FACULTY
OF MATHEMATICS
AND PHYSICS
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MASTER THESIS

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α-Symmetric Measures

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Study programme: Probability, Mathematical Statistics and Econometrics
Study branch: MPSP

Prague 2023

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I am sincerely grateful to my supervisor, Mgr. Stanislav Nagy, Ph.D., for his valuable insights and unwavering encouragement throughout this endeavor. Furthermore, I extend my heartfelt appreciation to my family, friends, and David for their enduring patience and invaluable assistance. I am immensely thankful for their unwavering support.

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Abstract: Spherically symmetric measures in $\mathbb{R}^n$ are rotationally invariant, indicating that their characteristic functions can be written as a composition of the Euclidean norm with a univariate function. If we replace the Euclidean norm with an ℓ_α norm, the resulting distributions are known as α-symmetric. This thesis aims to provide a general description of α-symmetric measures and explore various non-trivial examples. The existence of α-symmetric measures for a given α and dimension $n \in \mathbb{N}$ is discussed, along with the connection between the existence of α-symmetric measures and isometric embedding into L_p spaces through strictly stable distributions. One of the main properties explored in this thesis is the relationship between moments of non-integer order and α-symmetry in distributions. Additionally, several sufficient conditions for the existence and the form of α-symmetric measures are described. In the final chapter, a further generalization of α-symmetric distributions toward quasi-norms is discussed, along with the properties of the resulting concept of pseudo-isotropy.

Keywords: spherically symmetric measure, multivariate measure, characteristic function, positive definite function

Contents

Introduction

Spherical symmetry serves as a natural extension of normal distributions into more complicated structures by preserving invariance with respect to rotations as the defining characteristic. It can be shown that the characteristic functions of spherically symmetric distributions are composed of the Euclidean norm and a one-dimensional function. By substituting the Euclidean norm with an ℓ_α norm, we obtain α-symmetric distributions. However, it is important to note that not all properties of spherically symmetric distributions seamlessly translate into the realm of α-symmetry, as symmetric stable distributions assume the role of normal distributions in the context of α-symmetry.

The thesis is organized in the following way: Chapter 1 introduces mostly well-known tools which are utilized in order to establish and develop the theory of α-symmetric distributions. We aim to explore the connection between moments and characteristic functions and how stable distributions can be expressed via isometric embedding of quasi-normed spaces. Elementary properties of α-symmetric distributions are presented in Chapter 2 with emphasis on projections, mixtures, and density. Several examples of characteristic functions are established. Chapter 3 focuses on n-dimensional α-symmetric distributions for different pairs of $0 < \alpha \leq \infty$ and $n \geq 2$. Although only some examples and sufficient conditions are known for some pairs, there is a full characterization e.g. for 1-symmetric distributions. The thesis aims to present full proof of the fact that for some α the only α-symmetric distribution is the trivial one (concentrated at the origin). Chapter 4 generalizes α-symmetry by replacing ℓ_α norm by a quasi-norm and discusses some properties of such distributions. Multivariate symmetric stable distributions are again the only well-established examples of pseudo-isotropic distributions.

The aim of the thesis is to find new examples of α-symmetric distributions and explore the existing examples. Further, moments of α-symmetric and pseudo-isotropic distributions are unfolded as non-trivial α-symmetric distributions have power-heavy tails.

1. Preliminaries

This chapter aims to elucidate the theory used in further chapters, and its contents can be split into two parts. Section 1.1 covers the properties of characteristic functions as characteristic functions are naturally more suitable for dealing with α-symmetry in Chapter 2 than the density or the cumulative distribution function. The attention is brought to the relation between the characteristic function of a random variable and its absolute moments of non-integer order by Laue [1980]. The completely and m-times monotone functions and their integral representations through the Laplace transform will serve a different purpose in this thesis as shown in Section 3.3 and in Section 3.4.

The second part of this chapter presents stable distributions and their properties. The main result of Section 1.2 connects probability theory and convex geometry. The isometric embedding between quasi-metric spaces is defined and used to characterize characteristic functions of multivariate symmetric stable distributions. The representation derived in Subsection 1.2.1 generalizes α-symmetry in Chapter 4.

1.1 Integral Transformations

Several integral transformations are used throughout the thesis. This section aims to summarize the results. Most theorems of this chapter are presented without proof.

1.1.1 Characteristic Function

Definition 1. *Let* $\mathbf{X}$ *be a random vector in* $\mathbb{R}^n$*. The function* $\varphi : \mathbb{R}^n \to \mathbb{C}$ *defined as* $\varphi(\mathbf{t}) = E\, e^{i\mathbf{t}'\mathbf{X}}$ *is called the* characteristic function *of the random vector* $\mathbf{X}$.[1]

Results in further chapters, mostly by Zastavnyi [1992], Koldobsky [1991] were formulated in terms of positive definite functions which are closely related. The connection between characteristic functions and positive definite functions is known as Bochner's theorem (the proof in Lukacs [1970], Theorem 4.2.2).

Definition 2. *A function* $\varphi : \mathbb{R}^n \to \mathbb{C}$ *is* positive definite *if the inequality*

$$\sum_{i=1}^{k}\sum_{j=1}^{k} c_i \bar{c}_j \varphi(\mathbf{t}_i - \mathbf{t}_j) \geq 0$$

holds for any $k \in \mathbb{N}$*, any* $\mathbf{t}_1, \ldots, \mathbf{t}_k \in \mathbb{R}^n$ *and any constants* $c_1, \ldots, c_k \in \mathbb{C}$.

Theorem 1. *A function* $\varphi : \mathbb{R}^n \to \mathbb{C}$ *is a characteristic function of some random vector in* $\mathbb{R}^n$ *if and only if* φ *is positive definite, continuous at origin and* $\varphi(\mathbf{0}) = 1$.

The last condition can be taken into account by transforming[2] $\varphi(\cdot) \mapsto \frac{1}{\varphi(\mathbf{0})}\varphi(\cdot)$ which means the terms *continuous positive definite function* and a *characteristic*

[1] By $\mathbf{t}'\mathbf{x}$ we denote the dot product $\mathbf{t}'\mathbf{x} = t_1x_1 + \cdots + t_nx_n$ for $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$, $\mathbf{t} = (t_1, \ldots, t_n)' \in \mathbb{R}^n$.

[2] See Lemma 3.

function of a random vector will be almost exchangeable. Results that are originally formulated in terms of continuous positive definite functions will be thus taken into the context of random vectors. Integrable characteristic functions are linked to the density through the inversion theorem (the proof can be found in Lukacs [1970], Theorem 3.2.2).

Theorem 2. *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a characteristic function of a random vector $\mathbf{X}$. Then if φ is integrable, the density $f_{\mathbf{X}}$ of $\mathbf{X}$ exists and is given by*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\mathbf{t}'\mathbf{x}} \varphi(\mathbf{t})\ d\mathbf{t}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Several other properties of characteristic functions are needed throughout the thesis. The proofs of all properties can be found in Lukacs [1970], Chapter 2:

Lemma 3. *Let $\varphi, \varphi_n : \mathbb{R}^n \to \mathbb{C}$ be characteristic functions of random vectors $\mathbf{X}$, $\mathbf{X}_n$, $n \in \mathbb{N}$. Then the following statements are true:*

(i) *φ is uniformly continuous.*

(ii) $1 = \varphi(\mathbf{0}) \geq |\varphi(\mathbf{t})|$ *for any* $\mathbf{t} \in \mathbb{R}^n$.

(iii) *φ is a real function if and only if $\mathbf{X}$ is symmetric.*

(iv) *Let A be an $m \times n$ real matrix and $\mathbf{b} \in \mathbb{R}^m$ then the characteristic function of $A\mathbf{X} + \mathbf{b}$ is equal to $\varphi(A'\mathbf{t})e^{i\mathbf{b}'\mathbf{t}}$, $\mathbf{t} \in \mathbb{R}^m$.*

(v) $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ *if and only if* $\varphi_n(\mathbf{t}) \to \varphi(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$.

As mentioned in further sections the density does not have to be analytically expressible even for some simple characteristic functions.[3] Since the characteristic function fully characterizes the distribution of a random variable, the function can be used to derive its characteristics. The proof of the following theorem can be found in Lukacs [1970], Section 2.3.

Theorem 4. *Let X be a real random variable with a characteristic function φ.*

(i) *If $\mathsf{E}\, X^k$, $k \in \mathbb{N}$, exists the function φ is k-times differentiable at zero and*

$$\mathsf{E}\, X^k = i^{-k} \varphi^{(k)}(0).$$

The converse implication holds if k is an even integer.

(ii) *If additionally $X \geq 0$ a.s., then the k-th differentiability of φ at $t = 0$ is equivalent to the existence of $\mathsf{E}\, X^k$.*

The following example (mentioned also in Laue [1980]) shows a counterexample for a random variable with a diverging first moment.

[3] E.g. for some stable distributions, see Section 1.2.

Example 1. Let X be a random variable with a density

$$f(x) = \frac{C}{x^2 \log|x|} \mathbb{1}\{|x| \geq 2\}, \quad x \in \mathbb{R},$$

where $C > 0$ is a constant. The first absolute moment of X is infinite since

$$\begin{aligned} \mathsf{E}\,|X| &= \int_{\mathbb{R}} |x| f(x)\; dx \\ &= 2C \int_2^\infty \frac{x}{x^2 \log x}\; dx \\ &\overset{y=\log x}{=} 2C \int_{\log 2}^\infty \frac{1}{y}\; dy = \infty. \end{aligned}$$

However, its characteristic function (due to the symmetry of the density) is equal to

$$\begin{aligned} \mathsf{E}\, e^{itX} &= \int_{\mathbb{R}} e^{itx} f(x)\; dx \\ &= 2C \int_2^\infty \frac{\cos(tx)}{x^2 \log|x|}\; dx. \end{aligned}$$

Let us denote the characteristic function φ and find its derivative at zero. For $t \in \left(0, \frac{1}{2}\right)$

$$\begin{aligned} \frac{\varphi(t) - \varphi(0)}{t} &= 2C \int_2^\infty \frac{\cos(tx) - 1}{tx^2 \log x}\; dx \\ &= 2C \left(\int_2^{\frac{1}{t}} \frac{\cos(tx) - 1}{tx^2 \log x}\; dx + \int_{\frac{1}{t}}^\infty \frac{\cos(tx) - 1}{tx^2 \log x}\; dx \right). \end{aligned}$$

First, the in the second part $|\cos(tx) - 1| \leq 2$ and

$$\begin{aligned} \left| \int_{\frac{1}{t}}^\infty \frac{\cos(tx) - 1}{tx^2 \log x}\; dx \right| &\leq \frac{1}{t} \int_{\frac{1}{t}}^\infty \frac{2}{x^2 \log x}\; dx \\ &\overset{y=1/x}{=} -\frac{2}{t} \int_0^t \frac{1}{\log y}\; dy \\ &\overset{u=y/t}{=} -2 \int_0^1 \frac{1}{\log t + \log u}\; du \overset{t \to 0+}{\to} 0 \end{aligned}$$

since the integrand tends to 0 monotonously. For the first part we estimate $|\cos(tx) - 1| \leq (tx)^2$

$$\begin{aligned} \left| \int_2^{\frac{1}{t}} \frac{\cos(tx) - 1}{tx^2 \log x}\; dx \right| &\leq \int_2^{\frac{1}{t}} \frac{t^2 x^2}{tx^2 \log x}\; dx \\ &= \int_2^{\frac{1}{t}} \frac{t}{\log x}\; dx \\ &= \frac{1}{\frac{1}{t}} \int_2^{\frac{1}{t}} \frac{1}{\log x}\; dx \end{aligned}$$

and the limit $t \to 0+$ is equivalent to $\frac{1}{t} \to \infty$ which means we can use L'Hôpital's Rule (Rudin [1976], Theorem 5.13)

$$\lim_{s \to \infty} \frac{1}{s} \int_2^s \frac{1}{\log x}\; dx = \lim_{s \to \infty} \frac{1}{\log(s)} = 0.$$

We have found that $\mathsf{E}\,|X| = \infty$ and the derivative of the characteristic function at zero is equal to zero which concludes the counterexample to the first part of Theorem 4.

Distributions studied in this thesis will usually have infinite variance (or even expectations) with the exception of some trivial cases.[4] Thus, the following generalization of Theorem 4 was proposed by Laue [1980] using the *Marchaud fractional derivative.*

Definition 3. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function and $m \geq 0$ is decomposed as $m = k + \lambda$, where k is a non-negative integer and $\lambda \in (0,1)$. The m-th* Marchaud fractional derivative *of f at point $t \in \mathbb{R}$ is defined as*

$$\frac{\partial^m}{\partial t^m} f(t) = \frac{\partial^\lambda}{\partial t^\lambda} f^{(k)}(t) = \frac{\lambda}{\Gamma(1-\lambda)} \int_{-\infty}^{t} \frac{f^{(k)}(t) - f^{(k)}(u)}{(t-u)^{1+\lambda}}\, du.$$

The following generalization is not the only one, e.g. Wolfe [1975] found similar conditions using the Laplace transform of a random variable. The approach of Laue is more convenient in our case.

Theorem 5. *Let X be a non-negative random variable with a characteristic function φ and $m \geq 0$ is decomposed $m = k + \lambda$ as in Definition 3. Then $\mathsf{E}\, X^m$ exists if and only if both $\mathsf{E}\, X^k$ and*

$$\Re\left[i^k \frac{\partial^m}{\partial t^m} \varphi(t) \Big|_{t=0} \right]$$

exist.[5] *Then*

$$\mathsf{E}\, X^m = \frac{1}{\cos(\frac{1}{2}\lambda\pi)} \Re\left[(-i)^k \frac{\partial^m}{\partial t^m} \varphi(t) \Big|_{t=0} \right].$$

Similar results with imaginary parts

$$\mathsf{E}\, X^m = \frac{1}{\sin(\frac{1}{2}\lambda\pi)} \Im\left[(-i)^k \frac{\partial^m}{\partial t^m} \varphi(t) \Big|_{t=0} \right]$$

as well as the proof are available in Laue [1980], Theorem 2.1 and 2.2.

Theorem 6. *Let X be a random variable with a characteristic function φ and $m \geq 0$ is decomposed as $m = k + \lambda$ as in Theorem 5.*

(i) Let k be an even integer. Then $\mathsf{E}\,|X|^m$ is finite if and only if $\mathsf{E}\,|X|^k$ is finite and

$$\Re\left[\frac{\partial^m}{\partial t^m} \varphi(t) \Big|_{t=0} \right]$$

exists. Then

$$\mathsf{E}\,|X|^m = \frac{1}{\cos(\frac{1}{2}\lambda\pi)} \Re\left[(-1)^{\frac{k}{2}} \frac{\partial^m}{\partial t^m} \varphi(t) \Big|_{t=0} \right].$$

[4] See Theorem 21.

[5] By $\Re$ and $\Im$ we mean the real and imaginary parts of a complex number, respectively.

(ii) Let k be an odd integer. Then $\mathsf{E}\,|X|^m$ is finite if and only if

$$\Re\left[\frac{\partial^m}{\partial t^m}\varphi(t)\,\Big|_{t=0}\right]$$

exists and for $\varphi_k(t) = (-1)^{\frac{k-1}{2}}\varphi^{(k-1)}(t)/\,\mathsf{E}\,X^{k-1}$ the limit

$$\lim_{t\to 0}\frac{1-\Re\,\varphi_k(t)}{t^{1+\lambda}}$$

exists. The moment is equal to

$$\mathsf{E}\,|X|^m = \frac{1}{\sin(\frac{1}{2}\lambda\pi)}\Re\left[(-1)^{\frac{k+1}{2}}\frac{\partial^m}{\partial t^m}\varphi(t)\,\Big|_{t=0}\right].$$

As the computations are not simple, the following corollary presents equivalent conditions for the existence of non-integer moments. The characteristic functions in the thesis will mostly be real and symmetric, thus the existence of moments of order less than two depends solely on the behavior of the characteristic function near the origin since all characteristic functions are bounded and continuous in addition.

