FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Šimon Vrba

Magnetic fields of current loops around black holes

Institute of Theoretical Physics

Supervisor of the master thesis: doc. RNDr. Oldřich Semerák, DSc.
Study programme: Theoretical Physics
Study branch: Theoretical Physics

Prague 2023

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank professor Semerák for his supervision and willingness to help me anytime it was needed. A deep thanks also belongs to Dr. Kofroň for helping me understand his work on the Debye potential, for sharing his Kerr spacetime Mathematica notebook, and for introducing me to useful Mathematica packages.

Title: Magnetic fields of current loops around black holes

Author: Šimon Vrba

Institute: Institute of Theoretical Physics

Supervisor: doc. RNDr. Oldřich Semerák, DSc., Institute of Theoretical Physics

Abstract: We summarize and explain the mathematical procedure that allows us to find the closed form of the magnetic field generated by a test current loop in Kerr spacetime. We consider axisymmetric placement of the loop for all three cases of the Kerr background: below-extreme black hole, extreme black hole, and naked singularity. The field is obtained by differentiating the effective Green function of the Debye potential, which is expressed in terms of elliptic integrals.

Keywords: general theory of relativity, black holes, magnetic fields, accretion discs

Contents

Introduction

The existence of black holes was one of the most shocking predictions of Einstein's theory of general relativity. The initial seeming contradiction came from them being vacuum solutions which nevertheless generate such a strong gravitational field that even light cannot escape from within the horizon.

Since then, many years have passed and physicists have grown accustomed to their concept, found new exact solutions, and even obtained experimental confirmation of their existence. The most famous evidence includes the detection of gravitational waves by LIGO and images captured by the Event Horizon Telescope.

However, black holes are not isolated from the rest of the universe. Through their gravitational field, they attract matter which gathers around and creates an accretion disc. Even if the overall charge of the accretion disc is small, its presence and orbits can create strong currents and therefore strong magnetic fields. The description of such a system plays an important role in astrophysics.

However, the task is very complex as it requires solving both the Einstein's and Maxwell's equations. It seems almost impossible to accomplish this analytically but one can always make simplifications to the problem until it becomes solvable. Sufficiently simple model of an accretion disc is a massless current loop on a fixed black hole background.

The first paper essential for solving this problem was written by Newman and Penrose in which they introduced the Newman-Penrose (NP) formalism [1]. It is a special tetrad formalism which uses complex null vectors. The use of complex quantities is very convenient since they can store twice as much information as real quantities. The six degrees of freedom of the electromagnetic tensor are thus stored in three complex scalars.

The second essential paper came from Teukolsky [2]. He used the NP formalism to study test electromagnetic fields on Kerr background and successfully obtained decoupled equations for two of the three complex electromagnetic scalars.

This result of Teukolsky was then used to find the electromagnetic field of an axisymmetrically placed current loop in a below-extreme Kerr background. It was presented in terms of an infinite series by Chitre and Vishveshwara [3], Petterson [4], Bičák and Dvořák [5], Znajek [6], and Moss [7]. Vlasáková then showed that these results are equivalent and extended them to the case of an extreme Kerr background [8]. Further extension to the background of a Kerr naked singularity was presented in [9]. There, it can also be viewed that the infinite sum has unsatisfactory convergence properties on the radius of the loop everywhere, not just at the plane where the loop is located. It would thus be useful to have a closed form of the solution.

For this task, the essential paper came from Cohen and Kegeles [10] who were able to generalize the notion of Hertzian and Debye potentials of the electromagnetic fields to curved backgrounds. Linet was then able to use this result to find the Debye potential for an axially symmetric source in Kerr background [11]. The central object is an integral expression called the superpotential. When multiplied by a simple functional factor, it can give the Debye potential of a ring source by plain differentiation. The result of this integration for a below-extreme Kerr background was presented in terms of elliptic integrals by Kofroň and Kotlařík [12]. They further analyzed the superpotential and visualized the electromagnetic fields of an axisymmetric current loop and a charged ring.

In this work, we shall present a largely self-contained text that takes the most important parts of the papers mentioned above, explains all the necessary mathematical tools, and then uses them to obtain the desired electromagnetic fields in a closed form. In chapter 1, we summarize the necessary differential geometry. In chapter 2, we explain the NP formalism, reproduce the derivation of the Teukolsky's equation, introduce the Debye potential, and then express all the defined objects and obtained equations on Kerr background. In chapter 3, we go through the relevant aspects of the axially symmetric potential theory and find the integral expression for the superpotential. In chapter 4, we give the explicit form of the superpotential on a below-extreme Kerr background in terms of elliptic integrals. We also present original results in providing the superpotential on an extreme Kerr background and also an extension to the background of a naked Kerr singularity. We also visualize the magnetic field of a current loop around a below-extreme Kerr black hole. In chapter 5, we discuss how we obtained the superpotential and address other questions the reader might have.

1. Differential geometry

In order to make this text as self-contained as possible, we shall summarize most of the necessary mathematics. Assuming the reader is at least slightly familiar with the basics of differential geometry, this chapter aims to provide a logically advancing framework of the most relevant concepts, which help to naturally understand not just the theoretical aspect of the Newman-Penrose formalism as a tetrad formalism, but also its practical use. It will also give the reader the necessities required to tackle the Hertzian and Debye potentials.

We start by briefly reminding the definitions of covariant derivative, torsion, and curvature operator. Then we move into a coordinate frame of our Lorentzian manifold equipped with a Levi-Civita covariant derivative. Afterwards, we transition into a non-coordinate basis – a tetrad. We also define important operations on differential forms and state the Poincaré's theorem.

Most of the definitions and statements presented in the subchapters on covariant derivatives and differential forms can be found in [13], which the reader can also consult for some of the proofs. In the subchapter on Poincaré's theorem, we closely follow [13].

We assume a 4-dimensional manifold equipped with a metric (M, g). $\mathcal{F}(M)$ denotes smooth functions on M, $\mathcal{X}(M)$ is the set of vector fields, $\mathcal{T}^q_r(M)$ denotes the set of tensor fields of type (q, r), and $\Omega^r(M)$ is the set of r-forms on M.

1.1 Covariant derivative, curvature, and torsion

Let us have vectorfields X, Y, tensor fields of arbitrary type T_1, T_2, and a scalar function f. Then the covariant derivative along X is a map $\nabla_X : \mathcal{T}^q_r(M) \to \mathcal{T}^q_r(M)$ that commutes with contraction and satisfies the following properties

$$\begin{aligned} \nabla_X(T_1 + rT_2) &= \nabla_X T_1 + r\nabla_X T_2, \\ \nabla_X(T_1 \otimes T_2) &= (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2), \\ \nabla_X f &= X[f], \\ \nabla_{X+rY} T_1 &= \nabla_X T_1 + r\nabla_Y T_1, \end{aligned} \tag{1.1}$$

where $r \in \mathbb{R}$, $\otimes$ is the tensor product, and $X[f]$ denotes the directional derivative of f along the vectorfield X. Since the covariant derivative ∇_X is a linear map in X as well, we can write

$$\nabla_X = X^m \nabla_m. \tag{1.2}$$

This definition of the covariant derivative is not unique. Consider a general covariant derivative ∇_m. Let us also have another covariant derivative $\tilde{\nabla}_m$ compatible with the basis e_n, meaning

$$\tilde{\nabla}_m e_n = 0. \tag{1.3}$$

As a result, the computation of $\tilde{\nabla}_m X$ in the basis e_n reduces to taking the partial derivative of the components of X in this basis.

The difference between two covariant derivatives is given by connection coefficients

$$\nabla_m = \tilde{\nabla}_m + \tilde{\boldsymbol{\Gamma}}_m. \tag{1.4}$$

These specify how the basis changes with respect to ∇_m

$$\nabla_m e_n = \tilde{\Gamma}^k{}_{mn} e_k. \tag{1.5}$$

Thus, if we act with ∇_m on a vectorfield decomposed into the basis $X = X^m e_m$, with the help of (1.3)–(1.5) we obtain the rule for computing $\nabla_m X$. Using abstract index notation we get

$$\nabla_m X^k = \partial_m X^k + \Gamma^k{}_{mn} X^n. \tag{1.6}$$

We could obtain the rule for the covariant differentiation of a form ω by computing $\nabla_m (X^n \omega_n)$. On the one hand, we apply the commutation of the covariant derivative with contraction and also the Leibniz rule. On the other hand, the contraction itself is a scalar function. Afterwards, the generalization to an arbitrary tensor field is trivial. We shall give the formulas for specific covariant derivatives later.

The Lie bracket of vectorfields is also a vectorfield given by

$$[X, Y][f] = X\,[Y[f]] - Y\,[X[f]]\,. \tag{1.7}$$

Next we define torsion

$$T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.8}$$

It is a multilinear map, and so we can write

$$T^m(X, Y) \equiv T^m_{kl} X^k Y^l. \tag{1.9}$$

Using its definition, it is straightforward to prove that it provides the commutator of covariant derivatives acting on functions

$$-T^m_{kl} \nabla_m f = \nabla_k \nabla_l f - \nabla_l \nabla_k f. \tag{1.10}$$

Clearly, one only needs $\nabla f = \partial f$, and $(\partial f) g = \partial (f g) - f \partial g$. Writing out the covariant derivative as in (1.4), we find that the torsion can be expressed by the antisymmetric part of the connection coefficients

$$T^k_{mn} = \tilde{\Gamma}^k{}_{mn} - \tilde{\Gamma}^k{}_{nm}. \tag{1.11}$$

The curvature operator is defined as

$$\boldsymbol{R}(X, Y) \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{1.12}$$

It is a multilinear map, and a simple calculation can show that it is related to torsion

$$\boldsymbol{R}_{kl} = \nabla_k \nabla_l - \nabla_l \nabla_k + T^m_{kl} \nabla_m. \tag{1.13}$$

Its action on vectorfields is expressed by the Riemann tensor

$$\boldsymbol{R}_{kl} X^m = R^m{}_{nkl} X^n. \tag{1.14}$$

The action on forms is a simple consequence of $\boldsymbol{R}_{kl} f = 0$ combined with the Leibniz rule $\boldsymbol{R}_{kl}(AB) = (\boldsymbol{R}_{kl} A) B + A (\boldsymbol{R}_{kl} B)$. Next, we define the Ricci tensor

$$R_{nl} \equiv R^m{}_{nml}, \tag{1.15}$$

and at last using the metric g, we obtain the scalar curvature

$$R \equiv g^{nl} R_{nl}. \tag{1.16}$$

The Riemann tensor can be decomposed into its traces and a traceless part. The traceless part is called the Weyl tensor $C^m{}_{nkl}$. A direct calculation of contractions can show that the decomposition is given by

$$R_{mnkl} = C_{mnkl} + \frac{1}{2}\left(g_{mk}R_{nl} + g_{nl}R_{mk} - g_{nk}R_{ml} - g_{ml}R_{nk}\right) - \frac{R}{6}\left(g_{mk}g_{nl} - g_{nk}g_{ml}\right). \tag{1.17}$$

1.2 Levi-Civita covariant derivative

We shall assume a coordinate frame of basis vector ∂_μ and basis 1-forms $\mathrm{d}x^\mu$. The Levi-Civita covariant derivative (from now on always denoted by ∇) is a metric covariant derivative without torsion. The first condition means $\nabla g = 0$.

The partial derivative is essentially a covariant derivative compatible with the coordinate basis. Thus it can be related to the Levi-Civita covariant derivative via connection coefficients

$$\nabla_\mu = \partial_\mu + \mathbf{\Gamma}_\mu. \tag{1.18}$$

We define the connection coefficients by

$$\nabla_\mu \partial_\nu = \Gamma^\lambda{}_{\mu\nu} \partial_\lambda. \tag{1.19}$$

This gives us the rule for computing the covariant derivative of an arbitrary tensor

$$\nabla_\mu T^{\nu\ldots}{}_{\lambda\ldots} = \partial_\mu T^{\nu\ldots}{}_{\lambda\ldots} + \Gamma^\nu{}_{\mu\sigma} T^{\sigma\ldots}{}_{\lambda\ldots} + \ldots - \Gamma^\sigma{}_{\mu\lambda} T^{\nu\ldots}{}_{\sigma\ldots} - \ldots. \tag{1.20}$$

The condition on vanishing torsion means that the connection coefficients are symmetric in the last two indices

$$\Gamma^\sigma{}_{[\mu\nu]} = 0, \tag{1.21}$$

which is a consequence of (1.11).

Evaluating $0 = \nabla_\mu g_{\sigma\lambda}$, we find a relation between the metric and the connection coefficients

$$\partial_\mu g_{\sigma\lambda} = \Gamma_{\lambda\mu\sigma} + \Gamma_{\sigma\mu\lambda}. \tag{1.22}$$

Writing this equation three times with cyclic exchange of indices, and then summing them as $-\mathrm{I} + \mathrm{II} + \mathrm{III}$, we can express the connection coefficients in terms of the first derivatives of the metric

$$\Gamma^\kappa{}_{\sigma\lambda} = \frac{1}{2} g^{\kappa\mu}\left(\partial_\sigma g_{\lambda\mu} + \partial_\lambda g_{\mu\sigma} - \partial_\mu g_{\sigma\lambda}\right). \tag{1.23}$$

If we contract the Riemann tensor with a vector and express this via the curvature operator (1.14), we can then rewrite the Levi-Civita covariant derivatives in terms of partial derivatives and connection coefficients (1.18). We thus obtain the Riemann tensor expressed fully in terms of the connection coefficients and their partial derivatives

$$R^\mu{}_{\nu\sigma\kappa} = \partial_\sigma \Gamma^\mu{}_{\kappa\nu} - \partial_\kappa \Gamma^\mu{}_{\sigma\nu} + \Gamma^\mu{}_{\sigma\tau}\Gamma^\tau{}_{\kappa\nu} - \Gamma^\mu{}_{\kappa\tau}\Gamma^\tau{}_{\sigma\nu}. \tag{1.24}$$

1.3 Tetrad covariant derivative

Assume a non-coordinate basis of vectors e_a. They have the coordinate components e_a^μ, which we will not explicitly write most of the time. Contraction in the coordinate indices carried out by the metric will often be denoted by a dot $X \cdot \omega \equiv X^\mu \omega_\mu$.

The dual basis of 1-forms is defined by the condition

$$e_a \cdot e^b = \delta_a^b. \tag{1.25}$$

In this basis, the metric coefficients are denoted by η_{ab}

$$g = \eta_{ab} e^a e^b. \tag{1.26}$$

They can be calculated in a simple manner

$$e_a \cdot e_b = \eta_{ab}. \tag{1.27}$$

We choose the tetrad in such a way that $\eta_{ab} = \text{const}$.

Further, we define the inverse tetrad metric

$$\eta^{ab} \eta_{bc} = \delta_c^a. \tag{1.28}$$

Contracting the equation (1.25) with η, and then comparing it to (1.27), we learn how to raise and lower tetrad indices

$$\begin{aligned} e^a &= \eta^{ab} e_b, \\ e_a &= \eta_{ab} e^b. \end{aligned} \tag{1.29}$$

Next, we define a tetrad covariant derivative $\eth_a e_b = 0$ and the corresponding connection coefficients

$$\nabla_a = \eth_a + \boldsymbol{\gamma}_a. \tag{1.30}$$

The connection coefficients are also referred to as the Ricci rotation coefficients. We shall define them as

$$\nabla_a e_b = \gamma_b{}^c{}_a e_c, \tag{1.31}$$

which means they can be calculated by evaluating the covariant derivative of our tetrad

$$\gamma_{abc} = e_b \cdot \nabla_c e_a. \tag{1.32}$$

The Levi-Civita covariant derivative of a tensor can then be written in terms of our tetrad derivative

$$\nabla_a T^{b\ldots}{}_{c\ldots} = \eth_a T^{b\ldots}{}_{c\ldots} + \gamma_d{}^b{}_a T^{d\ldots}{}_{c\ldots} + \ldots - \gamma_c{}^d{}_a T^{b\ldots}{}_{d\ldots} - \ldots. \tag{1.33}$$

Since the Levi-Civita covariant derivative is a metric covariant derivative, and the tetrad metric components are constant, using (1.30) we learn that the Ricci rotation coefficients are antisymmetric in the first two indices

$$\gamma_{(ab)c} = 0. \tag{1.34}$$

We shall denote the tetrad torsion by τ. Then in complete analogy with (1.10) and (1.11) we get the commutator of the tetrad covariant derivatives

$$\eth_a \eth_b f - \eth_b \eth_a f = -\tau_{ab}^c \eth_c f, \tag{1.35}$$

and the expression of torsion in terms of the Ricci rotation coefficients

$$\tau^a_{bc} = \gamma_b{}^a{}_c - \gamma_c{}^a{}_b. \tag{1.36}$$

Next, we could write the action of the Riemann tensor on a vector field via the curvature tensor (1.14), express the Levi-Civita covariant derivatives using (1.33), and thus find the Riemann tensor fully in terms of the tetrad covariant derivatives, Ricci rotation coefficients, and the tetrad torsion

$$R_{abcd} = \eth_c \gamma_{bad} - \eth_d \gamma_{bac} + \tau^e_{cd}\gamma_{bae} + \gamma_{eac}\gamma_b{}^e{}_d - \gamma_{ead}\gamma_b{}^e{}_c. \tag{1.37}$$

Combining now computed relation (1.37) with (1.17), we can write the relationship between Weyl, Ricci, the curvature scalar, and the Ricci rotation coefficients

$$\begin{aligned} C_{abcd} + \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) - \frac{R}{6}\left(g_{ac}g_{bd} - g_{bc}g_{ad}\right) \\ = \eth_c \gamma_{bad} - \eth_d \gamma_{bac} + \tau^e_{cd}\gamma_{bae} + \gamma_{eac}\gamma_b{}^e{}_d - \gamma_{ead}\gamma_b{}^e{}_c. \end{aligned} \tag{1.38}$$

1.4 Operators acting on differential forms

In this part, we will consider a manifold of an arbitrary dimension m. We allow it to be both Riemannian and Lorentzian. Because of that, we define a special symbol s_g, whose value is $+1$ for Riemannian manifolds and 0 for Lorentzian manifolds.

