

The construction of the counterexamples is in detail described in Section 4 and 5 of Paper II. These examples are also used in the case that we loosen the assumptions on K even more and ask it to be in L^p_{loc} only. In that case, the problem still stands open. We are at least able to formulate a conjecture and support it by the counterexamples in the planar case (see Section 1.4 in Paper II).

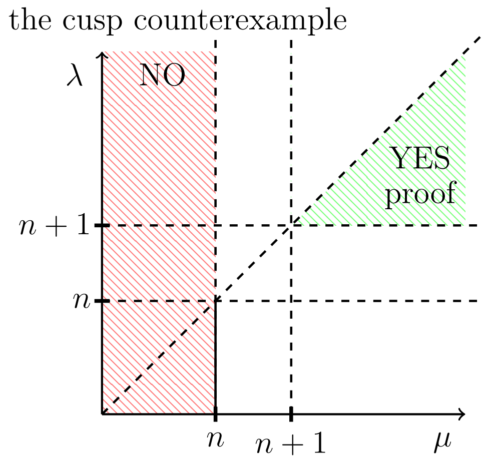

Figure C: Continuity in the case $n = 2$, $\exp(\lambda K) \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ and $\Sigma \log^{\mu}(e + \Sigma) \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$.

Theorem F. *Let $p, q \in (1, \infty)$. If $p^{-1} + q^{-1} \geq 1$, then there exists a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x \to 0} |f(x)| = \infty$ satisfying (2.1) with*

$$K \in L^p(\Omega) \qquad \text{and} \qquad \frac{\Sigma}{K} \in L^q(\Omega). \tag{2.2}$$

Conjecture G. *Let $1 \leq p, q \leq \infty$. Suppose that $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies (2.1) with $K \geq 1$, $\Sigma \geq 0$,*

$$K \in L^p_{loc}(\Omega), \qquad \text{and} \qquad \frac{\Sigma}{K} \in L^q_{loc}(\Omega), \qquad \text{where } \frac{1}{p} + \frac{1}{q} < 1\,.$$

Then f has a continuous representative.

Note that in the borderline cases $p = 1, q = \infty$ and $p = \infty$, $q = 1$ we have discontinuous examples: the triple logarithm construction from above and the spiral counterexample.

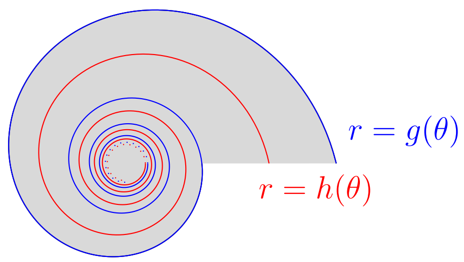

Figure D: The two spirals.

Roughly speaking, we cut the spiral in Figure D along the blue line and stretch it (that creates the blow-up at 0). To preserve continuity elsewhere, we fold the spiral along the red line and slightly stretch one side. The precise definition can be found in Section 5 of Paper II.

We also investigate the following version of the inequality

$$|Df(x)|^n \leq K(x) J_f(x) + \Sigma(x) |f(x) - y_0|,$$

where $y_0 \in \mathbb{R}^n$, and obtain analogies of the aforementioned results.

3. On the (INV) condition

3.1 Introduction

To prevent the interpenetration of matter, we want to impose some assumptions on our classes of mappings. A necessary assumption for using the Calculus of Variations approach is that this class is weakly closed so that the minimizer is still in the class. Optimally it would also allow for cavitations as they appear in Nonlinear Elasticity. Therefore e.g. the class of Sobolev homeomorphisms is not a good class, as it it nor weakly closed not it allows for cavitations. However, we can find an inspiration in their properties as illustrated in Figure E: Whenever we take two disjoint closed balls, their images are also disjoint and the inside of the ball is mapped into the image of the sphere. There are also no big jumps on the boundary. This makes sense for homeomorphisms as they are very nice mappings; we translate this property into more general language.

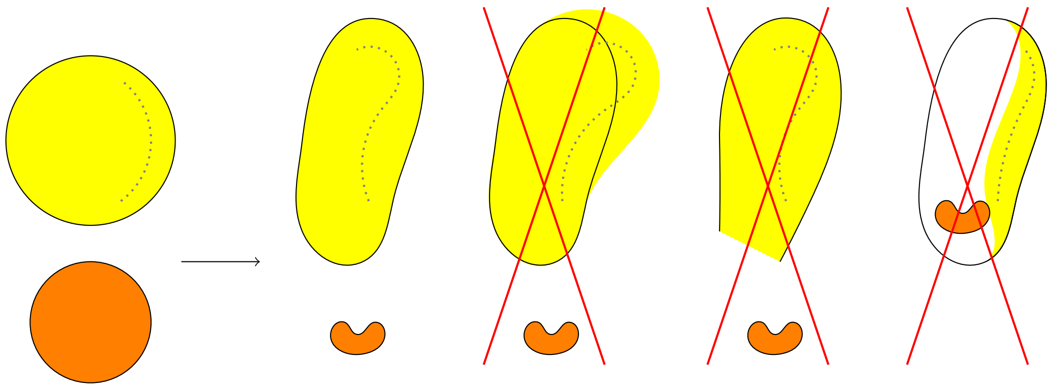

Figure E: An illustration of the desired and undesired behaviour. It is possible to create cavitations, but we don't want to fill them with material from another part of the body. Similarly we want to avoid big jumps on the spheres or spilling out of the material.

To talk about the image of a ball and inside of the image of a sphere we use the classical topological degree. We define the topological image of an open set A under a continuous mapping $f : \Omega \to \mathbb{R}^n$ as

$$\operatorname{im}_T(f, A) = \{y \in \mathbb{R}^n : \deg(f, A, y) \neq 0\}.$$

For general Sobolev mappings $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, this is not enough. For $p > n-1$ we have continuity on almost every sphere, so in that case we can use the classical degree. We then define the (INV) condition:

Definition H. *Let $f \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$, $p > n-1$. We say that f satisfies (INV) in the ball $B \subset\subset \Omega$ if*

(i) its trace on ∂B is in $W^{1,p} \cap L^\infty(\partial B, \mathbb{R}^n)$;

(ii) $f(x) \in \operatorname{im}_T(f, B)$ for a.e. $x \in B$;

(iii) $f(x) \notin \operatorname{im}_T(f, B)$ for a.e. $x \in \Omega \setminus B$.

We say that f satisfies the (INV) condition if for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{H}^1$-a.e. $r \in (0, r_a)$ it satisfies (INV) in $B(a,r)$.

The theory for $p > n-1$ was developed by Müller and Spector in [25]. They showed that the class of Sobolev mappings with (INV) is weakly closed and that mappings with nonzero Jacobian satisfying (INV) exhibit other good properties like being one-to-one a.e. or having degree only 1, 0 or -1 with respect to a.e. sphere.

However, in some models the energy functional contains the term $|Df|^{n-1}$ (the Dirichlet energy $|Df|^2$ in the physically most relevant case $n = 3$). This case was studied by Conti and De Lellis in [6]. As we do not have continuity on a.e. sphere, they used a generalization of the classical topological degree. For a ball $B \subset \Omega$ and $f \in W^{1,n-1} \cap L^\infty(\Omega, \mathbb{R}^n)$ it is defined as the distribution satisfying

$$\int_{\mathbb{R}^n} \mathrm{Deg}(f, K, y)\psi(y)\, dy = \int_{\partial K} (\mathbf{u} \circ f) \cdot (\Lambda_{n-1} D_\tau f)\nu \, d\mathcal{H}^{n-1}$$

for every test function $\psi \in C_c^\infty(\mathbb{R}^n)$ and every C^∞ vector field $\mathbf{u}$ on $\mathbb{R}^n$ with $\operatorname{div} \mathbf{u} = \psi$. It is possible to show that it is actually a BV function. As it is defined only up to a set of measure zero, we define the topological image as

$$\mathrm{im}_T(f, A) = \{z \in \mathbb{R}^n : \text{ density of the set } \{y \in \mathbb{R}^n : \mathrm{Deg}(f, A, y) \neq 0\} \text{ at } z \text{ is } 1\}.$$

That enables us to work with the (INV) condition even in the borderline case $p = n-1$ when we replace deg by Deg. Conti and De Lellis showed that mappings with (INV) and positive Jacobian a.e. again have many nice properties, but the crucial one is missing: In $W^{1,n-1}(\Omega, \mathbb{R}^n)$, it is possible to find a sequence of bilipschitz homeomorphisms which converges weakly to a mapping violating the (INV) condition. Without the class being weakly closed we cannot use the standard machinery of Calculus of Variations. The question of additional assumptions which would define a weakly closed subclass of mappings with (INV) in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ gave rise to Papers III and IV. Another approach was taken in the research of Barchiesi, Henao, Mora-Corral and Rodiac [3, 4], where they investigate axially symmetric mappings.

3.2 Results of Paper III

In Paper III, we work with the following energy functional

$$\mathcal{E}(f) = \int_\Omega \left(|Df|^{n-1} + \varphi(J_f) \right),$$

where φ satisfies

$$\varphi \text{ is a positive convex function on } (0,\infty) \text{ with } \lim_{t \to 0^+} \varphi(t) = \infty,\ \varphi(t) = \infty \text{ for } t \leq 0$$

and there exists $A > 0$ with

$$A^{-1}\varphi(t) \leq \varphi(2t) \leq A\varphi(t), \qquad t \in (0, \infty).$$

The first condition corresponds to the intuition that squeezing material from all sides and thus shrinking the body costs energy. As a special case of more general setting, we show that for $\varphi(t) = t^{-a}$ there is a sharp exponent a such that either the weak limit of homeomorphisms satisfy the (INV) condition (if we are above the critical exponent) or that we have a counterexample which violates the (INV) condition (below the critical exponent). Note that the assumptions $J_{f_m} > 0$ a.e. and the boundary condition makes intuitively sense as they correspond to not changing the orientation of matter and to the observed data on the surface of the deformed body.

Theorem I. *Let $n \geq 3$, $a = \frac{n-1}{n^2-3n+1}$ and $\Omega, \Omega' \subset \mathbb{R}^n$ be Lipschitz domains. Let $\varphi = t^{-a}$ for $t \in (0, \infty)$. Let $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $m = 0, 1, 2 \dots,$ be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega}'$ with $J_{f_m} > 0$ a.e. such that*

$$\sup_m \mathcal{E}(f_m) < \infty.$$

Assume further that $f_m = f_0$ on $\partial\Omega$ for all $m \in \mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$, then f satisfies the (INV) condition.

Theorem J. *Let $n = 3$ and $a < 2$. Then there exist homeomorphisms f_m of $\overline{B}(0, 10)$ to $\overline{B}(0, 10)$ such that $f_m \in W^{1,2}(B(0, 10), B(0, 10))$, f_m is an identity mapping on $\partial B(0, 10)$ with $J_{f_m} > 0$ a.e. and*

$$\sup_m \int_\Omega \left(|Df_m|^{n-1} + \frac{1}{(J_{f_m})^a} \right) dx < \infty,$$

whose weak limit f does not satisfy the (INV) condition.

This counterexample is in principle different from the one in [6]. We also show that the strong and weak closures of homeomorphisms in $W^{1,2}(\Omega, \mathbb{R}^n)$ for $n = 3$ are different, which stands in contrast to the planar case (see [14] and [19]).

Theorem K. *Let $n = 3$. There is a mapping $f \in W^{1,2}(B(0, 10), B(0, 10))$ which is a weak limit of Sobolev $W^{1,2}$ homeomorphisms f_m of $\overline{B}(0, 10)$ to $\overline{B}(0, 10)$ with $f_m(x) = x$ on $\partial B(0, 10)$ and $J_{f_m} > 0$ a.e., but there are no homeomorphisms h_m of $\overline{B}(0, 10)$ to $\overline{B}(0, 10)$ such that $h_m \to f$ strongly in $W^{1,2}(B(0, 10), \mathbb{R}^3)$.*

3.3 Results of Paper IV

In Paper IV we continued our research of (INV) in the case $W^{1,n-1}(\Omega, \Omega')$. Both known examples of limits of homeomorphisms which violates (INV) have inverses a.e., however, those are only BV mappings. Therefore we presented an assumption on the adjoint of the derivative which ensures that the inverses of the homeomorphisms converge weakly in $W^{1,1}(\Omega', \Omega)$. Namely we take the energy functional

$$\mathcal{E}(f) = \int_\Omega \Big(|Df|^{n-1} + A(|\operatorname{cof} Df|) + \varphi(J_f) \Big),$$

where A is a positive convex function on $(0, \infty)$ with superlinear growth and φ is as before, i.e. a positive convex function on $(0, \infty)$ which has a blow-up at 0. We proved that if the energy of the sequence is bounded, the limit satisfies the (INV) condition.

Theorem L. *Let $n \geq 3$ and $\Omega,\ \Omega' \subset \mathbb{R}^n$ be bounded domains. Let φ and A satisfy the conditions mentioned above. Let $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $m = 0, 1, 2 \dots$, be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_m} > 0$ a.e., such that f_m satisfies the Luzin (N) condition and*

$$\sup_m \mathcal{E}(f_m) < \infty.$$

Assume further that $f_m = f_0$ on $\partial\Omega$ for all $m \in \mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$, then f satisfies the (INV) condition.

Moreover, under the additional assumption $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ our f satisfies the Luzin (N) condition and we have lower semicontinuity of energy

$$\mathcal{E}(f) \leq \liminf_{m\to\infty} \mathcal{E}(f_m).$$

Assuming further that $|\partial\Omega'| = 0$ we have

for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$,

where h is a weak limit of (some subsequence of) f_k^{-1} in $W^{1,1}(\Omega', \mathbb{R}^n)$.

We also showed that with some additional assumptions on the energy functional, the functional in question both in Paper III and in Paper IV is weakly lower semicontinuous, and therefore we can use the standard methods of Calculus of Variations.

3.4 Results of Paper V

In Paper V, we keep working with the regularity of the inverses. We proved that if f_m are homeomorphisms in $W^{1,n-1}(\Omega, \Omega')$ and the inverses f_m^{-1} are in $W^{1,p}(\Omega', \Omega)$ for some $p > n-1$, the differentiability of both the weak limit and its a.e. inverse is guaranteed. This result complements the already-known fact that any Sobolev homeomorphism from $W^{1,p}$ for $p > n-1$ is differentiable a.e. (see [31] and [26]) and that in the borderline case $p = n-1$ there are several sets of assumptions which guarantee that, too (see [18], [30] and [32]). Notably, there exists a Sobolev homeomorphism f such that both f and f^{-1} are in $W^{1,n-1}$ but both of them are nowhere differentiable [8]. That justifies our stronger assumption on the integrability of the inverse.

Theorem M. *Let $n \geq 2$, $p > n-1$, $\Omega,\ \Omega' \subset \mathbb{R}^n$ be bounded domains and $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $m = 0, 1, 2 \dots$, be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$ a.e. and*

$$\sup_m \mathcal{E}(f_m) = \sup_m \int_\Omega |Df_m(x)|^{n-1} dx + \int_{\Omega'} |Df_m^{-1}(y)|^p dy < \infty.$$

Assume that $f\colon \Omega \to \mathbb{R}^n$ is a weak limit of $\{f_m\}_{m\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. and $h\colon \Omega' \to \mathbb{R}^n$ is a weak limit of $\{f_m^{-1}\}_{m\in\mathbb{N}}$ in $W^{1,p}(\Omega', \mathbb{R}^n)$ with $J_h > 0$ a.e. Then for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$, and both f and h are differentiable almost everywhere.

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List of Figures

Paper I

HAUSDORFF MEASURE OF CRITICAL SET FOR LUZIN N CONDITION

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Abstract. It is well-known that there is a Sobolev homeomorphism $f \in W^{1,p}([-1,1]^n, [-1,1]^n)$ for any $p < n$ which maps a set C of zero Lebesgue n-dimensional measure onto a set of positive measure. We study the size of this critical set C and characterize its lower and upper bounds from the perspective of Hausdorff measures defined by a general gauge function.

1. Introduction

1.1. **Motivation and history.** By $\Omega \subset \mathbb{R}^n$ we denote a nonempty domain of finite measure, n stands for dimension and $\mathcal{L}_n$ denotes Lebesgue n-dimensional measure. A function $f\colon \Omega \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, is said to satisfy the *Luzin N condition* if, for every $E \subseteq \Omega$, we have

$$\mathcal{L}_n(E) = 0 \implies \mathcal{L}_n(f(E)) = 0.$$

Analogously, f fulfils the *Luzin N^{-1} condition* if, for every $E \subseteq \Omega$, we have

$$\mathcal{L}_n(f(E)) = 0 \implies \mathcal{L}_n(E) = 0.$$

These are crucial properties in models of mechanics of solids and other physical models. The Luzin N condition (also known as the Luzin property or the N property) prohibits the "creation of matter" by deformation and the Luzin N^{-1} condition prohibits the "disappearance of matter". From the mathematical point of view, these conditions are bound to the question of validity of the change of variables formula with minimal regularity requirements, see [5, Theorem 8.4], [18, Theorem 2.5, Chapter 5] and [21]. Also, for Sobolev spaces the validity of the Luzin N condition is equivalent to the validity of the area formula, see [49] and [39], for connections of conditions with the co–area formula in Sobolev spaces see [38], [36] or [24, Section A.8].

Concerning the characterization of the validity of the Luzin N condition, Reshetnyak [48] proved the validity of the condition N for Sobolev homeomorphisms in $W^{1,n}$, Marcus and Mizel [40] proved its validity for Sobolev mappings in $W^{1,p}$ for $p > n$. To show the optimality of these results, Ponomarev [46] (see also the later paper [47]) provided a Sobolev homeomorphism violating the Luzin N condition for $W^{1,p}$, $1 \le p < n$ and Malý and Martio [37] used the older Cesari construction [9] to get a continuous $W^{1,n}$ mapping violating the Luzin N condition. The results concerning validity of the condition N on finer scales such as the Sobolev–Lorentz spaces of the spaces with derivatives in Banach function spaces is studied in [27] by Kauhanen, Koskela, and Malý.

The characterization of the validity of the Luzin N^{-1} condition differs a lot from the N condition case. It is possible to construct a homeomorphism that compresses a set in order to map a set of positive measure onto a set of zero measure in any $W^{1,p}$, i.e., to violate the N^{-1} condition. The Sobolev norm is not crucial, so the concept of distortion and the class of the mappings with finite distortion is needed. The positive result and its optimality are given by Kauhanen, Koskela, and Malý in [26] and [32], some border cases are further covered by Kleprlík in [30].

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Let us mention that the counterexample constructions violating the Luzin N condition by Ponomarev and by Cesari are fundamentally different. The counterexample violating the Luzin N^{-1} condition is based on the construction by Ponomarev.

1.2. **Ponomarev construction and its refinements.** We focus on the example given by Ponomarev, i.e., the homeomorphism in $W^{1,p}([-1,1]^n,[-1,1]^n)$ for $p<n$, which maps the Cantor type set C_A of measure zero onto the Cantor type set C_B of positive measure. The detailed construction is presented in Section 3. The original construction considers the Lebesgue n-dimensional measure and the Sobolev space $W^{1,p}$. However, this may be refined, we may ask about the size of the small Cantor type set C_A with respect to the Hausdorff measure $\mathcal{H}^h$ based on a gauge function h. We also may consider homeomorphisms in some other spaces, in general, in some spaces strictly bigger than $W^{1,n}$ and defined in finer than Sobolev scales, such as grand Sobolev spaces $W^{1,n)}$ (see 2. Preliminaries for its definition) or Sobolev–Orlicz spaces. The choice of the grand Sobolev space $W^{1,n)}$ is optimal in some sense in the perspective of spaces based on the integrability of weak derivative, see [26].

It is well-known that the Hausdorff dimension of C_A may be zero (see [37]), but Kauhanen [28] also studied the largest possible dimension of C_A. Obviously, the Hausdorff measure for the gauge function $h(t)=t^n$ should be still zero, otherwise the example does not violate the Luzin N condition. It was shown in [37, Theorem G] that for a mapping in $W^{1,n}$ we can always find a critical set C of Hausdorff dimension 0 such that outside of C the Luzin N condition holds. On the other hand, Kauhanen [28] showed that for any number $d<n$ there exists f for which it is necessary to omit a set of Hausdorff dimension d to be sure that the Luzin N condition holds outside of this exceptional set. His homeomorphism f belongs to the grand Sobolev space $W^{1,n)}$ and the result is obtained by the choice of the gauge function $h_s(t)=t^n\log^s\log(4+1/t)$ for $s>0$. No optimality of the choice of h_s is discussed, but the result still answers the question of the possible Hausdorff dimension of the exceptional set, as there is no universal constant $d<n$ such that for each $f\in W^{1,n)}$ the Luzin N condition holds if we omit a set of Hausdorff dimension d. We study the Hausdorff measure of the critical set in more general scales, not only the powers, resulting in the following statement:

Theorem 1.1. *Let $Q_0=[-1,1]^n$, $\tau:(0,\infty)\to[1,\infty)$ be a continuous decreasing function such that $\lim_{t\to 0+}\tau(t)=\infty$ and for all $p\in(0,1]$ there exists $x_p\in(0,1)$ such that for all $t\in(0,x_p)$ we have*
$$\frac{1}{\tau(pt)}>t^n.$$
Let $h:[0,\infty)\to[0,\infty)$ be a gauge function, i.e., a continuous non-decreasing function such that $h(0)=0$, satisfying $h(t)=t^n\tau(t)$ on $(0,\infty)$. Then there exists a homeomorphism $f:Q_0\to Q_0$ such that

(1) f is the identity on the boundary of Q_0,
(2) $f\in W^{1,n)}(Q_0,Q_0)$,
(3) $Jf>0$ a.e. in Q_0,
(4) if $E\subseteq Q_0$ with $\mathcal{H}^h(E)=0$, then $\mathcal{L}_n(f(E))=0$,
(5) there exists a set C_A such that $\mathcal{H}^h(C_A)\in(0,\infty)$, $\mathcal{L}_n(C_A)=0$ and $\mathcal{L}_n(f(C_A))>0$.

This is especially interesting for $\tau(t)$ being a slowly decreasing function for small t, such as $\log\log\log\dots(1/t)$. We can get as close to the power-type gauge function $h(t)=t^n$ as desired. This theorem extends the result from [28] in two ways: The statement holds for more general gauge functions, and whereas the previous result does not rule out the possibility that there exists a set of zero Hausdorff measure which is mapped onto a set of positive Lebesgue measure, we show that this is not possible (cf. [28, Theorem 1.1, property (d)] and property (4) here). In other words, we prove that there exists a mapping such that the set where the Luzin N condition is broken must be of positive Hausdorff measure.

We also study the other endpoint of the Hausdorff scale. Past results claim the size of the exceptional set to be possibly very small, but up to our knowledge, the results consider only gauge functions in the form of power $h(t) = t^\alpha$. We prove that the exceptional set C_A can be small in any possible scale of gauge functions.

Theorem 1.2. *Let $Q_0 = [-1,1]^n$ and let $h : [0,\infty) \to [0,\infty)$ be a gauge function, i.e., a continuous non-decreasing function such that $h(0) = 0$. Then there exists a homeomorphism $f : Q_0 \to Q_0$ such that*

(1) f is the identity on the boundary of Q_0,
(2) $f \in W^{1,n)}(Q_0, Q_0)$,
(3) $Jf > 0$ a.e. in Q_0,
(4) there exists a set $C_A \subseteq Q_0$ such that $\mathcal{H}^h(C_A) = 0$, $\mathcal{L}_n(C_A) = 0$ and $\mathcal{L}_n(f(C_A)) > 0$.

This theorem is interesting for h very rapidly increasing near 0, typically with a non-finite one-sided derivative. We can construct a Ponomarev-type homeomorphism such that the critical set violating the Luzin N condition is of measure 0 for the corresponding Hausdorff measure. This theorem extends previously known result (see [37]) that the dimension of the critical set may have Hausdorff dimension 0.

1.3. **Further applications of the Luzin N condition and related questions.** Let us introduce some closely related topics, applications, and development. We intend to promote papers and books that are essential for the topic, but we also point out some less known recent results.

From the historical point of view, the Peano curve [45] presented in 1890 is probably the oldest and the most known case of violating the Luzin N condition in some sense. The Cesari construction [9] can be interpreted as the Peano curve.

A question close to the Luzin N^{-1} condition is the validity of the Morse–Sard theorem in various settings, based on works of Morse [42] and Sard [52]. In a simplified version, it states that for a sufficiently smooth function, the image of the set where its Jacobian is zero has to be of zero Lebesgue measure. This principle has been extended, relaxed, and developed in many directions and applications. Naturally, one wishes to state the size of the image more subtly using the Hausdorff dimension. One can transfer the case from the Euclidean space into manifolds, see [53]. Also, the assumption of C^k smoothness may be relaxed, so Lipschitz mappings [2], Hölder spaces [6], Sobolev spaces [10, 16], or BV spaces [7] are also studied. Note that this list is picking just some highlights, and many other particular settings and applications were published recently, such as the application to PDEs in chemistry [56] or the application in studies of the Besicovitch–Federer projection theorem [17, 19].

The other closely related question is the problem of the composition of operators and the regularity of the inverse operator. The composition may produce outcomes with unexpected and unusual properties if the Luzin N or N^{-1} condition is not met, as it is often exploited to construct counterexamples. The boundedness and integrability of the distortion are studied to provide the validity of the Luzin N^{-1} condition. We recommend the following classical books on this topic [3], [24], [50], [54], and [49].

Another topic involving the Luzin N condition is the question of the equivalence between the pointwise Jacobian and the distributive Jacobian, first asked by Ball [4]. It is interesting since this equivalence is often assumed. By its characterization, we may either replace this assumption in a statement or alternate its proof. This question was addressed by Müller [43], by Iwaniec and Sbordone [25], and by Greco [20] mostly by integrability properties. The integrability requirements may be significantly relaxed in case of the validity of the Luzin N condition, as shown by D'Onofrio, Hencl, Malý, and Schiattarella [11] based on the previous research by Henao and Mora-Corral [22].

There is also a very interesting way to fail both of these conditions with such a restrictive setting as a Sobolev or even bi-Sobolev homeomorphism satisfying $Jf = 0$ a.e. Such examples can map a full measure set to a zero measure set and a zero measure set to a full measure set.

Also, these mappings provide a tool to construct other homeomorphisms with highly counter-intuitive properties concerning the preservation of matter or orientation, the change of the sign of the Jacobian and others, see [23], [12], [8], [14], [44] or [35].

At the end of this section, we shortly present recent development concerning the research of the Luzin N condition itself. For a survey of the development, see Koskela, Malý, and Zürcher [33]. For refinement by studying the modulus of continuity and the size of the critical set, see [34]. The paper concerning the failure of the Luzin N condition by Kauranen and Koskela [29] extended the classical result [37] and it was also later used by Zapadinskaya [55] to transfer the knowledge from the Euclidian case into more general metric measure spaces. Also, the counterexample of Ponomarev is refined with additional regularity such that it still violates the Luzin N condition (see [51]) or the N^{-1} condition (see [31]). In papers studying the Luzin N condition in view of Hausdorff dimension, the term $(\alpha-\beta)$ N *condition* is used, see [1, 15].

2. Preliminaries

By a *gauge function* $h:[0,\infty)\to[0,\infty)$ we denote a function satisfying

(1) h is non-decreasing,
(2) $h(0)=0$,
(3) h is continuous.

By the *Hausdorff measure* $\mathcal{H}^h(A)$ of a set $A\subseteq\mathbb{R}^n$ we understand

$$\mathcal{H}^h(A)=\lim_{\delta\to 0_+}\left(\inf\left\{\sum_{i=1}^{\infty}h(\operatorname{diam}U_i):A\subseteq\bigcup_i U_i;\operatorname{diam}(U_i)<\delta\right\}\right).$$

The definition may slightly differ in literature. The limit can be replaced by the supremum over positive δ. Some authors choose the covering system U_i to consist of general open sets, and others use the definition with the coverings just by open balls. Note that for the most classical case $h(t)=t^\alpha$ we write $\mathcal{H}^\alpha$ instead of $\mathcal{H}^{t^\alpha}$.

By the *Hausdorff dimension* of a set A we understand

$$\dim_H(A)=\inf_{d\geq 0}\{\mathcal{H}^d(A)=0\}.$$

We claim that our examples belong to the grand Sobolev space $W^{1,n)}$. This space is introduced in [25] by Iwaniec and Sbordone, we refer to [13] for a survey of the notion. The *grand Lebesgue norm* is

$$\|f\|_{q)}=\sup_{0<\varepsilon<q-1}\left(\frac{\varepsilon}{|\Omega|}\int_\Omega|f|^{q-\varepsilon}\right)^{\frac{1}{q-\varepsilon}}.$$

This norm defines the *grand Lebesgue space* $L^{q)}(\Omega)$, a Banach function space that is very close to L^q, the sharp inclusions explaining the relations between function spaces of interest are

$$L^q(\Omega)\subsetneq L^q\log^{-1}(L)(\Omega)\subsetneq L^{q)}(\Omega)\subsetneq\bigcap_{\alpha>1}L^q\log^{-\alpha}(L)(\Omega)\subsetneq\bigcap_{1<p<q}L^p(\Omega),$$

for the proofs of the inclusions see [20, Section 3]. The last inclusion is not proven there, but it may be easily proven by inequalities of the corresponding Young functions, the sharpness may be verified by the choice of function such as $f(t)=t^{-\frac{1}{q}}$ on $\Omega=(0,1)$. The *grand Sobolev space* is a set of such functions that the function itself and all its partial derivatives up to the desired rank belong to the corresponding grand Lebesgue space. We emphasize that usage of this modern tool allows for sharpening our result and extending the possibilities of the Ponomarev construction in the same way it was presented also in [28], as previously less fine Lebesgue scales were used in the foundation papers [46, 48]. The finer Lorentz scale was used in [27] in the study of the different case, the validity of the Luzin N condition for the Sobolev mappings.

In this text we use the notation $A\lesssim B$ and $A\approx B$. By $A\lesssim B$ we denote that there exists a constant K independent of parameters and depending only on the dimension and the gauge function h such that $A \leq KB$. $A \approx B$ denotes both $A \lesssim B$ and $B \lesssim A$ hold. We use the notation $\|x\|_\infty$ for the maximum norm of the vector x and $Q_{a,r} = \{x \in \mathbb{R}^n : \|x - a\|_\infty < r\}$ for the open n-dimensional cube of center a and edge length $2r$.

3. Ponomarev construction

We describe the Ponomarev construction in a general way with notation consistent with its description in [24, Theorem 4.10]. We obtain a Sobolev homeomorphism $f : (-1,1)^n \to (-1,1)^n$ with $Jf > 0$ a.e. violating the Luzin N condition.

Let $\mathbb{V}$ be the vertices of the cube $[-1,1]^n$. Let $\mathbb{V}^k = \mathbb{V} \times \mathbb{V} \times \cdots \times \mathbb{V}$, $k \in \mathbb{N}$, be a set of indices and let us consider two strictly decreasing sequences a_k and b_k such that

(1) $a_0 = 1$, $b_0 = 1$,
(2) $\lim_{k\to\infty} a_k = 0$,
(3) $\lim_{k\to\infty} b_k > 0$.

Note that this setting aims to break the Luzin N condition. In order to break the Luzin N^{-1} condition we demand $\lim_{k\to\infty} a_k > 0$ and $\lim_{k\to\infty} b_k = 0$ instead. However, in order to make the resulting mapping interesting, we have to set a_k and b_k carefully and check the crucial property, the integrability of the distortion.

Let us define $z_0 = \tilde{z}_0 = 0$ and

$$r_k = 2^{-k} a_k \text{ and } \tilde{r}_k = 2^{-k} b_k.$$

We start with $Q(z_0, r_0) = (-1,1)^n$ and proceed by induction. For $\mathbf{v} = [v_1, v_2, v_3, \dots, v_k] \in \mathbb{V}^k$ we denote $\mathbf{w}(\mathbf{v}) = [v_1, v_2, \dots, v_{k-1}] \in \mathbb{V}^{k-1}$ and we define

$$z_{\mathbf{v}} = z_{\mathbf{w}}(\mathbf{v}) + \frac{r_{k-1}}{2} v_k = z_0 + \sum_{i=1}^{k} \frac{r_{i-1}}{2} v_i.$$

For simplicity we write $\mathbf{w}$ instead of $\mathbf{w}(\mathbf{v})$. Around the center $z_{\mathbf{v}}$ we define an outer and inner cube

$$Q'_{\mathbf{v}} = Q\left(z_{\mathbf{v}}, \frac{r_{k-1}}{2}\right) \text{ and } Q_{\mathbf{v}} = Q\left(z_{\mathbf{v}}, r_k\right), \text{ respectively.}$$

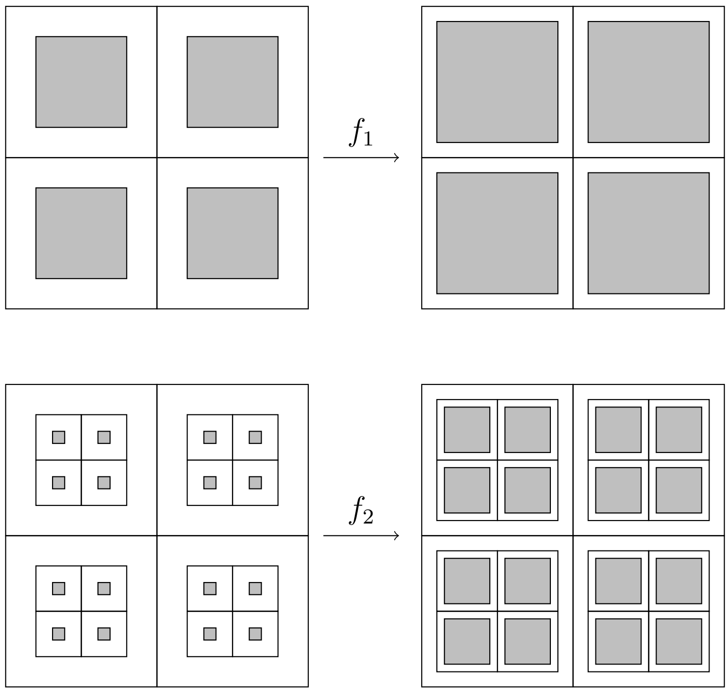


Figure 1. First two steps in the Ponomarev construction of f. Above: f_1 maps $\bigcup_{\mathbf{v}\in\mathbb{V}} Q_{\mathbf{v}}$ onto $\bigcup_{\mathbf{v}\in\mathbb{V}} \tilde{Q}_{\mathbf{v}}$. Below: f_2 maps $\bigcup_{\mathbf{v}\in\mathbb{V}^2} Q_{\mathbf{v}}$ onto $\bigcup_{\mathbf{v}\in\mathbb{V}^2} \tilde{Q}_{\mathbf{v}}$.

In the k-th step of the construction, we use indices $\mathbf{v} \in \mathbb{V}^k$ and produce 2^{nk} cubes $Q_{\mathbf{v}}$, which are copies of the same cube.

We get a Cantor-type set C_A defined as

$$C_A = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{v}\in\mathbb{V}^k} Q_{\mathbf{v}} = C_a \times C_a \times \cdots \times C_a,$$

where C_a is a Cantor-type set contained in the line segment $[-1, 1]$. Its construction is illustrated on the left-hand side of Figure 1.

Analogously for the image we define the first cube as $\tilde{Q}(z_0, r_0) = (-1, 1)^n$ and centers as

$$\tilde{z}_{\mathbf{v}} = \tilde{z}_{\mathbf{w}} + \frac{\tilde{r}_{k-1}}{2} v_k = \tilde{z}_0 + \sum_{i=1}^{k} \frac{\tilde{r}_{i-1}}{2} v_i,$$

and we define a structure of cubes by

$$\tilde{Q}'_{\mathbf{v}} = Q(\tilde{z}_{\mathbf{v}}, \frac{\tilde{r}_{k-1}}{2}) \text{ and } \tilde{Q}_{\mathbf{v}} = Q(\tilde{z}_{\mathbf{v}}, \tilde{r}_k).$$

We further define C_B as

$$C_B = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{v}\in\mathbb{V}^k} \tilde{Q}_{\mathbf{v}} = C_b \times C_b \times \cdots \times C_b,$$

where C_b is again a Cantor-type set contained in the line segment $[-1, 1]$.

Concerning the Lebesgue measure of both C_A and C_B, we obtain

$$\begin{aligned}
\mathcal{L}_n(C_A) &= \lim_{k\to\infty} \mathcal{L}_n\left(\bigcup_{\mathbf{v}\in\mathbb{V}^k} Q_{\mathbf{v}}\right) = \lim_{k\to\infty} 2^{nk}(2r_k)^n = \lim_{k\to\infty} 2^{nk-nk}2^n a_k^n = 0,\\
\mathcal{L}_n(C_B) &= \lim_{k\to\infty} 2^{nk}(2\tilde{r}_k)^n = \lim_{k\to\infty} 2^{nk-nk}2^n b_k^n = 2^n(\lim_{k\to\infty} b_k)^n > 0.
\end{aligned}$$

Our goal is to define a sequence of homeomorphism $f_k : [-1, 1]^n \to [-1, 1]^n$ such that its limit f is a homeomorphism mapping C_A onto C_B, as we demonstrate in Figure 1. We start with $f_0(x) = x$. To define f_1, we map $Q_{\mathbf{v}}$ onto $\tilde{Q}_{\mathbf{v}}$ homogenously with respect to the centres $z_{\mathbf{v}}$ and $\tilde{z}_{\mathbf{v}}$ for all $\mathbf{v} \in \mathbb{V}$. We define f_1 from $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ onto $\tilde{Q}'_{\mathbf{v}} \setminus \tilde{Q}_{\mathbf{v}}$ radially for the supremum norm with respect to the centres $z_{\mathbf{v}}$ and $\tilde{z}_{\mathbf{v}}$. In the general step, we keep $f_k = f_{k-1}$ on $[-1, 1]^n \setminus (\bigcup_{\mathbf{v}\in\mathbb{V}^k} Q'_{\mathbf{v}})$. It remains to define f_k inside the copies of $Q'_{\mathbf{v}}$. We use the homogeneous mapping of $Q_{\mathbf{v}}$ onto $\tilde{Q}_{\mathbf{v}}$ and the radial mapping of $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ onto $\tilde{Q}'_{\mathbf{v}} \setminus \tilde{Q}_{\mathbf{v}}$, both with respect to centres $z_{\mathbf{v}}$ and $\tilde{z}_{\mathbf{v}}$, see Figure 2.

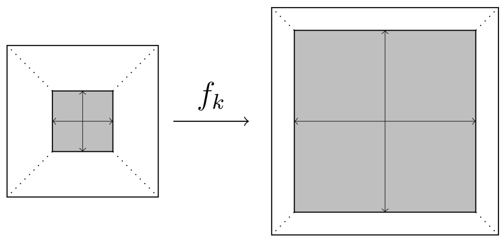


FIGURE 2. The mapping f_k transforms $Q_{\mathbf{v}}$ onto $\tilde{Q}_{\mathbf{v}}$ (the gray area) and $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ onto $\tilde{Q}'_{\mathbf{v}} \setminus \tilde{Q}_{\mathbf{v}}$ (the white area), $\mathbf{v} \in \mathbb{V}^k$.

Formally, we define

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{for } x \notin \bigcup_{\mathbf{v}\in\mathbb{V}^k} Q'_{\mathbf{v}},\\ f_{k-1}(z_{\mathbf{v}}) + (\alpha_k \|x - z_{\mathbf{v}}\|_\infty + \beta_k)\frac{x - z_{\mathbf{v}}}{\|x - z_{\mathbf{v}}\|_\infty} & \text{for } x \in Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}, \mathbf{v} \in \mathbb{V}^k,\\ f_{k-1}(z_{\mathbf{v}}) + \frac{\tilde{r}_k}{r_k}(x - z_{\mathbf{v}}) & \text{for } x \in Q_{\mathbf{v}}, \mathbf{v} \in \mathbb{V}^k, \end{cases}$$

where α_k and β_k are chosen for f_k to map the annulus $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ onto the annulus $\tilde{Q}'_{\mathbf{v}} \setminus \tilde{Q}_{\mathbf{v}}$, i.e., such that

$$\alpha_k r_k + \beta_k = \tilde{r}_k \text{ and } \alpha_k \frac{r_{k-1}}{2} + \beta_k = \frac{\tilde{r}_{k-1}}{2}. \tag{3.1}$$

Note that such f_k maps

$$\bigcup_{\mathbf{v}\in\mathbb{V}^j} Q_{\mathbf{v}} \text{ onto } \bigcup_{\mathbf{v}\in\mathbb{V}^j} \tilde{Q}_{\mathbf{v}} \tag{3.2}$$

for all $j \leq k$. Since f_k is continuous and a one-to-one mapping between compact spaces, it is a homeomorphism.

We need to estimate the derivatives of f_k in $Q_{\mathbf{v}}$ and $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ for $\mathbf{v} \in \mathbb{V}^k$. For $x \in Q_{\mathbf{v}}$ we get

$$|Df_k| = \frac{\tilde{r}_k}{r_k} = \frac{b_k}{a_k}.$$

For $x \in Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ we should consider two possible directions of partial derivatives, based on which coordinate determines the norm $\|x - z_{\mathbf{v}}\|_\infty$. Without loss of generality, suppose it is the first coordinate, the set containing points with more coordinates like that is of zero Lebesgue measure and does not change further calculations and estimates. For $x \in Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ we estimate

$$\begin{aligned} |D_{x_1} f_k| &= \left| D_{x_1} \left((\alpha_k \|x - z_{\mathbf{v}}\|_\infty + \beta_k) \frac{x - z_{\mathbf{v}}}{\|x - z_{\mathbf{v}}\|_\infty} \right) \right| \leq \alpha_k, \\ |D_{x_i} f_k| &= \left| D_{x_i} \left((\alpha_k \|x - z_{\mathbf{v}}\|_\infty + \beta_k) \frac{x - z_{\mathbf{v}}}{\|x - z_{\mathbf{v}}\|_\infty} \right) \right| \leq \alpha_k + \frac{\beta_k}{\|x - z_{\mathbf{v}}\|_\infty}, \quad i \neq 1. \end{aligned} \tag{3.3}$$

Therefore, each mapping f_k belongs to $W^{1,\infty}$ (however, the sequence is not bounded there).

The limit mapping f is absolutely continuous on almost all lines which are parallel to the coordinate axes, since almost all lines do not intersect the Cantor set C_A. Hence f is Lipschitz on such lines. Also, f maps C_A onto C_B, based on (3.2). Its pointwise partial derivatives on $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ for $\mathbf{v} \in \mathbb{V}^k$ are the same as those of f_k. In the end, we estimate

$$\|Df\|_p^p = \sum_{k=1}^{\infty} \sum_{\mathbf{v}\in\mathbb{V}^k} \int_{Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}} |Df|^p.$$

We should also check that the Jacobian is positive almost everywhere. Since J_f is equal to J_{f_k} on the sets $Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$ and the union of these sets has full measure, it is enough to verify the positivity of J_{f_k}, which can be done by a straightforward calculation.

Remark 3.1. The choice $a_k = \frac{1}{k+1}$ and $b_k = \frac{1}{2}(1 + \frac{1}{k+1})$ provides a pointwise estimate $|Df| \lesssim k$ for $x \in Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$, $\mathcal{L}_n(Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}) \approx 2^{-nk} \frac{1}{k^{n+1}}$, and $|Df| \in L^p$ if $p < n$. Note that these estimates can be adjusted to the special choice of a_k and b_k and they differ in literature.

4. Proof of Theorem 1.1 and Theorem 1.2

We now present the estimate for the norm of the derivative for a fairly general choice of a_k and b_k. We show that for this choice, the resulting mapping belongs to the grand Sobolev space $W^{1,n)}(Q_0, Q_0)$.

Let a_k be an arbitrary monotone positive sequence with $a_0 = 1$ and $\lim_{k\to\infty} a_k = 0$ and set

$$b_k = \frac{1}{2}(1 + a_k).$$

This together with (3.1) implies

$$\alpha_k = 2^{-1} \text{ and } \beta_k = 2^{-k-1}$$

for $k \geq 1$. For further use we prepare the pointwise estimate of $|Df_k(x)|$ for $x \in Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{V}^k$ based on (3.3). We get

$$|Df_k(x)| \approx \max_{i \in \{1,\dots,n\}} \{|D_{x_i} f_k|\} = \max \left\{ \alpha_k, \alpha_k + \frac{\beta_k}{\|x - z_v\|_\infty} \right\} \lesssim \frac{\beta_k}{\|x - z_v\|_\infty}.$$

The following estimate is universal for both Theorem 1.1 and Theorem 1.2 and may be used for any a_k, b_k satisfying the properties above. Using the fact that for each k we have 2^{nk} annuli with the same size, between which the function differs only by translation, we calculate

$$\begin{aligned}
\sup_{0<\varepsilon\leq n-1} \varepsilon \int_{(-1,1)^n} |Df|^{n-\varepsilon} &= \sup_{0<\varepsilon\leq n-1} \varepsilon \left(\sum_{k=1}^{\infty} \sum_{\mathbf{v} \in \mathbb{V}^k} \int_{Q'_{\mathbf{v}} \setminus Q_{\mathbf{v}}} |Df_k|^{n-\varepsilon} \right) \\
&\lesssim \sup_{0<\varepsilon\leq n-1} \varepsilon \sum_{k=1}^{\infty} 2^{nk} \int_{Q\left(0, \frac{r_{k-1}}{2}\right) \setminus Q(0, r_k)} \left(\frac{\beta_k}{\|x\|_\infty} \right)^{n-\varepsilon} dx \\
&\lesssim \sup_{0<\varepsilon\leq n-1} \varepsilon \sum_{k=1}^{\infty} 2^{nk} \int_{2^{-k} a_k}^{2^{-k} a_{k-1}} \left(\frac{2^{-k-1}}{t} \right)^{n-\varepsilon} t^{n-1} \, dt \\
&= \sup_{0<\varepsilon\leq n-1} \varepsilon \sum_{k=1}^{\infty} 2^{nk} \int_{2^{-k} a_k}^{2^{-k} a_{k-1}} 2^{(-k-1)(n-\varepsilon)} t^{-1+\varepsilon} \, dt \\
&\lesssim \sup_{0<\varepsilon\leq n-1} \varepsilon \sum_{k=1}^{\infty} \left(2^{(k+1)\varepsilon} \left[\varepsilon^{-1} t^{\varepsilon} \right]_{2^{-k} a_k}^{2^{-k} a_{k-1}} \right) = \sup_{0<\varepsilon\leq n-1} \sum_{k=1}^{\infty} 2^{(k+1)\varepsilon} \left(2^{-\varepsilon k} a_{k-1}^{\varepsilon} - 2^{-\varepsilon k} a_k^{\varepsilon} \right) \\
&= \sup_{0<\varepsilon\leq n-1} 2^{\varepsilon} \sum_{k=1}^{\infty} (a_{k-1}^{\varepsilon} - a_k^{\varepsilon}) \lesssim \sup_{0<\varepsilon\leq n-1} \left(a_0^{\varepsilon} - \lim_{k\to\infty} a_k^{\varepsilon} \right) = \sup_{0<\varepsilon\leq n-1} a_0^{\varepsilon} = 1 < \infty,
\end{aligned}$$

since the limit of a_k^{ε} is zero. Therefore $f \in W^{1,n)}(Q_0, Q_0)$.

Proof of Theorem 1.2. We choose a_k satisfying the conditions above (i.e., monotone positive with limit 0 and $a_0 = 1$) such that for every integer $k \geq 1$ we have

$$h(c_n 2^{-k} a_k) < 2^{-2nk},$$

where $c_n = 2\sqrt{n}$; we can do so, since h is non-decreasing continuous and $\lim_{t\to 0_+} h(t) = 0$. Set $b_k = (1 + a_k)/2$ as before. The Ponomarev type construction described in Section 3 ensures the properties (1) and (3) and the choice of parameters gives us (2). It remains to prove (4). Since

$$C_A \subseteq \bigcup_{\mathbf{v} \in \mathbb{V}^k} Q_{\mathbf{v}}$$

for an arbitrary k, from the definition of Hausdorff measure we have

$$\mathcal{H}^h(C_A) \leq \limsup_{k\to\infty} \sum_{\mathbf{v} \in \mathbb{V}^k} h(\operatorname{diam} Q_{\mathbf{v}}) = \limsup_{k\to\infty} 2^{nk} h(c_n r_k) \leq \lim_{k\to\infty} 2^{-nk} = 0.$$

Also $\mathcal{L}_n(C_A) = 0$ and $\mathcal{L}_n(f(C_A)) = \mathcal{L}_n(C_B) > 0$ as was shown in Section 3. □

Proof of Theorem 1.1. The proof is divided into several steps.

(i) Choice of a_k
We claim that we can find a decreasing sequence a_k satisfying the properties from Section 3 such that $a_k^n \tau(2^{-k} c_n a_k) \approx 1$. Since τ is continuous and bounded by 1 from below, for every parameter p there has to be a point $t_p \in (0, 1]$ such that $1/\tau(p t_p) = t_p^n$ and $1/\tau(pt) > t^n$ on $(0, t_p)$. We set $a_k = t_{2^{-k}}$. To show that it is a monotone sequence, let us have $p_1 > p_2$ and elaborate. From the monotonicity of τ we have

$$t_{p_1}^n = \frac{1}{\tau(p_1 t_{p_1})} > \frac{1}{\tau(p_2 t_{p_1})}.$$

This implies that t_{p_2} must be smaller than t_{p_1}, since $1/\tau(p_2 t) > t^n$ on $(0, t_{p_2})$. Now choose $\varepsilon > 0$ and find p small enough such that $1/\tau(pt) < \varepsilon^n$ for $t \in (0,1]$. Since $t_p \in (0,1]$, we have $t_p^n < \varepsilon^n$. This ensures that the limit of a_k is zero. With this choice of the sequence a_k, for any $\mathbb{U}$ subsystem of $\mathbb{V}^k$ we obtain

$$\sum_{\mathbf{u}\in\mathbb{U}} h(\operatorname{diam} Q_{\mathbf{u}}) = \#\mathbb{U} h(2^{-k} c_n a_k) \approx \#\mathbb{U} 2^{-nk} a_k^n \tau(2^{-k} c_n a_k) \approx 2^{-nk} \#\mathbb{U}, \tag{4.1}$$

where $\#\mathbb{U}$ denotes the number of elements of the system.

(ii) Properties (1) – (3)
By setting $b_k = (1 + a_k)/2$ and proceeding as in Section 3, we obtain a Sobolev homeomorphism f which satisfies properties (1) – (3).

(iii) $0 < \mathcal{H}^h(C_A) < \infty$
We immediately see from (4.1) that the Hausdorff measure of C_A is finite, since

$$\mathcal{H}^h(C_A) \le \limsup_{k\to\infty} \sum_{\mathbf{v}\in\mathbb{V}^k} h(\operatorname{diam} Q_{\mathbf{v}}) \approx \lim_{k\to\infty} 2^{-nk} \#\mathbb{V}^k = 1.$$

The other inequality is proven in several steps. We mimic the proof from [28, Lemma 3.2], which is inspired by [41, Section 4.10]. Since C_A is a compact set, it is enough to prove that for any finite open covering $\{U_j\}$ of C_A we have

$$\sum_j h(\operatorname{diam} U_j) \gtrsim 1. \tag{4.2}$$

We may assume that there exists $x_j \in C_A \cap U_j$ for each j. Therefore $B(x_j, \operatorname{diam} U_j) = B_j \supseteq U_j$ and

$$\begin{aligned}\sum_j h(\operatorname{diam} U_j) &= \sum_j h(\operatorname{diam} B_j/2) = \sum_j 2^{-n} (\operatorname{diam} B_j)^n \tau(\operatorname{diam} B_j/2)\\ &\ge \sum_j 2^{-n} (\operatorname{diam} B_j)^n \tau(\operatorname{diam} B_j) = \sum_j 2^{-n} h(\operatorname{diam} B_j),\end{aligned}$$

so we may consider only coverings by balls in (4.2). We now wish to show that for every $l \in \mathbb{N}$ and j we have

$$\sum_{\substack{\mathbf{v}\in\mathbb{V}^l,\\ Q_{\mathbf{v}}\subseteq B_j}} h(\operatorname{diam} Q_{\mathbf{v}}) \lesssim h(\operatorname{diam} B_j).$$

This can be proven by taking $Q_{\mathbf{v}_0} \subseteq B_j$ for some $\mathbf{v}_0 \in \mathbb{V}^l$ and m the smallest integer such that $Q_{\mathbf{u}_0} \subseteq B_j$ for some $\mathbf{u}_0 \in \mathbb{V}^m$ (obviously, $m \le l$). Set

$$\mathbb{U} = \{\mathbf{u} \in \mathbb{V}^m : Q_{\mathbf{u}} \cap B_j \ne \emptyset\}.$$

Since B_j is centered at a point from C_A, from the definition of m we obtain

$$r_m \lesssim \operatorname{diam} B_j \lesssim r_{m-1}.$$

Therefore there exists an upper bound for the number of pairwise disjoint cubes of side length r_{m-1}, which have a non-empty intersection with B_j, and this upper bound is independent of j and m. Since the size of $\mathbb{U}$ is at most 2^n times this number, we have an (independent) upper bound for $\#\mathbb{U}$, too.

Together with (4.1) it provides

$$\begin{aligned} h(\operatorname{diam} B_j) &\geq h(\operatorname{diam} Q_{\mathbf{u}_0}) \gtrsim \sum_{\mathbf{u}\in\mathbb{U}} h(\operatorname{diam} Q_{\mathbf{u}}) \approx 2^{-nm}\#\mathbb{U} \\ &= 2^{-nl}\#\{\mathbf{v}\in\mathbb{V}^l : Q_{\mathbf{v}}\subseteq Q_{\mathbf{u}}, \mathbf{u}\in\mathbb{U}\} \\ &\approx \sum_{\mathbf{u}\in\mathbb{U}} \sum_{\substack{\mathbf{v}\in\mathbb{V}^l,\\ Q_{\mathbf{v}}\subseteq Q_{\mathbf{u}}}} h(\operatorname{diam} Q_{\mathbf{v}}) \geq \sum_{\substack{\mathbf{v}\in\mathbb{V}^l,\\ Q_{\mathbf{v}}\subseteq B_j}} h(\operatorname{diam} Q_{\mathbf{v}}). \end{aligned}$$

Finally, since C_A is compact, there exists k_0 such that for every $Q_{\mathbf{v}} \in \mathbb{V}^k$, $k \geq k_0$, we can find j such that $Q_{\mathbf{v}} \subseteq B_j$. For such k we have

$$\sum_j h(\operatorname{diam} B_j) \gtrsim \sum_j \sum_{\substack{\mathbf{v}\in\mathbb{V}^k,\\ Q_{\mathbf{v}}\subseteq B_j}} h(\operatorname{diam} Q_{\mathbf{v}}) \geq \sum_{\mathbf{v}\in\mathbb{V}^k} h(\operatorname{diam} Q_{\mathbf{v}}) \approx 1.$$

This combined gives us the desired property that $\mathcal{H}^h(C_A) > 0$. Combined with the fact that $f(C_A) = C_B$ we have (5) (the Lebesgue measure properties are obvious from previous sections).

(iv) Construction of mapping z
For each point $x \in C_A$ we can find $\mathbf{v}_x$ from $\mathbb{V}^{\mathbb{N}}$ such that

$$x = \bigcap_j Q_{(\mathbf{v}_x)_j}.$$

The correspondence between x and $\mathbf{v}_x$ is one-to-one. Let π_i denote the projection of $\mathbf{v} \in \mathbb{V}$ to its i-th coordinate. Define $c_i : C_A \to \{-1, 1\}^{\mathbb{N}}$ which (in each coordinate) tells whether we chose a cube "on the right-hand side or on the left-hand side", i.e.,

$$c_i(x) = \{\pi_i((\mathbf{v}_x)_j)\}_{j=1}^{\infty}.$$

Next consider a function $\operatorname{Bin} : \{-1, 1\}^{\mathbb{N}} \to [0, 1]$, which takes a sequence $\mathbf{u}$ and interprets it as the number $0.\frac{\mathbf{u}_1+1}{2}\frac{\mathbf{u}_2+1}{2}\dots$ written in the binary system. This is obviously onto, however, it is not injective (because for example both $(0, 1, 1, 1, \dots)$ and $(1, 0, 0, 0, \dots)$ are mapped to $1/2$). We denote

$$z(x) = (\operatorname{Bin}(c_1(x)), \dots, \operatorname{Bin}(c_n(x))) : C_A \to [0, 1]^n.$$

Then z is onto and it is injective outside of the set

$$\begin{aligned} S &= \{x \in C_A : \operatorname{Bin}(c_i(x)) = k/2^j \text{ for some } i \in \{1, \dots, n\}, j \in \mathbb{N}_0 \text{ and } k \in \{0, \dots, 2^j\}\} \\ &= \{x \in C_A : c_i(x) \text{ is constant from some index } j_0 \in \mathbb{N} \text{ for some } i \in \{1, \dots, n\}\}, \end{aligned}$$

which consists of the preimages of boundaries of dyadic cubes in $[0, 1]^n$.

(v) Image of $\mathcal{H}^h$ under z
We start with showing that $\mathcal{H}^h(S) = 0$ and $\mathcal{L}_n(z(S)) = 0$. The second statement follows simply from the fact that the boundary of a dyadic cube is a set of (Lebesgue) measure zero and $z(S)$ is their countable union. The first statement is proven in a similar way since S is a countable union of the sets in the form

$$S_{i,j,k} = \{x \in C_A : \operatorname{Bin}(c_i(x)) = k/2^j\}$$

for $i \in \{1, \dots, n\}, j \in \mathbb{N}_0$ and $k \in \{0, \dots, 2^j\}$. These are (up to a permutation of coordinates and a translation) equal to $\{0\} \times C_a \times \dots \times C_a$ and $\mathcal{H}^h(S_{i,j,k}) = 0$, because $\mathcal{H}^h(C_A) < \infty$ and C_A contains uncountably many pairwise disjoint copies of $\{0\} \times C_a \times \dots \times C_a$.

Now we show the equality of the measures

$$z(\mathcal{H}^h) = \left(\mathcal{H}^h(C_A)\right) \mathcal{L}_n.$$

For any open dyadic cube D of edge length 2^{-j} take the corresponding $\mathbf{v} \in \mathbb{V}^j$. Then $\mathcal{L}_n(D) = 2^{-jn}$ and $\mathcal{H}^h(z^{-1}(D)) = \mathcal{H}^h(Q_{\mathbf{v}} \cap C_A) = 2^{-jn}\mathcal{H}^h(C_A)$, because $z^{-1}(D) = Q_{\mathbf{v}} \cap C_A \setminus S'$, where S' is a suitable subset of S.

The system

$$\mathcal{D} = \{D \cup S' : D \text{ is an open dyadic cube and } S' \text{ is a measurable subset of } S\}$$

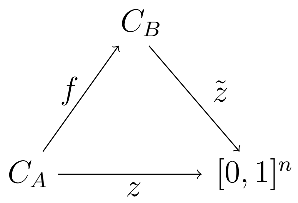
is closed under finite intersections and the sigma algebra generated by $\mathcal{D}$ contains all Borel sets in $[0,1]^n$. Since $z(\mathcal{H}^h) = \big(\mathcal{H}^h(C_A)\big)\, \mathcal{L}_n$ on elements from $\mathcal{D}$, they are the same on $[0,1]^n$.

(vi) Property (4)

We can analogously construct $\tilde{z} : C_B \to [0,1]^n$ for which

$$\tilde{z}(\mathcal{L}_n) = (\mathcal{L}_n(C_B))\, \mathcal{L}_n.$$

Then from the fact that $f(Q_{\mathbf{v}}) = \tilde{Q}_{\mathbf{v}}$ for an arbitrary $\mathbf{v}$ it follows that $\tilde{z} \circ f = z$, i.e., the following diagram commutes:



The injectivity of z is broken only on S and $\mathcal{H}^h(S) = 0$ and $\mathcal{L}_n(z(S)) = 0$ (analogously for $\tilde{z}$). We conclude that for an arbitrary measurable $E \subseteq C_A$ we have

$$\mathcal{H}^h(E) = 0 \iff \mathcal{L}_n(z(E)) = 0 \iff \mathcal{L}_n(\tilde{z}(f(E))) = 0 \iff \mathcal{L}_n(f(E)) = 0.$$

The Luzin N condition holds outside of C_A since f is locally Lipschitz there, and any set with finite measure $\mathcal{H}^h$ is of zero Lebesgue measure. Therefore property (4) holds.

□

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Paper II

MAPPINGS OF GENERALIZED FINITE DISTORTION AND CONTINUITY

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Abstract. We study continuity properties of Sobolev mappings $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geq 2$, that satisfy the following generalized finite distortion inequality

$$|Df(x)|^n \leq K(x) J_f(x) + \Sigma(x)$$

for almost every $x \in \mathbb{R}^n$. Here $K \colon \Omega \to [1, \infty)$ and $\Sigma \colon \Omega \to [0, \infty)$ are measurable functions. Note that when $\Sigma \equiv 0$, we recover the class of mappings of finite distortion, which are always continuous. The continuity of arbitrary solutions, however, turns out to be an intricate question. We fully solve the continuity problem in the case of bounded distortion $K \in L^\infty(\Omega)$, where a sharp condition for continuity is that Σ is in the Zygmund space $\Sigma \log^\mu(e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\mu > n - 1$. We also show that one can slightly relax the boundedness assumption on K to an exponential class $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ with $\lambda > n+1$, and still obtain continuous solutions when $\Sigma \log^\mu(e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega)$ with $\mu > \lambda$. On the other hand, for all $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, we construct a discontinuous solution with $K \in L^p_{\mathrm{loc}}(\Omega)$ and $\Sigma/K \in L^q_{\mathrm{loc}}(\Omega)$, including an example with $\Sigma \in L^\infty_{\mathrm{loc}}(\Omega)$ and $K \in L^1_{\mathrm{loc}}(\Omega)$.

1. Introduction

Let Ω be a connected, open subset of $\mathbb{R}^n$ with $n \geq 2$. Recall that a *differential inclusion* is a condition requiring that, for almost every (a.e.) $x \in \Omega$, a weakly differentiable mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ satisfies $Df(x) \in F(x, f(x))$ where F is a function from $\Omega \times \mathbb{R}^m$ to subsets of $m \times n$-matrices. Here, we are searching for differential inclusions under which a Sobolev map $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a continuous representative. More specifically, we are interested in ones which are motivated by the Geometric Function Theory, with connections to mathematical models of Nonlinear Elasticity.