Corollary 1. For $m \in (0, 2)$ and an arbitrary random variable X with a characteristic function φ Kawata et al. [1972] (Theorem 11.4.3) derived a simple necessary and sufficient condition for the existence of $\mathsf{E}\,|X|^m$. The moment exists if and only if

$$\int_0^\infty \frac{1-\Re\,\varphi(t)}{t^{1+m}}\,dt < \infty.$$

1.1.2 Laplace Transform

Definition 4. *Let X be a random variable. The function $L_X(t) = \mathsf{E}\,e^{-tX}$ is called the* Laplace transform *of a random variable X for any $t \in \mathbb{R}$ where $\mathsf{E}\,e^{-tX} < \infty$.*

Remark 1. Contrary to the characteristic function of a random variable, the Laplace transform does not have to be defined for all $t \in \mathbb{R}$.

Similarly, as the characteristic functions and continuous positive definite functions are connected there is a link between Laplace transforms and the completely monotone functions. The theorem is known as *Bernstein's theorem on monotone functions* (Bernstein [1929]).

Definition 5. *A function $f : [0, \infty) \to \mathbb{R}$ is said to be* completely monotone[6] *on $[0, \infty)$ if the function f is continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$ and for each $n \in \mathbb{N} \cup \{0\}$ and $t > 0$ we have $(-1)^n f^{(n)}(t) \geq 0$.*

Theorem 7. *The function $f : [0, \infty) \to \mathbb{R}$ is completely monotone on $[0, \infty)$ with $f(0) = 1$ if and only if f can be written as a Laplace transform of some non-negative random variable X with a cumulative distribution function F, i.e.*

$$f(t) = L_X(t) = \int_0^\infty e^{-ts}\,dF(s) < \infty, \quad t > 0.$$

[6] Also called absolutely monotone/monotonous.

Corollary 2. Similarly, for a function $g : (0, \infty) \to \mathbb{R}$ whose Laplace transform $L_g(t) = \int_0^\infty e^{-ts} g(s)\, ds < \infty$ for all $t > 0$ then $g \geq 0$ if and only if L_g is completely monotone. In this case, g represents the density (or its multiple) of some random variable.

Definition 6. *A function* $f \in \mathcal{C}^{m-1}(0, \infty)$ *is called* m-times monotone *if for each* $k = 0, 1, \ldots, m-1$ *the function* $(-1)^k f^{(k)}(t)$ *is non-negative, decreasing and convex on* $(0, \infty)$.[7]

Remark 2. It is sufficient to check the condition only for the last derivative, i.e. a function $f \in \mathcal{C}^{m-1}(0, \infty)$ is m-times monotone if and only if $(-1)^{m-1} f^{(m-1)}(t)$ is non-negative, decreasing, convex, and there exists a finite non-negative limit $\lim_{t\to\infty} f(t)$ (Williamson [1955]).

As completely monotone functions are characterized as Laplace transforms of finite measures on $(0, \infty)$ a similar transformation for m-times monotone is available. The proof can be found in Williamson [1955].

Theorem 8. *A function* $f : [0, \infty) \to \mathbb{R}$ *is* m*-times monotone with* $f(0) = 1$ *if and only if there exists a non-negative random variable* X *with a characteristic function* F *which satisfies*

$$f(t) = \int_0^\infty (1 - st)_+^m \, dF(s), \quad t \geq 0,$$

where $f_+ = \max\{f, 0\}$.

Any completely monotone function is m-times monotone for any $m \in \mathbb{N}$ which means completely monotone functions are a generalization as $m \to \infty$. Both classes of functions serve as examples in Section 3.3 and Section 3.4.

1.2 Stable Distributions

This section summarizes the results about stable distributions which will be useful in further sections as α-symmetry can be perceived as a generalization of stability. More details on stable distributions can be found in Uchaikin and Zolotarev [1999].

Definition 7. *A distribution of a random variable is called* stable *if for any* $a, b > 0$ *there are* $c > 0$, $\tilde{c} \in \mathbb{R}$ *such that for stable* X_1 X_2, X *independent identically distributed the following equality*[8] *holds:*

$$aX_1 + bX_2 \stackrel{d}{=} cX + \tilde{c}. \tag{1.1}$$

A random variable is called symmetric stable *if additionally* $X \stackrel{d}{=} -X$.

For any stable distribution, we may find its *index* $\alpha \in (0, 2]$ which satisfies $c = (|a|^\alpha + |b|^\alpha)^{\frac{1}{\alpha}}$ in the definition above (Uchaikin and Zolotarev [1999], Section 3.2). The symmetry of the distribution implies $\tilde{c} = 0$. Equality (1.1) can be rewritten

[7] A function $f \in \mathcal{C}^k(S)$ if it has k-continuous derivatives on S, $\mathcal{C}^0$ are all continuous functions.

[8] The symbol $\stackrel{d}{=}$ denotes equality in distribution.

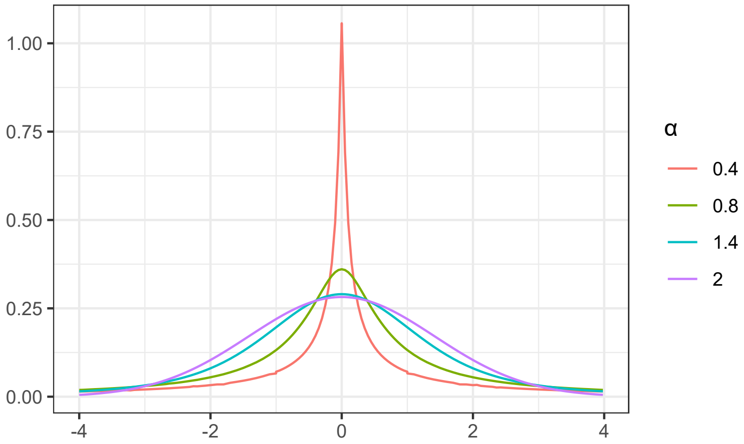


Figure 1.1: Densities of stable random variables for $\alpha \in \{0.4, 0.8, 1.4, 2\}$.

(for symmetric α-stable distributions) in terms of characteristic functions φ of X: For any $a, b, t \in \mathbb{R}$

$$\varphi(at)\varphi(bt) = \varphi\left((|a|^\alpha + |b|^\alpha)^{\frac{1}{\alpha}}\ t\right).$$

The characteristic function of a symmetric α-stable distribution is $e^{-C|t|^\alpha}$ for some $C > 0$ (Uchaikin and Zolotarev [1999], Section 3.2). If it is not specifically mentioned, the parameter C is set as 1.

For $\alpha > 2$ the characteristic function $e^{-|t|^\alpha}$ would not be positive definite.[9] The special cases include $\alpha = 2$ which are centered normal distributions and $\alpha = 1$ the Cauchy distributions. For other symmetric stable distributions, i.e. $\alpha \neq 1, 2$, there are no closed expressions for density, however, asymptotics for their tails is available (Koldobsky [2005], Chapter 6). Denote $\gamma_\alpha(x)$ the density of a symmetric α-stable variable, $\alpha < 2$. Then we have

$$\lim_{x\to\infty} x^{\alpha+1}\gamma_\alpha(x) = 2\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)$$

as can be seen in Figure 1.1.

The asymptotic of the density suggests that at least some fractional moments exist. The moments of order less than α do exist and can be found using Theorem 6.

Example 2. Using the Marchaud fractional derivative (Theorem 6) we may find the moments $\mathsf{E}\,|X|^r$ for symmetric α-stable random variables with a characteristic function $e^{-|t|^\alpha}$, $\alpha \in (0, 2]$.

From Corollary 1 the moments $\mathsf{E}\,|X|^r$, $r \in (0, 2)$, exist if and only if

$$\int_0^\infty \frac{1 - e^{-t^\alpha}}{t^{1+r}}\, dt < \infty,$$

which converges for $r \in (0, \alpha)$ since $1 - e^{-t^\alpha} \approx t^\alpha$ in the neighborhood of zero. Higher moments exist only for the normal distribution.

[9] A simple way to see that $e^{-|t|^\alpha}$ cannot be a characteristic function is to apply Theorem 4. For $\alpha > 2$ the second derivative of $e^{-|t|^\alpha}$ at zero equals zero which would imply the triviality of the corresponding distribution (almost surely constant).

For $r \in (0, \min\{\alpha, 1\})$ the moment (Theorem 6) is equal to

$$\begin{aligned}
\mathsf{E}\,|X|^r &= \frac{1}{\cos(r\frac{\pi}{2})}\frac{r}{\Gamma(1-r)}\int_0^\infty \frac{1-e^{-t^\alpha}}{t^{1+r}}\,dt \\
&= \frac{1}{\cos(r\frac{\pi}{2})}\frac{r}{\Gamma(1-r)}\frac{1}{r}\int_0^\infty \alpha t^{\alpha-1}e^{-t^\alpha}t^{-r}\,dt \\
&\overset{s=t^\alpha}{=} \frac{1}{\cos(r\frac{\pi}{2})\Gamma(1-r)}\int_0^\infty e^{-s}s^{-\frac{r}{\alpha}}\,ds \\
&= \frac{\Gamma(1-\frac{r}{\alpha})}{\cos(r\frac{\pi}{2})\Gamma(1-r)}
\end{aligned}$$

where per partes was used in the first equality and the definition of the Gamma function in the last.

Similarly, for $r \in [1, \alpha)$ we have to use the second part of Theorem 6

$$\begin{aligned}
\mathsf{E}\,|X|^r &= \frac{1}{\sin((r-1)\frac{\pi}{2})}\frac{r-1}{\Gamma(2-r)}\int_0^\infty \frac{\alpha t^{\alpha-1}e^{-t^\alpha}}{t^r}\,dt \\
&\overset{s=t^\alpha}{=} \frac{1}{\cos(r\frac{\pi}{2})\Gamma(1-r)}\int_0^\infty e^{-s}s^{-\frac{r}{\alpha}}\,ds \\
&= \frac{\Gamma(1-\frac{r}{\alpha})}{\cos(r\frac{\pi}{2})\Gamma(1-r)}.
\end{aligned}$$

Based on a stochastic decomposition derived by Shanbhag and Sreehari [1977], the moments of symmetric α-stable distributions are usually written as

$$\mathsf{E}\,|X|^r = \frac{2^r\Gamma(\frac{1+r}{2})\Gamma(1-\frac{r}{\alpha})}{\Gamma(1-\frac{r}{2})\Gamma(\frac{1}{2})}$$

which equals our derived terms (through the Euler's reflection formula[10] and other properties of the Gamma function).

Generally, the characteristic function φ of any stable distribution[11] is

$$\varphi(t) = \exp\{i\mu t - C|t|^\alpha(1 - i\beta\operatorname{sgn}(t)\Phi)\}$$

where $\mu \in \mathbb{R}$ is the location parameter, $C > 0$ is the scale parameter, $\alpha \in (0, 2]$ is the index, $\beta \in [-1, 1]$ is the skewness[12] parameter and

$$\Phi = \begin{cases} \tan(\frac{1}{2}\pi\alpha), & \alpha \neq 1, \\ -\frac{2}{\pi}\log|t|, & \alpha = 1. \end{cases}$$

The symmetric α-stable distribution satisfies $\mu = 0$, $\beta = 0$. The distribution for $\alpha < 1$ and $\beta = 1$ and $\mu = 0$ is concentrated on $[0, \infty)$. For the latter class of distributions the Laplace transform (Definition 4) is equal to

$$\mathsf{E}\,e^{-tX} = e^{-t^\alpha}, \quad t > 0. \tag{1.2}$$

[10] For any non-integer z we have $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(z\pi)}$.

[11] Thoroughly derived in Uchaikin and Zolotarev [1999], 3.2.12.

[12] Skewness in a traditional way is not defined for $\alpha < 2$.

Such distributions are useful as mixtures and will be referred to as *non-negative α-stable* random variables.

The definition of symmetric α-stable random variables can be extended into random vectors in $\mathbb{R}^n$ using the same property as in Definition 7.

Definition 8. *A random vector* $\mathbf{X}$ *in* $\mathbb{R}^n$ *is called* symmetric stable *if* $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$ *and for any* $a, b > 0$ *there is* $c > 0$ *such that*

$$a\mathbf{X}_1 + b\mathbf{X}_2 \stackrel{d}{=} c\mathbf{X} \tag{1.3}$$

where $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}$ *are i.i.d.*

For any symmetric stable random vector, we may again find an *index* $\alpha \in (0, 2]$ such that each linear combination of its components $\sum_{i=1}^{n} t_i X_i$ is a symmetric α-stable random variable (as explained in Theorem 9).

The characteristic functions of symmetric stable vectors will be described using the isometric embedding in the next subsection.

1.2.1 Isometric Embedding

This section introduces methods of isometric embedding into L_p-spaces in order to characterize the stable vectors in Theorem 10. Theorem 12 describes all two-dimensional symmetric 1-stable random vectors.

First, let us define a generalisation of a norm which will include the α-norm in $\mathbb{R}^n$

$$\|\mathbf{x}\|_\alpha = \left(|x_1|^\alpha + \cdots + |x_n|^\alpha\right)^{\frac{1}{\alpha}}, \quad \mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n, \tag{1.4}$$

for any $0 < \alpha < \infty$. In case of $\alpha = \infty$ the ∞-norm is the maximum norm, $\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$. The definition of α-norms can be extended even further into quasi-norms.

Definition 9. *Let* E *be a linear space over* $\mathbb{R}$. *A continuous function* $\rho : E \to [0, \infty)$ *is called a* quasi-norm *if*

(i) $\rho(\mathbf{x}) = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$,

(ii) $\rho(t\mathbf{x}) = |t|\rho(\mathbf{x})$ *for* $t \in \mathbb{R}$, $\mathbf{x} \in E$,

(iii) there exists $K > 0$ *such that* $\rho(\mathbf{x} + \mathbf{y}) \leq K(\rho(\mathbf{x}) + \rho(\mathbf{y}))$, *for each* $\mathbf{x}, \mathbf{y} \in E$.

The pair (E, ρ) *is called a* quasi-normed space. *If moreover* $K = 1$, *the function* ρ *is a* norm.

A quasi-norm is a norm if and only if the unit ball $B_\rho = \{\mathbf{x} \in E : \rho(\mathbf{x}) \leq 1\}$ is convex (Koldobsky [2005], Chapter 2). A quasi-norm may be also defined through its unit ball B_ρ using the *Minkowski functional* $\|\mathbf{x}\|_B = \inf\{u \geq 0 : \mathbf{x} \in uB\}$ where B can be any closed bounded origin symmetric star body (B is a *star body* if for every $x \in B$ the segment $[0, x)$ is a subset of the interior of B Koldobsky [2005], Chapter 2).

The characteristic functions of symmetric α-stable random vectors are connected with the problems of *isometric embedding* into L_p-spaces. For our purpose, the following definition of isometric embedding will be used:

Definition 10. *Let* (E, ρ), (F, σ) *be quasi-normed spaces. The space* (E, ρ) *is* isometrically embedded *in* (F, σ) *if there is a linear operator* $T : E \to F$ *which for each* $\mathbf{t} \in E$ *satisfies* $\rho(\mathbf{t}) = \sigma(T\mathbf{t})$.

If $E \subset F$ and ρ is a restriction of σ to E then (E, ρ) isometrically embeds into (F, σ) where the linear operator T is an identity. Example 3 shows isometry between spaces with the 1-norm and the ∞-norm.

Example 3. Normed spaces $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ are isometric (each isometrically embeds into the other) since there is a unique relationship between 1-norm and ∞-norm in $\mathbb{R}^2$:

$$\max\{|t_1|, |t_2|\} = \frac{|t_1 + t_2| + |t_1 - t_2|}{2}, \quad t_1, t_2 \in \mathbb{R}.$$

The linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by a matrix $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\|T\mathbf{t}\|_\infty = \|\mathbf{t}\|_1$, $\mathbf{t} \in \mathbb{R}^2$, due to the equality of norms stated above. The converse isometric embedding can be shown similarly. This example is limited to $n = 2$.

The main focus of this subsection is the isometric embedding into L_p-spaces as in Definition 11. By Ω we typically mean the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$.