At first, we define the totally antisymmetric tensor $\varepsilon_{\mu_1 \dots \mu_m}$ with the condition $\varepsilon_{12\dots m} = 1$. If we raise the indices we obtain

$$\varepsilon^{\mu_1 \dots \mu_m} = g^{\mu_1 \nu_1} \dots g^{\mu_m \nu_m} \varepsilon_{\nu_1 \dots \nu_m} = \det g^{-1} \varepsilon_{\mu_1 \dots \mu_m}. \tag{1.39}$$

Next, we define the generalized Kronecker delta in terms of the antisymmetrized Kronecker deltas

$$\delta^{\mu_1 \dots \mu_r}_{\alpha_1 \dots \alpha_r} \equiv r! \delta^{\mu_1}_{[\alpha_1} \dots \delta^{\mu_r}_{\alpha_r]} = r! \delta^{[\mu_1}_{\alpha_1} \dots \delta^{\mu_r]}_{\alpha_r}. \tag{1.40}$$

The latter acts in the same way as an antisymmetrization operator

$$\sigma_{\mu_1 \dots \mu_r} \delta^{\mu_1}_{[\alpha_1} \dots \delta^{\mu_r}_{\alpha_r]} = \sigma_{[\alpha_1 \dots \alpha_r]}. \tag{1.41}$$

When we completely contract the antisymmetrized Kronecker deltas, we get a binomial coefficient

$$\delta^{\alpha_1}_{[\alpha_1} \dots \delta^{\alpha_r}_{\alpha_r]} = \binom{m}{r}. \tag{1.42}$$

Thus, the general contraction identity is

$$\delta^{\mu_1}_{[\kappa_1} \dots \delta^{\mu_s}_{\kappa_s} \delta^{\nu_{s+1}}_{\nu_{s+1}} \dots \delta^{\nu_r}_{\nu_r]} = \frac{\binom{m}{r}}{\binom{m}{s}} \delta^{\mu_1}_{[\kappa_1} \dots \delta^{\mu_s}_{\kappa_s]}. \tag{1.43}$$

If the generalized Kronecker delta has the maximal number of indices, it can be related to the totally antisymmetric tensor

$$\varepsilon^{\mu_1 \dots \mu_r \mu_{r+1} \dots \mu_m} \varepsilon_{\alpha_1 \dots \alpha_r \alpha_{r+1} \dots \alpha_m} = \det g^{-1} \delta^{\mu_1 \dots \mu_r \mu_{r+1} \dots \mu_m}_{\alpha_1 \dots \alpha_r \alpha_{r+1} \dots \alpha_m}. \tag{1.44}$$

Next, we express an anti-symmetric r-form in the coordinate basis

$$\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_r}, \tag{1.45}$$

and define the Hodge star operator by

$$*\omega = \frac{\sqrt{|\det g|}}{r!(m-r)!}\omega_{\mu_1\dots\mu_r}\varepsilon^{\mu_1\dots\mu_r}{}_{\nu_{r+1}\dots\nu_m}\mathrm{d}x^{\nu_{r+1}}\wedge\dots\wedge\mathrm{d}x^{\nu_m}. \tag{1.46}$$

Using all the mentioned properties of the generalized Kronecker delta, it is trivial to prove

$$**\omega = (-1)^{r(m-r)+1-s_g}\omega. \tag{1.47}$$

The action of the exterior derivative in the coordinate frame can be expressed as

$$\begin{aligned}\mathrm{d}\omega &= \frac{1}{r!}\partial_\alpha\omega_{\mu_1\dots\mu_r}\mathrm{d}x^\alpha\wedge\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_r},\\ &= \frac{1}{r!}\nabla_\alpha\omega_{\mu_1\dots\mu_r}\mathrm{d}x^\alpha\wedge\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_r}.\end{aligned} \tag{1.48}$$

Note that thanks to the symmetry of the last two indices of the Levi-Civita connection coefficients, it does not matter whether we compute the exterior derivative using partial derivative or Levi-Civita covariant derivative. Next we define the co-derivative

$$\delta \equiv (-1)^{mr+m+s_g} * \mathrm{d} * . \tag{1.49}$$

We remind the reader, that one of the defining properties of the exterior derivative is $\mathrm{d}^2 = 0$. Thanks to (1.47), it clearly also holds $\delta^2 = 0$. The co-derivative acts as a divergence over the first index up to a sign

$$\delta\omega = -\frac{1}{(r-1)!}\nabla^\alpha\omega_{\alpha\alpha_1\dots\alpha_{r-1}}\mathrm{d}x^{\alpha_1}\wedge\dots\wedge\mathrm{d}x^{\alpha_{r-1}}. \tag{1.50}$$

At last we define the Laplacian

$$\Delta_L \equiv -(\mathrm{d}+\delta)^2 = -\mathrm{d}\delta - \delta\mathrm{d}. \tag{1.51}$$

Its action on a function has a simple form

$$\Delta_L f = (-1)^{s_g+1}\frac{1}{\sqrt{|\det g|}}\partial_\mu\left(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu f\right). \tag{1.52}$$

Having already defined the Hodge dual, we can introduce the self-dual and the anti-self-dual of an antisymmetric 2-form F by

$$F_S = F + i*F, \tag{1.53}$$

$$F_A = F - i*F. \tag{1.54}$$

These have the special property of being the eigentensors to the Hodge dual

$$*F_S = -iF_S, \qquad *F_A = iF_A. \tag{1.55}$$

1.5 Poincaré's lemma

In this part, we briefly introduce the Poincaré's lemma. First, we will define a few terms connected to the deformation of maps, then we shall define (co-)closed and (co-)exact forms, and eventually give the Poincaré's lemma. We shall also discuss its practical consequences in the simplest case of its use.

First assume that we have some topological space X, and let R be its non-empty subspace. If there exists a continuous map $f : X \to R$ such that $f|_R = \mathrm{id}_R$, then R is called a retract of X and f a retraction. Here id_R denotes the identity map on R.

Next, if there exists such a continuous map $H : X \times I \to X$ that

$$\begin{aligned} H(x,0) &= x \quad \& \quad H(x,1) \in R \quad \text{for any } x \in X, \\ H(x,t) &= x \quad \text{for any } x \in R \text{ and any } t \in I, \end{aligned} \tag{1.56}$$

then the space R is said to be a deformation retract of X. Moreover, if a point $a \in X$ is a deformation retract of X, then X is called contractible to a point.

Note that the parameter t in $H(x,t)$ functions as a deformation parameter. By moving in its parameter space from 0 to 1, we continuously shift/deform $H(x,t)$ from the identity map id_X to a retraction.

Contractibility to a point can be easily demonstrated on the example of a disc $\mathcal{D}_a \equiv \{re^{i\phi} : \ \phi \in [0, 2\pi),\ r \in [0, a]\}$. Then it can be easily viewed, that the point $a = 0$ is a deformation retract of $\mathcal{D}$. Simply notice the existence of the continuous map $H_{\mathcal{D}_a}(re^{i\phi}, t)$, which can be taken as

$$H_{\mathcal{D}_a}(re^{i\phi}, t) \equiv (1 - t)re^{i\phi}.$$

Indeed, $H_{\mathcal{D}_a}(re^{i\phi}, 0) = re^{i\phi}$, $H_{\mathcal{D}_a}(re^{i\phi}, 1) = 0$, and $H_{\mathcal{D}_a}(0, t) = 0$. Obviously, this can be generalized to an arbitrary point from the disc.

As a counterexample, consider the thick ring $\mathcal{D}_a \setminus \mathcal{D}_b$, where $a > b$. We will not be able to find a continuous deformation of the identity map into a retraction to a point. The reason is the existence of a hole in the space. In this case, the deformation retract is – for example – a circle.

Next, an r-form ω is called closed if $\mathrm{d}\omega = 0$, and it is called exact if there exists an $(r - 1)$-form σ such that $\omega = \mathrm{d}\sigma$. In the same manner, we define the terms co-closed and co-exact but with the exchange $\mathrm{d} \leftrightarrow \delta$.

Theorem 1 (Poincaré's lemma)**.** *If a coordinate neighbourhood U of a manifold M is contractible to a point $p \in M$, any closed r-form on U is also exact.*

Any sufficiently "physical" spacetime allows us to locally choose a neighbourhood U in the shape of a four-dimensional box or a ball. These are contractible, and thus any closed form ω is exact $\omega = \mathrm{d}\sigma$. The same applies to an overlapping neighbourhood U' where $\omega = \mathrm{d}\sigma'$. Since it is still the same form ω, it must be that σ and σ' differ on the intersection at most by an exact form $\sigma - \sigma' = \mathrm{d}\gamma$. The exact form can be found by a coordinate transformation from U' to U or vice-versa. Thus in practice, we shall assume we work on a neighbourhood contractible to a point. Problems may occur if we use multiple neighbourhoods and need to perform coordinate transformations on their overlaps.

Similar statement applies to co-closed and co-exact forms.

2. Equations for electromagnetic fields on curved background

In this chapter, we start by explaining the Newman-Penrose formalism as a special case of the tetrad formalism. It was first introduced by Newman and Penrose in [1], where they also elaborate on its spinor formulation. Afterwards, we follow in the footsteps of Teukolsky [2], and use the NP formalism on the Maxwell equations in order to find a test electromagnetic field on a curved background. Then we move on to the Hertzian and Debye potentials. They were nicely discussed in the classical (3+1)-formalism by Nisbet [15]. We follow the subsequent generalization to curved background in covariant formalism, which was presented by Cohen and Kegeles [10]. At the end, we summarize the metric and some important properties of the Kerr spacetime, and express all the Teukolsky, NP, and Debye equations explicitly in the Boyer-Lindquist coordinates.

2.1 Newman-Penrose formalism

The NP formalism is a tetrad formalism that uses two real null vectors l and n in combination with two complex null vectors: m and its complex conjugate $\bar{m}$. The only non-zero scalar products are

$$l \cdot n = -m \cdot \bar{m} = 1. \tag{2.1}$$

Thanks to (1.26) and (1.27), we can express the metric of our manifold in terms of the tetrad

$$g = ln + nl - m\bar{m} - \bar{m}m. \tag{2.2}$$

We shall sometimes denote the tetrad vectors by $e_a = (l, n, m, \bar{m})$ for $a = 1, 2, 3, 4$. Due to (1.25) and (2.1), the dual tetrad is $e^a = (n, l, -\bar{m}, -m)$. Using (1.27) we obtain the tetrad metric

$$\eta_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{2.3}$$

The inverse determined by (1.28) has identical matrix entries $\eta_{ab} = \eta^{ab}$.

The tetrad covariant derivatives are assigned special symbols $\eth_a = (D, \Delta, \delta, \bar{\delta})$. In practice, we use the spin coefficients instead of the Ricci rotation coefficients. For our purposes, this means we give special symbols to all the Ricci rotation coefficients or their combinations

$$\begin{aligned} \epsilon &= \frac{1}{2}(\gamma_{121} - \gamma_{341}), \quad \kappa = \gamma_{131}, \quad \pi = -\gamma_{241}, \\ \gamma &= \frac{1}{2}(\gamma_{122} - \gamma_{342}), \quad \tau = \gamma_{132}, \quad \nu = -\gamma_{242}, \\ \alpha &= \frac{1}{2}(\gamma_{124} - \gamma_{344}), \quad \sigma = \gamma_{133}, \quad \mu = -\gamma_{243}, \\ \beta &= \frac{1}{2}(\gamma_{123} - \gamma_{343}), \quad \rho = \gamma_{134}, \quad \lambda = -\gamma_{244}. \end{aligned} \tag{2.4}$$

The rest of the relations can be obtained by using antisymmetry of the Ricci rotation coefficients in the first two indices (1.34) and by complex conjugation, which corresponds to the interchange of indices $3 \leftrightarrow 4$. These relations can be easily inverted.

The tetrad projections of the Weyl tensor shall be referred to as the Weyl scalars and will be defined as

$$\begin{aligned} \Psi_0 &= -C_{1313}, \quad \Psi_2 = -C_{1342}, \quad \Psi_4 = -C_{2424}. \\ \Psi_1 &= -C_{1213}, \quad \Psi_3 = -C_{1242}, \end{aligned} \tag{2.5}$$

To express the rest of the Weyl components in terms of the Weyl scalars, we have to consider two conditions [16]

$$\eta^{ac} C_{abcd} = 0, \tag{2.6}$$

$$C_{1234} + C_{1342} + C_{1423} = 0, \tag{2.7}$$

which in turn give

$$\begin{aligned} C_{1343} &= C_{1213}, & C_{3434} &= C_{1212}, \\ C_{2434} &= C_{1242}, & C_{1314} &= 0, \\ C_{1234} &= C_{1432} - C_{1342}, & C_{2324} &= 0, \\ C_{1212} &= C_{1432} + C_{1342}, & C_{1332} &= 0. \end{aligned} \tag{2.8}$$

Depending on the Petrov type of our spacetime, there can exist such a tetrad that some of the Weyl scalars and spin coefficients vanish. In a type D spacetime – for us most importantly Kerr – we can choose the tetrad so that [2]

$$\begin{aligned} \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \\ \kappa = \sigma = \nu = \lambda = 0. \end{aligned} \tag{2.9}$$

For a short introduction into the Petrov classification, one can consult [17] for the tetrad approach, and [18] for the spinor approach.

We continue in the same manner with the Ricci tensor and define the complex Ricci scalars

$$\begin{aligned} \Phi_{00} &= \frac{1}{2} R_{11}, \quad \Phi_{11} = \frac{1}{4} \left(R_{12} + R_{34} \right), \\ \Phi_{01} &= \frac{1}{2} R_{13}, \quad \Phi_{12} = \frac{1}{2} R_{32}, \\ \Phi_{02} &= \frac{1}{2} R_{33}, \quad \Phi_{22} = \frac{1}{2} R_{22}, \end{aligned} \tag{2.10}$$

which satisfy

$$\Phi_{AB} = \bar{\Phi}_{BA}. \tag{2.11}$$

We also rename the scalar curvature

$$\Lambda = \frac{R}{24}. \tag{2.12}$$

Let us point out that the condition on the trace of the Ricci tensor means

$$R_{12} = 2\Phi_{11} + 6\Lambda, \quad R_{34} = 2\Phi_{11} - 6\Lambda. \tag{2.13}$$

Next, we introduce the complex electromagnetic scalars as projections of the electromagnetic tensor F [2]

$$\phi_0 = F_{13}, \qquad \phi_1 = \frac{1}{2}\left(F_{12} + F_{43}\right), \qquad \phi_2 = F_{42}. \tag{2.14}$$

We can re-express F in terms of the complex scalars

$$F = \left[\phi_1 \left(n \wedge l + m \wedge \bar{m}\right) + \phi_2 l \wedge m + \phi_0 \bar{m} \wedge n\right] + c.c., \tag{2.15}$$

where c.c. stands for complex conjugation of the preceding expression. Interestingly, if we do not add the complex conjugate part, we end up with the anti-self-dual of the electromagnetic tensor

$$\mathcal{F} \equiv 2\left[\phi_1 \left(n \wedge l + m \wedge \bar{m}\right) + \phi_2\, l \wedge m + \phi_0\, \bar{m} \wedge n\right]. \tag{2.16}$$

It can be shown to satisfy the anti-self-dual condition (1.55). Most of the calculation is simple and based on the use of symmetries and on rewriting the metric that comes up from the Hodge dual in terms of the tetrad (2.2). The single problematic step might be proving the following relation

$$\sqrt{-g}\, l^\mu n^\nu m^\lambda \bar{m}^\tau \epsilon_{\mu\nu\lambda\tau} = i, \tag{2.17}$$

which, however, can be swiftly shown to be true by combining once again (2.2) with a little help from a computation software. Note that the utility of the anti-self-dual of the electromagnetic tensor lies in

$$\mathcal{F}_{0j} = E_j - iB_j, \qquad j = 1, 2, 3. \tag{2.18}$$

In what follows, we shall need the 31 component of the tetrad derivative commutator (1.35). We express the torsion via the Ricci rotation coefficients (1.36) and then replace the Ricci rotation coefficients with the spin coefficients (2.4)

$$\delta D - D\delta = \left(\bar{\alpha} + \beta - \bar{\pi}\right) D + \kappa\Delta - \sigma\bar{\delta} - \left(\bar{\rho} + \epsilon - \bar{\epsilon}\right)\delta. \tag{2.19}$$

We will also need several components of the expression (1.38) relating the Weyl tensor, the Ricci tensor, the scalar curvature, and the Ricci rotation coefficients. Using the definitions for projections of all the required objects (2.5), (2.10), (2.12), and (2.4), the needed relations are

$$D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1, \tag{2.20}$$

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{2.21}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}. \tag{2.22}$$

Equation (2.20) is the difference of components 3413 and 1213. Equations (2.21) and (2.22) are the components 1312 and 1334 respectively. The last commutation identity we need is special for type D spacetimes [2]

$$\begin{aligned}\left[D - (p+1)\epsilon + \bar{\epsilon} + q\rho - \bar{\rho}\right]\left(\delta - p\beta + q\tau\right)& \\ - \left[\delta - (p+1)\beta - \bar{\alpha} + \bar{\pi} + q\tau\right]\left(D - p\epsilon + q\rho\right) &= 0,\end{aligned} \tag{2.23}$$

where p and q are arbitrary numbers. It can be proved directly by using the commutator (2.19) with the equations (2.20), (2.21), and (2.22) while assuming the conditions (2.9). From now on, we shall always assume a type D spacetime and the satisfaction of these conditions.

2.2 Maxwell equations in NP - Teukolsky equation

Solving the Einstein-Maxwell equations is a difficult task. However, the situation gets simplified once we decide to deal with this problem only perturbatively. Consider the fixed background to be a type D vacuum spacetime. Next, we introduce a first-order perturbation in the form of an electromagnetic field. Considering that the electromagnetic stress-energy tensor is quadratic in the field, the corrections to the fixed spacetime background will be of second order. As a consequence, if we wish to work only up to the first order of the perturbation series, it suffices to take a fixed background spacetime, and solve the Maxwell equations for the first order perturbations.

The Maxwell equations are

$$\nabla_{[a}F_{bc]} = 0, \tag{2.24}$$

$$-\nabla^a F_{ab} = 4\pi J_b. \tag{2.25}$$

We write out the Levi-Civita covariant derivative in terms of the tetrad derivative and spin coefficients. The constraint equations (2.24) become

$$\begin{aligned} &\Big[(D-2\rho)\phi_1 - (\bar{\delta}+\pi-2\alpha)\phi_0\Big] - c.c. = 0, \\ &[(\delta-\tau+2\beta)\phi_2 - (\Delta+2\mu)\phi_1] - c.c. = 0, \\ &\quad(\delta-2\tau)\phi_1 - (\Delta+\mu-2\gamma)\phi_0 = (D-\bar{\rho}+2\bar{\epsilon})\bar{\phi}_2 - (\delta+2\bar{\pi})\bar{\phi}_1, \end{aligned} \tag{2.26}$$

whereas the source equations (2.25) are

$$\begin{aligned} \Big[(D-2\rho)\phi_1 - (\bar{\delta}+\pi-2\alpha)\phi_0\Big] + c.c. &= 4\pi J_1, \\ [(\delta-\tau+2\beta)\phi_2 - (\Delta+2\mu)\phi_1] + c.c. &= 4\pi J_2, \\ (\delta-2\tau)\phi_1 - (\Delta+\mu-2\gamma)\phi_0 + (D-\bar{\rho}+2\bar{\epsilon})\bar{\phi}_2 - (\delta+2\bar{\pi})\bar{\phi}_1 &= 4\pi J_3, \\ (D-\rho+2\epsilon)\phi_2 - (\bar{\delta}+2\pi)\phi_1 + (\bar{\delta}-2\bar{\tau})\bar{\phi}_1 - (\Delta+\bar{\mu}-2\bar{\gamma})\bar{\phi}_0 &= 4\pi J_4. \end{aligned} \tag{2.27}$$

Using the constraints (2.26) the source equations (2.27) simplify to

$$(D-2\rho)\phi_1 - (\bar{\delta}+\pi-2\alpha)\phi_0 = 2\pi J_1, \tag{2.28}$$

$$(\delta-\tau+2\beta)\phi_2 - (\Delta+2\mu)\phi_1 = 2\pi J_2, \tag{2.29}$$

$$(\delta-2\tau)\phi_1 - (\Delta+\mu-2\gamma)\phi_0 = 2\pi J_3, \tag{2.30}$$

$$(D-\rho+2\epsilon)\phi_2 - (\bar{\delta}+2\pi)\phi_1 = 2\pi J_4. \tag{2.31}$$

To obtain a decoupled equation for the scalar ϕ_0, we multiply the equation (2.28) by $(\delta-\beta+\bar{\alpha}+\bar{\pi}-2\tau)$ from the left. We do the same with the equation (2.30) and the expression $(D-\epsilon+\bar{\epsilon}-\bar{\rho}-2\rho)$. Next, we subtract these two equations and use the commutation relation (2.23) with the choice $p=0$ and $q=-2$. As a result, we have the decoupled equation for ϕ_0

$$\begin{aligned} &\Big[(D-\epsilon+\bar{\epsilon}-\bar{\rho}-2\rho)(\Delta+\mu-2\gamma) \\ &\qquad - (\delta-\beta-\bar{\alpha}+\bar{\pi}-2\tau)(\bar{\delta}+\pi-2\alpha)\Big]\phi_0 = 2\pi\mathcal{J}_0, \end{aligned} \tag{2.32}$$

with the source $\mathcal{J}_0$ defined as

$$\mathcal{J}_0 = (\delta - \beta - \bar{\alpha} + \bar{\pi} - 2\tau)J_1 - (D - \epsilon + \bar{\epsilon} - \bar{\rho} - 2\rho)J_3. \tag{2.33}$$

At this point, we can notice that the defining conditions of the NP tetrad (2.1) are symmetric with respect to the exchange $l \leftrightarrow n$ and $m \leftrightarrow \bar{m}$. Thus, performing this exchange in our definitions (2.14) and (2.4) gives us the following correspondence

$$\begin{aligned} D &\leftrightarrow \Delta, & \tau &\leftrightarrow -\pi, \\ \delta &\leftrightarrow \bar{\delta}, & \rho &\leftrightarrow -\mu, \\ \epsilon &\leftrightarrow -\gamma, & \phi_0 &\leftrightarrow -\phi_2. \\ \alpha &\leftrightarrow -\beta, \end{aligned} \tag{2.34}$$

Carrying out the substitutions (2.34) on (2.32) and (2.33), we obtain the decoupled equation for ϕ_2

$$\Big[(\Delta + \gamma - \bar{\gamma} + \bar{\mu} + 2\mu)(D - \rho + 2\epsilon) \\ (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau} + 2\pi)(\delta - \tau + 2\beta)\Big]\phi_2 = 2\pi\mathcal{J}_2, \tag{2.35}$$

where

$$\mathcal{J}_2 = (\Delta + \gamma - \bar{\gamma} + \bar{\mu} + 2\mu)J_4 - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau} + 2\pi)J_2. \tag{2.36}$$

2.3 Hertzian and Debye potentials in NP

The Maxwell equations consist of two types. We have the constraints (2.24) that restrict what our fields can look like irrespective of any sources. These can be identically satisfied by expressing the electromagnetic tensor via suitable potentials – the usual choice being the scalar potential φ and the vector potential A^j. Then we have the equations that express the dependence of the field on the sources (2.25). We can thus solve the Maxwell equations by expressing the field in terms of potentials that identically solve the constraints and then by substituting the potentials into the source equations. Using the remaining gauge freedom, we can impose the Lorentz condition on these potentials and obtain separate wave equations for both the scalar and the vector potential.