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This leads us to consider the differential inclusions given by the set functions

$$\mathcal{M}_n(K,\Sigma)\colon x \mapsto \{A \in \mathbb{R}^{n\times n} \colon |A|^n \le K(x)\det A + \Sigma(x)\}\,, \tag{1.1}$$

where $K\colon \Omega \to [1,\infty)$ and $\Sigma\colon \Omega \to [0,\infty)$ are given measurable functions. Here and in what follows, $|A|$ stands for the operator norm of matrix $A \in \mathbb{R}^{n\times n}$; that is, $|A| = \sup\{|Ah| : h \in \mathbb{S}^{n-1}\}$. We also use the shorthand $G \in \mathcal{M}_n(K,\Sigma)$ if $G\colon \Omega \to \mathbb{R}^{n\times n}$ satisfies $G(x) \in \mathcal{M}_n(K,\Sigma)(x)$ for a.e. $x \in \Omega$. Now, our continuity problem reads as follows.

Problem 1.1. *Find a necessary and sufficient condition on the functions K and Σ which guarantees that if $f \in W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ with $Df \in \mathcal{M}_n(K,\Sigma)$, then f has a continuous representative.*

A necessary condition for Problem 1.1 is that Σ must at least to lie in the Zygmund space $L\log^\mu L_{\mathrm{loc}}(\Omega)$ for some $\mu > n-1$: that is,

$$\Sigma \log^\mu(e+\Sigma) \in L^1_{\mathrm{loc}}(\Omega) \qquad \mu > n-1\,. \tag{1.2}$$

Indeed, the mapping $f\colon \mathbb{B}^n(0,1) \to \mathbb{R}^n$ defined by

$$f(x) = \left(\log\log\log \frac{e^e}{|x|}, 0, \dots, 0\right) \tag{1.3}$$

has $\det Df \equiv 0$ and $|Df|^n \log^{n-1}(e+|Df|^n) \in L^1(\mathbb{B}^n(0,1))$, but $\lim_{x\to 0} |f(x)| = \infty$.

1.1. Results for bounded K.

When $\Sigma \equiv 0$ and $K \in L^\infty(\Omega)$, $\mathcal{M}_n(K,0)$ recovers the *mappings of bounded distortion*, also known as *quasiregular* mappings; a mapping $f\colon \Omega \to \mathbb{R}^n$ is *K-quasiregular* for $K \in [1,\infty)$ if $f \in W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ with $|Df(x)|^n \le K \det Df(x)$ for a.e. $x \in \Omega$. Homeomorphic K-quasiregular mappings are called *K-quasiconformal*. The first breakthrough in the theory of mappings of bounded distortion was Reshetnyak's theorem on Hölder continuity: a K-quasiregular mapping is locally $1/K$-Hölder continuous, see [23] and [24, Corollary II.1]. Such Hölder continuity properties of quasiconformal mappings in the plane were earlier established by Morrey [21].

Other differential inclusions of the type $\mathcal{M}_n(K,\Sigma)$ with $K \in L^\infty(\Omega)$ have also arisen naturally in different contexts. For instance, Simon [26] developed a local regularity theory for minimal graphs of functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ such that the Gauss map of the graph of u satisfies

$$|Df(x)|^2 \le K \det Df(x) + \Sigma \tag{1.4}$$

where $1 \le K < \infty$ and $0 \le \Sigma < \infty$ are given constants. Recall that the *Gauss map* takes the points of a surface $S \subset \mathbb{R}^n$ to the unit normal vector in $\mathbb{S}^{n-1}$. In particular, the Gauss map automatically satisfies (1.4) when u is a solution of any equation of mean curvature type [26, (1.9) (ii)]. Similar results for simply connected surfaces embedded in $\mathbb{R}^3$ are due to Schoen and Simon [25]. The main result in [26] enabling the regularity theory states that a local $W^{1,2}$-solution to (1.4) between embedded 2D-surfaces is Hölder continuous; see also [9, Ch. 12].

This Hölder continuity result has been generalized for unbounded Σ as well. Precisely, if $K \in L^\infty$ and $\Sigma \in L^p_{\text{loc}}(\Omega)$ for some $p > 1$, then a mapping $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with $Df \in \mathcal{M}_n(K, \Sigma)$ has a Hölder continuous representative. For the planar case, see the proof of [3, Theorem 8.5.1] by Astala, Iwaniec and Martin, and for the more general case $n \geq 2$, see the argument in [18, Section 3] by Kangasniemi and Onninen. While the planar argument of Astala, Iwaniec and Martin relies on complex potential theory, the higher dimensional proof is closer to that of Simon [26], mimicking the lines of reasoning by Morrey [21] and Reshetnyak [23] in the case of mappings of bounded distortion.

However, despite yielding sharp results on the L^p-scale, the Morrey-type decay argument used in [18, Section 3] does not give a sharp result if one moves to the Zygmund space setting $\Sigma \in L \log^\mu L_{\text{loc}}(\Omega)$. In particular, the decay argument shows continuity when $\mu > n$, but the optimal regularity assumption for Σ is in fact $\mu > n - 1$, precisely the minimal necessary condition stated in (1.2). This optimal regularity theorem is our first main result.

Theorem 1.2. *Suppose that $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $Df(x) \in \mathcal{M}_n(K, \Sigma)(x)$ a.e. in Ω, with*

$$K \in L^\infty_{\text{loc}}(\Omega) \qquad \text{and} \qquad \Sigma \log^\mu (e + \Sigma) \in L^1_{\text{loc}}(\Omega),$$

for some $\mu > n - 1$. Then f has a continuous representative.

Furthermore, under the assumptions of Theorem 1.2, the local modulus of continuity

$$\omega_f(x_0, r) = \sup\{|f(x_0) - f(x)| : x \in \Omega\,, \ \ |x - x_0| \leq r\} \tag{1.5}$$

is majorized by $C \log^{-(\mu-n+1)/n}(1/r)$ for $x_0 \in \Omega$ and small $r > 0$. By considering functions of the form $f(x) = (\log^{-\alpha} |x|^{-1}, 0, \ldots, 0)$ with $\alpha > 0$, it is easy to see that the above exponent $(\mu - n + 1)/n$ is sharp.

Theorem 1.2 is obtained by proving the following sharp higher integrability result for Df on the Zygmund scale.

Theorem 1.3. *Suppose that $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$K \in L^\infty_{\text{loc}}(\Omega) \qquad \text{and} \qquad \Sigma \log^\mu (e + \Sigma) \in L^1_{\text{loc}}(\Omega),$$

for some $\mu \geq 0$. Then $|Df|^n \log^\mu(e + |Df|) \in L^1_{\text{loc}}(\Omega)$.

It is worth noting that the sharp local $1/K$-Hölder continuity result for spatial K-quasiregular mappings cannot be obtained from known higher integrability results. Indeed, while K-quasiregular mappings have been shown to belong to the Sobolev space $W^{1,pn}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some $p > 1$ [8, 20], the sharp exponent $p = p(n, K)$ remains unknown when $n \geq 3$. A well-known conjecture asserts that

$$p(n, K) = \frac{K}{K-1}\,. \tag{1.6}$$

In a seminal work, Astala [2] established the sharp exponent in the planar case.

This conjecture also has a counterpart for mappings $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $Df \in \mathcal{M}_n(K, \Sigma)$. Indeed, if $\|K\|_{L^\infty(\Omega)} \leq K_\circ$, we expect that $f \in W^{1,pn}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ whenever $\Sigma \in L^p_{\mathrm{loc}}(\Omega)$ for all $p \leq p(n, K_\circ)$, where $p(n, K_\circ)$ is as in (1.6). This is the maximal amount of higher integrability of Df possible when $\Sigma \in L^p_{\mathrm{loc}}(\Omega)$, which can be seen by taking $f = (g, 0, \ldots, 0)$ and $\Sigma = |\nabla g|^n$, where g is any function in $W^{1,pn}_{\mathrm{loc}}(\Omega) \setminus \bigcup_{q>n} W^{1,qn}_{\mathrm{loc}}(\Omega)$. However, similar to the quasiregular theory, current tools are only enough to prove a result like this with an unknown value of $p(n, K_\circ)$.

Theorem 1.4. *For given $n \geq 2$ and $K_\circ \in [1, \infty)$, there exists a value $p(n, K_\circ) > 1$, such that if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$\|K\|_{L^\infty(\Omega)} \leq K_\circ \qquad \textit{and} \qquad \Sigma \in L^p_{\mathrm{loc}}(\Omega),$$

for some $p \in [1, p(n, K_\circ))$, then $|Df|^n \in L^p_{\mathrm{loc}}(\Omega)$.

1.2. **Results for general K.** In the last 20 years, systematic studies of *mappings of finite distortion* have emerged in the field of geometric function theory. Recall that a mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has finite distortion if $|Df(x)|^n \leq K(x) \det Df(x)$ a.e. on Ω for some measurable $K\colon \Omega \to [1, \infty)$: that is, if $Df \in \mathcal{M}_n(K, 0)$. Thus, the class of mappings of finite distortion extends the theory of mappings of bounded distortion to the degenerate elliptic setting, [14, 12]. There one finds applications in materials science, particularly in nonlinear elasticity. The mathematical models of nonlinear elasticity have been pioneered by Antman [1], Ball [4] and Ciarlet [6].

In general, some bounds on the distortion are needed to obtain a full theory, analogous to the theory of quasiregular maps. The continuity property, however, follows without any restriction on the distortion function K. Precisely, if $K\colon \Omega \to [1, \infty)$ is any measurable function, then a Sobolev mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $Df \in \mathcal{M}_n(K, 0)$ has a continuous representative [10, 15].

Surprisingly, the continuity problem becomes a lot more challenging when $\Sigma \not\equiv 0$. Our next result shows that the solutions need not be continuous even in the case of bounded Σ if the distortion K is just a measurable function.

Theorem 1.5. *There exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$\Sigma \in L^\infty(\Omega) \qquad \textit{and} \qquad K \in L^1(\Omega)\,.$$

On the other hand, it is well known that mappings of exponentially integrable distortion behave in many ways like quasiregular mappings [12]. For instance, if a nonconstant Sobolev mapping $f\colon \Omega \to \mathbb{R}^n$ satisfies $Df \in \mathcal{M}_n(K, 0)$ with $\exp(\lambda K) \in L^1(\Omega)$ and $\lambda > 0$, then f is both discrete and open [19]. Moreover, the local modulus of continuity $\omega_f(x_0, r)$ of f is majorized up to a multiplicative constant by $\log^{-\lambda/n}(1/r)$ if $x_0 \in \Omega$ and $r > 0$ is sufficiently small [15]. This raises a natural question in the general case $Df \in \mathcal{M}_n(K, \Sigma)$: is there a version of the continuity result of Theorem 1.2 where the boundedness assumption $K \in L^\infty_{\mathrm{loc}}(\Omega)$ has been relaxed to $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\lambda > 0$. The next result shows that this is not the case for arbitrary $\lambda > 0$.

Theorem 1.6. *For every $\mu \in (0, 2)$, there exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$\exp(\lambda K) \in L^1(\Omega) \qquad \textit{and} \qquad \Sigma \log^\mu(e + \Sigma) \in L^1(\Omega)$$

for every $\lambda > 0$.

Nevertheless, it is possible to obtain a modulus of continuity in the case with $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ and $\Sigma \log^\mu(e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega)$, if one assumes λ and μ to be sufficiently large.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma \log^\mu (e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu > \lambda > n + 1$. Then f has a continuous representative.

In particular, for all $x_0 \in \Omega$ and sufficiently small $r > 0$, we have the following local modulus of continuity estimate:

$$\omega_f(x_0, r) \le C \log^{-\alpha}(1/r) \quad \text{where } \alpha = \frac{\lambda - n - 1}{n}.$$

1.3. **Single-value theory.** Understanding the pointwise behavior of quasiregular mappings motivates us to study a variant of the differential inclusion of $\mathcal{M}_n(K, \Sigma)$. In particular, given $K, \Sigma\colon \Omega \to \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n$, we define a map $\mathcal{M}_n(K, \Sigma, y_0)$ from $\Omega \times \mathbb{R}^n$ to subsets of $\mathbb{R}^{n\times n}$ by

$$\text{(1.7)} \quad \mathcal{M}_n(K, \Sigma, y_0)\colon \quad (x, y) \mapsto \{A \in \mathbb{R}^{n\times n} \colon |A|^n \le K(x) \det A + |y - y_0|^n \Sigma(x)\}.$$

Consequently, we obtain a differential inclusion by requiring that $Df(x) \in \mathcal{M}_n(K, \Sigma, y_0)(x, f(x))$ for a.e. $x \in \Omega$, which we again denote by the shorthand $Df \in \mathcal{M}_n(K, \Sigma, y_0)$.

For $K \in L^\infty(\Omega)$, the differential inclusion $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ leads to the theory of *quasiregular values* developed by the last two authors in [18] and [17]. This term is motivated by the fact that for bounded $K \in L^\infty(\Omega)$, solutions of $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ satisfy a single-value version of the celebrated

Reshetnyak's theorem at y_0. Precisely, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is non-constant and $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $K \in [1, \infty)$ constant and $\Sigma \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$, then f is continuous, $f^{-1}\{y_0\}$ is discrete, the local index $i(x, f)$ is positive in $f^{-1}\{y_0\}$, and every neighborhood of a point of $f^{-1}\{y_0\}$ is mapped to a neighborhood of y_0: see [17, Theorem 1.2].

Notably, the additional term $|f - y_0|^n$ in the differential inclusion $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ causes no additional difficulty in our continuity problem on the L^p-scale. Indeed, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $\Sigma \in L^p_{\mathrm{loc}}(\Omega)$, $p > 1$, one can define $\Sigma_0 = |f - y_0|^n \Sigma$ and conclude using the Sobolev embedding theorem that $\Sigma_0 \in L^q_{\mathrm{loc}}(\Omega)$ for every $q \in [1, p)$. The question then reduces to the continuity of solutions of $Df \in \mathcal{M}_n(K, \Sigma_0)$.

The sharpness of such an approach, however, becomes an issue when one moves to the Zygmund space scale of (1.2). Indeed, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $\Sigma \log^\mu(e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega)$, then it can be shown using the Moser-Trudinger inequality that $\Sigma_0 = |f - y_0|^n \Sigma$ satisfies $\Sigma_0 \log^{\mu - n + 1}(e + \Sigma_0) \in L^1_{\mathrm{loc}}(\Omega)$. Theorem 1.2 hence yields that f has a continuous representative if $\mu - n + 1 > n - 1$, i.e. $\mu > 2n - 2$.

This result for $\mu > 2n - 2$, however, turns out to be far from optimal. This is because, by an iteration argument using Theorem 1.3, this gap from (1.2) can be entirely eliminated. Again the mapping $f(x) = (\log \log \log(e^e / |x|), 0, \dots, 0)$ on $\mathbb{B}^n(0, 1)$ shows that the following theorem is sharp.

Theorem 1.8. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that a Sobolev mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $K \colon \Omega \to [1, \infty)$, $\Sigma \colon \Omega \to [0, \infty)$ and $y_0 \in \mathbb{R}^n$. If*

$$K \in L^\infty_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma \log^\mu (e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu > n - 1$, then f has a continuous representative.

However, in the case of $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with exponentially integrable K, the use of this trick is prevented as our results are not based on higher integrability. Hence, the current best bound in this case is the following result, given by the above Moser-Trudinger -argument combined with Theorem 1.7.

Theorem 1.9. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that a Sobolev mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $K \colon \Omega \to [1, \infty)$, $\Sigma \colon \Omega \to [0, \infty)$ and $y_0 \in \mathbb{R}^n$. If*

$$\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma \log^\mu (e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu > \lambda + n - 1 > 2n$, then f has a continuous representative.

In particular, for all $x_0 \in \Omega$ and sufficiently small $r > 0$, we have the following local modulus of continuity estimate:

$$\omega_f(x_0, r) \le C \log^{-\alpha}(1/r) \quad \text{where } \alpha = \frac{\lambda - n - 1}{n}.$$

1.4. **L^p-integrable K.** In the case where $K \in L^p_{\mathrm{loc}}(\Omega)$ with $p \in [1, \infty]$, we conjecture that Problem 1.1 has a positive answer if $\Sigma \in L^q_{\mathrm{loc}}(\Omega)$ for any $q > p^*$, where p^* is the Hölder conjugate of p. In fact, we conjecture that a stronger statement is true, where $\Sigma \in L^q_{\mathrm{loc}}(\Omega)$ can be replaced by the hypothesis $\Sigma/K \in L^q_{\mathrm{loc}}(\Omega)$.

Conjecture 1.10. *Let $1 \le p, q \le \infty$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $Df \in \mathcal{M}_n(K, \Sigma)$ with $K \ge 1$, $\Sigma \ge 0$,*

$$K \in L^p_{\mathrm{loc}}(\Omega), \qquad \textit{and} \qquad \frac{\Sigma}{K} \in L^q_{\mathrm{loc}}(\Omega), \qquad \text{where } \frac{1}{p} + \frac{1}{q} < 1\,.$$

Then f has a continuous representative.

In order to justify the assumption $p^{-1} + q^{-1} < 1$ of Conjecture 1.10, we point out that we have a discontinuous example in the case $p = 1, q = \infty$ due to Theorem 1.5. Moreover, in the case $q = 1, p = \infty$, the triple logarithm map (1.3) provides a discontinuous example. The necessity of the assumption for the remaining cases $1 < p < \infty$ is then given by the following example.

Theorem 1.11. *Let $p, q \in (1, \infty)$. If $p^{-1} + q^{-1} \ge 1$, then there exists a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x \to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$K \in L^p(\Omega) \qquad \textit{and} \qquad \frac{\Sigma}{K} \in L^q(\Omega). \tag{1.8}$$

Furthermore, we give several versions of Theorem 1.11 where (1.8) is replaced by a condition of the type

$$K \in L^p(\Omega) \qquad \text{and} \qquad \Sigma \in L^s(\Omega),$$

see Theorems 4.1 and 5.1 for details.

2. Results based on higher integrability

In this section, we prove the continuity results that are based on higher integrability: Theorems 1.2, 1.3, 1.4, and 1.8.

2.1. **Higher integrability on the L^p-scale.** The higher integrability result of Theorem 1.4 is essentially the same as [17, Lemma 6.1], with only minor tweaks to account for the non-constant K. We regardless recall the argument for the convenience of the reader, as we require the reverse Hölder inequality proven during the argument for our later proof of Theorem 1.3.

If $B = \mathbb{B}^n(x, r)$ is a ball and $c \in (0, \infty)$, then we denote $cB = \mathbb{B}^n(x, cr)$. Similarly, if $Q = x + (-r, r)^n \subset \mathbb{R}^n$ is a cube and $c \in (0, \infty)$, we denote $cQ = x + (-cr, cr)^n$.

Lemma 2.1. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$K \in L^\infty(\Omega) \qquad \textit{and} \qquad \Sigma \in L^1_{\mathrm{loc}}(\Omega).$$

Then for every cube Q such that $\overline{2Q} \subset \Omega$, we have the reverse Hölder inequality

$$\left(\fint_Q |Df|^n \right)^{\frac{n}{n+1}} \leq C(n) \, \|K\|^{\frac{n}{n+1}}_{L^\infty(\Omega)} \left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} + \left(\fint_{2Q} \Sigma \right)^{\frac{n}{n+1}} \right).$$

Proof. Let Q be such a cube. Choose a cutoff function $\eta \in C_0^\infty(2Q)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on Q, and $|\nabla \eta| \leq C_1(n) \, |2Q|^{-1/n}$. By using the distortion estimate $|Df|^n \leq K J_f + \Sigma$, the assumption that $K \geq 1$, and a Caccioppoli-type inequality given for instance in [14, Lemma 8.1.1], we have

$$\begin{aligned}
\fint_Q |Df|^n &\leq \frac{\|K\|_{L^\infty}}{|Q|} \int_\Omega \frac{|Df|^n \, \eta^n}{K} \\
&\leq \frac{\|K\|_{L^\infty(\Omega)}}{|Q|} \int_\Omega J_f \eta^n + \frac{\|K\|_{L^\infty(\Omega)}}{|Q|} \int_\Omega \frac{\Sigma \eta^n}{K} \\
&\leq \frac{C_2(n) \, \|K\|_{L^\infty(\Omega)}}{|Q|} \int_\Omega |Df|^{n-1} \, \eta^{n-1} \, |f - c| \, |\nabla \eta| + \frac{\|K\|_{L^\infty(\Omega)}}{|Q|} \int_\Omega \Sigma \eta^n.
\end{aligned}$$

By using $|\nabla \eta| \leq C_1(n) \, |2Q|^{-1/n}$, $\eta \leq 1$, and $|2Q| = 2^n \, |Q|$, we hence obtain that

$$\fint_Q |Df|^n \leq C_3(n) \, \|K\|_{L^\infty} \left(\frac{1}{|Q|^{\frac{1}{n}}} \fint_{2Q} |Df|^{n-1} \, |f - c| + \fint_{2Q} \Sigma \right).$$

Hölder and Sobolev-Poincaré inequalities then yield that

$$\begin{aligned}
&\frac{1}{|Q|^{\frac{1}{n}}} \fint_{2Q} |Df|^{n-1} \, |f - c| \\
&\qquad \leq \left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} \frac{1}{|Q|^{\frac{1}{n}}} \left(\fint_{2Q} |f - c|^{n^2} \right)^{\frac{1}{n^2}} \\
&\qquad \leq \left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} C_4(n) \left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}} \\
&\qquad = C_4(n) \left(\fint_{2Q} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n}}.
\end{aligned}$$

Thus, the claimed estimate follows by using the elementary inequality $a+b \leq (a^{1/p} + b^{1/p})^p$ for $a, b \geq 0$, $p \geq 1$ □

We then recall the statement of Theorem 1.4 and give the short remaining parts of the proof.

Theorem 1.4. *For given $n \geq 2$ and $K_\circ \in [1, \infty)$, there exists a value $p(n, K_\circ) > 1$, such that if $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$\|K\|_{L^\infty(\Omega)} \leq K_\circ \qquad \text{and} \qquad \Sigma \in L^p_{\text{loc}}(\Omega),$$

for some $p \in [1, p(n, K_\circ))$, then $|Df|^n \in L^p_{\text{loc}}(\Omega)$.

Proof. Due to f satisfying the reverse Hölder inequality given in Lemma 2.1, the claimed result follows immediately from the version of Gehring's lemma given in [13, Proposition 6.1]. The upper bound of higher integrability given there depends only on the constants of the reverse Hölder inequality, which in turn depend only on n and $\|K\|_{L^\infty(\Omega)}$. □

2.2. **Higher integrability on the Zygmund space scale.** For the Zygmund space version of our main result, we need a corresponding variant of Gehring's lemma. We expect this to be known, but are not aware of any references that would directly give the version we need. Hence, we provide a proof here of the relevant version of Gehring's lemma, with the proof modeled on the arguments used in [13, Section 3].

Lemma 2.2. *Let $G, H \in L^p(\mathbb{R}^n)$ be non-negative functions satisfying the reverse Hölder inequality*

$$\left(\fint_Q G^p\right)^{\frac{1}{p}} \leq C\left(\fint_{2Q} G^q\right)^{\frac{1}{q}} + \left(\fint_{2Q} H^p\right)^{\frac{1}{p}},$$

for all cubes $Q \subset \mathbb{R}^n$, where $1 \leq q < p < \infty$ and $C \geq 1$ is a constant. Then for every $\mu > 0$, we have

$$\int_{\mathbb{R}^n} G^p \log^\mu(e+G) \leq a\int_{\mathbb{R}^n} G^p + b\int_{\mathbb{R}^n} H^p \log^\mu(e+H),$$

with $a = a(C, n, \mu, p, q) \geq 1$ and $b = b(C, n, \mu, p, q) \geq 1$.

We start the proof with the following estimate which directly follows from [13, Section 3].

Lemma 2.3. *Let $G, H \in L^p(\mathbb{R}^n)$ be non-negative functions satisfying the reverse Hölder inequality*

$$\left(\fint_Q G^p\right)^{\frac{1}{p}} \leq C\left(\fint_{2Q} G^q\right)^{\frac{1}{q}} + \left(\fint_{2Q} H^p\right)^{\frac{1}{p}},$$

for all cubes $Q \subset \mathbb{R}^n$, where $1 \leq q < p < \infty$ and $C \geq 1$ is a constant. Then for every $t > 0$, we have

$$\int_{G^{-1}(t,\infty)} G^p \leq \alpha t^{p-q}\int_{G^{-1}(t,\infty)} G^q + \beta\int_{H^{-1}(t,\infty)} H^p, \tag{2.1}$$

with $\alpha = \alpha(n, C, p, q) > 1$ and $\beta = \beta(n, C, p, q) > 0$.

Proof. This estimate is [13, Proof of Lemma 3.1, estimate (3.11)], where in the notation used therein we've chosen $\Phi(t) = tF(t) = t^{p/q}$ with $F(t) = t^{p/q-1}$, $g = G^q$, and $h = H^q$. □

Proof of Lemma 2.2. We assume first that $\mu \neq 1$; for $\mu = 1$, see the remark at the end of the proof. We define an auxiliary function

$$A_\mu(t) = \frac{p-q}{\mu}\log^\mu(t) + \frac{\mu}{\mu-1}\log^{\mu-1}(t).$$

The purpose of this specific choice is that

$$t^{p-q}\log(t)A'_\mu(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t^{p-q}\log^\mu(t)). \tag{2.2}$$

We may select a constant $M > 1$ large enough that A_μ and A'_μ are positive on $[M,\infty)$, and also large enough that

$$A_\mu(t) - \frac{\alpha}{\log(M)}\log^\mu(t) \geq \frac{p-q}{2\mu}\log^\mu(e+t) \qquad \text{for all } t \in [M,\infty), \tag{2.3}$$

where α is from (2.1). Let $L > M$. We multiply both sides of (2.1) with $A'_\mu(t)$, and integrate over $[M,L]$ with respect to t. By a use of the Fubini-Tonelli theorem, the left hand side yields

$$\begin{aligned}
&\int_M^L A'_\mu(t)\int_{G^{-1}(t,\infty)} G^p(x)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{G^{-1}(L,\infty)} G^p(x)\int_M^L A'_\mu(t)\,\mathrm{d}t\,\mathrm{d}x + \int_{G^{-1}[M,L]} G^p(x)\int_M^{G(x)} A'_\mu(t)\,\mathrm{d}t\,\mathrm{d}x\\
&\quad = \int_{G^{-1}(L,\infty)} A_\mu(L)G^p + \int_{G^{-1}[M,L]} G^p A_\mu(G) - \int_{G^{-1}(M,\infty)} A_\mu(M)G^p.
\end{aligned}$$

By the same computation for the H^p-term, we get the upper bound

$$\begin{aligned}
&\int_M^L A'_\mu(t)\int_{H^{-1}(t,\infty)} H^p(x)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{H^{-1}(L,\infty)} A_\mu(L)H^p + \int_{H^{-1}[M,L]} H^p A_\mu(H) - \int_{H^{-1}(M,\infty)} A_\mu(M)H^p\\
&\leq \int_{H^{-1}(L,\infty)} A_\mu(H)H^p + \int_{H^{-1}[M,L]} H^p A_\mu(H) \leq 2\int_{H^{-1}[M,\infty)} H^p A_\mu(H).
\end{aligned}$$

For the G^q-term, we use (2.2) and similar computations to obtain that

$$\begin{aligned}
&\int_M^L A'_\mu(t)t^{p-q}\int_{G^{-1}(t,\infty)} G^q(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \leq \frac{1}{\log(M)}\int_M^L A'_\mu(t)t^{p-q}\log(t)\int_{G^{-1}(t,\infty)} G^q(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \frac{1}{\log(M)}\int_M^L \frac{\mathrm{d}}{\mathrm{d}t}(t^{p-q}\log^\mu(t))\int_{G^{-1}(t,\infty)} G^q(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\qquad \leq \int_{G^{-1}(L,\infty)} \frac{L^{p-q}\log^\mu(L)G^q}{\log(M)} + \int_{G^{-1}[M,L]} \frac{G^p\log^\mu(G)}{\log(M)}.
\end{aligned}$$

In total, we have

$$\begin{aligned}\int_{G^{-1}(L,\infty)} A_\mu(L)G^p &+ \int_{G^{-1}[M,L]} G^p A_\mu(G)\\ &\leq A_\mu(M)\int_{G^{-1}(M,\infty)} G^p + \frac{\alpha}{\log(M)}\int_{G^{-1}(L,\infty)} L^{p-q}\log^\mu(L)G^q\\ &\quad + \frac{\alpha}{\log(M)}\int_{G^{-1}[M,L]} G^p\log^\mu(G) + 2\beta\int_{H^{-1}[M,\infty)} H^p A_\mu(H).\end{aligned}$$

Note that on $G^{-1}(L,\infty)$, we have $L^{p-q} \leq G^{p-q}$. By applying this and subtracting the $\alpha/\log(M)$-terms from both sides of the above estimate, we obtain

$$\begin{aligned}\int_{G^{-1}(L,\infty)} \left(A_\mu(L) - \frac{\alpha\log^\mu(L)}{\log(M)}\right) G^p + \int_{G^{-1}[M,L]} \left(A_\mu(G) - \frac{\alpha\log^\mu(G)}{\log(M)}\right) G^p\\ \leq A_\mu(M)\int_{G^{-1}(M,\infty)} G^p + 2\beta\int_{H^{-1}[M,\infty)} H^p A_\mu(H).\end{aligned}$$

We then apply (2.3), and conclude that

$$\begin{aligned}\int_{G^{-1}[M,L]} & G^p\log^\mu(e+G)\\ &\leq \int_{G^{-1}(L,\infty)} G^p\log^\mu(e+L) + \int_{G^{-1}[M,L]} G^p\log^\mu(e+G)\\ &\qquad \leq \frac{2\mu A_\mu(M)}{p-q}\int_{G^{-1}(M,\infty)} G^p + \frac{4\mu\beta}{p-q}\int_{H^{-1}[M,\infty)} H^p A_\mu(H).\end{aligned}$$

Notably, this upper bound is independent on L. Since we have $0 \leq A_\mu(H) \leq A_\mu(e+H) \leq ((p-q)/\mu + \mu/|\mu-1|)\log^\mu(e+H)$ in $H^{-1}[M,\infty)$, letting $L \to \infty$ gives us

$$\int_{G^{-1}[M,\infty)} G^p\log^\mu(e+G) \leq a_0\int_{\mathbb{R}^n} G^p + b\int_{\mathbb{R}^n} H^p\log^\mu(e+H),$$

with a_0, b dependent only on α, β, p, q, μ. The final desired claim then follows by combining the previous estimate with

$$\int_{G^{-1}[0,M)} G^p\log^\mu(e+G) \leq \log^\mu(e+M)\int_{\mathbb{R}^n} G^p.$$

We finally comment on the case $\mu = 1$. In this case, we must instead define $A_1(t) = (p-q)\log(t) + \log\log(t)$, which yields (2.2) for $\mu = 1$. The rest of the proof goes through essentially similarly in this case. □

With Lemma 2.2 proven, we may proceed to prove Theorem 1.3. We again recall the statement.

Theorem 1.3. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K,\Sigma)$ with*

$$K \in L^\infty_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma\log^\mu(e+\Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu \geq 0$. *Then* $|Df|^n \log^\mu(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. We select a ball $B = \mathbb{B}^n(x_0, r)$ with $r \leq 1$ such that $\overline{B} \subset \Omega$. By using Lemma 2.1, we obtain that $|Df|$ and Σ satisfy a reverse Hölder inequality for all cubes Q with $2Q \subset B$. It was shown in the proof of [13, Proposition 6.1] that in this case, the functions $G, H \colon \mathbb{R}^n \to [0, \infty)$ defined by

$$\begin{aligned} G(x) &= \operatorname{dist}(x, \mathbb{R}^n \setminus B)\, |Df(x)|^{\frac{n^2}{n+1}} \\ H(x) &= \operatorname{dist}(x, \mathbb{R}^n \setminus B) \Sigma^{\frac{n}{n+1}} + \chi_B(x) \left(\int_B \Sigma \right)^{\frac{n}{n+1}} \end{aligned}$$

satisfy a reverse Hölder inequality in all of $\mathbb{R}^n$. In particular, it follows from Lemma 2.2 that

$$\int_{\mathbb{R}^n} G^{\frac{n+1}{n}} \log^\mu(e + G) \leq a \int_{\mathbb{R}^n} G^{\frac{n+1}{n}} + b \int_{\mathbb{R}^n} H^{\frac{n+1}{n}} \log^\mu(e + H). \tag{2.4}$$

We then assume $0 < \varepsilon < r$, and denote $B_\varepsilon = \{x \in B : \operatorname{dist}(x, R^n \setminus B) > \varepsilon\}$. We note that for $t \geq 1$, and $p \geq 1$, we may estimate using Bernoulli's inequality that

$$e + \varepsilon t^p = e(1 + \varepsilon e^{-1} t^p) \geq e^\varepsilon (1 + e^{-1} t^p)^\varepsilon = (e + t^p)^\varepsilon \geq (e + t)^\varepsilon.$$

Hence, for every point $x \in B_\varepsilon$, we have either $|Df(x)| \leq 1$ and thus also $|Df(x)|^n \log^\mu(e + |Df(x)|) \leq \log^\mu(e + 1)$, or

$$\begin{aligned} G^{\frac{n+1}{n}} \log^\mu(e + G) &\geq \varepsilon^{\frac{n+1}{n}} |Df|^n \log^\mu(e + \varepsilon\, |Df|^{\frac{n^2}{n+1}}) \\ &\geq \varepsilon^{\frac{n+1}{n} + \mu} |Df|^n \log^\mu(e + |Df|). \end{aligned}$$

Consequently,

$$\int_{B_\varepsilon} |Df|^n \log^\mu(e + |Df|) \leq |B_\varepsilon| \log^\mu(e + 1) + \varepsilon^{-\frac{n+1}{n} - \mu} \int_{\mathbb{R}^n} G^{\frac{n+1}{n}} \log^\mu(e + G).$$

On the other hand, $G^{(n-1)/n} \leq |Df|^n \chi_B \in L^1(\mathbb{R}^n)$. For the H-term of (2.4), we have $H \equiv 0$ outside B and $H \leq \Sigma^{n/(n+1)} + C$ in B with $C = \|\Sigma\|^{n/(n+1)}_{L^1(B)} < \infty$. We recall that we have the elementary inequality $\log(e + a + b) \leq \log(e + a) + \log(e + b)$ for $a, b \geq 0$, and that $\Sigma^{n/(n+1)} \leq 1 + \Sigma$. Hence, we may estimate

$$\begin{aligned} \int_{\mathbb{R}^n} H^{\frac{n+1}{n}} \log^\mu(e + H) &\leq \int_B (\Sigma^{\frac{n}{n+1}} + C)^{\frac{n+1}{n}} \log^\mu\big(e + \Sigma^{\frac{n}{n+1}} + C\big) \\ &\leq \int_B 2^{\frac{n+1}{n} + \mu} \big(\Sigma + C^{\frac{n+1}{n}}\big) \big(\log^\mu(e + \Sigma) + \log^\mu(e + C + 1)\big) < \infty. \end{aligned}$$

It follows that $|Df|^n \log^\mu(e + |Df|)$ has finite integral over B_ε, which completes the proof of the claim. $\square$

2.3. **Embedding theorems.** As stated in the introduction, Theorem 1.2 is a direct corollary of combining Theorem 1.3 with a suitable version of Morrey's inequality for Zygmund spaces. Recall that the classical Morrey's inequality implies that if $p > n$, then elements of $W^{1,p}_{\mathrm{loc}}(\Omega)$ have a locally Hölder continuous representative, with Hölder exponent $1 - n/p$. For a Zygmund space version, we refer to e.g. [16, Theorem 3.1], which gives us the following.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega)$ satisfy*

$$|Df|^n \log^\mu(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega),$$

where $\mu > n - 1$. Then f has a continuous representative. In particular, whenever $0 < r < R$ and $\overline{\mathbb{B}^n(x, R)} \subset \Omega$, the modulus of continuity $\omega_f(x_0, r)$ defined in (1.5) *satisfies*

$$\omega_f(x, r) \le C(Df, \mu, x, R) \log^{\frac{\mu-n+1}{n}} \left(1 + \frac{2R}{r}\right),$$

where

$$C(Df, \mu, x, R) = \fint_{\mathbb{B}^n(x,R)} |Df|^n \log^\mu \left(e + \frac{|\mathbb{B}^n(x, r)|\,|Df|}{\|Df\|^n_{L^n(\mathbb{B}^n(x,R))}} \right).$$

Hence, by combining Theorems 1.3 and 2.4, the proof of Theorem 1.2 is complete.

Due to us requiring it in the following subsection, we also recall the corresponding result for $\mu \in [0, n-1)$. In this case, f is not necessarily continuous, but does satisfy an exponential Sobolev embedding theorem. We refer to e.g. [5, Theorem 2, Example 1] for the following result; note also that the case $\mu = 0$ corresponds to the classical Moser-Trudinger inequality.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega)$ satisfy*

$$|Df|^n \log^\mu(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega),$$

where $0 \le \mu < n - 1$. Then there exists $\lambda > 0$ such that

$$\exp(\lambda\, |f|^{\frac{n}{n-1-\mu}}) \in L^1_{\mathrm{loc}}(\Omega).$$

2.4. **Continuity for** (1.7) **with bounded** K**.** The final result we prove in this section is Theorem 1.8. For the proof, we require the following lemma on the integrability of products of functions.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be measurable, let $\mu, \nu, \lambda > 0$ be such that $\nu \le \mu$, and let $f, g \colon \Omega \to [0, \infty]$ be measurable functions such that*

$$f \log^\mu(e + f) \in L^1_{\mathrm{loc}}(\Omega), \qquad \exp\bigl(\lambda g^{\frac{1}{\nu}}\bigr) \in L^1_{\mathrm{loc}}(\Omega).$$

Then $fg \log^{\mu-\nu}(e + fg) \in L^1_{\mathrm{loc}}(\Omega)$.

We begin the proof of Lemma 2.6 by recalling the proof of the following elementary inequality. See e.g. [12, Lemmas 2.7, 6.2] for similar results and proofs.

Lemma 2.7. *Let $a, b \geq 0$, and $\kappa, \lambda > 0$. Then*

$$ab < \exp\left(\lambda a^{\frac{1}{\kappa}}\right) + C(\kappa, \lambda) b \log^{\kappa}(e + b),$$

where $C(\kappa, \lambda) \geq 0$.

Proof. Note that there exists a constant $A = A(\kappa)$ such that

$$\exp(t) \geq A t^{2\kappa}. \tag{2.5}$$

If $ab \leq \exp(a^{1/\kappa}\lambda)$, then the claim is clear. Hence, we assume that $ab > \exp(a^{1/\kappa}\lambda)$, with a goal of showing that $ab < C(\kappa, \lambda) b \log^{\kappa}(e + b)$.

By combining this assumption with (2.5), we have

$$a = a^{-1}a^2 \leq a^{-1}\left(\frac{\exp(a^{1/\kappa}\lambda)}{A\lambda^{2\kappa}}\right) < a^{-1}\left(\frac{ab}{A\lambda^{2\kappa}}\right) = \frac{b}{A\lambda^{2\kappa}}.$$

Consequently, we have

$$\exp(a^{1/\kappa}\lambda) < ab < \frac{b^2}{A\lambda^{2\kappa}} < \frac{(e+b)^2}{A\lambda^{2\kappa}}.$$

Taking logarithms yields

$$a^{1/\kappa}\lambda < \log \frac{(e+b)^2}{A\lambda^{2\kappa}} = \log \frac{1}{A\lambda^{2\kappa}} + 2\log(e+b).$$

In particular

$$a < \left|\frac{1}{\lambda} \log \frac{1}{A\lambda^{2\kappa}} + \frac{2}{\lambda}\log(e+b)\right|^{\kappa} \leq \frac{2^\kappa}{\lambda^\kappa}\left|\log \frac{1}{A\lambda^{2\kappa}}\right|^{\kappa} + \frac{4^\kappa}{\lambda^\kappa}\log^{\kappa}(e+b).$$

And hence, we obtain the desired estimate

$$\begin{aligned} ab &< \left(\frac{2^\kappa}{\lambda^\kappa}\left|\log \frac{1}{A\lambda^{2\kappa}}\right|^{\kappa}\right) b + \left(\frac{4^\kappa}{\lambda^\kappa}\right) b \log^{\kappa}(e+b) \\ &\leq \left(\frac{2^\kappa}{\lambda^\kappa}\left|\log \frac{1}{A\lambda^{2\kappa}}\right|^{\kappa} + \frac{4^\kappa}{\lambda^\kappa}\right) b \log^{\kappa}(e+b). \end{aligned}$$

□

Proof of Lemma 2.6. We first observe that $fg \in L^1_{\text{loc}}(\Omega)$. Indeed, Lemma 2.7 yields that $fg \leq \exp(\lambda g^{1/\nu}) + Cf \log^{\nu}(e + f)$, where both terms on the right hand side are integrable by $\nu \leq \mu$.

We then further estimate using Lemma 2.7

$$\begin{aligned} &fg \log^{\mu-\nu}(e + fg) \\ &\quad \leq \exp\left(2^{-1}\lambda g^{\frac{1}{\nu}}\right) \log^{\mu-\nu}(e + fg) + C_1 f \log^{\nu}(e + f) \log^{\mu-\nu}(e + fg). \end{aligned} \tag{2.6}$$

We have $\exp\left(2^{-1}\lambda g^{\frac{1}{\nu}}\right) \in L^2_{\text{loc}}(\Omega)$, and also $\log^{\mu}(e + f) \in L^2_{\text{loc}}(\Omega)$. It follows that the first term on the right hand side of (2.6) is locally integrable. For the second term, we estimate $e + fg \leq (e + f)(e + g)$, and hence

$$\text{(2.7)}\quad \begin{aligned} &f \log^{\nu}(e + f) \log^{\mu-\nu}(e + fg) \\ &\qquad \leq C_2 \left(f \log^{\mu}(e + f) + f \log^{\nu}(e + f) \log^{\mu-\nu}(e + g) \right). \end{aligned}$$

The first term on the right hand side of (2.7) is locally integrable by assumption. For the second term, we again use Lemma 2.7, this time with $\kappa = \mu - \nu$ and $\lambda = 1$. We get

$$\begin{aligned} &f \log^{\nu}(e + f) \log^{\mu-\nu}(e + g) \\ &\quad \leq e + g + C_3 f \log^{\nu}(e + f) \log^{\mu-\nu}(e + f \log^{\mu-\nu}(e + f)) \\ &\quad \leq e + g + C_4 f \log^{\mu}(e + f) \\ &\qquad\quad + C_4 f \log^{\nu}(e + f) \log^{\mu-\nu}(e + \log^{\mu-\nu}(e + f)), \end{aligned}$$

where the right hand side is locally integrable by the local integrability of g and $f \log^{\mu}(e + f)$. Hence, the claim follows. ☐

We're now ready to prove Theorem 1.8. We again recall the statement for convenience.

Theorem 1.8. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that a Sobolev mapping $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $K \colon \Omega \to [1, \infty)$, $\Sigma \colon \Omega \to [0, \infty)$ and $y_0 \in \mathbb{R}^n$. If*

$$K \in L^{\infty}_{\text{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma \log^{\mu}(e + \Sigma) \in L^1_{\text{loc}}(\Omega),$$

for some $\mu > n - 1$, then f has a continuous representative.

Proof. Let $q = \mu - (n - 1) > 0$. By slightly shrinking μ, we may assume that $n - 1$ is not an integer multiple of q. By our assumption, we have $|Df|^n \leq K J_f + \Sigma'$, where $\Sigma' = \Sigma\, |f - y_0|^n$.

By the Moser-Trudinger inequality (case $q = 0$ of Theorem 2.5), there exists $\lambda_0 > 0$ such that

$$\exp\left(\lambda_0 \, |f - y_0|^{\frac{n}{n-1}} \right) \in L^1_{\text{loc}}(\Omega).$$

Combining this with our assumption that $\Sigma \log^{\mu}(e + \Sigma) \in L^1_{\text{loc}}(\Omega)$ and recalling that $q = \mu - (n - 1)$, we can thus use Lemma 2.6 to conclude that

$$\Sigma' \log^{q}(e + \Sigma') \in L^1_{\text{loc}}(\Omega).$$

Using Theorem 1.3, we hence conclude that

$$|Df|^n \log^{q}(e + |Df|) \in L^1_{\text{loc}}(\Omega).$$

If $q > n - 1$, we are now done, since Theorem 2.4 implies that f has a continuous representative. Otherwise, we proceed to iterate this argument. Indeed, since $|Df|^n \log^{q}(e + |Df|) \in L^1_{\text{loc}}(\Omega)$, Theorem 2.5 yields us a slightly better estimate

$$\exp\left(\lambda_1 \, |f - y_0|^{\frac{n}{(n-1)-q}} \right) \in L^1_{\text{loc}}(\Omega)$$

for some $\lambda_1 > 0$. Lemma 2.6 then yields that

$$\Sigma' \log^{2q}(e + \Sigma') = \Sigma' \log^{\mu-((n-1)-q)}(e + \Sigma') \in L^1_{\mathrm{loc}}(\Omega),$$

from which we get that

$$|Df|^n \log^{2q}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega).$$

Then next iteration of this argument then yields $|Df|^n \log^{3q}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$, the next iteration after that yields $|Df|^n \log^{4q}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$, et cetera.

We may continue this iteration until $|Df|^n \log^{kq}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$, where k is the smallest positive integer such that $kq > n - 1$. Indeed, we assumed $n - 1$ not to be an integer multiple of q, so $(k - 1)q$ is a valid exponent for Theorem 2.5. Moreover, we also must have $kq < \mu$, since $kq = (k - 1)q + q < (n - 1) + q = \mu$. Hence, it follows that f has a continuous representative by Theorem 2.4. □

3. Direct continuity results

In this section, we prove Theorems 1.7 and 1.9. The method is a generalization of the approach used in [18, Section 3]. In particular, we prove a decay estimate for the integral of $|Df|^n$ over balls, which then implies continuity by using a chain of balls argument as in [11].

We begin by recalling an estimate that is used in the proofs of similar continuity results for mappings of finite distortion; see e.g. [22, Section 3] or [12, Theorem 5.18]. We give the proof for the convenience of the reader.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\exp(K) \in L^\lambda(\Omega)$ with $\lambda > 0$. Then there exist constants $C = C(\Omega, K, \lambda) > 0$ and $R_0 = R_0(\Omega, K, \lambda)$ as follows: if $x \in \Omega$, $R < \min(R_0, d(x, \partial\Omega))$ and $r \in (0, R/e^3)$, then*

$$\int_r^R s^{-1} \left(\fint_{\partial\mathbb{B}^n(x,s)} K^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}s \geq \frac{\lambda}{n} \left(\log\log \frac{C}{r^n} - \log\log \frac{Ce^2}{R^n} \right).$$

Proof. We denote $B_s = \mathbb{B}^n(x, s)$. We let k be the largest integer such that $re^k \leq R$. Since $r < R/e^3$, we must have $k \geq 3$. We define the function $\tilde{K} = \max(K, (n-2)\lambda^{-1})$, where we still have $\exp(\tilde{K}) \in L^\lambda_{\mathrm{loc}}(\Omega)$. We estimate $K \leq \tilde{K}$, perform a change of variables, and split the integral into a sum as follows:

$$\int_r^R s^{-1} \left(\fint_{\partial B_s} K^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}s \ \geq \ \sum_{i=1}^{k-1} \int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \tilde{K}^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}t.$$

We then use Jensen's inequality a total of three times, with the convex functions $\tau \mapsto \tau^{-1}$, $\tau \mapsto \exp(\lambda\tau^{\frac{1}{n-1}})$ and $\tau \mapsto \exp(\tau)$. Note that $\tau \mapsto$

$\exp(\lambda\tau^{\frac{1}{n-1}})$ is only convex for $\tau \geq ((n-2)\lambda^{-1})^{n-1}$, but the range of $\tilde{K}^{n-1}$ is in this region. The resulting estimate is

$$\int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \tilde{K}^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}t \geq \left(\int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \tilde{K}^{n-1} \right)^{\frac{1}{n-1}} \mathrm{d}t \right)^{-1}$$
$$\geq \lambda \left(\int_{i+\log r}^{i+1+\log r} \log \left(\fint_{\partial B_{e^t}} \exp(\lambda \tilde{K}) \right) \mathrm{d}t \right)^{-1}$$
$$\geq \lambda \log^{-1} \int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \exp(\lambda \tilde{K}) \right) \mathrm{d}t.$$

Now we may estimate

$$\log^{-1} \int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \exp(\lambda \tilde{K}) \right) \mathrm{d}t = \log^{-1} \int_{re^i}^{re^{i+1}} \left(\fint_{\partial B_s} \exp(\lambda \tilde{K}) \right) \frac{\mathrm{d}s}{s}$$
$$= \log^{-1} \int_{re^i}^{re^{i+1}} \frac{1}{\omega_{n-1} s^n} \left(\int_{\partial B_s} \exp(\lambda \tilde{K}) \right) \mathrm{d}s \geq \log^{-1} \frac{\|\exp(\lambda \tilde{K})\|_{L^1(\Omega)}}{\omega_{n-1}(re^i)^n}.$$

We then select $C = \|\exp(\lambda \tilde{K})\|_{L^1(\Omega)}/\omega_{n-1}$. The sum of the above terms over i can now be estimated by

$$\sum_{i=1}^{k-1} \int_{i+\log r}^{i+1+\log r} \left(\fint_{\partial B_{e^t}} \tilde{K}^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}t \geq \lambda \sum_{i=1}^{k-1} \log^{-1} \frac{C}{(re^i)^n}$$
$$\geq \lambda \int_0^{k-1} \log^{-1} \frac{C}{(re^t)^n} \mathrm{d}t \geq \lambda \int_r^{R/e^2} s^{-1} \log^{-1} \frac{C}{s^n} \mathrm{d}s$$
$$= \frac{\lambda}{n} \left(\log\log \frac{C}{r^n} - \log\log \frac{Ce^2}{R^n} \right).$$

The claim hence holds, assuming that $\log\log(Ce^2/R^n)$ is well defined; this is the case if we select $R_0^n = Ce$. □

We then consider the following abstract differential inequality of real functions, and show that it yields a decay condition. This is a more general version of [18, Lemma 3.2], which is essentially given by the case $\Psi(r) = r$ and $\Gamma(r) = Cr^\alpha$.

Lemma 3.2. *Let $A > 0$, and let $\Phi\colon [0,R] \to [0,S]$, $\Psi\colon [0,R] \to [0,\infty)$, and $\Gamma\colon [0,R] \to [0,\infty)$ be absolutely continuous increasing functions such that $\Phi(0) = 0$. Suppose that*

$$\Phi(r) \leq A \frac{\Psi(r)}{\Psi'(r)} \Phi'(r) + \Gamma(r)$$

for a.e. $r \in (0,R)$*, where* $A > 0$*. Then there exists a constant* $C = C(A,R,S,\Psi,\Gamma) \geq 0$ *such that we have*

$$\Phi(r) \leq \Gamma(r) + C\Psi^{A^{-1}}(r)\left(1 + \int_r^R \frac{\Gamma'(s)}{\Psi^{A^{-1}}(s)}\,\mathrm{d}s\right)$$

for all $r \in [0,R]$*.*

Proof. We find an integrating factor for the terms involving Φ:

$$\begin{aligned}-\frac{d}{ds}(\Psi^{-A^{-1}}(s)\Phi(s)) &= \left(\Phi(s) - A\frac{\Psi(s)}{\Psi'(s)}\Phi'(s)\right)\left(A^{-1}\Psi^{-A^{-1}-1}(s)\Psi'(s)\right)\\ &\leq -\Gamma(s)\frac{d}{ds}\Psi^{-A^{-1}}(s).\end{aligned}$$

We then integrate on both sides over $[r,R]$, and use integration by parts:

$$\begin{aligned}\Psi^{-A^{-1}}(r)\Phi(r) - \Psi^{-A^{-1}}(R)\Phi(R) &\leq -\int_r^R \Gamma(s)\left(\frac{d}{ds}\Psi^{-A^{-1}}(s)\right)\mathrm{d}s\\ &= \Gamma(r)\Psi^{-A^{-1}}(r) - \Gamma(R)\Psi^{-A^{-1}}(R) + \int_r^R \frac{\Gamma'(s)}{\Psi^{A^{-1}}(s)}\,\mathrm{d}s.\end{aligned}$$

Multiplying by $\Psi^{A^{-1}}(r)$ and moving the negative term to the right hand side yields the desired

$$\Phi(r) \leq \Gamma(r) + \frac{S - \Gamma(R)}{\Psi^{A^{-1}}(R)}\Psi^{A^{-1}}(r) + \Psi^{A^{-1}}(r)\int_r^R \frac{\Gamma'(s)}{\Psi^{A^{-1}}(s)}\,\mathrm{d}s.$$

□

Combining Lemmas 3.1 and 3.2 allows us to show the following decay estimate.

Lemma 3.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a connected domain. Let* $f \in W^{1,n}(\Omega,\mathbb{R}^n)$ *and* $Df \in \mathcal{M}_n(K,\Sigma)$ *with* $\exp(\lambda K) \in L^1(\Omega)$ *and* $\Sigma \log^\mu(e+\Sigma) \in L^1(\Omega)$*, where* $\mu > \lambda > 0$*. Then there exists* $R_0 = R_0(\Omega,\lambda,K) > 0$ *as follows: for any choice of* $x \in \Omega$ *and* $R \in (0,R_0)$ *such that* $\mathbb{B}^n(x,R) \subset \Omega$*, we have*

$$\int_{\mathbb{B}^n(x,r)} \frac{|Df|^n}{K} \leq C\log^{-\lambda} r^{-1}.$$

for all $r \in (0,R/e^3)$*, where* $C = C(\Omega,\lambda,\mu,K,\Sigma,R,\|Df\|_{L^n(\Omega)})$*. Notably,* C *is independent of* x*.*

Proof. We choose $R_0 < e^{-1}$ such that Lemma 3.1 holds for $R < R_0$: note that this choice depends only on Ω, λ, and K. We fix a point $x \in \Omega$ and a radius $R < R_0$ such that $\mathbb{B}^n(x,R) \subset \Omega$, and we denote $B_r = \mathbb{B}^n(x,r)$ for all $r \in [0,R]$. We then define a function $\Phi\colon [0,R] \to [0,\infty)$ by

$$\Phi(r) = \int_{B_r} \frac{|Df|^n}{K}.$$

By using the definition of $\mathcal{M}_n(K,\Sigma)$, we may estimate $\Phi(r)$ by

$$\int_{B_r} \frac{|Df|^n}{K} \leq \int_{B_r} J_f + \int_{B_r} \frac{\Sigma}{K}.$$

For the first term on the right hand side, we apply the isoperimetric inequality of Sobolev maps, see e.g. [24, Lemma II.1.2 and (II.1.7)], followed by a use of Hölder's inequality. The result is

$$\begin{aligned}\int_{B_r} J_f &\leq \frac{1}{n\sqrt[n-1]{\omega_{n-1}}} \left(\int_{\partial B_r} |Df|^{n-1}\right)^{\frac{n}{n-1}} \\ &\leq \frac{1}{n\sqrt[n-1]{\omega_{n-1}}} \left(\int_{\partial B_r} K^{n-1}\right)^{\frac{1}{n-1}} \int_{\partial B_r} \frac{|Df|^n}{K} \\ &= \frac{r}{n} \left(\fint_{\partial B_r} K^{n-1}\right)^{\frac{1}{n-1}} \int_{\partial B_r} \frac{|Df|^n}{K}\end{aligned}$$

for a.e. $r \in [0,R]$. For the other term, using $K \geq 1$, $\Sigma \log_+^\mu(\Sigma) \in L^1(\Omega)$, and $r \leq R < R_0 < e^{-1}$, we estimate that

$$\begin{aligned}\int_{B_r} \frac{\Sigma}{K} \leq \int_{B_r} \Sigma &\leq \int_{\{z\in B_r : \Sigma(z) \leq r^{-1}\}} \Sigma + \int_{\{z\in B_r : \Sigma(z) > r^{-1}\}} \Sigma \\ &\leq \int_{\{z\in B_r : \Sigma(z) \leq r^{-1}\}} r^{-1} + \int_{\{z\in B_r : \Sigma(z) > r^{-1}\}} \frac{\Sigma \log^\mu \Sigma}{\log^\mu r^{-1}} \\ &\leq \frac{\mathrm{vol}_n(B_r)}{r} + \log^{-\mu} r^{-1} \int_{B_r} \Sigma \log^\mu (e+\Sigma) \leq C_1 \log^{-\mu} r^{-1}\end{aligned}$$

for some $C_1 = C_1(n,\mu,\Sigma,R) \geq 0$. In conclusion, we have

$$\Phi(r) \leq \frac{r}{n} \left(\fint_{\partial B_r} K^{n-1}\right)^{\frac{1}{n-1}} \Phi'(r) + C_1 \log^{-\mu} r^{-1} \tag{3.1}$$

for all $r \in (0,R)$.

We then define

$$\Psi(r) = \exp\left(-\int_r^R s^{-1} \left(\fint_{\partial B_s} K^{n-1}\right)^{-\frac{1}{n-1}} \mathrm{d}s\right).$$

A simple computation by chain rule hence reveals that

$$\Psi'(r) = \Psi(r) r^{-1} \left(\fint_{\partial B_r} K^{n-1}\right)^{-\frac{1}{n-1}}$$

for all $r \in (0,R)$. In particular, (3.1) now reads as

$$\Phi(r) \leq \frac{1}{n} \frac{\Psi(r)}{\Psi'(r)} \Phi'(r) + C_1 \log^{-\mu} r^{-1}.$$

We also note that since $K \geq 1$, we have $\Phi(r) \leq S$ for all $r \in [0,R]$ with $S = \|Df\|_{L^n(\Omega)}$. Hence, we are in position to apply Lemma 3.2, which yields that

$$\Phi(r) \leq C_1 \log^{-\mu} r^{-1} + C_2 \left(\Psi^n(r) + \int_r^R \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} \right) \tag{3.2}$$

when $r \in [0, R/e^3]$, for some $C_2 = C_2(\Omega, \lambda, \mu, K, \Sigma, R, \|Df\|_{L^n(\Omega)})$.

Lemma 3.1 yields that for $r \in (0, R/e^3)$, we have

$$\begin{aligned}\Psi^n(r) &\leq \exp\left(-\frac{n\lambda}{n} \left(\log\log \frac{C_3}{r^n} - \log\log \frac{C_3 e^2}{R^n} \right) \right) \\ &= \left(\frac{\log(C_3 e^2 R^{-n})}{\log(C_3 r^{-n})} \right)^{\lambda} \leq C_4 \log^{-\lambda} r^{-1},\end{aligned}$$

where $C_3 = C_3(\Omega, \lambda, K)$ and $C_4 = C_4(\Omega, \lambda, K, R)$. Since $\mu > \lambda$, we also have

$$\log^{-\mu} r^{-1} \leq C_5 \log^{-\lambda} r^{-1}$$

for all $r \in (0, R/e^3]$, where $C_5 = C_5(\mu, \lambda, R)$. Hence, in order to obtain the claimed decay estimate for Φ from (3.2), it remains to estimate the term with the integral.

For this, we let $r \in (0, R/e^3)$, and split the integral into two parts:

$$\begin{aligned}&\int_r^R \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} \\ &\qquad = \int_r^{e^3 r} \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} + \int_{e^3 r}^R \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})}.\end{aligned}$$

In the range of the latter integral, we have $r < s/e^3$, which allows us to use Lemma 3.1 again to estimate

$$\begin{aligned}\frac{\Psi^n(r)}{\Psi^n(s)} &= \exp\left(-n \int_r^s t^{-1} \left(\fint_{\partial B_t} K^{n-1} \right)^{-\frac{1}{n-1}} \mathrm{d}t \right) \\ &\leq \left(\frac{\log(C_3 e^2 s^{-n})}{\log(C_3 r^{-n})} \right)^{\lambda}.\end{aligned}$$

Hence, by using the fact that $\mu - \lambda > 0$, and the fact that $s^{-1} \log^{-1-t}(s^{-1})$ is integrable for $t > 0$, the second integral can now be estimated by

$$\begin{aligned}\int_{e^3 r}^R \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} &\leq \int_{e^3 r}^R \left(\frac{\log(C_3 e^2 s^{-n})}{\log(C_3 r^{-n})} \right)^{\lambda} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} \\ &\leq \left(\int_0^R \frac{\mathrm{d}s}{s \log^{-\lambda}(C_3 e^2 s^{-n}) \log^{\mu+1}(s^{-1})} \right) \log^{-\lambda} \frac{C_3}{r^n} \leq C_6 \log^{-\lambda} r^{-1}\end{aligned}$$

where $C_6 = C_6(\Omega, \lambda, \mu, K, R)$. On the other hand, for the first integral, we may merely use the fact that Ψ is increasing to estimate that $\Psi^n(s) \geq \Psi^n(r)$, which again combined with $\mu > \lambda$ yields that

$$\int_r^{e^3 r} \frac{\Psi^n(r)}{\Psi^n(s)} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})} \leq \int_r^{e^3 r} \frac{\mathrm{d}s}{s \log^{\mu+1}(s^{-1})}$$
$$= \frac{1}{\mu}\left(\log^{-\mu}\frac{1}{e^3 r} - \log^{-\mu}\frac{1}{r}\right) \leq C_7 \log^{-\lambda} r^{-1}$$

where $C_7 = C_7(\mu, \lambda, R)$. The proof of the claimed estimate is hence complete. □

We then proceed to prove Theorem 1.7. We again begin by recalling the statement.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $Df \in \mathcal{M}_n(K, \Sigma)$ with*

$$\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad \Sigma \log^{\mu}(e + \Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu > \lambda > n+1$. Then f has a continuous representative.

In particular, for all $x_0 \in \Omega$ and sufficiently small $r > 0$, we have the following local modulus of continuity estimate:

$$\omega_f(x_0, r) \leq C \log^{-\alpha}(1/r) \quad \text{where } \alpha = \frac{\lambda - n - 1}{n}.$$

Proof. Fix a ball $B = \mathbb{B}^n(x, R)$ such that B is compactly contained in Ω and $R < R_0$, with R_0 given by Lemma 3.3. Let $A \subset B$ be the set of all Lebesgue points of f in B. We show first that the restriction of f to $A \cap \mathbb{B}^n(x, R/(4e^3))$ is continuous. For this, let $y, z \in A \cap \mathbb{B}^n(x, R/(4e^3))$.