Definition 11. *Let* $(\Omega, \mathcal{A}, \nu)$ *be a measure space. Denote* $L_p(\Omega, \mathcal{A}, \nu)$ *the space of all integrable functions* $f : \Omega \to \mathbb{R}$ *such that*

$$\int_\Omega |f(x)|^p d\nu(x) < \infty.$$

If $\mathcal{A}$ *is the Borel* σ*-algebra, the notation is simplified to* $L_p(\Omega, \nu)$.

The function

$$f \mapsto \left(\int_\Omega |f(x)|^p d\nu(x) \right)^{\frac{1}{p}}, \quad f \in L_p(\Omega, \mathcal{A}, \nu) \tag{1.5}$$

satisfies (ii) and (iii) of Definition 9 (Rudin [1991], 1.47), the triangular inequality is satisfied for $p \geq 1$ (i.e. $K = 1$ in Definition 9, (iii)).[13] A subspace of $L_p(\Omega, \mathcal{A}, \nu)$ endowed with a quasi-norm defined by (1.5) will be used as a target space of isometric embeddings (Definition 10).

The following theorems connect symmetric α-stable measures with the theory of isometric embedding, such connection has been known to Lévy.

Theorem 9. *Let* μ *be a finite symmetric measure on* $\mathbb{R}^n$ *and* $\alpha \in (0, 2]$ *such that for each* $\mathbf{t} \in \mathbb{R}^n$ *the integral*

$$\int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^\alpha d\mu(\mathbf{x})$$

is finite. Then

$$\phi(\mathbf{t}) = \exp\left\{ -\int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^\alpha d\mu(\mathbf{x}) \right\}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{1.6}$$

defines a characteristic function of some symmetric α*-stable random vector.*

Conversely, for any symmetric α*-stable random vector in* $\mathbb{R}^n$*, there exists a positive finite measure* μ *on* $\mathbb{R}^n$ *such that its characteristic function can be represented as in (1.6).*

[13] Such functions are called *(quasi-)semi-norms*.

Proof. The proof of the first part can be found in Misiewicz [1996], Theorem II.1.3.

As mentioned below Definition 8 a symmetric α-stable random vector $\mathbf{X}$ satisfies that for any $\mathbf{t} \in \mathbb{R}^n$ the linear combination $\mathbf{t}'\mathbf{X}$ has symmetric α-stable distribution. If we denote $Y_\mathbf{t} = \mathbf{t}'\mathbf{X}$, then the characteristic function of $Y_\mathbf{t}$ must be equal to $e^{-c_\mathbf{t}|u|^\alpha}$, $u \in \mathbb{R}$, where $c_\mathbf{t} \geq 0$ depends on $\mathbf{t} \in \mathbb{R}^n$. Set $c_\mathbf{t} = c^\alpha(\mathbf{t})$ where $c : \mathbb{R}^n \to [0, \infty)$. Then the characteristic function of $\mathbf{X}$ can be rewritten as $\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} = e^{-c^\alpha(\mathbf{t})}$, $\mathbf{t} \in \mathbb{R}^n$, and $\mathbf{t}'\mathbf{X} \overset{d}{=} c(\mathbf{t})Y$ for any $\mathbf{t} \in \mathbb{R}^n$ where Y has a characteristic function $e^{-|u|^\alpha}$, $u \in \mathbb{R}$.[14]

For any $r \in (0, \alpha)$ and $\mathbf{t} \in \mathbb{R}^n$ we have $\mathsf{E}\, |\mathbf{t}'\mathbf{X}|^r = c^r(\mathbf{t})\, \mathsf{E}\, |Y|^r < \infty$ which can be rewritten as

$$c(\mathbf{t}) = \left(\mathsf{E}\, |Y|^r\right)^{-\frac{1}{r}} \left(\mathsf{E}\, |\mathbf{t}'\mathbf{X}|^r\right)^{\frac{1}{r}}, \quad \mathbf{t} \in \mathbb{R}^n.$$

Now plugging it into the characteristic function of $\mathbf{X}$ results in

$$\begin{aligned}
\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} &= e^{-c^\alpha(\mathbf{t})} \\
&= \exp\left\{-\left(\mathsf{E}\, |Y|^r\right)^{-\frac{\alpha}{r}} \left(\mathsf{E}\, |\mathbf{t}'\mathbf{X}|^r\right)^{\frac{\alpha}{r}}\right\} \\
&= \exp\left\{-\left(\mathsf{E}\, |Y|^r\right)^{-\frac{\alpha}{r}} \left(\int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^r\ P_\mathbf{X}(\mathbf{x})\right)^{\frac{\alpha}{r}}\right\}
\end{aligned}$$

where $P_\mathbf{X}$ is the distribution of $\mathbf{X}$.

Transform $P_\mathbf{X}$ from $\mathbb{R}^n$ to a measure μ_r on $S_\infty^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty = 1\}$ such that for any Borel subset A of S_∞^{n-1}

$$\mu_r(A) = \int_{\left\{\mathbf{x}\in\mathbb{R}^n : \frac{\mathbf{x}}{\|\mathbf{x}\|_\infty} \in A\right\}} \frac{1}{\mathsf{E}\, |Y|^r} \|\mathbf{x}\|_\infty^r\ dP_\mathbf{X}(\mathbf{x}). \tag{1.7}$$

A function defined by (1.7) is a finite measure as it is non-negative, σ-additive, $\mu_r(\emptyset) = 0$, and $\mu_r(S_\infty^{n-1}) < \infty$ (the last property will be checked later, others follow $P_\mathbf{X}$). Then for any bounded measurable function $g : S_\infty^{n-1} \to \mathbb{R}$ we have

$$\int_{S_\infty^{n-1}} g(\mathbf{y})\ d\mu_r(\mathbf{y}) = \frac{1}{\mathsf{E}\, |Y|^r} \int_{\mathbb{R}^n} g\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_\infty}\right) \|\mathbf{x}\|_\infty^r\ dP_\mathbf{X}(\mathbf{x}).$$

We shall use the previous result for the function $g_\mathbf{t}(\mathbf{y}) = |\mathbf{t}'\mathbf{y}|^r$ which is bounded on S_∞^{n-1}, thus

$$\begin{aligned}
c^r(\mathbf{t}) &= \frac{1}{\mathsf{E}\, |Y|^r} \int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^r\ P_\mathbf{X}(\mathbf{x}) \\
&= \frac{1}{\mathsf{E}\, |Y|^r} \int_{\mathbb{R}^n} \left|\mathbf{t}'\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_\infty}\right)\right|^r \|\mathbf{x}\|_\infty^r\ P_\mathbf{X}(\mathbf{x}) \\
&= \int_{S_\infty^{n-1}} |\mathbf{t}'\mathbf{y}|^r\ d\mu_r(\mathbf{y}), \quad \mathbf{t} \in \mathbb{R}^n.
\end{aligned}$$

[14]Similar properties are revisited in Chapter 4.

Let us use the previous formula to bound $\mu_r(S^{n-1}_\infty)$ using elementary vectors as

$$\begin{aligned}\sum_{k=1}^{n} c^r(\mathbf{e}_k) &= \sum_{k=1}^{n}\int_{S^{n-1}_\infty} |y_k|^r \; d\mu_r(\mathbf{y}) \\ &= \int_{S^{n-1}_\infty} \sum_{k=1}^{n} |y_k|^r \; d\mu_r(\mathbf{y}) \\ &\geq \inf_{(x_1,\ldots,x_n)'\in S^{n-1}_\infty} \left[\sum_{k=1}^{n} |x_k|^r\right]\cdot \mu_r(S^{n-1}_\infty)\end{aligned}$$

which means $\mu_r(S^{n-1}_\infty) \leq \left(\inf_{\mathbf{x}\in S^{n-1}_\infty} \|\mathbf{x}\|_r^r\right)^{-1} \sum_{k=1}^{n} c^r(\mathbf{e}_k)$, since the infimum is positive. By that we have $\sup_{r<\alpha} \mu_r(S^{n-1}_\infty) < \infty$.

For any positive sequence $\{r_k\}$ satisfying $r_k \nearrow \alpha$ the sequence of measures $\{\mu_{r_k}\}$ is tight (by the previous result) and we may find a weakly convergent subsequence. If we denote the limit μ and extend μ as a measure on $\mathbb{R}^n$ (defining $\mu(A) = \mu(A \cap S^{n-1}_\infty)$, for $A \subset \mathbb{R}^n$), we have

$$c^\alpha(\mathbf{t}) = \int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^\alpha \; d\mu(\mathbf{x})$$

which implies (1.6).

□

The standard *Lévy spectral representation for symmetric stable vectors* is defined using a measure ν over a unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$ as in the following instances. Let us now combine the theory of characteristic functions with isometric embedding into $L_\alpha(S^{n-1}, \nu)$.

The quasi-normed space $(\mathbb{R}^n, \rho)$ embeds isometrically into $L_\alpha(S^{n-1}, \nu)$, $\alpha \in (0, 2]$, via the linear operator $T : \mathbb{R}^n \to L^\alpha(S^{n-1}, \nu)$ which maps a vector $\mathbf{t} \in \mathbb{R}^n$ to a linear function $f_\mathbf{t} : S^{n-1} \to \mathbb{R}$ where $f_\mathbf{t}(\mathbf{x}) = \mathbf{t}'\mathbf{x}$, $\mathbf{x} \in S^{n-1}$, if

$$\rho(\mathbf{t}) = \left(\int_{S^{n-1}} |\mathbf{t}'\mathbf{x}|^\alpha d\nu(\mathbf{x})\right)^{\frac{1}{\alpha}}, \quad \mathbf{t} \in \mathbb{R}^n. \tag{1.8}$$

This representation is known as *Blaschke-Lévy representation* of a norm (details and the inverse formula in Koldobsky [1997a]). We further assume that ν is not concentrated on any sub-sphere in order to ensure that $\rho(\mathbf{x}) > 0$ outside origin (so ρ may satisfy the definition of a quasi-norm). The relationship between isometric embedding and stability is summarized in Theorem 10 and relies on Theorem 9 and Misiewicz [1996], Remark II.1.1.

Theorem 10. *An n-dimensional random vector* $\mathbf{X}$ *is symmetric* α*-stable,* $\alpha \in (0, 2]$*, if and only if there exists a symmetric finite measure* ν *over* $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$ *such that*

$$E\, e^{i\mathbf{t}'\mathbf{X}} = \exp\left\{-\int_{S^{n-1}} |\mathbf{t}'\mathbf{x}|^\alpha d\nu(\mathbf{x})\right\}, \quad \mathbf{t} \in \mathbb{R}^n.$$

The measure ν *is unique for* $\alpha < 2$.

Utilizing the Blaschke-Lévy representation (1.8)*, a quasi-normed space* $(\mathbb{R}^n, \rho)$ *isometrically embeds into* $L_\alpha(S^{n-1}, \nu)$ *(via the operator* $T : \mathbf{t} \in \mathbb{R}^n \mapsto f_\mathbf{t} \in L_\alpha(S^{n-1}, \nu)$*,* $f_\mathbf{t}(\mathbf{x}) = \mathbf{t}'\mathbf{x}$*,* $\mathbf{x} \in S^{n-1}$*) if and only if* $e^{-\rho^\alpha(\mathbf{t})}$*,* $\mathbf{t} \in \mathbb{R}^n$*, is a characteristic function.*

The proof of Theorem 9 is partially covered by the proof of Theorem 10 (with the exception of uniqueness of the measure ν). However, we may find simple examples where the uniqueness is violated for $\alpha = 2$. For that we may take $\mathbf{X}$ with a standard normal distribution with a characteristic function $e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}}$, $\mathbf{t} \in \mathbb{R}^n$, and the spectral measure ν satisfies

$$\frac{1}{2}\mathbf{t}'\mathbf{t} = \int_{S^{n-1}} |\mathbf{t}'\mathbf{x}|^2 \ \nu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n.$$

Both a multiple of the Lebesgue measure on S^{n-1} and a measure concentrated on $\pm\mathbf{e}_k$, $k = 1, \ldots, n$, satisfy the representation (Misiewicz [1996], Example II.1.1).

Let us introduce a convention. From here when an L_p-space is mentioned we mean $L_p(S^{n-1}, \nu)$ for some finite symmetric measure ν on S^{n-1}. The following lemma clarifies the possibility of embedding of an L_p-space into another L_q-space.

Lemma 11. *Let ρ be a quasi-norm such that the space $(\mathbb{R}^n, \rho)$ embeds isometrically into L_p for some $p \in (0, 2]$. Then it also embeds into L_q for any $q \in (0, p)$.*

Proof. Our aim is to find a random vector with a characteristic function $\exp\{-\rho(\mathbf{t})^q\}$ if we know that $\exp\{-\rho(\mathbf{t})^p\}$ is a characteristic function of a random vector which will be denoted $\mathbf{X}_p$.

Let Z be a non-negative r-stable random vector where $r = q/p < 1$ independent of $\mathbf{X}_p$. Let us find the characteristic function of $Z^{1/p} \cdot \mathbf{X}_p$. By the law of total probability and the fact that the Laplace transform of Z is e^{-t^r} thanks to (1.2), we have

$$\begin{aligned} \mathsf{E}\ e^{i\mathbf{t}'(Z^{1/p}\mathbf{X}_p)} &= \mathsf{E}\ \mathsf{E}\left[e^{i(Z^{1/p}\mathbf{t})'\mathbf{X}_p}|Z\right] = \mathsf{E}\ e^{-\rho(Z^{1/p}\mathbf{t})^p} \\ &= \mathsf{E}\ e^{-\rho(\mathbf{t})^p Z} = e^{-\rho(\mathbf{t})^{pr}} = e^{-\rho(\mathbf{t})^q}, \quad \mathbf{t} \in \mathbb{R}^n \end{aligned}$$

since $Z \geq 0$ a.s. and ρ is homogeneous by Definition 9.

□

Remark 3. The construction in the previous lemma is called *substability.* A symmetric α-stable random vector $\mathbf{X}$ is called β*-substable*, $0 < \alpha < \beta \leq 2$, if $\mathbf{X} \stackrel{d}{=} Z^{\frac{1}{\beta}}\mathbf{Y}$ where $\mathbf{Y}$ is symmetric β-stable, Z is non-negative $\frac{\alpha}{\beta}$-stable, $\mathbf{Y}$ and Z are independent (Misiewicz and Takenaka [2002]).

A symmetric α-stable random vector $\mathbf{X}$ is called *maximal* if for any $\beta \geq \alpha$ the previous decomposition $\mathbf{X} \stackrel{d}{=} Z^{\frac{1}{\beta}}\mathbf{Y}$ with any symmetric β-stable $\mathbf{Y}$ implies $\alpha = \beta$ and Z is almost surely constant. Misiewicz and Takenaka [2002] further characterized the maximal symmetric α-stable distributions with a characteristic function $e^{-\rho^\alpha(\mathbf{t})}$ using the spectral measure of the Blaschke-Lévy representation (1.8). If the spectral measure is purely atomic, the symmetric α-stable random vector is maximal (Misiewicz and Takenaka [2002]).

By that the vector $(X_1, \ldots, X_n)'$ where X_i are i.i.d. symmetric α-stable, $\alpha \in (0, 2]$, is maximal since its spectral measure is concentrated on $\pm\mathbf{e}_i$, its characteristic function is equal to $e^{-(|t_1|^\alpha + \cdots + |t_n|^\alpha)}$, $\mathbf{t} = (t_1, \ldots, t_n)' \in \mathbb{R}^n$.

The embedding into L_p-spaces was thoroughly studied in Koldobsky [1991]. In the most simple case, any two-dimensional normed space embeds into L_1-space. Theory of isometric embedding states that $e^{-\rho(t_1, t_2)}$ must be a characteristic function of a symmetric 1-stable distribution for any norm (Ferguson [1962]).