However, it can be advantageous to choose a different route. We could express the potentials (φ, A^j) via two Hertzian vector potentials in such a way, that they automatically satisfy the Lorentz condition. Moreover, we could use the remaining gauge freedom to try to obtain the same equation for the two independent degrees of freedom of a free electromagnetic field – the two Debye potentials.

This is most easily done in the language of differential forms. We express the electromagnetic field in terms of the 2-form

$$F = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.37}$$

Then, using (1.48) and (1.50) we can write the source-free Maxwell equations as

$$\mathrm{d}F = 0, \tag{2.38}$$

$$\delta F = 0. \tag{2.39}$$

Assuming that Poincaré's lemma (Th. 1) holds, we can first solve the constraint (2.38) by setting

$$F = \mathrm{d}A, \tag{2.40}$$

where A is the standard electromagnetic potential 1-form. The satisfaction of the Lorentz condition can be expressed via the co-derivative

$$\delta A = 0. \tag{2.41}$$

This can be achieved by setting the electromagnetic potential equal to the co-derivative of the Hertzian potential $\tilde{P}$

$$A = \delta \tilde{P}. \tag{2.42}$$

So far we have the electromagnetic 2-form given by

$$F = \mathrm{d}\delta \tilde{P}. \tag{2.43}$$

In order to satisfy the remaining equation (2.39), we could require

$$F = -\delta \mathrm{d}\tilde{P}, \tag{2.44}$$

which then implies that the Hertzian potential can be found as a solution to the Laplacian

$$\Delta_L \tilde{P} = 0. \tag{2.45}$$

Here, we can exploit the remaining gauge freedom

$$\mathrm{d}F = \mathrm{d}(\mathrm{d}\delta\tilde{P}) = \mathrm{d}(\mathrm{d}\delta P - \mathrm{d}G)), \tag{2.46}$$

$$\delta F = \delta(-\delta\mathrm{d}\tilde{P}) = \delta(\delta W - \delta \mathrm{d}P), \tag{2.47}$$

where G is an arbitrary 1-form, and W is an arbitrary 3-form. Put together, we obtain

$$\begin{aligned} F &= \mathrm{d}\delta P - \mathrm{d}G, \\ &= \delta W - \delta \mathrm{d}P, \\ \Delta_L P &= -\mathrm{d}G - \delta W. \end{aligned} \tag{2.48}$$

For a special selection of the non-zero components of P and for a suitable choice of the gauge terms W and G, we could be able to obtain a single wave equation for the Debye potentials. However, the existence of this path is restricted by the properties of the spacetime. This is discussed in detail in the original article [10], which we are following.

Note that using the complex NP formalism enables us to encode twice as much information in any complex quantity. Thus, we will only need a single complex scalar Debye potential.

Now we write out the equations (2.48) using tensor notation in a coordinate frame, and express the Laplacian in terms of exterior derivative and co-derivative (1.51). Then, thanks to (1.48) and (1.50), the exterior derivative becomes an antisymmetrized Levi-Civita covariant derivative, and the co-derivative becomes a minus divergence over the first index. Put together the equation for the potential is

$$\begin{aligned} &- \nabla_\lambda \nabla^\lambda P_{\mu\nu} + \left(\nabla^\lambda \nabla_\mu - \nabla_\mu \nabla^\lambda\right) P_{\lambda\nu} + \left(\nabla_\nu \nabla^\lambda - \nabla^\lambda \nabla_\nu\right) P_{\lambda\mu} = \\ &\qquad \nabla_\mu G_\nu - \nabla_\nu G_\mu - \nabla^\lambda W_{\lambda\mu\nu}, \end{aligned} \tag{2.49}$$

and for the electromagnetic tensor we get

$$\begin{aligned} F_{\mu\nu} &= -\nabla_\mu \nabla^\lambda P_{\lambda\nu} + \nabla_\nu \nabla^\lambda P_{\lambda\mu} - \nabla_\mu G_\nu + \nabla_\nu G_\mu, \\ &= \nabla_\lambda \nabla^\lambda P_{\mu\nu} - \nabla^\lambda \nabla_\mu P_{\lambda\nu} + \nabla^\lambda \nabla_\nu P_{\lambda\mu} - \nabla^\lambda W_{\lambda\mu\nu}. \end{aligned} \tag{2.50}$$

At this point, we define the components of the potential and the gauge terms

$$\begin{aligned} &P_{24} \equiv \psi, && W_{243} \equiv -2\tau\psi + 2\bar{\tau}\bar{\psi}, \\ &G_2 \equiv 2\bar{\tau}\bar{\psi} + 2\tau\psi, && W_{124} \equiv -2\rho\psi, \\ &G_4 \equiv 2\rho\psi, \end{aligned} \tag{2.51}$$

where ψ is the complex Debye potential (we note that there is a misprint in the definition of W in the original article [10]). All other components are zero unless they can be obtained by complex conjugation or antisymmetry. Now we are ready to move the equations (2.49) and (2.50) into the NP tetrad frame with the help of (1.33) and (2.4) remembering our type D conditions (2.9). First, we write the equations for the Debye potential. For the 24 component we obtain

$$\begin{aligned} -2\Big[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})(D - \rho + 2\epsilon) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta - \tau + 2\beta)\Big]\psi = \\ 4\Big[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})\rho - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})\tau\Big]\psi, \end{aligned} \tag{2.52}$$

and half of the sum of components 12 and 43 gives us

$$\begin{aligned} -\Big[(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(\delta - \tau + 2\beta) + (-\delta + \bar{\alpha} - \beta - \bar{\pi} - \tau)(D - \rho + 2\epsilon)\Big]\psi = \\ 2\Big[(D + \epsilon + \bar{\epsilon} - \bar{\rho})\tau - (\delta + \bar{\pi} - \bar{\alpha} + \beta)\rho\Big]\psi. \end{aligned} \tag{2.53}$$

Other components either vanish or are obtained by complex conjugation or antisymmetry. Using the NP commutator (2.19) and the equations (2.20), (2.21), (2.22), it can be easily shown that the second Debye equation (2.53) is identically satisfied. Simplifying (2.52), we obtain a single NP equation for the Debye potential

$$\Big[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})(D + \rho + 2\epsilon) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + \tau + 2\beta)\Big]\psi = 0. \tag{2.54}$$

For the complex electromagnetic scalars (2.14), we get the following expressions in terms of the Debye potential

$$\phi_0 = -(D - \epsilon + \bar{\epsilon} - \bar{\rho})(D + \bar{\rho} + 2\bar{\epsilon})\bar{\psi}, \tag{2.55}$$

$$\phi_1 = \Big[(\pi + \bar{\tau})(D + \bar{\rho} + 2\bar{\epsilon}) - (D + \bar{\epsilon} + \epsilon)(\bar{\delta} + \bar{\tau} + 2\bar{\beta})\Big]\bar{\psi}, \tag{2.56}$$

$$\phi_2 = -(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + \bar{\tau} + 2\bar{\beta})\bar{\psi}. \tag{2.57}$$

In order to make them look as simple as possible, we had to subtract some of the wave equations (2.49). In the case of ϕ_1, we had to subtract half of the component 43, and in the case of ϕ_2, we added one half of the component 24.

2.4 Kerr spacetime

The metric of the Kerr spacetime (K, g_K) can be expressed in the Boyer-Lindquist (BL) coordinates as

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4Mar}{\Sigma}\sin^2\theta \mathrm{d}t\mathrm{d}\phi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 - \frac{A\sin^2\theta}{\Sigma}\mathrm{d}\phi^2, \quad (2.58)$$

with the auxiliary functions defined by

$$\begin{aligned} \Delta &\equiv r^2 - 2Mr + a^2, \\ \Sigma &\equiv r^2 + a^2\cos^2\theta, \\ A &\equiv \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta. \end{aligned} \quad (2.59)$$

The parameter M is the mass of the Kerr center, and a is the rotation parameter – in particular, the angular momentum per unit mass. The Kerr metric has a curvature singularity at

$$\Sigma = 0 \iff r = 0 \quad \& \quad \theta = \frac{\pi}{2}. \quad (2.60)$$

Thus we observe that the singularity has a ring-like character since we only encounter it at the equatorial plane. For $a < M$, the roots of Δ give the outer r_+ and inner r_- horizons

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \quad (2.61)$$

In the special case $a = M$, we obtain a single horizon of the extreme Kerr spacetime. Whenever $a > M$, there are no real roots, and we end up with the naked Kerr singularity. Another important surface is given by $g_{tt} = 0$. There cannot exist a stationary timelike observer below this surface – the observer is forced to corotate with the geometry. Thus it may be called the static limit. The condition becomes $\Sigma = 2Mr_E$, which then gives

$$r_E = M + \sqrt{M^2 - a^2\cos^2\theta}. \quad (2.62)$$

It might sometimes be advantageous to transform certain expressions into Weyl coordinates defined by

$$\begin{aligned} z &= (r - M)\cos\theta, \\ \rho &= \sqrt{\Delta}\sin\theta. \end{aligned} \quad (2.63)$$

It is useful to point out the character of these coordinates. If $a \leq M$, the horizon is located at $\rho = 0$ and $z \in (-\sqrt{M^2 - a^2}, \sqrt{M^2 - a^2})$. Therefore, the Weyl coordinates only cover the region above the horizon of the Kerr black hole. Moreover, the horizon is represented by a single straight line or – in the extreme case – by a single point. If $a > M$, the Weyl coordinates cover also the singularity that is located at $\rho = a$ and $z = 0$, thus further pointing towards its ring-like character. We draw the BL coordinate lines in the Weyl coordinates for $a = 0.8\,M$ in Fig. 2.1, for $a = M$ in Fig. 2.2, and for $a = 1.2\,M$ in Fig. 2.3.

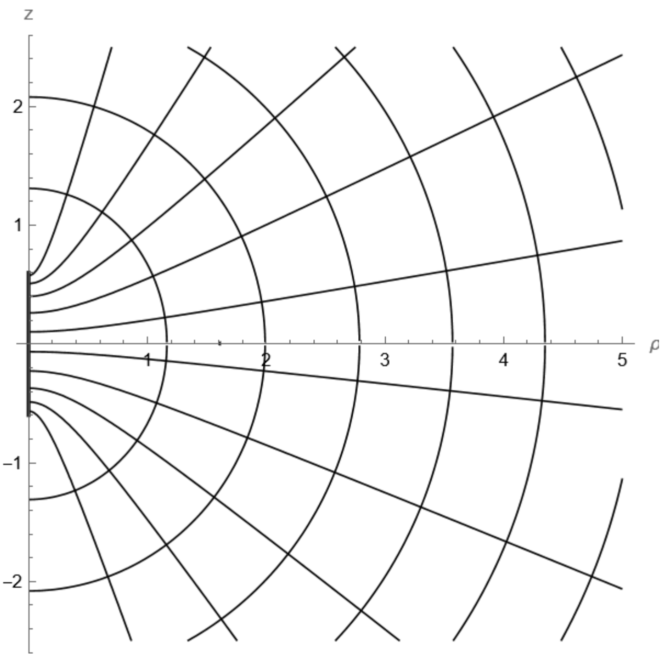

Figure 2.1: Curves of constant BL coordinates r and θ depicted in the Weyl coordinates for Kerr spacetime $M = 1$, $a = 0.8$. The lines that become circular away from the center are $r = \text{const.}$, whereas the lines that become radial are $\theta = \text{const.}$ The thick line at $\rho = 0$ is the horizon.

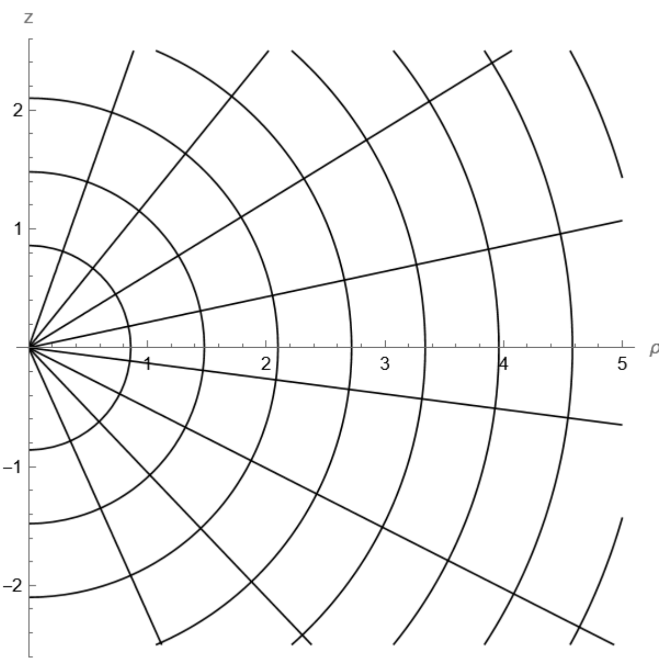

Figure 2.2: Curves of constant BL coordinates r and θ depicted in the Weyl coordinates for extreme Kerr spacetime $M = 1$, $a = 1$. The circular lines are $r = \text{const.}$, whereas the radial lines are $\theta = \text{const.}$ The point $\rho = 0$ and $z = 0$ is the extreme horizon.

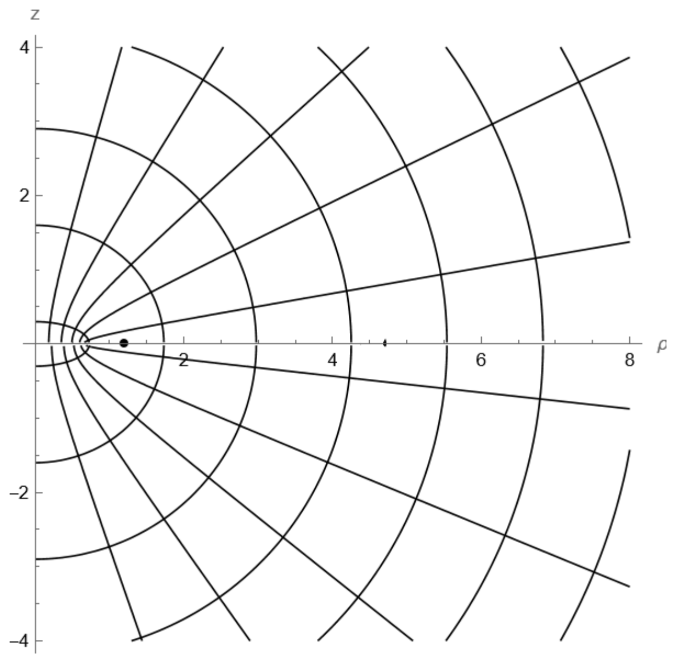

Figure 2.3: Curves of constant BL coordinates r and θ depicted in the Weyl coordinates for hyper-extreme Kerr spacetime $M = 1$, $a = 1.2$. The lines that become circular away from the center are r = const., whereas the lines that become radial are θ = const. The thick point at $\rho = 1.2$ is the ring singularity.

Turning to the NP formalism, we will be using the same tetrad as Teukolsky [2]

$$\begin{aligned} l^\mu &= \left[\frac{r^2+a^2}{\Delta},\, 1,\, 0,\, \frac{a}{\Delta}\right], \\ n^\mu &= \frac{1}{2\Sigma}\left[r^2+a^2,\, -\Delta,\, 0,\, a\right], \\ m^\mu &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left[ia\sin\theta,\, 0,\, 1,\, \frac{i}{\sin\theta}\right]. \end{aligned} \tag{2.64}$$

We can find the Ricci rotation coefficients dirrectly from (1.32) by computing the Levi-Civita covariant derivative of our NP tetrad. The spin coefficients then become

$$\begin{aligned} \rho &= -\frac{1}{r-ia\cos\theta}, & \beta &= -\bar{\rho}\frac{\cot\theta}{2\sqrt{2}}, \\ \pi &= ia\rho^2\frac{\sin\theta}{\sqrt{2}}, & \tau &= -ia\rho\bar{\rho}\frac{\sin\theta}{\sqrt{2}}, \\ \mu &= \rho^2\bar{\rho}\frac{\Delta}{2}, & \gamma &= \mu + \rho\bar{\rho}\frac{r-M}{2}. \\ \alpha &= \pi - \bar{\beta}, \end{aligned} \tag{2.65}$$

Since Kerr spacetime is a vacuum spacetime without cosmological constant, it has a vanishing Ricci tensor and a zero scalar curvature. Thus, the Weyl tensor is directly the Riemann tensor, and we can find the Weyl scalars by its projections onto our NP tetrad. The only non-zero component is

$$\Psi_2 = M\rho^3. \tag{2.66}$$

This confirms that the Kerr spacetime is type D, and that $\Sigma = 0$ is a curvature singularity.

The choice of our tetrad allows for some useful properties. The NP Maxwell equations (2.28)–(2.31) take on the following form

$$\left(\frac{\partial}{\partial r} - 2\rho\right)\phi_1 + \frac{\rho}{\sqrt{2}}\left(\frac{\partial}{\partial \theta} + ia\rho\sin\theta + \cot\theta\right)\phi_0 = 2\pi J_1, \quad (2.67)$$

$$\frac{\rho^*}{\sqrt{2}}\left(-\frac{\partial}{\partial \theta} + ia\rho\sin\theta - \cot\theta\right)\phi_2 + \frac{\Delta\rho\rho^*}{2}\left(\frac{\partial}{\partial r} - 2\rho\right)\phi_1 = 2\pi J_2, \quad (2.68)$$

$$-\frac{\rho^*}{\sqrt{2}}\left(\frac{\partial}{\partial \theta} - 2ia\rho\sin\theta\right)\phi_1 + \frac{\rho\rho^*}{2}\left(\Delta\frac{\partial}{\partial r} + \rho\Delta + 2(r-M)\right)\phi_0 = 2\pi J_3, \quad (2.69)$$

$$\left(\frac{\partial}{\partial r} - \rho\right)\phi_2 + \frac{\rho}{\sqrt{2}}\left(\frac{\partial}{\partial \theta} - 2ia\rho\sin\theta\right)\phi_1 = 2\pi J_4. \quad (2.70)$$

Let us assume that our source only has the t and ϕ components

$$J^\mu = \left(J^t, 0, 0, J^\phi\right). \quad (2.71)$$

Then, the NP projections onto l and n are the same up to a factor

$$J_2 = \frac{\Delta\rho\rho^*}{2}J_1. \quad (2.72)$$

Thanks to this, if we multiply the equation (2.67) by $\Delta\rho\bar{\rho}/2$, and subtract it from (2.68), we find a simple relationship between ϕ_0 and ϕ_2 for this special source

$$\phi_2 = -\frac{\rho^2\Delta}{2}\phi_0. \quad (2.73)$$

Next we would like to write down the NP decoupled equations for the scalars ϕ_0 (2.32) and ϕ_2 (2.35) in Kerr spacetime. Interestingly, it can be given in terms of the single Teukolsky's master equation [2]

$$\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi_s}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi_s}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi_s}{\partial\phi^2} \\
&- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi_s}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_s}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi_s}{\partial\phi} \\
&- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi_s}{\partial t} + (s^2\cot^2\theta - s)\psi_s = 4\pi\Sigma S_s, \quad (2.74)
\end{aligned}$$

where s is the so-called spin weight and

$$\begin{aligned}
\psi_1 &\equiv \phi_0, & \psi_{-1} &\equiv \rho^{-2}\phi_2, \\
S_1 &\equiv \mathcal{J}_0 & S_{-1} &\equiv \rho^{-2}\mathcal{J}_2.
\end{aligned} \quad (2.75)$$

In fact, this equation also describes a test scalar field ($s = 0$), a test neutrino field ($s = \pm 1/2$), and even gravitational perturbations ($s = \pm 2$). However, we are only interested in the test electromagnetic fields. In particular, we will only care about the fields, which are stationary and axially symmetric. In this case, the master equation reduces to

$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi_s}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_s}{\partial\theta}\right)+(s^2\cot^2\theta-s)\psi_s=4\pi\Sigma S_s, \quad (2.76)$$

Next, we write down the equation for the Debye potential (2.54) and the expressions for the electromagnetic scalars (2.55) in Kerr spacetime. Remember that these are only for a free electromagnetic field. Assuming stationarity and axial symmetry, the Debye equation is

$$-\Delta\frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\psi=0. \quad (2.77)$$

We can notice that it is the same as the Teukolsky equation (2.76) for the spin weight $s=-1$. This correspondence holds also in the non-stationary non-axially symmetric case. The electromagnetic scalars can be obtained from the Debye potential via differentiation

$$\phi_0=-\frac{\partial^2\bar{\psi}}{\partial r^2}, \quad (2.78)$$

$$\phi_1=\frac{\rho}{\sqrt{2}}\left[\rho\left(\frac{\partial}{\partial\theta}+\cot\theta+ia\sin\theta\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial r\partial\theta}+\cot\theta\frac{\partial}{\partial r}\right]\bar{\psi}. \quad (2.79)$$

The scalar ϕ_2 is determined by ϕ_0 (2.73).