We may select a two-sided sequence balls $B_i \subset B, i \in \mathbb{Z}$ in the following way: $B_0 = \mathbb{B}^n((y+z)/2, r_0)$ with $r_0 = |y - z| \in (0, R/(2e^3))$, $B_i = \mathbb{B}^n(y, e^{-|i|} r_0)$ for $i \in \mathbb{Z}_{>0}$, $B_i = \mathbb{B}^n(z, e^{-|i|} r_0)$ for $i \in \mathbb{Z}_{<0}$. We denote the integral average of f over B_i by $f_{B_i} \in \mathbb{R}^n$; since y and z are Lebesgue points, we have

$$\lim_{i \to \infty} f_{B_i} = f(y), \qquad \lim_{i \to -\infty} f_{B_i} = f(z). \tag{3.3}$$

Moreover, since $\mathbb{B}^n(y, R/2)$ and $\mathbb{B}^n(z, R/2)$ and $\mathbb{B}^n((y+z)/2, R/2)$ are all contained in B and $r_i < (R/2)/e^3$, Lemma 3.3 yields for every $i \in \mathbb{Z}$ that

$$\int_{B_i} \frac{|Df|^n}{K} \leq C_1 \log^{-\lambda} \frac{1}{r_i} = C_1 \left(\log \frac{1}{|y-z|} + |i|\right)^{-\lambda}, \tag{3.4}$$

with $C_1 = C_1(B, \lambda, \mu, K, \Sigma, R/2, \|Df\|_{L^n(B)})$ independent of y, z, and i.

We then estimate $|f_{B_{i+1}} - f_{B_i}|$. We present the case $i \geq 0$, as the case $i < 0$ is similar but with i and $i-1$ switched. Since $B_{i+1} \subset B_i$ and the radius of B_{i+1} is e^{-1} times the radius of B_i, we have by the Sobolev-Poincaré inequality that

$$\left|f_{B_{i+1}} - f_{B_i}\right| \leq \fint_{B_{i+1}} |f - f_{B_i}| \leq e^{-n} \fint_{B_i} |f - f_{B_i}| \\ \leq C_2(n) r_i \left(\fint_{B_i} |Df|^{n-1}\right)^{\frac{1}{n-1}}.$$

We then use Hölder's inequality to estimate that

$$r_i \left(\fint_{B_{r_i}} |Df|^{n-1}\right)^{\frac{1}{n-1}} \leq r_i \left(\fint_{B_{r_i}} \frac{|Df|^n}{K}\right)^{\frac{1}{n}} \left(\fint_{B_{r_i}} K^{n-1}\right)^{\frac{1}{n^2-n}} \\ = \frac{1}{\sqrt[n]{\omega_n}} \left(\int_{B_{r_i}} \frac{|Df|^n}{K}\right)^{\frac{1}{n}} \left(\fint_{B_{r_i}} K^{n-1}\right)^{\frac{1}{n^2-n}}.$$

Applying the decay estimate (3.4), we hence have that

$$\left|f_{B_{i+1}} - f_{B_i}\right| \leq C_3 \left(\fint_{B_{r_i}} K^{n-1}\right)^{\frac{1}{n^2-n}} \left(\log \frac{1}{|y-z|} + |i|\right)^{-\frac{\lambda}{n}},$$

where $C_3 = C_3(B, \lambda, \mu, K, \Sigma, \|Df\|_{L^n(B)})$. We then estimate the average integral term. For this, we again define $\tilde{K} = \max(K, (n-2)\lambda^{-1})$ as in Lemma 3.1, and use Jensen's inequality with the function $\tau \mapsto \exp(\lambda \tau^{\frac{1}{n-1}})$. This yields the estimate

$$\left(\fint_{B_{r_i}} K^{n-1}\right)^{\frac{1}{n^2-n}} \leq \lambda^{-\frac{1}{n}} \log^{\frac{1}{n}} \left(\fint_{B_{r_i}} \exp(\lambda \tilde{K})\right) \\ \leq \lambda^{-\frac{1}{n}} \log^{\frac{1}{n}} \left(\frac{\|\exp(\lambda \tilde{K})\|_{L^1(B)}}{\omega_n r_i^n}\right) \\ \leq C_4 \log^{\frac{1}{n}} \frac{1}{r_i} = C_4 \left(\log \frac{1}{|y-z|} + |i|\right)^{\frac{1}{n}}, \tag{3.5}$$

where $C_4 = C_4(B, \lambda, K)$.

Now, by (3.3) and a telescopic sum argument, we obtain that

$$|f(y) - f(z)| \leq \sum_{i=-\infty}^{\infty} \left|f_{B_{i+1}} - f_{B_i}\right| \leq 2 \sum_{i=0}^{\infty} C_5 \left(\log \frac{1}{|y-z|} + i\right)^{\frac{1-\lambda}{n}}, \tag{3.6}$$

with $C_5 = C_5(B, \lambda, \mu, K, \Sigma, \|Df\|_{L^n(B)})$. We then denote $a = \log(1/|y-z|)$, noting that $a > 1$ since $|y-z| < R < R_0 < e^{-1}$. Since we also assume that

$\lambda > n+1$, we have that $i \mapsto (a+i)^{(1-\lambda)/n}$ is decreasing, and we may estimate

$$\sum_{i=0}^{\infty} (a+i)^{\frac{1-\lambda}{n}} \leq a^{\frac{1-\lambda}{n}} + \int_0^{\infty} (a+t)^{\frac{1-\lambda}{n}} \, \mathrm{d}t$$
$$= a^{-\frac{\lambda-1}{n}} + \frac{n}{\lambda-n-1} a^{-\frac{\lambda-n-1}{n}} \leq \frac{\lambda-1}{\lambda-n-1} a^{-\frac{\lambda-n-1}{n}}.$$

In conclusion,

$$(3.7) \quad |f(y)-f(z)| \leq 2C_5 \sum_{i=0}^{\infty} (a+i)^{\frac{1-\lambda}{n}}$$
$$\leq 2C_5 \frac{\lambda-1}{\lambda-n-1} a^{-\frac{\lambda-n-1}{n}} = C_6 \log^{-\frac{\lambda-n-1}{n}} \frac{1}{|y-z|},$$

with $C_6 = C_6(B, \lambda, \mu, K, \Sigma, \|Df\|_{L^n(B)})$.

We hence have obtained the desired modulus of continuity for all Lebesgue points $y, z \in A \cap \mathbb{B}^n(x, R/(4e^3))$. Now, if $y \in \mathbb{B}^n(x, R/(4e^3)) \setminus A$, we can then use the fact that A has full measure in $\mathbb{B}^n(x, R/(4e^3))$ to select $y_j \in A \cap \mathbb{B}^n(x, R/(4e^3))$ such that $y_j \to y$ as $j \to \infty$. By (3.7), $(f(y_j))$ is a Cauchy sequence, and therefore convergent. We select $f(y) = \lim_{j\to\infty} f(y_j)$; doing this for all $y \in \mathbb{B}^n(x, R/(4e^3)) \setminus A$ only changes the values of f in a set of measure zero, and doesn't change $f(y)$ in points y where f is continuous. Now, by passing to the limit, we see that (3.7) applies to all $y, z \in \mathbb{B}^n(x, R/(4e^3))$. Hence, f has a continuous representative with the desired modulus of continuity. □

Theorem 1.9 then follows as an immediate corollary of already proven results.

Theorem 1.9. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that a Sobolev mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies $Df \in \mathcal{M}_n(K, \Sigma, y_0)$ with $K\colon \Omega \to [1,\infty)$, $\Sigma\colon \Omega \to [0,\infty)$ and $y_0 \in \mathbb{R}^n$. If*

$$\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \Sigma \log^{\mu}(e+\Sigma) \in L^1_{\mathrm{loc}}(\Omega),$$

for some $\mu > \lambda + n - 1 > 2n$, then f has a continuous representative.

In particular, for all $x_0 \in \Omega$ and sufficiently small $r > 0$, we have the following local modulus of continuity estimate:

$$\omega_f(x_0, r) \leq C \log^{-\alpha}(1/r) \quad \text{where } \alpha = \frac{\lambda-n-1}{n}.$$

Proof. Since $\Sigma \log^{\mu}(e+\Sigma) \in L^1_{\mathrm{loc}}(\Omega)$, and since $\exp(A\,|f|^{n/(n-1)}) \in L^1_{\mathrm{loc}}(\Omega)$ for some $A > 0$ by Theorem 2.5, we have $\Sigma\,|f|^n \log^{\mu-n+1}(e+\Sigma\,|f|^n) \in L^1_{\mathrm{loc}}(\Omega)$ by Lemma 2.6, where $\mu - n + 1 > \lambda > n+1$. Hence, the claim follows by applying Theorem 1.7. □

4. Counterexamples based on cusps

In this section, we consider our first type of counterexample, which yields Theorems 1.6 and 1.11. Our construction will be in a planar disk $\mathbb{D}(r_0)$ with center at the origin and radius r_0. Our constructed mapping $f\colon \mathbb{D}(r_0) \to \mathbb{R}^2$ has a first coordinate function of $-\log\log|z|^{-1}$, which is well defined in $\mathbb{D}(r_0) \setminus \{0\}$ as long as r_0 is small enough. We split the disk $\mathbb{D}(r_0)$ into two regions $\mathbb{D}(r_0) = A \cup B$ with different definitions of the second coordinate function, where in A we aim to have $J_f(x) \geq 0$ with $|Df(x)|^2 \leq K J_f(x)$, and in B we try to obtain $J_f(x) \leq 0$ and $|Df(x)|^2 + K|J_f(x)| \leq \Sigma$. The region B will form a cusp at the origin.

4.1. **The two regions.** Let $\Omega = \mathbb{D}(r_0)$, where we assume that $r_0 \leq e^{-e}$. We begin by assuming that γ is an absolutely continuous increasing function $\gamma\colon [0, r_0) \to [0, 1)$ such that $\gamma(0) = 0$. We specify γ later in the text, as we use different choices of γ to prove different theorems. We will use polar coordinates (r, θ) on the domain side in Ω, where $\theta \in (-\pi, \pi]$.

The regions $A, B \subset \Omega$ will consist of two sub-regions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ each. We let B_1 be the cusp-like region of Ω bounded by the curves $\theta = \gamma(r)$ and $\theta = -\gamma(r)$. Similarly, we let A_1 be the region bounded by the curves $\theta = \gamma(r)$ and $\theta = \pi - \gamma(r)$. The region B_2 is the reflection $-B_1$ of B_1 across the origin, and similarly $A_2 = -A_1$. See Figure 1 for an illustration.

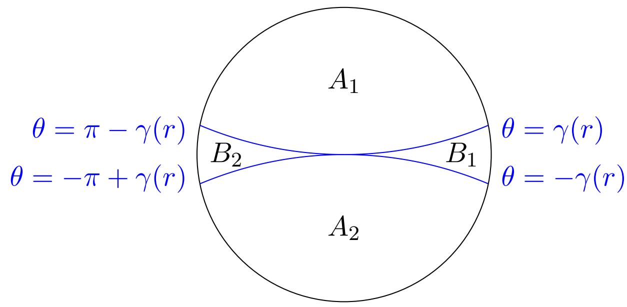

Figure 1. The regions A_1, A_2, B_1 and B_2.

4.2. **The function f in the region A.** We first define f in the region A_1. There, using polar coordinates on the domain side and Cartesian coordinates on the target side, we have

$$f(r, \theta) = (-\log\log r^{-1}, h(r)\theta) \tag{4.1}$$

for some absolutely continuous increasing function $h\colon [0, \infty) \to [0, \infty)$. We again specify h later.

Hence, we obtain a matrix of derivatives

$$\begin{bmatrix} \partial_r f_1 & r^{-1}\partial_\theta f_1 \\ \partial_r f_2 & r^{-1}\partial_\theta f_2 \end{bmatrix} = \begin{bmatrix} r^{-1}\log^{-1} r^{-1} & 0 \\ h'(r)\theta & r^{-1}h(r) \end{bmatrix}.$$

In particular,

$$|Df(r,\theta)|^2 \le \frac{1}{r^2\log^2 r^{-1}} + [h'(r)\theta]^2 + \frac{h^2(r)}{r^2} \tag{4.2}$$

and

$$J_f(r,\theta) = \frac{h(r)}{r^2\log r^{-1}} \ge 0. \tag{4.3}$$

We then simply pick $K = |Df|^2/J_f$ and $\Sigma \equiv 0$.

On A_2, we define $f(z) = f(-z)$. Since $z \mapsto -z$ is an orientation-preserving isometry in the plane, it follows that (4.2) and (4.3) remain true in A_2.

4.3. **The function f in the region B.** We wish that our function f is continuous outside the origin. Hence, our boundary values in B_1 must match the ones given by A_1 and A_2. For this, we define the second coordinate of f as a linear interpolation of these boundary values. That is, we define in B_1 that

$$f(r,\theta) = \left(-\log\log r^{-1}, h(r)\left(\frac{\pi + 2\theta - \pi\theta/\gamma(r)}{2}\right)\right). \tag{4.4}$$

Indeed, in the cases $\theta = \gamma(r)$ and $\theta = -\gamma(r)$, the second coordinate has the correct boundary values of $h(r)\gamma(r)$ and $h(r)(\pi - \gamma(r))$, respectively.

The derivatives of the first coordinate of f remain unchanged from domain A. For the other terms in the matrix of derivatives, we first get

$$\partial_r f_2 = \left(\frac{\pi}{2} + \theta\right) h'(r) - \frac{\pi\theta}{2}\frac{d}{dr}\left(\frac{h(r)}{\gamma(r)}\right). \tag{4.5}$$

Then, by $\gamma(r) < 1$, we get

$$r^{-1}\partial_\theta f_2 = \frac{h(r)}{r}\left(1 - \frac{\pi}{2\gamma(r)}\right) < -\left(\frac{\pi}{2} - 1\right)\frac{h(r)}{r\gamma(r)}. \tag{4.6}$$

In particular, we have that $r^{-1}\partial_\theta f_2 < 0$, and consequently $J_f < 0$ in B_1. Hence, in B_1, we select $K = -|Df|^2/J_f \ge 1$, and $\Sigma = 2|Df|^2$.

Similarly as for A_2, we may define f in B_2 by $f(z) = f(-z)$, and all our considerations will also apply to B_2.

4.4. **Fixing the parameters.** We have now outlined the construction, but have left the functions h and γ undetermined. The theorems we wish to prove follow with different choices of h and γ.

Throughout the rest of this paper, given two functions $f, g\colon X \to \mathbb{R}$, we use the notation $f \lesssim g$ if there exists a constant $C > 0$ such that $f \le Cg$. We also denote $f \approx g$ if $f \lesssim g \lesssim f$. Several of the uses of these symbols are based on the elementary fact that if $f, g\colon [a,\infty) \to (0,\infty)$, $a \in \mathbb{R}$, are continuous and $\limsup_{t\to\infty} f(t)/g(t) < \infty$, then $f \lesssim g$.

We now recall the statement of Theorem 1.11, and then give its proof.

Theorem 1.11. *Let $p, q \in (1, \infty)$. If $p^{-1} + q^{-1} \geq 1$, then there exists a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$K \in L^p(\Omega) \qquad \text{and} \qquad \frac{\Sigma}{K} \in L^q(\Omega).$$

Proof. Let $p, q \in (1, \infty)$, and let $\varepsilon > 0$. We select $r_0 = e^{-e}$ and

$$h(r) = r^{2p^{-1}}, \qquad \gamma(r) = \log^{-\varepsilon} r^{-1}.$$

Indeed, when $r < e^{-e}$, we have $0 \leq \gamma(r) < e^{-\varepsilon} < 1$.

In A_1, we have by (4.2) that

$$|Df(r,\theta)|^2 \leq \frac{1}{r^2 \log^2 r^{-1}} + \left(\frac{4\pi^2}{p^2} + 1\right) r^{-2+2p^{-1}} \lesssim \frac{1}{r^2 \log^2 r^{-1}}.$$

Hence, $|Df| \in L^2(A)$. By also referring to (4.3), we have in A_1 the estimate

$$K(r,\theta) = \frac{|Df(r,\theta)|^2}{J_f(r,\theta)} \lesssim \frac{1}{r^{2p^{-1}} \log r^{-1}}.$$

We hence estimate that

$$\begin{aligned}\int_A K^p &= 2\int_0^{e^{-e}} \int_{\gamma(r)}^{\pi-\gamma(r)} K^p(r,\theta) r \, \mathrm{d}\theta \, \mathrm{d}r \\ &\lesssim \int_0^{e^{-e}} r^{-1} \log^{-p} r^{-1} \, \mathrm{d}r < \infty,\end{aligned}$$

showing that $K \in L^p(A)$.

We then consider points (r, θ) in B_1. By (4.5) and $|\theta| \leq \gamma(r) < 1 \leq \pi$, we have

$$\begin{aligned}|\partial_r f_2(r,\theta)|^2 &= \left| \frac{2\theta + \pi}{p} r^{2p^{-1}-1} + \frac{\pi\theta \log^\varepsilon r^{-1}}{2} r^{2p^{-1}-1} \left(\frac{2}{p} - \frac{\varepsilon}{\log r^{-1}}\right) \right|^2 \\ &\lesssim r^{4p^{-1}-2} \left(1 + \log^{-2+2\varepsilon} r^{-1}\right).\end{aligned}$$

Furthermore, by (4.6),

$$\left|r^{-1} \partial_\theta f_2(r,\theta)\right|^2 \lesssim r^{4p^{-1}-2} \log^{2\varepsilon} r^{-1}.$$

The exponent $4p^{-1} - 2$ in the above bounds is greater than -2. Hence, we have the overall estimate

$$|Df(r,\theta)|^2 \approx \frac{1}{r^2 \log^2 r^{-1}} \tag{4.7}$$

whenever $(r, \theta) \in B_1$. In particular, we have $|Df| \in L^2(B_1)$, and consequently $|Df| \in L^2(\Omega)$. Moreover, we have

$$-J_f(r,\theta) = \left(\frac{\pi}{2} \log^\varepsilon r^{-1} - 1\right) r^{2p^{-1}-2} \log^{-1} r^{-1},$$

so by $\log^\varepsilon r^{-1} > 1$ we get the two-sided estimate

$$-J_f(r,\theta) \approx r^{2p^{-1}-2} \log^{\varepsilon-1} r^{-1}. \tag{4.8}$$

Combined with (4.7), this yields

$$\frac{|Df(r,\theta)|^2}{-J_f(r,\theta)} = K(r,\theta) \approx \frac{1}{r^{2p^{-1}} \log^{1+\varepsilon} r^{-1}}.$$

Since $p(2p^{-1}) = 2$ and $p(1+\varepsilon) > 1$, we see that $K \in L^p(B)$, and hence $K \in L^p(\Omega)$.

It hence remains to consider the integral $(\Sigma/K)^q$ over B. Since we chose $\Sigma = |Df|^2$ and $K = |Df|^2/(-J_f(x))$, we have $\Sigma/K = -J_f(x)$. Hence, by (4.8), we have $\Sigma/K \lesssim r^{2p^{-1}-2} \log^{\varepsilon-1} r^{-1}$. We note that since B is a cusp, this majorant in fact has a better degree of integrability over B than it has over Ω. In particular, we may estimate that

$$\begin{aligned}\int_B \frac{\Sigma^q}{K^q} &\lesssim \int_0^{e^{-e}} \int_{-\gamma(r)}^{\gamma(r)} \frac{r \,\mathrm{d}\theta \,\mathrm{d}r}{r^{2q-2p^{-1}q} \log^{q-q\varepsilon} r^{-1}} \\ &\leq 2\int_0^{e^{-e}} \frac{\mathrm{d}r}{r^{2q-2p^{-1}q-1} \log^{q-(q-1)\varepsilon} r^{-1}}.\end{aligned}$$

For integrability, we require $2q - 2p^{-1}q - 1 \leq 1$, which is equivalent to $q^{-1} \geq 1 - p^{-1}$. Moreover, in the extremal case $q^{-1} + p^{-1} = 1$, we also require $q - (q-1)\varepsilon > 1$, which is equivalent to $\varepsilon < 1$. Hence, any choice of $\varepsilon \in (0,1)$ will give us the desired example. □

Our next result is the version of this example with the highest degree of integrability for Σ. This is by a different choice of h and γ, and hence this gain in the regularity of Σ comes at a cost in the regularity of Σ/K.

Theorem 4.1. *Let $p, s \in (1,\infty)$. If $(p+1)^{-1} + s^{-1} \geq 1$, then there exists a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K,\Sigma)$ with*

$$K \in L^p(\Omega) \qquad \text{and} \qquad \Sigma \in L^s(\Omega).$$

Proof. Let $p, q \in (1,\infty)$, and let $\varepsilon > 0$. This time we choose

$$h(r) = r^{2p^{-1}}, \qquad \gamma(r) = r^{2p^{-1}} \log r^{-1}.$$

We may select an $r_0 \leq e^{-e}$ such that γ is increasing on $[0, r_0]$ and $\gamma(r_0) < 1$.

The verification that $K \in L^p(A)$ is unchanged from the previous lemma. The difference arises when applying (4.5) and (4.6). Indeed, since $|\theta| \leq r^{2p^{-1}} \log r^{-1} \leq 1$, we obtain.

$$\begin{aligned}|\partial_r f_2(r,\theta)|^2 &= \left| \frac{2\theta+\pi}{p} r^{2p^{-1}-1} + \frac{\pi\theta}{2} r^{-1} \log^{-2} r^{-1} \right|^2 \\ &\lesssim r^{4p^{-1}-2} + r^{-2} \log^{-4} r^{-1},\end{aligned}$$

and

$$\left| r^{-1} \partial_\theta f_2(r,\theta) \right|^2 \lesssim r^{-2} \log^{-2} r^{-1}.$$

In all of the previously computed terms, either the exponent of r is greater than -2, or the exponent of r is -2 and the exponent of the logarithm is at most -2. Hence, we still have (4.7) unchanged. For J_f, we compute similarly as in the last lemma, and instead get

$$-J_f(r,\theta) \approx r^{-2}\log^{-2} r^{-1} \tag{4.9}$$

when $(r,\theta) \in B$. In particular, $K = |Df|^2/(-J_f) \in L^\infty(B)$.

It remains to estimate the integral of $\Sigma^s = (2|Df|)^s$ over B. Computing similarly as in the previous lemma, we get

$$\begin{aligned}\int_B \Sigma^s &\lesssim \int_0^{r_0}\int_{-\gamma(r)}^{\gamma(r)} \frac{r\,\mathrm{d}\theta\,\mathrm{d}r}{r^{2s}\log^{2s} r^{-1}}\\ &\leq 2\int_0^{r_0} \frac{\mathrm{d}r}{r^{2s-2p^{-1}-1}\log^{2s-1} r^{-1}}.\end{aligned}$$

For this to converge, since the exponent of the logarithm satisfies $2s-1>1$ due to $s>1$, we only require $2s-2p^{-1}-1 \leq 1$. Rearranging yields $s \leq 1+p^{-1} = (p+1)^*$, where $(p+1)^*$ is the Hölder conjugate of $p+1$. In particular, this is equivalent with $(p+1)^{-1}+s^{-1} \geq 1$. □

The remaining result which relies on this example type is Theorem 1.6. This is achieved by selecting both h and γ to be powers of logarithms, with a suitable choice of corresponding exponents.

Theorem 1.6. *For every $\mu \in (0,2)$, there exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega,\mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\},\mathbb{R}^2)$, $\lim_{x\to 0}|f(x)| = \infty$, and $Df \in \mathcal{M}_2(K,\Sigma)$ with*

$$\exp(\lambda K) \in L^1(\Omega) \qquad \textit{and} \qquad \Sigma\log^\mu(e+\Sigma) \in L^1(\Omega)$$

for every $\lambda > 0$.

Proof. Let $\lambda \in (0,\infty)$. We may assume $\mu > 1$, as an example for a given μ also works for all smaller μ. We choose

$$h(r) = \log^{-\nu} r^{-1}, \qquad \gamma(r) = \log^{1-\nu} r^{-1},$$

Where $\nu \in (\mu,2)$. Since $\nu > \mu > 1$ by assumption, γ is increasing, and we may hence choose $r_0 = e^{-e}$ as $\gamma(e^{-e}) = e^{1-\nu} < 1$.

In A_1, (4.2) yields due to $\nu > 1$ that

$$|Df(r,\theta)|^2 \leq \frac{1}{r^2\log^2 r^{-1}} + \frac{\nu}{r^2\log^{2\nu+2} r^{-1}} + \frac{1}{r^2\log^{2\nu} r^{-1}} \lesssim \frac{1}{r^2\log^2 r^{-1}}.$$

Hence, clearly $|Df| \in L^2(A)$. Moreover, $J_f = r^{-2}\log^{-1-\nu} r^{-1}$, so

$$K(r,\theta) \leq C\log^{\nu-1} r^{-1}$$

for some $C > 0$. The exponential integrals of K are all finite by the estimate

$$\begin{aligned}\int_A \exp(\lambda K) &\leq 2\pi \int_0^{e^{-e}} \exp(C\lambda \log^{\nu-1} r^{-1}) r \,\mathrm{d}r \\ &= 2\pi \int_0^{e^{-e}} r^{1-C\lambda \log^{\nu-2} r^{-1}} \,\mathrm{d}r < \infty,\end{aligned}$$

since $\lim_{r\to 0^+} C\lambda \log^{\nu-2} r^{-1} = 0$ due to $\nu < 2$.

In B_1, (4.5) and (4.6) combined with $|\theta| \leq \log^{1-\nu} r^{-1} \leq 1 \leq \pi$ result in

$$\begin{aligned}|\partial_r f_2(r,\theta)|^2 &= \left| \frac{2\nu\theta + \pi\nu}{2} r^{-1} \log^{-\nu-1} r^{-1} + \frac{\pi\theta}{2} r^{-1} \log^{-2} r^{-1} \right|^2 \\ &\lesssim r^{-2} \left(\log^{-2-2\nu} r^{-1} + \log^{-4} r^{-1} \right)\end{aligned}$$

and

$$\left| r^{-1} \partial_\theta f_2(r,\theta) \right|^2 \lesssim r^{-2} \log^{-2} r^{-1}.$$

As $\log^t r^{-1}$ is increasing with respect to t when $r < e^{-e}$, we again have

$$|Df(r,\theta)|^2 \leq \frac{C'}{r^2 \log^2 r^{-1}}$$

in B_1 for some $C' > 0$. For the Jacobian, we instead get

$$-J_f(r,\theta) \approx r^{-2} \log^{-2} r^{-1}. \tag{4.10}$$

In particular, our choice $K = |Df|^2 / (-J_f)$ is in $L^\infty(B)$, concluding exponential integrability of K in all of Ω for all choices of λ.

The last step is to estimate the integral of $\Sigma \log^\mu(e + \Sigma)$ over B_1, where $\Sigma = 2|Df|^2$. We estimate using $(e + ab) \leq (e + a)(e + b)$ for $a, b \geq 0$ that

$$\log(e + \Sigma) \leq \log\left(e + \frac{2C'}{r^2 \log^2 r^{-1}} \right) \leq \log\left(e + \frac{2C'}{\log^2 r^{-2}} \right) + 2\log(e + r^{-1}),$$

and hence, as $\gamma(r) = \log^{1-\nu} r^{-1}$, we get

$$\begin{aligned}\int_B \Sigma \log^\mu(e + \Sigma) &= 2 \int_0^{e^{-e}} \int_{-\gamma(r)}^{\gamma(r)} \Sigma(r,\theta) \log^\mu(e + \Sigma(r,\theta)) r \,\mathrm{d}\theta \,\mathrm{d}r \\ &\lesssim \int_0^{e^{-e}} \frac{\log^\mu(e + 2C' \log^{-2} r^{-1}) + \log^\mu(e + r^{-1})}{r \log^{2-(1-\nu)} r^{-1}} \,\mathrm{d}r.\end{aligned}$$

When $r \to 0$, we have $\log^{-2} r^{-1} \to 0$. Hence, for small r, the largest term in the numerator is $\log^\mu(e + r^{-1})$. Since $r^{-1} > e^e$, we have $r^{-2} - r^{-1} - e \geq 0$. Hence, we may estimate

$$\frac{\log^\mu(e + r^{-1})}{r \log^{2-(1-\nu)} r^{-1}} \leq \frac{2^\mu}{r \log^{1+(\nu-\mu)} r^{-1}},$$

which is integrable over $[0, e^{-e}]$ due to our assumption $\nu > \mu$. Thus, $\Sigma \log^\mu(e + \Sigma) \in L^1(B)$, and consequently $\Sigma \log^\mu(e + \Sigma) \in L^1(\Omega)$. □

5. Counterexamples based on spirals

In this section, we construct a counterexample built around the case $\Sigma \in L^\infty(\Omega)$, which will give us Theorem 1.5. Furthermore, if $K \in L^p_{\mathrm{loc}}(\Omega)$ with $p \in [1,2]$, then this counterexample also yields an alternate proof of Theorem 1.11. In exchange for failing when $p > 2$, this alternate counterexample has a better optimal integrability for Σ when $p < \sqrt{2}$; that is, it improves Theorem 4.1 for such values of p. Moreover, this improved integrability of Σ is achieved simultaneously with the optimal integrability of Σ/K, whereas the construction of Theorem 1.11 involves a trade-off between the integrabilities of Σ and Σ/K.

Theorem 5.1. *Suppose that $p \in [1,2]$, $q \in [1,\infty]$, and $p^{-1} + q^{-1} \geq 1$. Then there exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x\to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$K \in L^p(\Omega), \qquad \frac{\Sigma}{K} \in L^q(\Omega), \qquad \textit{and} \qquad \Sigma \in L^{\frac{q}{2}}(\Omega).$$

We again construct our example in a planar region $\Omega \subset \mathbb{R}^2$ with a point of discontinuity at the origin, and we retain our strategy from the previous section of splitting Ω into two regions A and B, where $|Df|^2 \leq K J_f$ in A and $|Df|^2 + K|J_f| \leq \Sigma$ in B. Notably, when Σ is bounded from above by a constant, f ends up being Lipschitz under the path length metric in B. Hence, if we wish that f escapes to infinity along B, the region B must somehow be infinitely long. This pushes us towards a construction where A and B are two interlocking infinitely long spirals centered at the origin.

5.1. Preliminaries: Lambert's W-function.

We begin by recalling a special function that is of great use to us in our construction. Namely, *Lambert's W-function* is the inverse function $W = \psi^{-1}$ of the function $\psi(t) = te^t$. The W-function has two branches on the real line. In this paper, we assume W to be the positive branch:

$$W\colon [-e^{-1}, \infty) \to [-1, \infty), \quad W(t)e^{W(t)} = t.$$

We collect into the following lemma the elementary properties of the W-function that we use. For a general reference on the W-function, see e.g. [7].

Lemma 5.2. *The W-function satisfies the following.*

1. *W is strictly increasing on $[-e^{-1}, \infty)$.*
2. *$W(0) = 0$, and hence $W(t) > 0$ if $t > 0$.*
3. *We have $W(t \log t) = \log t$ if $t \geq e^{-1}$.*
4. *The derivative of W is given on $(-e^{-1}, \infty)$ by*

$$W'(t) = \frac{W(t)}{t(1 + W(t))} = \frac{1}{t + e^{W(t)}}.$$

5.2. **Construction.** We define two spirals in polar coordinates. The first one is the spiral $r = g(\theta)$, where

$$g(\theta) = \frac{1}{\theta \log \theta}, \qquad \theta \in [\theta_0, \infty),$$

where $\theta_0 \geq 2\pi$ is some starting angle. The second one is given by $r = h(\theta)$, where

$$h(\theta) = \frac{g(\theta) + g(\theta + 2\pi)}{2}, \qquad \theta \in [\theta_0, \infty);$$

that is, the spiral $r = h(\theta)$ lies exactly halfway between the successive points where the spiral $r = g(\theta)$ meets a specific ray from the origin. Using the standard symbol i for the complex imaginary unit, we define our domain $\Omega \subset \mathbb{C}$ by

$$\Omega = \{re^{i\theta} : 0 \leq r < g(\theta), \theta \in (\theta_0, \theta_0 + 2\pi]\}.$$

See Figure 2 for an illustration of Ω and the spirals.

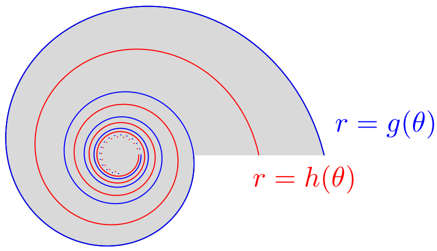

FIGURE 2. The two spirals $r = g(\theta)$ and $r = h(\theta)$, with the domain Ω highlighted in gray.

We parametrize Ω in the following way: let

$$U = \{(r, \theta) \in \mathbb{R}^2 : \theta \geq \theta_0, g(\theta + 2\pi) \leq r < g(\theta)\},$$

in which case the map $(r, \theta) \mapsto re^{i\theta}$ maps U bijectively to $\Omega \setminus \{0\}$. Let $\alpha \in (0, 1]$. We define $f \colon \Omega \setminus \{0\} \to \mathbb{C}$ on the two regions between the spirals $r = g(\theta)$ and $r = h(\theta)$ in terms of polar coordinates $(r, \theta) \in U$: when $h(\theta) \leq r < g(\theta)$, we define the complex-valued output of f by

$$f(r, \theta) = \varphi(r) - i \log \log \theta,$$

where $\varphi(r) \colon [0, r_0] \to \mathbb{R}$ is an increasing absolutely continuous function to be fixed later, with $r_0 > \theta_0^{-1} \log^{-1} \theta_0$. In the other region where $g(\theta + 2\pi) \leq r < h(\theta)$, we instead define

$$f(r, \theta) = \varphi(r) - i \log W\left(\frac{1}{2r - g(\theta + 2\pi)}\right).$$

This defines f on all of $\Omega \setminus \{0\}$.

We briefly verify that f is indeed continuous on $\Omega \setminus \{0\}$. If $r = h(\theta)$, then

$$\begin{aligned}\log W\left(\frac{1}{2r - g(\theta + 2\pi)}\right) &= \log W\left(\frac{1}{2h(\theta) - g(\theta + 2\pi)}\right) \\ &= \log W\left(\frac{1}{g(\theta)}\right) = \log W(\theta \log \theta) = \log\log\theta,\end{aligned}$$

which verifies that f is continuous on the spiral $r = h(\theta)$. On the other hand, if $r = g(\theta + 2\pi)$, then

$$\begin{aligned}\log W\left(\frac{1}{2r - g(\theta + 2\pi)}\right) &= \log W\left(\frac{1}{g(\theta + 2\pi)}\right) \\ &= \log W((\theta + 2\pi)\log(\theta + 2\pi)) = \log\log(\theta + 2\pi),\end{aligned}$$

which verifies continuity of f on the spiral $r = g(\theta)$. Hence, f is continuous on $\Omega \setminus \{0\}$.

5.3. **The first region.** We then compute $|Df|$ and J_f in the region $B \subset \Omega$ where $h(\theta) < r < g(\theta)$ in terms of our polar coordinate parametrization. In polar coordinates, the derivative matrix of f becomes

$$\begin{bmatrix}\partial_r \operatorname{Re}(f) & r^{-1}\partial_\theta \operatorname{Re}(f) \\ \partial_r \operatorname{Im}(f) & r^{-1}\partial_\theta \operatorname{Im}(f)\end{bmatrix} = \begin{bmatrix}\varphi'(r) & 0 \\ 0 & -r^{-1}\theta^{-1}\log^{-1}\theta\end{bmatrix} = \begin{bmatrix}\varphi'(r) & 0 \\ 0 & -r^{-1}g(\theta)\end{bmatrix}.$$

Since we moreover have $r \geq h(\theta) = (g(\theta) + g(\theta + 2\pi))/2 \geq g(\theta)/2$, we obtain the upper bound

$$r^{-1}g(\theta) \leq 2.$$

Hence, we have the estimate

$$|Df(r,\theta)|^2 \leq 4 + (\varphi'(r))^2.$$

Note especially that $|Df|$ is bounded in B when φ is Lipschitz. Moreover, we have $|Df| \in L^2(B)$ as long as $r(\varphi'(r))^2 \in L^1([0, r_0])$.

On the other hand, $J_f(r,\theta) = -r^{-1}\varphi'(r)g(\theta)$, which is negative since $\varphi(r)$ is increasing. Furthermore, $-J_f(r,\theta)$ is bounded from above by $2\varphi'(r)$. Hence, in order to achieve the desired condition $|Df|^2 + K|J_f| \leq \Sigma$, we arrive at the following valid choices for Σ and K:

$$\Sigma(r,\theta) = 6 + 3(\varphi'(r))^2, \qquad K(r,\theta) = \max(\varphi'(r), 1). \tag{5.1}$$

5.4. **The second region.** Next, we consider the region $A \subset \Omega$ where we have $g(\theta + 2\pi) \leq r < h(\theta)$ in terms of our polar coordinate parametrization. We still have $\partial_r \operatorname{Re}(f) = \varphi'(r)$ and $\partial_\theta \operatorname{Re}(f) = 0$, which are square integrable whenever $r(\varphi'(r))^2 \in L^1([0, r_0])$. The next step is then to compute $\partial_r \operatorname{Im}(f)$ and $\partial_\theta \operatorname{Im}(f)$. We use the shorthands

$$\tau = \theta + 2\pi, \qquad u = (2r - g(\tau))^{-1}.$$

For $\partial_r \operatorname{Im}(f)$, we have $\partial_r u = -2u^2$, and we may hence use Lemma 5.2 to obtain

$$\partial_r \left(-\log W(u)\right) = -\frac{1}{W(u)} \frac{W(u)}{u(1+W(u))} \left(-2u^2\right) = \frac{2u}{1+W(u)}.$$

For $\partial_\theta \operatorname{Im}(f)$, we have $\partial_\theta u = -u^2 \cdot (-g'(\tau)) = -u^2(1+\log(\tau))/(\tau^2 \log^2 \tau)$, and hence Lemma 5.2 similarly yields

$$\partial_\theta \left(-\log W(u)\right) = \frac{(1+\log\tau)u}{(1+W(u))\tau^2 \log^2 \tau}.$$

We thus arrive at the derivative matrix

$$\begin{bmatrix} \partial_r \operatorname{Re}(f) & r^{-1}\partial_\theta \operatorname{Re}(f) \\ \partial_r \operatorname{Im}(f) & r^{-1}\partial_\theta \operatorname{Im}(f) \end{bmatrix} = \begin{bmatrix} \varphi'(r) & 0 \\ \dfrac{2u}{1+W(u)} & \dfrac{(1+\log\tau)u}{r(1+W(u))\tau^2\log^2\tau} \end{bmatrix}.$$

To estimate these derivatives, we note that $g(\tau) \leq 2r - g(\tau) \leq g(\theta)$ in our region. Inverting all terms, it follows that $\theta \log\theta \leq u \leq \tau \log \tau$. Since W is increasing and $W(t \log t) = \log t$, we hence have $\log\theta \leq W(u) \leq \log\tau$. Moreover, recalling the notation from the beginning of Section 4.4, it is reasonably easy to see that $\log(\theta) \approx \log(\tau)$ and $g(\theta) \approx g(\tau)$ for $\theta \in [\theta_0, \infty)$. In particular, we have

$$u \approx \theta\log\theta \qquad \text{and} \qquad W(u) \approx \log\theta.$$

We can then estimate $|\partial_r \operatorname{Im} f(r,\theta)|$ from both sides by

$$\frac{2\theta\log\theta}{1+\log\tau} \leq |\partial_r \operatorname{Im} f(r,\theta)| \leq \frac{2\tau\log\tau}{1+\log\theta},$$

implying that

$$|\partial_r \operatorname{Im} f(r,\theta)| \approx \theta. \tag{5.2}$$

To bound $r^{-1}\partial_\theta \operatorname{Im}(f)$, we first use the above estimates to obtain

$$\frac{1}{\theta\log\theta} \lesssim \frac{\theta\log\theta}{\tau^2\log^2\tau} \leq \partial_\theta \operatorname{Im} f(r,\theta) \leq \frac{1+\log\tau}{\tau\log\tau(1+\log\theta)} \lesssim \frac{1}{\theta\log\theta}.$$

That is, $\partial_\theta \operatorname{Im} f(r,\theta) \approx g(\theta)$. Then, since $g(\tau) \leq r < h(\theta) \leq g(\theta)$ in our domain, and since $g(\tau) \approx g(\theta)$, we in fact have $r \approx g(\theta)$. Hence,

$$\frac{\partial_\theta \operatorname{Im} f(r,\theta)}{r} \approx 1. \tag{5.3}$$

In particular, the function $r^{-1}\partial_\theta \operatorname{Im}(f)$ is bounded and hence clearly square integrable over A.

Next, we check the square integrability of $\partial_r \operatorname{Im}(f)$. We begin by investigating the integral of an arbitrary function of θ over A under our chosen parametrization. Letting $F\colon [\theta_0, \infty) \to [0,\infty)$, we use polar integration to get

$$\int_A F(\theta) = \int_{\theta_0}^\infty \int_{g(\tau)}^{h(\theta)} F(\theta) r \, \mathrm{d}r \, \mathrm{d}\theta \leq \int_{\theta_0}^\infty (h(\theta) - g(\tau)) \frac{F(\theta)}{\theta\log\theta} \, \mathrm{d}\theta.$$

Moreover, we have

$$\begin{aligned} h(\theta) - g(\tau) &= \frac{g(\theta) + g(\theta + 2\pi)}{2} - g(\theta + 2\pi) = \frac{g(\theta) - g(\theta + 2\pi)}{2} \\ &= \frac{1}{2}\left(\frac{1}{\theta \log \theta} - \frac{1}{(\theta + 2\pi)\log(\theta + 2\pi)}\right) = \frac{(\theta + 2\pi)\log(\theta + 2\pi) - \theta \log \theta}{2\theta(\theta + 2\pi)\log\theta \log(\theta + 2\pi)} \\ &= \frac{2\pi \log(\theta + 2\pi) + \theta \log(1 + 2\pi/\theta)}{2\theta(\theta + 2\pi)\log\theta\log(\theta + 2\pi)} \lesssim \frac{1}{\theta^2 \log \theta}. \end{aligned}$$

Note in particular that in the last step of the above computation, we have $\theta \log(1 + 2\pi/\theta) = \log((1 + 2\pi/\theta)^\theta) \to \log \exp(2\pi) = 2\pi$ as $\theta \to \infty$, so hence the dominant term in the numerator is $2\pi \log(\theta + 2\pi)$. We thus finish our estimate as follows:

$$\int_A F(\theta) \leq \int_{\theta_0}^{\infty} (h(\theta) - g(\tau)) \frac{F(\theta)}{\theta \log \theta} \, \mathrm{d}\theta \lesssim \int_{\theta_0}^{\infty} \frac{F(\theta)}{\theta^3 \log^2 \theta}. \tag{5.4}$$

Now, since $|\partial_r \operatorname{Im}(f)|^2 \lesssim \theta^2$ by (5.2), we conclude that $|\partial_r \operatorname{Im}(f)| \in L^2(A)$ by taking $F(\theta) = \theta^2$ in (5.4), and observing that the resulting integrand $\theta^{-1} \log^{-2} \theta$ has a finite integral. We thus conclude that if $r(\varphi'(r))^2 \in L^1([0, r_0])$, then $f \in W^{1,2}(\Omega \setminus \{0\}, \mathbb{C})$, and consequently $f \in W^{1,2}(\Omega, \mathbb{C})$ by removability of isolated points for planar $W^{1,2}$-spaces.

It remains to find suitable choices of K and Σ. We choose $\Sigma \equiv 0$ in this region, in which case we require $K \geq |Df|^2 / J_f$. By (5.2) and (5.3), we have

$$|Df(r, \theta)|^2 \lesssim (\varphi'(r))^2 + \theta^2.$$

On the other hand, we have

$$J_f(r, \theta) \gtrsim \varphi'(r).$$

Consequently, we may choose Σ and K so that

$$\Sigma(r, \theta) = 0, \qquad K(r, \theta) \approx \varphi'(r) + \frac{\theta^2}{\varphi'(r)} + 1. \tag{5.5}$$

5.5. **The results.** It remains now to state our choices of φ and the resulting counterexamples. We begin with Theorem 1.5, recalling first its statement.

Theorem 1.5. *There exist a domain $\Omega \subset \mathbb{R}^2$ and a Sobolev map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $0 \in \Omega$, $f \in C(\Omega \setminus \{0\}, \mathbb{R}^2)$, $\lim_{x \to 0} |f(x)| = \infty$, and $Df \in \mathcal{M}_2(K, \Sigma)$ with*

$$\Sigma \in L^\infty(\Omega) \qquad \text{and} \qquad K \in L^1(\Omega).$$

Proof. We use the above construction with $\theta_0 = 2\pi$, $r_0 = 1$, and

$$\varphi(r) = r.$$

Notably, φ is Lipschitz, and consequently the resulting map f is Lipschitz in B. This choice indeed satisfies $r(\varphi'(r))^2 = r \in L^1([0, 1])$, so $|Df| \in L^2(\Omega \setminus \{0\})$. Moreover, by (5.1), both Σ and K are constant in the region

B. Since $\Sigma \equiv 0$ in the other region A, we have $\Sigma \in L^\infty(\Omega)$. For $K \in L^1(\Omega)$, since φ' is constant, it suffices by (5.5) to show that

$$\int_A \theta^2 < \infty.$$

But this is true by (5.4) with yet again $F(\theta) = \theta^2$. Finally, as $x \to 0$, the imaginary part of $f(x)$ clearly tends to infinity. □

The remaining result to prove is Theorem 5.1.

Proof of Theorem 5.1. The case $q = \infty$ is exactly the result of Theorem 1.5. Hence, we may assume that $q \in [1, \infty)$.

We use the above construction, this time with the choice

$$\varphi(r) = \int_0^r t^{2p^{-1}-2} \log^{-7/4+p^{-1}} t^{-1} \,\mathrm{d}t.$$

Note that by $p \in [1, 2]$, we have $2p^{-1}-2 \geq -1$. Moreover, the case $2p^{-1}-2 = -1$ corresponds to $p = 2$, in which case $-7/4 + p^{-1} = -5/4 < -1$. Hence, the integral used to define $\varphi(r)$ is finite for all $r > 0$ small enough, and we may hence choose r_0 and θ_0 so that $\varphi(r)$ is a finite-valued increasing function on $[0, r_0]$. By our choice of φ, we have

$$\varphi'(r) = r^{2p^{-1}-2} \log^{-7/4+p^{-1}} r^{-1}. \tag{5.6}$$

We first determine the degree of integrability of $\varphi'(|x|)$ over Ω, as this is used for many parts in the verification that our example is as desired. Indeed, if $s \in [1, \infty)$, we have by (5.6) that

$$\int_\Omega \left(\varphi'(|x|)\right)^s \lesssim \int_0^{r_0} \frac{\mathrm{d}r}{r^{2s-2sp^{-1}-1} \log^{7s/4-sp^{-1}} r^{-1}}. \tag{5.7}$$

This integral is finite if $2s - 2sp^{-1} - 1 \leq 1$, which is equivalent to $p^{-1} + s^{-1} \geq 1$. Note that in the extremal case $p^{-1} + s^{-1} = 1$, the finiteness of the integral also requires that $7s/4 - sp^{-1} > 1$; however, this condition rearranges to $p^{-1} + s^{-1} < 7/4$, which holds in the extremal case since $p^{-1} + s^{-1} = 1$.

We have $\Sigma(x) \leq 6 + 3(\varphi'(|x|))^2$ and $\Sigma(x)/K(x) \leq 6 + 3\varphi'(|x|)$ in B by (5.1), and we also have $\Sigma = \Sigma/K \equiv 0$ in A. Hence, (5.7) with $s = q$ yields that $\Sigma/K \in L^q(\Omega)$ and $\Sigma \in L^{q/2}(\Omega)$ if $p^{-1} + q^{-1} \geq 1$. Moreover, $|Df| \in L^2(\Omega)$ was shown to be equivalent with $r(\varphi'(r))^2 \in L^1([0, r_0])$: referring to (5.7) with $s = 2$, this is true if $p^{-1} + 2^{-1} \geq 1$, which holds due to our assumption that $p \leq 2$. As our last application of (5.7), we have by (5.1) that $K \in L^p(B)$ if $\varphi'(r) \in L^p(B)$: this is true if $2p \leq 4$, which again holds by our assumption that $p \leq 2$.

It remains to show that $K \in L^p(A)$. For this, it suffices by (5.5) to show the L^p-integrability of $\varphi'(r)$ and $\theta^2/\varphi'(r)$ over A. Since $K(r, \theta) \geq \varphi'(r)$ in B, the $\varphi'(r)$-term is covered by the same argument as used previously for

$K \in L^p(B)$. For the other term, we again use (5.4). Indeed, we have

$$\left(\frac{\theta^2}{\varphi'(r)}\right)^p = \frac{\theta^{2p} r^{2p-2}}{\log^{-7p/4+1} r^{-1}}$$
$$\leq \frac{\theta^{2p}(\theta \log \theta)^{2-2p}}{\log^{-7p/4+1}((\theta + 2\pi)\log(\theta + 2\pi))} \lesssim \frac{\theta^2}{\log^{p/4-1}(\theta)}$$

for all $\theta \in [\theta_0, \infty)$. Selecting $F(\theta) = \theta^2 \log^{1-p/4}(\theta)$, the resulting integrand $\theta^{-1} \log^{-1-p/4}(\theta)$ in (5.4) is integrable whenever $-1 - p/4 < -1$, which is clearly true. We conclude that $K \in L^p(\Omega)$, completing the proof. □

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Paper III

WEAK LIMIT OF HOMEOMORPHISMS IN $W^{1,n-1}$ AND (INV) CONDITION

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Abstract. Let $\Omega, \Omega' \subset \mathbb{R}^3$ be Lipschitz domains, let $f_m : \Omega \to \Omega'$ be a sequence of homeomorphisms with prescribed Dirichlet boundary condition and $\sup_m \int_\Omega (|Df_m|^2 + 1/J_{f_m}^2) < \infty$. Let f be a weak limit of f_m in $W^{1,2}$. We show that f is invertible a.e., more precisely it satisfies the (INV) condition of Conti and De Lellis and thus it has all the nice properties of mappings in this class.

Generalization to higher dimensions and an example showing sharpness of the condition $1/J_f^2 \in L^1$ are also given. Using this example we also show that unlike the planar case the class of weak limits and the class of strong limits of $W^{1,2}$ Sobolev homeomorphisms in $\mathbb{R}^3$ are not the same.

1. Introduction

In this paper, we study classes of mappings that might serve as classes of deformations in Nonlinear Elasticity models. Let $\Omega \subset \mathbb{R}^n$ be a domain set and let $f\colon \Omega \to \mathbb{R}^n$ be a mapping. Following the pioneering papers of Ball [3] and Ciarlet and Nečas [10] we ask if our mapping is in some sense injective as the physical 'non-interpenetration of the matter' indicates that a deformation should be one-to-one. We are led to study nonlinear classes of mappings based on integrability of gradient minors ([2], [3], [36], [17], [33]), on distortion ([34], [23], [20]) or on finiteness of some energy functional ([2], [16], [11], [19], [14]). The list of citations is far from being representative and the reader is encouraged to see also references therein. Our aim is to study injectivity properties of the mapping f. One can follow the ideas of Ball [3] and assume that our mapping has finite energy and that the energy functional $\int_\Omega W(Df)$ contains special terms (like powers of Df, $\operatorname{adj} Df$ and J_f). Under quite strong assumptions, any mapping with finite energy and reasonable boundary data is a homeomorphism ([3], [20]). However, is not realistic to insist on this requirement as in some real physical deformations cavitations or even fractures may occur. Thus we need conditions which guarantee that our mapping is injective a.e. but on some small set cavities may arise.

This was nicely settled in the the work of Müller and Spector [31] where they studied a class of mappings that satisfy $J_f > 0$ a.e. together with the (INV) condition (see also e.g. [6, 14, 19, 32, 37, 38, 39]). Informally speaking, the (INV) condition means that the ball $B(x, r)$ is mapped inside the image of the sphere $f(S(a, r))$ and the complement $\Omega \setminus \overline{B(x, r)}$ is mapped outside $f(S(a, r))$ (see Preliminaries for the formal definition). From

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[31] we know that mappings in this class are one-to-one a.e. and that this class is weakly closed which makes it suitable for variational approach. Moreover, any mapping in this class has many desirable properties, it maps disjoint balls into essentially disjoint sets, $\deg(f, B, \cdot) \in \{0, 1\}$ for a.e. ball B, its distributional determinant equals to the absolutely continuous part J_f plus a countable sum of positive multiples of Dirac measures (these corresponds to created cavities) and so on.

In all results in the previous paragraph the authors assume that $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ for some $p > n - 1$. However in some real models for $n = 3$ one often works with integrands containing the classical Dirichlet term $|Df|^2$ and thus this assumption is too strong. Therefore Conti and De Lellis [11] introduced the concept of (INV) condition also for $W^{1,n-1} \cap L^\infty$ (see also [4] and [5] for some recent work) and studied Neohookean functionals of the type

$$\int_\Omega \big(|Df(x)|^2 + \varphi(J_f(x))\big)\, dx \tag{1.1}$$

for $n = 3$, where φ is convex, $\lim_{t\to 0+} \varphi(t) = \infty$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. They proved that mappings in the (INV) class that satisfy $J_f > 0$ a.e. have nice properties like mappings in [31] but unfortunately this class is not weakly closed and therefore cannot be used in variational models easily. Our main aim is to fix this and to show that for suitable φ the validity of the (INV) condition is preserved also for the weak limit. Note that for $p > n-1$ we know that f is continuous on a.e. sphere and thus it is easy to define what is "inside of the image of the sphere $f(S(a, r))$" and we can define (INV) easily. In the situation $f \in W^{1,n-1} \cap L^\infty$ mappings do not need to be continuous on spheres and it is necessary to use topological degree for some classes of discontinuous mappings (in our case, $W^{1,n-1}$ on a sphere) which was introduced by Brezis and Nirenberg [7] (see also [11]).

Let us note that homeomorphisms clearly satisfy the (INV) condition and their weak limits in $W^{1,p}$, $p > n - 1$, also satisfy (INV) (see [31, Lemma 3.3]). Moreover, cavitation can be written as a weak limit (even strong limit) of homeomorphisms. Therefore the class of weak limits of Sobolev homeomorphisms is a suitable class for variational models involving cavitation and we can expect some invertibility properties in this class. This is clear for $p > n - 1$ because of the (INV) condition but it can fail in the limiting case of limit of $W^{1,n-1}$ homeomorphisms as shown e.g. in Bouchala, Hencl and Molchanova [8]. The class of weak limits of Sobolev homeomorphisms was recently characterized in the planar case by Iwaniec and Onninen [24, 25] and De Philippis and Pratelli [13]. The situation in higher dimension is much more difficult and deserves further study.

Our main result is the following theorem which shows that weak limits of $W^{1,n-1}$ homeomorphisms are invertible a.e. (and much better) under suitable integrability of $1/J_f$. We denote

$$\mathcal{F}(f) = \int_\Omega \big(|Df|^{n-1} + \varphi(J_f)\big)\, dx, \tag{1.2}$$

where
(1.3)

$$\varphi \text{ is a positive convex function on } (0, \infty) \text{ with } \lim_{t\to 0^+} \varphi(t) = \infty,\ \varphi(t) = \infty \text{ for } t \le 0$$

and there is $A > 0$ with

$$A^{-1}\varphi(t) \le \varphi(2t) \le A\varphi(t), \qquad t \in (0, \infty). \tag{1.4}$$

We need to further assume that all f_m have the same Dirichlet boundary condition.

Theorem 1.1. *Let $n \geq 3$, $a = \frac{n-1}{n^2-3n+1}$ and $\Omega, \Omega' \subset \mathbb{R}^n$ be Lipschitz domains. Let φ satisfy* (1.3)*,* (1.4) *and*

$$\varphi(t) \geq \frac{1}{t^a} \text{ for every } t \in (0,\infty).$$

Let $f_m \in W^{1,n-1}(\Omega, \Omega')$, $m = 0, 1, 2 \dots,$ be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega}'$ with $J_{f_m} > 0$ a.e. such that

$$\sup_m \mathcal{F}(f_m) < \infty. \tag{1.5}$$

Assume further that $f_m = f_0$ on $\partial\Omega$ for all $m \in \mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$, then f satisfies the (INV) condition.

As a corollary this weak limit f satisfies all the nice properties (see [11, Section 3 and 4]): it is one-to-one a.e., it maps disjoint balls into essentially disjoint sets, $\deg(f, B, \cdot) \in \{0, 1\}$ for a.e. ball B, its distributional determinant equals to the absolutely continuous part J_f plus a countable sum of positive multiples of Dirac measures and so on.

In fact our result is even more general and instead of integrability of $1/J_f^a$ it is enough to assume that its distortion $K_f = |Df|^n/J_f$ is integrable with power $\frac{1}{n-1}$ (see Theorem 3.1 below). This seems to be connected with the result of Koskela and Malý [27] about the validity of Lusin (N^{-1}) condition for mappings of finite distortion.

In our new paper [12] we use Theorem 1.1 as a main step in showing that we can use Calculus of Variations approach in this context. We show that there is an energy minimizer of certain polyconvex functional in the class of weak limits of $W^{1,n-1}$ homeomorphisms.

In our next result we improve the counterexample from [11, Theorem 6.3.]. Similarly to [11] we show that the weak limit of homeomorphisms (that automatically satisfy (INV)) can fail to satisfy (INV). Moreover, the degree of the limit f can be -1 on a set of positive measure even though $J_{f_m} > 0$ a.e. and $\deg(f_m, B, \cdot) \in \{0, 1\}$. Let us note here that we automatically have $J_f \geq 0$ a.e. for the weak limit once $J_{f_m} > 0$ a.e. (see Hencl and Onninen [22]) and since our φ tends to ∞ at 0 we also have $J_f > 0$ a.e. from Lemma 2.3 in [12]. Primarily we show that at least in dimension $n = 3$ our condition $J_f^{-2} \in L^1$ from Theorem 1.1 is sharp (example in [11] gives smaller integrability of $1/J_f$). We expect that our result is sharp in all dimensions but we have not pursued this as $n = 3$ is the physically relevant dimension.

Theorem 1.2. *Let $n = 3$ and $a < 2$. Then there exist homeomorphisms f_m of $\overline{B}(0, 10)$ to $\overline{B}(0, 10)$ such that $f_m \in W^{1,2}(B(0, 10), B(0, 10))$, f_m is an identity mapping on $\partial B(0, 10)$ with $J_{f_m} > 0$ a.e. and*

$$\sup_m \int_\Omega \left(|Df_m|^{n-1} + \frac{1}{(J_{f_m})^a} \right) dx < \infty,$$

whose weak limit f does not satisfy the (INV) condition.

It was shown by Iwaniec and Onninen in [25] that in the planar case $n = 2$ the class of weak limits of homeomorphisms and the class of strong limits of homeomorphisms are the same for any $p \geq 2$. The same result for $1 \leq p < 2$ was later shown by De Philippis and Pratelli in [13]. This is very useful in Calculus of Variations as we can approximate the minimizer of the energy not only in the weak but also in the strong convergence. We show that the situation is much more difficult in higher dimension and the result is not true in general. We show (see Theorem 3.1 (b) below) that each strong limit of $W^{1,n-1}$ homeomorphisms satisfies the (INV) condition. Together with Theorem 1.2 this implies the following result.

Theorem 1.3. *Let $n = 3$. There is a mapping $f \in W^{1,2}(B(0,10), B(0,10))$ which is a weak limit of Sobolev $W^{1,2}$ homeomorphisms f_m of $\overline{B}(0,10)$ to $\overline{B}(0,10)$ with $f_m(x) = x$ on $\partial B(0,10)$ and $J_{f_m} > 0$ a.e., but there are no homeomorphisms h_m of $\overline{B}(0,10)$ to $\overline{B}(0,10)$ such that $h_m \to f$ strongly in $W^{1,2}(B(0,10), \mathbb{R}^3)$.*

Again we expect that there are similar examples in $W^{1,n-1}$ in higher dimension. However, we do not see any simple way to generalize this counterexample to other Sobolev spaces $W^{1,p}$ for $p \neq n-1$.

2. Preliminaries

2.1. **Convention.** In what follows, we assume that Sobolev mappings are represented in the best way for our purposes. For example, if we consider a trace of a Sobolev mapping f on a k-dimensional surface S, then we assume that f is represented as the trace on S. If f has a continuous representative $\bar{f}$ on S, then we assume that $f = \bar{f}$ on S.

2.2. **Shapes and Figures.** In what follows we will work with open sets of a simple geometric nature called *shapes*. Our shapes will be of three types

(a) balls,
(b) full cuboids,
(c) a *hollowed cuboid* is the difference $Q \setminus \overline{B}$, where Q is a full cuboid and $B \subset\subset Q$ is a ball (see the set H on Fig. 1 below).

A *figure* is the interior of a finite union of closed full cuboids with pairwise disjoint interiors; it may be disconnected.

2.3. **Boundary gradient and cofactors.** Let $K \subset \mathbb{R}^n$ be a shape and f be a smooth mapping on a neighbourhood of $\overline{K}$. Then we can derive a useful degree formula (cf. (2.2) below) involving $(\operatorname{cof} Df(x))\nu(x)$, $x \in \partial K$; here $\nu(x)$ denotes the exterior normal to B at x and $\operatorname{cof} A$ is the cofactor matrix of A (which is the transpose of the adjugate matrix $\operatorname{adj} A$ and which satisfies $\operatorname{cof} A\ A^T = (\det A)\mathrm{Id}$). If f is a Sobolev mapping only, we can find "good shapes" with the property that f is Sobolev regular on their boundaries. Let $x \in \partial K$, in case of a cuboid we exclude points of edges. Then, instead of the expression $(\operatorname{cof} Df(x))\nu(x)$ we need a replacement relying on the tangential gradient $D_\tau f$. This can be given using some tools from the multilinear algebra, for details see [15] and [33]. We consider a linear subspace $\mathbb{V}$ of $\mathbb{R}^n$ and a linear mapping $L\colon \mathbb{V} \to \mathbb{R}^n$. Then the operator $\Lambda_{n-1}L$ is defined by

$$\Lambda_{n-1}L(\mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_{n-1}) = L\mathbf{y}_1 \wedge L\mathbf{y}_2 \wedge \cdots \wedge L\mathbf{y}_{n-1}. \tag{2.1}$$

If the dimension k of $\mathbb{V}$ is less that $n-1$, this operator is trivial. If $k = n$, and A is the matrix representing L, it can be shown that $\Lambda_{n-1}L$ is represented by $\operatorname{cof} A$. Therefore both sides of (2.1) depend only on values of L on the linear hull of $\mathbf{y}_1, \mathbf{y}_2, \dots \mathbf{y}_{n-1}$. We may identify the wedge product with the cross product through the Hodge star operator. Thus, in our case when $\mathbb{V}$ is the tangent space $T_x(\partial K)$, $\Lambda_{n-1}T_x(\partial K)$ is the one-dimensional space of multiples of $\nu(x)$ and $L = D_\tau f(x)$, the required expression $(\Lambda_{n-1}L)\nu$ can be computed (avoiding exterior algebra objects) as $(\operatorname{cof} A)\nu$ where A is a matrix of any operator $\bar{L}\colon \mathbb{R}^n \to \mathbb{R}^n$ which extends L, i.e.

$$\bar{L}\mathbf{y} = L\mathbf{y}, \qquad \mathbf{y} \in T_x(\partial K).$$

This also shows compatibility with the formula for smooth mappings where the extension $\bar{L}$ appears naturally as the full gradient $Df(x)$.