Theorem 12. *Let $\rho : \mathbb{R}^2 \to [0,\infty)$ be such that $\rho(t_1,t_2) = 0$ if and only if $t_1 = t_2 = 0$. Then $e^{-\rho(t_1,t_2)}$ is a characteristic function of a symmetric 1-stable random vector if and only if ρ is a norm.*

Proof. Let $e^{-\rho(t_1,t_2)}$ be a characteristic function of $\mathbf{X} = (X_1, X_2)'$ which is symmetric 1-stable, thus by the Blaschke-Lévy representation (1.8)

$$\rho(t_1,t_2) = \int_{S^1} |t_1x_1 + t_2x_2| \, d\mu(x_1,x_2), \quad (t_1,t_2)' \in \mathbb{R}^2,$$

the function ρ is non-negative, positive homogeneous of degree 1 and satisfies the triangle inequality which is evident as the integrand satisfies $|(\mathbf{t}_1 + \mathbf{t}_2)'\mathbf{x}| \le |\mathbf{t}_1'\mathbf{x}| + |\mathbf{t}_2'\mathbf{x}|$ for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^2$ and $\mathbf{x} \in S^1$. This concludes the first implication.

Conversely, let ρ be a non-trivial[15] norm and $B_\rho = \{(t_1,t_2)' \in \mathbb{R}^2 : \rho(t_1,t_2) \le 1\}$ its unit ball which is convex. Let us denote the distance from the origin to the contour $\rho(\cdot,\cdot) = 1$ in direction given by angle $\theta \in [-\pi,\pi)$ by $r(\theta)$ and its reciprocal by $h(\theta) = \rho(\cos(\theta), \sin(\theta))$ which is positive and π-periodic.

Our aim is to approximate the unit ball B_ρ from the inside by convex origin-symmetric polygons with vertices on the boundary of B_ρ as in Figure 1.2. For $n \in \mathbb{N}$ let $-\frac{1}{2}\pi = \theta_0^{(n)} < \theta_1^{(n)} < \cdots < \theta_{n-1}^{(n)} < \theta_n^{(n)} = \frac{1}{2}\pi$ be a partition of $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ which will define the vertices of the polygon P_n as intersection of $\rho(\cdot,\cdot) = 1$ and the line in the direction $(\cos(\theta_i^{(n)}), \sin(\theta_i^{(n)}))'$, i.e. $\{(t_1,t_2)' \in \mathbb{R}^2 : t_1 \sin(\theta_i^{(n)}) - t_2 \cos(\theta_i^{(n)}) = 0\}$, $i = 1, \ldots, n$.

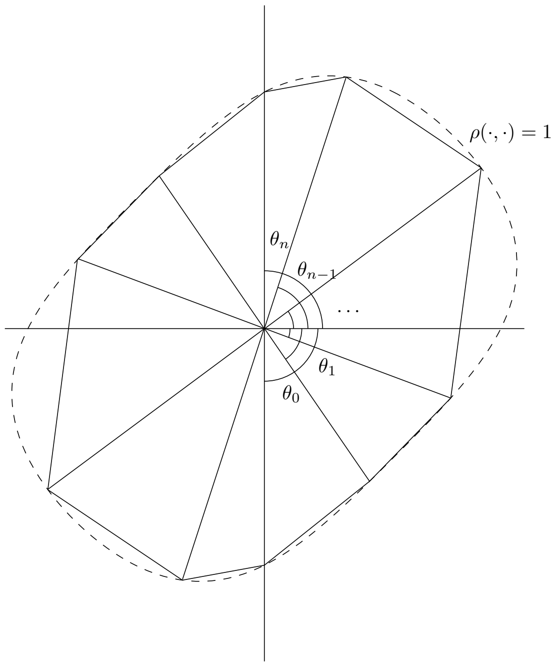


Figure 1.2: Approximation of B_ρ using symmetric convex polygons.

Let us show that we can write the polygon with such vertices as a lower-level set of a convex function:

$$P_n = \left\{ (t_1,t_2)' \in \mathbb{R}^2 : \sum_{i=1}^{n} |a_i^{(n)} t_1 + b_i^{(n)} t_2| \le 1 \right\}.$$

[15]For a trivial norm, i.e. $\rho(t_1,t_2) = 0$ for each t_1, t_2, the function $e^{-\rho}$ is equal to 1, the trivial characteristic function.

Since vertices are defined by directions $\theta_i^{(n)}$, $i = 1, \ldots, n$, the coefficients satisfy $a_i^{(n)} = r_i^{(n)} \sin(\theta_i^{(n)})$ and $b_i^{(n)} = -r_i^{(n)} \cos(\theta_i^{(n)})$ and we aim to show that $r_i^{(n)} \geq 0$, $i = 1, \ldots, n$. For that let us work with the equation for the boundary of P_n and plug the vertices with coordinates $(t_{1,k}, t_{2,k})' = (r(\theta_k^{(n)}) \cos(\theta_k^{(n)}), r(\theta_k^{(n)}) \sin(\theta_k^{(n)}))'$. Then for any fixed $k = 1, \ldots, n$

$$\sum_{i=1}^{n} |a_i^{(n)} t_{1,k} + b_i^{(n)} t_{2,k}| = 1$$

$$\sum_{i=1}^{n} r_i^{(n)} r(\theta_k^{(n)}) |\sin(\theta_i^{(n)}) \cos(\theta_k^{(n)}) - \cos(\theta_i^{(n)}) \sin(\theta_k^{(n)})| = 1$$

$$\sum_{i=1}^{n} r_i^{(n)} |\sin(\theta_i^{(n)} - \theta_k^{(n)})| = h(\theta_k^{(n)})$$

$$\sum_{i=1}^{k} r_i^{(n)} \sin(\theta_k^{(n)} - \theta_i^{(n)}) + \sum_{i=k+1}^{n} r_i^{(n)} \sin(\theta_i^{(n)} - \theta_k^{(n)}) = h(\theta_k^{(n)}).$$

For $r_i^{(n)}$, $i = 1, \ldots, n$ the equation defines a system of linear equations whose solution is non-negative (explicitly written in Ferguson [1962]).

As $n \to \infty$, the mesh of the partition tends to 0 and

$$\sum_{i=1}^{n} |a_i^{(n)} t_1 + b_i^{(n)} t_2| \to \rho(t_1, t_2)$$

and by that

$$\exp\left\{ -\sum_{i=1}^{n} |a_i^{(n)} t_1 + b_i^{(n)} t_2| \right\} \to \exp\{-\rho(t_1, t_2)\}. \tag{1.9}$$

Since $e^{-\rho(t_1,t_2)}$ is a continuous function it remains to show that the left-hand side is a characteristic function. For that denote $Z_1, \ldots, Z_n$ i.i.d. Cauchy distributed random variables and set $Y_1 = \sum_{i=1}^n a_i^{(n)} Z_i$, $Y_2 = \sum_{i=1}^n b_i^{(n)} Z_i$. The left-hand side of (1.9) is the characteristic function of $(Y_1, Y_2)'$ since

$$\mathsf{E}\, e^{i(t_1 Y_1 + t_2 Y_2)} = \mathsf{E} \exp\left\{ i \sum_{i=1}^{n} \left(a_i^{(n)} t_1 + b_i^{(n)} t_2 \right) Z_i \right\} = \exp\left\{ -\sum_{i=1}^{n} |a_i^{(n)} t_1 + b_i^{(n)} t_2| \right\}$$

for $(t_1, t_2)' \in \mathbb{R}^2$ which concludes the proof.

□

Remark 4. The first condition $\rho(t_1, t_2) = 0$ if and only if $t_1 = t_2 = 0$ is not necessary in order to define a characteristic function of a symmetric 1-stable random vector. However, it prevents degenerate solutions. As an example we can take the singular random vector $\mathbf{X} = (X_1, X_1)'$ where X_1 has a Cauchy distribution. The characteristic function of $\mathbf{X}$ is $e^{-|t_1+t_2|}$. The function

$$(t_1, t_2) \mapsto |t_1 + t_2|$$

is positive homogeneous of degree 1 and satisfies the triangle inequality. The spectral representation $\mu = \frac{1}{2}\delta_{(1,1)'} + \frac{1}{2}\delta_{(-1,-1)'}$ is concentrated on a sub-sphere.

The assumption of $\rho(t_1, t_2) = 0$ if and only if $t_1 = t_2 = 0$ is added in order to avoid similar degenerate cases.

Remark 5. The measure μ from the Blaschke-Lévy representation (1.8) can be written explicitly (Ferguson [1962], Misiewicz and Ryll-Nardzewski [1987]) using the function $h(\theta) = \rho(\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$, if h is twice differentiable. The measure μ satisfies

$$\begin{aligned}\rho(t_1, t_2) &= \int_{S^1} |t_1 x_1 + t_2 x_2| \; d\mu(x_1, x_2) \\ &= \int_0^{2\pi} |t_1 \cos(\phi) + t_2 \sin(\phi)| \; d\tilde{\mu}(\phi), \quad t_1, t_2 \in \mathbb{R},\end{aligned}$$

by changing into spherical coordinates $t_1 = r\cos(\theta)$, $t_2 = r\sin(\theta)$ and rewriting $\rho(t_1, t_2) = r \cdot \rho(\cos(\theta), \sin(\theta)) = rh(\theta)$. The measure $\tilde{\mu}$ corresponds to μ through the transformation of spherical coordinates from S^1 to $[0, 2\pi)$. Then

$$\begin{aligned}\rho(t_1, t_2) &= \int_0^{2\pi} |t_1 \cos(\phi) + t_2 \sin(\phi)| \; d\tilde{\mu}(\phi) \\ r \cdot h(\theta) &= r \cdot \int_0^{2\pi} |\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)| \; d\tilde{\mu}(\phi) \\ h(\theta) &= \int_0^{2\pi} |\cos(\phi - \theta)| \; d\tilde{\mu}(\phi) \qquad (1.10)\end{aligned}$$

where the last equality holds due to a common trigonometric identity. Let us show that the density of $\tilde{\mu}$ is equal to

$$\frac{1}{4}\left(h''\left(\phi - \frac{\pi}{2}\right) + h\left(\phi - \frac{\pi}{2}\right)\right) \qquad (1.11)$$

which is positive due to convexity of B_ρ. The right-hand side of (1.10) is equal to

$$\begin{aligned}\int_0^{2\pi} |\cos(\phi - \theta)| \; d\tilde{\mu}(\phi) &= \int_0^{2\pi} |\cos(\phi - \theta)| \frac{1}{4}\left(h''\left(\phi - \frac{\pi}{2}\right) + h\left(\phi - \frac{\pi}{2}\right)\right) \; d\phi \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos(\phi - \theta)\left(h''\left(\phi - \frac{\pi}{2}\right) + h\left(\phi - \frac{\pi}{2}\right)\right) d\phi.\end{aligned}$$

Both integrals (the summands) are solved by per partes:

$$\begin{aligned}\int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos(\phi - \theta) h''\left(\phi - \frac{\pi}{2}\right) \; d\phi &= \left[\cos(\phi - \theta) h'\left(\phi - \frac{\pi}{2}\right)\right]_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \\ &\quad + \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin(\phi - \theta) h'\left(\phi - \frac{\pi}{2}\right) \; d\phi, \qquad (1.12) \\ \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \cos(\phi - \theta) h\left(\phi - \frac{\pi}{2}\right) \; d\phi &= \left[\sin(\phi - \theta) h'\left(\phi - \frac{\pi}{2}\right)\right]_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \\ &\quad - \int_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta} \sin(\phi - \theta) h'\left(\phi - \frac{\pi}{2}\right) \; d\phi. \qquad (1.13)\end{aligned}$$

The remaining integrals on the right-hand sides of (1.12) and (1.13) cancel out as well as the term containing

$$\left[\cos(\phi - \theta) h'\left(\phi - \frac{\pi}{2}\right)\right]_{-\frac{\pi}{2}+\theta}^{\frac{\pi}{2}+\theta}$$

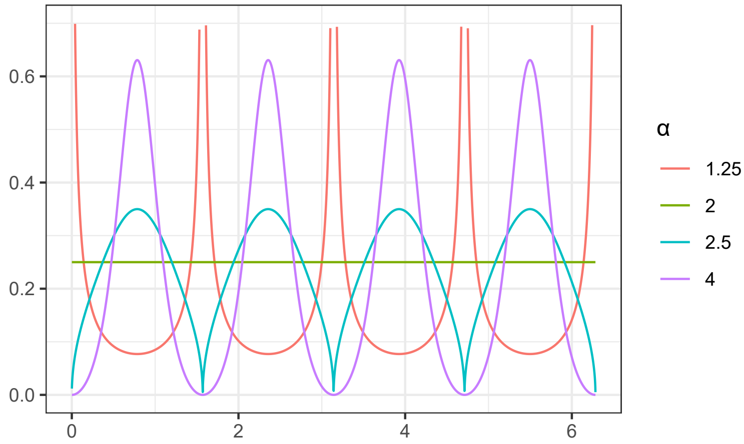


Figure 1.3: Densities of $\tilde{\mu}$ for $\alpha \in \{1.25, 2, 2.5, 4\}$.

since both cosines are zeros. Thus, $\int_0^{2\pi} |\cos(\phi-\theta)|\, d\tilde{\mu}(\phi)$ is equal to $\frac{1}{2}(\sin(\frac{\pi}{2})h(\theta) - \sin(\frac{-\pi}{2})h(\theta-\pi)) = h(\theta)$ from the periodicity of h.

If h is not twice differentiable, the measure $\tilde{\mu}$ can be approximated (Misiewicz and Ryll-Nardzewski [1987]).

For $\alpha \geq 1$ the function $\|\cdot\|_\alpha$ is a norm and we may directly compute the density of $\tilde{\mu}$.

Example 4. Let $\rho(t_1, t_2) = \|(t_1, t_2)'\|_\alpha$ be the α-norm, $\alpha > 1$. Then

$$h(\theta) = \|(\cos\theta, \sin\theta)'\|_\alpha = (|\cos(\theta)|^\alpha + |\sin(\theta)|^\alpha)^{\frac{1}{\alpha}} = h(\theta - \pi/2)$$

and the density of $\tilde{\mu}$ (1.11) is equal to

$$\frac{\alpha-1}{4} |\sin(\theta)\cos(\theta)|^{\alpha-2} \left(|\cos(\theta)|^\alpha + |\sin(\theta)|^\alpha\right)^{\frac{1}{\alpha}-2}, \quad \theta \in (0, 2\pi). \tag{1.14}$$

For $\alpha = 1$ the measure μ is not absolutely continuous but it can be taken as $\mu = \frac{1}{4}(\delta_{(1,0)'} + \delta_{(-1,0)'} + \delta_{(0,1)'} + \delta_{(0,-1)'})$. In this case, as the measure is purely atomic, the random vector with a characteristic function $e^{-\|(t_1,t_2)'\|_1}$ is maximal[16] which means $e^{-\|(t_1,t_2)'\|_1^\alpha}$ is a characteristic function if and only if $\alpha \leq 1$.

Figure 1.3 shows the density (1.14) for several values of α. It can be seen that as $\alpha \to 1$ and $\alpha \to \infty$ the limit in both cases would be atomic which corresponds to the discussion for $\alpha = 1$ ($\alpha = \infty$ is linked to $\alpha = 1$ via the isometry from Example 3). The measure is uniform for $\alpha = 2$.

The isometric embeddings are further used in Chapter 4 which includes further generalization[17] and necessary conditions for embedding into L_p-spaces using partial derivatives of the norm.

[16] See Remark 3.

[17] Such as an embedding into L_0, see Theorem 48.

2. α-Symmetry

This chapter includes known results about general α-symmetry in $\mathbb{R}^n$ for any $\alpha > 0$ and $n \in \mathbb{N}$. Fang et al. [1990] present some of the properties in Chapter 7. Other properties for specific pairs of $n \in \mathbb{N}$ and $\alpha > 0$ are discussed in Chapter 3.

The main result of the first section, Lemma 13, offers a key result in terms of α-symmetry. Definition 12 and Lemma 13 resemble symmetric stable random vectors and the connection is described in Theorem 15 and Example 6. Section 2.2 derives an integral expression for the density of an α-symmetric random vector. The multivariate integration given by Theorem 2 is reduced to univariate integration.