When searching for electromagnetic fields, we may encounter some unwanted electric and magnetic charges, which arise as an artifact of the integration process. However, we would like these charges to be removed. Thus, consider the Maxwell equations in terms of differential forms

$$\mathrm{d}F=4\pi J_M, \quad (2.80)$$

$$\delta F=*\,\mathrm{d}*F=4\pi J, \quad (2.81)$$

where J_M is the magnetic current 3-form. Applying the Hodge dual on the second equation, we get

$$\mathrm{d}*F=-4\pi*J.$$

We can thus combine these two equations into a single equation using the anti-self-dual

$$\mathrm{d}\mathcal{F}=4\pi(J_M+i*J). \quad (2.82)$$

Now we simply integrate over the volume Ω with the boundary $\partial\Omega$ and use the Stokes theorem

$$\begin{gathered}\frac{1}{4\pi}\int_\Omega\mathrm{d}\mathcal{F}=\int_\Omega(J_M+i*J)\,,\\ \downarrow\\ \frac{1}{4\pi}\int_{\partial\Omega}\mathcal{F}=Q_M+iQ,\end{gathered} \quad (2.83)$$

where Q_M and Q are the magnetic and electric charges respectively. Our choice of the volume Ω shall be such that

$$\partial\Omega=\{t=t_0,\,r=\tilde{r},\,\theta\in(0,\pi),\,\phi\in(0,2\pi)\}.$$

Then, the integral becomes

$$\frac{1}{4\pi}\int_{\partial\Omega}\mathcal{F} = \frac{1}{4\pi}\int_{\partial\Omega}\frac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathcal{F}_{\theta\phi}|_{r\to\tilde{r}}\,\mathrm{d}\theta\mathrm{d}\phi. \tag{2.84}$$

Next, we integrate over the coordinate ϕ since we have axial symmetry. We express the component of the anti-self-dual in terms of the complex scalars (2.16) and also use the relation between ϕ_0 and ϕ_2 (2.73)

$$Q_M + iQ = -\int_0^{\pi}\left(i\sin\theta(r^2+a^2)\phi_1 + \frac{a}{\sqrt{2}}\sin^2\theta\rho\Delta\phi_0\right)\bigg|_{r\to\tilde{r}}\mathrm{d}\theta. \tag{2.85}$$

The monopole field in Kerr geometry with the charge $Q_M + iQ$ is given only by a single non-zero electromagnetic scalar [19]

$$\phi_1 = -\frac{1}{2}\frac{Q - iQ_M}{(r - ia\cos\theta)^2}. \tag{2.86}$$

This can be easily checked by evaluating the integral (2.85). As a consequence, the scalar ϕ_0 holds no information about the monopole field.

When we try to visualize our results, it is important to choose a reasonable observer. An especially privileged one is the ZAMO (zero angular momentum observer), also sometimes referred to as the locally non-rotating observer. His worldlines draw out a circular orbit $r = \text{const.}$, $\theta = \text{const.}$, $\phi = \omega t + \text{const.}$, where $\omega = -g_{\phi t}/g_{\phi\phi}$. In Kerr spacetime, the ZAMO tetrad becomes [20]

$$\begin{aligned}
e^{\mu}_{(t)} &\equiv \left[\sqrt{\frac{A}{\Sigma\Delta}},\,0,\,0,\,\frac{2Mar}{\sqrt{A\Sigma\Delta}}\right],\\
e^{\mu}_{(r)} &\equiv \left[0,\,\sqrt{\frac{\Delta}{\Sigma}},\,0,\,0\right],\\
e^{\mu}_{(\theta)} &\equiv \left[0,\,0,\,\frac{1}{\sqrt{\Sigma}},\,0\right],\\
e^{\mu}_{(\phi)} &\equiv \left[0,\,0,\,0,\,\sqrt{\frac{\Sigma}{A}}\frac{1}{\sin\theta}\right].
\end{aligned} \tag{2.87}$$

The meaning of this frame is best captured in particle motion. If a particle at infinity with zero angular momentum with respect to the black hole starts to freely fall towards the black hole, its momentum is conserved, and thus it is forced to co-rotate around the black hole with the angular velocity ω.

The electromagnetic field with respect to this tetrad can be obtained by

$$E_{(j)} - iB_{(j)} = \mathcal{F}_{\mu\nu}e^{\mu}_{(t)}e^{\nu}_{(j)}. \tag{2.88}$$

3. Solutions to electromagnetic fields on Kerr background

The goal of this chapter is to show the reader how to obtain all the information about a test electromagnetic field of a ring source in Kerr background in terms of a single complex Debye potential. This was first done by Linet [11]. We go step by step over his strategy and discuss all the details.

First, we need to solve the axially symmetric stationary Teukolsky equation in the BL coordinates. We show that after a simple transformation, the resulting equation can be interpreted as a Laplacian in a Kerr-like family of spacetimes. We then follow Linet and transform it into the Weyl coordinates. Afterwards, we follow the work of Heins [21] – we interpret the equation as a cylindrical Laplacian, derive the solution, and prove that it is completely determined by the values on the symmetry axis. At last, we get to expressing the Debye potential. There, we only discuss the details of Linet's approach.

3.1 Formulating the problem

Let us introduce G_s as the Green function of the Teukolsky equation (2.76) for an axisymmetric and stationary solution ψ_s

$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{1+s}\frac{\partial G_s}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G_s}{\partial\theta}\right)+\left[\cot^2\theta+s(s-1)\right]G_s = \delta(r-r_0)\delta(\cos\theta-\cos\theta_0). \quad (3.1)$$

Then, we can express ψ_s for an arbitrary axisymmetric and stationary source S_s via convolution

$$\psi_s = 4\pi\int G_s(r,\theta,r_0,\theta_0)\Sigma(r_0,\theta_0)S_s(r_0,\theta_0,r_J,\theta_J)\sin\theta_0\mathrm{d}\theta_0\mathrm{d}r_0, \quad (3.2)$$

where r_J and θ_J are the BL coordinates of the source in Kerr spacetime. Using the substitution

$$G_s = -\frac{\sin^s\theta}{\sin^s\theta_0\Sigma_0}\mathcal{G}_s, \quad (3.3)$$

the equation (3.1) takes on the following form

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{1+s}\frac{\partial \mathcal{G}_s}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathcal{G}_s}{\partial\theta}\right)+2s\cot\theta\frac{\partial \mathcal{G}_s}{\partial\theta} = \Sigma\delta(r-r_0)\delta(\cos\theta-\cos\theta_0) \quad (3.4)$$

We shall sometimes denote the differential operator of the left-hand-side (LHS) by T_s

$$T_s[\mathcal{G}_s] \equiv \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{1+s}\frac{\partial \mathcal{G}_s}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathcal{G}_s}{\partial\theta}\right)+2s\cot\theta\frac{\partial \mathcal{G}_s}{\partial\theta}. \quad (3.5)$$

Next, we transform the equation for $\mathcal{G}_s$ (3.4) into the Weyl coordinates (2.63). For its right-hand-side (RHS) we obtain

$$
\begin{aligned}
\mathrm{RHS} &= \Sigma\delta(r-r_0)\delta(\cos\theta-\cos\theta_0),\\
&= \frac{\Sigma}{\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0),\\
&= \frac{\Sigma}{\sqrt{\Delta}\sin\theta}\left(\Delta\cos^2\theta+\frac{1}{4}(\Delta')^2\sin^2\theta\right)\delta(\rho-\rho_0)\delta(z-z_0),
\end{aligned}
\tag{3.6}
$$

whereas for the LHS we get

$$
\left(\Delta\cos^2\theta+\frac{1}{4}(\Delta')^2\sin^2\theta\right)\left(\frac{\partial^2\mathcal{G}_s}{\partial\rho^2}+\frac{\partial^2\mathcal{G}_s}{\partial z^2}+\frac{1+2s}{\rho}\frac{\partial\mathcal{G}_s}{\partial\rho}\right). \tag{3.7}
$$

Therefore, the transformed equation takes on a simple form

$$
\frac{\partial^2\mathcal{G}_s}{\partial z^2}+\frac{\partial^2\mathcal{G}_s}{\partial\rho^2}+\frac{1+2s}{\rho}\frac{\partial\mathcal{G}_s}{\partial\rho}=\frac{\Sigma}{\rho_0}\delta(\rho-\rho_0)\delta(z-z_0). \tag{3.8}
$$

The associated differential operator shall be denoted by W_s

$$
W_s[\mathcal{G}_s]\equiv\frac{\partial^2\mathcal{G}_s}{\partial z^2}+\frac{\partial^2\mathcal{G}_s}{\partial\rho^2}+\frac{1+2s}{\rho}\frac{\partial\mathcal{G}_s}{\partial\rho}. \tag{3.9}
$$

To finish this part, we note that there is a relationship between the fundamental solutions of T_s (W_s) for s and their counterparts given by the exchange $s\to -s$. This can be proved by direct computation. It holds

$$
\Delta^s T_s\left[\frac{1}{\Delta^s}\mathcal{G}_{-s}\right]=\delta \quad\Longleftrightarrow\quad T_{-s}[\mathcal{G}_{-s}]=\delta \tag{3.10}
$$

In the Weyl coordinates the relation is just the same.

3.2 Interpretation of T_s and W_s

With the help of (1.52), it can be easily shown that the operator T_0/Σ reduces to the Laplace operator in Kerr spacetime (K, g_K) when acting on $f(r,\theta)$. We could thus try to find a generalization of the Kerr spacetime (K_s, g_{K_s}) for $s\neq 0$. For simplicity of notation, we shall omit the index K_s for the generalized metric in the following calculations.

We start by demanding that the only non-zero r and θ components of the metric are g_{rr} and $g_{\theta\theta}$. Then it clearly holds

$$
g^{rr}=\frac{1}{g_{rr}},\qquad g^{\theta\theta}=\frac{1}{g_{\theta\theta}}.
$$

Next, we calculate the action of the Laplacian Δ_{K_s} on a function $f(r,\theta)$. We obtain the following expression

$$
\begin{aligned}
\Delta_{K_s}f(r,\theta) &= -\frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\,g^{\mu\nu}\partial_\nu f(r,\theta)\right),\\
&= -\frac{\partial_{rr}f}{g_{rr}}-\frac{1}{\sqrt{-\det g}}\partial_r\left(\frac{\sqrt{-\det g}}{g_{rr}}\right)\partial_r f\\
&\quad -\frac{\partial_{\theta\theta}f}{g_{\theta\theta}}-\frac{1}{\sqrt{-\det g}}\partial_\theta\left(\frac{\sqrt{-\det g}}{g_{\theta\theta}}\right)\partial_\theta f.
\end{aligned}
\tag{3.11}
$$

Thus, all we need to find are two metric components g_{rr}, $g_{\theta\theta}$, and the metric determinant. By comparing (3.11) to T_s/Σ (3.4) we quickly identify

$$\begin{aligned} g_{rr} &= -\frac{\Sigma}{\Delta}, \\ g_{\theta\theta} &= -\Sigma. \end{aligned} \tag{3.12}$$

We shall denote the unknown part of the metric determinant by γ

$$\det g \equiv -g_{rr} g_{\theta\theta} \gamma = -\frac{\Sigma^2}{\Delta}\gamma. \tag{3.13}$$

Now, we substitute this back into (3.11), compare it to T_s/Σ, and obtain the following set of conditions

$$\begin{aligned} 2(1+s)\frac{r-M}{\Sigma} &= \frac{r-M}{\Sigma} + \frac{\Delta}{2\Sigma}\frac{\partial_r \gamma}{\gamma}, \\ \frac{1+2s}{\Sigma}\cot\theta &= \frac{1}{2\Sigma}\frac{\partial_\theta \gamma}{\gamma}, \end{aligned}$$

which are equivalently expressed as

$$\begin{aligned} (1+2s)\partial_r \ln \Delta &= \partial_r \ln \gamma, \\ 2(1+2s)\partial_\theta \ln \sin\theta &= \partial_\theta \ln \gamma. \end{aligned}$$

We can easily write down their solution in the following form

$$\gamma = \Delta^{(1+2s)} \sin^{2(1+2s)} \theta. \tag{3.14}$$

We thus conclude that T_s/Σ acting on a function $f(r,\theta)$ can be interpreted as a Laplacian in a class of Kerr-like Lorentzian spacetimes (K_s, g_{K_s}) determined only by the r and θ metric components in the BL coordinates and by the metric determinant

$$\begin{aligned} g_{rr} &= -\frac{\Sigma}{\Delta}, \\ g_{\theta\theta} &= -\Sigma, \\ g_{r\mu} &= 0, \;\; \mu \neq r, \\ g_{\theta\mu} &= 0, \;\; \mu \neq \theta, \\ \det g &= -\Sigma^2 \Delta^{2s} \sin^{2(1+2s)} \theta. \end{aligned} \tag{3.15}$$

The dimension is arbitrary as there can be other non-zero components of the metric. However, since we are only interested in acting on axisymmetric stationary functions, the Laplacian of a metric diagonal in r and θ components cannot distinguish between the additional degrees of freedom. They just have to give the computed determinant. Although we might add that from the upcoming paragraphs, we will see that $4+2s$ spacetime dimensions would seem most natural in our context.

The interpretation of W_s is very simple, and the reader might see it at first glance. However, for completeness, let us write the Laplacian in cartesian coordinates in an m-dimensional Euclidean space

$$\sum_{i=1}^{m} \frac{\partial^2 \varphi}{\partial x_i^2}.$$

We define the spherical coordinates as follows

$$\begin{aligned}
x_1 &= r\sin\theta_1 ... \sin\theta_{m-1},\\
x_2 &= r\sin\theta_1 ... \cos\theta_{m-1},\\
x_3 &= r\sin\theta_1 ... \cos\theta_{m-2},\\
&\vdots\\
x_m &= r\cos\theta_1,
\end{aligned}$$

where $r^2 = \sum_{i=1}^m x_i^2$, the bounds are given by $\theta_{m-1} \in (0, 2\pi)$ and for the rest $\theta_j \in (0, \pi)$. We transform the cartesian Laplacian into the spherical coordinates, demand spherical symmetry of the solution, and obtain

$$\frac{\partial^2\varphi}{\partial^2 r} + \frac{m-1}{r}\frac{\partial\varphi}{\partial r}.$$

The last step is to realize, that cylindrical coordinates in $3+2s$ dimensions would consist of a z coordinate along the symmetry axis and of $(2+2s)$-dimensional spherical coordinates.

We thus conclude that W_s acting on a function $f(\rho, z)$ can be interpreted as a cylindrical Laplacian in $3+2s$ Euclidean dimensions.

3.3 Finding the axisymmetric Green function of the Laplace equation

The fundamental solution to the cartesian Laplacian in $m \geq 3$ dimensions

$$\nabla^2_{x^i}\varphi = \delta^{(m)}(\vec{x}), \tag{3.16}$$

can be obtained by standard procedures – for example by Fourier transformation. It can be expressed as

$$\varphi = -\frac{1}{(m-2)\mathcal{S}_{m-1}}\left[\sum_{i=1}^m x_i^2\right]^{1-\frac{m}{2}},$$

where $\mathcal{S}_{m-1}$ is the measure of a unit $(m-1)$-dimensional sphere embedded in an m-dimensional Euclidean space.

The equation (3.16) is invariant with respect to translations in the constant direction of $\vec{\xi}$

$$\nabla^2_{x^i} = \nabla^2_{x^i-\xi^i},$$

and thus we can write the equality

$$\nabla^2_{x^i}\left[\sum_{i=1}^m (x_i-\xi_i)^2\right]^{1-\frac{m}{2}} = -(m-2)\mathcal{S}_{m-1}\delta^{(m)}(\vec{x}-\vec{\xi}). \tag{3.17}$$

Now, we consider an $(m-1)$-dimensional subspace. There we define the vectors $\vec{\rho} \equiv (x_1, ..., x_{m-1})$ and $\vec{\rho}_0 = (\xi_1, ..., \xi_{m-1})$. The equation (3.17) thus can be further rewritten as

$$\nabla^2_{x^i}\left[(z-z_0)^2 + (\vec{\rho}(x^i)-\vec{\rho}_0)^2\right]^{1-\frac{m}{2}} = -(m-2)\mathcal{S}_{m-1}\delta(z-z_0)\delta^{(m-1)}(\vec{\rho}-\vec{\rho}_0), \tag{3.18}$$

where we have also used the notation $z \equiv x_m$, $z_0 \equiv \xi_m$. Next, we denote the angle between $\vec{\rho}$ and $\vec{\rho}_0$ as $\theta \equiv \theta_1$. This corresponds to a special selection of our coordinate system. As a consequence, we have $\vec{\rho} \cdot \vec{\rho}_0 = \rho\rho_0 \cos\theta$. Now, we transform the $(m-1)$-dimensional subspace into spherical coordinates

$$\left(\nabla^2_{C_m} + \frac{1}{\rho^2}\nabla^2_{S_{m-2}}\right)\left[(z-z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\theta\right]^{1-\frac{m}{2}} \\ = -(m-2)\mathcal{S}_{m-1}\delta(z-z_0)\frac{1}{\mathcal{M}}\delta(\rho-\rho_0)\delta(\theta)\,. \quad (3.19)$$

Here, $\nabla^2_{C_m}$ is the cylindrical Laplacian

$$\nabla^2_{C_m} = W_{\frac{m-3}{2}},$$

and $\nabla^2_{S_{m-2}}$ is the Laplacian on a unit $(m-2)$-dimensional sphere. The constant $\mathcal{M}$ is chosen so that

$$\int_{B_{m-1}(R)} \frac{1}{\mathcal{M}}\delta(\rho-\rho_0)\delta(\theta)\mathrm{d}V = 1, \quad (3.20)$$

with the integral being computed over an $(m-1)$-dimensional ball of radius $R = \rho_0 + \varepsilon$ for $\varepsilon > 0$. Therefore the volume element is

$$\mathrm{d}V = \sqrt{\det g_{B_{m-1}}}\mathrm{d}r\mathrm{d}\theta...\mathrm{d}\theta_{m-2} = r^{m-2}\sin^{m-3}\theta\sin^{m-4}\theta_2...\sin\theta_{m-3}\mathrm{d}r\mathrm{d}\theta...\mathrm{d}\theta_{m-2}.$$

Our equation still has angular dependence which we shall remove by integration over a unit $(m-2)$-dimensional sphere. Let us notice that the spherical part of the Laplacian vanishes upon this integration. This can be easily demonstrated since the only angular dependence is on θ, which means the only non-trivial part of the spherical Laplacian will be

$$\nabla^2_{S_{m-2}} f(\cos\theta) = \frac{1}{\sqrt{\det g_{S_{m-2}}}}\partial_\theta\left(\frac{\sqrt{\det g_{S_{m-2}}}}{\left(g_{S_{m-2}}\right)_{\theta\theta}}\partial_\theta f(\cos\theta)\right).$$

Now, since the angular measure is $\mathrm{d}\Omega = \sqrt{\det g_{S_{m-2}}}\mathrm{d}\theta...\mathrm{d}\theta_{m-2}$, we obtain

$$\begin{aligned}\int_{S_{m-2}} \nabla^2_{S_{m-2}} f(\cos\theta)\mathrm{d}\Omega &\propto \int_0^\pi \partial_\theta\left(\frac{\sqrt{\det g_{S_{m-2}}}}{\left(g_{S_{m-2}}\right)_{\theta\theta}}\partial_\theta f(\cos\theta)\right)\mathrm{d}\theta, \\ &= \left[-\sin^{m-2}\theta\sin^{m-4}\theta_2...\sin\theta_{m-3}\partial_x f(x)\right]_0^\pi, \\ &= 0.\end{aligned}$$

In the case of $m = 3$, the integration bounds of θ are of course $(0, 2\pi)$.