2.4. **Degree for continuous mappings.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given a smooth map $f\colon \overline{\Omega} \to \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n \setminus f(\partial\Omega)$ such that $J_f(x) \neq 0$ for each $x \in \Omega \cap f^{-1}(y_0)$, we can define the *topological degree* as

$$\deg(f, \Omega, y_0) = \sum_{\Omega \cap f^{-1}(y)} \operatorname{sgn}(J_f(x)).$$

By uniform approximation, this definition can be extended to an arbitrary continuous mapping $f\colon \overline{\Omega} \to \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n \setminus f(\partial\Omega)$. Note that the degree depends only on values of f on $\partial\Omega$.

If $f\colon \overline{\Omega} \to \mathbb{R}^n$ is a homeomorphism, then either $\deg(f, \Omega, y) = 1$ for all $y \in f(\Omega)$ (f is *sense preserving*), or $\deg(f, \Omega, y) = -1$ for all $y \in f(\Omega)$ (f is *sense reversing*). If, in addition, $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, then this topological orientation corresponds to the sign of the Jacobian. More precisely, we have

Proposition 2.1 ([21])**.** *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a homeomorphism on $\overline{\Omega}$ with $J_f > 0$ a.e. Then*

$$\deg(f, \Omega, y) = 1, \qquad y \in f(\Omega).$$

2.5. **Degree for $W^{1,n-1} \cap L^\infty$ mappings.** If K is a shape or a figure, $f \in W^{1,n-1}(\partial K) \cap C(\partial K)$, $|f(\partial K)| = 0$, and $\mathbf{u} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \deg(f, K, y) \operatorname{div} \mathbf{u}(y)\, dy = \int_{\partial K} (\mathbf{u} \circ f) \cdot (\Lambda_{n-1} D_\tau f)\nu \, d\mathcal{H}^{n-1}, \tag{2.2}$$

see [31, Proposition 2.1].

Let $\mathcal{M}(\mathbb{R}^n) = C_0(\mathbb{R}^n)^*$ be the space of all signed Radon measures on $\mathbb{R}^n$. By (2.2) we see that $\deg(f, K, \cdot) \in BV(\mathbb{R}^n)$ and

$$\|D \deg(f, K, \cdot)\|_{\mathcal{M}(\mathbb{R}^n)} \leq C \|\Lambda_{n-1} D_\tau f\|_{L^1(\partial K)} \leq C \|D_\tau f\|^{n-1}_{L^{n-1}(\partial K)}. \tag{2.3}$$

Following [11] (see also [7]) we need a more general version of the degree on the boundary of a shape which works for mappings in $W^{1,n-1} \cap L^\infty$ that are not necessarily continuous. Although only the three dimensional case on balls is discussed on [11], the arguments pass in the general case as well. The definition is in fact based on (2.2).

Definition 2.2. Let $K \subset \mathbb{R}^n$ be a shape and let $f \in W^{1,n-1}(\partial K, \mathbb{R}^n) \cap L^\infty(\partial K, \mathbb{R}^n)$. Then we define $\operatorname{Deg}(f, K, \cdot)$ as the distribution satisfying

$$\int_{\mathbb{R}^n} \operatorname{Deg}(f, K, y)\psi(y)\, dy = \int_{\partial K} (\mathbf{u} \circ f) \cdot (\Lambda_{n-1} D_\tau f)\nu \, d\mathcal{H}^{n-1} \tag{2.4}$$

for every test function $\psi \in C_c^\infty(\mathbb{R}^n)$ and every C^∞ vector field $\mathbf{u}$ on $\mathbb{R}^n$ satisfying $\operatorname{div} \mathbf{u} = \psi$.

As in [11] it can be verified that the right hand side does not depend on the way ψ is expressed as $\operatorname{div}\mathbf{u}$. Indeed, this works if f is smooth. If we approximate f by a sequence $(f_m)_m$ of smooth functions in the usual mollification way, then $\mathbf{u}\circ f_m \to \mathbf{u}\circ f$ weakly* in L^∞, $\Lambda_{n-1}D_\tau f_m \to \Lambda_{n-1}D_\tau f$ strongly in L^1, hence the right hand side of (2.2) converges well. In fact, the distribution $\operatorname{Deg}(f,K,\cdot)$ can be represented as a BV function by the following lemma.

Lemma 2.3. *Let K be a shape. Let $(f_m)_m$ be a sequence of continuous Sobolev mappings which converges to a limit function f strongly in $W^{1,n-1}(\partial K,\mathbb{R}^n)$ and is bounded in $L^\infty(\partial K,\mathbb{R}^n)$. Then $\operatorname{Deg}(f,K,\cdot)$ is an integer valued function in $BV(\mathbb{R}^n)$ and $\deg(f_m,K,\cdot)\to \operatorname{Deg}(f,K,\cdot)$ strongly in $L^1(\mathbb{R}^n)$.*

Proof. Let ψ be a smooth test function and $\mathbf{u}$ be a smooth vector field such that $\operatorname{div}\mathbf{u}=\psi$. As above we observe that $\mathbf{u}\circ f_m\to\mathbf{u}\circ f$ weakly* in L^∞ and $\Lambda_{n-1}D_\tau f_m\to\Lambda_{n-1}D_\tau f$ strongly in L^1. Hence we observe that $\deg(f_m,K,\cdot)\to\operatorname{Deg}(f,K,\cdot)$ in distributions. By (2.3), the sequence $\deg(f_m,K,\cdot)$ is bounded in $BV(\mathbb{R}^n)$, so that the limit is in BV as well and the convergence is weak* in BV. By the compact embedding and the L^∞ bound of f_m we have $\deg(f_m,K,\cdot)\to\operatorname{Deg}(f,K,\cdot)$ in $L^1(\mathbb{R}^n)$. It also follows that $\operatorname{Deg}(f,K,\cdot)$ is integer valued. □

Remark 2.4. Let K be a shape and $f\in W^{1,n-1}(\partial K)\cap C(\overline{K})$. If $|f(\partial K)|=0$, then $\operatorname{Deg}(f,K,y)=\deg(f,K,y)$ for a.e. $y\in\mathbb{R}^n$. We use different symbols to distinguish and emphasize that deg is defined pointwise on $\mathbb{R}^n\setminus f(\partial K)$, whereas Deg is determined only up to a set of measure zero.

Assume that $f,g\in W^{1,n-1}(\partial K,\mathbb{R}^n)\cap L^\infty(\partial K,\mathbb{R}^n)$. From the embedding of BV spaces into $L^{\frac{n}{n-1}}$ [1, Theorem 3.47], the definition of BV norm [1, Definition 3.4] and (2.4) (note that by approximation it must hold also for $\psi=\operatorname{div}\mathbf{u}$, $\mathbf{u}\in C^1_0(\mathbb{R}^n)$) we obtain

$$
\begin{aligned}
\bigl|\{y\in\mathbb{R}^n:\ \operatorname{Deg}(f,K,y)\neq\operatorname{Deg}(g,K,y)\}\bigr|^{\frac{n-1}{n}} &\le \|\operatorname{Deg}(f,K,\cdot)-\operatorname{Deg}(g,K,\cdot)\|_{L^{\frac{n}{n-1}}}\\
&\le C\bigl\|D(\operatorname{Deg}(f,K,\cdot)-\operatorname{Deg}(g,K,\cdot))\bigr\|_{\mathcal{M}(\mathbb{R}^n)}\\
&= C\sup\Bigl\{\int_{\mathbb{R}^n}(\operatorname{Deg}(f,K,y)-\operatorname{Deg}(g,K,y))\operatorname{div}\mathbf{u}(y)\,dy:\\
&\qquad\qquad \mathbf{u}\in C^1_0(\mathbb{R}^n),\ \|\mathbf{u}\|_{L^\infty}\le 1\Bigr\}\\
&\le C\int_{\partial K\cap\{f\neq g\}}\bigl(|D_\tau f(x)|^{n-1}+|D_\tau g(x)|^{n-1}\bigr)d\mathcal{H}^{n-1}(x).
\end{aligned}
\tag{2.5}
$$

2.6. **(INV) condition.** Analogously to [11] (see also [31]) we define the (INV) class.

Definition 2.5. Let $B\subset\mathbb{R}^n$ be a ball and let $f\in W^{1,n-1}(\partial B,\mathbb{R}^n)\cap L^\infty(\partial B,\mathbb{R}^n)$. We define the topological image of B under f, $\operatorname{im}_T(f,B)$ as the set of all points where the density of the set $\{y\in\mathbb{R}^n:\ \operatorname{Deg}(f,B,y)\neq 0\}$ is one.

Definition 2.6. Let $f\in W^{1,n-1}(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$. We say that f satisfies (INV) in the ball $B\subset\subset\Omega$ if

(i) its trace on ∂B is in $W^{1,2}\cap L^\infty$;
(ii) $f(x)\in\operatorname{im}_T(f,B)$ for a.e. $x\in B$;
(iii) $f(x)\notin\operatorname{im}_T(f,B)$ for a.e. $x\in\Omega\setminus B$.

We say that f satisfies (INV) if for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{H}^1$-a.e. $r \in (0, r_a)$ it satisfies (INV) in $B(a, r)$.

Remark 2.7. If f, in addition, satisfies $J_f > 0$ a.e., then preimages of sets of zero measure have zero measure and thus we can characterize the (INV) condition in a simpler way. Namely, such a mapping satisfies the (INV) condition in the ball $B \subset\subset \Omega$ if and only if

(i) its trace on ∂B is in $W^{1,2} \cap L^\infty$;
(ii) $\operatorname{Deg}(f, B, f(x)) \neq 0$ for a.e. $x \in B$;
(iii) $\operatorname{Deg}(f, B, f(x)) = 0$ for a.e. $x \in \Omega \setminus B$.

2.7. **Estimates of measure of preimages.**

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure and $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ satisfy $J_f \neq 0$ a.e. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set $F \subset \mathbb{R}^n$ we have*

$$|F| < \delta \implies |f^{-1}(F)| < \varepsilon.$$

Proof. Assume for contradiction that there are $\varepsilon > 0$ and F_j with $|F_j| < \frac{1}{2^j}$ such that $|f^{-1}(F_j)| \geq \varepsilon$, $j = 1, 2, \dots$. Then the set

$$E := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} F_j \text{ with } f^{-1}(E) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} f^{-1}(F_j)$$

satisfies $|E| = 0$ but $|f^{-1}(E)| \geq \varepsilon$. We can find a set $A \subset f^{-1}(E)$ of full measure such that $J_f \neq 0$ on A and such that change of variables formula

$$\int_A |J_f(x)|\, dx = \int_{f(A)} N(f, \Omega, y)\, dy.$$

holds on A (see [15] or the proof of [20, Theorem A.35] for $\eta = \chi_{f(A)}$). Now the left hand is positive as $J_f \neq 0$ and $|A| > 0$ and the right side is zero as $|f(A)| \subset E$ and $|E| = 0$. This gives us a contradiction. ☐

Let $\Omega \subset \mathbb{R}^n$ be open, $A \subset \Omega$ be measurable and let $g \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ be one-to-one. Without any additional assumption we have (see e.g. [20, Theorem A.35] for $\eta = \chi_{g(A)}$)

$$\int_A |J_g(x)| dx \leq |g(A)|. \tag{2.6}$$

Lemma 2.9. *Given $C_1 < \infty$, there exists a function $\Phi\colon (0, \infty) \to (0, \infty)$ with*

$$\lim_{s \to 0^+} \Phi(s) = 0$$

such that the following holds: Let $g \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a one-to-one mapping with $\|g\|_{L^\infty} \leq C_1$ and $\mathcal{F}(g) \leq C_1$, where $\mathcal{F}$ is as in (1.2) with φ satisfying (1.3). Then for each measurable set $A \subset \Omega$ we have

$$\Phi(|A|) \leq |g(A)|. \tag{2.7}$$

Proof. Choose $t_0 > 0$ such that φ is decreasing on $(0, t_0)$ and write $t = \varphi_L^{-1}(s)$ if

$$\varphi(t) = s \text{ and } 0 < t < t_0. \tag{2.8}$$

Then, either $\frac{|g(A)|}{|A|} \geq t_0$, or we use that φ is decreasing on $(0, t_0]$, area formula (2.6) and the Jensen inequality to obtain

$$\varphi\Big(\frac{|g(A)|}{|A|}\Big) \leq \varphi\Big(\frac{\int_A |J_g|}{|A|}\Big) \leq \fint_A \varphi(J_g)\, dx \leq \frac{C_1}{|A|}.$$

This implies that

$$|g(A)| \geq \Phi(|A|), \tag{2.9}$$

where

$$\Phi(s) = \begin{cases} s\varphi_L^{-1}\Big(\frac{C_1}{s}\Big), & s < \frac{C_1}{\varphi(t_0)}, \\ t_0 s, & s \geq \frac{C_1}{\varphi(t_0)}, \end{cases}$$

and φ_L^{-1} is the left partial inverse function defined by (2.8).

□

We need the following observation from [12, Lemma 2.3] to show that the limit mapping in Theorem 1.1 satisfies $J_f \neq 0$.

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $f_k \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a sequence of homeomorphisms with $J_{f_k} > 0$ a.e. such that $f_k \to f \in W^{1,1}(\Omega, \mathbb{R}^n)$ pointwise a.e. Assume further that*

$$\sup_k \int_\Omega \varphi(J_{f_k}(x))\, dx < \infty,$$

where φ satisfies (1.3)*. Then $J_f \neq 0$ a.e.*

2.8. **Minimizers of the tangential Dirichlet integral.** In our main proof we have a sphere (or cuboid) K in $\mathbb{R}^n$ and on this sphere we have a small $(n-2)$-dimensional circle which is a boundary of an open spherical cap $S \subset K$. Our map f is in $W^{1,n-1}$ so we can choose the sets so that f is continuous on the small circle $\overline{S} \setminus S$. Our mapping f can have a big oscillation on S so we need to replace it by a reasonable mapping. We do this by choosing a minimizer of the tangential Dirichlet energy over this cap S which has the same value on the circle $\overline{S} \setminus S$. In fact we need this even for more general shapes than spheres and circles.

Let $K \subset \mathbb{R}^n$ be a shape. We say that a relatively open set $S \subset \partial K$ satisfies the *exterior ball condition* if for each $z \in \overline{S} \setminus S$ there exists a ball $B(z', r)$ with $z' \in \partial K$ such that $z \in \partial B(z', r)$ and $B(z', r) \cap S = \emptyset$.

Theorem 2.11. *Let $K \subset \mathbb{R}^n$ be a shape. Let $S \subset \partial K$ be a connected relatively open subset of ∂K which does not contain points of edges. Let T be the relative boundary of S with respect to K. Suppose that* $\operatorname{diam} S < \frac{r}{4n}$ *and that S satisfies the exterior ball condition. Let $f = (f^1, \dots, f^n) \in W^{1,n-1}(\partial K, \mathbb{R}^n)$ be continuous on T. Then there exists a unique function $h = (h^1, \dots, h^n) \in C(\overline{S}) \cap W^{1,n-1}(S, \mathbb{R}^n)$ such that each coordinate h^i minimizes $\int_S |D_\tau u|^{n-1}\, d\mathcal{H}^{n-1}$ among all functions $u \in f^i + W_0^{1,n-1}(S)$. We have $h = f$ on T, the function h satisfies the estimate*

$$\operatorname{diam} h(S) \leq \sqrt{n} \operatorname{diam} f(T). \tag{2.10}$$

and we have $\mathcal{L}^n(h(S)) = 0$. Moreover, if f_m are continuous and converge to f uniformly on T, then h_m converge to h uniformly on S, where h_m are minimizers corresponding to boundary values f_m.

Proof. We give the proof for the case of a ball $K = B$. In case of a full cuboid everything is much simpler as S is flat, and the same references for properties of minimizers are valid. In case of a hollowed cuboid, S is a part of the boundary of a cuboid or of a sphere. The part $\mathcal{L}^n(h(S)) = 0$ is proven in [12].

Choose $z = (z_1, \dots, z_n) \in S$. We may assume that $B = B(0,1)$ and that $z_n \geq \frac{1}{\sqrt{n}}$. Let Π be the projection $x \mapsto \hat{x} := (x_1, \dots, x_{n-1})$. For each $x \in S$ we have $x_n > 0$ and

$$|\hat{x}| \leq |\hat{z}| + |\hat{x} - \hat{z}| \leq \sqrt{1 - \frac{1}{n}} + \frac{1}{4n} \leq 1 - \frac{1}{4n}.$$

If $u \in W^{1,n-1}(S)$ and $\hat{u}(\hat{x}) = u(\hat{x}, \sqrt{1 - |\hat{x}|^2})$, then

$$|D_\tau u|^2 = |D\hat{u}|^2 - (\hat{x} \cdot D\hat{u})^2.$$

Indeed, we can extend u to the neighbourhood of S in $\mathbb{R}^n$ as $u(x) = u(\hat{x}, \sqrt{1 - |\hat{x}|^2})$ and then $Du = (D_1\hat{u}, \dots, D_{n-1}\hat{u}, 0)$. We clearly have

$$|Du|^2 = |D_\tau u|^2 + |D_\nu u|^2 = |D_\tau u|^2 + |(Du \cdot \nu)|^2,$$

and for the unit ball we have $Du \cdot \nu = D\hat{u} \cdot \hat{x}$ as $\frac{\partial u}{\partial x_n} = 0$.

Note that

$$\xi \mapsto |\xi|^2 - (\hat{x} \cdot \xi)^2, \qquad \xi \in \mathbb{R}^{n-1},$$

is a positive definite quadratic form whenever $|\hat{x}| < 1$ and

$$|\xi|^2 - (\hat{x} \cdot \xi)^2 \geq (1 - |\hat{x}|^2)|\xi|^2.$$

The functional

$$\int_S |D_\tau u|^{n-1} \, d\mathcal{H}^{n-1} = \int_{\Pi(S)} (|D\hat{u}|^2 - (\hat{x} \cdot D\hat{u})^2)^{\frac{n-1}{2}} \, d\hat{x}$$

thus satisfies the axioms of Chapter 5 in [18]. The existence and uniqueness of the minimizer follows from [18, Theorem 5.28]. The continuity up to the boundary follows from [18, Theorem 6.6 and Theorem 6.31]. The oscillation estimate (2.10) follows from the maximum principle [18, Theorem 6.5]. The uniform convergence of a sequence of solutions can be obtained from the comparison principle [18, Lemma 3.18], namely, if u, v are continuous scalar minimizers and $u \leq v$ on T, then $u \leq v$ on S. □

2.9. **Mappings of finite distortion.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a domain. The mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \widetilde{R}^n)$ is said to be a mapping of finite distortion if $J_f(x) \geq 0$ a.e. in Ω, $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and $Df(x)$ vanishes a.e. in the zero set of $J_f(x)$ (note that the last condition automatically holds if $J_f > 0$ a.e.). With such a mapping f we may associate the distortion function as

$$K_f(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{if } J_f(x) > 0 \\ 1 & \text{if } J_f(x) = 0. \end{cases}$$

See [23] and [20] and references given there for the introduction to the theory of mappings of finite distortion.

Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a mapping of finite distortion with $K_f \in L^{\frac{1}{n-1}}$ and $u \in C^1(\mathbb{R}^n)$. Then the following crucial estimate from Koskela and Malý [27, (2.1)] holds (see [27] for detailed proof)

$$
\begin{aligned}
\int_\Omega |D(u\circ f(x))|\,dx &\leq \int_\Omega |Du(f(x))|\,|Df(x)|\,dx\\
&\leq \int_\Omega |Du(f(x))|\,(K_f(x)J_f(x))^{\frac{1}{n}}\,dx\\
&\leq \Bigl(\int_\Omega |Du(f(x))|^n J_f(x)\,dx\Bigr)^{\frac{1}{n}}\Bigl(\int_\Omega K_f^{\frac{1}{n}\frac{n}{n-1}}(x)\,dx\Bigr)^{\frac{n-1}{n}}\\
&\leq \|Du\|_{L^n(f(\Omega))}\|K_f^{\frac{1}{n-1}}\|_{L^1(\Omega)}^{\frac{n-1}{n}}.
\end{aligned}
\tag{2.11}
$$

2.10. **Extension properties of Lipschitz domains.** It is well known that Lipschitz domains are Sobolev extension domains, see Calderón [9] and Stein [35]. The Sobolev extension property holds even for so called uniform domains, see Jones [26]. For nice recent progress in the field of Sobolev extension see Koskela, Rajala and Zhang [28].

Much less is known if we want to extend a Sobolev homeomorphism on $\overline{\Omega}$ ($\Omega\subset\mathbb{R}^n$ Lipschitz) and require injectivity at least on a neighbourhood of $\overline{\Omega}$. Such a result would simplify the proof of our main theorem. Unfortunately, we are aware only of planar result and thus we bypass the absence of such a tool in a series of auxiliary results (Lemma 2.12, Theorem 2.13, Lemma 3.6). Note that the planar result due to Koski and Onninen [29] deals in fact with a more difficult problem of extension from the boundary. If we do not start from a function given on the interior, we cannot use any kind of reflection.

Lemma 2.12. *Let $\Omega'\subset\mathbb{R}^n$ be a Lipschitz domain. Then there exist a Lipschitz mapping $\ell\colon\overline{\Omega'}\to\mathbb{R}^n$ and $\delta>0$ with the following properties:*
(a) $x\in\partial\Omega' \implies \ell(x)=x$,
(b) $\operatorname{dist}(x,\partial\Omega')<\delta \implies \ell(x)\notin\Omega'$.

Proof. By the definition of a Lipschitz domain, there exist open sets $U_i\subset\mathbb{R}^n$, unit vectors $\mathbf{v}_i\in\mathbb{R}^n$, Lipschitz mappings $\Pi_i\colon U_i\to\partial\Omega'\cap U_i$, $i=1,\dots,m$, and $R,\rho>0$ with the following properties

(i) for each $x\in U_i$ there exists $\lambda\in(-R,R)$ such that $x=\Pi_i(x)+\lambda\mathbf{v}_i$,
(ii) for each $x\in U_i\cap\partial\Omega'$ and $t\in(0,2R)$ we have $\Pi_i(x)=x$, $x+t\mathbf{v}_i\in\mathbb{R}^n\setminus\overline{\Omega'}$, $x-t\mathbf{v}_i\in\Omega'$.
(iii) $\{x\in\mathbb{R}^n\colon \operatorname{dist}(x,\partial\Omega')\leq\rho\}\subset\bigcup_i U_i$.

For each $z\in\overline{\Omega'}$ with $\operatorname{dist}(z,\partial\Omega')\leq\rho$ find $B_z=B(z,r_z)$ such that there exists $i=i(j)\in\{1,\dots,m\}$ with $\overline{B}(z,(m+1)r_z)\subset U_i$. Using compactness of $\{z\in\mathbb{R}^n\colon \operatorname{dist}(z,\partial\Omega')\leq\rho\}$ select finite covering of this sets by balls $B(z_j,r_j)$, $j=1,\dots,p$ with the property that $r_j=r_{z_j}$ and find a smooth partition of unity $(\omega_j)_j$ on $\{z\in\mathbb{R}^n\colon \operatorname{dist}(z,\partial\Omega')\leq\rho\}$ such that $\{\omega_j>0\}=B(z_j,r_j)$, $j=1,\dots,p$. Set $r=\min\{R/(m+1),\rho,r_1,\dots,r_p\}$ and find $\delta>0$ such that for each $x\in U_i$, $i=1,\dots,m$, we have

$$\operatorname{dist}(x,\partial\Omega)<\delta \implies |x-\Pi_i(x)|<r.$$

Set

$$\eta_i(x)=\sum_{j\colon i(j)=i}\omega_j,\qquad i=1,\dots,m,$$

$$\ell(x)=x+m\sum_{i=1}^m\eta_i(x)(\Pi_i(x)-x)\text{ if }x\in\overline{\Omega}'\text{ and }\operatorname{dist}(x,\partial\Omega')\leq\delta$$

and extend ℓ in a Lipschitz way to $\overline{\Omega'}$. Then ℓ is a Lipschitz mapping which is identity on $\partial\Omega'$.

Fixing $x \in \overline{\Omega'}$ with $\operatorname{dist}(x, \partial\Omega') \leq \delta$, we must prove that $\ell(x) \notin \Omega'$. We find $i_0 \in \{1, \dots, m\}$ such that

$$\eta_{i_0}(x) \geq 1/m.$$

We may assume that $i_0 = 1$ and that $x \in \operatorname{supp} \eta_i$ if and only if $i \in \{1, \dots, k\}$ for some $k \in \{1, \dots, m\}$. Write

$$x_1 := x + m\eta_1(x)(\Pi_1(x) - x) = \Pi_1(x) + (m\eta_1(x) - 1)(|\Pi_1(x) - x|)\mathbf{v}_1.$$

Since

$$(m\eta_1(x) - 1)(|\Pi_1(x) - x|) \leq mr \leq R,$$

we have $x_1 \notin \Omega'$. We have $x_1 \in B(x, m\eta_1(x)r)$. If $k = 1$, we are done.

Now, we proceed by induction. Write

$$x_q = x + m \sum_{i \leq q} \eta_i(x)(\Pi_i(x) - x), \qquad q \leq k.$$

By induction hypothesis we have $x_{q-1} \notin \Omega'$, Further,

$$x_{q-1} \in B(x, m(\eta_1(x) + \dots + \eta_{q-1}(x))r) \subset B(x, mr).$$

We find j_q such that $i(j_q) = q$ and $|x - z_{j_q}| \leq r_{j_q}$. Then

$$|x_{q-1} - z_{j_q}| \leq mr + r_{j_q} \leq (m + 1)r_{j_q}$$

and thus we have $x_{q-1} \in U_q$. This means that x_{q-1} is of the form $\Pi_q(x_{q-1}) + \lambda\mathbf{v}_q$ with $\lambda < R$. Since $m\eta_q(x)(\Pi_q(x) - x) = \lambda'\mathbf{v}_q$ with $0 \leq \lambda' \leq mr \leq R$, we have $x_q = \Pi_q(x_{q-1}) + (\lambda + \lambda')\mathbf{v}_q$ with $\lambda + \lambda' < 2R$ and it follows that $x_q \notin \Omega'$. We conclude that $\ell(x) = x_k \notin \Omega'$. □

Theorem 2.13. *Let Ω, Ω' be Lipschitz domains and f be a $W^{1,p}$-homeomorphism of $\overline{\Omega}$ onto $\overline{\Omega'}$. Then there exist $\Omega_0 \supset \overline{\Omega}$, $\Omega_0' \supset \overline{\Omega'}$, and a continuous $W^{1,p}$-mapping $\tilde{f} \colon \Omega_0 \to \Omega_0'$ such that $\tilde{f} = f$ on $\overline{\Omega}$ and $\tilde{f}$ maps $\Omega_0 \setminus \Omega$ to $\Omega_0' \setminus \Omega'$.*

Proof. We use Lemma 2.12 to Ω' and we keep the notation from Lemma 2.12. Let $f^* \colon \mathbb{R}^n \to \mathbb{R}^n$ be the usual $W^{1,p}$-extension of f, by its construction it follows that f^* is continuous. Find $\tau > 0$ such that

$$\operatorname{dist}(x, \overline{\Omega}) < \tau \implies \operatorname{dist}(f^*(x), \partial\Omega') < \delta.$$

Set

$$\begin{aligned} \Omega_0 &= \{x \colon \operatorname{dist}(x, \overline{\Omega}) < \tau\}, \\ \tilde{f}(x) &= \begin{cases} f^*(x), & x \in \overline{\Omega} \text{ or } f^*(x) \notin \Omega', \\ \ell(f^*(x)), & x \in \Omega_0 \setminus \overline{\Omega} \text{ and } f^*(x) \in \Omega', \end{cases} \\ \text{and } \Omega_0' &= \tilde{f}(\Omega_0). \end{aligned}$$

It is easily verified that $\tilde{f}$ has the desired properties. We use the chain rule (see e.g. [40, Theorem 2.1.11] or [1, Theorem 3.16]) to prove the Sobolev regularity of the composition. □

3. Limit of homeomorphisms satisfies (INV)

Recall that our energy (1.1) is given by

$$\mathcal{F}(f)=\int_\Omega (|Df|^{n-1}+\varphi(J_f))\,dx,$$

where φ is a positive convex function on $(0,\infty)$ that satisfies (1.3) and (1.4).

Theorem 3.1. *Let $n\geq 3$, $\Omega,\Omega'\subset\mathbb{R}^n$ be Lipschitz domains and let φ satisfy* (1.3) *and* (1.4).

Let $f_m\in W^{1,n-1}(\Omega,\Omega')$, $m=0,1,2,\dots$, be a sequence of homeomorphisms with $J_{f_m}>0$ a.e. Let f_m converge weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ to a limit function f. Assume further that we have either

(a) *f_m are homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ such that $f_m=f_0$ on $\partial\Omega$, for all $m\in\mathbb{N}$,*

$$\mathcal{F}(f_m)\leq C_1 \tag{3.1}$$

and

$$\|K_{f_m}^{\frac{1}{n-1}}\|_1\leq C_1, \tag{3.2}$$

or

(b) *f_m converge strongly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ to f and $J_f>0$ a.e.*

Then f satisfies (INV).

Our main theorem follows easily from this more general result.

Proof of Theorem 1.1. Assumption (1.5) clearly implies (3.1) and by the Young inequality

$$ab\leq \frac{1}{p}a^p+\frac{1}{p'}b^{p'} \text{ for } a\geq 0, b\geq 0, p>1$$

used for $p=\frac{(n-1)^2}{n}$ (and thus $p'=\frac{(n-1)^2}{n^2-3n+1}$) we obtain

$$\int_\Omega K_{f_m}^{\frac{1}{n-1}}\,dx=\int_\Omega |Df_m|^{\frac{n}{n-1}}\frac{1}{J_{f_m}^{\frac{1}{n-1}}}\,dx\leq \frac{1}{p}\int_\Omega |Df_m|^{n-1}\,dx+\frac{1}{p'}\int_\Omega \frac{1}{J_{f_m}^{\frac{n-1}{n^2-3n+1}}}\,dx.$$

The conclusion now follows from Theorem 3.1. □

Remark 3.2. Using the Young inequality with $p=\frac{(n-1)^2}{n(1-\varepsilon)}$ (and thus $p'=\frac{(n-1)^2}{n^2-3n+1+n\varepsilon}$) we obtain a similar inequality for lower powers of K_{f_m}, i.e., the counterexample from Theorem 1.2 shows that assuming

$$\|K_{f_m}^{\frac{1-\varepsilon}{n-1}}\|_1\leq C_1$$

is also not enough to preserve the (INV) condition under weak limits.

Definition 3.3. Let $\Omega\subset\mathbb{R}^n$ be open and let $f_m\in W^{1,n-1}(\Omega,\mathbb{R}^n)$ be homeomorphisms that converge to a limit function f weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$. We say that a shape $K\subset\subset\Omega$ is a *good shape (in particular, good ball or good cuboid) with respect to* $(f_m)_m$ if the following properties are satisfied.

(i) The trace of f on ∂K is in $W^{1,n-1}(\partial K,\mathbb{R}^n)$. In what follows we assume that f is represented to coincide with this trace on ∂K.

(ii) If K is a full cuboid or a hollowed cuboid, the the trace of f on each $(n-2)$-dimensional edge E of K is in $W^{1,n-1}(E)$ and the trace representative of f on the closed $(n-2)$-dimensional skeleton of K is continuous.
(iii) $|f_m(\partial K)| = 0$ for all $m \in \mathbb{N}$.
(iv) There is a subsequence of f_m such that the convergence $f_{m_k} \to f$ occurs weakly in $W^{1,n-1}(\partial K, \mathbb{R}^n)$ and $\mathcal{H}^{n-1}$-a.e. on ∂K, (and therefore $\deg(f_{m_k}, K, \cdot)$ forms a bounded sequence in BV.)

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be homeomorphisms that converge to a limit function f weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. Let $B(x_0, r_0) \subset \Omega$. Then $B(x_0, r)$ is a good ball with respect to $(f_m)_m$ for a.e. $r \in (0, r_0)$.*

Proof. By slicing analogous to the proof of the ACL property we obtain that the trace of f on ∂B is in $W^{1,n-1}(\partial B(x_0, r), \mathbb{R}^n)$ for a.e. $r > 0$. Images of spheres by f_m are disjoint as f_m are one-to-one and thus $|f_m(\partial B(x_0, r))| = 0$ for a.e. $r > 0$. The fact that $\deg(f_m, B, \cdot)$ forms a bounded sequence in BV follows from Section 2.5.

By the Fubini theorem and by the Fatou theorem

$$\int_0^{r_0} \liminf_{m\to\infty} \Big(\int_{\partial B(x_0,r)} |D_\tau f_m|^{n-1} \, d\mathcal{H}^{n-1} \Big) \, dr \leq \liminf_{m\to\infty} \int_{B(x_0,r_0)} |Df_m|^{n-1} \leq C_1. \tag{3.3}$$

The last inequality implies that for a.e. r

$$\liminf_{m\to\infty} \Big(\int_{\partial B(x_0,r)} |D_\tau f_m|^{n-1} \, d\mathcal{H}^{n-1} \Big) < \infty$$

and we can choose a subsequence for which the limes inferior turns to the limit. Thus, we have a bounded sequence in $W^{1,n-1}(\partial B(x_0, r))$ and we select a weakly convergent subsequence. Since $W^{1,n-1}$ is compactly embedded into L^{n-1} we obtain that this subsequence converge to f in L^{n-1}. Up to a subsequence we can thus assume that it converges to f pointwise $\mathcal{H}^{n-1}$-a.e. on ∂B. ☐

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open and let $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be homeomorphisms that converge to a limit function f weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. Let $\delta > 0$. Then there exist partitions*

$$\begin{aligned} &t_1^0 < t_1^1 < \cdots < t_1^{m_1}, \\ &t_2^0 < t_2^1 < \cdots < t_2^{m_2}, \\ &\dots \\ &t_n^0 < t_n^1 < \cdots < t_1^{m_n} \end{aligned}$$

such that

$$\overline{\Omega} \subset (t_1^0, t_1^{m_1}) \times \cdots \times (t_n^0, t_n^{m_n}),$$

each $t_i^j - t_i^{j-1} < \delta$ and each

$$Q = (t_1^{j_1-1}, t_1^{j_1}) \times (t_1^{j_2-1}, t_1^{j_2}) \times \cdots \times (t_1^{j_n-1}, t_1^{j_n})$$

with $1 \leq j_i \leq m_i$, $i = 1, \dots, n$, is a good cuboid with respect to $(f_m)_m$ provided that $\overline{Q} \subset \Omega$.

Proof. The proof is analogous to the proof of Lemma 3.4 with the additional difficulty that we must take care of the $W^{1,n-1}$-regularity (which implies continuity for a suitable representative by the Morrey estimates) on the $(n-2)$-dimensional edges. Therefore we select the partition points t_i^j subsequently for $i = 1, 2, \dots, n$ in such a way that the Sobolev regularity on the intersections of $\{x \in \Omega\colon x_i = t_i^j\}$ with all $\{x \in \Omega\colon x_i = t_{i'}^{j'}\}$ for all $i' < i$ and $j' \in \{1, \dots, m_{i'}\}$ is controled. $\square$

In the main proof we assume that the (INV) condition fails. Hence we can find a ball $B \subset \Omega$ such either something from outside of B is mapped into the topological image of the ball or something from inside of B is mapped outside of topological image, i.e. that the set

$$\{x \in \Omega \setminus B : \ f(x) \in \operatorname{im}_T(f, B)\} (\text{or } \{x \in B : \ f(x) \notin \operatorname{im}_T(f, B)\}) \text{ has positive measure.}$$

At the same time we need to show that also something from inside of B is mapped inside the topological image and something from outside of B is mapped outside, i.e. that the following set have positive measure

$$\{x \in \Omega \setminus B : \ f(x) \notin \operatorname{im}_T(f, B)\} (\text{or } \{x \in B : \ f(x) \in \operatorname{im}_T(f, B)\}).$$

This second condition seems to be believable but unfortunately we need the following technical Lemma to show its existence (note that we replace $f(x) \in \operatorname{im}_T(f, B)$ by $\operatorname{Deg}(f, B, f(x)) \neq 0$ in (3.6)). The main idea to show this is simple, we extend our $f, f_m : \Omega \to \Omega'$ to mappings $f^*, f_m^* : \Omega_1 \to \Omega_1'$ with $\Omega_1 \supsetneq \Omega$ so that other conditions holds for these extensions. Now it is not difficult to see that for many points

$$x \in \Omega_1 \setminus \Omega \text{ we have } f(x) \notin \operatorname{im}_T(f, B).$$

Moreover, we do another important observation there. In the proof of the main theorem we assume that (INV) condition fails and thus either *(ii)* or *(iii)* of Definition 2.6 fail. We show that if *(ii)* fails for some ball then *(iii)* fails for some other shape. It follows that we can assume in the proof of main theorem that *(iii)* fails.

Lemma 3.6. *Let $n \geq 3$, $\Omega, \Omega' \subset \mathbb{R}^n$ be Lipschitz domains, let φ satisfy* (1.3) *and* (1.4). *Let $f_m \in W^{1,n-1}(\Omega, \Omega')$, $m = 0, 1, 2, \dots,$ be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega}'$ such that $f_m = f_0$ on $\partial\Omega$, $J_{f_m} > 0$ a.e., and*

$$\mathcal{F}(f_m) \leq C_1, \tag{3.4}$$

which converges weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ to a limit function f. Assume that f does not satisfy (INV).

Then we can find domains $\Omega_1 \supset \Omega, \Omega_1' \subset \mathbb{R}^n$, such that f_m extend to $W^{1,n-1}$-homeomorphisms $f_m^*\colon \overline{\Omega}_1 \to \overline{\Omega}_1'$ with $f_m^* = f_0^*$ on $\overline{\Omega}_1 \setminus \Omega$,

$$\sup_{m \in \mathbb{N}} \int_{\Omega_1} \varphi(J_{f_m^*})\, dx < \infty \text{ and } \sup_{m \in \mathbb{N}} \int_{\Omega_1} K_{f_m^*}^{\frac{1}{n-1}}\, dx < \infty, \tag{3.5}$$

and we can find a good shape $K \subset\subset \Omega_1$ for the sequence f_m^* and the limit function f^* such that both sets

$$\{x \in \Omega_1 \setminus K : \ \operatorname{Deg}(f^*, K, f^*(x)) \neq 0\} \text{ and } \{x \in \Omega_1 \setminus K : \ \operatorname{Deg}(f^*, K, f^*(x)) = 0\} \tag{3.6}$$

have positive measure.

Proof. Denote $\mathbb{H} = \{x \in \mathbb{R}^n\colon x_1 < 0\}$. Denote the reflection $(x_1, x_2 \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ by R. Since domains Ω, Ω' are Lipschitz and we can find a joint localization of both, there exist k pairs of open sets $U_i, V_i \subset \mathbb{R}^n$ and of bilipschitz mappings $\Phi_i\colon U_i \to \mathbb{R}^n$, $\Psi_i\colon V_i \to \mathbb{R}^n$ with $i \in \{1, \dots, , k\}$ such that

(i) the sets U_i cover $\partial\Omega$,

(ii) $f_0(U_i) \subset V_i$,
(iii) for each $x \in \overline{U}_i$ we have $x \in \Omega$ iff $\Phi_i(x) \in \mathbb{H}$,
(iv) for each $x \in \overline{V}_i$ we have $x \in \Omega'$ iff $\Psi_i(x) \in \mathbb{H}$,
(v) $x \in \Phi_i(U_i) \setminus \mathbb{H} \implies R(x) \in \Phi_i(U_i)$,
(vi) $x \in \Psi_i(V_i) \cap \mathbb{H} \implies R(x) \in \Psi_i(V_i)$.

Then we can construct “Lipschitz reflections” near $\partial\Omega$ and $\partial\Omega'$,

$$R_i^\Phi = \Phi_i^{-1} \circ R \circ \Phi_i,$$
$$R_i^\Psi = \Psi_i^{-1} \circ R \circ \Psi_i.$$

Let us fix $i \in \{1, \dots, k\}$. Then for any m we can extend f_m, $j = 0, 1, \dots,$ to a Sobolev homeomorphism ${f^*}_m \colon \overline{\Omega \cup U_i} \to \mathbb{R}^n$ setting

$${f^*}_m(x) = \begin{cases} f_m(x), & x \in \overline{\Omega}, \\ R_i^{\Psi_i}(f_0(R_i^\Phi(x))), & x \in U_i \setminus \overline{\Omega}. \end{cases}$$

Also we use the limit function

$$f^* = \begin{cases} f(x), & x \in \overline{\Omega}, \\ f_0^*(x), & x \in U_i \setminus \overline{\Omega}. \end{cases} \tag{3.7}$$

The Sobolev regularity and continuity are preserved by composition with the bilipschitz mappings. We use the property (1.4) of φ and $\mathcal{F}(f_0) < \infty$ to verify that

$$\sup_{m\in\mathbb{N}} \int_{\Omega_1} \varphi(J_{{f^*}_m})\,dx < \infty \text{ and } \sup_{m\in\mathbb{N}} \int_{\Omega_1} K_{{f^*}_m}^{\frac{1}{n-1}}\,dx < \infty.$$

Now we look for a good shape K as in the statement of the theorem. Since f does not satisfy (INV) on Ω we can use Lemma 3.4 and Remark 2.7 to find an (arbitrarily small) good ball $B(c, r) \subset \Omega$ such that either

$$\{x \in \Omega \setminus B : \ \operatorname{Deg}(f, B, f(x)) \neq 0\}$$

or

$$A := \{x \in B : \ \operatorname{Deg}(f, B, f(x)) = 0\} \tag{3.8}$$

have positive measure. Since B is small we can assume that $3\sqrt{n}\operatorname{diam} B < \operatorname{dist}(B, \partial\Omega)$.

In the first case we find $y_0 \in \partial\Omega'$ such that y is a boundary point of the convex hull $\widehat{\Omega'}$ of Ω', find $i \in \{1, \dots, k\}$ such that $y_0 \in V_i$ and extend f_m and f to $\overline{\Omega}_1$ as in (3.7), where $\Omega_1 = \Omega \cup U_i$. Notice that for all $y \in f(\Omega_1) \setminus \overline{\widehat{\Omega'}}$ we have $\operatorname{Deg}(f, B, y) = 0$. This can be obtained by approximation. Namely, by the Mazur lemma there exist convex combinations g_m of f_m such that $g_m \to f$ strongly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. All the functions g_m have values in $\widehat{\Omega'}$ and therefore

$$\deg(g_m, B, y) = 0, \qquad y \notin \widehat{\Omega'}.$$

Now we use Lemma 2.3 and (2.2) to deduce that

$$\operatorname{Deg}(f^*, B, f^*(x)) = 0 \qquad \text{for a.e. } x \in \Omega_1 \setminus (f_0^*)^{-1}(\widehat{\Omega'}). \tag{3.9}$$

Hence our conclusion holds for this ball B, but of course for Ω_1 instead of Ω.

The second case, namely that the set A has positive measure, is more tricky. By Theorem 2.13 there exist $\Omega_0 \supset \overline{\Omega}$, $\Omega_0' \supset \overline{\Omega}'$, and a continuous $W^{1,n-1}$-mapping $\tilde{f}_0 \colon \Omega_0 \to \Omega_0'$ such that $\tilde{f}_0 = f_0$ on $\overline{\Omega}$ and $\tilde{f}_0$ maps $\Omega' \setminus \Omega$ to $\Omega_0' \setminus \Omega_0$. We set

$$\tilde{f}_m(x) = \begin{cases} \tilde{f}_0(x), & x \in \Omega_0 \setminus \Omega, \\ f_m(x), & x \in \Omega \end{cases} \quad \text{and } \tilde{f}(x) = \begin{cases} \tilde{f}_0(x), & x \in \Omega_0 \setminus \Omega, \\ f(x), & x \in \Omega. \end{cases}$$

We use Lemma 3.5 to construct a partition $\mathcal{Q}$ of a neighbourhood of $\overline{\Omega}$ into good cuboids with respect to the extended functions. Moreover, we may assume that each cuboid $Q \in \mathcal{Q}$ intersects $\overline{\Omega}$ and is so small that it satisfies

$$\operatorname{diam} Q < \operatorname{diam} B \quad \text{and} \quad Q \cap \partial\Omega \neq \emptyset \implies Q \subset \Omega_0 \cap U_i \text{ for some } i. \tag{3.10}$$

We define the figure F as (see Fig. 1)

$$\overline{F} = \bigcup_{Q \in \mathcal{Q}} \overline{Q}.$$

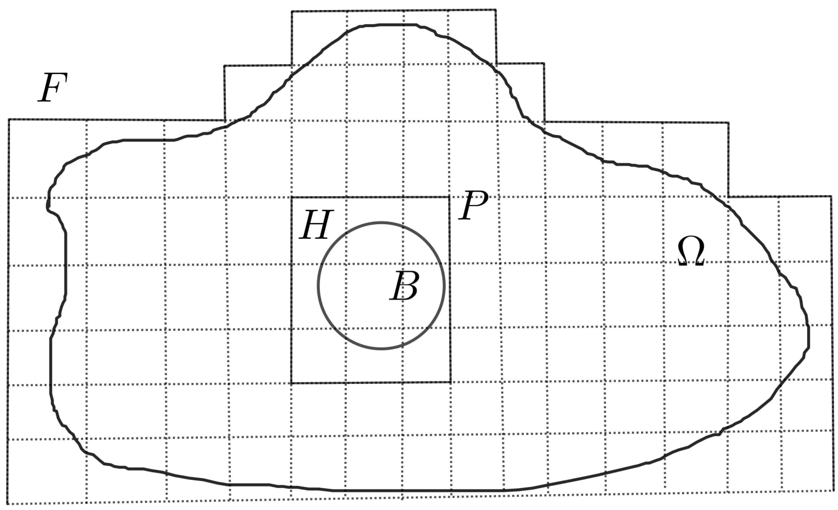

FIGURE 1. We cover Ω by a set of good cuboids F and B by full cuboids P.

Using $\operatorname{dist}(B, \partial\Omega) > 3\sqrt{n} \operatorname{diam} B$ and (3.10) we find $\mathcal{Q}' \subset \mathcal{Q}$ such that the figure P with

$$\overline{P} := \bigcup_{Q \in \mathcal{Q}'} \overline{Q}$$

is itself a full cuboid and

$$\overline{B} \subset P \subset\subset \Omega.$$

We will consider the hollowed cuboid

$$H = P \setminus B.$$

Denote

$$\mathcal{Q}'' = \{Q \in \mathcal{Q} \colon Q \cap P = \emptyset\}.$$

We have (see Fig. 1)

$$\deg(\tilde{f}, F, y) = 1, \qquad y \in \Omega'. \tag{3.11}$$

Indeed, $\tilde{f} = \tilde{f}_0$ on ∂F, f_0 is a sense preserving homeomorphism on Ω and

$$\deg(\tilde{f}_0, F, y) = \deg(f_0, \Omega, y) + \deg(\tilde{f}_0, F \setminus \overline{\Omega}, y).$$

Here we use the additivity property of the degree and the fact that $\deg(\tilde{f}_0, F \setminus \overline{\Omega}, y) = 0$ as

$$y \notin \tilde{f}_0(F \setminus \overline{\Omega}).$$

Note that the additivity property of the degree defined by (2.4) follows from the fact that the normals on boundary parts of adjacent surfaces are opposite and thus cancellation occurs. Further by (3.8)

$$\operatorname{Deg}(f, B, y) = 0, \qquad y \in f(A).$$

Now, by (3.11)

(3.12)
$$1 = \operatorname{Deg}(\tilde{f}, F, y) = \operatorname{Deg}(f, B, y) + \operatorname{Deg}(f, H, y) + \sum_{Q \in \mathcal{Q}''} \operatorname{Deg}(\tilde{f}, Q, y), \qquad \text{for a.e. } y \in f(A).$$

By (3.12) there exists a good shape K such that $K \cap B = 0$ and $\deg(\tilde{f}, B, y) \neq 0$ for a.e. $y \in f(A)$, namely either $K = H$ or $K \in \mathcal{Q}''$. If $K \subset\subset \Omega$, we can proceed as in the preceding case (see (3.9)) and add a suitable U_i to Ω. Let $K \cap \partial\Omega \neq \emptyset$. Then we find i such that $K \subset\subset U_i$ and as in the preceding case extend f_m as ${f_m}^*$ and f as f^* to $\overline{\Omega \cup U_i}$ as in (3.7). We now claim that that

$$\operatorname{Deg}(f^*, K, f(x)) \neq 0 \quad \text{for a.e.} \quad x \in A. \tag{3.13}$$

We start with showing that

$$\operatorname{Deg}(f^*, K, y) = \operatorname{Deg}(\tilde{f}, K, y) \neq 0 \quad \text{for a.e.} \quad y \in f(A). \tag{3.14}$$

To this end we first use a homotopy

$$h(y,t) = \Psi_i^{-1}\Big(\Psi_i(\tilde{f}_0(y)) + t\big(\Psi_i(f_0^*(y)) - \Psi_i(\tilde{f}_0(y))\big)\Big)$$

to prove that

$$\deg(\tilde{f}_0, K, y) = \deg(f_0^*, K, y), \qquad y \in \Omega'. \tag{3.15}$$

Let ψ be a smooth function supported in Ω' and $\mathbf{u}$ be a smooth function satisfying $\operatorname{div} \mathbf{u} = \psi$. Then by (3.15) we have

$$\begin{aligned}
&\int_{\partial K \setminus \Omega} (\mathbf{u} \circ f_0^*) \cdot (\Lambda_{n-1} D_\tau f_0^*)\nu \, d\mathcal{H}^{n-1} \\
&\quad = \int_{\partial K} (\mathbf{u} \circ f_0^*) \cdot (\Lambda_{n-1} D_\tau f_0^*)\nu \, d\mathcal{H}^{n-1} - \int_{\partial K \cap \Omega} (\mathbf{u} \circ f_0) \cdot (\Lambda_{n-1} D_\tau f_0)\nu \, d\mathcal{H}^{n-1} \\
&\quad = \int_{\partial K} (\mathbf{u} \circ \tilde{f}_0) \cdot (\Lambda_{n-1} D_\tau \tilde{f}_0)\nu \, d\mathcal{H}^{n-1} - \int_{\partial K \cap \Omega} (\mathbf{u} \circ f_0) \cdot (\Lambda_{n-1} D_\tau f_0)\nu \, d\mathcal{H}^{n-1} \\
&\quad = \int_{\partial K \setminus \Omega} (\mathbf{u} \circ \tilde{f}_0) \cdot (\Lambda_{n-1} D_\tau \tilde{f}_0)\nu \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\partial K} (\mathbf{u}\circ f^*)\cdot(\Lambda_{n-1}D_\tau f^*)\nu\, d\mathcal{H}^{n-1}\\
&\quad = \int_{\partial K\setminus\Omega} (\mathbf{u}\circ f_0^*)\cdot(\Lambda_{n-1}D_\tau f_0^*)\nu\, d\mathcal{H}^{n-1} + \int_{\partial K\cap\Omega} (\mathbf{u}\circ f)\cdot(\Lambda_{n-1}D_\tau f)\nu\, d\mathcal{H}^{n-1}\\
&\quad = \int_{\partial K\setminus\Omega} (\mathbf{u}\circ \tilde{f}_0)\cdot(\Lambda_{n-1}D_\tau \tilde{f}_0)\nu\, d\mathcal{H}^{n-1} + \int_{\partial K\cap\Omega} (\mathbf{u}\circ f)\cdot(\Lambda_{n-1}D_\tau f)\nu\, d\mathcal{H}^{n-1}\\
&\quad = \int_{\partial K} (\mathbf{u}\circ \tilde{f})\cdot(\Lambda_{n-1}D_\tau \tilde{f})\nu\, d\mathcal{H}^{n-1}.
\end{aligned}$$

We thus proved (3.14). Then, by Lemma 2.8 we have

$$\operatorname{Deg}(f^*, K, f(x)) = \operatorname{Deg}(\tilde{f}, K, f(x)) \neq 0 \quad \text{ for a.e. } x\in A.$$

which establishes (3.13).

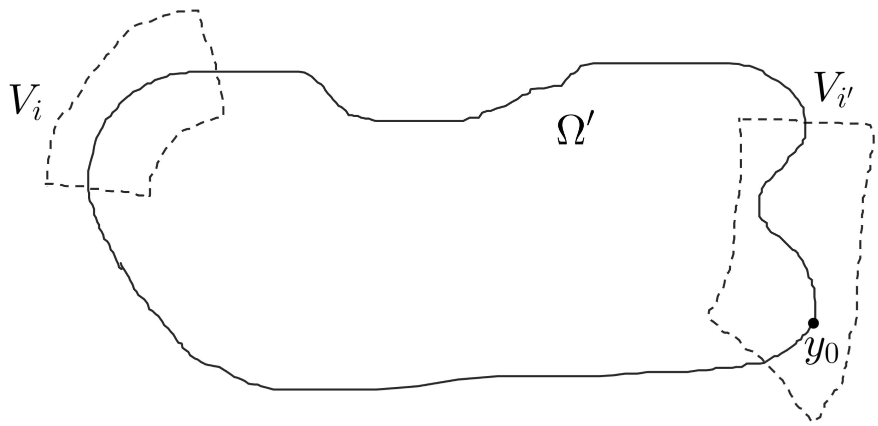

FIGURE 2. We add to Ω' two disjoint sets V_i and V_i'.

Now, we need to extend the function to a still larger set Ω_1 (see Fig. 2). Similarly to the first step, we find $i'\in\{1,\dots,k\}$ and a point $y_0\in\partial\Omega'\cap\partial\widehat{\Omega}'$ such that $y_0\in V_{i'}$ (recall that $\widehat{\Omega}'$ denotes the convex hull of Ω'). We set $\Omega_1=\Omega\cup U_i\cup U_{i'}$ and extend f_m and f to ${f^*}_m$ and f^* using "Lipschitz reflection" on both U_i and $U_{i'}$. To make it possible, we require in addition that $V_i\cap V_{i'}=\emptyset$. This can be achieved if the covering of the boundary is chosen fine enough. If we consider the strongly converging convex combinations ${g_m}^*$, we observe that ${g_m}^*(x)\in\widehat{\Omega}'$ if $x\in\Omega$ and ${g_m}^*(x)\in V_{i'}$ if $x\in U_{i'}\setminus\Omega$. Therefore ${g_m}^*(x)\notin V_{i'}\setminus\widehat{\Omega}'$ for $x\in K$ and thus

$$\operatorname{Deg}(f^*, K, f(x)) = 0 \qquad \text{for a.e. } x\in U_{i'}\setminus (f_0^*)^{-1}(\widehat{\Omega}').$$

□

Proof of Theorem 3.1. Step 1. Outline of the proof: We give here a short informal summary of the proof first. Case (b) is proven by a simplified version of the proof of Case (a), as thanks to the strong convergence we do not need the assumption on integrability of the distortion and Jacobian of f_m.

We start by assuming that f violates the (INV) condition and that "something from outside is mapped inside the topological image". We find a good shape K with respect to $(f_m)_m$ such that

$$U \setminus K = \{x \in \Omega \setminus K\colon \ \mathrm{Deg}(f, K, f(x)) \neq 0\}$$

is of positive measure. (In the most simple case, K may be one of the balls which violate the (INV) condition.) Those are the points which originally were outside of K but f mapped them into the topological image of K. We cover the boundary of K by a $(n-2)$-dimensional "cage" or "skeleton" made of parts of $(n-2)$-dimensional circles. On this skeleton our functions are Hölder continuous. On the rest of the boundary of K we replace them by g_m and g which are continuous. One can think of it as of having prescribed deformation of the skeleton and g_m and g being a suitable continuous extensions of it on ∂K. The differences between the topological images of f_m and g_m (or [and g) create bubbles of some kind, through which the material can leave the topological image of K or enter it from the outside (see Figure 3). The neck of such bubble must be getting thinner and thinner as m grows, since in the end the topological image "skips" it completely (see Figure 4). We find two balls B_U and B_V of the same sizes outside of K such that a big parts of them lie in U and V, respectively. Since most of B_U is then mapped inside the topological image of K but B_V is mapped outside of it, the lines connecting these two balls must pass through the thin neck of the bubble. That gives a contradiction with our assumption on the integrability of the distortion, as the necks are getting smaller and smaller, but the material of the lines cannot be deformed that much.

Step 2. Finding a good shape K: We assume for contradiction that f does not satisfy the (INV) condition. Assume first Case (a). Since (INV) fails for f, by Lemma 3.6 we may assume (passing if necessary to a different domain and different mapping) that there is a good shape K with respect to $(f_m)_m$ such that both sets $U \setminus K$ and $V \setminus K$ have positive measure, where

$$U = \{x \in \Omega\colon \ \mathrm{Deg}(f, K, f(x)) \neq 0\}, \quad V = \{x \in \Omega\colon \ \mathrm{Deg}(f, K, f(x)) = 0\}. \tag{3.16}$$

In Case (b), we find a good ball K such that (INV) is violated on K. Since either (ii) or (iii) from Definition 2.6 fails, by Remark 2.7 we have that either $U \setminus K$ has positive measure, or $V \cap K$ has positive measure. We will handle the former case, the latter one being similar.

Step 3. Finding a skeleton of ∂K: Now, we handle Cases (a) and (b) together. Since f_m converge weakly in $W^{1,n-1}$ and $W^{1,n-1}$ is compactly embedded into L^{n-1} on each ball $B \subset\subset \Omega$, we obtain that f_m converge to f in L^{n-1} at least locally. Up to a subsequence we can thus assume that $f_m \to f$ pointwise a.e. Using Lemma 2.10 we thus obtain that $J_f \neq 0$ a.e. (for Case (a), as we assume it in Case (b)). Passing if necessary to a subsequence we find a constant C_2 such that

$$\int_{\partial K} (|D_\tau f|^{n-1} + |D_\tau f_m|^{n-1})\, d\mathcal{H}^{n-1} < C_2, \qquad m \in \mathbb{N}.$$

Choose $\varepsilon > 0$ small enough whose exact value is specified later. Find $\rho \in (0, \frac{1}{16n} r_0)$ such that for each $z \in \partial K$ we have

$$\int_{\partial K \cap B(z, 2\rho)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-1} < \varepsilon^{n-1}. \tag{3.17}$$

Now, we distinguish three possibilities according to the form of the shape K. We define sets $T_j \subset \partial K$ which form a "skeleton" of ∂K. Their key property will be that the diameter of their image under f is small, namely

$$\operatorname{diam} f(T_j) \leq C_3\varepsilon. \tag{3.18}$$

First, let K be a ball. For each $z \in \partial K$ we find $\rho_z \in (\rho, 2\rho)$ such that

$$\rho \int_{\partial K \cap \partial B(z,\rho_z)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-2} < \varepsilon^{n-1}.$$

Analogously to the definition of the good ball we can also assume that $f_m \to f$ occurs $\mathcal{H}^{n-2}$-a.e. on $\partial K \cap \partial B(z, \rho_z)$ and that

$$\liminf_{m\to\infty} \|f_m\|_{W^{1,n-1}(\partial K \cap \partial B(z,\rho_z))} < \infty.$$

It follows that up to a subsequence (see e.g. [31, Lemma 2.9])

$$f_m \to f \text{ weakly in } W^{1,n-1} \text{ and also uniformly on } \partial K \cap \partial B(z, \rho_z). \tag{3.19}$$

Note that on the $(n-2)$ dimensional space $\partial K \cap \partial B(z, \rho_z)$ we have embedding into Hölder functions $W^{1,n-1} \hookrightarrow C^{0,1-\frac{n-2}{n-1}}$ and thus f is continuous there and we have the estimate

$$\operatorname{diam} f(\partial K \cap \partial B(z, \rho_z)) \leq C(\rho_z)^{1-\frac{n-2}{n-1}} \Big(\int_{\partial K \cap \partial B(z,\rho_z)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-2}\Big)^{\frac{1}{n-1}} \leq C_3\varepsilon. \tag{3.20}$$

Using a Vitali type covering, we find $B_j = B(z_j, \rho_j)$ such that $\rho_j = \rho_{z_j}$,

$$\partial K \subset \bigcup_j B(z_j, \rho_j)$$

and the balls $B(z_j, \frac{1}{5}\rho_j)$ are pairwise disjoint. Here $j = 1, \dots, j_{\max}$. Note that the multiplicity of the covering is estimated by a constant N_1 depending only on the dimension since $\rho_z \in (\rho, 2\rho)$ for every z. Furthermore, the balls in the Vitali covering theorem are chosen inductively so we can also assume using (3.19) that for a subsequence (chosen in a diagonal argument)

$$f_m \to f \text{ weakly in } W^{1,n-1} \text{ and uniformly on } \partial K \cap \partial B(z_j, \rho_j) \text{ for each } j. \tag{3.21}$$

Given j, denote

$$S_j = \partial K \cap B_j \setminus \bigcup_{l<j} \overline{B}_l.$$

Note that S_j obviously satisfies the exterior ball condition of Subsection 2.8. Let T_j denote the relative boundary of S_j with respect to ∂K. From (3.20) we have (3.18).

If K is a full cuboid, similarly to the proof of Lemma 3.5 we find partitions of each face of K to $(n-1)$-dimensional (full) cuboids S_j such that, denoting the relative boundaries of S_j with respect to K by T_j, we have $\operatorname{diam} S_j < \rho$ and

$$\rho \int_{T_j} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-2} \leq \varepsilon^{n-1}.$$

We can also assume that $f_m \to f$ occurs $\mathcal{H}^{n-2}$-a.e. on T_j and that

$$\liminf_{m\to\infty} \|f_m\|_{W^{1,n-1}(T_j)} < \infty.$$

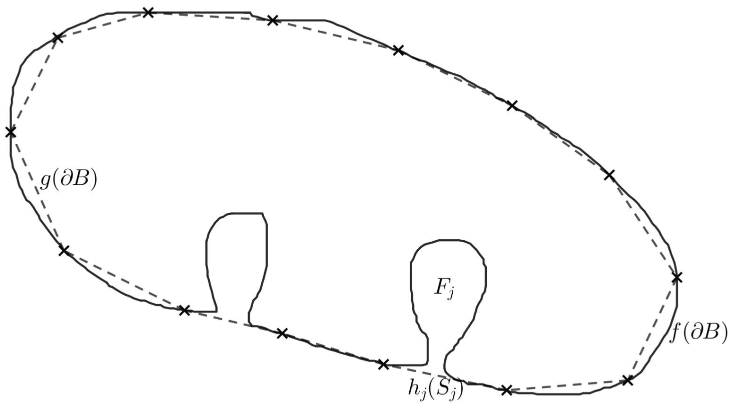

FIGURE 3. 2D representation of the sets F_j. T_j corresponds to points on $f(\partial K)$ (of course in $\mathbb{R}^n$ they are $(n-2)$-dimensional), h_j is represented by dashed lines connecting these points (of course these are minimizers of $(n-1)$-energy in higher dimensions and not lines) and F_j is created "between" $h_j(S_j)$ and $f(S_j)$.