2.1 Definitions

Definition 12. *A random vector* $\mathbf{X} = (X_1, \dots, X_n)'$ *has an* α-symmetric *distribution,* $\alpha > 0$*, if its characteristic function can be written as a univariate function of the* α*-norm* (1.4)*, i.e.*

$$\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} = \psi\left((|t_1|^\alpha + \cdots + |t_n|^\alpha)^{\frac{1}{\alpha}}\right) = \psi(\|\mathbf{t}\|_\alpha), \quad \mathbf{t} \in \mathbb{R}^n.$$

The function $\psi : [0, \infty) \to \mathbb{R}$ *is called the* characteristic generator *of the random vector* $\mathbf{X}$*.*

The class of characteristic generators of n-dimensional α-symmetric distributions is denoted by $S(n, \alpha)$. Since the characteristic function fully determines the distribution and the relationship is bijective, the set of n-dimensional α-symmetric distributions will be referred to simply by $S(n, \alpha)$ for brevity. When necessary we may write $\mathbf{X} \sim S(n, \alpha, \psi)$.

Remark 6. Alternatively, we may express the characteristic function of an α-symmetric random vector as $\varphi(|t_1|^\alpha + \cdots + |t_n|^\alpha)$. The connection between ψ and φ is imminent. Using the norm in the definition allows $\alpha = \infty$ where $\|\mathbf{t}\|_\infty = \max\{|t_1|, \dots, |t_n|\}$ or even a generalization into other quasi-norms.[1]

The most notable examples of α-symmetric distributions are:

Example 5. For $\alpha = 2$ the so-called *spherically symmetric* distributions were extensively discussed in Chapter 2 of Fang et al. [1990] and Ranošová [2021]. The random vector is spherically symmetric if and only if

$$\mathbf{X} \stackrel{d}{=} R\mathbf{U}$$

where $R \geq 0$ is independent of $\mathbf{U}$ uniformly distributed on the unit sphere S^{n-1} in $\mathbb{R}^n$.

Example 6. Let $X_1, \dots, X_n$ be i.i.d. symmetric α-stable variables (1.1), $\alpha \in (0, 2]$. Then the vector $\mathbf{X} = (X_1, \dots, X_n)'$ has an α-symmetric distribution, since the characteristic function of a symmetric α-stable distribution is $e^{-C|t|^\alpha}$ for some $C > 0$, the characteristic function of $\mathbf{X}$ is $e^{-C\|\mathbf{t}\|_\alpha^\alpha}$, $\mathbf{t} \in \mathbb{R}^n$, for some $C > 0$.

[1] See Chapter 4.

Other α-symmetric distributions which are also symmetric stable are further discussed in the Schoenberg problem (established in Example 7) which connects stability and α-symmetry.

The following lemma summarizes the crucial property of α-symmetric distributions and is taken from Fang et al. [1990] (Theorem 7.1).

Lemma 13. *The random vector* $\mathbf{X} = (X_1, \ldots, X_n) \sim S(n, \alpha, \psi)$ *if and only if* $\mathbf{c}'\mathbf{X} \sim S(1, \alpha, \psi)$ *for all* $\mathbf{c} \in \mathbb{R}^n$ *with* $\|\mathbf{c}\|_\alpha = 1$*. The* α*-symmetric random vector is fully determined by any of its marginals.*

Proof. The proof follows Lemma 3: If A is an $m \times n$ matrix and $\varphi_{\mathbf{X}}$ is a characteristic function of an n-dimensional random vector $\mathbf{X}$, then the characteristic function of $A\mathbf{X}$ is $\varphi_{A\mathbf{X}}(\mathbf{s}) = \varphi_{\mathbf{X}}(A'\mathbf{s})$, $\mathbf{s} \in \mathbb{R}^m$. If the characteristic function of $\mathbf{X}$ is $\psi(\|\mathbf{t}\|_\alpha)$ then the characteristic function of $\mathbf{c}'\mathbf{X}$ for any $\mathbf{c} \in \mathbb{R}^n$, $\|\mathbf{c}\|_\alpha = 1$, is $\psi(\|t\mathbf{c}\|_\alpha) = \psi(|t|)$, $t \in \mathbb{R}$.

Conversely, if $\mathbf{c}'\mathbf{X}$ has the same distribution for any $\mathbf{c} \in \mathbb{R}^n$,$\|\mathbf{c}\|_\alpha = 1$, we have $\mathbf{c}'\mathbf{X} \stackrel{d}{=} -\mathbf{c}'\mathbf{X}$ and the distribution is symmetric. Denote $\psi(|\cdot|)$ the characteristic function corresponding to the distribution of $\mathbf{c}'\mathbf{X}$ for any $\mathbf{c} \in \mathbb{R}^n$, $\|\mathbf{c}\|_\alpha = 1$, and we have $\mathbf{c}'\mathbf{X} \sim S(1, \alpha, \psi)$ for all $\mathbf{c} \in \mathbb{R}^n$ with $\|\mathbf{c}\|_\alpha = 1$. Then for $\mathbf{t} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we have

$$\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} = \mathsf{E}\, e^{i\|\mathbf{t}\|_\alpha \left(\frac{\mathbf{t}}{\|\mathbf{t}\|_\alpha}\right)'\mathbf{X}} = \psi(\|\mathbf{t}\|_\alpha)$$

and the random vector is α-symmetric since $\mathbf{t}/\|\mathbf{t}\|_\alpha$ has an α-norm equal to 1. □

Following the proof of Lemma 13 any marginal vector of an α-symmetric random vector is also α-symmetric (with the same characteristic generator) which for $m < n$ implies $S(n, \alpha) \subset S(m, \alpha)$ and we may define $S(\infty, \alpha)$ as $\cap_{n=1}^{\infty} S(n, \alpha)$. The class $S(\infty, \alpha)$ are characteristic generators in any dimension, alternatively, it can be seen as characteristic generators in the spaces of sequences $\ell_\alpha = \{(x_1, x_2, \ldots)' \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^\alpha < \infty\}$. On the other hand, the class $S(1, \alpha)$ includes all symmetric one-dimensional distributions for any α.

Corollary 3. The random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ is α-symmetric if and only if $\mathbf{c}'\mathbf{X} \stackrel{d}{=} \|\mathbf{c}\|_\alpha X_1$, for each $\mathbf{c} \in \mathbb{R}^n$.

As an extension to the preceding corollary a generalization of the α-symmetry was proposed by Eaton [1981]. The distribution of a random vector $\mathbf{X}$ in $\mathbb{R}^n$ is the *n-dimensional version* of a random variable Y if there exists a function $\gamma : \mathbb{R}^n \to [0, \infty)$ such that

$$\mathbf{t}'\mathbf{X} \stackrel{d}{=} \gamma(\mathbf{t})Y, \quad \mathbf{t} \in \mathbb{R}^n, \tag{2.1}$$

where $\gamma(\mathbf{t}) > 0$ for $\mathbf{t} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Clearly, Y is a scaled version of the marginal random variable X_1. The random vector satisfying (2.1) for some γ will be called *pseudo-isotropic*[2] and the function γ is called a *standard*. The pseudo-isotropic distributions are further discussed in Chapter 4. The next lemma shows the properties of $S(n, \alpha)$ in the space of characteristic functions.

[2]The name is due to Misiewicz as isotropy refers to spherical symmetry.

Lemma 14. *For each $\alpha \in (0, \infty]$ and $n \in \mathbb{N}$ the set $S(n, \alpha)$ is a convex closed subset of all characteristic functions on $\mathbb{R}$.*

Proof. A convex linear combination of characteristic functions is a characteristic function with a generator given by the linear combination of the generators.

Similarly, by Lemma 3 let ψ_n converge to ψ in $\mathbb{R}$ such that $\psi(|\cdot|)$ is a characteristic function. Then $\psi_n(\|\cdot\|_\alpha)$ converges $\psi(\|\cdot\|_\alpha)$ in $\mathbb{R}^n$, since $\psi(\|\cdot\|_\alpha)$ is continuous at zero (Lemma 3).

□

The components of an α-symmetric random vector may be independent only in a very special case, thus vectors of i.i.d. random variables are among the most researched examples among α-symmetry.

Theorem 15. *The α-symmetric random vector $\mathbf{X}$ contains two independent subvectors if and only if the marginal random variables are i.i.d. symmetric α-stable.*

Proof. Denote X_1 and X_2 two independent marginal random variables, each from one of the independent subvectors and ψ is the characteristic generator. Then the characteristic function of $(X_1, X_2)'$ has the same generator and satisfies for $t_1, t_2 \in \mathbb{R}$

$$\psi\left((|t_1|^\alpha + |t_2|^\alpha)^{\frac{1}{\alpha}}\right) = \psi(|t_1|)\psi(|t_2|)$$

since X_1 and X_2 are independent. Substitute $g(u) = \psi\left(u^{\frac{1}{\alpha}}\right)$ and $u_i = |t_i|^\alpha$, then for $u_1, u_2 > 0$ we have

$$g(u_1 + u_2) = \psi\left((u_1 + u_2)^{\frac{1}{\alpha}}\right) = \psi\left(u_1^{\frac{1}{\alpha}}\right)\psi\left(u_2^{\frac{1}{\alpha}}\right) = g(u_1)g(u_2).$$

The continuous positive solutions to *Cauchy's multiplicative functional equation*[3] are $g(u) = e^{cu}$ for some $c \in \mathbb{R}$, which implies $\psi(t) = e^{c|t|^\alpha}$. Since the characteristic function is bounded the constant c is negative. Thus $\psi(t) = e^{-C|t|^\alpha}$, $C > 0$, which is a characteristic function of the symmetric α-stable distribution.

The converse implication is trivial.

□

The following theorem gives general instructions on how to create other α-symmetric distributions. The process is used multiple times throughout the thesis when constructing examples of α-symmetric distributions.

Theorem 16. *Let $\mathbf{X} \stackrel{d}{=} R\mathbf{Y}$ where the random variable $R > 0$ and the random vector $\mathbf{Y}$ are independent. Then if $\mathbf{Y}$ is α-symmetric, $\mathbf{X}$ is α-symmetric as well. Conversely, if $\mathbf{X}$ is α-symmetric and $\mathsf{E}\left(R^{it}\right) \neq 0$ for almost all t with respect to the Lebesgue measure, then $\mathbf{Y}$ is α-symmetric.*

[3] By additionally applying logarithm on both sides $h(u_1 + u_1) = h(u_1) + h(u_2)$ where $h = \log(g)$ the equation becomes *Cauchy's functional equation* whose continuous solutions are known to be linear functions.

Proof. Let us find a characteristic function of $R\mathbf{Y}$ where $\mathbf{Y} \sim S(n, \alpha, \psi)$ is independent of R. By the law of total expectation

$$\begin{aligned} \mathsf{E}\left(e^{i\mathbf{t}'\mathbf{X}}\right) &= \mathsf{E}\left(e^{i\mathbf{t}'(R\mathbf{Y})}\right) = \mathsf{E}\left(e^{i(R\mathbf{t})'\mathbf{Y}}\right) \\ &= \mathsf{E}\left[\mathsf{E}\left(e^{i(R\mathbf{t})'\mathbf{Y}} \mid R\right)\right] = \mathsf{E}\left[\psi(\|R\mathbf{t}\|_\alpha)\right] \end{aligned}$$

which is a function of $\|\mathbf{t}\|_\alpha$ and the random vector $\mathbf{X} \stackrel{d}{=} R\mathbf{Y}$ is α-symmetric since its characteristic function is a univariate function of the norm. If R has a density g on $(0, \infty)$, the characteristic generator of $R\mathbf{Y}$ is equal to

$$u \mapsto \int_0^\infty \psi(ru)g(r)\, dr, \quad u \geq 0.$$

If $\mathbf{X}$ is α-symmetric and $\mathbf{X} \stackrel{d}{=} R\mathbf{Y}$ for some independent R and $\mathbf{Y}$, then for $\mathbf{c}$ such that $\|\mathbf{c}\|_\alpha = 1$ we have

$$R\mathbf{c}'\mathbf{Y} \stackrel{d}{=} \mathbf{c}'(R\mathbf{Y}) \stackrel{d}{=} \mathbf{c}'\mathbf{X} \stackrel{d}{=} X_1 \stackrel{d}{=} RY_1.$$

For the final argument that $R\mathbf{c}'\mathbf{Y} \stackrel{d}{=} RY_1$ implies $\mathbf{c}'\mathbf{Y} \stackrel{d}{=} Y_1$ denote

$$W_X(t) = \begin{pmatrix} \mathsf{E}\,|X|^{it} & 0 \\ 0 & \mathsf{E}\,|X|^{it} \cdot \operatorname{sign} X \end{pmatrix}, \quad t \in \mathbb{R}$$

the *characteristic transform* (as defined by Uchaikin and Zolotarev [1999], Section 5.7) of a random variable X. The characteristic transform exists for any random variable and determines its distribution uniquely (Uchaikin and Zolotarev [1999], Section 5.7). Moreover, for independent random variables U and V the product of characteristic transforms $W_U(t)$ and $W_V(t)$ is the characteristic transform of UV. The characteristic transform applied on both sides of the equality is $W_R(t)W_{\mathbf{c}'\mathbf{Y}}(t) = W_R(t)W_{Y_1}(t)$, $t \in \mathbb{R}$, and since $\mathsf{E}\,(R^{it}) \neq 0$ for almost all t and $R > 0$ we obtain $W_{\mathbf{c}'\mathbf{Y}}(t) = W_{Y_1}(t)$ which concludes $\mathbf{c}'\mathbf{Y} \stackrel{d}{=} Y_1$. Finally, Corollary 3 is applied.

□

The following lemma (Misiewicz [1996], Theorem II.2.3, for pseudo-isotropy and $n = 2$) shows that α-symmetric random vector in $\mathbb{R}^n$ cannot be concentrated in any hyperplane.

Lemma 17. *Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be an α-symmetric random vector in $\mathbb{R}^n$ such that $\mathsf{P}(\mathbf{X} = \mathbf{0}) = 0$. Then for each hyperplane $H_{\boldsymbol{\xi},p} = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\xi}'\mathbf{x} = p\}$, $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $p \in \mathbb{R}$, we have $\mathsf{P}(\mathbf{X} \in H_{\boldsymbol{\xi},p}) = 0$.*

Consequently, the distribution of $\mathbf{X}$ has no atoms.

Proof. First let us show that for each point $Q \in \mathbb{R}^n$ we have $\mathsf{P}(\mathbf{X} = Q) = 0$.

By contradiction let there be Q such that $\mathsf{P}(\mathbf{X} = Q) = q > 0$. Then there exists a one-dimensional projection which maps Q onto the origin, thus the projected vector has an atom of size at least q at the origin. Since all one-dimensional projections have the same distribution (up to a scale factor), all have an atom at zero. The kernel of a one-dimensional projection $\mathbf{x} \mapsto \boldsymbol{\xi}'\mathbf{x}$ defined by $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$

is $\{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\xi}'\mathbf{x} = 0\}$, i.e. a hyperplane containing the origin which means each hyperplane $H_{\boldsymbol{\xi},0}$ has a probability at least q. For any line ℓ passing through the origin we may find infinitely many hyperplanes such that $\ell = \bigcap H_{\boldsymbol{\xi},0}$. Then $\mathsf{P}(\mathbf{X} \in \ell)$ is positive (otherwise the distribution would not be a probability measure).

The space $\mathbb{R}^n \setminus \{\mathbf{0}\}$ can be decomposed into uncountably many distinct lines intersecting at the origin. Each having a non-zero probability is a contradiction with $\mathsf{P}(\mathbf{X} \in \mathbb{R}^n \setminus \{\mathbf{0}\}) = 1$.

Now let us prove again by contradiction that $\mathsf{P}(\mathbf{X} \in H_{\boldsymbol{\xi},p}) = 0$. For contradiction find $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $p \neq 0$ (the case $p = 0$ is covered above) such that $\mathsf{P}(\mathbf{X} \in H_{\boldsymbol{\xi},p}) = q > 0$. Which means $\mathsf{P}(\mathbf{X} \in H_{\boldsymbol{\xi},p}) = \mathsf{P}(\boldsymbol{\xi}'\mathbf{X} = p) = \mathsf{P}(\|\boldsymbol{\xi}\|_\alpha X_1 = p) = q > 0$ and each one-dimensional projection has an atom outside zero of mass q.

Let us show that if X_1 has an atom outside the origin, then $\mathbf{X} = (X_1, \ldots, X_n)'$ cannot be α-symmetric for any $\alpha > 0$, $n > 1$. It suffices to prove this only for $n = 2$. If each one-dimensional measure has an atom outside zero then in each direction there is a line with non-zero probability q. From that, it is possible to find a point with a positive probability which is a contradiction with the first part of the proof.