And so the equation (3.19) turns into the following equality

$$\nabla^2_{C_m}\int_{S_{m-2}}\left[(z-z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\theta\right]^{1-\frac{m}{2}}\mathrm{d}\Omega \\ = -(m-2)\mathcal{S}_{m-1}\delta(z-z_0)\delta(\rho-\rho_0)\int_{S_{m-2}}\frac{1}{\mathcal{M}}\delta(\theta)\,\mathrm{d}\Omega. \quad (3.21)$$

Now we rewrite the RHS. Using the definition of $\mathcal{M}$ (3.20)

$$\begin{aligned} 1 &= \int_{B_{m-1}} \frac{1}{\mathcal{M}} \delta(\rho - \rho_0)\delta\left(\theta\right) \mathrm{d}V, \\ &= \int_{\rho=0}^{\rho_0+\varepsilon} \int_{S_{m-2}} \frac{1}{\mathcal{M}} \delta(\rho - \rho_0)\delta\left(\theta\right) \rho^{m-2} \mathrm{d}\rho \mathrm{d}\Omega, \\ &= \rho_0^{m-2} \int_{S_{m-2}} \frac{1}{\mathcal{M}} \delta\left(\theta\right) \mathrm{d}\Omega, \end{aligned}$$

we obtain the following simplification

$$\int_{S_{m-2}} \frac{1}{\mathcal{M}} \delta\left(\theta\right) \mathrm{d}\Omega = \rho_0^{2-m}. \tag{3.22}$$

Carrying out partial integration of the LHS of (3.19) and using the equality (3.22), we obtain the final form

$$\begin{aligned} \nabla^2_{C_m} \int_0^\pi \left[(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\theta\right]^{1-\frac{m}{2}} \sin^{m-3}\theta \mathrm{d}\theta & \\ = -\frac{(m-2)\mathcal{S}_{m-1}}{\rho_0^{m-2}\mathcal{S}_{m-3}} \delta(z - z_0)\delta(\rho - \rho_0). & \end{aligned} \tag{3.23}$$

Since the area of a unit m-dimensional sphere is given by

$$\mathcal{S}_m = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}, \tag{3.24}$$

the fundamental solution to the equation

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\varphi}{\partial \rho^2} + \frac{m-2}{\rho}\frac{\partial\varphi}{\partial\rho} = -\frac{2\pi}{\rho_0^{m-2}} \delta(z - z_0)\delta(\rho - \rho_0), \tag{3.25}$$

can be written as

$$\varphi = \int_0^\pi \frac{\sin^{m-3}\theta}{\left[(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\theta\right]^{\frac{m}{2}-1}} \mathrm{d}\theta. \tag{3.26}$$

This in turn means that the solution to (3.8) is given by

$$\mathcal{G}_s(\rho, z) = -\frac{\Sigma_0 \rho_0^{2s}}{2\pi} \int_0^\pi \frac{\sin^{2s}\theta}{\left[(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\theta\right]^{\frac{1+2s}{2}}} \mathrm{d}\theta. \tag{3.27}$$

Transforming $\mathcal{G}_s$ into the BL coordinates we obtain the solution to (3.4).

3.4 Testing the fundamental solution

To make sure our solutions have the correct normalizations, we can use the fact, that they are the Green functions to the Laplacian, on which we can apply the generalized Stokes theorem. First we do this in the cylindrical coordinates in Euclidean space. We omit the integration across angular degrees of freedom, since they trivially cancel out from both sides of the equation. We shall call the effective volume $\tilde{\mathrm{d}}V$. We obtain

$$\begin{aligned}\int W_s[\mathcal{G}_s(\rho,z)]\tilde{\mathrm{d}}V &= \int \Delta_{W_s}\mathcal{G}_s\tilde{\mathrm{d}}V,\\ &= \int \nabla_{W_s}\cdot\nabla_{W_s}\mathcal{G}_s\tilde{\mathrm{d}}V,\\ &= \oint \nabla_{W_s}\mathcal{G}_s\cdot\vec{n}\tilde{\mathrm{d}}S,\\ &= \int_{S_+}\partial_z\mathcal{G}_s\rho^{m-2}\mathrm{d}\rho - \int_{S_-}\partial_z\mathcal{G}_s\rho^{m-2}\mathrm{d}\rho + \int_{S_\mathrm{I}}\partial_\rho\mathcal{G}_s\rho^{m-2}\mathrm{d}z.\end{aligned} \tag{3.28}$$

For the integration set, we used a cylinder with the upper and lower cap denoted by $S_\pm$, whereas the side is given by S_I.

In the case of the BL coordinates, we integrate in the spacetime (K_s, g_{K_s}). Once again, we omit the integration across trivial degrees of freedom and only keep r and θ. We get

$$\begin{aligned}\int \frac{1}{\Sigma}T_s[\mathcal{G}_s(r,\theta)]\mathrm{d}V &= \int \Delta_{K_s}\mathcal{G}_s(r,\theta)\mathrm{d}V,\\ &= \oint \nabla_{K_s}\mathcal{G}_s\cdot\mathrm{d}\Sigma,\\ &= \oint\left(\frac{1}{g_{rr}}\partial_r\mathcal{G}_s\hat{x}_r + \frac{1}{g_{\theta\theta}}\partial_\theta\mathcal{G}_s\hat{x}_\theta\right)\cdot\mathrm{d}\Sigma,\\ &= \int_0^\pi \partial_r\mathcal{G}_s\frac{\sqrt{-g}}{g_{rr}}\mathrm{d}\theta,\end{aligned} \tag{3.29}$$

where $\hat{x}_\mu$ denotes the unit vector in the given coordinate direction. The chosen integration surface is a sphere.

3.5 Axial uniqueness of the axisymmetric Green function

We shall explicitly show that an axisymmetric solution to the Laplace equation is fully determined by the values on the symmetry axis.

Let us have an axisymmetric solution to the cylindrical Laplacian (3.25)

$$\varphi(\rho,z) \equiv \int_0^\pi \frac{\sin^{m-3}\theta}{\left[(z-z_0)^2+\rho^2+\rho_0^2-2\rho\rho_0\cos\theta\right]^{\frac{m}{2}-1}}\mathrm{d}\theta.$$

We perform the substitution $t=\cos^2(\theta/2)$ which yields

$$\frac{2^{m-2}}{(4\rho\rho_0)^2}\int_0^1 \frac{[t(1-t)]^{\frac{m}{2}-2}}{[Z-t]^{\frac{m}{2}-1}}\mathrm{d}t.$$

Here, Z is defined by

$$Z = \frac{(z-z_0)^2-(\rho+\rho_0)^2}{4\rho\rho_0}.$$

We shall now look at this as an Abelian integral in the complex t-plane

$$\frac{2^{m-2}}{(4\rho\rho_0)^{\frac{m}{2}-1}}\int_{\phi(t)}\frac{[t(1-t)]^{\frac{m}{2}-2}}{[Z-t]^{\frac{m}{2}-1}}\mathrm{d}t,$$

where $\phi(t) = t$ for $t \in (0,1)$. We perform a bilinear transformation into the λ-plane via the holomorphic map

$$\lambda = \frac{A+Bt}{C+Dt}.$$

We would like to rewrite this transformation in terms of different parameters α, σ_1, $\sigma_2 \in \mathbb{R}$, and $\sigma \equiv \sigma_1 + i\sigma_2$, which satisfy

$$\begin{aligned} t(0,0) &\leftrightarrow \lambda(-\alpha), \\ t(1,0) &\leftrightarrow \lambda(\alpha), \\ t(Z,0) &\leftrightarrow \lambda(\sigma), \\ t(\infty) &\leftrightarrow \lambda(-\bar{\sigma}). \end{aligned}$$

Here, $t(z)$ denotes the point z in the t-plane and similarly for the $\lambda(z)$. Let us notice that this choice is not arbitrary, but is motivated by the roots and singularities of the integrand. As a consequence, the result will look fairly simple. These conditions give us a system of four equations

$$\begin{aligned} \frac{A}{C} &= -\alpha, \\ \frac{A+B}{C+D} &= \alpha, \\ \frac{A+ZB}{C+ZD} &= \sigma, \\ \frac{B}{D} &= -\bar{\sigma}, \end{aligned}$$

whose solution can be easily found to be

$$\begin{aligned} A &= \frac{\alpha+\bar{\sigma}}{2}D, \\ B &= -\bar{\sigma}D, \\ C &= -\frac{\alpha+\bar{\sigma}}{2\alpha}D. \end{aligned}$$

At the same time, we obtain an expressions for Z

$$Z = \frac{(\alpha+\sigma_1)^2+\sigma_2^2}{4\alpha\sigma_1}.$$

Put together, the transformation takes on the form

$$\lambda = \frac{\frac{\alpha+\bar{\sigma}}{2}-\bar{\sigma}t}{-\frac{\alpha+\bar{\sigma}}{2\alpha}+t},$$

with the inverse transformation being

$$t = \frac{\alpha+\bar{\sigma}}{2\alpha}\frac{\alpha+\lambda}{\bar{\sigma}+\lambda}.$$

We can find the differential in a straightforward manner

$$\mathrm{d}t = \frac{(\bar\sigma+\alpha)(\bar\sigma-\alpha)}{2\alpha(\bar\sigma+\lambda)^2}\mathrm{d}\lambda.$$

Moving into the λ-plane the integral can be written as

$$\frac{1}{\alpha^{\frac{m}{2}-2}}\left(\frac{\sigma_1}{\rho\rho_0}\right)^{\frac{m}{2}-1}\int_{-\alpha}^{\alpha}\frac{(\alpha^2-\lambda^2)^{\frac{m}{2}-2}}{[(\sigma-\lambda)(\bar\sigma+\lambda)]^{\frac{m}{2}-1}}\mathrm{d}\lambda.$$

Next, we perform another substitution $\lambda = \alpha\cos\theta$ which gives

$$\int_0^{\pi}\left(\frac{\sigma_1\alpha}{\rho\rho_0}\right)^{\frac{m}{2}-1}\frac{\sin^{m-3}\theta\mathrm{d}\theta}{[\sigma_1^2+(\sigma_2+i\alpha\cos\theta)^2]^{\frac{m}{2}-1}}.$$

At last, we shall specify the transformation parameters. Apparently, a nice choice would be $\sigma_1=\rho_0$, $\sigma_2 = z-z_0$, and $\alpha=\rho$, because then the integral becomes

$$\varphi(\rho,z) = \int_0^{\pi}\frac{\sin^{m-3}\theta\mathrm{d}\theta}{[\rho_0^2+(z-z_0+i\rho\cos\theta)^2]^{\frac{m}{2}-1}}.$$

As a consequence, we have

$$\varphi(\rho,z) = \frac{\int_0^{\pi}\varphi(0,z+i\rho\cos\theta)\sin^{m-3}\mathrm{d}\theta}{\int_0^{\pi}\sin^{m-3}\theta\mathrm{d}\theta}. \tag{3.30}$$

This proves, that the Green function is determined by the values on the symmetry axis. Here

$$\int_0^{\pi}\sin^{m-3}\theta\mathrm{d}\theta = \pi^{1/2}\Gamma\left(\frac{m-2}{2}\right)\Gamma^{-1}\left(\frac{m-1}{2}\right),$$

and so

$$\varphi(\rho,z) = \pi^{-1/2}\Gamma^{-1}\left(\frac{m-2}{2}\right)\Gamma\left(\frac{m-1}{2}\right)\int_0^{\pi}\varphi(0,z+i\rho\cos\theta)\sin^{m-3}\mathrm{d}\theta.$$

3.6 Debye potential from ϕ_0

As we have already found the solution (3.27) to (3.8), using the substitution (3.3) we can obtain the Green function to the Teukolsky equation (3.1)

$$G_s = \frac{\rho_0^{2s}\sin^s\theta}{2\pi\sin^s\theta_0}\int_0^{\pi}\frac{\sin^{2s}\theta}{\left[(z-z_0)^2+\rho^2+\rho_0^2-2\rho\rho_0\cos\theta\right]^{\frac{1+2s}{2}}}\mathrm{d}\theta. \tag{3.31}$$

The Debye potential can be expressed from (2.78) as the twice r-integrated scalar ϕ_0, namely

$$\bar\psi = -I_{rr}(\phi_0), \tag{3.32}$$

while the zeroth scalar can be expressed via its Green function. In (3.2) we simply set $s=1$

$$\phi_0 = 4\pi\int G(r,\theta,r_0,\theta_0)\Sigma(r_0,\theta_0)\mathcal{J}_0(r_0,\theta_0)\sin\theta_0\mathrm{d}\theta_0\mathrm{d}r_0. \tag{3.33}$$

As a consequence, integrating this expression twice we obtain the Debye potential

$$\bar{\psi} = -4\pi \int I_{rr}(G)\Sigma \mathcal{J}_0 \sin\theta_0 \mathrm{d}\theta_0 \mathrm{d}r_0. \tag{3.34}$$

At this moment, we can identify $-I_{rr}(G)$ as the effective Green function of the Debye potential

$$\psi_G \equiv -I_{rr}(G) = \frac{\sin\theta}{\Sigma_0 \sin\theta_0} I_{rr}(\mathcal{G}). \tag{3.35}$$

Should we have the solution of T_1 – meaning that the application of T_1 gives 0 everywhere up to a set of measure 0 with respect to $\mathrm{d}r\mathrm{d}\theta$ – we can use the axial uniqueness of axisymmetric solutions. However, the derivation does not hold for $s = -1$ since in the cylindrical coordinates this corresponds to dimension $m = 1$. We can overcome this using (3.10) which holds also for the Teukolsky equation. And thus the expression

$$\frac{1}{\Delta}\psi_G(r,\theta) = \frac{1}{\Delta\Sigma_0}\frac{\sin\theta}{\sin\theta_0} I_{rr}\left(\mathcal{G}(r,\theta)\right), \tag{3.36}$$

solves the Teukolsky equation for $s = 1$. This means that

$$\Xi = \frac{1}{\Delta\Sigma_0}\frac{1}{\sin\theta_0} I_{rr}\left(\mathcal{G}(r,\theta)\right), \tag{3.37}$$

solves T_1 in the BL coordinates and W_1 in the Weyl coordinates. This object has been referred to as the superpotential. It clearly holds that upon knowing the superpotential, we obtain the effective Green function simply by

$$\psi_G = \Delta \sin\theta \Xi. \tag{3.38}$$

We note that ψ_G is not a Green function because it does not yield δ when inserted into the Teukolsky equation. Nevertheless, we can treat it as an effective Green function in the sense, that the Debye potential obtained by (3.34) gives us the desired ϕ_0 scalar.

3.7 Superpotential from values on the symmetry axis

We know the expression for $\mathcal{G}$ from (3.27) by setting $s = 1$

$$\mathcal{G} = -\frac{\Sigma_0\rho_0^2}{2\pi}\int_0^\pi \frac{\sin^2\theta}{\left[(z-z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\theta\right]^{\frac{3}{2}}}\mathrm{d}\theta.$$

Now, we need to write it in the BL coordinates and integrate it with respect to r twice. This is no easy task. The problem would become much simpler if we could integrate it on the symmetry axis in the Weyl coordinates and then spread the solution to the rest of the space. We shall show that this step is indeed legitimate.

Since Ξ solves T_1, it can be expressed by the values on the symmetry axis

$$\Xi_W(\rho, z) = \mathcal{C}\int_0^\pi \Xi_W(0, z + i\rho\cos\alpha)\sin^2\alpha \mathrm{d}\alpha.$$

Without any problems we can write this in the BL coordinates as well

$$\Xi_{BL}(r,\theta) = \mathcal{C}\int_0^\pi \Xi_W\left(0,(r-M)\cos\theta + i\sqrt{\Delta(r)}\sin\theta\cos\alpha\right)\sin^{2s}\alpha\mathrm{d}\alpha.$$

Therefore, we obtain the following equality on the symmetry axis

$$\Xi_{BL}(r,0) = \mathcal{C}\int_0^\pi \Xi_W(0,r-M)\sin^{2s}\alpha\mathrm{d}\alpha = \Xi_W(0,r-M) = \Xi_W(0,z).$$

We can also see that the relation

$$\Xi_{BL}(r,\theta) = \frac{1}{\Delta(r)}I_{r'r'}\left(\mathcal{G}_{BL}(r',\theta)\right)|_{r'=r},$$

has to hold for $\forall\theta$ including $\theta = 0$, and so

$$\Xi_{BL}(r,0) = \frac{1}{\Delta(r)}I_{r'r'}\left(\mathcal{G}_{BL}(r',0)\right)|_{r'=r}.$$

Using the above equalities combined with a linear transformation of the integration variable $r' \to r'' = r' - M$, we get the desired result

$$\begin{aligned}\Xi_W(0,z) &= \frac{1}{\Delta(z+M)}I_{r'r'}\left(\mathcal{G}_{BL}(r',0)\right)|_{r'=z+M},\\ &= \frac{1}{\Delta_W(z)}I_{r'r'}\left(\mathcal{G}_W(0,r'-M)\right)|_{r'=z+M},\\ &= \frac{1}{\Delta_W(z)}I_{r''r''}\left(\mathcal{G}_W(0,r'')\right)|_{r''=z},\\ &= \frac{1}{\Delta_W(z)}I_{zz}\left(\mathcal{G}_W(0,z)\right).\end{aligned}$$

Having shown our plans are justified, we now evaluate $\mathcal{G}$ on the symmetry axis

$$\mathcal{G}(0,z) = -\frac{\Sigma_0\rho_0^2}{4}\frac{1}{\left[(z-z_0)^2+\rho_0^2\right]^{\frac{3}{2}}}.$$

Next, we integrate twice ignoring the integration constants

$$I_{zz}\left(\mathcal{G}(0,z)\right) = -\frac{\Sigma_0}{4}\sqrt{(z-z_0)^2+\rho_0^2},$$

which means

$$\Xi(0,z) = -\frac{1}{4\sin\theta_0}\frac{\sqrt{(z-z_0)^2+\rho_0^2}}{z^2-M^2+a^2}.$$

The last step is to spread the superpotential away from the symmetry axis using (3.30)

$$\Xi(\rho,z) = \frac{1}{\mathcal{N}}\int_0^\pi \frac{\sqrt{(z-z_0+i\rho\cos\theta)^2+\rho_0^2}}{(z+i\rho\cos\theta)^2-M^2+a^2}\sin^2\theta\mathrm{d}\theta, \tag{3.39}$$

where

$$\mathcal{N} = -2\pi\sin\theta_0.$$

3.8 Debye potential of a ring source

In the Kerr spacetime the Teukolsky source for $s = 1$ (2.33) simplifies thanks to the special properties of the chosen tetrad

$$\mathcal{J}_0 = (m \cdot \partial - 2\tau) l \cdot J - (l \cdot \partial - 2\rho - \rho^*) m \cdot J. \tag{3.40}$$

Given our choice of the four-current (2.71), we define the Teukolsky source for an axisymmetric time-independent current whose only component is J^ϕ

$$\mathcal{J}^{(\phi)} = (m^\theta \partial_\theta - 2\tau) l_\phi J^\phi - (l^r \partial_r - 2\rho - \rho^*) m_\phi J^\phi. \tag{3.41}$$

Similarly, we can define the Teukolsky source for an axially symmetric time-independent charge $\mathcal{J}^t$ by the exchange $\phi \to t$.

