Report on the Ph.D. Thesis
"Nonlinear classes of mappings: properties and approximation"
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This is a report on the Ph.D. thesis by Anna Doležalová, who finished her Ph.D. program at the Charles University in Prague under the supervision of Stanislav (Standa) Hencl. The thesis collects the five papers written by the candidate, two of which already accepted at the time when I was sent the thesis.

Before being asked to write this report, I had already read with great interest the paper III, which was written by Anna and Standa together with the late Jan Maly, and I had the opportunity to discuss about it with Anna and Standa, and to ask some details and explanations. This is really an exceptional and deep paper, and I have to say that I was admired by its results. This work deals with the celebrated "INV condition" for mappings; without entering too much in the technical details, we can quickly say that a Sobolev map from an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfies the INV condition if for almost every ball B the image of the interior of B lies "inside" the image of ∂B. This is a natural condition in the study of the nonlinear elasticity, where the map should describe the deformation of the original body under some forces. It is known that the INV condition is easier to define and to deal with in the Sobolev space $W^{1,p}(\Omega)$ with $p > n-1$, while for $p = n-1$ already the definition requires some additional care, and many properties are missing. The paper III deals with the critical case $p = n-1$ for $n \geq 3$, and considers the natural energy $\mathcal{E}(f) = \int |Df|^{n-1} + |J_f|^{-a}$ for any deformation f. The first main result of the paper is about the closure of the INV condition: more precisely, if $\{f_m\}$ is a sequence of homeomorphisms, all coinciding on $\partial\Omega$, and with equibounded energy, then any weak limit satisfies the INV condition as soon as a is greater than or equal to an explicit coefficient $a(n)$. The second result is the converse, in dimension $n = 3$; more precisely, for every $a < a(3) = 2$ an explicit sequence $\{f_m\}$ is exhibited such that the above assumptions remain true, but the weak limit of f_m does not satisfy the INV condition. The last result shows that for $n = 3$ the weak and strong closures of homeomorphisms in $W^{1,2}(\Omega)$ are different, while it is known that weak and strong closures coincide for $n = 2$.

In the last weeks, I have read also the other four papers (though, not as in detail as the paper III). Paper IV is the continuation of the study of paper III, and finds a condition which implies the $W^{1,1}$ property for the inverse. More precisely, adding to the above functional the additional term $A(|\mathrm{cof} Df|)$, where A is a positive and convex function with superlinear growth, then if f is a weak limit of a sequence $\{f_m\}$ of functions with equibounded energy, coinciding on $\partial\Omega$, and satisfying the Luzin (N) condition, then f fulfills the INV condition, the lower semicontinuity of the energy holds, and the inverse of f (in a suitable sense) is $W^{1,1}$. Paper V, written in collaboration with Anastasia Molchanova, proves the differentiability almost everywhere of weak limits of homeomorphisms, considering this time the energy given by $\int |Df|^{n-1} + \int |Df^{-1}|^p$ with $p > n - 1$.

Papers I and II deal with different questions, though closely related to the arguments of papers III, IV and V. More precisely, paper I considers the important Luzin (N) condition of a map, which simply means that the image of negligible sets must be negligible, or in other words, "mass cannot be created from nothing". This condition is classically known to hold for free for any map in $W^{1,p}$ with $p > n$, and even for $p = n$ if the map is a homeomorphism, and both these results are sharp in the realm of Sobolev spaces. Nevertheless, there is still room for improvement by using Grand Lebesgue spaces $W^{1,p)}$, or even using gauge functions to generalise the notion of Hausdorff measures. Two nontrivial results of this kind are the content of paper I.

Finally, paper II considers mappings of "generalised distortion", which is defined by mixing the usual notion of mappings of finite distortion and the boundedness of the L^p norm of Df for $p > n$. The content of the paper is to find conditions on this generalised distortion which ensure the continuity of the map, and to find a class of counterexamples to show the sharpness of the assumptions.

To conclude, I have no doubts in saying that the work contained in this thesis is enough for Anna Doležalová to positively conclude the Ph.D. studies. But such a statement would not be enough. Indeed, I want to add that this is for sure one of the most interesting Ph.D. theses that I have read, and I hope that the candidate will be able to continue her mathematical work with the same depth and success as it happened up to now.

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