It follows that up to a subsequence (see e.g. [31, Lemma 2.9])

$$f_m \to f \text{ weakly in } W^{1,n-1}(T_j) \text{ and also uniformly on } T_j. \tag{3.22}$$

By embedding we also have continuity and Hölder estimates similar to (3.20) of f on T_j, in particular (3.18).

If K is a hollowed cuboid, we construct the skeleton of flat and round parts of the boundary combining the methods used for a ball and a cuboid, obtaining sets T_j with the desired property (3.18).

Step 4. Replacing f by g with similar degree: Now we consider the shapes together. For each j we define h_j on S_j such that h_j minimizes coordinate-wise the tangential $(n-1)$-Dirichlet integral among functions with boundary data f on T_j (see Theorem 2.11). We define $h_j = f$ on $\partial K \setminus S_j$. Also we define the function g on ∂K as $g = h_j$ on each $\overline{S}_j$. Set (see Fig. 3)

$$\begin{aligned} F &= \{y \in \Omega' \colon \operatorname{Deg}(f, K, y) \neq \deg(g, K, y)\}, \\ F_j &= \{y \in \Omega' \colon \operatorname{Deg}(f, K, y) \neq \operatorname{Deg}(h_j, K, y)\}. \end{aligned}$$

Then

$$y \in \bigcup_j F_j \quad \text{for a.e. } y \in F$$

(this can be viewed e.g. by using (2.2)) and, by (2.5), (3.17), and the minimizing property $\int_{S_j} |D_\tau h_j|^{n-1}\, d\mathcal{H}^{n-1} \leq C \int_{S_j} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-1}$ we have

$$\begin{aligned}\sum_j |F_j| &\leq C \sum_j \Big(\int_{S_j} (|D_\tau f|^{n-1} + |D_\tau h_j|^{n-1})\, d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}}\\ &\leq C \sum_j \Big(\int_{S_j} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}}\\ &\leq C\varepsilon \sum_j \int_{S_j} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-1} \leq CC_2\varepsilon.\end{aligned}$$

Step 5. Concluding the proof for Case (b): Now, we distinguish the cases again. Assume (b). Since $J_f \neq 0$ a.e. we can choose ε small enough so that using Lemma 2.8 we obtain

$$|f^{-1}(F)| \leq \Big| f^{-1}\Big(\bigcup_j F_j\Big)\Big| \leq \kappa. \tag{3.23}$$

We can find $\delta = \delta(\kappa) > 0$ such that there exists a set $Z \subset \Omega$ such that $J_f > 2\delta$ on $\Omega \setminus Z$ and $|Z| < \kappa/2$. Since (up to a subsequence) $J_{f_m} \to J_f$ pointwise a.e., we can find m big enough such that $J_{f_m} > \delta$ on $\Omega \setminus Z'$, where $|Z'| < \kappa$. Then we can pass to a subsequence so that we have $J_{f_m} > 1/\delta$ on $\Omega' \setminus Z'$

$$|f_m^{-1}(F)| \leq |Z'| + |f_m^{-1}(F \setminus f_m(Z'))| \leq \kappa + \Big| f_m^{-1}\Big(\bigcup_j F_j \setminus f_m(Z')\Big)\Big| \leq \kappa + \frac{CC_2\varepsilon}{\delta} \leq 2\kappa \tag{3.24}$$

for all $m \in \mathbb{N}$ when ε is chosen small enough. Fix $m \in \mathbb{N}$ and note that

for every $x \in U \setminus K$ we have $\deg(f_m, K, f_m(x)) = 0$ since f_m is a homeomorphism.

Using the definitions of U (3.16)

$$\begin{aligned} U \setminus K \subset\ & \{\deg(f_m, K, f_m(x)) = 0,\ \mathrm{Deg}(f, K, f(x)) \neq 0\}\\ &\subset \{\deg(f_m, K, f_m(x)) \neq \mathrm{Deg}(f, K, f_m(x))\} \cup \{\mathrm{Deg}(f, K, f_m(x)) \neq \deg(g, K, f_m(x))\}\\ &\cup \{\deg(g, K, f_m(x)) \neq \deg(g, K, f(x))\} \cup \{\mathrm{Deg}(g, K, f(x)) \neq \mathrm{Deg}(f, K, f(x))\}.\end{aligned}$$

We already know by (3.23) and (3.24) that

$$\{\mathrm{Deg}(f, K, f_m(x)) \neq \mathrm{Deg}(g, K, f_m(x))\} \cup \{\mathrm{Deg}(g, K, f(x)) \neq \mathrm{Deg}(f, K, f(x))\} < 3\kappa.$$

Now, since the components of $\mathbb{R}^n \setminus g(\partial K)$ are open and $f_m \to f$ a.e., we can assume that m is so large that

$$|\{\deg(g, K, f_m(x)) \neq \deg(g, K, f(x))\}| < \kappa.$$

Finally, since $f_m \to f$ strongly, for m large enough we have by Lemma 2.3

$$|\{\deg(f_m, K, \cdot) \neq \mathrm{Deg}(f, K, \cdot)\}| < \Phi(\kappa),$$

so that by (2.7)

$$|\{\deg(f_m, K, f_m(\cdot)) \neq \mathrm{Deg}(f, K, f_m(\cdot))\}| < \kappa.$$

Altogether, $|U \setminus K| < 5\kappa$. Since $|U \setminus K| > 0$ we can choose κ small enough, so that we have a contradiction. The case $|V \cap K| > 0$ is done analogously.

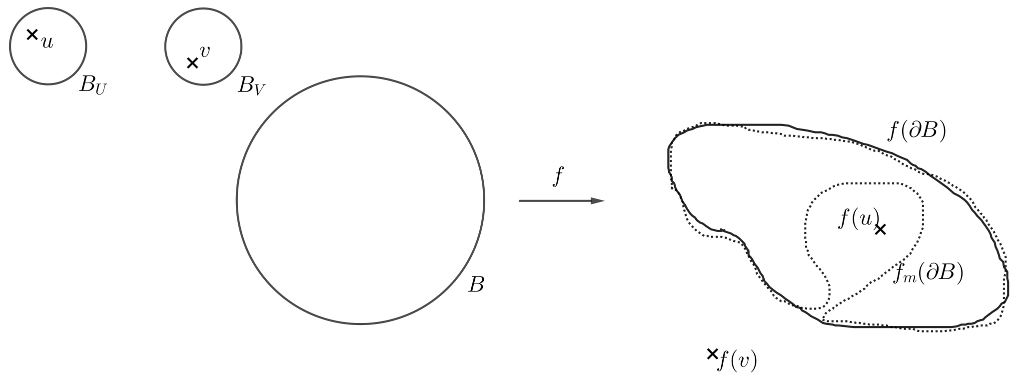

FIGURE 4. Definition of B_U and B_V. Images of most of the points $u \in B_U$ lies inside $f(\partial B)$. On the other hand, for $f_m(\partial B)$ (see dotted line) both points $f(u)$ and $f(v)$ are outside for high enough m and most of the points in B_U and B_V.

Step 6. Finding balls B_U and B_V which are mostly in U and V: From now on, we consider only Case (a). We can use Lemma 3.6 and (3.16) and we can thus assume that both $U \setminus K$ and $V \setminus K$ have positive measure.

Let Π be the orthogonal projection onto the hyperplane $\{x \in \mathbb{R}^n : \ x_1 = 0\}$. Assume that $\kappa \in (0, \frac{1}{6})$ is so small that for each ball B and each measurable set E we have

$$|B \setminus E| < 5\kappa |B| \implies |\Pi(E)| \geq \frac{7}{8} |\Pi(B)|. \tag{3.25}$$

Using Lebesgue density arguments, we find $r > 0$ small enough and balls $B_U = B(x_U, r)$ and $B_V = B(x_V, r)$ such that

$$\begin{aligned} |B_U \setminus U| &\leq \kappa |B_U|, \\ |B_V \setminus V| &\leq \kappa |B_V| \end{aligned} \tag{3.26}$$

and the convex hull of $\overline{B}_U \cup \overline{B}_V$ is contained in $\Omega \setminus K$. We may assume that $x_V - x_U$ is a multiple of $\mathbf{e}_1$ (see Fig. 4).

Choosing ε small enough we can assume using Lemma 2.8 that

$$|f^{-1}(F)| \leq \Big| f^{-1}\Big(\bigcup_j F_j\Big)\Big| \leq \kappa |B_U|. \tag{3.27}$$

Step 7. Replacing f_m by g_m with similar degree: Find a compact set $H \subset \Omega' \setminus g(\partial K)$ such that

$$\Omega' \setminus H < \Phi(\kappa |B_U|). \tag{3.28}$$

For each $m \in \mathbb{N}$ and $j \in \{1, \dots, j_{\max}\}$ let $g_{m,j}$ be defined in S_j as the coordinate-wise minimizer of the $(n-1)$-Dirichlet integral among functions with boundary data f_m on T_j. We define $g_{m,j}$ as f_m on $\partial K \setminus S_j$. We also define g_m on ∂K as $g_{m,j}$ on each $\overline{S_j}$.

Since $f_m \to f = g$ uniformly on T_j by (3.21) (or (3.22)), we have $g_m \to g$ uniformly on ∂K using Theorem 2.11. Hence we find $m \in \mathbb{N}$ such that $g_m(\partial K)$ does not intersect H and

$$\deg(g_m, K, \cdot) = \deg(g, K, \cdot) \quad \text{in } H. \tag{3.29}$$

Also, we require

$$|f_m - f| = |f_m - g| < \varepsilon \quad \text{on all } T_j.$$

Similarly as in Fig. 3 (but using f_m instead of f) we define

$$E = \{y \in \Omega' \colon \deg(f_m, K, y) = 0 \neq \deg(g_m, K, y)\},$$
$$E_j = \{y \in \Omega' \colon \deg(f_m, K, y) = 0 \neq \deg(g_{m,j}, K, y)\}.$$

Then

$$y \in \bigcup_j E_j \quad \text{for a.e. } y \in E.$$

Using (2.5) and the minimizing property $\int_{S_j} |D_\tau g_{m,j}|^{n-1}\, d\mathcal{H}^{n-1} \leq C \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1}$, we obtain

$$|E_j|^{1-\frac{1}{n}} \leq C \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1}.$$

Step 8. Not that many big bubbles where f_m and g_m have different degree: Choose $a > 0$ and set

$$J^+ = \{j \colon \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} > a\},$$
$$J^- = \{j \colon \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} \leq a\}.$$

Hence

$$\begin{aligned} \sum_{j\in J^-} |E_j| &\leq C \sum_{j\in J^-} \Big(\int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}} \\ &\leq C a^{\frac{1}{n-1}} \sum_{j\in J^-} \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} \\ &\leq C a^{\frac{1}{n-1}} \int_{\partial K} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} \leq C_4 a^{\frac{1}{n-1}}, \end{aligned} \tag{3.30}$$

where $C_4 = CC_2$. We fix a such that

$$C_4 a^{\frac{1}{n-1}} \leq \Phi(\kappa |B_U|). \tag{3.31}$$

We set

$$W = f_m^{-1}\Big(\bigcup_{j\in J^-} E_j\Big).$$

and using (2.9), (3.30) and (3.31) we obtain

$$|W| < \kappa |B_U|. \tag{3.32}$$

We have

$$\# J^+ \leq M := \frac{C_2}{a}. \tag{3.33}$$

Step 9. A big part of B_U is mapped into big bubbles, a big part of B_V stays away from them: Now, consider the situation in B_U and B_V. Set

$$X = \{x \in B_U \setminus W \colon \deg(g_m, K, f_m(x)) \neq 0\},$$
$$Y = \{x \in B_V \setminus W \colon \deg(g_m, K, f_m(x)) = 0\}.$$

Using definition of X, definition of U (3.16) and (3.29)

$$\begin{aligned}B_U\setminus X &\subset W\cup(B_U\setminus U)\cup\{x\in B_U\colon \deg(g_m,K,f_m(x))=0,\ \mathrm{Deg}(f,K,f(x))\neq 0\}\\ &\subset W\cup(B_U\setminus U)\cup\{x\in B_U\colon \deg(g,K,f_m(x))=0,\ \mathrm{Deg}(f,K,f(x))\neq 0\}\cup\{f_m(x)\notin H\}\\ &\subset W\cup(B_U\setminus U)\cup\{\deg(g,K,f(x))\neq \mathrm{Deg}(f,K,f(x))\}\cup\\ &\quad\cup\{\deg(g,K,f_m(x))\neq\deg(g,K,f(x))\}\cup\{f_m(x)\notin H\}.\end{aligned}$$

Then by (3.26), (3.32) and (3.27)

$$|W\cup(B_U\setminus U)|<2\kappa|B_U| \text{ and}$$
$$|\{\deg(g,K,f(x))\neq \mathrm{Deg}(f,K,f(x))\}|<\kappa|B_U|.$$

Since the set $\{y\colon \deg(g,K,y)=0\}$ is open and $f_m\to f$ a.e., we can take m so large that

$$|\{\deg(g,K,f_m(x))\neq\deg(g,K,f(x))\}|<\kappa|B_U|.$$

Finally using (3.28) and (2.7) (for f_m since $\int\varphi(J_{f_m}))\leq C_1$) we obtain

$$|\{f_m(x)\notin H\}|\leq\kappa|B_U|$$

and all these inequalities together give us

$$|B_U\setminus X|\leq 5\kappa|B_U|.$$

Similarly using

$$\begin{aligned}B_V\setminus Y &\subset W\cup(B_V\setminus V)\cup\{x\in B_V\colon \deg(g_m,K,f_m(x))\neq 0,\ \mathrm{Deg}(f,K,f(x))=0\}\\ &\subset W\cup(B_V\setminus V)\cup\{\deg(g,K,f(x))\neq\mathrm{Deg}(f,K,f(x))\}\cup\\ &\quad\cup\{\deg(g,K,f_m(x))\neq\deg(g,K,f(x))\}\cup\{f_m(x)\notin H\}.\end{aligned}$$

we obtain

$$|B_V\setminus Y|\leq 5\kappa|B_V|.$$

Step 10. Concluding the proof for Case (a): By (3.25) we have

$$|\Pi(B_U\cap X)|>\frac{7}{8}|\Pi(B_U)|,\quad |\Pi(B_V\cap Y)|>\frac{7}{8}|\Pi(B_V)|,$$

so that

$$|P|>\frac{3}{4}|\Pi(B_V)|, \tag{3.34}$$

where

$$P=\Pi(B_U\cap X)\cap\Pi(B_V\cap Y).$$

Consider the segment parallel to the x_1-axis that connects $x'\in B_U\cap X$ with $x''\in B_V\cap Y$. We have

$$\deg(g_m,K,f_m(x'))\neq\deg(g_m,K,f_m(x''))=0.$$

Since x'', $x'\notin W=f_m^{-1}(\bigcup_{j\in J^-}E_j)$ there exists $j\in J^+$ such that (see Fig. 5)

$$\deg(g_{m,j},K,f_m(x'))\neq\deg(g_{m,j},K,f_m(x''))).$$

Hence there exists x between x'' and x' such that $f_m(x)\in\partial E_j$ (see Fig. 5). Since $\partial E_j\subset f_m(\overline{S}_j)\cup g_m(\overline{S}_j)$ and $f_m(x)\notin f_m(\partial K)$ as $x\in\Omega\setminus\overline{K}$ and f_m is a homeomorphism, it follows that $f_m(x)\in g_m(\overline{S}_j)$. Using (3.34) and (3.33) we can fix $j_0\in J^+$ such that

$$\Big|\Pi\big(\{x\in\Omega\setminus K\colon f_m(x)\in g_m(\overline{S}_{j_0})\}\big)\Big|\geq\frac{3}{4M}|\Pi(B_U)|.$$

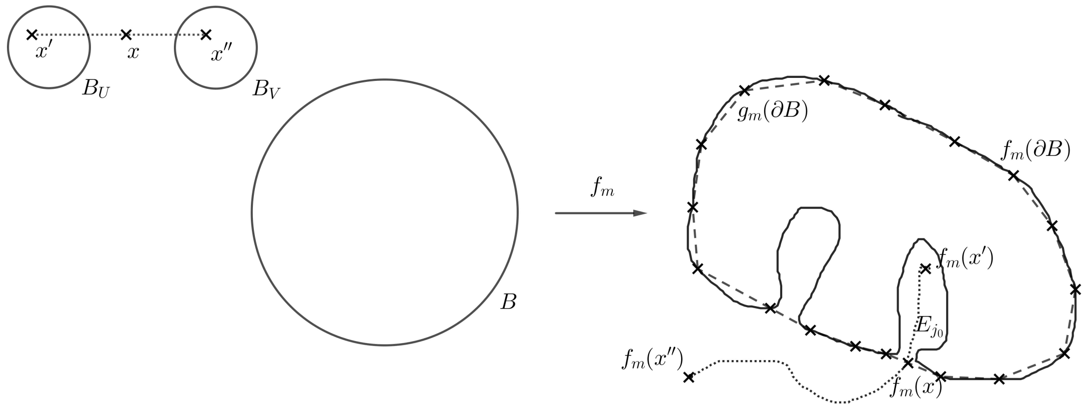

FIGURE 5. 2D representation of the segment $[x', x'']$ and its image (see dotted curves). $f_m(\partial K)$ is a full curve, T_j corresponds to points on $f_m(\partial K)$, g_m is represented by dashed lines connecting these points.

Note that $\operatorname{diam} g_m(\overline{S}_{j_0}) < C\varepsilon$ by (3.18) and Theorem 2.11. Now, choose $\beta > 0$ and use $\varepsilon > 0$ so small that the $W_0^{1,n}(\Omega')$-capacity of $g_m(\overline{S}_{j_0})$ in Ω' is smaller than β^n. It follows that we can find smooth $u \in W_0^{1,n}(\Omega')$ such that u has compact support, $u \equiv 1$ on $g_m(\overline{S}_{j_0})$ and

$$\int_{\Omega'} |Du|^n \, dy \leq \beta^n.$$

It is clear that for each $a \in \Pi\big(\{x \in \Omega \setminus K \colon f_m(x) \in g_m(\overline{S}_{j_0})\}\big)$, where f_m is absolutely continuous on the segment $\Pi^{-1}(a) \cap \Omega$, we have

$$\int_{\Pi^{-1}(a)\cap\Omega} |Du \circ f_m| \geq 1$$

since the function is changing value from 0 to 1. Therefore, by (2.11),

$$\frac{3}{4M}|\Pi(B_U)| \leq \int_\Omega |D(u \circ f_m)| \, dx \leq \|Du\|_{L^n(\Omega')} \|K_{f_m}^{\frac{1}{n-1}}\|_{L^1(\Omega)}^{\frac{n-1}{n}} \leq \beta C_1^{\frac{n-1}{n}}. \tag{3.35}$$

Given $\beta > 0$, in the course of the construction we derive ε, then ρ and m. On the other hand, B_U, κ, a, M and thus all the left hand side of (3.35) do not depend on β. Thus, by a suitable choice of β we obtain a contradiction. □

4. Counterexample - sharpness of the condition $\frac{1}{J_f^2} \in L^1$

We use the notation $A \lesssim B$ for $A \leq C \cdot B$, where C is a positive constant which may depend on the dimension n and exponents a and p, but not on ε nor any of the variables. By $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

We first recall some elementary inequalities that we use often in this section. For every $y \in [0,1]$ and $p \in (\frac{1}{2}, 1)$ we have

$$1 - y^p \leq 1 - y$$

and since the function y^p is concave and its derivative is p at 1

$$y^p \leq 1 + p(y-1).$$

Therefore for every $p \in (\frac{1}{2}, 1)$ we have

$$1 - y^p \approx 1 - y \text{ for every } y \in [0,1]. \tag{4.1}$$

We also use the fact that

(4.2)

$$\sin(\alpha) \approx \alpha \text{ on } [0,\pi/2], \sin(\alpha) \approx \alpha(\pi-\alpha) \text{ on } [0,\pi] \text{ and } \cos(\pi/2-\alpha) \approx \alpha \text{ on } [0,\pi/2].$$

Note that for $\alpha \in (0,\pi)$ we have the following elementary estimate

$$\frac{1}{\sin\alpha} \lesssim \frac{1}{\alpha(\pi-\alpha)} = \frac{1}{\pi}\left(\frac{1}{\alpha} + \frac{1}{\pi-\alpha}\right). \tag{4.3}$$

Proof of Theorem 1.2. **Step 1. Geometrical explanation:** We fix a parameter $\varepsilon > 0$ small enough, we construct a homeomorphism f_ε and later we choose f_m as f_ε for $\varepsilon = 1/m$. We define the mapping from spherical coordinates (r,α,β) to spherical coordinates. We first define it on $B(0,2)$, i.e. for $r \in (0,2)$, $\alpha \in (0,\pi)$ and $\beta \in (-\pi,\pi)$. Then we extend it to $B(0,10) \setminus B(0,2)$ so that $f(x,y,z) = (x,y,-z)$ on $\partial B(0,10)$ and then we compose it with a proper reflection. The mapping has the form

$$f_\varepsilon((r,\alpha,\beta)) = (\tilde{r}(r,\alpha,\varepsilon), \tilde{\alpha}(r,\alpha,\varepsilon), \beta),$$

i.e. it is enough to define it in the xz-plane and then rotate the picture around the z-axis both in the domain and in the target.

To improve the readability we first give the informal idea about the behaviour of the mapping using pictures and later we give exact formulas. In Figure 6 we show the behaviour of f_ε for $\varepsilon = 1/m$ on different spheres in the xz-plane. The outer sphere $\partial B(0,2)$ is mapped onto some drop-shape with $[0,0,0]$ at the very top and this shape is actually the same for all $\varepsilon > 0$. The behaviour on spheres inside is described for spheres of radius $\frac{1}{2}$ and $\frac{3}{2}$. Each sphere $\partial B(0,r)$ inside is divided into two parts - the inner part I_r denoted in a dotted curve and the outer part O_r denoted by a full curve. The boundary between these two regions W is denoted by the thin blue dashed curve and is very important for the behaviour of our map. The image $f_m(O_r)$ is some outer half-drop (denoted by a full curve on the right part of the picture) and the image $f_m(I_r)$ is some inner half-drop (denoted by a dotted curve) so that the image $f_m(B(0,r))$ looks like a "horseshoe". These horseshoes are nested, i.e. $f_m(B(0,r_1)) \subset f_m(B(0,r_2))$ for $r_1 < r_2$, so that the whole map f_m could be a homeomorphism. Let us describe what happens for $\varepsilon \to 0+$, that is, $m \to \infty$. The tips of all horseshoes (the upper two parts) are approaching the point $[0,0,0]$ on the very top. At the same time W (boundary between inner and outer parts of spheres) is changing drastically but only on $B(0,1)$. The small "pie" on the bottom has very small angle which disappears as $\varepsilon \to 0+$ so in the limit there are no outer parts O_r for $r < 1$. It is actually possible to do so with bounded $W^{1,2}$ energy - on each $\partial B(0,r)$, $0 < r < 1$, we map something like 2D ball or radius δ (in fact a small spherical cap) to something like 2D ball of radius 1 with energy $\int_{B^2(0,\delta)} |Dh|^2 \approx \mathcal{H}^2(B^2(0,\delta))|\frac{1}{\delta}|^2 \approx 1$.

The behaviour of limit mapping is depicted in Figure 7. We will show that f_m forms a bounded sequence in $W^{1,2}$ (and also that $\int \frac{1}{J_f^a}$ is bounded) so there is a subsequence which converges weakly to the pointwise limit f. All "horseshoes" $f(B(0,r))$ have two tips that go up to the point $[0,0,0]$. Let us describe in details the behaviour of f on $\overline{B(0,\frac{1}{2})}$ and why the limit fails to satisfy the (INV) condition there. The boundary $\partial B(0,\frac{1}{2})$ has only inner part $I_{\frac{1}{2}}$ and there is no outer part so the image $f(\partial B(0,\frac{1}{2}))$ consists only from the

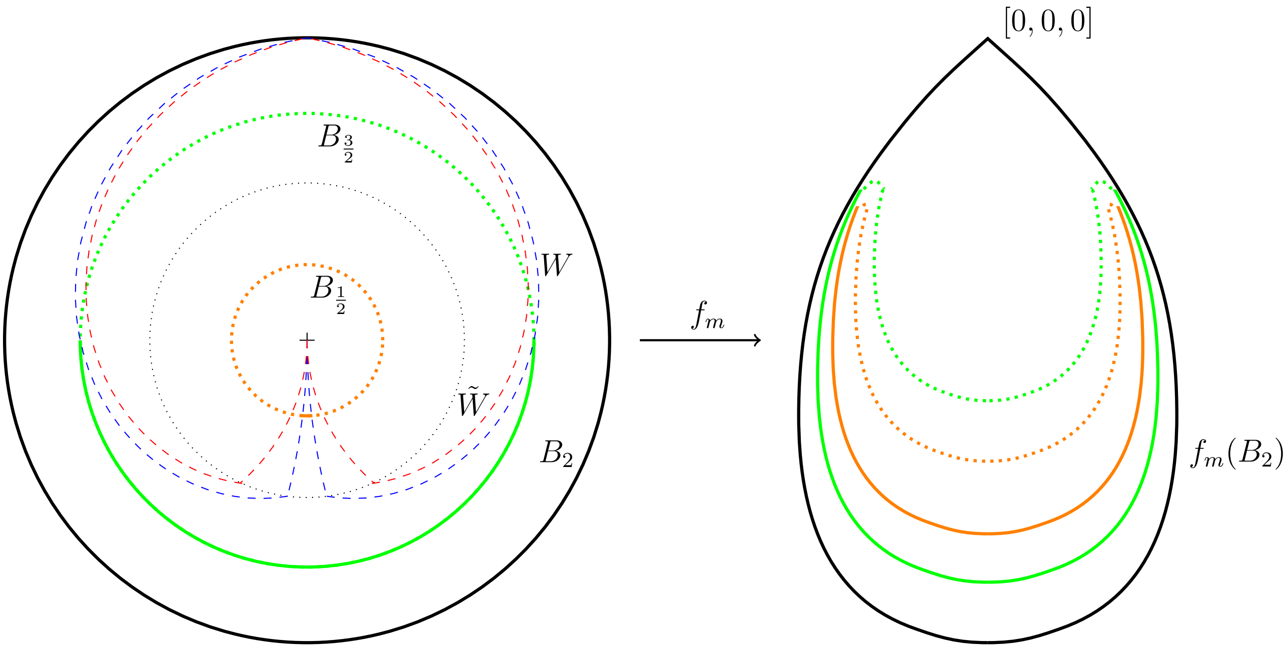

FIGURE 6. Mapping f_m and its behaviour on spheres of radius $\frac{1}{2}$, $\frac{3}{2}$ and 2.

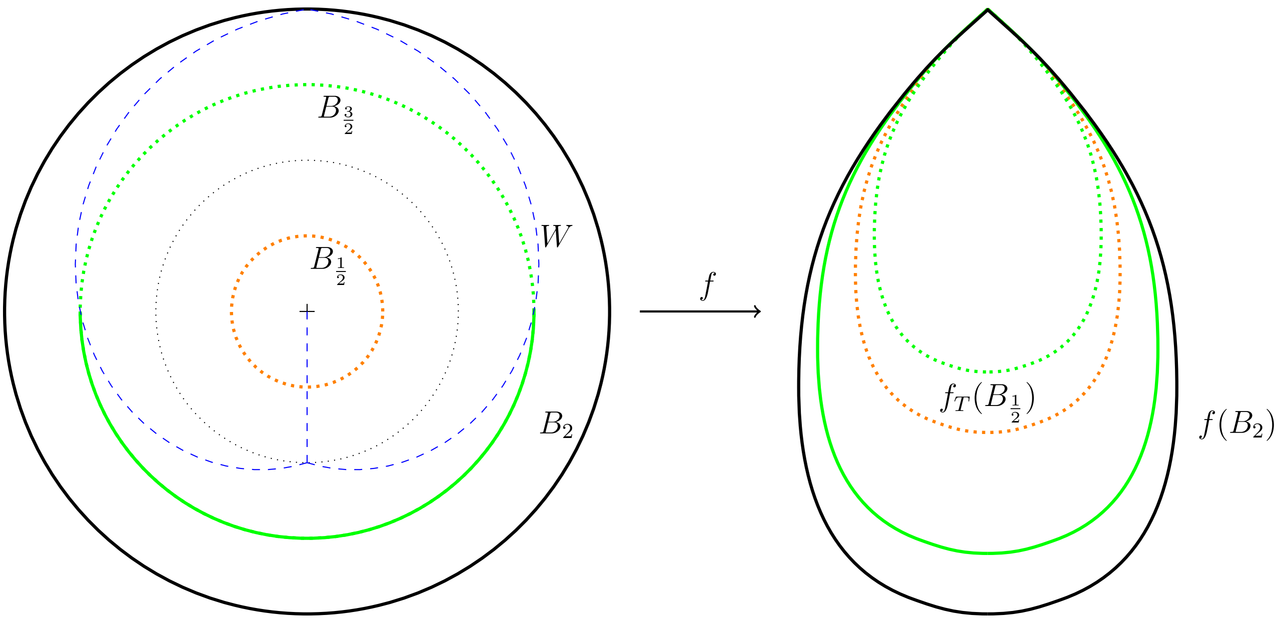

FIGURE 7. Limit mapping f and its behaviour on spheres of radius $\frac{1}{2}$, $\frac{3}{2}$ and 2.

dotted orange drop on the right-hand side of the picture. It follows that $f_T(B(0,\frac{1}{2}))$ is equal to the inner part of this (rotated) drop and it is not difficult to check that the degree actually equals -1 there as we have changed the orientation of the sphere. However for $x \in B(0,\frac{1}{2})$ we know that $f(x)$ does not belong to to $f_T(B(0,\frac{1}{2}))$ as it is mapped outside of this drop, in fact for f_m we had the outer drop $f_m(O_r)$ and $f_m(B(0,r))$ lies between $f_m(O_r)$ and $f_m(I_r)$ so in the limit outside of $f(I_r)$ (which is the limit of $f_m(I_r)$).

Step 2. Formal definitions: We first define the set W between the inner and outer parts of $\partial B(0,r)$ and then we divide $B(0,2)$ into different regions accordingly. We set $r^1_\varepsilon = 1+\frac{\varepsilon}{\pi-\varepsilon}$ and

$$S_\varepsilon = \begin{cases} \pi-\varepsilon r, & 0<r<r^1_\varepsilon, \\ (2-r)\pi, & r^1_\varepsilon<r<2 \end{cases}$$

and our W is defined as (see the blue curve in Fig. 8)

$$W := \Big\{(r,\alpha) : \alpha = S_\varepsilon\Big\}.$$

This formula corresponds to the blue curve on the right half of Fig. 8 while the blue curve on the left side is created by rotation around the z-axes. Note that $r^1_\varepsilon \to 1$ as $\varepsilon\to 0$. Given $a<2$ we fix $p\in\left(\frac{1}{2},1\right)$ such that

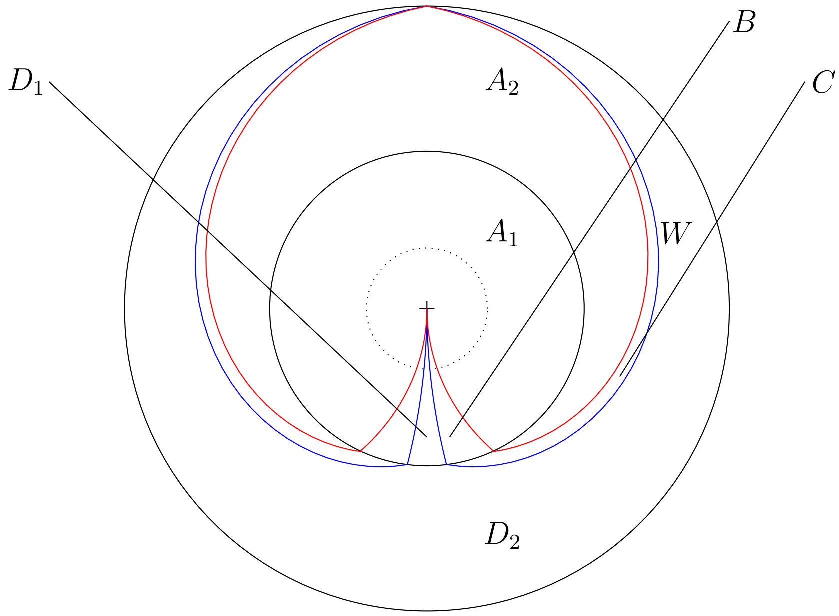

FIGURE 8. Definition of W and different areas.

$$a(1-3p) > -1. \tag{4.4}$$

Now we define the "thickness between the blue curve and the red curve" as

$$\delta(\varepsilon,r) = \begin{cases} \varepsilon^{1/p} r & 0<r<r^1_\varepsilon, \\ c_0\varepsilon^{1/p}(2-r)^\lambda & r^1_\varepsilon<r<2, \end{cases}$$

where

$$\lambda = \frac{2}{1+a-3ap} \geq \frac{2}{1+a-3a/2} = \frac{2}{1-a/2} > 2 \text{ and } c_0 = \frac{\pi}{\pi-\varepsilon}\cdot\frac{(\pi-\varepsilon)^\lambda}{(\pi-2\varepsilon)^\lambda} \approx 1, \tag{4.5}$$

so that δ is continuous at r^1_ε. Finally, we define the red curve on the picture as

$$\tilde W := \Big\{(r,\alpha) : \alpha = \tilde S_\varepsilon\Big\}. \text{ where } \tilde S_\varepsilon = S_\varepsilon - \delta(\varepsilon,r).$$

Now we can define the regions in Fig. 8 as

$$\begin{aligned} A_1 &:= \Big\{(r,\alpha):\ r\in\big(0,r^1_\varepsilon\big)\,,\ \alpha<\tilde S_\varepsilon\Big\}\,,\\ B &:= \Big\{(r,\alpha):\ r\in\big(0,r^1_\varepsilon\big)\,,\ \tilde S_\varepsilon<\alpha<S_\varepsilon\Big\}\,,\\ D_1 &:= \big\{(r,\alpha):\ r\in\big(0,r^1_\varepsilon\big)\,,\ S_\varepsilon<\alpha\big\}\,,\\ A_2 &:= \Big\{(r,\alpha):\ r\in\big(r^1_\varepsilon,2\big)\,,\ \alpha<\tilde S_\varepsilon\Big\}\,,\\ C &:= \Big\{(r,\alpha):\ r\in\big(r^1_\varepsilon,2\big)\,,\ \tilde S_\varepsilon<\alpha<S_\varepsilon\Big\}\ \text{and}\\ D_2 &:= \big\{(r,\alpha):\ r\in\big(r^1_\varepsilon,2\big)\,,\ S_\varepsilon<\alpha\big\}\,. \end{aligned}$$

Note that we always define only the part of the region in the right part of Fig. 8 and the corresponding left-part is created by rotation around the z-axes (or mirroring).

Our mapping $f_\varepsilon : (r,\alpha,\beta)\mapsto(\tilde r,\tilde\alpha,\tilde\beta)$ is defined as

$$\begin{aligned} \tilde r &= R_\varepsilon(r,\alpha)\cos(T_\varepsilon(r,\alpha))\\ \tilde\alpha &= R_\varepsilon(r,\alpha)T_\varepsilon(r,\alpha)\\ \tilde\beta &= \beta, \end{aligned} \tag{4.6}$$

where we define R_ε and T_ε below. Informally speaking, we deform a sphere into a horseshoe with inner and outer part. Were those half-circles, it would be natural to parametrize them in polar coordinates. However, as we work with half-drops, we use another way. Our $R_\varepsilon\in[0,1]$ could be viewed as some "radius of the drop in the image" and $T_\varepsilon\in[0,\frac{\pi}{2}]$ corresponds to some "angle or parametrization of the boundary of the drop", but instead of using $[R_\varepsilon\cos T_\varepsilon, R_\varepsilon\sin T_\varepsilon]$ as in the case of polar coordinates we use $[R_\varepsilon\cos T_\varepsilon, R_\varepsilon T_\varepsilon]$ as it fits us better. We want to keep our formulas as simple as possible: we define the functions piecewise on regions A_1, A_2, B, C, D_1, D_2. We keep R_ε to be the same as our limit mapping on A_1 and D_2 and very close on A_2 and D_1. We use B and C to continuously connect the values on these regions (by a linear convex combination).

We define our $R_\varepsilon\in[0,1]$ as

$$R_\varepsilon=\begin{cases} \frac{2-r}{3}, & \text{on } A_1,\\ \sqrt{\frac{\pi-2\varepsilon}{\pi-\varepsilon}}\cdot\frac{\sqrt{2-r}}{3}, & \text{on } A_2,\\ \frac{2}{3}+\frac{\varepsilon r}{3\pi}, & \text{on } D_1,\\ \frac{1+r}{3}, & \text{on } D_2,\\ \frac{2-r}{3}\cdot\frac{S-\alpha}{\delta(\varepsilon,r)}+\left(\frac{2}{3}+\frac{\varepsilon r}{3\pi}\right)\left(1-\frac{S-\alpha}{\delta(\varepsilon,r)}\right), & \text{on } B,\\ \sqrt{\frac{\pi-2\varepsilon}{\pi-\varepsilon}}\cdot\frac{\sqrt{2-r}}{3}\cdot\frac{S-\alpha}{\delta(\varepsilon,r)}+\frac{1+r}{3}\left(1-\frac{S-\alpha}{\delta(\varepsilon,r)}\right), & \text{on } C, \end{cases}$$

Note that R_ε is continuous and the values on boundaries between regions (like for $r=r^1_\varepsilon$) agree. To define T_ε we need two additional auxiliary functions. The first one $\xi_\varepsilon\in[0,1]$ measures how close we are to the critical strip between the blue line W and the red line $\tilde W$ and is equal to 0 exactly on the strip:

$$\xi_\varepsilon(r,\alpha)=\begin{cases} 1-\frac{\alpha}{\tilde S_\varepsilon}, & \text{on } A_1\cup A_2 \text{ (i.e., on } \alpha<\tilde S_\varepsilon),\\ 0, & \text{on } B\cup C \text{ (i.e., on } \tilde S_\varepsilon\le\alpha\le S_\varepsilon),\\ 1-\frac{\pi-\alpha}{\pi-S_\varepsilon}, & \text{on } D_1\cup D_2 \text{ (i.e., on } S_\varepsilon<\alpha). \end{cases}$$

Let us define $r^0_\varepsilon = \frac{\varepsilon-2\varepsilon^2}{1-\varepsilon^2} \approx \varepsilon$ so that functions $\frac{r}{\varepsilon}$ and $1-(2-r)\varepsilon$ are equal at this point. The second one (recall that $p \in (\frac{1}{2}, 1)$ and $\lambda > 2$ were chosen in (4.4) and (4.5))

$$\psi(\varepsilon, r) = \begin{cases} \frac{r}{\varepsilon} & \text{for } r \in [0, r^0_\varepsilon], \\ 1 - \varepsilon(2-r) & \text{for } r \in [r^0_\varepsilon, r^1_\varepsilon], \\ 1 - \left(\frac{\pi-2\varepsilon}{\pi-\varepsilon}\right)^{1-\lambda p} \varepsilon(2-r)^{\lambda p} & \text{for } r \in [r^1_\varepsilon, 2], \end{cases}$$

is influencing the shape of the "horseshoes" (see Fig. 6). For $\psi(\varepsilon, r) = 1$ the horseshoe is coming up to the point $[0, 0, 0]$ so we want $\lim_{\varepsilon\to 0+} \psi(\varepsilon, r) = 1$, but $\psi(\varepsilon, r) < 1$ (to have injectivity). Moreover, the definition $\psi(\varepsilon, 0) = 0$ and $\psi(\varepsilon, r)$ small for r small ensures that for really small r our horseshoes are small so that f_ε is continuous at the origin.

We set

$$T_\varepsilon = \frac{\pi}{2}(1 - \xi^p_\varepsilon)\psi(\varepsilon, r) \in \Big[0, \frac{\pi}{2}\Big].$$

Note that for ξ_ε close to 0 (i.e. close to blue-red strip) and for $\psi(\varepsilon, r)$ close to 1 we have T_ε close to $\frac{\pi}{2}$ and thus by (4.6) we obtain that $\tilde{r}$ is close to 0, i.e. the image of our point is close to $[0, 0, 0]$. Note that our f_ε is continuous up to the boundary of $\overline{B(0, 2)}$. For simplicity of notation we sometimes omit the subscript ε and we write only R, ξ and T and not R_ε, ξ_ε and T_ε.

Step 3. Continuity and injectivity: It is easy to check that our f_ε is continuous on all regions. Moreover, it is not difficult to check that on boundaries between two regions the values are the same from both sides and hence our f_ε is continuous.

It is also not difficult to check that f_ε restricted to each boundary ∂A_1, ∂A_2, ∂D_1, ∂D_2, ∂B and ∂C is a homeomorphism. For that purpose we will extend R_ε and T_ε on $S(0, 2) \cup \{[0, 0, 0]\}$:

$$\begin{aligned} &R_\varepsilon = 1, \psi(\varepsilon, r) = 1, \xi_\varepsilon(r, \alpha) = \frac{\alpha}{\pi} \text{ on } S(0, 2) \setminus \{[0, 0, 2]\} \\ &R_\varepsilon = 0, T_\varepsilon = 0 \text{ on } \{[0, 0, 2]\} \\ &R_\varepsilon = \frac{2}{3}, T_\varepsilon = 0 \text{ on } \{[0, 0, 0]\}. \end{aligned}$$

There are two points where the extension of ξ is not defined, points $[0, 0, 0]$ and $[0, 0, 2]$. Apart from them we have continuous functions.

Let us now prove the injectivity on the boundaries, firstly in the planar setting. Assume we have $(R^1, T^1) = (R^2, T^2)$, we want to show that the preimages are the same.

1. <u>$R^1 = 0$ or $R^1 = 2/3$:</u> The only possible preimages in those cases are the points $[0, 0, 2]$ or $[0, 0, 0]$, respectively.

2. <u>$R \in (0, 1] \setminus \{2/3\}$:</u> In this case we have R and T determined by the previously used formulas. Also we can uniquely describe the preimage by its polar coordinates $(r, \alpha) \in (0, 2] \times [0, \pi]$.

- $\partial A_1, \partial A_2, \partial D_1, \partial D_2$: Since R is independent of α and injective with respect to r, $R^1 = R^2$ implies $r^1 = r^2$. That gives $\psi^1 = \psi^2 \neq 0$. From that and $T^1 = T^2$ we have $\xi^1 = \xi^2$. Again, for fixed r on each of those domains we have that ξ is injective with respect to α. Together this gives $\alpha^1 = \alpha^2$.
- $\partial B, \partial C$: Here we have that T is independent of α and injective, as ψ is injective with respect to r. So we know that $r^1 = r^2$. Since R for fixed r is a convex

combination of two distinct numbers, it is therefore injective with respect to α and we are done.

Now we address the mapping $(R,T) \mapsto (\tilde r, \tilde\alpha) = (R\cos T, RT)$. We claim that it is injective for $(R,T) \in ((0,1] \times [0,\pi/2]) \cup \{(0,0)\}$. Let us have (R^1,T^1), (R^2,T^2) such that $(\tilde r^1, \tilde\alpha^1) = (\tilde r^2, \tilde\alpha^2)$. If $(\tilde r^1, \tilde\alpha^1) = (0,0)$, we know that $R^1 = R^2 = 0$, $T^1 = T^2 = 0$ and the result follows. If $\tilde\alpha^1 = 0$ and $\tilde r^1$ is positive, it follows that $T^1 = T^2 = 0$ and $R^1 = R^2 = \tilde r^1$. Otherwise since $\cos T$ is decreasing and T is increasing, we have that $\cos T/T : (0,\pi/2] \to [0,\infty)$ is strictly monotone. Since

$$\frac{\cos(T^1)}{T^1} = \frac{\tilde r^1}{\tilde\alpha^1} = \frac{\tilde r^2}{\tilde\alpha^2} = \frac{\cos(T^2)}{T^2},$$

we obtain $T^1 = T^2$, and so $R^1 = R^2$.

When we add the third dimension and rotate, the injectivity does not change and hence our f_ε is a homeomorphism on boundaries of different regions. Below we estimate the integrability of $J_{f_\varepsilon}^{-a}$ and in those estimates we show (as a by-product) that $J_{f_\varepsilon} \neq 0$ in all the regions. By Inverse Mapping Theorem it follows that f_ε is locally a homeomorphism and since it is a homeomorphism on the boundaries we obtain that it is a homeomorphism in each of the regions (see e.g. [30]). Moreover, it is a homeomorphism on $\partial B(0,2)$ and thus a homeomorphism on $\overline{B(0,2)}$.

Step 4. Integrability of $|Df_\varepsilon|^2$ and $J_{f_\varepsilon}^{-a}$ on $A_1 \cup A_2 \cup D_1 \cup D_2$:

Estimate from spherical to spherical coordinates: For mappings from spherical to spherical coordinates that are rotationally symmetric with respect to β, we have

(4.7)

$$\begin{aligned}\int_{B(0,2)} \|Df_\varepsilon\|^2 &= 2\pi \int_0^2 \int_0^\pi \Big[(\partial_r \tilde r)^2 + (\tilde r \partial_r \tilde\alpha)^2 + \Big(\frac{\partial_\alpha \tilde r}{r}\Big)^2 + \Big(\frac{\tilde r \partial_\alpha \tilde\alpha}{r}\Big)^2 + \Big(\frac{\tilde r \sin(\tilde\alpha)}{r\sin\alpha}\Big)^2\Big]\cdot \\ &\qquad\cdot r^2 \sin\alpha \, d\alpha\, dr \\ &\approx \int_0^2 \int_0^\pi \Big[r^2\alpha(\pi-\alpha)\left[(\partial_r(R\cos T))^2 + (R\cos T \partial_r(RT))^2\right] \\ &\quad + \alpha(\pi-\alpha)\left[(\partial_\alpha(R\cos T))^2 + (R\cos T\partial_\alpha(RT))^2\right] + \frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\Big]\, d\alpha\, dr\end{aligned}$$

and

$$\begin{aligned}\int_{B(0,2)} |J_{f_\varepsilon}|^{-a} &= 2\pi \int_0^2 \int_0^\pi |\partial_r \tilde r \cdot \partial_\alpha \tilde\alpha - \partial_r \tilde\alpha \cdot \partial_\alpha \tilde r|^{-a}\, |\tilde r^2 \sin(\tilde\alpha)|^{-a} |r^2 \sin\alpha|^{1+a}\, d\alpha\, dr \\ &= 2\pi \int_0^2 \int_0^\pi |\partial_r R \cdot \partial_\alpha T - \partial_r T \cdot \partial_\alpha R|^{-a}\, R^{-a} |\cos T + T\sin T|^{-a} \cdot \\ &\qquad \cdot |R^2 (\cos T)^2 \sin(RT)|^{-a} |r^2\sin\alpha|^{1+a}\, d\alpha\, dr \qquad (4.8)\\ &\approx \int_0^2\int_0^\pi |\partial_r R\cdot\partial_\alpha T - \partial_r T \cdot \partial_\alpha R|^{-a}\, R^{-3a} |(\cos T)^2 \sin(RT)|^{-a} |r^2 \sin\alpha|^{1+a}\, d\alpha\, dr.\end{aligned}$$

Note that the term $|\cos T + T\sin T|$ is bounded both from below and above for $T \in [0, \frac{\pi}{2}]$ so we can estimate it by a constant.

Estimate on $A_1 \cap \{r > r^0_\varepsilon\}$: On this set we have $0 < \alpha < \pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r$,

$$R_\varepsilon = \frac{2-r}{3} \text{ and } T_\varepsilon = \frac{\pi}{2}\left(1 - \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p\right)(1-(2-r)\varepsilon).$$

Let us first estimate

(4.9)
$$|\partial_\alpha T_\varepsilon| = \frac{\pi}{2} p \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^{p-1} \frac{1}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}(1-(2-r)\varepsilon) \approx (\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r - \alpha)^{p-1}$$

and

$$\begin{aligned}|\partial_r T_\varepsilon| &= \frac{\pi}{2}\left(1 - \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p\right)\varepsilon + \\ &\quad + \frac{\pi}{2} p \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^{p-1} \frac{\alpha}{(\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r)^2}(\varepsilon + \varepsilon^{1/p})(1-(2-r)\varepsilon) \\ &\lesssim \varepsilon(\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r - \alpha)^{p-1}.\end{aligned}$$

Using $\cos(\frac{\pi}{2} - y) \approx y$ we get

$$\begin{aligned}\cos T &= \cos\left[\frac{\pi}{2}\left(1 - \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p\right)(1-(2-r)\varepsilon)\right] \\ &\approx \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p + (2-r)\varepsilon.\end{aligned} \tag{4.10}$$

Now we use $R \approx 1$, the previous line, (4.2) and $RT \leq \pi/2$, (4.1) and $p > \frac{1}{2}$ to estimate

$$\begin{aligned}\frac{(R\cos T \sin(RT))^2}{\alpha(\pi-\alpha)} &\approx \frac{\left(\left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p + (2-r)\varepsilon\right)^2 T^2}{\alpha(\pi - \alpha)} \\ &\lesssim \frac{\left(\left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^{2p} + \varepsilon^2\right)}{(\pi-\alpha)} \frac{\left(1 - \left(1 - \frac{\alpha}{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}\right)^p\right)^2}{\alpha} \\ &\lesssim \frac{(\pi - \alpha - \varepsilon r - \varepsilon^{\frac{1}{p}} r)^{2p} + \varepsilon^2}{\pi - \alpha} \frac{\alpha^2}{\alpha} \leq 1 + \frac{\varepsilon^2}{\pi - \alpha}.\end{aligned}$$

With the help of these estimates, using (4.7) and $p > \frac{1}{2}$ we get that

$$\begin{aligned}\int_{A_1 \cap \{r > r^0_\varepsilon\}} \|Df_\varepsilon\|^2 &\lesssim \int_{r^0_\varepsilon}^{r^1_\varepsilon} \int_0^{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r} \Big[\alpha(\pi-\alpha)\left[|\partial_r R|^2 + |\partial_r T|^2 + |\partial_\alpha T|^2\right] + \\ &\qquad + \frac{(R\cos T \sin(RT))^2}{\alpha(\pi - \alpha)}\Big]\, d\alpha\, dr \\ &\lesssim \int_0^{r^1_\varepsilon} \int_0^{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r} \left(1 + (\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r - \alpha)^{2p-2} + \frac{\varepsilon^2}{(\pi - \alpha)}\right) d\alpha\, dr \\ &\lesssim 1 + \varepsilon^2 \int_0^{r^1_\varepsilon} -\log(\varepsilon r)\, dr \lesssim 1.\end{aligned}$$

It remains to estimate the Jacobian on A_1 using (4.8), $\partial_\alpha R = 0$ and $R \approx 1$

$$\int_{A_1\cap\{r>r_\varepsilon^0\}} |J_{f_\varepsilon}|^{-a} \lesssim \int_{r_\varepsilon^0}^{r_\varepsilon^1}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r} |\partial_\alpha T|^{-a}\left|(\cos T)^2\sin(RT)\right|^{-a}\left|r^2\sin\alpha\right|^{1+a}\, d\alpha\, dr.$$

We estimate using (4.1) and $R \approx 1$

$$\left|\frac{\sin\alpha}{\sin RT}\right|^a \lesssim \left|\frac{\alpha}{1-\big(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\big)^p}\right|^a \lesssim 1.$$

Further using (4.10) we obtain

$$\frac{1}{\cos T} \lesssim \left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^{-p} \approx (\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{-p}.$$

Together with (4.9) these estimates give us

$$\begin{aligned}\int_{A_1\cap\{r>r_\varepsilon^0\}} |J_{f_\varepsilon}|^{-a} &\lesssim \int_{r_\varepsilon^0}^{r_\varepsilon^1}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r} |\partial_\alpha T|^{-a}\left|\cos T\right|^{-2a}\, d\alpha\, dr\\ &\lesssim \int_0^{r_\varepsilon^1}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r} |(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{p-1}|^{-a}\left|(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^p\right|^{-2a}\, d\alpha\, dr\end{aligned}$$

and our choice of p in (4.4) implies that this integral is finite.

Estimate on $A_1\cap\{r<r_\varepsilon^0\}$: On this set we have $0<\alpha<\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r$, $0<r<r_\varepsilon^0\approx\varepsilon$ and

$$R_\varepsilon = \frac{2-r}{3} \text{ and } T_\varepsilon = \frac{\pi}{2}\left(1-\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^p\right)\frac{r}{\varepsilon}.$$

Again we first estimate

$$|\partial_\alpha T_\varepsilon| = \frac{\pi}{2}p\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^{p-1}\frac{1}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\cdot\frac{r}{\varepsilon} \approx \frac{r}{\varepsilon}(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{p-1}$$

and using (4.1)

$$\begin{aligned}|\partial_r T_\varepsilon| &= \frac{\pi}{2}\left(1-\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^p\right)\frac{1}{\varepsilon}+\\ &\quad+\frac{\pi}{2}p\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^{p-1}\frac{\alpha}{(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r)^2}(\varepsilon+\varepsilon^{\frac{1}{p}})\frac{r}{\varepsilon}\\ &\approx \frac{\alpha}{\varepsilon}+\left(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha\right)^{p-1}r\alpha \lesssim \frac{1}{\varepsilon}\left(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha\right)^{p-1}.\end{aligned}\tag{4.11}$$

Using (4.2), $R\approx 1$ and (4.1) we estimate the last term of the derivative

$$\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)} \lesssim \frac{R^2T^2}{\alpha(\pi-\alpha)} \lesssim \frac{\left(1-\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^p\right)^2\frac{r^2}{\varepsilon^2}}{\alpha(\pi-\alpha)} \lesssim \frac{r^2}{\varepsilon^2(\pi-\alpha)}.$$

With the help of these estimates, using (4.7) and $\partial_\alpha R = 0$ we get

$$\int_{A_1\cap\{r<r^0_\varepsilon\}} \|Df_\varepsilon\|^2 \lesssim \int_0^{r^0_\varepsilon}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r} \left[r^2\left[|\partial_r R|^2+|\partial_r T|^2\right]+|\partial_\alpha T|^2+\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\right]\,d\alpha\,dr$$

$$\lesssim \int_0^{r^0_\varepsilon}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\left[1+\frac{r^2(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{2p-2}}{\varepsilon^2}+\frac{r^2}{\varepsilon^2(\pi-\alpha)}\right]\,d\alpha\,dr$$

and the first part of the integral is finite since $p>\frac{1}{2}$ and $r\lesssim\varepsilon$. The second one we can estimate as

$$\frac{1}{\varepsilon^2}\int_0^{r^0_\varepsilon} r^2\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\frac{1}{\pi-\alpha}\,d\alpha\,dr \lesssim \frac{1}{\varepsilon^2}\int_0^{r^0_\varepsilon}\varepsilon^2(-\log(\varepsilon r))\,dr\lesssim 1.$$

It remains to estimate the Jacobian on A_2 using (4.8), $\partial_\alpha R=0$ and $R\approx 1$

$$\int_{A_1\cap\{r<r^0_\varepsilon\}}|J_{f_\varepsilon}|^{-a}\lesssim\int_0^{r_\varepsilon}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}|\partial_\alpha T|^{-a}\big|(\cos T)^2\sin(RT)\big|^{-a}|r^2\sin\alpha|^{1+a}\,d\alpha\,dr.$$

Using (4.11) we obtain

$$|\partial_\alpha T|\gtrsim(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{p-1}\alpha r$$

and using $R\approx 1$ and (4.1) we have

$$\sin(RT)\approx RT\approx T\approx\alpha\frac{r}{\varepsilon}.$$

Moreover, using again $\cos(\frac{\pi}{2}-y)\approx y$ we get

$$\cos T\approx\frac{\pi}{2}-T\approx\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^p+\left(1-\frac{r}{\varepsilon}\right)$$

$$\geq\left(1-\frac{\alpha}{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\right)^p\gtrsim(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^p.$$

Combining these estimates we obtain

$$\int_{A_1\cap\{r<r^0_\varepsilon\}}|J_{f_\varepsilon}|^{-a}\lesssim$$

$$\lesssim\int_0^{r^0_\varepsilon}\int_0^{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}\frac{1}{(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{ap-a}\alpha^a r^a}\cdot\frac{|r^2\sin\alpha|^{1+a}}{(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)^{2ap}}\cdot\frac{\varepsilon^a}{\alpha^a r^a}\,d\alpha\,dr.$$

As before (see (4.4)) the power of $(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r-\alpha)$ is bigger than -1 and this term is integrable. Using $\sin\alpha\leq\alpha$ we obtain that the power of α is $-a-a+1+a=1-a>-1$ and this term is also integrable. The power of r is $-a-a+2+2a$ and the power of ε is positive so the whole integral is bounded.

Estimate on A_2: We have $0<\alpha<\tilde{S}=(2-r)\pi-c_0\varepsilon^{1/p}(2-r)^\lambda$, $1<r^1_\varepsilon<r<2$,

$$R_\varepsilon=\sqrt{\frac{\pi-2\varepsilon}{\pi-\varepsilon}}\cdot\frac{\sqrt{2-r}}{3}\ \text{and}\ T_\varepsilon=\frac{\pi}{2}\Big(1-\Big(1-\frac{\alpha}{\tilde{S}}\Big)^p\Big)\psi,$$

where $\psi=1-\left(\frac{\pi-2\varepsilon}{\pi-\varepsilon}\right)^{1-\lambda p}\varepsilon(2-r)^{\lambda p}$ and $\lambda=\frac{2}{1+a-3ap}>2$.

Since

$$\partial_r\Big(\frac{\alpha}{\tilde{S}}\Big) = \frac{\alpha\left(\pi - \lambda c_0\varepsilon^{1/p}(2-r)^{\lambda-1}\right)}{\tilde{S}^2} \approx \frac{\alpha}{\tilde{S}^2},$$

and $\lambda p - 1 > 0$, we have

$$\begin{aligned}|\partial_r T| &= \frac{\pi}{2}p\Big(1-\frac{\alpha}{\tilde{S}}\Big)^{p-1}\frac{\alpha\left(\pi - c_0\varepsilon^{1/p}(2-r)^{\lambda-1}\right)}{\tilde{S}^2}\psi + \frac{\pi}{2}\Big(1-\Big(1-\frac{\alpha}{\tilde{S}}\Big)^p\Big)\partial_r\psi\\ &\lesssim \Big(\frac{\tilde{S}-\alpha}{\tilde{S}}\Big)^{p-1}\frac{\alpha}{\tilde{S}^2} + 1 \lesssim \Big(\frac{\tilde{S}-\alpha}{\tilde{S}}\Big)^{p-1}\frac{1}{\tilde{S}} + 1\end{aligned}$$

and

$$|\partial_\alpha T| = \frac{\pi}{2}\left|-p\left(1-\frac{\alpha}{\tilde{S}}\right)^{p-1}\frac{1}{\tilde{S}}\psi\right| \approx \left(\frac{\tilde{S}-\alpha}{\tilde{S}}\right)^{p-1}\frac{1}{\tilde{S}}.$$

Using (4.1) we know that

$$T \approx 1 - \left(1-\frac{\alpha}{\tilde{S}}\right)^p \approx \frac{\alpha}{\tilde{S}}$$

and

$$\frac{\pi}{2} - T \approx (1-\psi) + \left(1-\frac{\alpha}{\tilde{S}}\right)^p\psi \approx (1-\psi) + \left(1-\frac{\alpha}{\tilde{S}}\right)^p, \tag{4.12}$$

so together with (4.3), $R \le 1$ and $\alpha < \tilde{S} < \tilde{S} + c_0\varepsilon^{1/p}(2-r)^\lambda \le \pi$ we get

$$\begin{aligned}\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)} &\lesssim \frac{T^2(\frac{\pi}{2}-T)^2}{\alpha(\pi-\alpha)} \approx \frac{\frac{\alpha^2}{\tilde{S}^2}\Big((1-\psi)+\Big(1-\frac{\alpha}{\tilde{S}}\Big)^p\Big)^2}{\alpha(\pi-\alpha)}\\ &\lesssim \frac{\alpha^2}{\tilde{S}^2\alpha} + \frac{(1-\psi)^2 + \Big(\frac{\tilde{S}-\alpha}{\tilde{S}}\Big)^{2p}}{\pi-\alpha} \lesssim \frac{1}{\tilde{S}} + \frac{\varepsilon^2(2-r)^{2\lambda p}}{\pi-\alpha} + \frac{(\tilde{S}-\alpha)^{2p}}{\tilde{S}^{2p}(\pi-\alpha)}\\ &\lesssim \frac{1}{\tilde{S}} + \frac{\varepsilon^2(2-r)^{2\lambda p}}{\varepsilon^{1/p}(2-r)^\lambda} + \frac{(\tilde{S}-\alpha)^{2p}}{\tilde{S}^{2p}(\tilde{S}-\alpha)} \lesssim \frac{1}{\tilde{S}}.\end{aligned}$$

Therefore using (4.7), $\partial_\alpha R = 0$, $\alpha < \tilde{S} < S \approx (2-r)$ and $p > 1/2$ gives

$$\begin{aligned}\int_{A_2}\|Df_\varepsilon\|^2 &\lesssim \int_{r_\varepsilon^1}^2\int_0^{\tilde{S}}\left[\alpha(\pi-\alpha)\left[(\partial_r R)^2 + (\partial_r T)^2 + (\partial_\alpha T)^2\right] + \frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\right]d\alpha\, dr\\ &\lesssim \int_1^2\int_0^{\tilde{S}}\left[\alpha(\pi-\alpha)\left[\frac{1}{2-r} + \frac{(\tilde{S}-\alpha)^{2p-2}}{\tilde{S}^{2p}} + 1 + \frac{(\tilde{S}-\alpha)^{2p-2}}{\tilde{S}^{2p}}\right] + \frac{1}{\tilde{S}}\right]d\alpha\, dr\\ &\lesssim \int_1^2\left[1 + \frac{\tilde{S}}{S} + \int_0^{\tilde{S}}\alpha\frac{(\tilde{S}-\alpha)^{2p-2}}{\tilde{S}^{2p}}d\alpha\right]dr \lesssim \int_1^2\left[1 + \tilde{S}\frac{\tilde{S}^{2p-1}}{\tilde{S}^{2p}}\right]dr \lesssim 1.\end{aligned}$$

Considering the Jacobian estimate, due to the fact that $\partial_\alpha R = 0$ we can rewrite (4.8) as

$$\int_{A_2}|J_{f_\varepsilon}|^{-a} \approx \int_{r_\varepsilon^1}^2\int_0^{\tilde{S}}|\partial_r R\cdot\partial_\alpha T|^{-a}R^{-3a}|(\cos T)^2\sin(RT)|^{-a}|r^2\sin\alpha|^{1+a}d\alpha dr.$$

We estimate (using again (4.2) and (4.1))

$$\sin(RT) \approx RT \approx \sqrt{2-r}\Big(1-\Big(1-\frac{\alpha}{\tilde{S}}\Big)^p\Big) \approx \frac{\alpha}{\tilde{S}}\sqrt{2-r}$$

and from (4.12)

$$\cos(T) \approx \frac{\pi}{2} - T \approx \Big(1-\frac{\alpha}{\tilde{S}}\Big)^p + (1-\psi) \gtrsim \Big(1-\frac{\alpha}{\tilde{S}}\Big)^p.$$

Together this gives

$$\begin{aligned}
&\int_{A_2} |J_{f_\varepsilon}|^{-a} \lesssim \\
&\lesssim \int_{r_\varepsilon^1}^2 \int_0^{\tilde{S}} \Big|\frac{1}{\sqrt{2-r}}\Big(\frac{(\tilde{S}-\alpha)^{p-1}}{\tilde{S}^p}\Big)\Big|^{-a} \frac{1}{\sqrt{2-r}^{3a}}\Big(1-\frac{\alpha}{\tilde{S}}\Big)^{-2ap}\Big(\frac{\alpha}{\tilde{S}}\sqrt{2-r}\Big)^{-a}\alpha^{1+a} d\alpha\, dr \\
&\lesssim \int_1^2 \int_0^{\tilde{S}} (\tilde{S}-\alpha)^{a-3ap}\tilde{S}^{3ap+a}(2-r)^{\frac{-3a}{2}} d\alpha\, dr \\
&= \int_1^2 \tilde{S}^{3ap+a}(2-r)^{\frac{-3a}{2}} \int_0^{\tilde{S}} (\tilde{S}-\alpha)^{a-3ap} d\alpha\, dr \lesssim 1,
\end{aligned}$$

since $a-3ap > -1$ and $\tilde{S} < S \approx (2-r)$.