We shall need a few identities concerning the Dirac delta-function. First, whenever we multiply a function by the Dirac delta, we immediately evaluate the function

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) \equiv f_0 \delta(x - x_0). \tag{3.42}$$

Secondly, using the Leibniz rule we can find the identity for a derivative of the Dirac delta

$$f(x)\delta'(x - x_0) = (f(x)\delta(x - x_0))' - f'(x)\delta(x - x_0), \tag{3.43}$$

$$= f_0 \delta'(x - x_0) - f_0' \delta(x - x_0). \tag{3.44}$$

Combining these two properties, we obtain our final identity for arbitrary functions f, g, and h

$$(f\partial + g)h\delta = (fh)_0 \partial\delta + (gh - h\partial f)_0 \delta. \tag{3.45}$$

Now, we choose an infinitely thin ring as our axisymmetric current

$$J^\phi \equiv j\delta(r - r_J)\delta(\theta - \theta_J). \tag{3.46}$$

Using the identity (3.45) and the form of the Teukolsky source (3.41) we can find the following equality

$$\sin\theta \Sigma \mathcal{J}^{(\phi)} = \frac{\sin^2\theta_J \Sigma_J}{\sqrt{2}(r_J + ia\cos\theta_J)} \Big(-a\sin\theta_J \partial_\theta + i(r_J^2 + a^2)\partial_r + ir_J \Big) J^\phi. \tag{3.47}$$

This is exactly the effective source for the Debye potential (3.34).

Similarly, we repeat the steps with the axisymmetric charge. The only non-zero component of the four-current for a charged infinitely thin ring will be given by

$$J^t = q\delta(r - r_J)\delta(\theta - \theta_J). \tag{3.48}$$

The needed expression can be written in the following manner

$$\sin\theta \Sigma \mathcal{J}^{(t)} = \frac{\sin\theta_J \Sigma_J}{\sqrt{2}(r_J + ia\cos\theta_J)} \left(\partial_\theta - ia\sin\theta_J \partial_r - \cot\theta_J \right) J^t. \tag{3.49}$$

In view of earlier definitions, before we move further, we need to make the exchange $r \to r_0$. Then, our effective sources have the structure

$$\left(N_{\theta_J} \partial_{\theta_0} + N_{r_J} \partial_{r_0} + N \right) \delta(r_0 - r_J) \delta(\cos\theta_0 - \cos\theta_J).$$

We can thus explicitly compute the convolution for the Debye potential (3.34) which simplifies to plain differentiation

$$\begin{aligned}\psi(r,\theta,r_J,\theta_J) &= \int_0^\infty \int_0^\pi \psi_G(r,\theta,r_0,\theta_0) N_{\theta_J}\partial_{\theta_0}\delta(\theta_0-\theta_J)\delta(r_0-r_J)\mathrm{d}\theta_0\mathrm{d}r_0 + ... \\ &= -\int_0^\infty \int_0^\pi N_{\theta_J}\partial_{\theta_0}\psi_G(r,\theta,r_0,\theta_0)\delta(\theta_0-\theta_J)\delta(r_0-r_J)\mathrm{d}\theta_0\mathrm{d}r_0 + ... \\ &= \left(-N_{\theta_J}\partial_{\theta_J} - N_{r_J}\partial_{r_J} + N\right)\psi_G(r,\theta,r_J,\theta_J).\end{aligned} \tag{3.50}$$

As a result, we obtain the Debye potential for a current loop ψ_J

$$\psi_J \equiv \frac{j\sin^2\theta_J \Sigma_J}{\sqrt{2}(r_J+ia\cos\theta_J)}\Big(a\sin\theta_J\partial_{\theta_J} - i(r_J^2+a^2)\partial_{r_J} + ir_J\Big)\psi_G, \tag{3.51}$$

and for a charged loop ψ_C

$$\psi_C \equiv \frac{q\sin\theta_J \Sigma_J}{\sqrt{2}(r_J+ia\cos\theta_J)}\left(-\partial_{\theta_J} + ia\sin\theta_J\partial_{r_J} - \cot\theta_J\right)\psi_G. \tag{3.52}$$

There are two obvious ways to obtain the field of a current disc. One is to choose the source as $J^\phi = j\delta(\theta-\theta_J)\chi_{[r_a,r_b]}(r)$, and compute the Debye potential. Here, $\chi_{[r_a,r_b]}$ is the characteristic function of the interval $[r_a,r_b]$. However, the absence of delta function in r means that the result cannot be rewritten as a simple differentiation of the effective Green function. It will be given by its integration which is fairly problematic.

The second way works at the level of the field rather than the potential. Since we are working with the first order perturbations of the electromagnetic field on a fixed gravitational background, we can superpose the electromagnetic fields of multiple sources. We can thus approximate the field of a disc $B_d(r_a,r_b)$ located between r_a and r_b by a series of current loops at $r_j \in [r_a,r_b]$ with the field $B_l(r_j)$ as

$$B_d(r_a,r_b) \approx \sum_{j=1}^{N} B_l(r_j). \tag{3.53}$$

If we wanted an exact result, we would have to perform a limit $N\to\infty$. Since the disc appears as a loop at radial infinity, the infinite sum of loops has to be regularized by their number. Otherwise, the magnetic field would be divergent. Assuming the loops are identical, we obtain

$$\begin{aligned}B_d(r_a,r_b) &= \lim_{\substack{N\to\infty \\ \Delta r\to 0 \\ N\Delta r = r_b-r_a}} \frac{1}{N}\sum_{j=1}^{N} B_l(r_a+j\Delta r), \\ &= \lim_{\substack{N\to\infty \\ \Delta r\to 0 \\ N\Delta r = r_b-r_a}} \frac{\sum_{j=1}^N B_l(r_a+j\Delta r)\Delta r}{N\Delta r}, \\ &= \lim_{\substack{N\to\infty \\ \Delta r\to 0 \\ N\Delta r = r_b-r_a}} \frac{\sum_{j=1}^N B_l(r_a+j\Delta r)\Delta r}{\sum_{j=1}^N \Delta r}, \\ &= \frac{\int_{r_a}^{r_b} B_l(r_J)\mathrm{d}r_J}{\int_{r_a}^{r_b}\mathrm{d}r_J}.\end{aligned}$$

Thus, the field of a current disc as a limit of continuous distribution of identical loops can be written as

$$\frac{1}{r_b - r_a} \int_{r_a}^{r_b} B_l(r_J) \mathrm{d}r_J. \tag{3.54}$$

At the end, the second way still relies on integration. However, this time the integration gives the final result as opposed to giving an object one has to further differentiate and work with.

4. Superpotential for Kerr background

In this chapter, we shall give explicit expressions for the Debye superpotential on Kerr background in terms of elliptic integrals. We will also analyze the structure of the discontinuities that arise in the superpotential. At the end, we discuss how these discontinuities affect the magnetic field of a current loop and visualize the results.

4.1 Explicit expressions for superpotentials

First, we define a few auxiliary functions to shorten the desired expressions

$$g_\beta \equiv \rho - i(z+\beta) \qquad h_\beta \equiv i\bar{g}_\beta, \tag{4.1}$$

$$g_{\beta(0)} \equiv \rho_0 - i(z_0+\beta) \qquad h_{\beta(0)} \equiv i\bar{g}_{\beta(0)}, \tag{4.2}$$

$$g(\rho_0) \equiv \rho - \rho_0 + i(z-z_0), \qquad h(\rho_0) \equiv i\bar{g}(\rho_0). \tag{4.3}$$

The reason we use both g and h is to absorb imaginary units or signs in the arguments. We shall also need the following definitions

$$d(\rho_0) \equiv \sqrt{(\rho-\rho_0)^2 + (z-z_0)^2}, \tag{4.4}$$

$$m \equiv \frac{4\rho\rho_0}{(\rho+\rho_0)^2 + (z-z_0)^2}, \tag{4.5}$$

$$\mu \equiv \frac{1}{1-m}, \tag{4.6}$$

$$\beta \equiv \sqrt{M^2 - a^2}. \tag{4.7}$$

For elliptic integrals, we use these conventions

$$E(k) \equiv \int_0^{\pi/2} \sqrt{1 - k\sin^2\theta}\,\mathrm{d}\theta, \tag{4.8}$$

$$K(k) \equiv \int_0^{\pi/2} \frac{1}{\sqrt{1 - k\sin^2\theta}}\mathrm{d}\theta, \tag{4.9}$$

$$\Pi(t|k) \equiv \int_0^{\pi/2} \frac{1}{(1-t\sin^2\theta)\sqrt{1 - k\sin^2\theta}}\mathrm{d}\theta. \tag{4.10}$$

We will need several useful limits. First, the elliptic integral $\Pi(t|k)$ is discontinuous when crossing the real line from above in t for $t > 1$ [22]. However, it is continuous from below [23]

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \Pi(t+i\varepsilon|k) = \Pi(t|k) + \frac{\pi}{\sqrt{1-t}\sqrt{1-\frac{k}{t}}}, \\ &\lim_{\varepsilon\to 0^-} \Pi(t+i\varepsilon|k) = \Pi(t|k). \end{aligned} \tag{4.11}$$

Similar thing happens in the case of the argument k for $k > 1$ [24]. It is discontinuous from above but continuous from below

$$\lim_{\varepsilon\to 0^+} \Pi(t|k+i\varepsilon) = \frac{1}{\sqrt{k}}\Pi\left(\frac{t}{k}\,\middle|\,\frac{1}{k}\right) + i\frac{k}{k-t}\Pi\left(\frac{t(1-k)}{t-k}\,\middle|\,1-k\right). \tag{4.12}$$

In the case of the elliptic integral $K(k)$, its value at $k > 1$ exists only as a limit from the complex plane onto the real axis. We get a different sign of the imaginary part depending on whether we go from below or from above [25] (we notify the reader that this source uses different convention for the definitions of the elliptic integrals)

$$\lim_{\varepsilon\to 0^-} K(k+i\varepsilon) = \frac{1}{\sqrt{k}}\left[K\left(\frac{1}{k}\right) - iK\left(1-\frac{1}{k}\right)\right]. \tag{4.13}$$

The same applies for the elliptic integral $E(k)$ [25]

$$\begin{aligned}\lim_{\varepsilon\to 0^-} E(k+i\varepsilon) = \sqrt{k}\Bigg[iE\left(1-\frac{1}{k}\right) + E\left(\frac{1}{k}\right) \\ - \left(1-\frac{1}{k}\right)K\left(\frac{1}{k}\right) - \frac{i}{k}K\left(1-\frac{1}{k}\right)\Bigg].\end{aligned} \tag{4.14}$$

The superpotential (3.39) for a non-extremal Kerr black hole Ξ_K was obtained in terms of elliptic integrals by Kofroň and Kotlařík [12]. They found it can be written in the form

$$\mathcal{N}\Xi_{(K)} = f(\rho_0) + \bar{f}(-\rho_0), \tag{4.15}$$

where

$$\begin{aligned}f(\rho_0) = \frac{1}{\rho^2 d(\rho_0)}\Bigg[& - id^2(\rho_0)E(\mu) + 2\rho_0\left(4z - h(\rho_0)\right)K(\mu) \\ & - 4(z+z_0)\rho_0\Pi\left(\frac{h(-\rho_0)}{h(\rho_0)}\,\middle|\,\mu\right) \\ & - 2\rho_0\frac{\rho^2+(z+\beta)^2}{\beta}\Pi\left(\frac{h_{\beta(0)}\bar{h}(-\rho_0)}{\bar{h}_{\beta(0)}\bar{h}(\rho_0)}\,\middle|\,\mu\right) + (\beta\to-\beta)\Bigg].\end{aligned} \tag{4.16}$$

Here, the addition given by $(\beta \to -\beta)$ applies only to the terms in the same line. However, the argument μ is greater than 1 as can be seen from (4.5) and (4.6). It would be preferred to use arguments that are less than 1 – for example to have a well defined sign of the imaginary part of the integrals K and E. This can be achieved by considering the limits (4.12), (4.13), and (4.14). We set $\varepsilon = 0$ and use them as a transformation rule. We obtain the following expression completely in terms of the argument m

$$\begin{aligned}\mathcal{N}\Xi_{(K)} = & \frac{d(-\rho_0)}{\rho^2}E(m) - \frac{z^2+\rho^2+(z_0+i\rho_0)^2 - 2z(z_0-3i\rho_0)}{\rho^2 d(-\rho_0)}K(m) \\ & - 2i(z+z_0)\frac{\bar{g}(\rho_0)}{\rho^2 d(-\rho_0)}\Pi\left(\frac{2\rho}{g(-\rho_0)}\,\middle|\,m\right) \\ & + i\frac{g(\rho_0)}{\rho^2 d(-\rho_0)}\left[\frac{g_\beta g_{\beta(0)}}{\beta}\Pi\left(\frac{\bar{g}_{\beta(0)}}{\bar{g}_\beta}\frac{2\rho}{\bar{g}(-\rho_0)}\,\middle|\,m\right) + (\beta\to-\beta)\right].\end{aligned} \tag{4.17}$$

Due to (4.11), the superpotential has discontinuities that divide the region into three parts – inner i, northern n, and southern s. The boundaries are composed of a ray $\gamma_{(K)e}$ and two arcs $\gamma_{(K)n}$, $\gamma_{(K)s}$. For simplicity, we assume $z_0 \geq 0$ throughout. The ray can be written as

$$\gamma_{(K)e} : z = z_0 \quad \& \quad \rho \geq \rho_0. \tag{4.18}$$

The arcs are defined by

$$\begin{aligned}\gamma_{(K)n} : r_{(K)n} &= 0 \quad \& \quad \min\{z_0, \beta\} < z < \max\{z_0, \beta\}, \\ \gamma_{(K)s} : r_{(K)s} &= 0 \quad \& \quad -\beta < z < z_0,\end{aligned} \tag{4.19}$$

where

$$\begin{aligned} r_{(K)n} &\equiv \rho^2 + (z - z_{(K)n})^2 - R^2_{(K)n}, & r_{(K)s} &\equiv \rho^2 + (z - z_{(K)s})^2 - R^2_{(K)s}, \\ z_{(K)n} &\equiv \frac{1}{2}\frac{\rho_0^2 + z_0^2 - \beta^2}{z_0 - \beta}, & z_{(K)s} &\equiv \frac{1}{2}\frac{\rho_0^2 + z_0^2 - \beta^2}{z_0 + \beta}, \\ R_{(K)n} &\equiv \frac{1}{2}\frac{\rho_0^2 + (z_0 - \beta)^2}{z_0 - \beta}, & R_{(K)s} &\equiv \frac{1}{2}\frac{\rho_0^2 + (z_0 + \beta)^2}{z_0 + \beta}. \end{aligned} \tag{4.20}$$

If $z_0 = \beta$, the arc $\gamma_{(K)n}$ turns into a line segment and combines with $\gamma_{(K)e}$ into a single ray. Next, we define the region functions

$$\Theta_{(K)n} \equiv \Theta(z - z_0) - \text{sign}(\beta - z_0)\Theta(-r_{(K)n})\Theta\left((z - z_0)\text{sign}(\beta - z_0)\right), \tag{4.21}$$

$$\Theta_{(K)s} \equiv \Theta(-z + z_0) - \Theta(-r_{(K)s})\Theta(-z + z_0), \tag{4.22}$$

$$\Theta_{(K)i} \equiv \begin{cases} \Theta(-r_{(K)s})\left(1 - \Theta(-r_{(K)n})\right), & \text{Cond. A}, \\ \Theta(-r_{(K)s}) + \Theta(-r_{(K)n}) - \Theta(-r_{(K)s})\Theta(-r_{(K)n}), & \text{Cond. B}, \\ \Theta(-r_{(K)s})\Theta(-r_{(K)n}), & \text{Cond. C}, \end{cases} \tag{4.23}$$

where we have used

$$\text{Cond. A} \equiv z_0 > \beta, \tag{4.24}$$

$$\text{Cond. B} \equiv z_0 < \beta \;\;\&\;\; \beta^2 - \rho_0^2 - z_0^2 > 0, \tag{4.25}$$

$$\text{Cond. C} \equiv z_0 < \beta \;\;\&\;\; \beta^2 - \rho_0^2 - z_0^2 \leq 0. \tag{4.26}$$

In the following, $\Xi_\gamma \equiv \Xi_{\gamma^-} - \Xi_{\gamma^+}$ shall denote the jump across the curve γ. Using (4.11), we obtain

$$\mathcal{N}\Xi_{(K)e} = 2\pi\frac{z + z_0}{\rho^2}, \tag{4.27}$$

$$\mathcal{N}\Xi_{(K)n} = -\pi\sqrt{\rho_0^2 + (z_0 - \beta)^2}\frac{\sqrt{\rho^2 + (z - \beta)^2}}{\beta\rho^2}, \tag{4.28}$$

$$\mathcal{N}\Xi_{(K)s} = \pi\sqrt{\rho_0^2 + (z_0 + \beta)^2}\frac{\sqrt{\rho^2 + (z + \beta)^2}}{\beta\rho^2}. \tag{4.29}$$

All these can be added to $\Xi_{(K)}$ since they satisfy $W_1[\Xi] = 0$. They represent sources located beneath the horizon as can be shown by computing the charge. However, we point out that $\Xi_{(K)n}$ and $\Xi_{(K)s}$ diverge for $\beta \to 0$ which corresponds to the extreme limit.

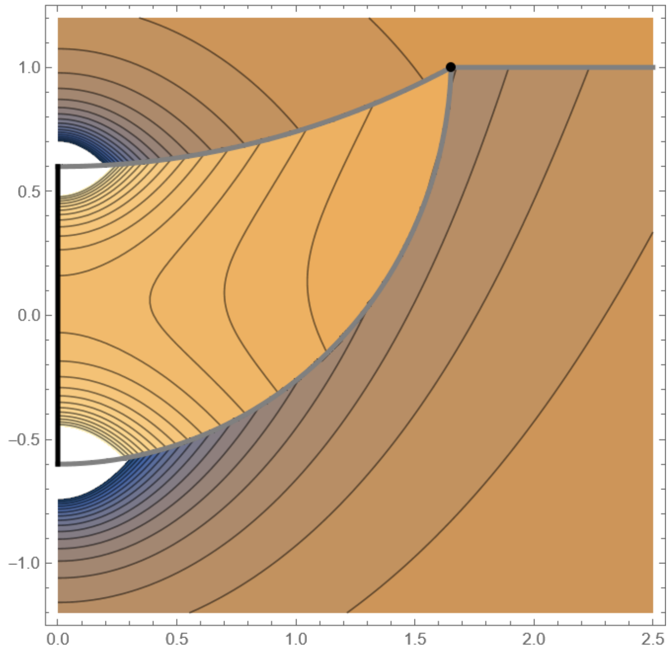

Figure 4.1: Contour-plot of the superpotential $\Xi_{(K)}$ in the Weyl coordinates for the values $M = 1$, $a = 0.8$, $r_0 = 3$, $\theta_0 = \pi/3$. The thick black line at $\rho = 0$ is the outer horizon, the gray lines are the discontinuities, and the dot at the intersection of the discontinuities is the position of the source.