□

Remark 7. Distributions without an atom at the origin will be referred to as *pure*. Lemma 17 shows that for pure α-symmetric distributions, $n > 1$, the probability of each affine subspace equals zero.

On the other hand, the Dirac $\delta_{\mathbf{0}}$ distribution concentrated at $\mathbf{0}$ is α-symmetric for each $\alpha \in (0, \infty]$, $n \in \mathbb{N}$, and each α-symmetric distribution may be written as a mixture of $\delta_{\mathbf{0}}$ and some distribution without an atom at the origin (which is consequently pure). Classes which satisfy $S(n, \alpha) = \{1\}$ (the only permissible distribution is $\delta_{\mathbf{0}}$) will be called *trivial*.

If we combine Example 6 with Lemma 11, we obtain an important class of α-symmetric distributions which are also symmetric stable.

Example 7. (Schoenberg problem) The function e^{-t^β} is a characteristic generator of an α-symmetric n-dimensional random vector if and only if $0 \leq \beta \leq \sigma(n, \alpha)$ where

$$\sigma(n, \alpha) = \begin{cases} 2 & n = 1, \quad 0 < \alpha \leq \infty, \\ \alpha & n \geq 2, \quad 0 < \alpha \leq 2, \\ 1 & n = 2, \quad 2 < \alpha \leq \infty, \\ 0 & n \geq 3, \quad 2 < \alpha \leq \infty. \end{cases}$$

The value $\sigma(n, \alpha) = \sup\{\beta \in [0, 2] : \exp\{-t^\beta\} \in S(n, \alpha)\}$ will be referred to as the *Schoenberg constant*. We may see that $\sigma(n, \alpha)$ is well-defined (applying Lemma 11) and using substability (Remark 3) we can obtain $\sigma(n, \alpha) = \alpha$, for $\alpha \leq 2$ and $n > 1$. The Schoenberg problem relates to isometric embedding as $e^{-t^\beta} \in S(n, \alpha)$ if and only if $(\mathbb{R}^n, \|\cdot\|_\alpha)$ isometrically embeds into L_β-space. Cases which are not covered by substability, e.g. $S(n, \alpha)$, for $\alpha > 2$ and $n \geq 3$, are described in further chapters (the two-dimensional problem in Section 3.1, and the three-dimensional problem in Section 3.3).

Zastavnyi [2000] found several sufficient conditions for a function $f : [0, \infty) \to \mathbb{R}$ such that $f(u^\lambda) \in S(n, \alpha)$ if $\lambda \in (0, \sigma(n, \alpha)]$. More details in Section 3.3 and Section 3.4.

Example 8. (Kuttner-Golubov problem) Another important class of generators is

$$\varphi_{\lambda,\delta}(u) = (1 - u^\lambda)_+^\delta, \quad u \geq 0,$$

such class of characteristic generators is useful in terms of creating sufficient conditions for the characteristic generators from $S(n, \alpha)$ (Gneiting [2000]).

The parameter structure is the following: if for some $\lambda > 0$ and $\delta > 0$ we have $\varphi_{\lambda,\delta} \in S(n, \alpha)$ then for any $\mu > 0$ we have $\varphi_{\lambda,\delta+\mu} \in S(n, \alpha)$ since for a possible α-symmetric random vector $\mathbf{X}$ with a generator $\varphi_{\lambda,\delta}$ we can find an independent random variable V with a density

$$\frac{1}{B(\delta + 1, \mu)} (v - 1)^{\mu-1} v^{-\delta-\mu-1} \mathbb{1}_{(1,\infty)}(v),$$

by $B(\cdot, \cdot)$ we denote the Beta function.[4] Then the characteristic generator ψ of $V^{\frac{1}{\lambda}}\mathbf{X}$ is equal to

$$\begin{aligned}
\psi(u) &= \frac{1}{B(\delta + 1, \mu)} \int_0^\infty (1 - su^\lambda)_+^\delta (s - 1)_+^{\mu-1} s^{-\delta-\mu-1}\, ds \\
&= \frac{1}{B(\delta + 1, \mu)} \int_1^{u^{-\lambda}} (1 - su^\lambda)^\delta (s - 1)^{\mu-1} s^{-\delta-\mu-1}\, ds \\
&\overset{z=1/s}{=} \frac{1}{B(\delta + 1, \mu)} \int_{u^\lambda}^1 (z - u^\lambda)^\delta (1 - z)^{\mu-1}\, dz \\
&= \frac{1}{B(\delta + 1, \mu)} \int_0^1 (1 - u^\lambda)^{\delta+\mu} z^\delta (1 - z)^{\mu-1} dz \\
&= (1 - u^\lambda)^{\delta+\mu}.
\end{aligned}$$

The first integral automatically permits only $u \in [0, 1)$, otherwise, the integral is equal to zero since the inner function includes a product of $(1 - su^\lambda)_+^\delta (s - 1)_+^{\mu-1}$. We may write $\psi(u) = (1 - u^\lambda)_+^{\delta+\mu}$.

Similarly, if for some $\lambda_0 > 0$ and $\delta > 0$ we have $\varphi_{\lambda_0,\delta} \in S(n, \alpha)$ then for $\lambda \in (0, \lambda_0]$ we have $\varphi_{\lambda,\delta} \in S(n, \alpha)$ as can be shown using the derived expression for density in Example 10.

Example 9. Let us find moments of a random variable X with a characteristic function $(1-|u|^\lambda)_+^\delta$ for $\delta \in \mathbb{N}$. For non-integer δ we may combine this result with Example 8. Using Theorem 6 for $r \in (0, \min\{\lambda, 1\})$ we have

$$\mathsf{E}\, |X|^r = \frac{1}{\cos(r\frac{\pi}{2})} \frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1 - (1 - |u|^\lambda)_+^\delta}{u^{1+r}}\, du.$$

[4] The density implies $\frac{1}{V} \sim \mathsf{Beta}(\delta + 1, \mu)$.

And the integral is equal to

$$\begin{aligned}
\int_0^\infty \frac{1-(1-|u|^\lambda)_+^\delta}{u^{1+r}} &= \frac{1}{r} + \int_0^1 \frac{1-(1-u^\lambda)^\delta}{u^{1+r}}\, du \\
&= \frac{1}{r} + \int_0^1 \frac{1-\sum_{k=0}^{\delta}(-1)^k\binom{\delta}{k}u^{\lambda k}}{u^{1+r}}\, du \\
&= \frac{1}{r} + \sum_{k=1}^{\delta}(-1)^{k+1}\binom{\delta}{k}\int_0^1 u^{\lambda k - r - 1}\, du \\
&= \sum_{k=0}^{\delta}(-1)^k\binom{\delta}{k}\frac{1}{r-k\lambda} = \frac{-\lambda^\delta \delta!}{\prod_{k=0}^{\delta}(r-k\lambda)}.
\end{aligned}$$

A similar result can be found for $r \in [1, \lambda)$ using the second part of Theorem 6. If we rewrite the product and factorial as Gamma functions, we have

$$\mathsf{E}\,|X|^r = \frac{\Gamma(1+\delta)\Gamma\left(1-\frac{r}{\lambda}\right)}{\cos(r\frac{\pi}{2})\Gamma(1-r)\Gamma\left(\delta+1-\frac{r}{\lambda}\right)}, \quad r < \lambda.$$

2.2 Density

The general expression for the density of an α-symmetric random vector can be given only in an integral form. This section derives a general result, special cases are discussed in Chapter 3. The results were first given by Richards [1986] through the Radon transform of the characteristic function and the following approach is due to Zastavnyi [2000]. Let us first present a very general lemma for Riemann-Stieltjes integration of a composite function to prove Theorem 20.

Lemma 18. *For a function $p : X \to [0, \infty)$ defined on a set X denote its lower level sets $B_t = \{x \in X : p(x) \le t\}$. Let $f \in \mathcal{C}([0, r])$ for some $r > 0$ and $(B_r, \mathcal{F}, \mu)$ be a measure space with finite complex sign measure μ and $f(p(\cdot))$ and p are $\mathcal{F}$-measurable. Denote $G(t) = \mu(B_t)$, $t \in [0, r]$, then*

$$\int_{B_r} f(p(x))\, d\mu(x) = f(0)G(0) + \int_0^r f(t)\, dG(t) \tag{2.2}$$

$$= f(r)G(r) - \int_0^r G(t)\, df(t) \tag{2.3}$$

where the integrals are Riemann-Stieltjes.

Proof. It suffices to prove only (2.2) as (2.3) follows from the integration by parts for Riemann-Stieltjes integrals (Rudin [1976], Theorem 6.22). Any finite complex sign measure can be decomposed into positive/negative real and imaginary parts. Hence, we may assume μ is finite and positive, the integrals exist, and $G(t) = \mu(B_t)$ is increasing. Let $T_n = \{t_0, \dots, t_{m_n}\}$ be a sequence of partitions such that $\delta(T_n) = \sup_{1\le k\le m_n}|t_k - t_{k-1}| \overset{n\to\infty}{\to} 0$ and $0 = t_0 < \dots < t_{m_n} = r$ and let $\omega(\delta) = \sup\{|f(s) - f(t)| : |s-t| \le \delta, s, t \in [0, r]\}$ be the continuity module of f.

Then

$$\begin{aligned}
&\left|\int_{B_r} f(p(x))\ d\mu(x) - f(0)G(0) - \sum_{k=0}^{m_n-1} f(t_k)(G(t_{k+1}) - G(t_k))\right| \\
=&\left|\sum_{k=0}^{m_n-1} \int_{B_{t_{k+1}} \setminus B_{t_k}} f(p(x)) - f(t_k)\ d\mu(x)\right| \\
\le& \sum_{k=0}^{m_n-1} \int_{B_{t_{k+1}} \setminus B_{t_k}} |f(p(x)) - f(t_k)|\ d\mu(x) \\
\le& \omega(\delta(T_n)) \sum_{k=0}^{m_n-1} \mu(B_{t_{k+1}} \setminus B_{t_k}) \\
\le& \omega(\delta(T_n))\ \mu(B_r \setminus B_0)
\end{aligned}$$

which tends to 0 as $n \to \infty$ ($\delta(T_n) \to 0$). The limit of

$$f(0)G(0) + \sum_{k=0}^{m_n-1} f(t_k)(G(t_{k+1}) - G(t_k))$$

is the right-hand side of (2.2).

□

In further use, p is a quasi-norm (e.g. α-norm) on $\mathbb{R}^n$ and μ has a complex density g with respect to the Lebesgue measure which means $G(r) = \int_{B_r} g(\mathbf{t})\ d\mathbf{t}$ for some integrable function g.

Lemma 19. *Let ρ be a quasi-norm on $\mathbb{R}^n$ and f a continuous function on $[0,r)$, $r > 0$, $h \in \mathcal{C}^1(\mathbb{R})$. For $\mathbf{x} \in \mathbb{R}^n$ define*

$$F(\mathbf{x}) = \int_{B_r} f(\rho(\mathbf{t}))h(\mathbf{x}'\mathbf{t})\ d\mathbf{t}$$

where $B_r = \{\mathbf{t} \in \mathbb{R}^n : \rho(\mathbf{t}) \le r\}$, $r \in (0,\infty)$. Using the function h set $I(\mathbf{x}) = \int_{B_1} h(\mathbf{x}'\mathbf{t})\ d\mathbf{t}$ and $J(\mathbf{x}) = \int_{B_1} (nh(\mathbf{x}'\mathbf{t}) + \mathbf{x}'\mathbf{t} \cdot h'(\mathbf{x}'\mathbf{t}))\ d\mathbf{t}$. Then F can be rewritten as

$$F(\mathbf{x}) = \int_0^r f(t)t^{n-1}J(t\mathbf{x})\ dt. \tag{2.4}$$

If f is absolutely continuous on $[0,r]$ then

$$F(\mathbf{x}) = f(r)r^n I(r\mathbf{x}) - \int_0^r f'(t)t^n I(t\mathbf{x})\ dt. \tag{2.5}$$

Proof. Fix $\mathbf{x} \in \mathbb{R}^n$. Use Lemma 18 with $G(u) = \int_{B_u} h(\mathbf{t}'\mathbf{x})\ d\mathbf{x}$, $u \ge 0$, then since ρ is a quasi-norm we have $B_u = uB_1$ which means

$$G(u) = \int_{B_u} h(\mathbf{x}'\mathbf{t})\ d\mathbf{t} = u^n \int_{B_1} h(\mathbf{x}'(u\mathbf{t}))\ d\mathbf{t} = u^n I(u\mathbf{x}),$$

$p = \rho$ and $d\mu(\mathbf{t}) = h(\mathbf{x}'\mathbf{t})\ d\mathbf{t}$. If f is absolutely continuous, we may use f' as a density in the Riemann-Stieltjes integral (2.3). Thus,

$$\begin{aligned}
F(\mathbf{x}) &= G(r)f(r) - \int_0^r G(t)f'(t)dt \\
&= f(r)r^n I(r\mathbf{x}) - \int_0^r f'(t)t^n I(t\mathbf{x})\ dt.
\end{aligned}$$

For the first equation (2.4) we have $G(0) = 0$ and

$$G'(u) = nu^{n-1} \int_{B_1} h(\mathbf{x}'(u\mathbf{t}))\ d\mathbf{t} + u^n \int_{B_1} \mathbf{x}'\mathbf{t} h'(\mathbf{x}'(u\mathbf{t}))\ d\mathbf{t}$$
$$= u^{n-1} J(u\mathbf{x})$$

which is absolutely continuous and the first equation holds.

□

The previous lemma is thus applied to $h(t) = e^{-it}$ and the α-norm. In case of $(\mathbb{R}^n, \|\cdot\|_\alpha)$, the function $J = J_{n,\alpha}$ can be linked to the Bessel function. As in our case the distributions are symmetric, i.e. for an α-symmetric random vector $\mathbf{X}$ for any $\alpha > 0$ and $n \in \mathbb{N}$ then $(X_1, \ldots, X_n)' \stackrel{d}{=} (\pm X_1, \ldots, \pm X_n)'$ over all possible combinations of signs. Thus, we may use $h(t) = \cos(t)$ in further theorems. The original statements were formulated by Zastavnyi [2000] with $h(t) = e^{it}$. The following definition formalizes the function J from the previous theorem for α-norms. As for $\alpha = 2$ the function is related to Bessel functions (Remark 9), the functions from Definition 13 are called as in Richards [1986]. The original definition by Richards [1986] was formulated in terms of surface integration (Remark 8).

Definition 13. *For $n \in \mathbb{N}$ and $\alpha > 0$ denote $J_{n,\alpha}$ the n-dimensional* α-Bessel *function as*

$$J_{n,\alpha}(\mathbf{x}) = \int_{B_\alpha^n} n\cos(\mathbf{t}'\mathbf{x}) - \mathbf{t}'\mathbf{x} \cdot \sin(\mathbf{t}'\mathbf{x})\ d\mathbf{t}$$

where $B_\alpha^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\alpha \leq 1\}$ is the unit α-ball.

Remark 8. Let $S_\alpha^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\alpha = 1\}$ be the unit α-sphere in $\mathbb{R}^n$. Then

$$J_{n,\alpha}(\mathbf{x}) = \int_{S_\alpha^{n-1}} e^{-i\mathbf{t}'\mathbf{x}}\ \omega(\mathbf{t}) \tag{2.6}$$

where the integration is given by

$$\omega(\mathbf{t}) = \sum_{i=1}^{n} (-1)^{i-1}\ t_i\ dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n.$$

The equality (2.6) is due to Stokes' theorem (Rudin [1976], Theorem 10.33) as

$$\int_{S_\alpha^{n-1}} \omega(\mathbf{t}) = \int_{B_\alpha^n} n\ dt_1 \cdots dt_n.$$

Remark 9. The functions are called Bessel as before Richards [1986], only the densities of spherically symmetric distributions ($\alpha = 2$) were known from the stochastic decomposition (Example 5). The characteristic function of the uniform distribution on $S^{n-1} \equiv S_2^{n-1}$ satisfies

$$\mathsf{E}\ e^{i\mathbf{t}'\mathbf{U}} = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{\|\mathbf{t}\|_2}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(\|\mathbf{t}\|_2), \quad \mathbf{t} \in \mathbb{R}^n$$

where J_k is the Bessel function of the first kind of order k (Bowman [1958]).