Estimate on $D_1 \cap \{r > r_\varepsilon^0\}$: On this set we have $S = \pi - \varepsilon r < \alpha < \pi$ and

$$R_\varepsilon = \frac{2}{3} + \frac{\varepsilon r}{3\pi} \text{ and } T_\varepsilon = \frac{\pi}{2}\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)(1-(2-r)\varepsilon).$$

Let us first estimate

$$|\partial_\alpha T_\varepsilon| = \frac{\pi}{2}p\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^{p-1}\frac{1}{\varepsilon r}(1-(2-r)\varepsilon) \approx (\alpha-\pi+\varepsilon r)^{p-1}\frac{1}{\varepsilon^p r^p} \tag{4.13}$$

and using $\pi - \alpha < \varepsilon r$

$$\begin{aligned}
|\partial_r T_\varepsilon| &= \left|\frac{\pi}{2}\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)\varepsilon - \frac{\pi}{2}p\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^{p-1}\frac{\pi-\alpha}{\varepsilon r^2}(1-(2-r)\varepsilon)\right| \\
&\lesssim \frac{\varepsilon^{1-p}(\alpha-\pi+\varepsilon r)^{p-1}}{r^p}.
\end{aligned}$$

Now using $\sin RT \le RT$, (4.1) and $\pi - \alpha < \varepsilon r$ we have

$$\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)} \lesssim \frac{R^2T^2}{\alpha(\pi-\alpha)} \lesssim \frac{\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)^2}{\alpha(\pi-\alpha)} \lesssim \frac{\frac{(\pi-\alpha)^2}{\varepsilon^2 r^2}}{\alpha(\pi-\alpha)} \le \frac{1}{\alpha\varepsilon r} \lesssim \frac{1}{\varepsilon r}.$$

With the help of these estimates we use (4.7), $\pi-\alpha<\varepsilon r$, $p>\frac{1}{2}$ and elementary integration to obtain

$$\begin{aligned}\int_{D_1\cap\{r>r_\varepsilon^0\}}\|Df_\varepsilon\|^2 &\lesssim \int_{r_\varepsilon^0}^{r_\varepsilon^1}\int_{\pi-\varepsilon r}^{\pi}\Big[\alpha(\pi-\alpha)\left[|\partial_r R|^2+|\partial_r T|^2+|\partial_\alpha T|^2\right]+\\ &\qquad\qquad+\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\Big]\,d\alpha\,dr\\ &\lesssim \int_0^{r_\varepsilon^1}\int_{\pi-\varepsilon r}^{\pi}\left[(\pi-\alpha)(\alpha-\pi+\varepsilon r)^{2p-2}\frac{1}{\varepsilon^{2p}r^{2p}}+\frac{1}{\varepsilon r}\right]\,d\alpha\,dr\\ &\lesssim \int_0^{r_\varepsilon^1}\frac{1}{\varepsilon^{2p-1}r^{2p-1}}\int_{\pi-\varepsilon r}^{\pi}(\alpha-\pi+\varepsilon r)^{2p-2}\,d\alpha\,dr+\int_0^{r_\varepsilon^1}\frac{1}{\varepsilon r}(\varepsilon r)\,dr\\ &\lesssim \int_0^{r_\varepsilon^1}\frac{1}{\varepsilon^{2p-1}r^{2p-1}}(\varepsilon r)^{2(p-1)+1}\,dr+1\approx 1.\end{aligned}$$

Now we estimate the Jacobian on D_1 using (4.8), $\partial_\alpha R=0$ and $R\approx 1$ as

$$\int_{D_1\cap\{r>r_\varepsilon^0\}}|J_{f_\varepsilon}|^{-a}\lesssim\int_0^{r_\varepsilon^1}\int_{\pi-\varepsilon r}^{\pi}|\partial_\alpha T|^{-a}\big|(\cos T)^2\sin(RT)\big|^{-a}|r^2\sin\alpha|^{1+a}\,d\alpha\,dr.$$

Using (4.13) we estimate $|\partial_\alpha T|$, further using (4.1)

$$\sin(RT)\approx T\approx 1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\approx\frac{\pi-\alpha}{\varepsilon r}.$$

As usual we estimate using $\cos(\frac{\pi}{2}-y)\approx y$ that

$$\cos T\approx\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p+(2-r)\varepsilon\gtrsim(\alpha-\pi+\varepsilon r)^p\,\frac{1}{\varepsilon^p r^p}.$$

Combining these estimates with $|\frac{\sin\alpha}{\pi-\alpha}|\le 1$ and $|\sin\alpha|\le\varepsilon r$ we get

$$\begin{aligned}\int_{D_1\cap\{r>r_\varepsilon^0\}}|J_{f_\varepsilon}|^{-a}&\lesssim\int_{r_\varepsilon^0}^{r_\varepsilon^1}\int_{\pi-\varepsilon r}^{\pi}\frac{\varepsilon^{ap}r^{ap}}{(\alpha-\pi+\varepsilon r)^{ap-a}}\frac{\varepsilon^{2ap}r^{2ap}}{(\alpha-\pi+\varepsilon r)^{2ap}}\frac{(\varepsilon r)^a}{(\pi-\alpha)^a}r^{2+2a}|\sin\alpha|^a\varepsilon r\,d\alpha\,dr\\ &\lesssim\varepsilon^{3ap+a+1}\int_0^{r_\varepsilon^1}r^{3ap+a+2+2a}\int_{\pi-\varepsilon r}^{\pi}(\alpha-\pi+\varepsilon r)^{a-3ap}\,d\alpha\,dr\end{aligned}$$

and this integral is bounded using (4.4).

Estimate on $D_1\cap\{r<r_\varepsilon^0\}$: On this set we have $\pi-\varepsilon r<\alpha<\pi$ and $0<r<r_\varepsilon^0\approx\varepsilon$ and

$$R_\varepsilon=\frac{2}{3}+\frac{\varepsilon r}{3\pi}\text{ and }T_\varepsilon=\frac{\pi}{2}\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)\frac{r}{\varepsilon}.$$

Let us first estimate

$$|\partial_\alpha T_\varepsilon|=\frac{\pi}{2}p\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^{p-1}\frac{1}{\varepsilon r}\cdot\frac{r}{\varepsilon}\approx(\alpha-\pi+\varepsilon r)^{p-1}\frac{r^{1-p}}{\varepsilon^{1+p}}.\tag{4.14}$$

and using $\pi-\alpha<\varepsilon r$

$$
\begin{aligned}
|\partial_r T_\varepsilon| &= \left|\frac{\pi}{2}\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)\frac{1}{\varepsilon}-\frac{\pi}{2}p\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^{p-1}\frac{\pi-\alpha}{\varepsilon r^2}\cdot\frac{r}{\varepsilon}\right| \\
&\lesssim \frac{1}{\varepsilon}+(\alpha-\pi+\varepsilon r)^{p-1}\varepsilon^{1-p}r^{1-p}\frac{1}{\varepsilon}=\frac{1}{\varepsilon}+(\alpha-\pi+\varepsilon r)^{p-1}\frac{r^{1-p}}{\varepsilon^p}.
\end{aligned}
$$

Now using (4.1), $\sin RT\leq RT$, $\alpha\approx 1$ and $\pi-\alpha<\varepsilon r$ we have

$$
\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\lesssim\frac{R^2T^2}{\alpha(\pi-\alpha)}\lesssim\frac{\left(1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right)^2\frac{r^2}{\varepsilon^2}}{\alpha(\pi-\alpha)}\lesssim\frac{\frac{(\pi-\alpha)^2}{\varepsilon^2r^2}\cdot\frac{r^2}{\varepsilon^2}}{\alpha(\pi-\alpha)}\lesssim\frac{r}{\varepsilon^3}.
$$

With the help of these estimates we use (4.7), $\pi-\alpha<\varepsilon r$, $p>\frac{1}{2}$, $r<r_\varepsilon^0\approx\varepsilon$ and elementary integration to obtain

$$
\begin{aligned}
\int_{D_1\cap\{r<r_\varepsilon^0\}}\|Df_\varepsilon\|^2 &\lesssim \int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r}^{\pi}\Big[\alpha(\pi-\alpha)\left[|\partial_r R|^2+|\partial_r T|^2+|\partial_\alpha T|^2\right]+ \\
&\qquad\qquad +\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\Big]\,d\alpha\,dra \\
&\lesssim \int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r}^{\pi}\left[1+\frac{\pi-\alpha}{\varepsilon^2}+(\pi-\alpha)(\alpha-\pi+\varepsilon r)^{2p-2}\frac{r^{2-2p}}{\varepsilon^{2+2p}}+\frac{r}{\varepsilon^3}\right]\,d\alpha\,dr \\
&\lesssim 1+\int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r}^{\pi}\left[\frac{\varepsilon r}{\varepsilon^2}+(\alpha-\pi+\varepsilon r)^{2p-2}\frac{r^{3-2p}}{\varepsilon^{1+2p}}\right]\,d\alpha\,dr+\int_0^{r_\varepsilon^0}\frac{r}{\varepsilon^3}(\varepsilon r)\,dr \\
&\lesssim 1+\int_0^{r_\varepsilon^0}\left[\frac{\varepsilon^2r^2}{\varepsilon^2}+\frac{r^{3-2p}}{\varepsilon^{1+2p}}(\varepsilon r)^{2p-2+1}\right]\,dr+1\approx 1.
\end{aligned}
$$

Now we estimate the Jacobian on D_1 using (4.8), $\partial_\alpha R=0$ and $R\approx 1$ as

$$
\int_{D_1\cap\{r<r_\varepsilon^0\}}|J_{f_\varepsilon}|^{-a}\lesssim\int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r}^{\pi}|\partial_\alpha T|^{-a}\left|(\cos T)^2\sin(RT)\right|^{-a}|r^2\sin\alpha|^{1+a}\,d\alpha\,dr.
$$

Using (4.14) we estimate $|\partial_\alpha T|$, further using (4.1) we get

$$
\sin(RT)\approx T\approx\left[1-\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p\right]\frac{r}{\varepsilon}\approx\frac{\pi-\alpha}{\varepsilon r}\cdot\frac{r}{\varepsilon}=\frac{\pi-\alpha}{\varepsilon^2}.
$$

As usual we estimate using $\cos(\frac{\pi}{2}-y)\approx y$ that

$$
\cos T\approx\left(1-\frac{\pi-\alpha}{\varepsilon r}\right)^p+\left(1-\frac{\varepsilon}{r}\right)\gtrsim\frac{(\alpha-\pi+\varepsilon r)^p}{\varepsilon^pr^p}.
$$

Combining these estimates with $|\frac{\sin\alpha}{\pi-\alpha}|\leq 1$ and $|\sin\alpha|\leq\varepsilon r$ we get

$$
\begin{aligned}
\int_{D_1\cap\{r<r_\varepsilon^0\}}|J_{f_\varepsilon}|^{-a} &\lesssim \int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r}^{\pi}\frac{\varepsilon^{ap+a}r^{ap-a}}{(\alpha-\pi-\varepsilon r)^{ap-a}}\frac{\varepsilon^{2ap}r^{2ap}}{(\alpha-\pi-\varepsilon r)^{2ap}}\frac{\varepsilon^{2a}}{(\pi-\alpha)^a}|r^2|^{1+a}|\sin\alpha|^a\varepsilon r\,d\alpha\,dr \\
&\lesssim \varepsilon^{3ap+3a+1}\int_0^{r_\varepsilon^0}r^{3ap-a+2+2a+1}\int_{\pi-\varepsilon r}^{\pi}(\alpha-\pi-\varepsilon r)^{a-3ap}\,d\alpha\,dr
\end{aligned}
$$

and this integral is bounded using (4.4).

Estimate on D_2: We have $S=(2-r)\pi<\alpha<\pi$, $r^1_\varepsilon<r<2$ and

$$R_\varepsilon=\frac{1+r}{3}\text{ and }T_\varepsilon=\frac{\pi}{2}\left(1-\left(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\right)^p\right)\psi,$$

where $\psi=1-\left(\frac{\pi-2\varepsilon}{\pi-\varepsilon}\right)^{1-\lambda p}\varepsilon(2-r)^{\lambda p}$ and $\lambda=\frac{2}{1+a-3ap}>2$.

First notice that similarly as before, since $\pi-\alpha<\pi-(2-r)\pi=(r-1)\pi$,

$$\begin{aligned}|\partial_r T|&=\frac{\pi}{2}\Big|-p\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^{p-1}\frac{\pi(\pi-\alpha)}{(\pi-(2-r)\pi)^2}\psi+\Big(1-\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^p\Big)\partial_r\psi\Big|\\&\lesssim\frac{(\alpha-(2-r)\pi)^{p-1}(\pi-\alpha)}{(r-1)^{p+1}}+1\lesssim\frac{(\alpha-(2-r)\pi)^{p-1}}{(r-1)^p}+1,\\|\partial_\alpha T|&=\frac{\pi}{2}\Big|-p\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^{p-1}\frac{1}{\pi-(2-r)\pi}\psi\Big|\approx\frac{(\alpha-(2-r)\pi)^{p-1}}{(r-1)^p}.\end{aligned}$$

To estimate the following term, we again use (4.2), (4.3) and (4.1) and $0<\alpha-(2-r)\pi<\pi-(2-r)\pi=(r-1)\pi$:

$$\begin{aligned}\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}&\lesssim\frac{T^2(\frac{\pi}{2}-T)^2}{\alpha(\pi-\alpha)}\approx\frac{\Big(1-\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^p\Big)^2\Big((1-\psi)+\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^p\psi\Big)^2}{\alpha(\pi-\alpha)}\\&\approx\frac{\Big(1-\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)\Big)^2}{\pi-\alpha}+\frac{\Big((1-\psi)^2+\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^{2p}\Big)}{\alpha}\\&\lesssim\frac{\pi-\alpha}{(\pi-(2-r)\pi)^2}+\frac{(1-\psi)^2+\Big(\frac{\alpha-(2-r)\pi}{\pi-(2-r)\pi}\Big)^{2p}}{\alpha}\\&\lesssim\frac{1}{r-1}+\frac{\varepsilon^2(2-r)^{2\lambda p}}{2-r}+\frac{(\alpha-(2-r)\pi)^{2p-1}}{(\pi-(2-r)\pi)^{2p}}\lesssim\frac{1}{r-1}.\end{aligned}$$

Now we can use all those estimates to integrate (4.7)

$$\begin{aligned}\int_{D_2}\|Df_\varepsilon\|^2&\lesssim\int_{r^1_\varepsilon}^2\int_{(2-r)\pi}^{\pi}\Big[\alpha(\pi-\alpha)\left[(\partial_r R)^2+(\partial_r T)^2+(\partial_\alpha T)^2\right]+\\&\qquad+\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\Big]d\alpha\,dr\\&\lesssim\int_{r^1_\varepsilon}^2\int_{(2-r)\pi}^{\pi}\left[\alpha(\pi-\alpha)\left[1+\frac{(\alpha-(2-r)\pi)^{2p-2}}{(r-1)^{2p}}\right]+\frac{1}{r-1}\right]d\alpha\,dr\\&\lesssim\int_{r^1_\varepsilon}^2\left[1+\frac{\pi-(2-r)\pi}{r-1}+\int_{(2-r)\pi}^{\pi}\frac{(\alpha-(2-r)\pi)^{2p-2}(\pi-\alpha)}{(r-1)^{2p}}d\alpha\right]dr\\&\lesssim\int_{r^1_\varepsilon}^2\left[1+\frac{(\pi-(2-r)\pi)^{2p}}{(r-1)^{2p}}\right]dr\lesssim 1.\end{aligned}$$

To integrate the Jacobian, we estimate (using again (4.2) and (4.1))

$$\sin(RT)\approx RT\approx\Big(1-\Big(1-\frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^p\Big)\approx\frac{\pi-\alpha}{r-1}$$

and

$$\cos(T) \approx \frac{\pi}{2} - T \gtrsim \Big(1 - \frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^p.$$

Since $\partial_\alpha R = 0$ and $R \approx 1$ we have using (4.8)

$$\begin{aligned}\int_{D_2} |J_{f_\varepsilon}|^{-a} &\approx \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi}^{\pi} |\partial_\alpha T|^{-a} \, |(\cos T)^2 \sin(RT)|^{-a} |r^2 \sin\alpha|^{1+a} \, d\alpha \, dr \\ &\lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi}^{\pi} \left|\frac{(\alpha-(2-r)\pi)^{p-1}}{(r-1)^p}\right|^{-a} \Big(1 - \frac{\pi-\alpha}{\pi-(2-r)\pi}\Big)^{-2ap} \Big(\frac{\pi-\alpha}{r-1}\Big)^{-a} (\pi-\alpha)^{1+a} \, d\alpha \, dr \\ &\lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi}^{\pi} (\alpha-(2-r)\pi)^{a-3ap} (r-1)^{a+3ap} \, d\alpha \, dr \lesssim \int_{r_\varepsilon^1}^2 (\pi-(2-r)\pi)^{1+a-3ap} \, dr \lesssim 1.\end{aligned}$$

Step 5. Integrability of $|Df_\varepsilon|^2$ and $J_{f_\varepsilon}^{-a}$ on $B \cup C$:

Estimate on $B \cap \{r > r_\varepsilon^0\}$: On this set we have $\tilde{S}_\varepsilon = \pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r < \alpha < \pi - \varepsilon r = S_\varepsilon$ and

$$R_\varepsilon = \left(\frac{2}{3} + \frac{\varepsilon r}{3\pi}\right) \frac{\alpha - (\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r)}{\varepsilon^{\frac{1}{p}} r} + \frac{2-r}{3} \cdot \frac{\pi - \varepsilon r - \alpha}{\varepsilon^{\frac{1}{p}} r} \text{ and } T_\varepsilon = \frac{\pi}{2}(1-(2-r)\varepsilon).$$

Let us first estimate

$$|\partial_\alpha R_\varepsilon| = \left(\frac{2}{3} + \frac{\varepsilon r}{3\pi}\right) \frac{1}{\varepsilon^{\frac{1}{p}} r} + \frac{2-r}{3} \cdot \frac{-1}{\varepsilon^{\frac{1}{p}} r} = \frac{\varepsilon^{1-\frac{1}{p}}}{3\pi} + \frac{1}{3\varepsilon^{\frac{1}{p}}} \approx \frac{1}{\varepsilon^{\frac{1}{p}}}. \tag{4.15}$$

and with the help of $\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r < \alpha < \pi - \varepsilon r$

$$\begin{aligned}|\partial_r R_\varepsilon| = \Big| &\frac{\varepsilon}{3\pi} \cdot \frac{\alpha - (\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r)}{\varepsilon^{\frac{1}{p}} r} + \left(\frac{2}{3} + \frac{\varepsilon r}{3\pi}\right) \frac{\pi - \alpha}{\varepsilon^{\frac{1}{p}} r^2} \\ &+ \frac{-1}{3} \cdot \frac{\pi - \varepsilon r - \alpha}{\varepsilon^{\frac{1}{p}} r} + \frac{2-r}{3} \cdot \frac{\alpha - \pi}{\varepsilon^{\frac{1}{p}} r^2}\Big| \lesssim \frac{\varepsilon}{\varepsilon^{\frac{1}{p}} r}.\end{aligned} \tag{4.16}$$

Further we estimate for $\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r < \alpha < \pi - \varepsilon r$

$$\frac{(R \cos T \sin(RT))^2}{\alpha(\pi-\alpha)} \lesssim \frac{(\frac{\pi}{2} - T)^2}{\pi - \alpha} \lesssim \frac{\varepsilon^2}{\varepsilon r}.$$

Now using (4.7) (note that each term with $\partial_\alpha R$ always contains also $\cos T \approx \frac{\pi}{2} - T \approx \varepsilon$) we obtain using $\pi - \alpha \approx \varepsilon r$ and $\frac{1}{2} < p < 1$

$$\begin{aligned}\int_{B \cap \{r > r_\varepsilon^0\}} \|Df_\varepsilon\|^2 &\lesssim \int_{r_\varepsilon^0}^{r_\varepsilon^1} \int_{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}^{\pi - \varepsilon r} \Big[\alpha(\pi-\alpha) \left[|\partial_r R|^2 + |\partial_r T|^2 + |\cos T \cdot \partial_\alpha R|^2\right] + \\ &\qquad + \frac{(R \cos T \sin(RT))^2}{\alpha(\pi-\alpha)}\Big] \, dr \, d\alpha \\ &\lesssim \int_0^{r_\varepsilon^1} (\varepsilon^{\frac{1}{p}} r)(\varepsilon r) \left[\frac{\varepsilon^2}{\varepsilon^{\frac{2}{p}} r^2} + 1 + \frac{\varepsilon^2}{\varepsilon^{\frac{2}{p}}}\right] \, dr + \int_0^{r_\varepsilon^1} \int_{\pi - \varepsilon r - \varepsilon^{\frac{1}{p}} r}^{\pi - \varepsilon r} \frac{\varepsilon}{r} \, dr \, d\alpha \\ &\lesssim 1 + \int_0^{r_\varepsilon^1} \varepsilon^{3 - \frac{1}{p}} \, dr + \int_0^{r_\varepsilon^1} \varepsilon^{\frac{1}{p}} r \cdot \frac{\varepsilon}{r} \, dr \approx 1.\end{aligned}$$

Now we estimate the Jacobian on B using (4.8), $\partial_\alpha T = 0$, $\cos T \approx \frac{\pi}{2} - T \approx \varepsilon$, $\sin RT \approx 1$ and (4.4) as

$$\begin{aligned}\int_{B\cap\{r>r_\varepsilon^0\}} |J_{f_\varepsilon}|^{-a} &\lesssim \int_{r_\varepsilon^0}^{r_\varepsilon^1} \int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r} |\partial_\alpha R\cdot\partial_r T|^{-a}\Big|(\cos T)^2\sin(RT)\Big|^{-a}|r^2\sin\alpha|^{1+a}\,d\alpha\,dr\\ &\lesssim \int_0^{r_\varepsilon^1}\int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r}\Big|\frac{1}{\varepsilon^{\frac{1}{p}}}\,\varepsilon\Big|^{-a}\varepsilon^{-2a}\,d\alpha\,dr = \int_0^{r_\varepsilon^1}(\varepsilon^{\frac{1}{p}}r)\varepsilon^{\frac{a}{p}-a}\varepsilon^{-2a}\,dr\\ &\lesssim \int_0^{r_\varepsilon^1}\varepsilon^{\frac{1+a}{p}-3a}\,dr\lesssim 1.\end{aligned}$$

Estimate on $B\cap\{r<r_\varepsilon^0\}$: On this set we have $\tilde{S}_\varepsilon = \pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r<\alpha<\pi-\varepsilon r = S_\varepsilon$ and

$$R_\varepsilon = \left(\frac{2}{3}+\frac{\varepsilon r}{3\pi}\right)\frac{\alpha-(\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r)}{\varepsilon^{\frac{1}{p}}r}+\frac{2-r}{3}\cdot\frac{\pi-\varepsilon r-\alpha}{\varepsilon^{\frac{1}{p}}r}\text{ and } T_\varepsilon = \frac{\pi}{2}\cdot\frac{r}{\varepsilon}.$$

We can use the same estimates (4.15) and (4.16) as before. Further we estimate for $\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r<\alpha<\pi-\varepsilon r$

$$\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\lesssim\frac{T^2}{\pi-\alpha}\lesssim\frac{\frac{r^2}{\varepsilon^2}}{\varepsilon r}=\frac{r}{\varepsilon^3}.$$

Now using (4.7), $\pi-\alpha\approx\varepsilon r$ and $r_\varepsilon^0\approx\varepsilon$ we obtain

$$\begin{aligned}\int_{B\cap\{r<r_\varepsilon^0\}}\|Df_\varepsilon\|^2 &\lesssim \int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r}\Big[\alpha(\pi-\alpha)\left[|\partial_r R|^2+|\partial_r T|^2+|\partial_\alpha R|^2\right]+\\ &\qquad\qquad+\frac{(R\cos T\sin(RT))^2}{\alpha(\pi-\alpha)}\Big]\,d\alpha\,dr\\ &\lesssim\int_0^{r_\varepsilon^0}(\varepsilon^{\frac{1}{p}}r)(\varepsilon r)\left[\frac{\varepsilon^2}{\varepsilon^{\frac{2}{p}}r^2}+\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon^{\frac{2}{p}}}\right]\,dr+\int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r}\frac{r}{\varepsilon^3}\,d\alpha\,dr\\ &\lesssim 1+\varepsilon^{1-\frac{1}{p}}\int_0^{r_\varepsilon^0}r^2\,dr+\int_0^{r_\varepsilon^0}(\varepsilon^{\frac{1}{p}}r)\frac{r}{\varepsilon^3}\,dr\approx 1.\end{aligned}$$

Now we estimate the Jacobian on B using (4.8), $\partial_\alpha T=0$, $\sin RT\approx T\approx\frac{r}{\varepsilon}$ and

$$\cos T\approx\frac{\pi}{2}-T\approx\frac{\varepsilon-r}{\varepsilon}\text{ and }\varepsilon-r_\varepsilon^0=\varepsilon-\frac{\varepsilon-2\varepsilon^2}{1-\varepsilon^2}\approx\varepsilon^2$$

as

$$\begin{aligned}\int_{B\cap\{r<r_\varepsilon^0\}}|J_{f_\varepsilon}|^{-a}&\lesssim\int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r}|\partial_\alpha R\cdot\partial_r T|^{-a}\Big|(\cos T)^2\sin(RT)\Big|^{-a}|r^2\sin\alpha|^{1+a}\,d\alpha\,dr\\ &\lesssim\int_0^{r_\varepsilon^0}\int_{\pi-\varepsilon r-\varepsilon^{\frac{1}{p}}r}^{\pi-\varepsilon r}\left|\frac{1}{\varepsilon^{\frac{1}{p}}}\cdot\frac{1}{\varepsilon}\right|^{-a}\frac{\varepsilon^{2a}}{(\varepsilon-r)^{2a}}\cdot\frac{\varepsilon^a}{r^a}|r^2|^{1+a}\,d\alpha\,dr\\ &\lesssim\varepsilon^{\frac{a}{p}+4a}\int_0^{r_\varepsilon^0}\frac{1}{(\varepsilon-r)^{2a}}dr\lesssim\varepsilon^{\frac{a}{p}+4a}(\varepsilon-r_\varepsilon^0)^{1-2a}\approx\varepsilon^{\frac{a}{p}+4a}\varepsilon^{2-4a}\lesssim 1.\end{aligned}$$

Estimate on C: We have $(2-r)\pi - \delta(\varepsilon, r) = \tilde{S} < \alpha < S = (2-r)\pi$, $r_\varepsilon^1 < r < 2$,

$$R_\varepsilon = \sqrt{\frac{\pi - 2\varepsilon}{\pi - \varepsilon}} \cdot \frac{\sqrt{2-r}}{3} \cdot \frac{(2-r)\pi - \alpha}{\delta(\varepsilon, r)} + \frac{1+r}{3}\left(\frac{\alpha - (2-r)\pi + \delta(\varepsilon, r)}{\delta(\varepsilon, r)}\right)$$

and

$$T_\varepsilon = \frac{\pi}{2}\psi = \frac{\pi}{2}\Big(1 - \Big(\frac{\pi - 2\varepsilon}{\pi - \varepsilon}\Big)^{1-\lambda p} \varepsilon(2-r)^{\lambda p}\Big),$$

where $\lambda = \frac{2}{1+a-3ap}$ and $\delta(\varepsilon, r) = c_0 \varepsilon^{1/p}(2-r)^\lambda$. Here it is crucial that

$$\lambda p \geq \frac{2}{1+a-3a/2} \cdot \frac{1}{2} \geq 1 \tag{4.17}$$

and

$$\cos T \approx \frac{\pi}{2} - T \approx 1 - \psi \approx \varepsilon(2-r)^{\lambda p} \approx \delta^p. \tag{4.18}$$

We first estimate

$$\partial_r \delta = -\lambda c_0 \varepsilon^{1/p}(2-r)^{\lambda - 1} = \frac{-\lambda}{2-r}\delta,$$

so

$$\partial_r\left(\frac{(2-r)\pi - \alpha}{\delta}\right) = \frac{-\pi\delta + ((2-r)\pi - \alpha)\frac{-\lambda\pi}{(2-r)\pi}\delta}{\delta^2} = -\pi\frac{1 + \lambda\frac{(2-r)\pi - \alpha}{(2-r)\pi}}{\delta}.$$

Then we can use it to estimate

$$|\partial_r R| = \Big|\sqrt{\frac{\pi - 2\varepsilon}{\pi - \varepsilon}} \cdot \frac{1}{6\sqrt{(2-r)}} \cdot \frac{(2-r)\pi - \alpha}{\delta} + \sqrt{\frac{\pi - 2\varepsilon}{\pi - \varepsilon}} \cdot \frac{\sqrt{2-r}}{3}\Big(-\pi\frac{1 + \lambda\frac{(2-r)\pi - \alpha}{(2-r)\pi}}{\delta}\Big)$$
$$+ \frac{\alpha - (2-r)\pi + \delta}{3\delta} + \frac{1+r}{3} \cdot \pi\frac{1 + \lambda\frac{(2-r)\pi - \alpha}{(2-r)\pi}}{\delta}\Big| \lesssim \frac{1}{\sqrt{2-r}} + \frac{1}{\delta} + 1 + \frac{1}{\delta} \lesssim \frac{1}{\sqrt{2-r}} + \frac{1}{\delta}$$

and

$$|\partial_\alpha R| = \frac{-\sqrt{\frac{\pi - 2\varepsilon}{\pi - \varepsilon}}\sqrt{(2-r)} + 1 + r}{3\delta} \approx \frac{1}{\delta}.$$

Since $|\partial_\alpha T| = 0$, using (4.7), (4.18), (4.17) and (4.3) we obtain

$$\int_C \|Df_\varepsilon\|^2 \lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi - \delta}^{(2-r)\pi} \Big[|\cos T \cdot \partial_r R|^2 + |\partial_r T|^2 + |\cos T \cdot \partial_\alpha R|^2 +$$
$$+ \frac{(R\cos T \sin(RT))^2}{\alpha(\pi - \alpha)}\Big] d\alpha\, dr$$
$$\lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi - \delta}^{(2-r)\pi} \left[\left(\frac{1}{2-r} + \frac{1}{\delta^2}\right)\delta^{2p} + \varepsilon^2(2-r)^{2\lambda p - 2} + \frac{\delta^{2p}}{\delta^2} + \frac{1}{\alpha} + \frac{1}{\pi - \alpha}\right] d\alpha\, dr$$
$$\lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi - \delta}^{(2-r)\pi} \left[\frac{\delta^{2p}}{2-r} + \frac{\delta^{2p}}{\delta^2} + 1 + \frac{1}{\alpha} + \frac{1}{\pi - \alpha}\right] d\alpha\, dr$$
$$\approx 1 + \int_{r_\varepsilon^1}^2 \left[\frac{\delta^{2p+1}}{2-r} + \delta^{2p-1} + \log\left(\frac{(2-r)\pi}{(2-r)\pi - \delta}\right) + \log\left(\frac{\pi - (2-r)\pi + \delta}{\pi - (2-r)\pi}\right)\right] dr,$$

now since $\delta \le (2-r)\pi/2$ and $\varepsilon < \varepsilon r < \pi - (2-r)\pi$ (recall that $\pi - \varepsilon r > (2-r)\pi$ for $r > r_\varepsilon^1$)

$$\lesssim 1 + \int_{r_\varepsilon^1}^2 \frac{(2-r)^{2\lambda p+\lambda}}{2-r} + 1 + \log\Bigl(\frac{(2-r)\pi}{(2-r)\pi/2}\Bigr) + \log\Bigl(1+\frac{\delta}{\varepsilon r}\Bigr) dr$$
$$\lesssim 1 + \int_{r_\varepsilon^1}^2 (2-r)^{2\lambda p+\lambda-1} + \log(2) + \log\Bigl(1+\frac{\varepsilon^{1/p}}{\varepsilon}\Bigr) dr \lesssim 1.$$

Considering the Jacobian estimate, due to the fact that $\partial_\alpha T = 0$ we can rewrite (4.8) as

$$\int_C |J_{f_\varepsilon}|^{-a} \approx \int_{(r,\alpha)} |\partial_\alpha R \cdot \partial_r T|^{-a} R^{-3a} |(\cos T)^2 \sin(RT)|^{-a} |r^2 \sin(\alpha)|^{1+a} dr d\alpha.$$

Also we need

$$\sin(RT) \approx RT \approx R = \sqrt{\frac{\pi-2\varepsilon}{\pi-\varepsilon}} \cdot \frac{\sqrt{2-r}}{3} \cdot \frac{(2-r)\pi-\alpha}{\delta} + \frac{1+r}{3}\Bigl(\frac{\alpha-(2-r)\pi+\delta}{\delta}\Bigr)$$
$$\ge C\Bigl[\sqrt{2-r}\cdot\frac{(2-r)\pi-\alpha}{\delta} + \sqrt{2-r}\Bigl(\frac{\alpha-(2-r)\pi+\delta}{\delta}\Bigr)\Bigr] = C\sqrt{2-r}.$$

Together with (4.18), $\delta \approx \varepsilon^{1/p}(2-r)^\lambda$ and (4.4) it yields

$$\int_C |J_{f_\varepsilon}|^{-a} \approx \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi-\delta}^{(2-r)\pi} \Bigl|\frac{\varepsilon(2-r)^{\lambda p-1}}{\delta}\Bigr|^{-a} R^{-4a}\delta^{-2ap}\sin^{1+a}\alpha \, d\alpha \, dr$$
$$\lesssim \int_{r_\varepsilon^1}^2 \int_{(2-r)\pi-\delta}^{(2-r)\pi} \frac{\delta^{a-2ap}}{\varepsilon^a}(2-r)^{a-\lambda ap}\left(\sqrt{2-r}\right)^{-4a} d\alpha \, dr \approx \int_{r_\varepsilon^1}^2 \frac{\delta^{1+a-2ap}}{\varepsilon^a(2-r)^{a+\lambda ap}} \, dr$$
$$\approx \int_{r_\varepsilon^1}^2 \frac{\varepsilon^{(1+a-2ap)/p}(2-r)^{\lambda(1+a-2ap)}}{\varepsilon^a(2-r)^{a+\lambda ap}} \, dr = \int_{r_\varepsilon^1}^2 \varepsilon^{(1+a-3ap)/p}(2-r)^{\lambda(1+a-3ap)-a} \, dr$$
$$\lesssim \int_{r_\varepsilon^1}^2 (2-r)^{2-a} dr \lesssim 1,$$

since $\lambda = \frac{2}{1+a-3ap}$.

Step 6. Extension to $\overline{B(0,10)} \setminus B(0,2)$: Our mapping $f_\varepsilon : \overline{B(0,2)} \to \mathbb{R}^3$ is defined on the sphere $\partial B(0,2)$ in polar coordinates as

$$f(2,\alpha,\beta) = (\cos T, T, \beta)$$

where

$$T = T(\varepsilon,\alpha) = \frac{\pi}{2}\left(1-\left(\frac{\alpha}{\pi}\right)^p\right).$$

We define it on $B(0,10) \setminus B(0,2)$ as a simple interpolation between $(r, \pi-\alpha, \beta)$ on $\partial B(0,10)$ and this mapping on $\partial B(0,2)$, i.e. for $r \in [2,10]$ we set

$$f_\varepsilon(r,\alpha,\beta) = \left(\frac{r-2}{8}10 + \frac{10-r}{8}\cos T, \frac{r-2}{8}(\pi-\alpha) + \frac{10-r}{8}T, \beta\right).$$

Note that this is independent of ε and that the mapping $(10, \pi-\alpha, \beta)$ on $\partial B(0,10)$ is actually the identity mapping up to reflection $(x,y,z) \to (x,y,-z)$, so after we compose it with this reflection as mentioned in Step 1 we obtain the identity on the boundary. Our mapping f_ε is a homeomorphism on $\partial B(0,10)$ and $\partial B(0,2)$, so similarly to Step 3 it is enough to show that $J_{f_\varepsilon} \neq 0$ to obtain that it is a homeomorphism on $B(0,10)\setminus B(0,2)$. Using first line of (4.8) it is enough to show that

$$\begin{aligned}
0 \neq & \partial_r\tilde{r}\partial_\alpha\tilde{\alpha} - \partial_\alpha\tilde{r}\partial_r\tilde{\alpha} \\
= & \frac{1}{8^2}\Big[(10-\cos T)\,(2-r+(10-r)\partial_\alpha T) - ((10-r)(-\sin T)\partial_\alpha T)\,(\pi-\alpha-T)\Big] \\
= & \frac{1}{8^2}\Big[(10-\cos T) + (\pi-\alpha-T)\sin T\Big](10-r)\partial_\alpha T + \frac{1}{8^2}(10-\cos T)(2-r).
\end{aligned}$$

Since $\partial_\alpha T < 0$ and $-\partial_\alpha T > \frac{p}{2} \geq \frac{1}{4}$ for $p \in [\frac{1}{2}, 1]$ (and $\alpha \in [0,\pi]$) we obtain

$$-8^2\left(\partial_r\tilde{r}\partial_\alpha\tilde{\alpha} - \partial_\alpha\tilde{r}\partial_r\tilde{\alpha}\right) \geq \Big[(10-1) + (\pi-\pi-\frac{\pi}{2})\Big](10-r)\frac{1}{4} + (10-1)(r-2) \geq 1.$$

To obtain integrability of $J_{f_\varepsilon}^{-a}$ it is thus enough to use first line of (4.8) and show the finiteness of

$$\int_{B(0,10)\setminus B(0,2)} \frac{\left|r^2\sin\alpha\right|^{1+a}}{\left|\tilde{r}\sin\tilde{\alpha}\right|^a}\,dr\,d\alpha\,d\beta.$$

Since

$$\tilde{r} \geq \cos T \geq C\alpha^p$$

and using (4.1)

$$\begin{aligned}
\tilde{\alpha} &= \frac{r-2}{8}(\pi-\alpha) + \frac{10-r}{8}\frac{\pi}{2}\left(1-\left(\frac{\alpha}{\pi}\right)^p\right) \\
&\approx \frac{r-2}{8}(\pi-\alpha) + \frac{10-r}{8}\left(1-\frac{\alpha}{\pi}\right) \approx \pi-\alpha
\end{aligned} \tag{4.19}$$

we obtain the convergence easily as $a < 2$ and $\frac{1}{2} < p < 1$.

It remains to show the finiteness of $\int |Df_\varepsilon|^2$ on $B(0,10)\setminus B(0,2)$ using first line of (4.7). Since $|\partial_r\tilde{r}| \leq C$ and $|\partial_r\tilde{\alpha}| \leq C$ it is easy to estimate first two terms. Further $r \approx 1$ and

$$|\partial_\alpha\tilde{r}| + |\partial_\alpha\tilde{\alpha}| \leq C + C|\partial_\alpha T| \leq C + C\alpha^{p-1}$$

give us boundedness of

$$\int_2^{10}\int_0^\pi \Big[\left(\frac{\partial_\alpha\tilde{r}}{r}\right)^2 + \left(\frac{\tilde{r}\partial_\alpha\tilde{\alpha}}{r}\right)^2\Big] r^2\sin\alpha\,d\alpha\,dr \leq C\int_0^\pi (\alpha)^{2p-2}\alpha\,d\alpha.$$

Using (4.19) it is easy to show convergence of the last term

$$\int_2^{10}\int_0^\pi \left(\frac{\tilde{r}\sin\tilde{\alpha}}{r\sin\alpha}\right)^2 r^2\sin\alpha\,d\alpha\,dr.$$

We thus showed that when we extend f_ε, the energy 1.2 stays uniformly bounded for the whole sequence.

Step 7. Violation of the (INV) condition: Our limit mapping f violates the (INV) condition on $B(0,r)$ for every $r \in (0,1)$. This can be easily seen as the mapping is continuous on $S(0,r)$, so we can consider the classical topological degree. We have

$$R = \frac{2-r}{3} \text{ and } T = \frac{\pi}{2}\left(1-\left(\frac{\pi-\alpha}{\pi}\right)^p\right)$$

on $B(0,1)\setminus\{0\}$. The image of $S(0,r)$ is only the inner drop, so its topological image is the inside of this drop. However, when we take $0 < r_1 < r_2 < 1$, we can see that the smaller sphere is mapped onto a bigger drop, which contains the smaller drop (=

image of the bigger ball). This shows that the material from $S(0,r_2)$ is ejected outside of $\mathrm{im}_T(f,B(0,r_2))$, which itself is enough to break the (INV) condition. Moreover, the material which is mapped into $\mathrm{im}_T(f,B(0,r_2))$ was originally outside of $B(0,r_2)$. □

Proof of Theorem 1.3. The mapping here is the same as mapping f from Theorem 1.2 which is a weak limit of homeomorphism f_m from that statement. However, it does not satisfy the (INV) condition and hence Theorem 3.1 (b) implies that it cannot be obtained as strong limit of homeomorphisms $h_m \in W^{1,2}$. □

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Paper IV

WEAK LIMIT OF HOMEOMORPHISMS IN $W^{1,n-1}$: INVERTIBILITY AND LOWER SEMICONTINUITY OF ENERGY

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Abstract. Let Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and let $f_m \colon \Omega \to \Omega'$ be a sequence of homeomorphisms with positive Jacobians $J_{f_m} > 0$ a.e. and prescribed Dirichlet boundary data. Let all f_m satisfy the Lusin (N) condition and $\sup_m \int_\Omega (|Df_m|^{n-1} + A(|\operatorname{cof} Df_m|) + \varphi(J_f)) < \infty$, where A and φ are positive convex functions. Let f be a weak limit of f_m in $W^{1,n-1}$. Provided certain growth behaviour of A and φ, we show that f satisfies the (INV) condition of Conti and De Lellis, the Lusin (N) condition, and polyconvex energies are lower semicontinuous.

1. Introduction

In this paper, we study classes of mappings that might serve as classes of deformations in Continuum Mechanics models. Let $\Omega \subset \mathbb{R}^n$ be a domain, i.e., a non-empty connected open set, and let $f \colon \Omega \to \mathbb{R}^n$ be a mapping with $J_f > 0$ a.e. Following the pioneering papers of Ball [2] and Ciarlet and Nečas [11] we ask if our mapping is in some sense injective as the physical “non-interpenetration of the matter” indicates that a deformation should be one-to-one. We continue our study from [17] and suggest studying the class of weak limits of Sobolev homeomorphism. We show that under natural assumptions on energy functional these limits are also invertible a.e. and that the energy functional is weakly lower semicontinuous which makes it a suitable class for variational approach.

Concerning invertibility we use the (INV) condition which was introduced for $W^{1,p}$-mappings, $p > n-1$, by Müller and Spector [33] (see also e.g. [4, 24, 25, 34, 36, 37, 38]). Informally speaking, the (INV) condition means that the ball $B(x,r)$ is mapped inside the image of the sphere $f(S(a,r))$ and the complement $\Omega \setminus \overline{B(x,r)}$ is mapped outside $f(S(a,r))$ (see Preliminaries for the formal definition). From [33] we know that mappings in this class with $J_f > 0$ a.e. are one-to-one a.e. and that this class is weakly closed. Moreover, any mapping in this class has many desirable properties, it maps disjoint balls into essentially disjoint balls, $\deg(f, B, \cdot) \in \{0, 1\}$ for a.e. ball B, under an additional assumption its distributional Jacobian equals to the absolutely continuous part of J_f plus a countable sum of positive multiples of Dirac measures (these corresponds to created cavities) and so on.

In all results in the previous paragraph the authors assume that $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ for some $p > n-1$. However, in some real models for $n = 3$ one often works with integrands

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containing the classical Dirichlet term $|Df|^2$ and thus this assumption is too strong. Therefore, for $n=3$, Conti and De Lellis [12] introduced the concept of (INV) condition also for $W^{1,2}\cap L^\infty$ (see also [5] and [6] for some recent work) and studied Neohookean functionals of the type

$$\int_\Omega \big(|Df(x)|^2+\varphi(J_f(x))\big)\, dx,$$

where φ is convex, $\lim_{t\to 0+}\varphi(t)=\infty$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty$. They proved that mappings in the (INV) class that satisfy $J_f>0$ a.e. have nice properties like mappings in [33], but this class is not weakly closed and hence cannot be used in variational models easily. To fix this problem we add an additional term to the energy functional and we work only with the class of weak limits of homeomorphisms.

Let us note that homeomorphisms clearly satisfy the (INV) condition and so do their weak limits in $W^{1,p}$, $p>n-1$ (see [33, Lemma 3.3]). Unfortunately, this is not true anymore in the limiting case of limit of $W^{1,n-1}$ homeomorphisms as shown by Conti and De Lellis [12] (see also Bouchala, Hencl and Molchanova [7]). Let us also note that the class of weak limits of Sobolev homeomorphisms was recently characterized in the planar case by Iwaniec and Onninen [28, 29] and De Philippis and Pratelli [16]. Our paper contributes to the study of this class in higher dimensions $n\geq 3$.

In our paper, we study the energy functional

$$\mathcal{F}(f)=\int_\Omega \big(|Df(x)|^{n-1}+A(|\operatorname{cof} Df(x)|)+\varphi(J_f(x))\big)\, dx$$

where

$$\varphi \text{ is a positive convex function on } (0,\infty) \text{ with } \lim_{t\to 0^+}\varphi(t)=\infty, \tag{1.1}$$

$$\lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty \tag{1.2}$$

and

$$A \text{ is a positive convex function on } (0,\infty) \text{ with } \lim_{t\to\infty}\frac{A(t)}{t}=\infty. \tag{1.3}$$

We further assume that our homeomorphisms have the same Dirichlet boundary data and that they satisfy the Lusin (N) condition, i.e. that for every $E\subset\Omega$ with $|E|=0$ we have $|f(E)|=0$. Our main result is the following:

Theorem 1.1. *Let $n\geq 3$ and Ω, $\Omega'\subset\mathbb{R}^n$ be bounded domains. Let φ and A satisfy* (1.1) *and* (1.3). *Let $f_m\in W^{1,n-1}(\Omega,\mathbb{R}^n)$, $m=0,1,2\dots$, be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_m}>0$ a.e., such that f_m satisfies the Lusin* (N) *condition and*

$$\sup_m \mathcal{F}(f_m)<\infty. \tag{1.4}$$

Assume further that $f_m=f_0$ on $\partial\Omega$ for all $m\in\mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega,\mathbb{R}^n)$, then f satisfies the (INV) *condition.*

Moreover, under the additional assumption (1.2) *our f satisfies the Lusin* (N) *condition and we have lower semicontinuity of energy*

$$\mathcal{F}(f)\leq\liminf_{m\to\infty}\mathcal{F}(f_m). \tag{1.5}$$

Assuming further that $|\partial\Omega'| = 0$ *we have*

$$\text{for a.e. } x \in \Omega \text{ we have } h(f(x)) = x \text{ and for a.e. } y \in \Omega' \text{ we have } f(h(y)) = y,$$

where h *is a weak limit of (some subsequence of)* f_k^{-1} *in* $W^{1,1}(\Omega', \mathbb{R}^n)$.

Let us comment on our assumptions. Each homeomorphism $f_m\colon \Omega \to \Omega'$, $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_{f_m} > 0$ a.e. satisfies $f_m^{-1} \in W^{1,1}(\Omega', \mathbb{R}^n)$ (see [13]). Moreover (see e.g. [13] or [18]), we have

$$\int_{\Omega'} |Df_m^{-1}(y)|\, dy \leq \int_{\Omega} |Df_m(x)|^{n-1}\, dx$$

and hence (1.4) implies that there is a subsequence of f_m^{-1} which converges weak-$*$ to some $h \in BV(\Omega', \mathbb{R}^n)$. Using

$$\sup_m \int_{\Omega} A(|\operatorname{cof} Df_m(x)|)\, dx < \infty \tag{1.6}$$

we get that Df_m^{-1} are equiintegrable (see Theorem 2.5 below) and hence (up to a subsequence) f_m^{-1} converge to $h \in W^{1,1}(\Omega', \mathbb{R}^n)$ weakly in $W^{1,1}(\Omega', \mathbb{R}^n)$. This assumption (1.6) is also crucial in our proof of the (INV) condition as it implies that image $f_m(A)$ of small set $A \subset \partial B(c, r)$ is "uniformly" small in m and therefore cannot enclose some big set that would like to escape from $f(\partial B(c, r))$ violating the (INV) condition.

The condition

$$\sup_m \int_{\Omega} \varphi(J_{f_m}(x))\, dx < \infty \text{ with } \lim_{t\to 0+} \varphi(t) = \infty \ \left(\text{resp.} \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty\right)$$

implies that small sets have uniformly small preimages (resp. small sets have uniformly small images) and these conditions are quite standard in the theory. Moreover, we need to assume that f_m maps null sets to null sets (by the Lusin (N) condition), which is again natural as our deformation cannot create a new material from "nothing". Let us note that this is crucial for the lower semicontinuity of our functional. In Lemma 4.5 below we construct a series of homeomorphisms that do not satisfy the Lusin (N) condition as they map some null set to a set of positive measure a, though they satisfy all other assumptions, converge weakly to $f(x) = x$ and

$$\int_{(0,1)^n} J_{f_m}(x)\, dx = 1 - a < 1 = \int_{(0,1)^n} J_f(x)\, dx$$

and hence lower semicontinuity fails at least for some polyconvex functionals. Similarly, if we omit the condition (1.2), we can construct a counterexample to semicontinuity of some functional if all f_m satisfy (N) but f does not. The lower semicontinuity of functionals below the natural $W^{1,n}$ energy has attracted a lot of attention in the past and we refer the reader e.g. to Ball and Murat [3], Malý [31], Dal Maso and Sbordone [15] and Celada and Dal Maso [9] for further information.

Let us note one disadvantage of our approach. In the previous models [12], [33] it was possible to model also the cavitation, i.e., the creation of small holes. Unfortunately, this is not possible for us as the condition (1.2) together with (1.4) tells us that f_m cannot map small sets onto big sets. However, this is exactly what is needed to be done by our approximating homeomorphisms around the point where the cavity is created by f. On the other hand, the condition (1.2) is crucial for us in order to prove the Lusin (N) condition for f and this condition is the key for the proof and the validity of the lower semicontinuity of our functional.

Let us briefly comment on the structure of this paper. We recall the definition of the degree and of the (INV) condition in the Preliminaries and we prove the equiintegrability of Df_m^{-1} there. Our proof of (INV) condition for f uses some techniques and results that we have developed in our previous paper [17] on this topic. We recall some of those in the Preliminaries and then we give a detailed proof of the (INV) condition using some of those techniques in Section 3. In Sections 4.1–4.3 we use the (INV) condition to prove that f satisfies the (N) condition and that h ($W^{1,1}$ weak limit of f_m^{-1}) is the "a.e. inverse" of f. Then we use the (N) condition to prove the lower semicontinuity of our polyconvex functional in Section 4.2 and we show some counterexamples to lower semicontinuity without our assumption (1.2) in Section 4.4. Finally, we return to the result of [17] where we have shown (INV) under different assumptions and we show that the lower semicontinuity of energy is valid also there if we additionally assume that f_m satisfy (N) and that we have (1.2). In the last Section 5 we give a quick application of our result in Calculus of Variations.

2. Preliminaries

2.1. **Change of variables estimates.**

Let $\Omega \subset \mathbb{R}^n$ be open, $A \subset \Omega$ be measurable and let $g \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^n)$ be one-to-one. Without any additional assumption we have (see e.g. [26, Theorem A.35] for $\eta = \chi_{g(A)}$)

$$\int_A |J_g(x)|\, dx \leq |g(A)|. \tag{2.1}$$

Moreover, for general g satisfying the Lusin (N) condition we have (see e.g. [26, Theorem A.35] for $\eta = \chi_{g(A)}$)

$$\int_A |J_g(x)|\, dx = \int_{\mathbb{R}^n} N(g, A, y)\, dy, \tag{2.2}$$

where $N(g, y, A)$ is defined as a number of preimages of y under g in A.

Analogous change of variables formula holds also for mappings $h\colon \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^{n-1}$. For Lipschitz h we have (see e.g. [20, Theorem 3.2.3])

$$\int_A J_{n-1}h(x)\; dx = \int_{\mathbb{R}^n} N(f, A, y)\; d\mathcal{H}^{n-1}(y), \tag{2.3}$$

where $A \subset \Omega$ is measurable, $N(f, A, y)$ denotes the number of preimages $f^{-1}(y)$ in a set A and $J_{n-1}h$ is the $(n-1)$-dimensional Jacobian of h, i.e. it consists of sizes of $(n-1) \times (n-1)$ subdeterminants. We know that each $h \in W^{1,1}(\Omega, \mathbb{R}^n)$ is approximatively differentiable a.e. (see e.g. [1, Theorem 3.83]) and for each approximately differentiable function we can exhaust Ω up to a set of measure zero by sets so that the restriction of h is Lipschitz continuous on those sets (see [20, Theorem 3.1.8 and Theorem 3.1.4]). It follows that (2.3) holds for Sobolev mapping $h \in W^{1,1}(\Omega, \mathbb{R}^n)$ if we know that for every $E \subset \Omega$ with $\mathcal{H}^{n-1}(E) = 0$ we have $\mathcal{H}^{n-1}(h(E)) = 0$. In general the area formula (2.3) holds for Sobolev h only up to a set of $(n-1)$-dimensional measure zero $E \subset \Omega$ (see also [22, Chapter 3, Section 1.5, Theorem 1 and Corollary 2]).

Lemma 2.1. *Given $C_1 < \infty$ and φ satisfying* (1.1)*, there exist monotone functions $\Phi,\ \Psi\colon (0,\infty) \to (0,\infty)$ with*

$$\lim_{s\to 0^+} \Phi(s) = 0 \text{ and } \lim_{s\to 0^+} \Psi(s) = 0$$

such that: Let $g \in W^{1,1}(\Omega,\mathbb{R}^n)$ be a one-to-one mapping with $\int_\Omega \varphi(J_g) \le C_1$. Then for each measurable set $A \subset \Omega$ we have

$$\Phi(|A|) \le |g(A)|. \tag{2.4}$$

If we moreover assume that the Lusin (N) *condition holds for g and that* (1.2) *holds, then also*

$$|g(A)| \le \Psi(|A|). \tag{2.5}$$

Proof. The proof of (2.4) can be found in the proof of [17, Lemma 2.9] (we omit here the assumption on $\|g\|_{L^\infty}$ as it is not necessary). The proof of (2.5) follows from De la Vallee Pousin theorem [30, Theorem B.103] applied to $|J_g|$ and the fact that the Lusin (N) condition implies an equality in (2.1). Note that we can assume that both Φ and Ψ are monotonous. □

The following lemma was shown in [17, Lemma 2.8].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set of finite measure and $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ satisfy $J_f \neq 0$ a.e. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set $F \subset \mathbb{R}^n$ we have*

$$|F| < \delta \implies |f^{-1}(F)| < \varepsilon.$$

In order to apply the previous lemma we use the following observation.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $f_k \in W^{1,1}(\Omega,\mathbb{R}^n)$ be a sequence of homeomorphisms with $J_{f_k} > 0$ a.e. such that $f_k \to f \in W^{1,1}(\Omega,\mathbb{R}^n)$ pointwise a.e. Assume further that*

$$\sup_k \int_\Omega \varphi(J_{f_k}(x))\, dx < \infty,$$

where φ satisfies (1.1)*. Then $J_f \neq 0$ a.e.*

Proof. Assume by contradiction that

$$E := \{x \in \Omega : \ J_f(x) = 0\} \text{ satisfies } |E| > 0.$$

As usual we find a set $E_0 \subset E$ with $|E_0| = |E|$ such that the (N) condition holds on E_0 for f (see e.g. [26, proof of Theorem A.35]). Moreover, we assume that $f_k(x) \to f(x)$ for every $x \in E_0$. By (2.2) we obtain

$$|f(E_0)| = 0.$$

We find an open set $G \subset \Omega$ such that

$$f(E_0) \subset G \text{ and } |G| < \tfrac{1}{2}\Phi\big(\tfrac{1}{2}|E_0|\big),$$

where Φ comes from Lemma 2.1. Since $f_k(x) \to f(x)$ we can find $k_0(x)$ such that for every $k \ge k_0(x)$ we have $f_k(x) \in G$. It follows that

$$E_0 = \bigcup_{k_0=1}^{\infty} E_{k_0}, \text{ where } E_{k_0} = \{x \in E_0 : \ f_k(x) \in G \text{ for every } k \ge k_0\}.$$

These sets are nested and hence we can fix k_0 such that $|E_{k_0}| > \frac{1}{2}|E_0|$. It follows that

$$f_{k_0}(E_{k_0}) \subset G \text{ with } |E_{k_0}| > \tfrac{1}{2}|E_0| \text{ and } |G| < \tfrac{1}{2}\Phi\big(\tfrac{1}{2}|E_0|\big)$$

which contradicts (2.4).

□

Theorem 2.4. *Let $B(c,R) \subset \mathbb{R}^n$ and let $g \in W^{1,n-1}(B(c,R),\mathbb{R}^n)$ be a homeomorphism. Then for a.e. $r \in (0,R)$ we know that $g \in W^{1,n-1}(\partial B(c,r),\mathbb{R}^n)$ and that g satisfies the Lusin* (N) *condition on the sphere $\partial B(c,r)$, i.e.,*

$$\textit{for every } E \subset \partial B(c,r) \textit{ with } \mathcal{H}^{n-1}(E) = 0 \textit{ we have } \mathcal{H}^{n-1}(g(E)) = 0.$$

Moreover, for such r and every relatively open set $E \subset \partial B(c,r)$ we have

$$\mathcal{H}^{n-1}(g(E)) \le C(r) \int_E |\operatorname{cof} Dg| \, d\mathcal{H}^{n-1}. \tag{2.6}$$

Proof. The fact that $g \in W^{1,n-1}(\partial B(c,r),\mathbb{R}^n)$ for a.e. r is standard and follows e.g. by using the ACL condition (on circles and not lines). The part about the validity of Lusin (N) condition on a.e. sphere follows from [13, Lemma 4.1].

Let us have a homeomorphism $h \in W^{1,1}(\mathbb{R}^{n-1},\mathbb{R}^n)$ which satisfies the Lusin (N) condition. Then the area formula (2.3) implies that for every measurable set $E \subset \mathbb{R}^{n-1}$ we have

$$\mathcal{H}^{n-1}(h(E)) = \int_E |J_{n-1}h(x)| \, dx,$$

where $J_{n-1}h$ is the $(n-1)$-dimensional Jacobian, i.e. it consists of all $(n-1) \times (n-1)$ subdeterminants. To obtain the wanted estimate we simply do a bilipschitz change of variables (locally) from round $\partial B(c,r)$ to flat $\mathbb{R}^{n-1}$ and the result for h implies our estimate (2.6) for g. Of course the constant in the bilipschitz change of variables might depend on r so our constant in (2.6) could depend on r. □

2.2. **Equiintegrability of Df_m^{-1}.** The following theorem tells us that our mappings f_m from Theorem 1.1 have equiintegrable Df_m^{-1}. It follows that up to a subsequence f_m^{-1} converge weakly to some $h \in W^{1,1}(\Omega',\mathbb{R}^n)$, see [30, Theorem B.103] and [14, Lemma 1.2 in Chapter 2.1].