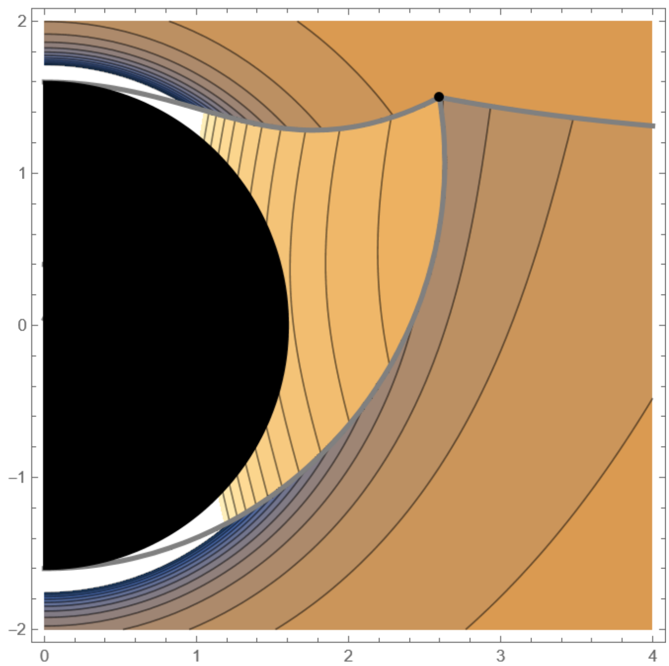

Figure 4.2: Contourplot of the superpotential $\Xi_{(K)}$ in the BL coordinates for the values $M = 1$, $a = 0.8$, $r_0 = 3$, $\theta_0 = \pi/3$. The outer horizon is depicted by a black half-disc, the gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

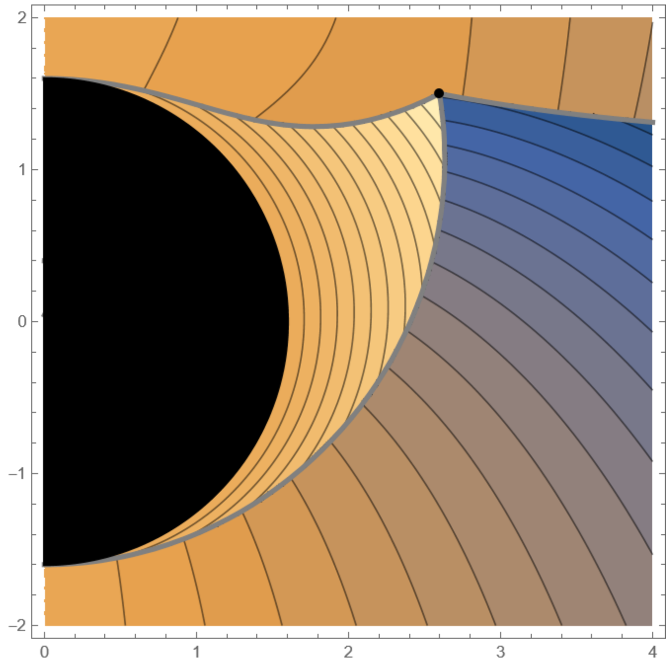

Figure 4.3: Contourplot of the Debye green function $\psi_{G(K)}$ in the BL coordinates for the values $M = 1$, $a = 0.8$, $r_0 = 3$, $\theta_0 = \pi/3$. The outer horizon is depicted by a black half-disc, the gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

We show the contour-plots of the superpotential for the non-extremal Kerr black hole background in both Weyl (Fig. 4.1) and BL coordinates (Fig. 4.2). We also visualize the effective Green function (Fig. 4.3). We shall point out one important fact. The source at (r_0, θ_0) looks like a ring, however, it is not the physical current-loop whose field we were originally looking for. When we look at how we derived the superpotential, we find that the (r_0, θ_0) originates from searching for the fundamental solution of the Teukolsky equation with Dirac distributions on the right-hand-side. The physical source, however, is given by (2.33). This is a more complicated object created from the four-current.

We also point out that the discontinuities in the plots appear as lines, however, we only plot the cross-section containing the symmetry axis. Every plot can thus be imagined to be rotated around the vertical axis at $\rho = 0$ and the discontinuities become surfaces.

We obtained the superpotential on an extreme Kerr background in the same manner as Kofroň and Kotlařík for the below-extreme case

$$
\begin{aligned}
\mathcal{N}\Xi_{(E)} = & 2\frac{d(-\rho_0)}{\rho^2}\left(E(m)-K(m)\right) \\
& + \frac{2\rho_0}{\rho^2 d(-\rho_0)}\left(2(\rho+iz)+\rho_0-iz_0-\frac{\rho^2+z^2}{\rho_0+iz_0}+2\frac{\rho^2+z^2}{\rho_0-iz_0}\right)K(m) \\
& - 2i(z+z_0)\frac{\bar{g}(\rho_0)}{\rho^2 d(-\rho_0)}\Pi\left(\frac{2\rho}{g(-\rho_0)}\,\middle|\, m\right) \\
& - 2i\frac{\rho_0^2 z+z_0(\rho^2+zz_0+z^2)}{\rho^2 d(-\rho_0)}\left[2\frac{\bar{g}(\rho_0)}{(\rho-iz)(\rho_0-iz_0)}\Pi\left(\frac{\rho_0-iz_0}{\rho-iz}\frac{2\rho}{g(-\rho_0)}\,\middle|\, m\right)\right. \\
& \left. + \frac{g(\rho_0)}{(\rho+iz)(\rho_0+iz_0)}\Pi\left(\frac{\rho_0+iz_0}{\rho+iz}\frac{2\rho}{\bar{g}(-\rho_0)}\,\middle|\, m\right)\right].
\end{aligned}
\tag{4.30}
$$

It can be numerically checked that $\Xi_{(K)} \to \Xi_{(E)}$ for $\beta \to 0$.

This time, we only have a northern and a southern region separated by the ray

$$\gamma_{(E)e} : z = z_0 \quad \& \quad \rho \geq \rho_0, \tag{4.31}$$

and the arc

$$\gamma_{(E)w} : r_{(E)} = 0 \quad \& \quad z \in (0, z_0), \tag{4.32}$$

where we defined

$$
\begin{aligned}
r_{(E)} &\equiv \rho^2 + (z - z_{(E)})^2 - R_{(E)}^2, \\
z_{(E)} &\equiv \frac{\rho_0^2 + z_0^2}{2z_0}, \\
R_{(E)} &= z_{(E)}.
\end{aligned}
\tag{4.33}
$$

For $z_0 = 0$, $\gamma_{(E)e}$ and $\gamma_{(E)w}$ combine into a single ray. The region functions can be found to be

$$\Theta_{(E)n} \equiv \begin{cases} \Theta(z-z_0) + \Theta(-r_{(E)}) - \Theta(z-z_0)\Theta(-r_{(E)}), & z_0 > 0, \\ \Theta(z-z_0), & z_0 = 0, \end{cases} \tag{4.34}$$

$$\Theta_{(E)s} \equiv 1 - \Theta_{(E)n}. \tag{4.35}$$

We only have one jump satisfying $W_1[\Xi] = 0$, and that is

$$\mathcal{N}\Xi_{(E)e} = 2\pi\frac{z+z_0}{\rho^2}. \tag{4.36}$$

This was expected as the other two below-extreme superpotential jumps (4.28) and (4.29) diverged in the extreme limit.

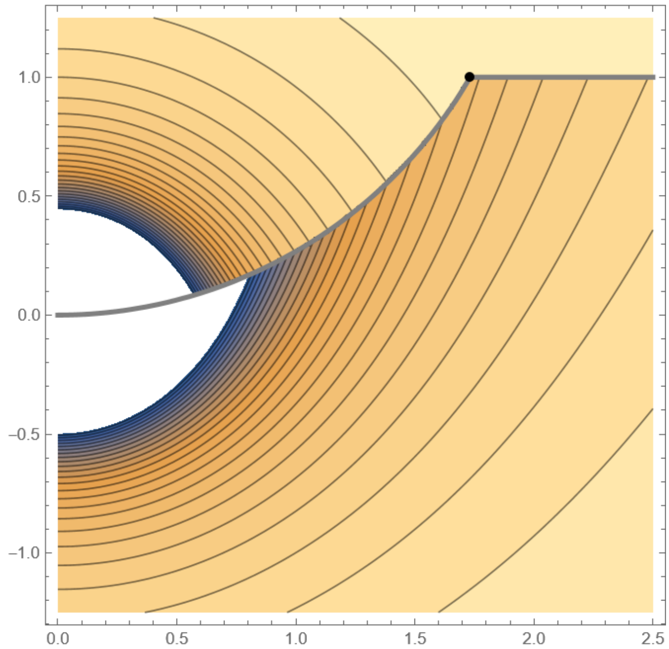

Figure 4.4: Contourplot of the superpotential $\Xi_{(E)}$ in the Weyl coordinates for the values $M = a = 1$, $r_0 = 3$, $\theta_0 = \pi/3$. The gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

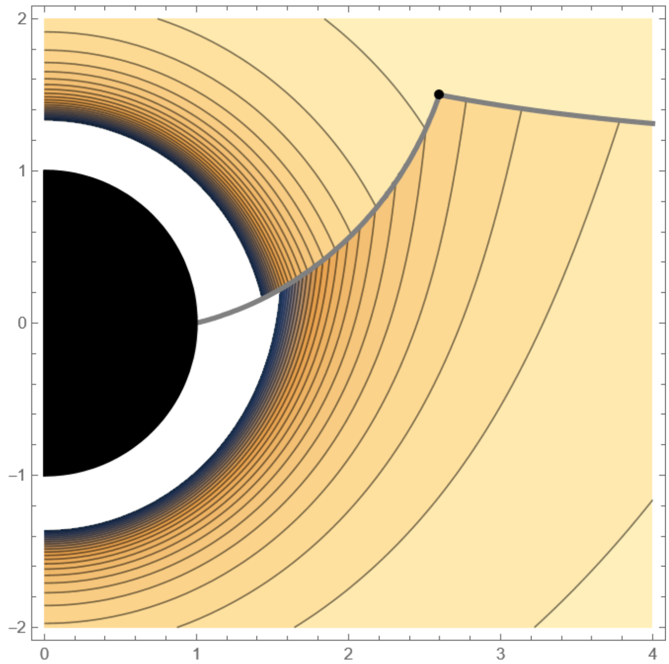

Figure 4.5: Contourplot of the superpotential $\Xi_{(E)}$ in the BL coordinates for the values $M = a = 1$, $r_0 = 3$, $\theta_0 = \pi/3$. The outer horizon is depicted by a black half-disc, the gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

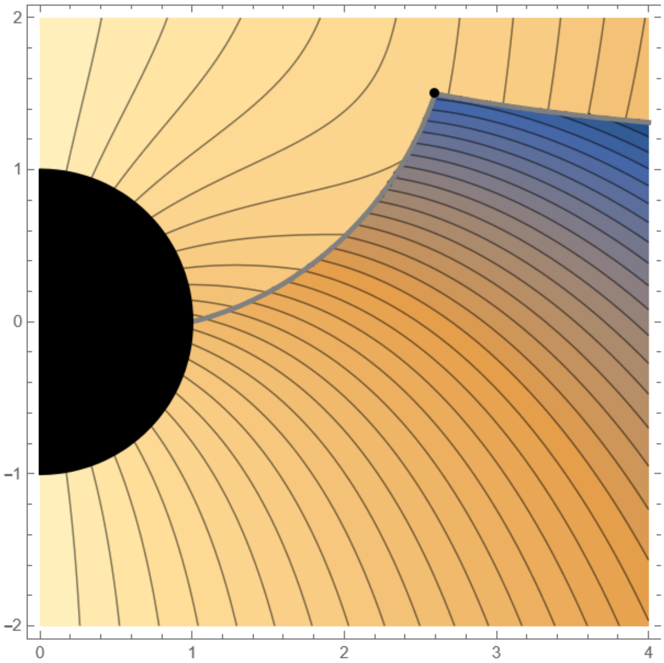

Figure 4.6: Contourplot of the Debye green function $\psi_{G(E)}$ in the BL coordinates for the values $M = a = 1$, $r_0 = 3$, $\theta_0 = \pi/3$. The outer horizon is depicted by a black half-disc, the gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

We show the contour-plots of the superpotential for extremal Kerr background in both Weyl (Fig. 4.4) and BL coordinates (Fig. 4.5). We also visualize the effective Green function (Fig. 4.6).

The superpotential in the case of the Kerr naked singularity can be easily obtained from the below-extreme case by

$$\Xi_{(N)} = \Xi_{(K)}|_{\beta\to i\beta}. \tag{4.37}$$

The discontinuity no longer divides the region. Its position can be built up by the ray

$$\gamma_{(N)e} : z = z_0 \quad \& \quad \rho \geq \min\{\rho_0, \beta\}, \tag{4.38}$$

and the arc given by

$$\gamma_{(N)w} : r_{(N)} = 0 \quad \& \quad z \in (0, z_0), \tag{4.39}$$

where we used the definitions

$$\begin{aligned} r_{(N)} &\equiv \rho^2 + (z - z_{(N)})^2 - R_{(N)}^2, \\ z_{(N)} &\equiv \frac{\rho_0^2 + z_0^2 - \beta^2}{2z_0}, \\ R_{(N)} &= \sqrt{z_{(N)}^2 + \beta^2}. \end{aligned} \tag{4.40}$$

For $z_0 = 0$, $\gamma_{(N)e}$ and $\gamma_{(N)w}$ combine into a single ray.

We show the contour-plots of the superpotential for the background of a Kerr naked singularity in both Weyl (Fig. 4.7) and BL coordinates (Fig. 4.8). We also visualize the effective Green function (Fig. 4.9).

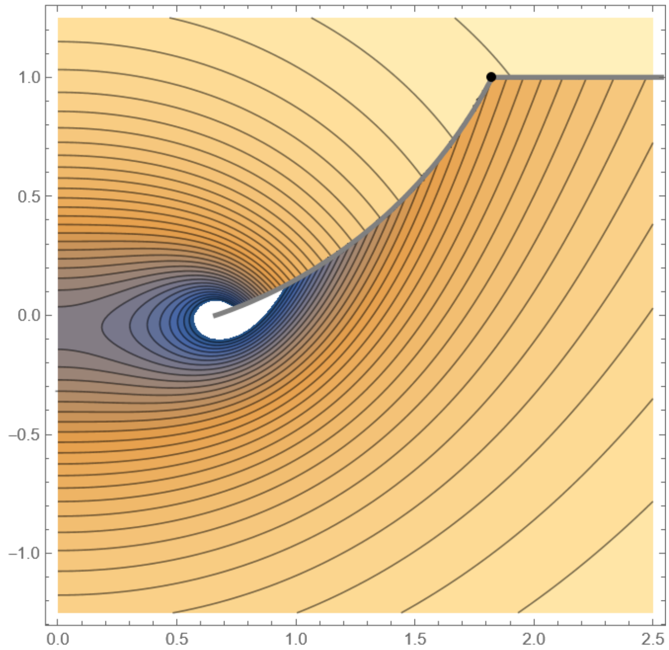

Figure 4.7: Contourplot of the superpotential $\Xi_{(N)}$ in the Weyl coordinates for the values $M = 1$, $a = 1.2$, $r_0 = 3$, $\theta_0 = \pi/3$. The gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

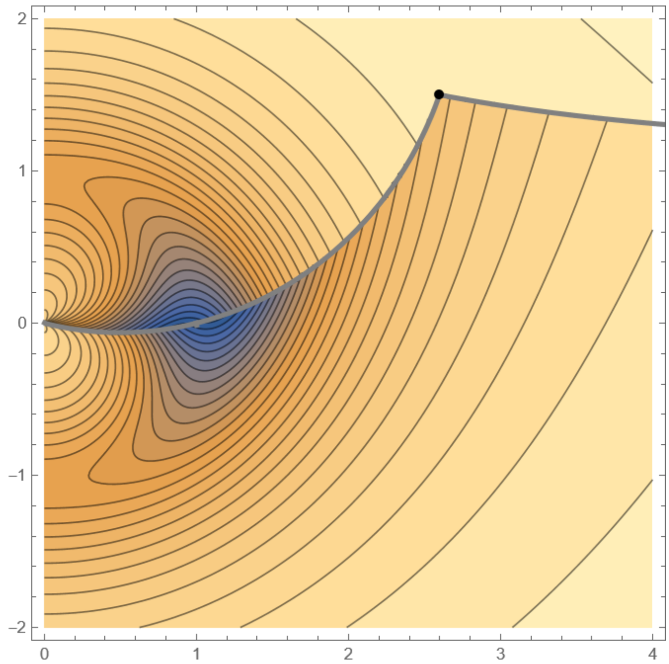

Figure 4.8: Contourplot of the superpotential $\Xi_{(N)}$ in the BL coordinates for the values $M = 1$, $a = 1.2$, $r_0 = 3$, $\theta_0 = \pi/3$. The gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

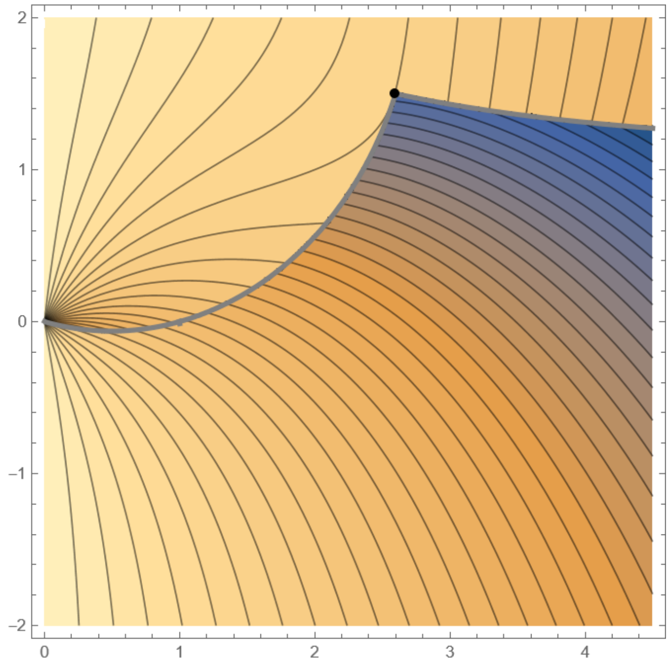

Figure 4.9: Contourplot of the Debye green function $\psi_{G(N)}$ in the BL coordinates for the values $M = 1$, $a = 1.2$, $r_0 = 3$, $\theta_0 = \pi/3$. The gray lines are the discontinuities and the dot at the intersection of the discontinuities is the position of the source.

We can notice something interesting about the superpotential for naked Kerr singularity in Fig. (4.7). We would be tempted to think that the most western point of the discontinuity $\gamma_{(N)w}$ is the ring singularity, but it is in fact given by $\sqrt{a^2 - M^2}$. The ring singularity does not produce any effect at the level of the superpotential.

4.2 Magnetic field of a current loop

Once we have the superpotential Ξ, we mutiply it by $\Delta \sin\theta$ to obtain the Debye effective Green function ψ_G (3.38). Next we perform differentiation in order to calculate the Debye potential for a current loop ψ_J (3.51). At this point, we can get to computing the Maxwell tensor. By further differentiation, we first get the electromagnetic scalars (2.78), (2.73), and then the anti-self-dual (2.16). At the end, we project this onto the ZAMO tetrad (2.88). For simplicity, we shall only consider the case of the equatorial plane $z_J = 0$.

If we compute the magnetic field directly corresponding to the superpotentials Ξ_K, Ξ_E, and Ξ_N, we find that it is discontinuous when crossing the equatorial plane for $r > r_J$. This can be viewed in Fig. 4.10. Moreover, if we calculate the charge at different points, we find that it is generally non-zero, dependent on r, and asymptotically constant. This points to the presence of sources other than the current loop.

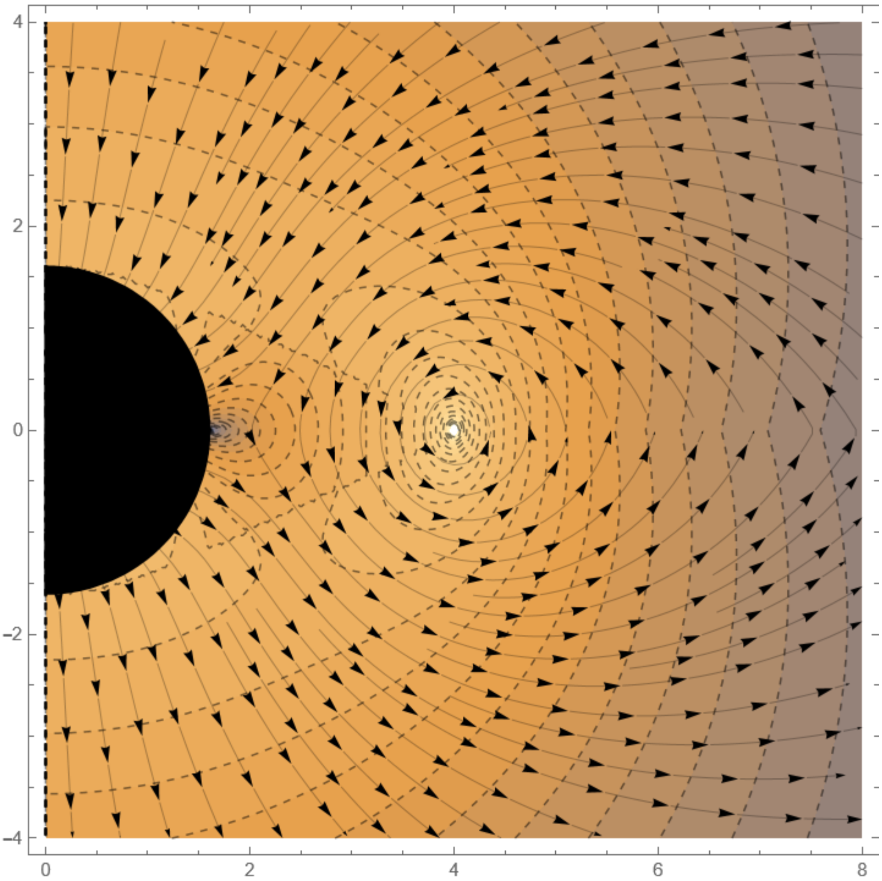

Figure 4.10: Plot of the magnetic field of a current loop in Kerr background computed from the superpotential $\Xi_{(K)}$. It is visualized with respect to the ZAMO observer for the values $M = 1$, $a = 0.8$, $r_0 = 4$, $\theta_0 = \pi/2$, $j = -1$. We present the streamlines of the vectorfield $\vec{B}$ and the contours of $|B|^2$ in a logarithmic scale.