Moreover, for any non-trivial 2-symmetric random vector $(X_1, \ldots, X_n)'$ the density of the marginal vector $(X_1, \ldots, X_m)'$ for any $m < n$ exists (details can be found in Ranošová [2021]).

Now we may combine the definitions with Lemma 19 in order to find the density of an α-symmetric random vector $\mathbf{X}$ with an integrable characteristic function $\psi(\|\mathbf{t}\|_\alpha)$, $\mathbf{t} \in \mathbb{R}^n$. The theorem is based on the inversion formula (Theorem 2).

Theorem 20. *Let $\psi \in S(n, \alpha)$ correspond to the random vector $\mathbf{X}$ and satisfy*

$$\int_0^\infty t^{n-1}|\psi(t)|\ dt < \infty.$$

Then the density of $\mathbf{X}$ is equal to

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_0^\infty t^{n-1}\psi(t) J_{n,\alpha}(t\mathbf{x})\ dt, \quad \mathbf{x} \in \mathbb{R}^n.$$

Proof. The assumption of $\int_0^\infty t^{n-1}|\psi(t)|\ dt < \infty$ ensures we may use Theorem 2. From Lemma 19 with $r \to \infty$ and by setting $h(t) = \cos(t)$ where

$$I(\mathbf{x}) = \int_{B_1} \cos(\mathbf{t}'\mathbf{x})\ d\mathbf{t},$$
$$J(\mathbf{x}) = \int_{B_1} n\cos(\mathbf{t}'\mathbf{x}) - \mathbf{t}'\mathbf{x}\sin(\mathbf{t}'\mathbf{x})\ d\mathbf{t}$$

we have

$$\int_{\mathbb{R}^n} \psi(\|\mathbf{t}\|_\alpha)\cos(\mathbf{t}'\mathbf{x})\ d\mathbf{t} = \int_0^\infty t^{n-1}\psi(t)J(t\mathbf{x})\ dt.$$

Therefore, $J(\mathbf{x}) = J_{n,\alpha}(\mathbf{x})$ and the theorem is proven.

□

As the function $J_{n,\alpha}$ is not easy to compute we may either use the second part of Lemma 19 as in Corollary 4 or find a different integral representation as in Remark 10.

Corollary 4. If the characteristic generator $\psi \in S(n, \alpha)$ is absolutely continuous then we can use the second part of Lemma 19 and the density of $\mathbf{X}$ is equal to

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_0^\infty -\psi'(t)\ t^n I(t\mathbf{x})\ dt, \quad \mathbf{x} \in \mathbb{R}^n.$$

Remark 10. Misiewicz [1996] (Lemma II.4.1) obtains another form of $J_{n,\alpha}(\mathbf{x})$ as

$$J_{n,\alpha}(\mathbf{x}) = \underset{\left\{\sum_{i=1}^{n-1}|u_i|^\alpha \le 1\right\}}{\int\cdots\int} (\cos(\mathbf{x}'\mathbf{u}_+) + \cos(\mathbf{x}'\mathbf{u}_-))\left(1 - \sum_{i=1}^{n-1}|u_i|^\alpha\right)^{\frac{1}{\alpha}-1} du_1\cdots du_{n-1}, \tag{2.7}$$

where by $\mathbf{u}_+$ and $\mathbf{u}_-$ denote the opposite vectors on S_α^{n-1}, i.e. are equal to $\mathbf{u}_+ = (u_1, \ldots, u_{n-1}, u_n)'$, $\mathbf{u}_- = (u_1, \ldots, u_{n-1}, -u_n)'$, where $u_n = \left(1 - \sum_{i=1}^{n-1}|u_i|^\alpha\right)^{\frac{1}{\alpha}}$.

The derivation of (2.7) directly computes the density relying on the fact that the norm $\|\cdot\|_\alpha$, the characteristic function $\psi(\|\cdot\|_\alpha)$ and, in conclusion, the density is sign-invariant.

Special cases of Theorem 20 for $\alpha = 1, \infty$ are discussed in Chapter 3, however with the exception of $\alpha = 2$ the density of an α-symmetric distribution cannot be generally written as a one-dimensional function of $\|\cdot\|_\alpha$. Misiewicz [1989] applied Theorem 20 to $S(3,\infty)$ to show triviality of $S(3,\infty)$. Zastavnyi [2000] uses Corollary 4 in order to establish a parameter structure in the Kuttner-Golubov problem. A different approach to the same result can be found in Zastavnyi and Manov [2017].

Example 10. (Kuttner-Golubov problem) Let us use the relationship from Corollary 4 with the function $\varphi_{\lambda,\delta}(u)$ defined in Example 8 in order to show that if for some $\lambda_0 > 0$ and $\delta > 0$ we have $\varphi_{\lambda_0,\delta} \in S(n,\alpha)$ then for any $\lambda \in (0,\lambda_0)$ the function satisfies $\varphi_{\lambda,\delta} \in S(n,\alpha)$.

The density of a random vector with a characteristic generator $\varphi_{\lambda_0,\delta}$ is equal to

$$f_{\lambda_0,\delta}(\mathbf{x}) = \frac{\lambda_0\delta}{(2\pi)^n}\int_0^1 (1-t^{\lambda_0})^{\delta-1}t^{\lambda_0-1}t^n I_{n,\alpha}(t\mathbf{x})\, dt, \quad \mathbf{x}\in\mathbb{R}^n,$$

where $\varphi'_{\lambda_0,\delta} = -\lambda_0\delta(1-t^{\lambda_0})_+^{\delta-1}t^{\lambda_0-1}$ and the function $I_{n,\alpha}$ is defined in Lemma 19

$$I_{n,\alpha}(\mathbf{x}) = \int_{B_\alpha^n} e^{-i\mathbf{t}'\mathbf{x}}\, d\mathbf{t}.$$

Kuttner [1944] showed that if for some function $A(z)$ the following integral

$$A_{\lambda_0,\delta}(z) = \lambda_0 z^{\lambda_0\delta}\int_0^1 (1-t^{\lambda_0})^{\delta-1}t^{\lambda_0-1}A(tz)\, dt$$

is non-negative for some $\lambda_0 > 0$, $\delta > 0$ and any $z \geq 0$ and $A_{\lambda_0,\delta}$ is not identically zero on the interval $[0,z]$, then for $\lambda \in (0,\lambda_0)$ we have $A_{\lambda,\delta}(z) > 0$.

Fix any $\mathbf{x}\in\mathbb{R}^n$ and set $A(t) = t^n I_{n,\alpha}(t\mathbf{x})$ and $A_{\lambda_0,\delta}(z) \geq 0$ since $f_{\lambda_0,\delta}(z\mathbf{x}) \geq 0$ for all $z \geq 0$. Since $f_{\lambda_0,\delta}(\mathbf{0}) > 0$ the condition is satisfied for $z = 1$ and $f_{\lambda,\delta}(\mathbf{x}) = (2\pi)^{-n}\delta A_{\lambda,\delta}(1)$ is positive for any $\lambda \in (0,\lambda_0)$. Theorem 1 implies that $\varphi_{\lambda,\delta} \in S(n,\alpha)$.

In conclusion, we may define an analogue of the Schoenberg constant as

$$\lambda(n,\alpha) = \sup\{\lambda \in (0,2] : (1-t^\lambda)_+^\delta \in S(n,\alpha) \text{ for some } \delta > 0\} \tag{2.8}$$

where the condition $\lambda \leq 2$ is given by the fact that $\varphi''_{\lambda,\delta}(0) = 0$ for $\lambda > 2$ which would imply triviality from Theorem 4. If $(1-u^\lambda)_+^\delta \notin S(n,\alpha)$ for any $\lambda,\delta > 0$, set $\lambda(n,\alpha) = 0$. Furthermore, we may define

$$\delta(\lambda;n,\alpha) = \inf\{\delta > 0 : (1-t^\lambda)_+^\delta \in S(n,\alpha)\}. \tag{2.9}$$

In the one-dimensional case the function

$$\delta(\lambda) = \delta(\lambda;1,2) = \inf\{\delta > 0 : (1-|t|^\lambda)_+^\delta \text{ is positive definite}\} \tag{2.10}$$

is known as the *Kuttner's function*, more details about the function $\delta(\lambda)$ can be found in Gneiting [2000] and Gneiting et al. [2001], however exact values for all λ are not available.

Example 11. Let us find $J_{2,\infty}(x_1, x_2)$ for $x_1, x_2 \neq 0$

$$\begin{aligned}
J_{2,\infty}(x_1, x_2) &= \int_{-1}^{1}\int_{-1}^{1} 2\cos(t_1x_1 + t_2x_2) - (t_1x_1 + t_2x_2)\sin(t_1x_1 + t_2x_2)\ dt_1dt_2 \\
&= \frac{2\sin(x_2) + 2x_2\cos(x_2)}{x_2}\int_{-1}^{1}\cos(t_1x_1)\ dt_1 \\
&- \frac{2\sin(x_2)}{x_2}\int_{-1}^{1} t_1x_1\sin(t_1x_1)\ dt_1 \\
&= \frac{2\sin(x_2) + 2x_2\cos(x_2)}{x_2}\frac{2\sin(x_1)}{x_1} - \frac{2\sin(x_2)}{x_2}\frac{2\sin(x_1) - 2x_1\cos(x_1)}{x_1} \\
&= \frac{4\cos(x_2)\sin(x_1)}{x_1} + \frac{4\cos(x_1)\sin(x_2)}{x_2}.
\end{aligned}$$

The class $S(2,\infty)$ is further discussed in Subsection 3.1.1 together with $S(2,1)$.

3. Classes $S(n, \alpha)$

The aim of this chapter is to present results about the existence of α-symmetric distributions and the properties of non-trivial classes.

The one-dimensional α-symmetric distributions are simply all symmetric distributions for each $\alpha \in (0, \infty]$. For other dimensions the cases are usually split into $\alpha \in (0, 2]$ and $\alpha \in (2, \infty]$. For $n \geq 2$ and $\alpha \leq 2$ the class $S(n, \alpha)$ was known to include random vectors of i.i.d. symmetric α-stable random variables, and the problem of the (non-)triviality of $S(n, \alpha)$, $\alpha > 2$, remained open until solved independently by Lisitskii and Zastavnyi [1992].

This chapter is divided into four sections. Section 3.1 discusses the two-dimensional α-symmetry where the isometric embedding of a two-dimensional normed space derived in Theorem 12 becomes useful, the class $S(2, 1)$ is discussed in detail. Section 3.2 focuses on the multivariate 1-symmetry as it is the only case of α-symmetry where a characterization using a stochastic decomposition is available. Other multivariate α-symmetric distributions are analyzed in Section 3.3 and Section 3.4. Firstly, the (non-)triviality of $S(n, \alpha)$ is proven and several sufficient conditions are established.

The theorems from Eaton [1981] and Misiewicz [1996] (Theorem II.2.2) show that spherically symmetric distributions ($\alpha = 2$) are the only distributions among α-symmetric which may have a finite variance or a bounded support.

Theorem 21. *Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be an α-symmetric random vector such that $0 < \mathsf{var}(X_1) < \infty$. Then $\alpha = 2$.*

Proof. Denote $\mathsf{var}(X_1) = \sigma^2$ and let $\mathsf{Var}(\mathbf{X}) = \Sigma$ be a finite positive definite matrix (all diagonal elements exist and are equal to σ^2, covariances between components are also finite by the Cauchy–Schwartz inequality). Then for any $\mathbf{c} \in \mathbb{R}^n$ the α-symmetry implies

$$\mathbf{c}'\Sigma\mathbf{c} = \mathsf{var}(\mathbf{c}'\mathbf{X}) = \mathsf{var}(\|\mathbf{c}\|_\alpha X_1) = \|\mathbf{c}\|_\alpha^2 \sigma^2.$$

Thus, for each $\mathbf{c} \in \mathbb{R}^n$ we have $\|\mathbf{c}\|_\alpha = \sqrt{\frac{1}{\sigma^2}\mathbf{c}'\Sigma\mathbf{c}}$ and the only α-norm induced by an inner product is the Euclidean, therefore $\alpha = 2$.

□

Theorem 22. *Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a non-trivial α-symmetric random vector. Then one of the following conditions hold:*

(i) For each open set $U \subset \mathbb{R}$ the probability $\mathsf{P}(X_1 \in U)$ is positive.

(ii) The random vector $\mathbf{X}$ is almost surely bounded and $\alpha = 2$.

Proof. It suffices to prove the statement only for $n = 2$.

Assume we may find $u, v \in \mathbb{R}$ such that $\mathsf{P}(u < X_1 < v) = 0$. Then by α-symmetry for any $(t_1, t_2)' \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$

$$\mathsf{P}\left(u < \frac{t_1 X_1 + t_2 X_2}{\|(t_1, t_2)'\|_\alpha} < v\right) = 0.$$

Denote $C_{(t_1,t_2)'} = \{(x_1,x_2)' \in \mathbb{R}^2 : \|(t_1,t_2)'\|_\alpha^{-1}(t_1x_1 + t_2x_2) \in (u,v)\}$, which is a subset of $\mathbb{R}^2$ bordered by two parallel lines, and $M = \max\{1, \|(\sqrt{2}/2, \sqrt{2}/2)'\|_\alpha\}$ the maximal value of $\|\cdot\|_\alpha$ on the unit circle. Then we may estimate the probability of B,

$$B = \{(x_1,x_2)' \in \mathbb{R}^2 : \|(x_1,x_2)'\|_2 > Mu\} \subset \bigcup_{(t_1,t_2)' \in \mathbb{R}^2 \setminus \{\mathbf{0}\}} C_{(t_1,t_2)'}.$$

For any compact $K \subset B$ there exists a finite covering by sets $C_{(t_1,t_2)'}$, therefore $\mathsf{P}((X_1,X_2)' \in K) = 0$, and consequently $\mathsf{P}((X_1,X_2)' \in B) = 0$. The random vector $(X_1,X_2)'$ has bounded support and a finite second moment $\mathsf{E}\, X_1^2 < \infty$. By Theorem 21 we have $\alpha = 2$.

□

The spherically symmetric distributions are by the previous theorems very special cases among α-symmetric distributions. Those properties and the stochastic decomposition mentioned in Example 5 means spherical symmetry will not be thoroughly discussed in the thesis. For the properties of spherically (and elliptically) symmetric distributions, we refer to Fang et al. [1990]. Conditions for the existence of spherically symmetric random vectors in $\mathbb{R}^n$ using Riemann-Liouville fractional calculus are discussed in Ranošová [2021].

3.1 Two-Dimensional α-Symmetry

The two-dimensional α-symmetry is the most simple multivariate example. Theorem 12 implies that any two-dimensional normed space embeds into some L_1-space. The positive definiteness of $e^{-\|\mathbf{t}\|_\alpha}$, $\mathbf{t} \in \mathbb{R}^2$, for $\alpha \geq 1$ can be proven even without the theory of isometric embedding, instead relying on the inversion formula from Theorem 2, since the function $e^{-\|\mathbf{t}\|_\alpha}$, $\mathbf{t} \in \mathbb{R}^2$, is integrable (the proof can be found in Shestakov and Kuritsyn [1985]). Moreover, Example 6 implies that vectors of i.i.d. symmetric α-stable random variables are α-symmetric. Thus, for any $0 < \alpha \leq \infty$ the class $S(2,\alpha)$ is non-trivial. Sufficient conditions for generators from $S(n,\alpha)$ for $\alpha \leq 2$ are presented in Section 3.4 and for $\alpha > 2$ in Example 23.

Theorem 23 by Zastavnyi [1992] provides additional necessary conditions for non-trivial functions from $S(2,\alpha)$ for $\alpha > 2$. Several examples of two-dimensional α-symmetric random vectors are shown below. Subsection 3.1.1 focuses on the class $S(2,1)$ where a stochastic decomposition was developed by Cambanis et al. [1983].