Theorem 2.5. *Let Ω, $\Omega' \subset \mathbb{R}^n$ be domains. Let functions φ and A satisfy* (1.1) *and* (1.3)*. Then there is a continuous monotone function B with $\lim_{t\to\infty} \frac{B(t)}{t} = \infty$ such that: Let $f_m \in W^{1,n-1}(\Omega,\Omega')$ be homeomorphisms with $J_{f_m}(x) > 0$ a.e., $J_{f_m^{-1}}(y) > 0$ a.e. and*

$$\sup_m \int_\Omega \big(|Df_m(x)|^{n-1} + A(|\operatorname{cof} Df_m(x)|) + \varphi(J_{f_m}(x))\big) \, dx < \infty.$$

Then

$$\sup_m \int_{\Omega'} B\big(|Df_m^{-1}(y)|\big) \, dy < \infty.$$

Proof. Let us write $A(t) = ta(t)$, $B(t) = tb(t)$ and assume that b is a suitable function such that $b(t) \le a(t)$,

$$b(st) \le b(s) + b(t), \tag{2.7}$$

and B is continuous and monotone with superlinear growth. We give a detailed construction of such b below. By differentiation of $f_m^{-1}(f_m(x)) = x$ we obtain

$$Df_m^{-1}(f_m(x))Df_m(x) = I \text{ and } J_{f_m^{-1}}(f_m(x))J_{f_m}(x) = 1$$

for a.e. x (see [21, Lemma 2.1]). Using the previous line, (2.7), (2.1) and $A\cdot\operatorname{cof} A = \det A\cdot I$ we have

$$\begin{aligned}
\int_{\Omega'} B(|Df_m^{-1}(y)|)\,dy &= \int_{\Omega'} |Df_m^{-1}(y)| b(|Df_m^{-1}(y)|)\frac{J_{f_m^{-1}}(y)}{J_{f_m^{-1}}(y)}\,dy\\
&\leq \int_{\Omega} |Df_m^{-1}(f_m(x))|\; b(|Df_m^{-1}(f_m(x))|)\frac{1}{J_{f_m^{-1}}(f_m(x))}\,dx\\
&= \int_{\Omega} |(Df_m(x))^{-1}|\; b(|(Df_m(x))^{-1}|)J_{f_m(x)}\,dx\\
&= \int_{\Omega} |\operatorname{cof} Df_m(x)|\; b\left(|\operatorname{cof} Df_m(x)|\frac{1}{J_{f_m}(x)}\right)\,dx\\
&\leq \int_{\Omega} |\operatorname{cof} Df_m(x)|\; b(|\operatorname{cof} Df_m(x)|)\,dx + \int_{\Omega} |\operatorname{cof} Df_m(x)|\; b\left(\frac{1}{J_{f_m}(x)}\right)\,dx.
\end{aligned}$$

From $B(t) = tb(t) \leq A(t)$ we obtain that the first term is uniformly bounded. By the Young inequality, we estimate the second term

$$\int_{\Omega} |\operatorname{cof} Df_m(x)|\; b\left(\frac{1}{J_{f_m}(x)}\right)\,dx \leq \int_{\Omega} A(|\operatorname{cof} Df_m(x)|)\,dx + \int_{\Omega} A'\left(b\left(\frac{1}{J_{f_m}(x)}\right)\right)\,dx$$

where A' is the fixed conjugate function to our convex function A (see [23, Chapter 2.4]). If we ask also for

$$b(t) \leq (A')^{-1}(\varphi(\tfrac{1}{t}))$$

for large t, we have

$$A'\left(b\left(\frac{1}{J_{f_m}(x)}\right)\right) \leq \varphi(J_{f_m}(x)) + C$$

for every t and we are finished.

Now we find such function b. We define auxiliary functions $\overline{\psi}$ and $\overline{b}$ and take ψ and b which are smaller than their counterparts and monotone continuous. Let us set

$$\overline{\psi}(t) = \frac{a(t)}{\log(t)}$$

for $t > 1$. It is continuous and from the continuity and positivity of a on $(0,\infty)$ we know that $\lim_{t\to 1+}\overline{\psi}(t) = \infty$. Therefore we can define

$$\psi(t) = \begin{cases} 1, & 1 \leq t < t_0 = \min\{s > 1 : \overline{\psi}(s) = 1\},\\ \min\{\overline{\psi}(s), s\in[t_0,t]\}, & t_0 \leq t,\end{cases}$$

which is a positive continuous nonincreasing function less or equal to $\overline{\psi}$.

Define

$$\overline{b}(t) = \begin{cases} 0, & 0 < t \leq 1,\\ \psi(t)\log(t), & 1 < t < \infty.\end{cases}$$

Since ψ is continuous nonincreasing bounded on $(1,\infty)$, $\overline{b}$ is also continuous and (2.7) holds for $s,t\geq 1$ (since ψ is nonincreasing and log satisfies (2.7)) and $s,t<1$ (since $\overline{b}(st)=0$). Moreover we have $\overline{b}(t)\leq a(t)$. Now we check that

$$\lim_{t\to\infty}\overline{b}(t)=\lim_{t\to\infty}\psi(t)\log(t)=\infty.$$

Either $\lim_{t\to\infty}\psi(t)>0$ and the statement holds, or $\lim_{t\to\infty}\psi(t)=0$. In the later case, we can find a sequence $t_0<t_1<t_2\dots$ such that

$$t_k=\min\{s>t_0:\overline{\psi}(s)=1/k\}$$

and $t_k\to\infty$ (since $\overline{\psi}$ is positive). Then on $[t_0,t_k]$ we have $\psi(t)\geq 1/k=\overline{\psi}(t_k)=a(t_k)/\log(t_k)$ and consequentially

$$\liminf_{t\to\infty}\overline{b}(t)=\liminf_{k\to\infty}\min_{t\in[t_k,t_{k+1}]}\overline{b}(t)=\liminf_{k\to\infty}\min_{t\in[t_k,t_{k+1}]}\psi(t)\log(t)\geq\liminf_{k\to\infty}\psi(t_{k+1})\log(t_k)$$
$$\geq\liminf_{k\to\infty}\frac{\log(t_k)}{k+1}\geq\frac{1}{2}\liminf_{k\to\infty}\frac{\log(t_k)}{k}=\frac{1}{2}\liminf_{k\to\infty}a(t_k)=\infty.$$

Now we want to resolve (2.7) for $s<1$, $t\geq 1$. If $st\leq 1$, it is clear. In the other case, we need $\overline{b}(st)\leq\overline{b}(t)$ which we do not have for $\overline{b}$ in general as it does not have to be monotonous. Therefore we define

$$b(t)=\inf_{s\in[t,\infty)}\overline{b}(s).$$

That function is clearly monotone, continuous, smaller than $\overline{b}$ and tends to ∞. For $s,t<1$ (2.7) is still trivial. For $s<1$, $t\geq 1$ it follows from monotonicity. For $s,t\geq 1$ we find $s_0=\max\{r:\overline{b}(r)=b(s)\}$ and t_0 analogously. Obviously from the definition of b we have $s\leq s_0$, $t\leq t_0$. Then we have

$$b(st)\leq b(s_0t_0)\leq\overline{b}(s_0t_0)\leq\overline{b}(s_0)+\overline{b}(t_0)=b(s)+b(t).$$

Now we know that $B(t)=tb(t)$ is continuous nonnegative non-decreasing with $B(0)=0$ and $\lim_{t\to\infty}\frac{B(t)}{t}=\infty$.

The last step is to show that we can ask

$$b(t)\leq(A')^{-1}(\varphi(\tfrac{1}{t}))$$

for large t. We can use a similar procedure as before (replacing a by $\min\{a,(A')^{-1}\}$), since $\lim_{t\to\infty}(A')^{-1}(t)=\infty$ (A' is convex, negative in 0 and positive for large values, so going to ∞ — and so does its inverse, too).

□

2.3. **Degree for continuous mappings.** Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Given a continuous map $f\colon\overline{\Omega}\to\mathbb{R}^n$ and $y\in\mathbb{R}^n\setminus f(\partial\Omega)$, we can define the *topological degree* as

$$\deg(f,\Omega,y)=\sum_{\Omega\cap f^{-1}(y)}\operatorname{sgn}(J_f(x))$$

if f is smooth in Ω and $J_f(x)\neq 0$ for each $x\in\Omega\cap f^{-1}(y)$. By uniform approximation, this definition can be extended to an arbitrary continuous mapping $f\colon\overline{\Omega}\to\mathbb{R}^n$. Note that the degree depends only on values of f on $\partial\Omega$.

If $f\colon\overline{\Omega}\to\mathbb{R}^n$ is a homeomorphism, then either $\deg(f,\Omega,y)=1$ for all $y\in f(\Omega)$ (f is *sense preserving*), or $\deg(f,\Omega,y)=-1$ for all $y\in f(\Omega)$ (f is *sense reversing*). If, in addition, $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, then this topological orientation corresponds to the sign of the Jacobian. More precisely, we have

Proposition 2.6 ([27]). *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a homeomorphism on $\overline{\Omega}$ with $J_f > 0$ a.e. Then*

$$\deg(f, \Omega, y) = 1, \qquad y \in f(\Omega).$$

2.4. **Degree for $W^{1,n-1} \cap L^\infty$ mappings.** Let B be a ball, $f \in W^{1,n-1}(\partial B, \mathbb{R}^n) \cap C(\partial B, \mathbb{R}^n)$, $|f(\partial B)| = 0$, and $\mathbf{u} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then (see [33, Proposition 2.1])

$$\int_{\mathbb{R}^n} \deg(f, B, y) \operatorname{div} \mathbf{u}(y)\, dy = \int_{\partial B} (\mathbf{u} \circ f) \cdot (\Lambda_{n-1} D_\tau f)\nu \, d\mathcal{H}^{n-1}, \tag{2.8}$$

where $D_\tau f$ denotes the tangential gradient and $\Lambda_{n-1} D_\tau f$ is the restriction of $\operatorname{cof} Df$ to the corresponding subspace (see [17] for details).

Let $\mathcal{M}(\mathbb{R}^n) = C_0(\mathbb{R}^n)^*$ be the space of all signed Radon measures on $\mathbb{R}^n$. By (2.8) we see that $\deg(f, B, \cdot) \in BV(\mathbb{R}^n)$ and

$$\|D \deg(f, B, \cdot)\|_{\mathcal{M}(\mathbb{R}^n)} \leq C \|\Lambda_{n-1} D_\tau f\|_{L^1(\partial B)} \leq C \|D_\tau f\|^{n-1}_{L^{n-1}(\partial B)}. \tag{2.9}$$

Following [12] (see also [8]) we need a more general version of the degree which works for mappings in $W^{1,n-1} \cap L^\infty$ that are not necessarily continuous. Although only the three-dimensional case is discussed in [12], the arguments pass in the general case as well. The definition is in fact based on (2.8).

Definition 2.7. Let $B \subset \mathbb{R}^n$ be a ball and let $f \in W^{1,n-1}(\partial B, \mathbb{R}^n) \cap L^\infty(\partial B, \mathbb{R}^n)$. Then we define $\operatorname{Deg}(f, B, \cdot)$ as the distribution satisfying

$$\int_{\mathbb{R}^n} \operatorname{Deg}(f, B, y)\psi(y)\, dy = \int_{\partial B} (\mathbf{u} \circ f) \cdot (\Lambda_{n-1} D_\tau f)\nu \, d\mathcal{H}^{n-1} \tag{2.10}$$

for every test function $\psi \in C_c^\infty(\mathbb{R}^n)$ and every C^∞ vector field $\mathbf{u}$ on $\mathbb{R}^n$ satisfying $\operatorname{div} \mathbf{u} = \psi$.

As in [12] (see also [17]) it can be verified that the right-hand side does not depend on the way ψ is expressed as $\operatorname{div} \mathbf{u}$ and that the distribution $\operatorname{Deg}(f, B, \cdot)$ can be represented as a BV function.

Assume that $f,\ g \in W^{1,n-1}(\partial B, \mathbb{R}^n) \cap L^\infty(\partial B, \mathbb{R}^n)$. As in [17, (2.5)] we obtain the following version of some "weak isoperimetric inequality"

$$\begin{aligned} &\bigl|\{y \in \mathbb{R}^n :\ \operatorname{Deg}(f, B, y) \neq \operatorname{Deg}(g, B, y)\}\bigr|^{\frac{n-1}{n}} \leq \\ &\qquad \leq C \int_{\partial B \cap \{f \neq g\}} \bigl(|D_\tau f(x)|^{n-1} + |D_\tau g(x)|^{n-1}\bigr)\, d\mathcal{H}^{n-1}(x). \end{aligned} \tag{2.11}$$

We need also the classical isoperimetric inequality (see e.g. [19, Theorem 2 in section 5.6.2 and Theorem 2 in section 5.7.3]). Let $E \subset \mathbb{R}^n$ be and open set with finite perimeter. Then

$$|E|^{1-\frac{1}{n}} \leq C \mathcal{H}^{n-1}(\partial E). \tag{2.12}$$

Remark 2.8. Let B be a ball and $f \in W^{1,n-1}(\partial B, \mathbb{R}^n) \cap C(\overline{B}, \mathbb{R}^n)$. If $|f(\partial B)| = 0$, then $\operatorname{Deg}(f, B, y) = \deg(f, B, y)$ for a.e. $y \in \mathbb{R}^n$. We use different symbols to distinguish and emphasize that deg is defined pointwise on $\mathbb{R}^n \setminus f(\partial B)$, whereas Deg is determined only up to a set of measure zero.

2.5. (INV) **condition.** Analogously to [12] (see also [33]) we define the (INV) class.

Let $A \subset \Omega \subset \mathbb{R}^n$. We say that $x \in \mathbb{R}^n$ is a *point of density one* (or just *point of density*) of a set A if

$$\lim_{r \to 0+} \frac{|B(x,r) \cap A|}{|B(x,r)|} = 1.$$

It is well-known that a.e. $x \in A$ is a point of density of A.

Definition 2.9 (geometrical image)**.** Let $\Omega \subset \mathbb{R}^n$ be open, $f\colon \Omega \to \mathbb{R}^n$ be a function which is approximately differentiable almost everywhere. Given a set $A \subset \Omega$ we call the geometrical image of A through f the set given by $f(\Omega_d \cap A)$, where Ω_d denotes the set where f is approximatively differentiable. Further on, we denote this geometrical image by $f(A)$ (since f is nevertheless defined only up to a set of measure zero).

Definition 2.10 (topological image)**.** Let $B \subset \mathbb{R}^n$ be a ball and let $f \in W^{1,n-1}(\partial B, \mathbb{R}^n) \cap L^\infty(\partial B, \mathbb{R}^n)$. We define the topological image of B under f, $\operatorname{im}_T(f, B)$, as the set of all points of density one of the set $\{y \in \mathbb{R}^n : \operatorname{Deg}(f, B, y) \neq 0\}$.

Definition 2.11 ((INV) condition)**.** Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. We say that f satisfies (INV) in a ball $B \subset\subset \Omega$ if

(i) its trace on ∂B is in $W^{1,n-1}(\partial B, \mathbb{R}^n) \cap L^\infty(\partial B, \mathbb{R}^n)$;
(ii) $f(x) \in \operatorname{im}_T(f, B)$ for a.e. $x \in B$;
(iii) $f(x) \notin \operatorname{im}_T(f, B)$ for a.e. $x \in \Omega \setminus B$.

We say that f satisfies (INV) if for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{H}^1$-a.e. $r \in (0, r_a)$ it satisfies (INV) in $B(a, r)$.

Remark 2.12. If f, in addition, satisfies $J_f > 0$ a.e., then preimages of sets of measure zero are of measure zero and thus we can characterize the (INV) condition in a simpler way. Namely, such a mapping satisfies the (INV) condition in the ball $B \subset\subset \Omega$ if and only if

(i) its trace on ∂B is in $W^{1,n-1}(\partial B, \mathbb{R}^n) \cap L^\infty(\partial B, \mathbb{R}^n)$;
(ii) $\operatorname{Deg}(f, B, f(x)) \neq 0$ for a.e. $x \in B$;
(iii) $\operatorname{Deg}(f, B, f(x)) = 0$ for a.e. $x \in \Omega \setminus B$.

Definition 2.13. Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. The distributional Jacobian of f is the distribution defined by setting

$$\operatorname{Det} Df(\varphi) := -\int_\Omega f_1(x) J(\varphi, f_2, ..., f_n)(x)\, dx \qquad \text{for all } \varphi \in C_C^\infty(\Omega),$$

where $J(\varphi, f_2, ..., f_n)$ is the classical Jacobian defined as the determinant of the Jacobi matrix Dg of $g = (\varphi, f_2, \dots, f_n)$.

We need the following lemmata from [12]. They are stated there only for $n = 3$ but it is clear from the proofs that everything works also in higher dimensions.

Lemma 2.14 (Lemma 3.8, [12])**.** *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$, with $J_f \neq 0$ on Ω_d, and choose $B \subset \Omega$ such that f satisfies* (INV) *on B. Then $f(B) \subset \operatorname{im}_T(f, B)$, and $f(\mathbb{R}^n \setminus B) \subset \mathbb{R}^n \setminus \operatorname{im}_T(f, B)$.*

Lemma 2.15 (Lemma 4.3, [12])**.** *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. Suppose that condition* (INV) *holds for f, and that $J_f > 0$ a.e. Then,*

(i) $\operatorname{Det} Df \geq 0$*, hence it is a Radon measure;*

(ii) *the absolutely continuous part of* $\operatorname{Det} Df$ *with respect to* $\mathcal{L}^n$ *has density* J_f*;*
(iii) *for every* $a \in \Omega$ *and for a.e.* $r \in (0, r_a)$*,*

$$\operatorname{Det} Df(B(a,r)) = |\operatorname{im}_T(f, B(a,r))|. \tag{2.13}$$

2.6. **Minimizers of the tangential Dirichlet integral.** In our main proof, we have a sphere ∂B in $\mathbb{R}^n$ and on this sphere we have a small $(n-2)$-dimensional circle which is a boundary of an open spherical cap $S \subset \partial B$. Our map f is in $W^{1,n-1}$, therefore we can choose the sets so that f is continuous on the $(n-2)$-dimensional circle $\overline{S} \setminus S$. Our mapping f can have a big oscillation on S so we need to replace it with a reasonable mapping. We do this by choosing a minimizer of the tangential Dirichlet energy over this cap S which has the same value on the circle $\overline{S} \setminus S$. In fact, we need this even for more general shapes than spheres and circles.

We say that a relatively open set $S \subset \partial B$ satisfies the *exterior ball condition* if for each $z \in \overline{S} \setminus S$ there exists a ball $B(z', r)$ with $z' \in \partial B$ such that $z \in \partial B(z', r)$ and $B(z', r) \cap S = \varnothing$. The following Theorem was shown in [17, Theorem 2.10]:

Theorem 2.16. *Let* $B \subset \mathbb{R}^n$ *be a ball. Let* $S \subset \partial B$ *be a connected relatively open subset of* ∂B*. Let* T *be the relative boundary of* S *with respect to* ∂B*. Suppose that* $\operatorname{diam} S < \frac{r}{4n}$ *and that* S *satisfies the exterior ball condition. Let* $f = (f^1, \dots, f^n) \in W^{1,n-1}(\partial B, \mathbb{R}^n)$ *be continuous on* T*. Then there exists a unique function* $h = (h^1, \dots, h^n) \in C(\overline{S}, \mathbb{R}^n) \cap W^{1,n-1}(S, \mathbb{R}^n)$ *such that each coordinate* h^i *minimizes* $\int_S |D_\tau u|^{n-1}\, d\mathcal{H}^{n-1}$ *among all functions* $u \in f^i + W_0^{1,n-1}(S, \mathbb{R}^n)$*. We have* $h = f$ *on* T*, the function* h *satisfies the estimate*

$$\operatorname{diam} h(\overline{S}) \leq \sqrt{n}\operatorname{diam} f(T) \tag{2.14}$$

and we have $\mathcal{L}^n(h(S)) = 0$*. Moreover, let* f_m *be continuous and converge to* f *uniformly on* T*, then* h_m *converge to* h *uniformly on* S*, where* h_m *are minimizers corresponding to boundary values* f_m*.*

Proof. Everything except $\mathcal{L}^n(h(S)) = 0$ was shown already in [17, Theorem 2.10].

It is standard that the change of variable formula holds for Sobolev mappings up to a null set (see (2.3) and the paragraph after) and hence using $h \in W^{1,n-1}$ there is $N \subset S$ with $\mathcal{H}^{n-1}(N) = 0$ such that

$$\mathcal{H}^{n-1}\big(h(S \setminus N)\big) < \infty \text{ and hence } \mathcal{L}^n\big(h(S \setminus N)\big) = 0.$$

We claim that h is pseudomonotone, i.e. there is $C > 0$ such that for each spherical cap $A \subset S$ we have

$$\operatorname{diam} h(\overline{A}) \leq C \operatorname{diam} f(\partial A)$$

(here ∂A is the relative $(n-2)$-dimensional boundary with respect to ∂B). This fact follows from (2.14), i.e., we consider the corresponding minimizer on A with respect to boundary data $h|_{\partial A}$. By the uniqueness of this minimizer we obtain that $h|_A$ is this minimizer and (2.14) holds for $\overline{A}$ and ∂A (instead of $\overline{S}$ and T) gives us the pseudomonotonicity. Let us now consider a mapping $g := P \circ h \colon S \to \mathbb{R}^{n-1}$, where P is the projection to the hyperplane $\{x_1 = 0\}$. It is easy to see that $g \in W^{1,n-1}$ and that g is continuous and pseudomonotone. By the result of Malý and Martio [32, Theorem A] and $\mathcal{H}^{n-1}(N) = 0$ we obtain that

$$\mathcal{H}^{n-1}\big(g(N)\big) = 0 \text{ and hence } \mathcal{L}^n\big(h(N)\big) = 0.$$

□

3. Proof of Theorem 1.1: (INV) condition

Proof of Theorem 1.1: (INV) *condition.* Assume on the contrary that f does not satisfy the (INV) condition. Then we can find a center c such that for $\mathcal{H}^1$-positively many radii $r > 0$ our f maps either something from $B(c,r)$ outside of $\operatorname{im}_T(f, B(c,r))$ or something outside of $B(c,r)$ inside of $\operatorname{im}_T(f, B(c,r))$. We treat the first case in detail and at the end we briefly explain the analogous second case.

Step 1. Outline of the proof: We assume that f does not satisfy (INV) and hence there is $c \in \Omega$ such that the set

(3.1)
$$\{r : \ B(c,r) \subset \Omega, \ \exists V_r \subset B(c,r) \text{ with } |V_r| > 0 \text{ and } \operatorname{Deg}(f, B(c,r), f(x)) = 0 \text{ for all } x \in V_r\}$$

has positive (one-dimensional) measure.

Let us now briefly explain the idea of the proof. We know that f_m converge to f weakly in $W^{1,n-1}$ and thus up to a subsequence strongly in L^{n-1} and a.e. We can thus imagine that f_m is really close to f both on some fixed $\partial B(c,r)$ and on V_r. The situation is illustrated in Fig. 1.

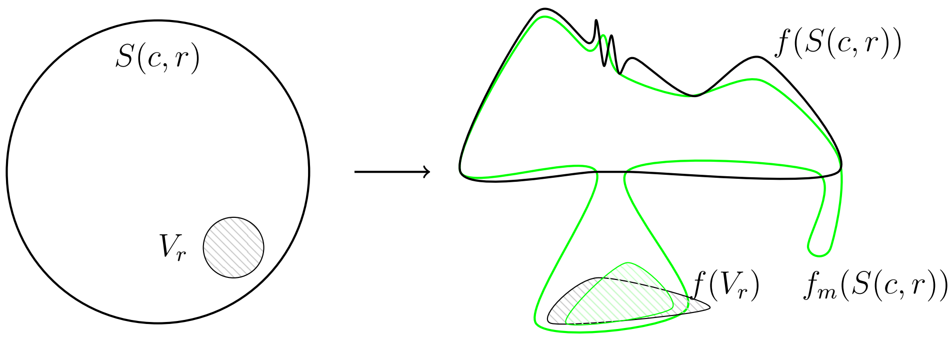

Figure 1. Behaviour of mappings f (in black) and f_m (in green) on $S(c,r)$ and V_r.

That means $f(V_r)$ is outside of $\operatorname{im}_T(f, B(c,r))$ but $f_m(V_r)$ lies inside $f_m(\partial B(c,r))$ since f_m is a homeomorphism. It follows that $f_m(\partial B(c,r))$ is close to $f(\partial B(c,r))$ in most of the places but it makes a "bubble" (or several bubbles) around $f_m(V_r)$ which is really close to $f(V_r)$.

Firstly, we define those bubbles and then we show that the number of bubbles that contain a big part of $f_m(V_r)$ is uniformly bounded and hence in one of them we have a big portion of the volume of $f_m(V_r)$ (and that $|f_m(V_r)| \geq C$ using (1.1) and (1.4)). Since $f_m(V_r)$ is quite big (in one of the big bubbles) we obtain that the boundary of the bubble has big $\mathcal{H}^{n-1}$ measure. However, this boundary is essentially image of some very small set (as small as we wish for m big enough) on ∂B under f_m and using (2.6) we obtain that the integral of $|\operatorname{cof} Df_m|$ over this small set is big. This contradicts the equiintegrability of $|\operatorname{cof} Df_m|$ which results in $\sup_m \int_{\partial B(c,r)} A(|\operatorname{cof} Df_m|)\, d\mathcal{H}^{n-1} < \infty$.

Step 2. Replacement of f on $\partial B(c,r)$ with continuous g that has similar degree: We need to apply plenty of techniques and results developed in [17]. For the convenience of the reader, we include most of the details in the current proof.

We fix $B(c,R) \subset\subset \Omega$. Since f_m converge weakly in $W^{1,n-1}$, which is compactly embedded into $L^{(n-1)^*}$, and so into L^{n-1}, we obtain that f_m converge to f in $L^{n-1}(B(c,R))$. Up to a subsequence we can thus assume that $f_m \to f$ pointwise a.e. and by Lemma 2.3 we obtain $J_f \neq 0$ a.e. We fix $r \in (0,R)$ (and pass to a subsequence if necessary, see e.g. [33, Lemma 2.9]) such that

$$f_m \to f \text{ weakly in } W^{1,n-1}(\partial B(c,r),\mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}\text{-a.e. on } \partial B(c,r)$$

and such that there exists a constant C_2 so that

$$\int_{\partial B(c,r)} (|D_\tau f|^{n-1} + |D_\tau f_m|^{n-1})\, d\mathcal{H}^{n-1} < C_2 \text{ for all } m \in \mathbb{N}. \tag{3.2}$$

Moreover, using Theorem 2.4 we can assume that all $f_m \in W^{1,n-1}(\partial B(c,r),\mathbb{R}^n)$ satisfy the (N) condition on $\partial B(c,r)$. Analogously to the proof of [33, Lemma 2.9] we use the Fatou Lemma and (1.4) to deduce

$$\int_0^R \liminf_{m\to\infty} \int_{\partial B(c,r)} A(|\operatorname{cof} Df_m|)\, d\mathcal{H}^{n-1}\, d\varrho \le \liminf_{m\to\infty} \int_0^R \int_{\partial B(c,r)} A(|\operatorname{cof} Df_m|)\, d\mathcal{H}^{n-1}\, d\varrho < \infty.$$

Choosing further r so that the lim inf on the lefthand side is finite and thus (passing again to a subsequence) we have

$$\int_{\partial B(c,r)} A\big(|\operatorname{cof} Df_m|\big)\, d\mathcal{H}^{n-1} < C_2 \text{ for all } m \in \mathbb{N}. \tag{3.3}$$

We set $B := B(c,r)$ and choose $\varepsilon > 0$ small enough with the exact value to be specified later. Find $\rho \in (0, \min\{\frac{1}{16n}r, \frac{\varepsilon}{2}\})$ such that for each $z \in \partial B$ we have

$$\int_{\partial B\cap B(z,2\rho)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-1} < \varepsilon^{n-1} \tag{3.4}$$

(we can do that since the integral over the whole ∂B is finite). For each fixed $z \in \partial B$ we find $\rho_z \in (\rho, 2\rho)$ such that

$$\rho \int_{\partial B\cap\partial B(z,\rho_z)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-2} < \varepsilon^{n-1}, \tag{3.5}$$

which is possible because the length of $(\rho, 2\rho)$ is ρ, combined with (3.4). Moreover, we can also choose ρ_z such that $f_m \to f$ occurs $\mathcal{H}^{n-2}$-a.e. on $\partial B \cap \partial B(z,\rho_z)$ and that

$$\liminf_{m\to\infty} \|f_m\|_{W^{1,n-1}(\partial B\cap\partial B(z,\rho_z),\mathbb{R}^n)} < \infty.$$

It follows that up to a subsequence (depending on z and ρ_z, see e.g. [33, Lemma 2.9])

$$f_m \to f \text{ weakly in } W^{1,n-1} \text{ and also uniformly on } \partial B \cap \partial B(z,\rho_z). \tag{3.6}$$

Note that on the $(n-2)$-dimensional space $\partial B\cap\partial B(z,\rho_z)$ we have embedding into Hölder functions $W^{1,n-1} \hookrightarrow C^{0,1-\frac{n-2}{n-1}}$, thus f is continuous there and we have the estimate

$$\operatorname{diam} f(\partial B \cap \partial B(z,\rho_z)) \le C(\rho_z)^{1-\frac{n-2}{n-1}} \left(\int_{\partial B\cap\partial B(z,\rho_z)} |D_\tau f|^{n-1}\, d\mathcal{H}^{n-2}\right)^{\frac{1}{n-1}} \le C_3\varepsilon. \tag{3.7}$$

Using the Vitali type covering, we find $B_j = B(z_j,\rho_j)$ such that $\rho_j = \rho_{z_j}$,

$$\partial B \subset \bigcup_j B(z_j,\rho_j)$$

and the balls $B(z_j, \frac{1}{5}\rho_j)$ are pairwise disjoint. Here $j = 1, \dots, j_{\max}$. Furthermore, the balls in the Vitali covering theorem are chosen inductively so we can also assume using (3.6) that for a subsequence (chosen in a diagonal argument)

$$f_m \to f \text{ weakly in } W^{1,n-1} \text{ and uniformly on } \partial B \cap \partial B(z_j, \rho_j) \text{ for each } j. \tag{3.8}$$

Given j, denote

$$S_j = \partial B \cap B_j \setminus \bigcup_{l<j} \overline{B_l}.$$

Note that S_j satisfies the exterior ball condition of Subsection 2.6. Let T_j denote the relative boundary of S_j with respect to ∂B.

For each j we define h_j on S_j such that h_j minimizes coordinate-wise the tangential $(n-1)$-Dirichlet integral among functions with boundary data f on T_j (see Theorem 2.16). We define $h_j = f$ on $\partial B \setminus S_j$. Also we define the function g on ∂B as $g = h_j$ on each $\overline{S_j}$. Set (see Fig. 2)

$$F = \{y \in \Omega' \colon \operatorname{Deg}(f, B, y) \neq \deg(g, B, y)\},$$
$$F_j = \{y \in \Omega' \colon \operatorname{Deg}(f, B, y) \neq \operatorname{Deg}(h_j, B, y)\}.$$

Let us recall that by Theorem 2.16 we have $\mathcal{L}^n(g(S_j)) = 0$ and hence Remark 2.8 gives us $\deg g = \operatorname{Deg} g$.

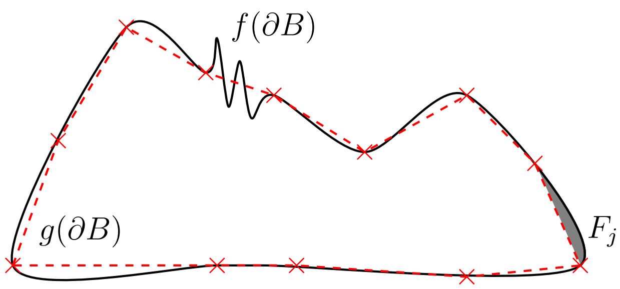

FIGURE 2. Behaviour of mappings f (in black) and g (red) on ∂B in 2D representation. T_j corresponds to points on $f(\partial B)$ (of course in $\mathbb{R}^n$ they are $(n-2)$-dimensional), g is represented by dashed lines connecting these points (of course these are minimizers of $(n-1)$-energy in higher dimensions and not lines) and the gray set F_j is created "between" $g(S_j)$ and $f(S_j)$.

It is not difficult to find out that

$$y \in \bigcup_j F_j \quad \text{for a.e. } y \in F$$

(this can be viewed e.g. by using (2.8)). Now, by (2.11), (3.4), and the minimizing property $\int_{S_j} |D_\tau h_j|^{n-1}\,d\mathcal{H}^{n-1} \leq C\int_{S_j} |D_\tau f|^{n-1}\,d\mathcal{H}^{n-1}$ we have

$$\begin{aligned}\sum_j |F_j| &\leq C\sum_j \Big(\int_{S_j} (|D_\tau f|^{n-1} + |D_\tau h_j|^{n-1})\,d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}}\\ &\leq C\sum_j \Big(\int_{S_j} |D_\tau f|^{n-1}\,d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}}\\ &\leq C\varepsilon\sum_j \int_{S_j} |D_\tau f|^{n-1}\,d\mathcal{H}^{n-1} \leq CC_2\varepsilon.\end{aligned}\tag{3.9}$$

It follows that f and g have the same degree up to a very small set. It is more convenient for us to work with g since for this continuous mapping on ∂B we can use the classical degree deg and not Deg as for f.

Step 3. Replacement of f_m on $\partial B(c, r)$ with continuous g_m that is close to g:

From Theorem 2.16 we know that $\mathcal{L}^n(h_j(S_j)) = 0$ for each j and thus $|g(\partial B)| = 0$. It follows that we can find a compact set $H \subset \Omega' \setminus g(\partial B)$ such that

$$\Omega' \setminus H < \Phi(\tfrac{1}{10}|V_r|),\tag{3.10}$$

where Φ comes from Lemma 2.1. For each $m \in \mathbb{N}$ and $j \in \{1, \dots, j_{\max}\}$ let $g_{m,j}$ be defined in S_j as the coordinate-wise minimizer of the $(n-1)$-Dirichlet integral among functions with boundary data f_m on T_j. We define $g_{m,j}$ as f_m on $\partial B \setminus S_j$. We also define g_m on ∂B as $g_{m,j}$ on each $\overline{S_j}$.

Since $f_m \to f = g$ uniformly on T_j by (3.8), we have $g_m \to g$ uniformly on ∂B using Theorem 2.16. Hence we find $m \in \mathbb{N}$ such that $g_m(\partial B)$ does not intersect H and

$$\deg(g_m, B, \cdot) = \deg(g, B, \cdot) \quad \text{in } H.\tag{3.11}$$

Also, we require

$$|f_m - f| = |g_m - g| < \varepsilon \quad \text{on all } T_j.\tag{3.12}$$

With the help of (2.14) and (3.7) (which holds also for T_j) this implies that

$$|g_m - g| < C\varepsilon \text{ on } \partial B.$$

Similarly as in Fig. 2 (but using f_m instead of f), we define

$$\begin{aligned}E &= \{y \in \Omega'\colon\ \deg(f_m, B, y) = 1 \neq \deg(g_m, B, y)\},\\ E_j &= \{y \in \Omega'\colon\ \deg(f_m, B, y) = 1 \neq \deg(g_{m,j}, B, y)\},\end{aligned}\tag{3.13}$$

see Fig. 3.

Let us note that these sets E_j are exactly those bubbles discussed in the first step (see Fig. 1). Then

$$y \in \bigcup_j E_j \quad \text{for a.e. } y \in E.$$

Using (2.11) and the minimizing property $\int_{S_j} |D_\tau g_{m,j}|^{n-1}\,d\mathcal{H}^{n-1} \leq \int_{S_j} |D_\tau f_m|^{n-1}\,d\mathcal{H}^{n-1}$, we obtain

$$|E_j|^{1-\frac{1}{n}} \leq C\int_{S_j} |D_\tau f_m|^{n-1}\,d\mathcal{H}^{n-1}.\tag{3.14}$$

Step 4. Not that many big bubbles where f_m and g_m have different degree:

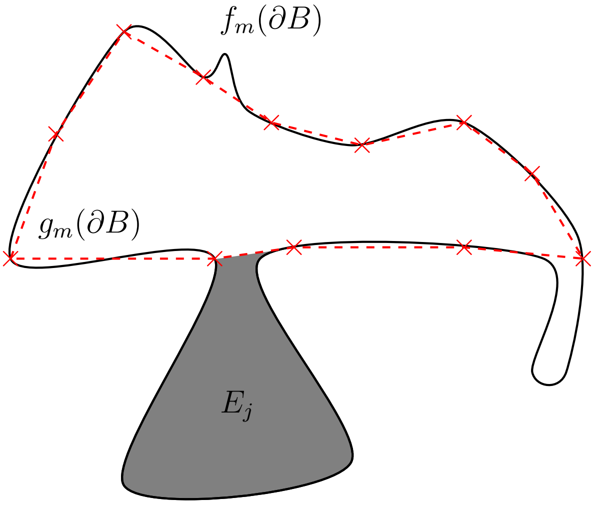

FIGURE 3. Behaviour of mappings f_m (in black) and g_m (red) on ∂B in 2D representation.

Choose $a > 0$ and set

$$J^+ = \{j\colon \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} > a\},$$
$$J^- = \{j\colon \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} \leq a\}.$$

Note that (3.14) implies that $|E_j|$ are small for $j \in J^-$. Hence using (3.2)

$$\begin{aligned}\sum_{j\in J^-} |E_j| &\leq C \sum_{j\in J^-} \Big(\int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1}\Big)^{\frac{n}{n-1}}\\ &\leq C a^{\frac{1}{n-1}} \sum_{j\in J^-} \int_{S_j} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1}\\ &\leq C a^{\frac{1}{n-1}} \int_{\partial B} |D_\tau f_m|^{n-1}\, d\mathcal{H}^{n-1} \leq C_4 a^{\frac{1}{n-1}},\end{aligned} \tag{3.15}$$

where $C_4 = CC_2$. We fix a such that

$$C_4 a^{\frac{1}{n-1}} \leq \Phi\left(\tfrac{1}{10}|V_r|\right). \tag{3.16}$$

We set

$$W = f_m^{-1}\Big(\bigcup_{j\in J^-} E_j\Big).$$

and using Lemma 2.1, (3.15) and (3.16) we obtain

$$\Phi(|W|) \leq |f_m(W)| = \Big|\bigcup_{j\in J^-} E_j\Big| \leq C_4 a^{\frac{1}{n-1}} \leq \Phi\left(\tfrac{1}{10}|V_r|\right).$$

From the monotonicity of Φ we get

$$|W| \leq \tfrac{1}{10}|V_r|. \tag{3.17}$$

From (3.2) we have

$$\#J^+ \le M := \frac{C_2}{a}. \tag{3.18}$$

It follows that we have only boundedly many E_j, $j \in J^+$, where the size of the bubble $|E_j|$ could be big. This bound depends on $|V_r|$ and Φ (and hence φ) and C_2 but it does not depend on m nor on ε. We could have plenty of other small bubbles E_j, $j \in J^-$, but (3.17) implies that the union of their preimages is really small.

Step 5. Big part of $f_m(V_r)$ lies in big bubbles:
Set

$$Y = \{x \in V_r \setminus W\colon \deg(g_m, B, f_m(x)) = 0\}.$$

With the help of (3.11) we have

$$\begin{aligned} V_r \setminus Y \subset & W \cup \{x \in V_r\colon \operatorname{Deg}(f, B, f(x)) \neq 0\} \cup \\ & \cup \{x \in V_r\colon \deg(g_m, B, f_m(x)) \neq 0,\ \operatorname{Deg}(f, B, f(x)) = 0\} \\ \subset & W \cup \{x \in V_r\colon \operatorname{Deg}(f, B, f(x)) \neq 0\} \cup \\ & \cup \{x \in V_r\colon \deg(g, B, f(x)) \neq \operatorname{Deg}(f, B, f(x))\} \cup \\ & \cup \{x \in V_r\colon \deg(g, B, f_m(x)) \neq \deg(g, B, f(x))\} \cup \{x \in V_r\colon f_m(x) \notin H\}. \end{aligned}$$

From (3.17) and (3.1) we obtain

$$\big|W \cup \{x \in V_r\colon \operatorname{Deg}(f, B, f(x)) \neq 0\}\big| \le \frac{1}{10}|V_r|$$

as the second set is empty. Using (3.9) we have

$$\big|\{y \in \Omega'\colon \deg(g, B, y) \neq \operatorname{Deg}(f, B, y)\}\big| = |F| \le CC_2\varepsilon.$$

Using Lemmata 2.2 and 2.3 we obtain that (for ε small enough)

$$\big|\{x \in V_r\colon \deg(g, B, f(x)) \neq \operatorname{Deg}(f, B, f(x))\}\big| \le \frac{1}{10}|V_r|.$$

Since the sets $\{y\colon \deg(g, B, y) = 0\}$ and $\{y\colon \deg(g, B, y) = 1\}$ are open and $f_m \to f$ a.e., we can take m so large that

$$\big|\{x \in V_r\colon \deg(g, B, f_m(x)) \neq \deg(g, B, f(x))\}\big| < \frac{1}{10}|V_r|. \tag{3.19}$$

Finally using (3.10) and (2.4) (as in (3.17)) we obtain

$$|\{x \in V_r\colon f_m(x) \notin H\}| \le \frac{1}{10}|V_r|$$

and all these inequalities together give us

$$|V_r \setminus Y| \le \frac{1}{2}|V_r|. \tag{3.20}$$

It follows that for many points $x \in V_r$ we have

$$\deg(g_m, B, f_m(x)) = 0,$$

but

$$\deg(f_m, B, f_m(x)) = 1$$

since f_m is a homeomorphism and $x \in V_r \subset B$. Therefore

$$\big|\{x \in V_r : \ f_m(x) \in E_j \text{ for some } j \in J^+\}\big| \geq \frac{1}{2}|V_r|. \tag{3.21}$$

Step 6. Integral $\int_{S_j} |\operatorname{cof} Df_m|$ is big on a small set S_j for some $j \in J^+$:
Using (3.18) and (3.21) we fix $j \in J^+$ such that for

$$U := \{x \in V_r : \ f_m(x) \in E_j\} \text{ we have } \big|U\big| \geq \frac{1}{2\# J^+}|V_r| \geq C|V_r|.$$

From Lemma 2.1 (2.4) we obtain that

$$|f_m(U)| \geq 2\delta, \tag{3.22}$$

where δ is constant which does not depend on m or ε.

From the definition of E_j (see (3.13) and Fig. 3) we obtain that E_j is an open set and

$$\partial E_j \subset f_m(S_j) \cup g_m(S_j).$$

We know that (see (3.7), (3.12) and (2.14))

$$\operatorname{diam}(g_m(\overline{S_j})) \leq C\varepsilon$$

and thus we can find a ball B_0 of radius $C\varepsilon$ such that $g_m(\overline{S_j}) \subset B_0$. Now the set

$$\tilde{E}_j := E_j \setminus \overline{B_0}$$

is open,

$$\partial \tilde{E}_j \subset f_m(S_j) \cup \partial B_0.$$

and using (3.22) we obtain that

$$|\tilde{E}_j| \geq 2\delta - C\varepsilon^n > \delta$$

once ε is small enough.

It is not difficult to show that the set $\tilde{E}_j$ has finite perimeter, and therefore we can use the isoperimetric inequality (2.12) and Theorem 2.4 to get

$$\begin{aligned}\delta^{1-\frac{1}{n}} \leq |\tilde{E}_j|^{1-\frac{1}{n}} &\leq C\mathcal{H}^{n-1}(\partial \tilde{E}_j) \leq C\big(\mathcal{H}^{n-1}(f_m(S_j)) + \mathcal{H}^{n-1}(\partial B_0)\big)\\ &\leq C(r)\Big(\int_{S_j} |\operatorname{cof} Df_m|\, d\mathcal{H}^{n-1} + C_0\varepsilon^{n-1}\Big).\end{aligned}$$

It follows that for ε sufficiently small we get

$$\frac{1}{2}\delta^{1-\frac{1}{n}} \leq C(r)\int_{S_j} |\operatorname{cof} Df_m|\, d\mathcal{H}^{n-1}$$

and this estimate on S_j (with $\operatorname{diam} S_j \leq \varepsilon$) clearly contradicts the uniform integrability of $|\operatorname{cof} Df_m|$ given by (3.3).

Step 7. Something from outside of $B(c,r)$ goes inside $\operatorname{im}_T(f, B(c,r))$:
This case works analogously. We can find a ball $B(c,r)$ and $V_r \subset \Omega \setminus B(c,r)$ such that its big part is mapped inside the topological images of $B(c,r)$. Therefore, f_m creates bubbles inside. We can define F_j, F, E_j and E in a similar way and conclude.

□

Recall that the symmetric difference of two sets $S, T \subset \mathbb{R}^n$ is defined as

$$S \triangle T := (S \setminus T) \cup (T \setminus S).$$

Lemma 3.1. *Let $n \geq 3$, $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains and let φ satisfy* (1.1) *and* (1.2). *Let $f_m \in W^{1,n-1}(\Omega, \Omega')$, $m = 0, 1, 2 \dots$, be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_m} > 0$ a.e. such that*

$$\sup_m \int_\Omega \Big(|Df_m|^{n-1} + \varphi(J_{f_m}) \Big)\, dx < \infty.$$

Assume further that $f_m = f_0$ on $\partial\Omega$ for all $m \in \mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $B \subseteq \Omega$ be a ball such that f satisfies (INV) *in B,*

- *$f_m \to f$ weakly in $W^{1,n-1}(\partial B, \mathbb{R}^n)$,*
- *$f_m \to f$ $\mathcal{H}^{n-1}$-a.e. on ∂B,*
- *$\int_{\partial B} \left(|D_\tau f|^{n-1} + |D_\tau f_m|^{n-1}\right) d\mathcal{H}^{n-1} < C$.*

Then it holds that

$$|f_k(B) \triangle \operatorname{im}_T(f, B)| \overset{k\to\infty}{\to} 0. \tag{3.23}$$

Proof. Let us assume by contradiction that we have a subsequence f_{m_k} such that

$$|f_{m_k}(B) \triangle \operatorname{im}_T(f, B)| > 4\lambda > 0.$$

We show that then f does not satisfy (INV).

For simplicity, we pass to that subsequence and keep the notation f_m. We consequentially choose a further subsequence such that $f_m(x) \to f(x)$ a.e. on B and, again passing to subsequences if neccessary, given $\varepsilon > 0$ we find g as in the proof of Theorem 1.1 such that

$$|\operatorname{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\}| \leq |\{y \in \Omega' : \deg(g, B, y) \neq \operatorname{Deg}(f, B, y)\}| \leq \varepsilon. \tag{3.24}$$

Let us split into two cases: either $|\operatorname{im}_T(f, B) \setminus f_m(B)| > 2\lambda$ or $|f_m(B) \setminus \operatorname{im}_T(f, B)| > 2\lambda$ for infinitely many m and thus we can assume that this is true for all m.

In the first case we find

$$U_m \subseteq \Omega \setminus B \text{ such that } f_m(U_m) \subseteq \operatorname{im}_T(f, B) \text{ and } |f_m(U_m)| > 2\lambda.$$

We can assume that $\varepsilon < \lambda$ and hence we can find $U'_m \subseteq U_m$ such that

$$f_m(U'_m) \cap \big(\operatorname{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\}\big) = \varnothing$$

and $|f_m(U'_m)| > \lambda$. We know from (2.5) that $|U'_m| > \Psi^{-1}(\lambda)$. Therefore,

$$|U| > \Psi^{-1}(\lambda)/2, \text{ where } U = \limsup U'_m = \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty U'_m.$$

Using Theorem 2.16 (as $|h(S)| = 0$ there) and (3.24) we have

$$\big|\partial\{y \in \Omega' : \deg(g, B, y) \neq 0\})\big| = 0 \text{ and } \big|\operatorname{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\}\big| < \varepsilon$$

and according to Lemmata 2.3 and 2.2 their preimages under f are arbitrarily small (depending on ε). Hence we can set ε to be small enough such that $|U'| > \Psi^{-1}(\lambda)/2$, where

$$U' = U \setminus \Big[f^{-1}\Big(\partial\{y \in \Omega' : \deg(g, B, y) \neq 0\} \cup \big(\operatorname{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\}\big)\Big)\Big].$$

For every $x \in U'$ we have a subsequence f_{m_k} such that $x \in U'_{m_k}$ and thus

$$f_{m_k}(x) \in \operatorname{im}_T(f, B) \cap \{y \in \Omega' : \deg(g, B, y) \neq 0\}.$$

Since $f_m \to f$ pointwise a.e., we have for a.e. $x \in U'$ that

$$f(x) \in \overline{\{y \in \Omega' : \deg(g, B, y) \neq 0\}}.$$

However, from the definition of U' we know that actually

$$f(x) \in \{y \in \Omega' : \deg(g, B, y) \neq 0\},$$

and since

$$x \notin f^{-1}\big(\mathrm{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\}\big),$$

we have that $f(x) \in \mathrm{im}_T(f, B)$ for every $x \in U'$. That contradicts (INV).

We deal with the second case analogously. We find

$$U_m \subseteq B \text{ such that } f_m(U_m) \cap \mathrm{im}_T(f, B) = \varnothing \text{ and } |f_m(U_m)| > 2\lambda.$$

We define the following sets in the same way and arrive to

$$f_{m_k}(x) \in \big(\Omega' \setminus \mathrm{im}_T(f, B)\big) \cap \big(\Omega' \setminus \{y \in \Omega' : \deg(g, B, y) \neq 0\}\big).$$

Again, x is not in the f-preimage of

$$\partial\{y \in \Omega' : \deg(g, B, y) \neq 0\} \text{ nor of } \mathrm{im}_T(f, B) \triangle \{y \in \Omega' : \deg(g, B, y) \neq 0\},$$

and therefore $f(x) \notin \mathrm{im}_T(f, B)$ for a set of a positive measure, which contradicts (INV). □

4. Proof of Theorem 1.1: (N) condition, lower semicontinuity and injectivity a.e.

4.1. **Lusin** (N) **condition.**

Lemma 4.1. *Let $f_m \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a sequence of homeomorphisms with $J_{f_m} > 0$ a.e. such that f_m satisfies the Lusin* (N) *condition and the sequence of Jacobians J_{f_m} is equiintegrable. Assume that $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ is a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ such that for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{H}^1$-a.e. $r \in (0, r_a)$ it satisfies* (3.23) *and* (INV) *in $B(a, r)$. Then,*

- *(i) the distributional Jacobian* $\mathrm{Det}\, Df \geq 0$ *is a Radon measure;*
- *(ii)* $\mathrm{Det}\, Df$ *is absolutely continuous w.r.t. Lebesgue measure: for any set $E \subset \Omega$ with $|E| = 0$, it holds that* $\mathrm{Det}\, Df(E) = 0$*;*
- *(iii) f satisfies the Lusin* (N) *condition.*

Proof. The first item (i) is stated in Lemma 2.15 (i).

Let $E \subset \Omega$ with $|E| = 0$ be given. Fix $\delta > 0$ and let $c(n)$ be a constant from Besicovitch covering theorem. Since E is a set of measure zero, there exists an open set $U \subset \Omega$ such that $E \subset A$ and $|U| < \frac{\delta}{c(n)}$. Consider a covering of E by balls $B(a, \tilde{r}_a)$ for all $a \in E$ such that $B(a, \tilde{r}_a) \subset U$, f satisfies (INV) in $B(a, \tilde{r}_a)$ and (3.23) holds on this ball. By the Bezicovitch Theorem (e.g. [26, Theorem A.2]), we can find at most countable collection of balls $B_k := B(a_k, \tilde{r}_k)$ such that

$$E \subset \bigcup_k B_k \subset U \text{ and } \bigcup_k B_k = \bigcup_{j=1}^{c(n)} \bigcup_{B_i \in A_j} B_i,$$

where subcollections A_j consists of disjoined balls B_i and a constant $c(n)$ depends only on the dimension n.

Note that since (3.23) is valid for one ball of the covering, it stays true for a finite union of such balls $\bigcup_{k=1}^M B_k$: for any $\varepsilon > 0$ and m big enough it holds that

$$\Big| f_m\Big(\bigcup_{k=1}^M B_k\Big) \triangle \Big(\bigcup_{k=1}^M \mathrm{im}_T\,(f, B_k)\Big)\Big| \leq \frac{\varepsilon}{3c(n)}. \tag{4.1}$$

Then, by Lemma 2.15 (iii) we have

$$\begin{aligned}\mathrm{Det}\,Df\Big(\bigcup_{k=1}^M B_k\Big) &= \mathrm{Det}\,Df\Big(\bigcup_{j=1}^{c(n)} \bigcup_{B_i\in A_j, i\leq M} B_i\Big) \leq \sum_{j=1}^{c(n)} \mathrm{Det}\,Df\Big(\bigcup_{B_i\in A_j, i\leq M} B_i\Big)\\ &\leq \sum_{j=1}^{c(n)}\Big|\bigcup_{B_i\in A_j, i\leq M} \mathrm{im}_T(f, B_i)\Big| \leq c(n)\Big|\bigcup_{k=1}^M \mathrm{im}_T(f, B_k)\Big|.\end{aligned}$$

To prove (ii), we fix $\varepsilon > 0$ and $\delta > 0$ such that

$$\Psi(t) < \frac{\varepsilon}{3\,c(n)} \quad \text{for any } t < \delta, \tag{4.2}$$

where Ψ is given by Lemma 2.1 (note that in the proof of this part of lemma we have used only equiintegrability of J_{f_m} and Lusin (N) condition for f_m). Since $\bigcup_k B_k \subset U$ and $|U| < \delta$ we have using (2.5)

$$\sum_{k\in\mathbb{N}} |B_k| < \delta, \text{ and therefore } \Big|f_m\Big(\bigcup_{k=1}^M B_k\Big)\Big| < \frac{\varepsilon}{3\,c(n)} \text{ for any } m\in\mathbb{N}.$$

Relying on (4.1)–(4.2), for M big enough there exists m such that

$$\begin{aligned}\mathrm{Det}\,Df(E) &\leq \mathrm{Det}\,Df\Big(\bigcup_{k\in\mathbb{N}} B_k\Big) \leq \mathrm{Det}\,Df\Big(\bigcup_{k=1}^M B_k\Big) + \frac{\varepsilon}{3} \leq c(n)\Big|\bigcup_{k=1}^M \mathrm{im}_T(f, B_k)\Big| + \frac{\varepsilon}{3}\\ &\leq c(n)\Big|f_m\Big(\bigcup_{k=1}^M B_k\Big)\Big| + c(n)\Big|f_m\Big(\bigcup_{k=1}^M B_k\Big)\triangle\Big(\bigcup_{k=1}^M \mathrm{im}_T\,(f, B_k)\Big)\Big| + \frac{\varepsilon}{3} \leq \varepsilon.\end{aligned}$$

For (iii) it is enough to notice that $|f(B)| \leq |\,\mathrm{im}_T\, f(B)|$ by Lemma 2.14. Hence

$$\begin{aligned}|f(E)| &\leq \Big|\bigcup_{k=1}^M \mathrm{im}_T(f, B_k)\Big| + \frac{\varepsilon}{3}\\ &\leq \Big|f_m\Big(\bigcup_{k=1}^M B_k\Big)\Big| + \Big|f_m\Big(\bigcup_{k=1}^M B_k\Big)\triangle\Big(\bigcup_{k=1}^M \mathrm{im}_T\,(f, B_k)\Big)\Big| + \frac{\varepsilon}{3} \leq \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ has been chosen arbitrary, we conclude $\mathrm{Det}\,Df(E) = |f(E)| = 0$.

□

4.2. **Lower semicontinuity.** The main obstacle to obtain the lower semicontinuity of $\mathcal{F}$ is to ensure weak convergence of Jacobians. One usually has to assume higher regularity of $f \in W^{1,n}$ (see e.g. [9, 15, 31]) but this is not available for us.

Lemma 4.2. *Let $f_m \in W^{1,n-1}(\Omega,\mathbb{R}^n)$ be a sequence of homeomorphisms with $J_{f_m} > 0$ a.e., such that f_m satisfies the Lusin* (N) *condition and the sequence of Jacobians J_{f_m} is equiintegrable. Assume that $f \in W^{1,n-1}(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$ is a weak limit of f_m in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ such that for every $a\in\Omega$ there is $r_a>0$ such that for $\mathcal{H}^1$-a.e. $r\in(0,r_a)$ it satisfies* (3.23) *and* (INV) *in $B(a,r)$. Then there exists a subsequence $\{J_{f_{k_m}}\}_{m\in\mathbb{N}}$ of $\{J_{f_k}\}_{k\in\mathbb{N}}$ such that $J_{f_{k_m}} \rightharpoonup J_f$ weakly in $L^1(\Omega)$.*

Proof. By the Dunford–Pettis Theorem (e.g. [30, Theorem B.103]) there exist a (non-relabeled) subsequence J_{f_m} and a function $j\in L^1(\Omega)$ s.t. $J_{f_m}\to j$ weakly in $L^1(\Omega)$, and hence for any measurable set $E\subset\Omega$

$$\int_E J_{f_m}(x)\,dx \to \int_E j(x)\,dx.$$

Let B be such that f satisfies the (INV) condition and (3.23) in B, then by the area formula (2.1) and (3.23) we have

$$\int_B J_{f_m}(x)\,dx = |f_m(B)| \to |\operatorname{im}_T(f,B)|.$$

On the other hand, by Lemma 2.15 (*ii*)–(*iii*) and Lemma 4.1 (*ii*), we know that

$$\operatorname{Det} f(B) = \int_B J_f(x)\,dx = |\operatorname{im}_T f(B)|.$$

Therefore, for all such balls, it holds that

$$\int_B j(x)\,dx = |\operatorname{im}_T(f,B)| = \int_B J_f(x)\,dx,$$

which in turn implies $j(x)=J_f(x)$ for a.e. $x\in\Omega$. □

Lemma 4.3. *Under conditions of Theorem 1.1, the functional $\mathcal{F}$ is lower semicontinuous with respect to weak convergence, i.e.,* (1.5) *holds.*

Proof. Weak convergence $f_k\to f$ in $W^{1,n-1}$ implies weak convergence (up to subsequence) in L^1 of all minors $\det_l(Df_m)$ of order $l\le n-2$ (see e.g. [35, Lemma 5.10]):

$$\|\det_l(Df_m)\|_{L^1(\Omega)} \le C\|Df_m\|^l_{L^{n-1}} \le CM^{\frac{l}{n-1}}.$$

Weak convergence of $\det_{n-1}(Df_k)=\operatorname{cof} Df_k$ in L^1 follows by the standard argument, provided uniform integrability (1.3) (see [2, Theorem 6.2] or [10, Theorem 7.5-1]). The equiintegrability of J_{f_m} follows from (1.2) by the de la Valée Poussin Theorem (e.g. [30, Theorem B.104]). Moreover, conditions of Lemma 3.1 are fulfilled (see Step 2 of the proof in Section 3). Therefore, Lemmata 3.1 and 4.2 ensure $J_{f_m}\to J_f$ weakly in L^1, which now allows us to use De Giorgi Theorem [14, Theorem 3.23] to conclude the lower semicontinuity (1.5) of $\mathcal{F}$. □

4.3. **Injectivity almost everywhere.** One of the main reasons to consider the (INV) condition is that it implies injectivity a.e. [12, Lemma 3.7], [33, Lemma 3.4], as discussed in Introduction. In the setting of this paper, we can say even more.

Lemma 4.4. *Let conditions of Theorem 1.1 be fulfilled and let h be a weak limit of f_m^{-1} in $W^{1,1}(\Omega',\mathbb{R}^n)$. Then $h(f(x))=x$ for a.e. $x\in\Omega$ and under additional assumption $|\partial\Omega'|=0$ we have $f(h(y))=y$ for a.e. $y\in\Omega'$.*

Proof. From Theorem 2.5 we obtain that there is a subsequence of f_m^{-1} which converges weakly in $W^{1,1}$ to some h and we work with this subsequence here.

Recall that we know that $J_f(x) > 0$ for a.e. $x \in \Omega$. Since f satisfies (N) we can use (2.2) (for $A = f^{-1}(E)$) to obtain that

$$|f^{-1}(E)| = 0 \text{ for every } E \subset \mathbb{R}^n \text{ with } |E| = 0. \tag{4.3}$$

Therefore, for a.e. $x \in \Omega$ we know that $f(x)$ is a Lebesgue point of h.

Moreover, we claim that for a.e. $x \in \Omega$ and every $\eta > 0$ there exists a ball $B = B(a,r)$ such that f satisfies (INV) and (3.23) in B and

$$x \in B,\ r < \eta, \text{ and } f(x) \in \operatorname{im}_T(f,B) \text{ is a point of density 1 of } \operatorname{im}_T(f,B). \tag{4.4}$$

Indeed, choose a countable set of balls

$$\begin{aligned}\mathcal{B} := \big\{B(c,r_i) : c \in \mathbb{Q}^n \cap \Omega,\ r_i \in [2^{-i-1}, 2^{-i}) \text{ for all } i \in \mathbb{N} \\ \text{and } f \text{ satisfies (INV) and (3.23) in } B(c,r_i)\big\}.\end{aligned}$$

For every $B_j \in \mathcal{B}$ we know that a.e. point of $\operatorname{im}_T(f,B_j)$ is a point of density 1 and with the help of (4.3) we can find a null set Σ_j such that

$$f(x) \in \operatorname{im}_T(f,B_j) \text{ for each } x \in B_j \setminus \Sigma_j \text{ and } f(x) \text{ is a point of density of } \operatorname{im}_T(f,B_j).$$

Then $\Sigma := \bigcup_j \Sigma_j$ is a null set and for every $x \in \Omega \setminus \Sigma$ we have (4.4) for some ball B_j.

Let us first prove that $h(f(x)) = x$ a.e. We pick x such that $f(x)$ is a Lebesgue point of h and $f(x)$ is a point of density of $\operatorname{im}_T(f,B)$ for some ball $B = B(a,r)$ satisfying (4.4). Now we can find a ball $\tilde{B}$ around $f(x)$ so that

$$\frac{1}{|\tilde{B}|}\int_{\tilde{B}} |h(z) - h(f(x))|\,dz < r/2 \qquad \text{and} \qquad \Big|\big\{y \in \tilde{B} :\ y \in \operatorname{im}_T(f,B)\big\}\Big| > 0.9|\tilde{B}|.$$

Using convergence f_m^{-1} to h in $L^1_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$ and (3.23), we fix m big enough so that

$$\int_{\tilde{B}} |f_m^{-1}(z) - h(z)|\,dz < r|\tilde{B}|/2 \qquad \text{and} \qquad |f_m(B) \triangle \operatorname{im}_T(f,B)| < 0.1|\tilde{B}|.$$

Combining the estimates, we obtain

$$\frac{1}{|\tilde{B}|}\int_{\tilde{B}} |f_m^{-1}(z) - h(f(x))|\,dz < r \qquad \text{and} \qquad \Big|\big\{y \in \tilde{B} :\ y \in f_m(B)\big\}\Big| > 0.8|\tilde{B}|.$$

We claim that this implies

$$h(f(x)) \in B(a,4r), \tag{4.5}$$

since otherwise we get a contradiction from

$$\begin{aligned}\frac{1}{|\tilde{B}|}\int_{\tilde{B}} |f_m^{-1}(z) - h(f(x))|\,dz &\geq \frac{1}{|\tilde{B}|}\int_{\{y\in\tilde{B}:\ y\in f_m(B)\}} \big|h(f(x)) - a - (f_m^{-1}(z) - a)\big|\,dz \\ &\geq 0.8(4r - r) > r.\end{aligned}$$

Since $r > 0$ is chosen arbitrary, we conclude from (4.5) that $h(f(x)) = x$ for a.e. $x \in \Omega$.