Here we bring attention to the discontinuities in the superpotential which we presented earlier. The superpotential jumps all have vanishing ϕ_0 and their charge is located beneath the horizon. Thus, the superpotentials have differential charges in different regions and this causes discontinuities in the magnetic field. Therefore, using the jumps and the region functions, we would like to remove some of the discontinuities so as to obtain a continuous magnetic field with charge contained below the horizon or near the singularity. This charge then can be removed by the addition of an appropriate monopole field.

In the work of Kofroň and Kotlařík, they used the superpotential

$$\Xi_{(K)} - \Xi_{(K)s}\Theta_{(K)s} + \Xi_{(K)n}\Theta_{(K)n}. \tag{4.41}$$

This solves the mentioned problems, however, the jumps in the extreme limit $\beta \to 0$ diverge, dominate the expression, and do not approach the extreme background solution. We present two different configurations that resolve the discontinuities and charge distribution

$$\Xi_s^{(K)} = \Xi_{(K)} + \Xi_{(K)s}\Theta_{(K)i} + \frac{1}{2}\Xi_{(K)e}\left(\Theta_{(K)n} - \Theta_{(K)s} - \Theta_{(K)i}\right), \tag{4.42}$$

$$\Xi_n^{(K)} = \Xi_{(K)} - \Xi_{(K)n}\Theta_{(K)i} + \frac{1}{2}\Xi_{(K)e}\left(\Theta_{(K)n} - \Theta_{(K)s} + \Theta_{(K)i}\right). \tag{4.43}$$

It is clear that these converge to the solution in the extreme background since the divergent part is located only in the inner region that vanishes in the extreme limit. Moreover, both of these choices give rise to the same magnetic field which can be verified numerically. The superpotentials for the extreme case and the case of a naked singularity can be chosen as

$$\Xi^{(E)} = \Xi_{(E)} + \frac{1}{2}\Xi_{(E)e}\left(\Theta(z) - \Theta(-z)\right), \tag{4.44}$$

$$\Xi^{(N)} = \Xi_{(N)} + \frac{1}{2}\Xi_{(N)e}\left(\Theta(z) - \Theta(-z)\right). \tag{4.45}$$

At last, all the charge is situated below the horizon or below $\tilde{r} < r_J$ in the case of naked singularity. We can compute the charge by (2.85) and subtract the monopole field given by the scalar (2.86).

We visualize the resulting magnetic field of a current loop on Kerr background in Fig. (4.11). In this case, we have used $\Xi_s^{(K)}$ and have also removed the excess charge.

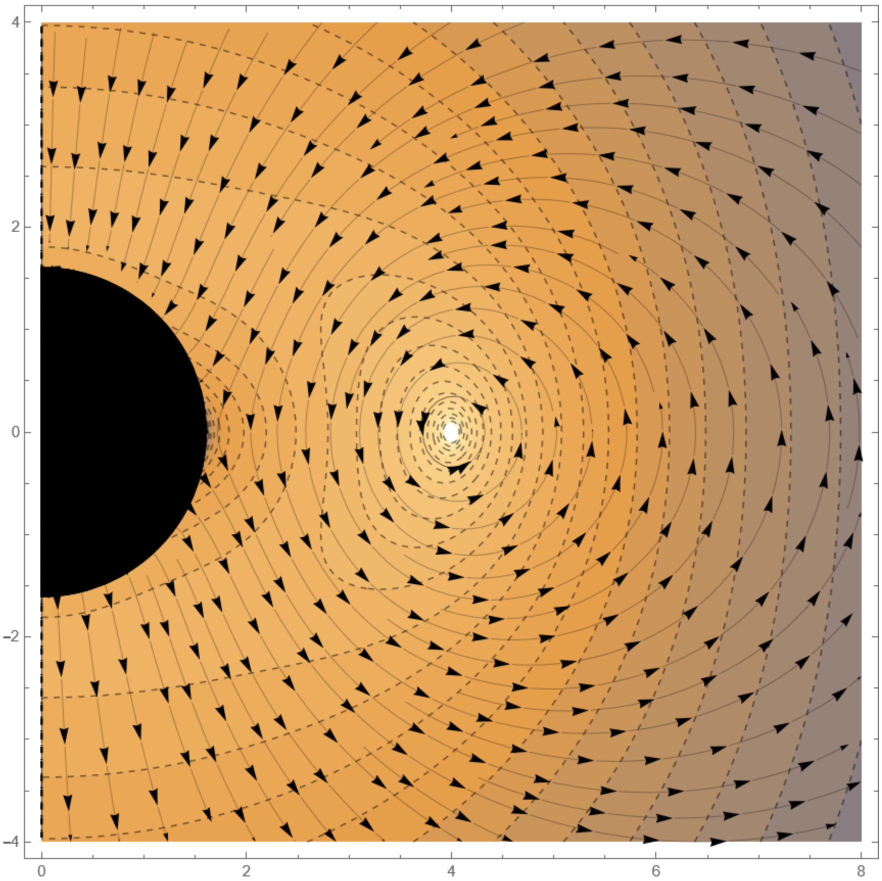

Figure 4.11: Plot of the magnetic field of a current loop in Kerr background computed from the superpotential $\Xi_s^{(K)}$. It is visualized with respect to the ZAMO observer for the values $M = 1$, $a = 0.8$, $r_0 = 4$, $\theta_0 = \pi/2$, $j = -1$. We present the streamlines of the vectorfield $\vec{B}$ and the contours of $|B|^2$ in a logarithmic scale.

5. Discussion

Here, we would like to address a few questions that might occur to the reader. First, we explain how exactly we obtained the superpotential, and we also mention what problems we encountered along the way. Then, we point out the difference between the numerical evaluation of the integral that gives the superpotential (3.39), and the explicit expressions for the superpotential in terms of elliptic integrals. At the end, we discuss alternative approaches.

5.1 Notes on finding the superpotential

When analytically integrating the expression for the superpotential (3.39) on an extreme Kerr background, we followed the same route as Kofroň and Kotlařík in [12]. Since Mathematica 13.1 cannot perform the integration (nor definite or indefinite), we had to carry it out in Maple 2019. As Maple gave the wrong result for the definite integral (the result was not real, nor did it satisfy W_1), we were forced to settle for an indefinite one. This produced a fairly long result containing elliptic integrals. Here, we came across another problem. When taking derivatives of the elliptic integrals in Maple, we obtained errors. Moreover, when plotting the $\Pi(t|k)$ integrals, there were additional discontinuities which were not supposed to be there. We thus had to translate the result into Mathematica. There, we were able to verify by differentiation that we had obtained the correct indefinite integral. What was left to do was to perform the desired limits of the primitive function. However, some limits of the elliptic integrals are not implemented correctly in Mathematica (a fact one is warned about on the Wolfram website). Therefore, the limits had to be done manually. At the end, after plenty of simplifications, we obtained the desired expression that was real and satisfied W_1.

5.2 Superpotential - analytical vs. numerical integration

Should we numerically compute the values of the superpotential given by (3.39), and compare them to our expressions in terms of elliptic integrals (4.17), (4.30), (4.37), we would find they do not always match. Recall that the superpotential has a structure of discontinuities. We find that the reason for the occasional inconsistency is that the location of the discontinuities got shifted upon expressing the original integral via elliptic integrals. Thus the solutions covering the different regions are the same, but the shapes of the regions change.

Below, we show plots of the numerically calculated superpotentials $\mathcal{N}\Xi$ for Kerr background (Fig. 5.1), extreme Kerr background (Fig. 5.2) and for the background of a naked Kerr singularity (Fig. 5.3). The position of the discontinuities is independent of the location of the source. This is quite the opposite when compared to the results of analytic integration where the location of the source served as a meeting point of different discontinuities.

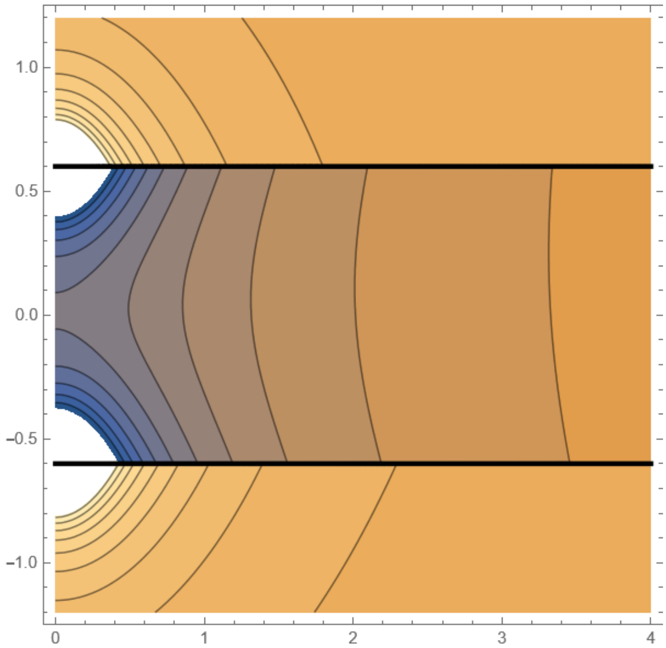

Figure 5.1: Contour-plot of the numerically computed superpotential $\mathcal{N}\Xi$ in the Weyl coordinates for the values $M = 1$, $a = 0.8$, $\rho_0 = 3$, $z_0 = 1$. The black lines are discontinuities.

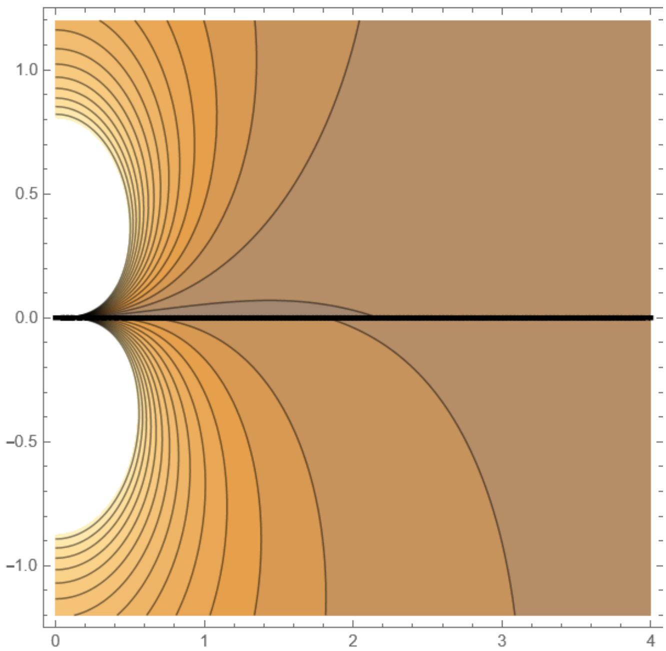

Figure 5.2: Contour-plot of the numerically computed superpotential $\mathcal{N}\Xi$ in the Weyl coordinates for the values $M = 1$, $a = 1$, $\rho_0 = 3$, $z_0 = 1$. The black line is a discontinuity.

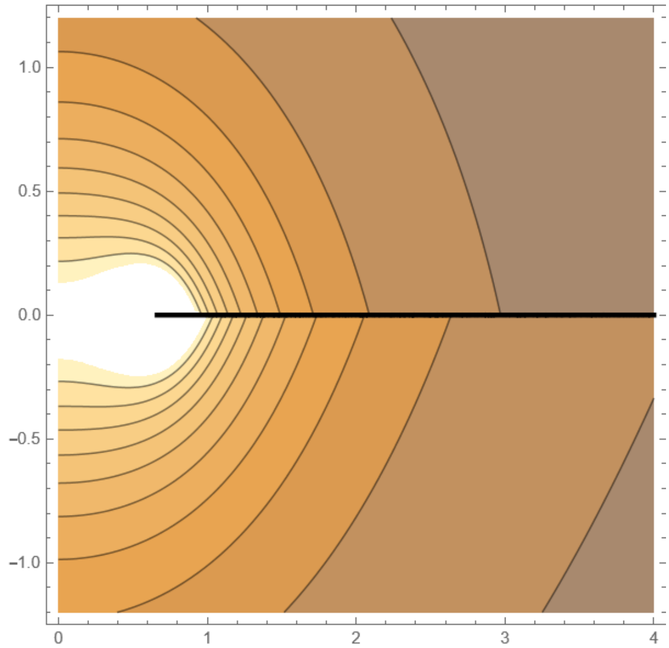

Figure 5.3: Contour-plot of the numerically computed superpotential $\mathcal{N}\Xi$ in the Weyl coordinates for the values $M = 1$, $a = 1.2$, $\rho_0 = 3$, $z_0 = 1$. The black line is a discontinuity.

5.3 Alternative approaches

First, let us address a seeming contradiction in our approach. We obtained the Debye potential by integrating ϕ_0 which was determined by the Teukolsky equation containing sources. However, the relationship between the Debye potential and the electromagnetic scalar assumed vacuum Maxwell's equations. This should not pose a problem since the source is an infinitesimal ring and thus the vacuum equations hold everywhere up to a single point in the cross-section spanned by r and θ. Thus, the sources of ϕ_0 effectively act as a boundary condition that gives us the Debye potential of the ring source.

An alternative approach would be to find the Debye potential using Maxwell's equations with sources J. These would then come into the Debye wave equation in the form of a source co-potential j given by $J = \delta j$. In the vacuum case, we were able to find such a gauge of the Debye potential that only a single independent component of the wave equation remained. This should not be possible whenever sources are present since only the vacuum electromagnetic field has two degrees of freedom. We would thus expect that the gauge freedom of the sources $\delta(j + \delta\sigma)$ can only nullify two independent components. We would subsequently require another Debye potential determined by a different equation. These were presented by Cohen and Kegeles in [10] – equations (CK:5.8) or (CK:5.10). However, we are not familiar with their Green functions in Kerr spacetime.

On a different note, we could also look at a similar problem regarding the gravitational field of a rotating thin disc around a Schwarzschild black hole treated by Čížek and Semerák in [26] (or in later review [27]). The scheme to obtaining the field relies on writing the metric in a form that satisfies all the necessary symmetries and boundary conditions. The metric is thus described by four functions

B, ω, ν, and ζ, which are determined by the Einstein's equations. The solution for B is simple and chosen in a conventional manner. A crucial step is finding the Green functions of ω and ν which were originally known only in the form of a series. They were able to show that the the infinite series can be written in a closed form using elliptic integrals. At last, ζ can be obtained by integration from ω and ν. One can see a similarity in the electromagnetic counterpart. There, the series solutions came from solving the Teukolsky equation for ϕ_0 from which one can swiftly obtain ϕ_2. The problem lies in calculating ϕ_1 from the Maxwell's equations. This was accomplished in the infinite series approach but remains fairly problematic in general. We thus performed the integration by using the Debye potential approach combined with the axisymmetric potential theory.

Conclusion

In this text, we have put together all the results from topics related to electromagnetic fields of current loops around black holes – which are necessary to find the magnetic field in a closed form – and presented them in a compact logical framework. We have also provided original results: the Debye superpotential for electromagnetic field of a stationary axially symmetric source in the extreme Kerr background, and an extension of the superpotential from the below-extreme background to the background of a Kerr naked singularity.

Bibliography

[1] E. T. Newman and R. Penrose. An approach to gravitational radiation by a method of spin coefficients. *J. Math. Phys.* 3, 566 (1962).

[2] S. A. Teukolsky. Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations. *Astrophys. J.* 185, 635 (1973).

[3] D. M. Chitre and C. V. Vishveshwara. Electromagnetic field of a current loop around a Kerr black hole. *Phys. Rev. D* 12, 1538 (1975).

[4] J. A. Petterson. Stationary axisymmetric electromagnetic fields around a rotating black hole. *Phys. Rev. D* 12, 2218 (1975).

[5] J. Bičák and L. Dvořák. Stationary electromagnetic fields around black holes. II. general solutions and the field of some special sources near a Kerr black hole. *Gen. Relativ. Gravit.* 7, 959 (1976).

[6] R. L. Znajek. Charged current loops around Kerr holes. *Mon. Not. R. Astr. Soc.* 182, 639 (1978).

[7] I. G. Moss. Black holes with current loops revisited. *Phys. Rev. D* 83, 123046 (2011).

[8] Z. Vlasáková. The fields of current loops around black holes. Master thesis, Charles University, Faculty of physics and mathematics, Prague, 2020.

[9] Š. Vrba. Magnetic field of current loops around black holes. Bachelor thesis, Charles University, Faculty of physics and mathematics, Prague, 2021.

[10] J. M. Cohen and L. S. Kegeles. Electromagnetic fields in curved spaces: A constructive procedure. *Phys. Rev. D* Vol. 10, No. 4 (1974).

[11] B. Linet. Stationary axisymmetric electromagnetic fields in the Kerr metric. *J. Phys. A: Math. Gen.,* Vol. 12, No. 6 (1979).

[12] D. Kofroň and P. Kotlařík. Debye superpotential for charged ring or circular current on Kerr BH. *Phys. Rev. D,* 106 no.10, 104022 (2022).

[13] M. Nakahara. Geometry, topology and physics. Institute of Physics Publishing, 2003.

[14] S. Kobayashi and K. Nomizu. Foundations of differential geometry, vol. 1. Interscience publishers, 1963.

[15] A. Nisbet. Hertzian electromagnetic potentials and associated gauge transformations. *Proc. R. Soc.* A231, 250 (1955).

[16] S. Chandrasekhar. The mathematical theory of black holes. Claredon Press, Oxford University press, 1983.

[17] J. B. Griffiths and J. Podolsky. Exact space-times in Einstein's general relativity. Cambridge University Press, 2009.

[18] A. Trautman, F. A. E. Pirani and H. Bondi. Lectures on general relativity. Prentice-Hall, 1965.

[19] E. D. Fackerell and J. R. Ipser. Weak electromagnetic fields around a rotating black hole. *Phys. Rev. D* 5, 2455 (1972).

[20] J. M. Bardeen, W. H. Press and S. A. Teukolsky. Rotating black holes: locally nonrotating frames, energy extraction, and scalar synchrotron radiation. *Astrophys. J.* 178, 347 (1972).

[21] A. E. Heins. The fundamental solution in axially symmetric potential theory. *Proc. London Math. Soc.,* (3) 29, 735-749 (1974).

[22] Wolfram Research - characteristics of elliptic integrals. *http://functions.wolfram.com/08.03.04.0013.01*

[23] Wolfram Research - characteristics of elliptic integrals. *http://functions.wolfram.com/08.03.04.0014.01*

[24] Wolfram Research - characteristics of elliptic integrals. *http://functions.wolfram.com/08.03.04.0017.01*

[25] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark. NIST handbook of mathematical functions. Cambridge University Press, 2010.

[26] P. Čížek and O. Semerák. Perturbation of a Schwarzschild black hole due to a rotating thin disk. *ApJS* 232, 14 (2017).

[27] O. Semerák and P. Čížek. Rotating Disc around a Schwarzschild Black Hole. *Universe* 6(2), 27 (2020).