It is not difficult to see that $f(\Omega) \subset \overline{\Omega'}$. From Lemma 4.2 and change of variables (2.2) we know that

$$|\Omega'| = \lim_{k\to\infty} |f_k(\Omega)| = \lim_{k\to\infty} \int_\Omega J_{f_k}(x)\,dx = \int_\Omega J_f(x)\,dx = \int_{\mathbb{R}^n} N(f,\Omega,y)\,dy.$$

From the a.e.-injectivity [12, Lemma 3.7] of f together with the (N) condition for f we now obtain

$$|\Omega'| = \int_{\mathbb{R}^n} N(f,\Omega,y)\,dy = \int_{f(\Omega)} 1\,dy = |f(\Omega)|.$$

Since $f(\Omega) \subset \overline{\Omega'}$ and $|\partial\Omega'| = 0$ we obtain that a.e. point $y \in \Omega'$ lies in $f(\Omega)$ and $N(f,\Omega,y) = 1$ there.

The other equality $f(h(y)) = y$ for a.e. $y \in \Omega'$ now follows easily. We know $h(f(x)) = x$ holds for a.e. $x \in \Omega$ and that f satisfies the (N) condition. Hence for a.e. $y \in f(\Omega)$ we can pick $x \in \Omega$ such that $f(x) = y$ and $h(f(x)) = x$. Now

$$f(h(y)) = f(h(f(x))) = f(x) = y.$$

Note that in this proof we do not need a Sobolev regularity but only $f_m^{-1} \to h$ in $L^1_{\rm loc}$. □

Proof of Theorem 1.1. Theorem 1.1 now follows from result in Section 3, Lemma 4.1, Lemma 4.4 and Lemma 4.3. □

4.4. **Counterexamples to lower semicontinuity.** The following example shows that one has to ask the condition (N) for f_m to conclude lower semicontinuity of a quasiconvex functional, even if φ and A satisfy (1.1) and (1.3).

Lemma 4.5 (Counterexample for lsc)**.** *Let $p < n$, then there exist φ and A that satisfy* (1.1) *and* (1.3) *and homeomorphisms $f_m, f\colon [0,1]^n \to [0,1]^n$ such that J_{f_m}, $J_f > 0$ a.e.,* (1.4) *is fulfilled, $f_m = id$ on $\partial([0,1]^n)$ and f_m converge to f weakly in $W^{1,p}([0,1]^n,\mathbb{R}^n)$. However, f_m does not satisfy the Lusin* (N) *condition for all $m \in \mathbb{N}$ and*

$$\int_{(0,1)^n} J_f(x)\,dx > \liminf_{m\to\infty} \int_{(0,1)^n} J_{f_m}(x)\,dx.$$

Proof. Take any $A(t) \le Ct^\beta$, where $C > 0$ and $\beta > 1$ are some constants, and take φ which behaves like an identity around 1 and satisfies (1.1). Consider a Ponomarev-type map $g\colon [0,1]^n \to [0,1]^n$, $g \in W^{1,p}([0,1]^n,\mathbb{R}^n)$ which maps a Cantor-set $\mathcal{C}_A$ of measure zero to a Cantor-set $\mathcal{C}_B = g(\mathcal{C}_A)$ of positive measure, and which is identical on $\partial([0,1]^n)$. Such a map can be found by the standard construction, see [26, Chapter 4.3] with

$$a_k = \frac{1}{k^\alpha} \text{ and } b_k = 1 + \frac{1}{k^{\alpha n}} \text{ where } 0 < \alpha < \min\left\{\frac{n}{p}, \frac{n}{(n-1)\beta}\right\}.$$

Referring the reader to [26, Chapter 4.3] for details, we just notice that on the k-th level we have

$$|Dg| \approx \max\left\{\frac{b_k}{a_k}, \frac{b_{k-1}-b_k}{a_{k-1}-a_k}\right\} \approx k^\alpha, \qquad J_g \approx \left(\frac{b_k}{a_k}\right)^{n-1} \cdot \frac{b_{k-1}-b_k}{a_{k-1}-a_k} \approx 1,$$
$$|\operatorname{cof} Dg| \le C|Dg|^{n-1} \approx k^{\alpha(n-1)} \quad \text{and} \quad |\{k\text{-th level}\}| \approx 2^{-kn}\frac{1}{k^{n+1}}.$$

It follows that the map g has finite energy since
(4.6)

$$\int_{(0,1)^n} |Dg(x)|^p\,dx \leq C\sum_{k=1}^{\infty} 2^{kn}2^{-kn}\frac{1}{k^{n+1}}k^{\alpha p} < \infty,$$

$$\int_{(0,1)^n} A(|\operatorname{cof} Dg(x)|)\,dx \leq C\int_{(0,1)^n} |\operatorname{cof} Dg(x)|^{\beta}\,dx \leq C\sum_{k=1}^{\infty} 2^{kn}2^{-kn}\frac{1}{k^{n+1}}k^{\alpha\beta(n-1)} < \infty,$$

$$\int_{(0,1)^n} \varphi(J_g(x))\,dx < \infty, \qquad \text{and} \qquad \int_{(0,1)^n} K_g^{\frac{1}{n-1}}(x)\,dx \leq \int_{(0,1)^n} |Dg(x)|^{\frac{n}{n-1}}\,dx < \infty.$$

We set $f_1 = g$ and we divide the cube $[0,1]^n$ into m^n equal cubes both in the domain and in the target. Fix one of those m^n small cubes $Q_z := \{x \in [0,1]^n : \|x - z\|_\infty < \frac{1}{2m}\}$ with a center point z and define $f_m|_{Q_z} : Q_z \to Q_z$ as a scaled and translated copy of g

$$f_m(x) := \frac{1}{m}g\big(m(x-z)\big) + z.$$

It is easy to see by change of variables that

$$\int_{(0,1)^n} |Df_m|^p\,dx = \int_{(0,1)^n} |Dg|^p\,dx$$

and analogously for other integrals in (4.6). It follows that $\sup_m \mathcal{F}(f_m) = \mathcal{F}(g) < \infty$ and hence there is a subsequence which converges weakly in $W^{1,n-1}$. Since $f_m \to f := \mathrm{id}$ pointwise, identity is a weak limit of f_m.

By construction, g maps the Cantor-set $\mathcal{C}_A$ to the Cantor-set $\mathcal{C}_B$, where $\mathcal{C}_A$ is a set where g fails the Lusin (N) condition. Hence, for every $m \in \mathbb{N}$ and for each Q_z it holds that

$$\int_{Q_z} J_{f_m}(x)\,dx = \frac{1 - |\mathcal{C}_B|}{m^n}.$$

Therefore,

$$\int_{(0,1)^n} J_{f_m}(x)\,dx = 1 - |\mathcal{C}_B| < 1 = \int_{(0,1)^n} J_f(x)\,dx,$$

so the lower semicontinuity fails at least for this quasiconvex functional. □

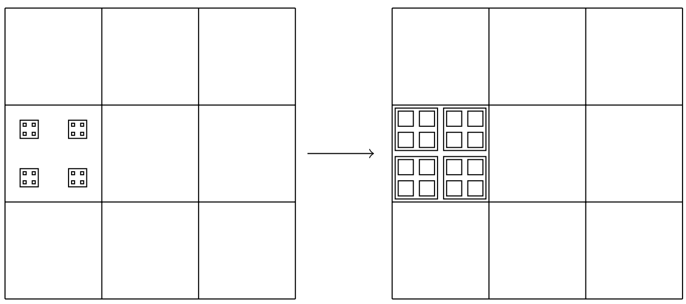

FIGURE 4. f_3 and its action on $Q_{(\frac{1}{6},\frac{1}{2})}$.

4.5. (N) **condition and lower semicontinuity in the context of** [17]**.** In our previous result [17] we have shown that limit of homeomorphisms in $W^{1,n-1}$ satisfy the (INV) condition under different assumptions. Instead of $\int A(|\operatorname{cof} Df_m|) \leq C$ we assumed the integrability of the distortion function. Here we show that even in that case we can obtain that the corresponding functional is lower semicontinuous and that f satisfies (N) under the additional assumptions that all f_m satisfy (N) and that (1.2) holds (these assumptions were not in [17]). As in [17] we assume that there is $A > 0$ with

$$A^{-1}\varphi(t) \leq \varphi(2t) \leq A\varphi(t), \qquad t \in (0, \infty). \tag{4.7}$$

Theorem 4.6. *Let $n \geq 3$, Ω, $\Omega' \subset \mathbb{R}^n$ be Lipschitz domains and let φ satisfy* (1.1) *and* (4.7)*. Let $f_m \in W^{1,n-1}(\Omega, \Omega')$, $m = 0, 1, 2 \dots$, be a sequence of homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_m} > 0$ a.e. such that*

$$\sup_m \int_\Omega \Big(|Df_m(x)|^{n-1} + \varphi(J_{f_m}(x)) + \Big(\frac{|Df_m(x)|^n}{J_{f_m(x)}}\Big)^{\frac{1}{n-1}} \Big)\, dx < \infty.$$

Assume further that $f_m = f_0$ on $\partial\Omega$ for all $m \in \mathbb{N}$. Let f be a weak limit of f_m in $W^{1,n-1}(\Omega, \mathbb{R}^n)$, then f satisfies the (INV) *condition.*

Moreover, under the additional assumptions (1.2) *and that all f_m satisfy the Lusin* (N) *condition we obtain that our f satisfies the Lusin* (N) *condition and we have lower semicontinuity of energy*

$$\mathcal{G}(f) := \int_\Omega \Big(|Df(x)|^{n-1} + \varphi(J_f(x)) + \Big(\frac{|Df(x)|^n}{J_f(x)}\Big)^{\frac{1}{n-1}} \Big) \leq \liminf_{m\to\infty} \mathcal{G}(f_m). \tag{4.8}$$

Further

for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$,

where h is a weak-$*$ limit of (some subsequence of) f_m^{-1} in $BV(\Omega', \mathbb{R}^n)$.

Proof. The fact that the limit f satisfies the (INV) condition follows from [17, Theorem 3.1 a)] and Lemma 2.3.

As in Step 2 of Section 3, we know that for every center and almost every radius the corresponding ball satisfies the conditions of Lemma 3.1. Thus,

$$|f_m(B) \triangle \operatorname{im}_T(f, B)| \overset{k\to\infty}{\to} 0. \tag{4.9}$$

Further, the (N) condition of f and a lower semicontinuity of $\mathcal{G}$ follow from Lemmata 4.1–4.2, provided with (4.9). To prove the lower semicontinuity of $\mathcal{G}$, we just note that the function

$$g(x, y) = x^{n-1} + x^{\frac{n}{n-1}} y^{-\frac{1}{n-1}} + \varphi(y)$$

is convex, and as in Lemma 4.3 use the De Giorgi Theorem [14, Theorem 3.23] again. Let us note that our functional depends only on $|Df|$ and $\det Df$, but not on $\operatorname{cof} Df$, so we do not need to care about convergence of $(n-1) \times (n-1)$ subdeterminants here.

Further, from [18, Corollary 4.2] we conclude that there exists a subsequence of f_m^{-1} which converges weakly-$*$ in BV to some h, in particular $f_m^{-1} \to h$ in L^1_{loc}. Injectivity almost everywhere of both f and h is then obtained by following the lines of proof of Lemma 4.4. □

5. Application to Calculus of Variations

Let $n \geq 3$ and $\Omega,\ \Omega' \subset \mathbb{R}^n$ be bounded domains, e.g. representing the reference and deformed configurations in nonlinear elasticity. Define the energy functional

$$\mathcal{E}(f) := \int_\Omega W(Df(x))\,dx, \tag{5.1}$$

where $W\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a *polyconvex* function, i.e., W can be expressed as a convex function of the minors of its argument, satisfying

$$W(F) \geq \begin{cases} C\big(|F|^{n-1} + \varphi(\det F) + A(|\operatorname{cof} F|) - 1\big), & \text{if } \det F > 0,\\ \infty, & \text{if } \det F \leq 0, \end{cases} \tag{5.2}$$

for some $C > 0$ and for some positive functions A and φ. Consider a homeomorphism f_0 from $\overline{\Omega}$ onto $\overline{\Omega'}$ such that $\mathcal{E}(f_0) < \infty$, and the following sets of admissible functions:

$$\mathcal{H}_{f_0}(\Omega,\mathbb{R}^n) := \Big\{ f : \overline{\Omega} \to \mathbb{R}^n : f \text{ is a homeomorphism of } \overline{\Omega} \text{ onto } \overline{\Omega'} \text{ satisfying the Lusin (N) condition, } f = f_0 \text{ on } \partial\Omega, \text{ and } \mathcal{E}(f) \leq \mathcal{E}(f_0) \Big\}$$

and

$$\overline{\mathcal{H}}^w_{f_0}(\Omega,\mathbb{R}^n) := \Big\{ f : \Omega \to \mathbb{R}^n : \text{ there are } f_m \in \mathcal{H}_{f_0}(\Omega,\mathbb{R}^n) \text{ with } f_m \rightharpoonup f \text{ weakly in } W^{1,n-1}(\Omega,\mathbb{R}^n) \Big\}.$$

Note that $\overline{\mathcal{H}}^w_{f_0}(\Omega,\mathbb{R}^n)$ is weakly (sequentially) closed and hence it is a suitable set of mappings for variational approach:

Proposition 5.1. *Let $g_m \in \overline{\mathcal{H}}^w_{f_0}(\Omega,\mathbb{R}^n)$ and assume that $g_m \rightharpoonup g$ weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$. Then $g \in \overline{\mathcal{H}}^w_{f_0}(\Omega,\mathbb{R}^n)$. In particular, there exists a sequence $f_m \in \mathcal{H}_{f_0}(\Omega,\mathbb{R}^n)$ such that $f_m \rightharpoonup g$ weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$, $g = f_0$ on $\partial\Omega$, and $\sup_m \mathcal{E}(f_m) \leq \mathcal{E}(f_0) < \infty$.*

Proof. Since $W^{1,n-1}$ is reflexive and separable we can find $\{L_i\}_{i\in\mathbb{N}} \subset (W^{1,n-1})^*$ which is dense. We can assume (passing to a subsequence) that

$$|L_i(g_m - g)| < \frac{1}{k} \text{ for every } i \in \{1,\dots,m\}.$$

For every g_m we can find a sequence in $\mathcal{H}_{f_0}(\Omega,\mathbb{R}^n)$ which converges weakly and thus we can fix $f_m \in \mathcal{H}_{f_0}(\Omega,\mathbb{R}^n)$ such that

$$|L_i(g_m - f_m)| < \frac{1}{m} \text{ for every } i \in \{1,\dots,m\}.$$

It follows that for every $i \in \mathbb{N}$ we have

$$\lim_{m\to\infty} L_i(f_m) = L_i(g).$$

Since $f_m(x) \in \Omega' \subset B(0,R)$ for all $x \in \Omega$, $\|Df_m\|_{L^{n-1}} \leq \mathcal{E}(f_m) \leq \mathcal{E}(f_0)$ result in $\|f_m\|_{W^{1,n-1}} \leq M$ for all m and some constant $M > 0$, so we easily obtain that

$$\lim_{m\to\infty} L(f_m) = L(g) \text{ for every } L \in (W^{1,n-1})^*.$$

Note further that the set $f_0 + W^{1,n-1}_0(\Omega,\mathbb{R}^n)$ is closed and convex and thus weakly closed, therefore, $g = f_0$ on $\partial\Omega$. ☐

Proposition 5.2. *Let g, $g_m \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)$ and assume that $g_m \rightharpoonup g$ weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. Then (up to subsequence) $J_{g_m} \rightharpoonup J_g$ weakly in $L^1(\Omega)$.*

Proof. Let us first prove that for every ball B, such that f satisfies the (INV) condition and (3.23) in B,

$$\lim_{m\to\infty} |\operatorname{im}_T(g_m, B)| = |\operatorname{im}_T(g, B)|. \tag{5.3}$$

Fix such a ball B for all $m \in \mathbb{N}$ and $\varepsilon > 0$. In view of Lemma 3.1, for any $m \in \mathbb{N}$ there exists a sequence $f_{m,k} \in \mathcal{H}_{f_0}(\Omega, \mathbb{R}^n)$ such that

$$|f_{m,k}(B) \triangle \operatorname{im}_T(g_m, B)| \leq \frac{\varepsilon}{2}.$$

Using the diagonal procedure as in Proposition 5.1, we find a sequence $f_{m,k(m)} \in \mathcal{H}_{f_0}(\Omega, \mathbb{R}^n)$ with $f_{m,k(m)} \rightharpoonup g$ weakly in $W^{1,n-1}$. Again by Lemma 3.1 it holds that

$$|f_{m,k(m)}(B) \triangle \operatorname{im}_T(g, B)| \leq \frac{\varepsilon}{2}$$

for m big enough. Combining these two inequalities, we obtain (5.3).

Now the proof follows proof of Lemma 4.2, since for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{H}^1$-a.e. $r \in (0, r_a)$ the mapping g satisfies (3.23) and (INV) in $B(a, r)$. □

Theorem 5.3. *Let $n \geq 3$ and Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains, let also $W: \mathbb{R}^{n\times n} \to \mathbb{R}$ be a polyconvex function satisfying* (5.2) *for some functions A and φ satisfying* (1.1)–(1.3) *and a constant $C > 0$. Assume further that f_0 is a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega'}$ such that $\mathcal{E}(f_0) < \infty$, where $\mathcal{E}$ is the energy defined by* (5.1)*. Then there exists $f \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)$ such that*

$$\mathcal{E}(f) = \inf\big\{\mathcal{E}(h) : \; h \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)\big\}.$$

Moreover, f satisfies the (INV) *condition and the Lusin* (N) *condition.*

Proof. Let f_m be a minimizing sequence for $\mathcal{E}$, then f_m form a bounded sequence in $W^{1,n-1}$, and hence using Proposition 5.1 there is $f \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)$ such that (up to a subsequence) $f_m \rightharpoonup f$ weakly in $W^{1,n-1}$. Provided with Proposition 5.2, we obtain that $\mathcal{E}$ is lower semicontinuous in $\overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)$ following the proof of Lemma 4.3.

Proposition 5.1 and Theorem 1.1 imply thus that f satisfies the (INV) and the (N) conditions and also that

$$\mathcal{E}(f) \leq \liminf_{m\to\infty} \mathcal{E}(f_m) = \lim_{m\to\infty} \mathcal{E}(f_m) = \inf\big\{\mathcal{E}(h) : \; h \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)\big\} \leq \mathcal{E}(f).$$

□

Remark 5.4. Let us note that it is not clear if the two following infima

$$\inf\big\{\mathcal{E}(h) : \; h \in \mathcal{H}_{f_0}(\Omega, \mathbb{R}^n)\big\} \quad \text{and} \quad \inf\big\{\mathcal{E}(h) : \; h \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)\big\},$$

are equal or not since the space $\mathcal{H}_{f_0}(\Omega, \mathbb{R}^n)$ is not compact.

Analogously we can use the results of [17] and Section 4.5 to obtain the following theorem.

Theorem 5.5. *Let $n \geq 3$ and Ω, $\Omega' \subset \mathbb{R}^n$ be Lipschitz domains, let also $W\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ be a polyconvex function satisfying*

$$W(F) \geq \begin{cases} C\Big(|F|^{n-1} + \varphi(\det F) + \Big(\frac{|F|^n}{\det F}\Big)^{\frac{1}{n-1}} - 1\Big), & \textit{if } \det F > 0,\\ \infty, & \textit{if } \det F \leq 0,\end{cases}$$

for some function φ satisfying (1.1)–(1.2) *and* (4.7)*, and a constant $C > 0$. We assume that W may be represented as a convex function of subdeterminants of order strictly less than $n-1$ and of $\det F$, i.e., it is not a function of subdeterminants of order $n-1$. Assume further that f_0 is a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega'}$ such that $\mathcal{E}(f_0) < \infty$, where $\mathcal{E}$ is the energy defined by* (5.1)*. Then there exists $f \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)$ such that*

$$\mathcal{E}(f) = \inf\big\{\mathcal{E}(h):\ h \in \overline{\mathcal{H}}^w_{f_0}(\Omega, \mathbb{R}^n)\big\}.$$

Moreover, f satisfies the (INV) *condition and the Lusin* (N) *condition.*

Proof. The proof is analogous to the proof of Theorem 5.3. The only difference is that in the proof of lower semicontinuity we do not have (1.3) and therefore we cannot prove weak convergence of cof Df_m as in the proof of Lemma 4.3. However, we do not need this as our W "does not depend" on $(n-1)\times(n-1)$ subdeterminants. □

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Paper V

DIFFERENTIABILITY ALMOST EVERYWHERE OF WEAK LIMITS OF BI-SOBOLEV HOMEOMORPHISMS

ANNA DOLEŽALOVÁ AND ANASTASIA MOLCHANOVA

[Reshetnyak's] synthesis of classical function theory and Sobolev function classes was so fruitful that it was given a special name: quasiconformal analysis.

A. D. Aleksandrov, 1999 Russ. Math. Surv. 54 1069

Abstract. This paper investigates the differentiability of weak limits of bi-Sobolev homeomorphisms. Given $p > n-1$, consider a sequence of homeomorphisms f_k with positive Jacobians $J_{f_k} > 0$ almost everywhere and $\sup_k(\|f_k\|_{W^{1,n-1}} + \|f_k^{-1}\|_{W^{1,p}}) < \infty$. We prove that if f and h are weak limits of f_k and f_k^{-1}, respectively, with positive Jacobians $J_f > 0$ and $J_h > 0$ a.e., then $h(f(x)) = x$ and $f(h(y)) = y$ both hold a.e. and f and h are differentiable almost everywhere.

1. Introduction

Let Ω and Ω' be domains, i.e. non-empty connected open sets, in $\mathbb{R}^n$ and $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a mapping from Ω to Ω'. According to classic results of Geometric analysis, if $p > n$, the mapping f is differentiable almost everywhere. This result was established in 1941 for $n = 2$ by Cesari [4] and later generalized to arbitrary n by Calderón [2]. The a.e.-differentiability of continuous and monotone mappings was studied from a geometrical perspective by Väisälä [29] and Reshetnyak [24, 25, 26]. This includes mappings with bounded distortion, also known as quasiregular mappings, and mappings with finite distortion (even for $p = n$). Further details on these results can be found in [25, 29]. The results also extend to $W^{1,1}$-homeomorphisms in dimension $n = 2$, as shown by Gehring and Lehto [10], and $W^{1,p}$-homeomorphisms with $p > n-1$ if $n \geq 3$, see Väisälä [29] (also Onninen [22, Theorem 1.2 and Example 1.3]).

For $W^{1,n-1}$-Sobolev homeomorphisms with $n \geq 3$, the a.e.-differentiability was established by considering the integrability of the inner distortion $K_I \in L^1(\Omega)$, where $J_f(x) := \det Df(x)$ is the Jacobian, $\operatorname{adj} Df$ is the adjugate matrix of Df and

$$K_I := \frac{|\operatorname{adj} Df|^n}{J_f(x)^{n-1}},$$

see [28]. This condition on integrability of distortion is sharp, meaning for any $\delta \in (0,1)$ and $n \geq 3$ there exists a homeomorphism $f \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$ such that $K_I \in L^\delta((-1,1)^n)$ and f is not classically differentiable on a set of positive measure [14]. The a.e.-differentiability of $W^{1,n-1}$-Sobolev maps also holds for continuous, open, and discrete mappings of finite distortion with nonnegative Jacobian if a particular weighted

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distortion function is integrable [30]. The condition $K_I \in L^1(\Omega)$ essentially means that $f^{-1} \in W^{1,n}(f(\Omega), \mathbb{R}^n)$ [20, Theorem 1.1]. Together with the oscillation estimate from [23, Lemma 2.1] we then obtain that for almost all $x \in \Omega$

$$\limsup_{r \to 0+} \frac{\operatorname{osc}_{B(x,r)} f}{r} < \infty,$$

and hence f is differentiable in x by the Stepanov Theorem. Thus, instead of assumptions for distortion, we can directly consider bi-Sobolev homeomorphisms. The inverse mapping theorem (see e.g. [12, Theorem A.29]) states that if $f \in W^{1,n-1}$, $J_f > 0$ a.e., and $f^{-1} \in W^{1,p}$ with $p > n-1$, then both f and f^{-1} are differentiable almost everywhere (for a more general approach, the reader is referred to [31]). However, Csörnyei, Hencl, and Malý constructed in Example 5.2 in [5] a homeomorphism $f \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$, $n \geq 3$, with $J_f > 0$ a.e. that is nowhere differentiable and its inverse $f^{-1} \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$ is also nowhere differentiable.

In this work, we examine the a.e.-differentiability of a class of *weak limits of homeomorphisms*. This class of mappings is well suited for the calculus of variations approach and may serve as deformations in Continuum Mechanics models. For further information, refer to [15, 17, 19]. Weak limits of Sobolev homeomorphisms have received significant attention in recent years, with various studies conducted, including [1, 3, 6, 7, 8, 9, 13, 16].

Here we consider the energy functional

$$\mathcal{E}(f) := \int_\Omega |Df(x)|^{n-1}\, dx + \int_{\Omega'} |Df^{-1}(y)|^p\, dy$$

for bi-Sobolev mappings $f\colon \Omega \to \Omega'$ such that f is invertible almost everywhere, $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, and $f^{-1} \in W^{1,p}(\Omega', \mathbb{R}^n)$ for some $p > n-1$.

The main result, which is proven in Section 1.2, reads as follows.

Theorem 1.1. *Let $n \geq 2$, $p > n-1$, Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $k = 0, 1, 2\dots$, be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$ a.e. and*

$$\sup_k \mathcal{E}(f_k) < \infty.$$

Assume that $f\colon \Omega \to \mathbb{R}^n$ is a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. and $h\colon \Omega' \to \mathbb{R}^n$ is a weak limit of $\{f_k^{-1}\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega', \mathbb{R}^n)$ with $J_h > 0$ a.e. Then for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$, and both f and h are differentiable almost everywhere.

Let us note the following result, which better suits the Calculus of Variations approach since it formulates the assumptions only for f_k.

Corollary 1.2. *Let $n \geq 2$, $p > n-1$, Ω, $\Omega' \subset \mathbb{R}^n$ be bounded domains and φ be a positive convex function on $(0, \infty)$ with*

$$\lim_{t\to 0^+} \varphi(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty. \tag{1.1}$$

Let $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $k = 0, 1, 2\dots$, be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$ a.e. such that $\sup_k \mathcal{F}(f_k) < \infty$, where

$$\mathcal{F}(f) := \int_\Omega |Df(x)|^{n-1} + \frac{|\operatorname{adj} Df(x)|^p}{J_f^{p-1}(x)} + \varphi(J_f(x))\, dx.$$

Assume that $f\colon \Omega \to \mathbb{R}^n$ is a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $h\colon \Omega' \to \mathbb{R}^n$ is a weak limit of $\{f_k^{-1}\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega', \mathbb{R}^n)$. Then for a.e. $x \in \Omega$ we have $h(f(x)) = x$

and for a.e. $y \in \Omega'$ *we have* $f(h(y)) = y$*, and both* f *and* h *are differentiable almost everywhere.*

2. Preliminaries

By $B(c, r)$, we denote the open euclidean ball with centre $c \in \mathbb{R}^n$ and radius $r > 0$, and $S(c, r)$ stands for the corresponding sphere.

2.1. **Topological image and** (INV) **condition.** Although a weak limit of homeomorphisms may not be a homeomorphism, it may possess an invertibility property known as the (INV) condition. The (INV) condition states, informally, that a ball $\overline{B(x, r)}$ is mapped inside the image of the sphere $f(S(x, r))$ and the complement $\Omega \setminus \overline{B(x, r)}$ is mapped outside $f(S(x, r))$. This concept was introduced for $W^{1,p}$-mappings, where $p > n - 1$, by Müller and Spector [21], although the fact that a ball $B(x, r)$ is mapped inside the image of a sphere $f(S(a, r))$ was known in literature before as *monotonicity*, see [25] and [32, §2]. Suppose that $f \colon S(y, r) \to \mathbb{R}^n$ is continuous, we define the *topological image* of $B(x, r)$ as

$$f^T(B(x, r)) := \{z \in \mathbb{R}^n \setminus f(S(x, r)) : \deg(f, S(x, r), z) \neq 0\} \tag{2.1}$$

and the *topological image* of x as

$$f^T(x) := \bigcap_{r>0, r \notin N_x} f^{*T}(B(x, r)) \cup f^*(S(x, r)),$$

where N_x is a null set from the definition just below.

Definition 2.1. A mapping $f \colon \Omega \to \mathbb{R}^n$ satisfies the (INV) *condition*, provided that for every $x \in \Omega$ there exist a constant $r_x > 0$ and an $\mathcal{L}^1$-null set N_x such that for all $r \in (0, r_x) \setminus N_x$, the restriction $f|_{S(x,r)}$ is continuous and

(i) $f(z) \in f^T(B(x, r)) \cup f(S(x, r))$ for a.e. $z \in \overline{B(x, r)}$,
(ii) $f(z) \in \mathbb{R}^n \setminus f^T(B(x, r))$ for a.e. $z \in \Omega \setminus B(x, r)$.

Let us note that for a particular representative of a Sobolev mapping, Definition 2.1 allows for some points to escape their destiny, e.g. a null-set inside the ball may be mapped outside the image of this ball. Thus, we also consider a stronger version of the (INV) condition.

Definition 2.2. A mapping $f \colon \Omega \to \mathbb{R}^n$ satisfies the *strong* (INV) *condition*, provided that for every $x \in \Omega$ there exist a constant $r_x > 0$ and an $\mathcal{L}^1$-null set N_x such that for all $r \in (0, r_x) \setminus N_x$ the restriction $f|_{S(x,r)}$ is continuous and

(i) $f(z) \in f^T(B(x, r)) \cup f(S(x, r))$ for every $z \in \overline{B(x, r)}$,
(ii) $f(z) \in \mathbb{R}^n \setminus f^T(B(x, r))$ for every $z \in \Omega \setminus B(x, r)$.

2.2. **Precise, super-precise, and hyper-precise representative of a Sobolev mapping.** Let $1 \leq p \leq n$ and $f \in W^{1,p}(\mathbb{R}^n)$, then the *precise representative* of f is given by

$$f^*(a) := \begin{cases} \lim\limits_{r \to 0^+} \dfrac{1}{|B(a, r)|} \displaystyle\int_{B(a,r)} f(x)\, dx & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that the representative f^* is p-quasicontinuous (see remarks after [21, Proposition 2.8]).

Let now $f \colon \Omega \to \mathbb{R}^n$ be a $W^{1,p}$-weak limit of homeomorphisms $f_k \colon \Omega \to \mathbb{R}^n$ with $p \in (n-1, n]$ for $n > 2$ or $p \in [1, 2]$ for $n = 2$. Then by [1, Theorem 5.2] there exists an

$\mathcal{H}^{n-p}$-null set $NC \subset \Omega$ and a representative f^{**} of f such that f^{**} is continuous at every $x \in \Omega \backslash NC$, a set-valued image $f^T(x)$ is a singleton for every $y \in \Omega \backslash NC$, $f^{**} = f^*$ cap_p-a.e., and f^{**} can be chosen so that $f^{**}(x) \in f^T(x)$ for every $x \in \Omega$. We will call f^{**} a *super-precise representative* of f.

The *hyper-precise* representative $\tilde{f}$ is defined as

$$\tilde{f}(a) := \limsup_{r \to 0^+} \frac{1}{|B(a,r)|} \int_{B(a,r)} f(x)\, dx. \tag{2.3}$$

We need the following monotonicity property of mappings satisfying the strong (INV) condition.

Lemma 2.3. *Let $n \geq 2$ and $\Omega' \subset \mathbb{R}^n$ be a bounded domain. If $h\colon \Omega' \to \mathbb{R}^n$ satisfies the strong* (INV) *condition, then h is monotone for almost all radii, i.e. for $y \in \Omega'$ there exists an $\mathcal{L}^1$-null set N_y such that for all $r \in (0, r_y) \setminus N_y$ it holds that $\mathrm{osc}_{B(y,r)}\, h \leq \mathrm{osc}_{S(y,r)}\, h$.*
If, moreover, $h \in W^{1,p}(\Omega', \mathbb{R}^n)$ with $p > n-1$, then for any $r \in \left(0, \frac{r_y}{2}\right)$ the following estimate holds

$$\mathrm{osc}_{B(y,r)}\, h \leq Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p}.$$

Proof. Let N_y be a set from Definition 2.2. Then for $y \in \Omega'$ and $r \in (0, r_y) \setminus N_y$ it holds that h is continuous on the sphere $S(y,r)$ and $h(z) \in h^T(B(y,r)) \cup h(S(y,r))$ for every $z \in \overline{B(y,r)}$. In this case, $h(S(y,r))$ is a compact set and $h^T(B(y,r)) \subseteq \mathbb{R}^n \setminus A$, where A is the unbounded component of $\mathbb{R}^n \setminus h(S(y,r))$ (since by the basic properties of the topological degree [12, p. 48(d)] we have $\deg(h, S(y,r), \xi) = 0$ for all $\xi \in A$), and therefore $\mathrm{osc}_{B(y,r)}\, h \leq \mathrm{osc}_{S(y,r)}\, h$.

Further, for $y \in \Omega'$ and $r > 0$, and for a.e. $t \in [r, 2r)$, it holds that

$$\mathrm{osc}_{B(y,r)}\, h \leq \mathrm{osc}_{B(y,t)}\, h \leq \mathrm{osc}_{S(y,t)}\, h.$$

Then by the Sobolev embedding theorem on spheres [12, Lemma 2.19], following the proof of [12, Theorem 2.24], we obtain that

$$\mathrm{osc}_{B(y,r)}\, h \leq \mathrm{osc}_{S(y,t)}\, h \leq Ct \left(t^{-n+1} \int_{S(y,t)} |Dh|^p \right)^{1/p} \leq Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p}.$$

□

Remark 2.4. In case $p > n$, $h^* = h^{**} = \tilde{h}$ is the continuous representative of h and h^* is differentiable almost everywhere [2] and satisfies the Lusin (N) condition in Ω [18]. Moreover, due to compact embedding of $W^{1,p}$ into the Hölder space $C^{0,\alpha}$, weak convergence in $W^{1,p}$ implies uniform convergence on compact sets. With these properties, the subsequent analysis becomes simplified, and the details are left to the reader.

3. A.e.-invertibility of f

Since a limit of homeomorphisms may not be a homeomorphism, we need to define a weaker notion of inverse mapping. First recall that a mapping $f\colon \Omega \to \Omega'$ is called *injective a.e. in domain* if there exists a null set $\Sigma \subset \Omega$, $|\Sigma| = 0$, such that the restriction $f|_{\Omega\setminus\Sigma}\colon \Omega \setminus \Sigma \to f(\Omega \setminus \Sigma)$ is injective. A mapping $f\colon \Omega \to \Omega'$ is called *injective a.e. in image* if there exists a null set $\Sigma' \subset \Omega'$, $|\Sigma'| = 0$, such that for any $y \in f(\Omega) \setminus \Sigma'$ the preimage $f^{-1}(y) := \{x \in \Omega : f(x) = y\}$ consists of only one point. Note that if f is injective a.e. in image and satisfies the $(\mathrm{N})^{-1}$ condition, then f is injective a.e. in domain. If instead f is injective a.e. in domain, f satisfies the (N) condition, and $|\Omega'| = |f(\Omega)|$ then f is injective a.e. in image. We say that $h\colon \Omega' \to \Omega$ is the a.e.-inverse to $f\colon \Omega \to \Omega'$ if for a.e. $x \in \Omega$ we have $h(f(x)) = x$ and for a.e. $y \in \Omega'$ we have $f(h(y)) = y$. Note that if f satisfies the (N)^{-1} condition, then f is injective a.e. in image if and only if there exists the a.e.-inverse to f.

The following lemma provides some additional conditions that guarantee the a.e.-invertibility of f in our setting.

Lemma 3.1. *Let $n \geq 2$, Ω and Ω' be bounded domains in $\mathbb{R}^n$, $p > n-1$, and let $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$. Let also $f\colon \Omega \to \mathbb{R}^n$ be a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. Assume also that the sequence $\{f_k^{-1}\}_{k\in\mathbb{N}}$ converges $W^{1,p}$-weakly to $h\colon \Omega' \to \mathbb{R}^n$ with $J_h > 0$ a.e. Then $h^{**}(f(x)) = x$ a.e. in Ω and $f(h^{**}(y)) = y$ a.e. in Ω'.*

Proof. Let $p > n-1$, and fix a representative of f, which we denote by the same symbol. If needed, we pass to a subsequence so that $f_k \to f$ and $f_k^{-1} \to h$ pointwise a.e. Since h is a $W^{1,p}$-weak limit of Sobolev homeomorphisms with $p > n-1$, the super-precise representative h^{**} satisfies the strong (INV) condition [1, Theorem 5.2 and Lemma 5.3]. Then there exists a set $G_1' \subset \Omega'$ of full measure $|G_1'| = |\Omega'|$: $J_{h^{**}}(y) > 0$ for all $y \in G_1'$, h^{**} is injective in G_1' (see [21, Lemma 3.4] and [1, Theorem 1.2]) and $f_k^{-1}(y) \to h^{**}(y)$ for all $y \in G_1'$.

<u>Step 1. $h^{**}(f(x)) = x$ a.e.</u>: By Lemma 2.3, we know that $\operatorname{osc}_{B(y,r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$ for a.e. $y \in \Omega'$. Since $J_f > 0$ a.e. (and therefore f satisfies the (N)^{-1} condition), $\operatorname{osc}_{B(f(x),r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$ for a.e. $x \in \Omega$.

Let $G_1 \subset f^{-1}(G_1')$ be a set such that $|G_1| = |\Omega|$ and for all $x \in G_1$ it holds that $f_k(x) \to f(x)$ and $\operatorname{osc}_{B(f(x),r)} h^{**} \underset{r\to 0}{\longrightarrow} 0$.

For $x \in G_1$ and $r > 0$, by the pointwise convergence of f_k in $x \in G_1$ and f_k^{-1} in $f(x) \in G_1'$, we can find $k_0 \in \mathbb{N}$ big enough such that

$$f_k(x) \in B(f(x), r) \quad \text{and} \quad f_k^{-1}(f(x)) \in B(h^{**}(f(x)), r)$$

for all $k \geq k_0$. Moreover, by [21, Lemma 2.9] (though it is formulated for the precise representative h^*, it holds also for the super-precise representative h^{**} with analogous proof), there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ (that depends on r) and a number $j_0 \in \mathbb{N}$ big enough such that

$$\operatorname{osc}_{S(f(x),r)} f_{k_j}^{-1} \leq \operatorname{osc}_{S(f(x),r)} h^{**} + r$$

for all $j \geq j_0$.

Then we have

$$\begin{aligned} |f_{k_j}^{-1}(f_{k_j}(x)) - h^{**}(f(x))| &\leq |f_{k_j}^{-1}(f_{k_j}(x)) - f_{k_j}^{-1}(f(x))| + |f_{k_j}^{-1}(f(x)) - h^{**}(f(x))| \\ &\leq \operatorname{osc}_{B(f(x),r)} f_{k_j}^{-1} + r \leq \operatorname{osc}_{S(f(x),r)} f_{k_j}^{-1} + r \\ &\leq \operatorname{osc}_{S(f(x),r)} h^{**} + r + r \leq \operatorname{osc}_{B(f(x),2r)} h^{**} + 2r. \end{aligned}$$

Therefore, by definition of G_1,

$$|x - h^{**}(f(x))| = |f_{k_j}^{-1}(f_{k_j}(x)) - h^{**}(f(x))| \leq \lim_{r\to 0} (\operatorname{osc}_{B(f(x),2r)} h^{**} + 2r) = 0$$

for all $x \in G_1$, which concludes Step 1.

<u>Step 2. $f(h^{**}(y)) = y$ a.e.</u>: We know that h^{**} is injective a.e. on G_1' and both f and h^{**} satisfies the (N)^{-1} condition, so when we set

$$G_2' := \big(G_1' \cap (h^{**})^{-1}(G_1)\big) \setminus (h^{**})^{-1}(f^{-1}(\Omega' \setminus G_1')),$$

we know it is a set of full measure. Let us take $y \in G_2'$. Since f_k^{-1} is a homeomorphism onto Ω, we can find $y_k \in \Omega'$ such that $f_k^{-1}(y_k) = h^{**}(y)$. Therefore,

$$y_k = f_k(f_k^{-1}(y_k)) = f_k(h^{**}(y)) \to f(h^{**}(y)),$$

so y_k converges to some $\tilde{y} = f(h^{**}(y))$. We apply h^{**} to both sides to get $h^{**}(\tilde{y}) = h^{**}(f(h^{**}(y)))$. From $y \in G_2'$ we have that $h^{**}(y) \in G_1$. Since $h^{**}(f(x)) = x$ on G_1 we get $h^{**}(\tilde{y}) = h^{**}(f(h^{**}(y))) = h^{**}(y)$. Now we can have either $\tilde{y} \in G_1'$ or $\tilde{y} \notin G_1'$. In the first case, $\tilde{y} = y$ as h^{**} is injective on G_1', so $f(h^{**}(y)) = y$. In the other case, $f(h^{**}(y)) \in \Omega' \setminus G_1'$, which is a contradiction to $y \in G_2'$.

□

Remark 3.2. If $p > n$, equality $h^{**}(f(x)) = x$ can be derived easily from

$$|x - h^{**}(f(x))| \le |f_k^{-1}(f_k(x)) - f_k^{-1}(f(x))| + |f_k^{-1}(f(x)) - h^{**}(f(x))|,$$

using uniform convergence $f_k^{-1} \rightrightarrows h^{**}$ (up to subsequence) and the Morrey inequality for f_k^{-1}. The other relation $f(h^{**}(y)) = y$ follows the same way as above.

Remark 3.3. Since both f and h satisfy the $(\mathrm{N})^{-1}$ condition, the identities $h(f(x)) = x$ a.e. in Ω and $f(h(y)) = y$ a.e. in Ω' hold for arbitrary representatives.

4. Differentiability

First, let us notice the following well-known fact.

Lemma 4.1. *Let $n \ge 2$, $p > n-1$ and Ω' be a bounded domain in $\mathbb{R}^n$. If $h \in W^{1,p}_{\mathrm{loc}}(\Omega', \mathbb{R}^n)$ satisfies the strong* (INV) *condition, then h is differentiable a.e. in Ω'.*

Proof. By Lemma 2.3 we have

$$\operatorname{osc}_{B(y,r)} h \le Cr \left(r^{-n} \int_{B(y,2r)} |Dh|^p \right)^{1/p},$$

which implies by setting $r = |z - y|$ that

$$\limsup_{z \to y} \frac{|h(z) - h(y)|}{|z - y|} \le C|Dh(y)| < \infty$$

for any Lebesgue point y of $|Dh|^p$ and, therefore, h is differentiable a.e. by the Stepanov theorem [27], see also [12, Theorem 2.23].

□

We also need the following modification of [12, Lemma A.29], which gives us the a.e.-differentiability of mapping f from Theorem 1.1 – but the derivative is only with respect to a set of full measure.

Lemma 4.2. *Let $n \ge 2$ and Ω, Ω' be bounded domains in $\mathbb{R}^n$. Let $\Lambda \subset \Omega$, $\Lambda' \subset \Omega'$ be sets of full measure and $h\colon \Omega' \to \Omega$ such that $h\colon \Lambda' \to \Lambda = h(\Lambda')$ is differentiable with respect to the relative topology in Λ', i.e. induced by the topology in $\mathbb{R}^n$, and $J_h(y) > 0$ for all $y \in \Lambda'$. Assume also that $h|_{\Lambda'}$ is injective, and the inverse mapping $f := h^{-1}$ is continuous in Λ with respect to the relative topology in Λ. Then f is differentiable on Λ with respect to the relative topology in Λ and $Df(x) = (Dh(f(x)))^{-1}$ for all $x \in \Lambda$.*

Proof. Since $h\colon \Lambda' \to \Lambda$ is a homeomorphism, the proof of this lemma follows the lines of the proof of [12, Lemma A.29]. We present it here for the convenience of the reader.

By the differentiability of h we know that for $y \in \Lambda'$

$$\lim_{\bar{y} \to y,\, \bar{y} \in \Lambda'} \frac{h(\bar{y}) - h(y) - Dh(y)(\bar{y} - y)}{|\bar{y} - y|} = 0. \tag{4.1}$$

For $\bar{x}$, $x \in \Lambda$ denote $\bar{y} = f(\bar{x})$, $y = f(x) \in \Lambda'$, then

$$h(\bar{y}) - h(y) = h(f(\bar{x})) - h(f(x)) = \bar{x} - x.$$

Since $J_h(y) > 0$ we obtain for $\bar{y}$ close enough to y that

$$|\bar{x} - x| = |h(\bar{y}) - h(y)| \approx |Dh(y)(\bar{y} - y)| \approx |\bar{y} - y|.$$

Then from (4.1) it follows

$$0 = \lim_{\bar{y}\to y,\, \bar{y}\in\Lambda'} \frac{(Dh(y))^{-1}\,(h(\bar{y}) - h(y) - Dh(y)(\bar{y} - y))}{|y' - y|} =$$
$$\lim_{\bar{y}\to y,\, \bar{y}\in\Lambda'} \frac{(Dh(y))^{-1}\,(h(\bar{y}) - h(y)) - (\bar{y} - y)}{|y' - y|} \approx$$
$$\lim_{\bar{x}\to x,\, \bar{x}\in\Lambda} \frac{(Dh(f(x)))^{-1}\,(\bar{x} - x) - (f(\bar{x}) - f(x))}{|\bar{x} - x|},$$

which concludes the proof.

□

The following proposition is a version of an inverse function theorem.

Proposition 4.3. *Let $n \geq 2$, $p > n-1$, Ω and Ω' be bounded domains in $\mathbb{R}^n$, $\Lambda \subset \Omega$ and $\Lambda' \subset \Omega'$ be sets of full measure and $h \in W^{1,p}(\Omega', \Omega)$ satisfy the strong* (INV) *condition and be differentiable with $J_h(y) > 0$ for any $y \in \Lambda'$. Assume also that the restriction $h|_{\Lambda'}\colon \Lambda' \to \Lambda$ is one-to-one for any $y \in \Lambda'$. Then for any $y_0 \in \Lambda'$ there exists a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that the topological image $h^T(B(y_0, r_m))$ contains $B\left(h(y_0), \frac{r_m}{3}\right)$.*

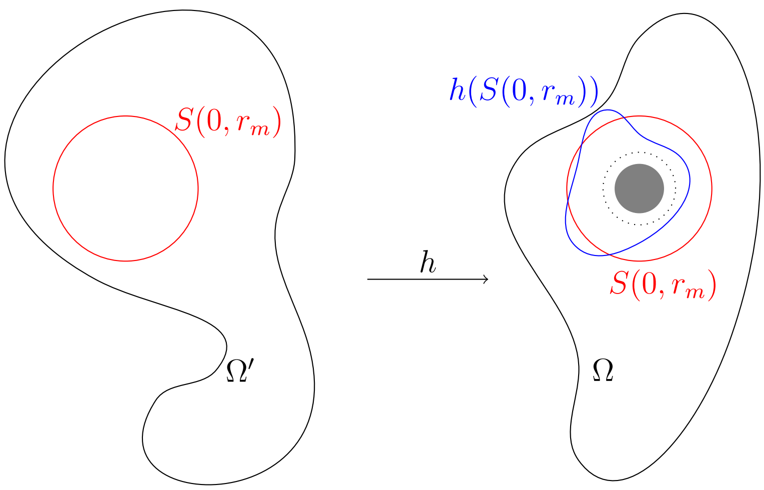

FIGURE 1. Mapping h maps the red sphere $S(0, r_m)$ to $h(S(0, r_m))$ (blue); the grey ball $B(0, r_m/3)$ does not intersect $h(S(0, r_m))$, since its distance from 0 is at least $r_m/2$ (denoted by the dotted sphere).

Proof. Without loss of generality, by a translation and a linear change of variables, we may assume that $y_0 = 0$, $h(y_0) = 0$, and $Dh(y_0) = Id$. Since h is differentiable at 0, it holds that $h(y) = y + o(|y|)$ if $y \to 0$. That means that there exists $r_0 > 0$ such that

$$|h(y) - y| \leq \frac{|y|}{2} \quad \text{for all } y \in B(0, r_0) \subset \Omega'. \tag{4.2}$$

Consider a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that h is continuous on $S(0, r_m)$ and Definition 2.2 (i–ii) is fulfilled. Let now $z \in B\left(0, \frac{r_m}{3}\right) \subset \Omega$, the inequality (4.2) implies $z \notin h(S(0, r_m))$. Since $\operatorname{dist}(z, S(0, r_m)) > r_m/2$, from (4.2) we know that $1 = \deg(z, Id, S(0, r_m)) = \deg(z, h, S(0, r_m))$. Therefore, $B\left(0, \frac{r_m}{3}\right) \subset h^T(B(0, r_m))$, see Figure 1 for illustration. □

The closing theorem of this section concludes the differentiability part of Theorem 1.1.

Theorem 4.4. *Let $n \geq 2$, $p > n - 1$, Ω and Ω' be bounded domains in $\mathbb{R}^n$ and $f_k \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be homeomorphisms of $\overline{\Omega}$ onto $\overline{\Omega'}$ with $J_{f_k} > 0$. Let $f : \Omega \to \mathbb{R}^n$ be a weak limit of $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. Assume also that the sequence $\{f_k^{-1}\}_{k\in\mathbb{N}}$ converges $W^{1,p}$-weakly to $h : \Omega' \to \mathbb{R}^n$ with $J_h > 0$ a.e. Then h^{**} is differentiable a.e. in Ω' and $\tilde{f}$ is differentiable a.e. in Ω.*

Proof. We again pass to a subsequence (if needed) so that $f_k \to f$ and $f_k^{-1} \to h$ pointwise a.e. Since h is a $W^{1,p}$-weak limit of Sobolev homeomorphisms with $p > n-1$, the super-precise representative h^{**} satisfies the strong (INV) condition [1, Theorem 5.2 and Lemma 5.3], is injective a.e. (see [21, Lemma 3.4] and [1, Theorem 1.2]) and continuous on almost all spheres [11, Lemma 2.19]. By Lemma 4.1, h is differentiable a.e. in Ω'. Moreover, since $J_h(y) > 0$ a.e. in Ω', by the change-of-variable formula we conclude that h satisfies the $(\mathrm{N})^{-1}$ condition.

Step 1. Finding sets Λ, Λ': Let f be an arbitrarily fixed representative, and let us introduce *good* sets $G \subset \Omega$, $G' \subset \Omega'$ as

$$G := \{x \in \Omega \,:\, h^{**}(f(x)) = x\} \subset \Omega \qquad \text{and} \qquad G' := \{y \in \Omega' \,:\, f(h^{**}(y)) = y\} \subset \Omega'.$$

It is easy to check that $f(G) = G'$, $h^{**}(G') = G$, and by Lemma 3.1, $|G| = |\Omega|$, $|G'| = |\Omega'|$. And we define *bad* sets $\Sigma \subset G$, $\Sigma' \subset G'$ as

$$\begin{aligned}
\Sigma &:= G \setminus \{x \in \Omega \,:\, J_f(x) > 0,\ f_k(x) \to f(x)\}, \\
\Sigma' &:= G' \setminus \{y \in \Omega' \,:\, h^{**} \text{ is differentiable in } y,\ J_{h^{**}}(y) > 0,\ f_k^{-1}(y) \to h^{**}(y)\}.
\end{aligned}$$

Clearly $|\Sigma| = |\Sigma'| = 0$. Then *very good* sets $\Lambda \subset G$, $\Lambda' \subset G'$ are defined by

$$\Lambda' := G' \setminus (\Sigma' \cup f^{-1}(\Sigma)) \qquad \text{and} \qquad \Lambda := h^{**}(\Lambda').$$

By Lemma 3.1 and $(\mathrm{N})^{-1}$ condition for f and h^{**}, it is not difficult to see that $|\Lambda'| = |G'| = |\Omega'|$, $|\Lambda| = |\Omega|$ and $f(\Lambda) = \Lambda'$.

Step 2. $f|_\Lambda$ is continuous: The restriction $f|_\Lambda : \Lambda \to \Lambda'$ is continuous with respect to the relative topology in Λ. Indeed, let $f|_\Lambda$ be not continuous in some point $x_0 \in \Lambda$, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Lambda$, $x_k \to x_0$, but $f(x_k) \not\to f(x_0)$. We set $y_k := f(x_k) \in \Lambda'$ and $y_0 := (h^{**})^{-1}(x_0) = f(x_0)$. Since $h^{**}|_{\Lambda'} = (f|_\Lambda)^{-1}$, we have $h^{**}(y_k) \to h^{**}(y_0)$, but $y_k \not\to y_0$.

By Proposition 4.3 there exists a sequence $\{r_m\}_{m\in\mathbb{N}} \searrow 0$ such that

$$B\left(h^{**}(y_0), \frac{r_m}{3}\right) \subset (h^{**})^T(B(y_0, r_m)).$$

Let m and $k_0 \in \mathbb{N}$ be big enough so that infinitely many y_k are outside of $B(y_0, r_m)$ for $k \geq k_0$ and $h^{**}(y_{k_0}) \in B\left(h^{**}(y_0), \frac{r_m}{6}\right)$. Passing to a subsequence, we can, for now, assume that $y_k \notin B(y_0, r_m)$ for all k. Then we can find $r > 0$ such that

$$B(y_{k_0}, r) \cap B(y_0, r_m) = \varnothing$$

and, since $h^{**}|_{\Lambda'}$ is continuous,

$$h^{**}(B(y_{k_0}, r) \cap \Lambda') \subset B\left(h^{**}(y_{k_0}), \frac{r_m}{6}\right).$$

Summarizing the above, we obtain

$$h^{**}(B(y_{k_0}, r) \cap \Lambda') \subset B\left(h^{**}(y_{k_0}), \frac{r_m}{6}\right) \subset B\left(h^{**}(y_0), \frac{r_m}{3}\right) \subset (h^{**})^T(B(y_0, r_m)).$$

Thus, for every

$$z \in (B(y_{k_0}, r) \cap \Lambda') \subset (\Omega' \setminus B(y_0, r_m))$$

it holds that $h^{**}(z) \in (h^{**})^T(B(y_0, r_m))$, the latter contradicts to the strong (INV) condition for h^{**}, since a set of positive measure $B(y_{k_0}, r) \cap \Lambda'$ from outside of the ball $B(y_0, r_m)$ is mapped inside the topological image of this ball.

Therefore, f is continuous on Λ with respect to the relative topology, and by Lemma 4.2, we conclude that f is differentiable on Λ with respect to the relative topology.

Step 3. $\tilde{f}$ is differentiable a.e.: It is left to show that a hyper-precise representative $\tilde{f}$, given by (2.3), is differentiable at $x_0 \in \Lambda$ with respect to Ω. Since Λ is a set of full measure and f is continuous on Λ with respect to the relative topology, any point $x \in \Lambda$ is a Lebesgue point of f, and therefore $\tilde{f} = f$ on Λ.

Fix $x_0 \in \Lambda$ and $\varepsilon > 0$. By differentiability of f on Λ with respect to the relative topology, there exists $s > 0$ such that for any $x \in B(x_0, s) \cap \Lambda$ it holds that

$$\frac{|f(x) - f(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} = \frac{|\tilde{f}(x) - \tilde{f}(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} < \frac{\varepsilon}{2}, \tag{4.3}$$

where $Df(x_0)$ denotes the derivative $Df|_\Lambda(x_0)$ with respect to the relative topology. To prove differentiability of $\tilde{f}$, we need to show that for an arbitrary x' close to x_0 it holds that

$$\frac{|\tilde{f}(x') - \tilde{f}(x_0) - Df(x_0)(x' - x_0)|}{|x' - x_0|} < \varepsilon. \tag{4.4}$$

If $x' \in \Lambda$, (4.4) follows immediately from (4.3). In the other case, roughly speaking, we want to find a point $z \in \Lambda$ such that $\frac{|\tilde{f}(x') - \tilde{f}(z)|}{|x' - x_0|}$ and $\frac{|x' - z|}{|x' - x_0|}$ are small, and so we can estimate

$$\begin{aligned} &\frac{|\tilde{f}(x') - \tilde{f}(x_0) - Df(x_0)(x' - x_0)|}{|x' - x_0|} \\ &\quad \leq \frac{|\tilde{f}(x') - \tilde{f}(z)| + |Df(x_0)(x' - z)|}{|x' - x_0|} + \frac{|\tilde{f}(z) - \tilde{f}(x_0) - Df(x_0)(z - x_0)|}{|x' - x_0|} < \varepsilon. \end{aligned}$$

Now we prove the above paragraph rigorously. Let $x' \in B\left(x_0, \frac{s}{2}\right)$. By (2.3), there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \searrow 0$ such that $r_k < 2^{-k}|x' - x_0|$ and

$$\left| \tilde{f}(x') - \frac{1}{|B(x', r_k)|} \int_{B(x', r_k) \cap \Lambda} \tilde{f}(x)\, dx \right| < 2^{-k}|x' - x_0|. \tag{4.5}$$

In the following, we proceed coordination-wise for $i \in \{1, \dots, n\}$. Denote by a_k^i and b_k^i points in $B(x', r_k) \cap \Lambda$ such that

$$\tilde{f}_i(a_k^i) \geq \frac{1}{|B(x', r_k)|} \int_{B(x', r_k) \cap \Lambda} f_i(x)\, dx - 2^{-k}|x' - x_0|, \tag{4.6}$$

$$\tilde{f}_i(b_k^i) \leq \frac{1}{|B(x', r_k)|} \int_{B(x', r_k) \cap \Lambda} f_i(x)\, dx + 2^{-k}|x' - x_0|. \tag{4.7}$$

If there is an equality in (4.6) or (4.7), we define x_k^i as a_k^i or b_k^i, correspondingly. Otherwise, by continuity of $\tilde{f}_i$ on Λ, there exist two balls $B(a_k^i, \rho(a_k^i))$ and $B(b_k^i, \rho(b_k^i))$, contained in $B(x', r_k)$, such that (4.6) holds for any $a \in B(a_k^i, \rho(a_k^i)) \cap \Lambda$ and (4.7) holds for any $b \in B(b_k^i, \rho(b_k^i)) \cap \Lambda$. Without loss of generality, we may assume $a_k^i = (0, \dots, 0)$ and $b_k^i = (b_1, 0, \dots, 0)$. Let us now consider the lines $l_d := (t, d_2, \dots, d_n)$ connecting $B(a_k^i, \rho(a_k^i))$ and $B(b_k^i, \rho(b_k^i))$. Since Λ is of full measure, for $\mathcal{L}^{n-1}$-a.e. $d := (d_2, \dots, d_n)$ a line l_d contains $x_a \in B(a_k^i, \rho(a_k^i)) \cap \Lambda$ and $x_b \in B(b_k^i, \rho(b_k^i)) \cap \Lambda$, and $\mathcal{L}^1(l_d \setminus \Lambda) = 0$. Moreover, $\tilde{f}_i \in W^{1,n-1}$ and hence $\tilde{f}_i$ is absolutely continuous on $\mathcal{L}^{n-1}$-a.e. l_d. Therefore, by the intermediate value property, there is a point $c_k^i \in l_d$ such that

$$\left| \tilde{f}_i(c_k^i) - \frac{1}{|B(x', r_k)|} \int_{B(x', r_k) \cap \Lambda} \tilde{f}_i(x)\, dx \right| \leq 2^{-k}|x' - x_0|. \tag{4.8}$$

Moreover, there exists $x_k^i \in l_d \cap \Lambda \subset B(x', r_k)$ such that

$$|\tilde{f}(c_k^i) - \tilde{f}(x_k^i)| \leq 2^{-k}|x' - x_0|. \tag{4.9}$$

Then, by (4.5), (4.8), and (4.9),

$$|\tilde{f}_i(x_k^i) - \tilde{f}_i(x')| \leq |\tilde{f}_i(x_k^i) - \tilde{f}_i(c_k^i)| + |\tilde{f}_i(c_k^i) - \tilde{f}_i(x')| < 2^{-k+2}|x' - x_0|. \tag{4.10}$$

Further,

$$\begin{aligned} &\frac{|\tilde{f}_i(x') - \tilde{f}_i(x_0) - Df_i(x_0)(x' - x_0)|}{|x' - x_0|} \\ &\quad \leq \frac{|\tilde{f}_i(x') - \tilde{f}_i(x_k^i)| + |Df_i(x_0)(x' - x_k^i)|}{|x' - x_0|} + \frac{|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x' - x_0|}. \end{aligned} \tag{4.11}$$

Since $x_k^i \in B(x', r_k)$ and (4.10) holds, the first term in (4.11) can be estimated as

$$\frac{|\tilde{f}_i(x') - \tilde{f}_i(x_k^i)| + |Df_i(x_0)(x' - x_k)|}{|x' - x_0|} \leq 2^{-k+2} + 2^{-k}|Df(x_0)|.$$

While to estimate the second term in (4.11), we note that

$$|x_k^i - x_0| \leq |x_k^i - x'| + |x' - x_0| \leq (1 + 2^{-k})|x' - x_0| \leq 2|x' - x_0| \leq s,$$

since $x_k^i \in B(x', r_k)$. And hence, by (4.3), we conclude

$$\frac{|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x' - x_0|} \leq \frac{2|\tilde{f}_i(x_k^i) - \tilde{f}_i(x_0) - Df_i(x_0)(x_k^i - x_0)|}{|x_k^i - x_0|} \leq \varepsilon.$$

Summarizing the above, we obtain that for $x_0 \in \Lambda$ and any $\varepsilon > 0$ there exists $s > 0$ such that for any $x' \in B\left(x_0, \frac{s}{2}\right)$ it holds

$$\frac{|\tilde{f}_i(x') - \tilde{f}_i(x_0) - Df_i(x_0)(x' - x_0)|}{|x' - x_0|} \leq \liminf_{k \to \infty} (2^{-k}(4 + |Df_i(x_0)|) + \varepsilon) = \varepsilon.$$

Therefore, $\tilde{f}_i$ is differentiable in any $x_0 \in \Lambda$ with respect to Ω and, moreover, $D\tilde{f}_i(x_0) = Df_i|_\Lambda(x_0)$.

□

5. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Lemma 3.1 and Theorem 4.4. □

Proof of Corollary 1.2. Let us first note that following the proof of [20, Theorem 1.1] with substituting n by p, we obtain

$$\int_{\Omega'} |Df_k^{-1}|^p(y)\, dy \leq \int_{\Omega} \frac{|\operatorname{adj} Df_k|^p(x)}{(J_{f_k}(x))^{p-1}}\, dx.$$

Hence, $\mathcal{E}(f_k) \leq \mathcal{F}(f_k)$ and the sequence $\{f_k^{-1}\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega', \mathbb{R}^n)$ and passing to a subsequence if needed, there exists a weak limit h. Moreover, by [8, Lemma 2.3] and (1.1), the inequality

$$\int_{\Omega} \varphi(J_f(x))\, dx \leq C$$

guarantees that $J_f > 0$ a.e. in Ω and $J_h > 0$ a.e. in Ω'. To finish the proof, we apply Theorem 1.1. □

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