The following theorem by Zastavnyi [1992] together with Theorem 37 is formulated for general normed spaces and pseudo-isotropy defined by (2.1) which is thoroughly discussed in Chapter 4. The proof of Theorem 23 is postponed after Theorem 37 as they share similar steps.

Theorem 23. *Let $(E, \|\cdot\|)$ be a two-dimensional normed space and $\mathbf{a}_1, \mathbf{a}_2 \in E$ linearly independent vectors. Define a function $G(t,y) = \frac{\partial}{\partial t}\|t\mathbf{a}_1 + y\mathbf{a}_2\|$, $t, y \in \mathbb{R}$, and assume that $H(y) = G(1,y)$ is integrable. If $\|\cdot\|$ is a standard of a pseudo-isotropic random vector $\mathbf{X}$ from (2.1) such that the characteristic function of X_1 satisfies $\lim_{t\to 0+} \psi'(t) = 0$ then $\mathbf{X}$ is trivial.*

Corollary 5. The conditions of Theorem 23 are satisfied for $(\mathbb{R}^2, \|\cdot\|_\alpha)$ where $\alpha = 1$ and $\alpha \in (2, \infty]$. The function

$$H(y) = (|y|^\alpha + 1)^{\frac{1}{\alpha}-1}, \quad y \in \mathbb{R},$$

is integrable if $1 - \alpha < -1$, i.e. $\alpha > 2$. Similarly for $\alpha = \infty$ the function $H(y) = \mathbb{1}_{(0,1)}(y)$ is integrable. The case $\alpha = 1$ is included from the isometry with $\alpha = \infty$ (Example 3).

Theorem 23 also solves the two-dimensional Schoenberg problem which asks for possible $\beta \in (0, 2]$ such that $e^{-u^\beta} \in S(n, \alpha)$ (see Example 7). Since we know that

$$\left(e^{-t^\beta}\right)' = -\beta t^{\beta-1} e^{-t^\beta}, \quad t \in (0, \infty),$$

is equal to zero as $t \to 0+$ for $\beta > 1$. Therefore, $\sigma(2, \alpha) = 1$ for $\alpha \in \{1\} \cup (2, \infty]$ as e^{-t} is a suitable characteristic generator.

Similarly, we can check the parameter λ in the Kuttner-Golubov problem (Example 8) as for any $\lambda > 0$, $\delta > 0$

$$\left((1 - t^\lambda)_+^\delta\right)' = \delta\lambda(1 - t^\lambda)^{\delta-1} t^{\lambda-1}, \quad t \in (0, 1),$$

which as $t \to 0+$ is equal to 0 if $\lambda > 1$. The parameter structure is further discussed in Section 3.4. Thus, as was proven for $\sigma(2, \alpha) = 1$ for any $\alpha \in \{1\} \cup (2, \infty]$ we may bound $\lambda(2, \alpha)$ from the Kuttner-Golubov problem (Example 10) as $\lambda(2, \alpha) \leq 1$ for $\alpha \in \{1\} \cup (2, \infty]$.

The relationship between $\lambda(n, \alpha)$ and $\sigma(n, \alpha)$ is studied in Section 3.4 in order to develop sufficient conditions for $S(n, \alpha)$.

Example 12. For any $\alpha \geq 1$ denote $\mathbf{X}$ the distribution with a characteristic function $e^{-\|\mathbf{t}\|_\alpha}$, $\mathbf{t} \in \mathbb{R}^2$. Then both marginal variables X_1, X_2 have a Cauchy (symmetric 1-stable) distribution (1.1). Moreover, the stochastic decomposition is known from Lemma 11 and Remark 3 if $\alpha \in (1, 2]$.

Denote Y_1, Y_2 i.i.d. symmetric α-stable random variables so that $\mathbf{Y} = (Y_1, Y_2)'$ has a characteristic function $e^{-\|\mathbf{t}\|_\alpha^\alpha}$, $\mathbf{t} \in \mathbb{R}^2$, and Z be a non-negative $\frac{1}{\alpha}$-stable variable independent of $\mathbf{Y}$. By Remark 3 we may write $\mathbf{X} \stackrel{d}{=} Z^{\frac{1}{\alpha}}\mathbf{Y}$.

Let us use the stochastic decomposition in order to find the mixed moment of $\mathbf{X}$. For $m_1, m_2 \in (0, 1)$, $m_1 + m_2 \leq 1$, we have

$$\begin{aligned} \mathsf{E}\,(|X_1|^{m_1}|X_2|^{m_2}) &= \mathsf{E}\left(Z^{\frac{m_1+m_2}{\alpha}}|Y_1|^{m_1}|Y_2|^{m_2}\right) \\ &= \mathsf{E}\,Z^{\frac{m_1+m_2}{\alpha}}\,\mathsf{E}\,|Y_1|^{m_1}\,\mathsf{E}\,|Y_2|^{m_2}. \end{aligned}$$

The moments of non-negative $\frac{1}{\alpha}$-stable random variables are equal to (Matsui and Pawlas [2016])

$$\mathsf{E}\,Z^r = \frac{\Gamma(1 - r\alpha)}{\Gamma(1 - r)}, \quad -\infty < r < \frac{1}{\alpha},$$

and the moments of symmetric stable distributions are found in Example 2. The mixed moments are thus equal to

$$\mathsf{E}\,(|X_1|^{m_1}|X_2|^{m_2}) = \frac{\Gamma(1 - m_1 - m_2)}{\Gamma\left(1 - \frac{m_1+m_2}{\alpha}\right)} \frac{\Gamma(1 - \frac{m_1}{\alpha})}{\cos(m_1\frac{\pi}{2})\Gamma(1 - m_1)} \frac{\Gamma(1 - \frac{m_2}{\alpha})}{\cos(m_2\frac{\pi}{2})\Gamma(1 - m_2)}.$$

Remark 11. In the case of $\alpha = 1$ the density of $(X_1, X_1)'$ with a characteristic function $e^{-|t_1|-|t_2|}$, $(t_1, t_2)' \in \mathbb{R}^2$, is

$$\frac{1}{\pi^2(1+x_1^2)(1+x_2^2)}, \quad (x_1, x_2)' \in \mathbb{R}^2.$$

If $\alpha = 2$, the random vector with a characteristic function $e^{-\sqrt{t_1^2+t_2^2}}$, $(t_1, t_2)' \in \mathbb{R}^2$, is the bivariate (spherically symmetric) extension to the Cauchy distribution with a density

$$\frac{\Gamma(\frac{3}{2})}{\pi^{\frac{3}{2}}(1+x_1^2+x_2^2)^{\frac{3}{2}}}, \quad (x_1, x_2)' \in \mathbb{R}^2.$$

For $\alpha = \infty$ the random vector with a characteristic function

$$e^{-\|\mathbf{t}\|_\infty} = e^{-\max\{|t_1|,|t_2|\}}, \quad (t_1, t_2)' \in \mathbb{R}^2,$$

can be defined through Example 3. Denote $(X_1, X_2)'$ where X_1,X_2 are i.i.d. random variables with the Cauchy distribution. Then the random vector $\frac{1}{2}(X_1 + X_2, -X_1 + X_2)'$ has a characteristic function $e^{-\max\{|t_1|,|t_2|\}}$ and the density can be found using the linear transformation as

$$\frac{2}{\pi^2(1+(x_1+x_2)^2)(1+(-x_1+x_2)^2)}, \quad (x_1, x_2)' \in \mathbb{R}^2.$$

Examples of two-dimensional α-symmetric distributions which are not stable (instead are Kuttner-Golubov) are discussed in Example 24 in Chapter 4. Example 24 also includes visualizations.

3.1.1 Two-Dimensional 1-Symmetry

The case of $\alpha = 1$ allows for characterization by stochastic decomposition (similarly as for the spherically symmetric distributions[1]) which was proven by Cambanis et al. [1983]. The properties and sufficient conditions of $S(2, 1)$ will be discussed together for any $n \geq 2$ in Section 3.2. The classes of characteristic generators coincide $S(2, 1) = S(2, \infty)$ and the results from this subsection hold for $\|\cdot\|_\infty$. This isometry between $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ is unique to $n = 2$ (Example 3).

Let us first prove some properties of the special random vector which will serve as a *primitive* distribution among the 1-symmetric distributions which means any other 1-symmetric distribution will be a mixture of the primitive.

Lemma 24. *Let $(U_1, U_2)'$ be a uniformly distributed random vector on the unit circle S^1 and $B \sim \mathsf{Beta}(\frac{1}{2}, \frac{1}{2})$ independent of $(U_1, U_2)'$. The random vector*

$$(Y_1, Y_2)' = \left(\frac{U_1}{\sqrt{B}}, \frac{U_2}{\sqrt{1-B}}\right)'$$

has a characteristic function

$$\psi(|t_1| + |t_2|) = \mathsf{E}\, e^{i(Y_1 t_1 + Y_2 t_2)} = \frac{2}{\pi} \int_{|t_1|+|t_2|}^{\infty} \frac{\sin v}{v}\, dv, \quad t_1, t_2 \in \mathbb{R}, \tag{3.1}$$

[1]See Fang et al. [1990], Theorem 2.3.

and a joint density

$$f_0(u_1, u_2) = \frac{1}{\pi^2 |u_1^2 - u_2^2|}, \quad |u_1| < 1 \leq |u_2| \text{ or } |u_2| < 1 \leq |u_1|, \tag{3.2}$$

and zero otherwise. The marginal densities of Y_1 *and* Y_2 *are*

$$h_0(u) = \frac{1}{\pi^2 |u|} \log \left| \frac{1 + |u|}{1 - |u|} \right|, \quad u \neq 0. \tag{3.3}$$

Proof. First, let us find the density function by transforming the random vector (B, θ), where $B \sim \mathsf{Beta}(\frac{1}{2}, \frac{1}{2})$ is independent of $\theta \sim \mathsf{Unif}(0, 2\pi)$, θ transforms into the uniform distribution on the unit circle via the map $(\cos(\theta), \sin(\theta))'$. The joint density of $(B, \theta)'$ is

$$f_{(B,\theta)'}(b, \vartheta) = \frac{1}{2\pi^2 \sqrt{b(1-b)}}, \mathbb{1}_{(0,1)}(b) \mathbb{1}_{(0,2\pi)}(\vartheta)$$

and

$$(Y_1, Y_2)' = G(B, \theta) := \left(\frac{\cos\theta}{\sqrt{B}}, \frac{\sin\theta}{\sqrt{1-B}} \right)'.$$

The Jacobian of G is

$$D_G(b, \vartheta) = -\frac{1}{2\sqrt{b(1-b)}} \left(\frac{\cos^2(\vartheta)}{b} - \frac{\sin^2(\vartheta)}{1-b} \right)$$

and the transformed density is equal to

$$f_0(u_1, u_2) = f_{(B,\theta)}(G^{-1}(u_1, u_2)) |D_G^{-1}(G^{-1}(u_1, u_2))|$$

where the first part $2b^{-\frac{1}{2}}(1-b)^{-\frac{1}{2}}$ of the Jacobian cancels with the density $f_{(B,\theta)'}$ and the second part is equal to $u_1^2 - u_2^2$. Now the constraints $b \in (0, 1)$ and $\theta \in (0, 2\pi)$ ensure that G is well defined and the support of $(Y_1, Y_2)'$ satisfies

$$\begin{aligned} |u_1| &\leq 1, \\ |\cos\vartheta| &\leq \sqrt{b}, \\ \cos^2\vartheta &\leq b, \\ 1 - b &\leq \sin^2\vartheta, \\ 1 &\leq |u_2|. \end{aligned}$$

In conclusion, the density is equal to

$$f_0(u_1, u_2) = \frac{1}{\pi^2 |u_1^2 - u_2^2|}, \quad |u_1| < 1 \leq |u_2| \text{ or } |u_2| < 1 \leq |u_1|.$$

Furthermore, the density of $(Y_1, Y_2)'$ can be rewritten using the difference of two indicator functions as

$$f_0(u_1, u_2) = \frac{\mathbb{1}_{[1,\infty)}(u_1) - \mathbb{1}_{[1,\infty)}(u_2)}{\pi^2 (u_1^2 - u_2^2)}, \quad (u_1, u_2)' \in \mathbb{R}^2. \tag{3.4}$$

The marginal density is easily obtained by integrating (shown only for positive values, since the marginals are symmetric). For $u_1 > 1$ we have to integrate

$$h(u_1) = \frac{2}{\pi^2}\int_0^1 \frac{1}{u_1^2 - u_2^2}\, du_2 = \frac{1}{\pi^2 u_1}\int_0^1 \frac{1}{u_1+u_2} + \frac{1}{u_1 - u_2}\, du_2$$

and for $0 < u_1 < 1$ the other interval is used

$$h(u_1) = \frac{2}{\pi^2}\int_1^\infty \frac{1}{u_2^2 - u_1^2}\, du_2 = \frac{2}{\pi^2 u_1}\int_{\frac{1}{u_1}}^\infty \frac{1}{t^2-1}\, dt.$$

For the characteristic function let us use several known facts about the Beta distribution and the uniform distribution on the unit circle. Firstly, the identity proven by Cambanis et al. [1983]: For $t, s \in \mathbb{R}$ the following equality holds

$$\frac{s^2}{B} + \frac{t^2}{1-B} \overset{d}{=} \frac{(|s|+|t|)^2}{B}. \tag{3.5}$$

Since B can be expressed as $\sin^2(\theta)$, $\theta \sim \mathsf{Unif}(0, \pi/2)$ and for $t, s > 0$, $s + t > 0$

$$\frac{s^2}{\sin^2(\theta)} + \frac{t^2}{\cos^2(\theta)} = (s+t)^2\left(1 + T^2_{\frac{s-t}{s+t}}(\theta)\right)$$

where $T_x(\theta) = (x + \cos 2\theta)/(\sin 2\theta)$ and its distribution does not depend on $x \in [-1, 1]$, thus it has the same distribution as for $\frac{s-t}{s+t} = 1$ $(t = 0)$.

And secondly, the Bessel function of the zero-order J_0 is the characteristic function of U_1 which means

$$\mathsf{E}\ e^{i(t_1U_1 + t_2U_2)} = J_0\left(\sqrt{t_1^2 + t_2^2}\right)$$

and $-J_0$ is the anti-derivative of the Bessel function of order one (Bowman [1958], 1.6), i.e. $J_0(ax) = 1 - x\int_0^a J_1(vx)\, dv$.

Let us now find the characteristic function of $(Y_1, Y_2)'$: For $(t_1, t_2)' \in \mathbb{R}^2$ denote $t = |t_1| + |t_2|$, then by the law of total probability and (3.5)

$$\begin{aligned}
\mathsf{E}\ e^{i(t_1Y_1+t_2Y_2)} &= \mathsf{E}\ \mathsf{E}\left[\exp\left\{i\left(\frac{t_1U_1}{\sqrt{B}} + \frac{t_2U_2}{\sqrt{1-B}}\right)\right\}\Big|B\right] \\
&= \mathsf{E}\ J_0\left(\sqrt{\frac{t_1^2}{B} + \frac{t_2^2}{1-B}}\right) = \mathsf{E}\ J_0\left(\frac{t}{\sqrt{B}}\right) \\
&= \int_0^1 J_0\left(\frac{t}{\sqrt{b}}\right)\frac{1}{\pi\sqrt{b(1-b)}}\, db \\
&\overset{x=1/b^2}{=} \int_1^\infty J_0(tx)\frac{2}{\pi x\sqrt{x^2-1}}\, dx \\
&= \int_1^\infty \frac{2}{\pi x\sqrt{x^2-1}}\, dx - \int_0^t \frac{2}{\pi}\int_1^\infty J_1(vx)\frac{1}{\sqrt{x^2-1}}\, dx\, dv.
\end{aligned}$$

The first integral is equal to 1 and for the second, the inner integral is equal to $\sin(v)/v$ (Cambanis et al. [1983]) which means

$$\mathsf{E}\ e^{i(t_1Y_1+t_2Y_2)} = 1 - \frac{2}{\pi}\int_0^{|t_1|+|t_2|}\frac{\sin v}{v}\, dv = \frac{2}{\pi}\int_{|t_1|+|t_2|}^\infty \frac{\sin v}{v}\, dv$$

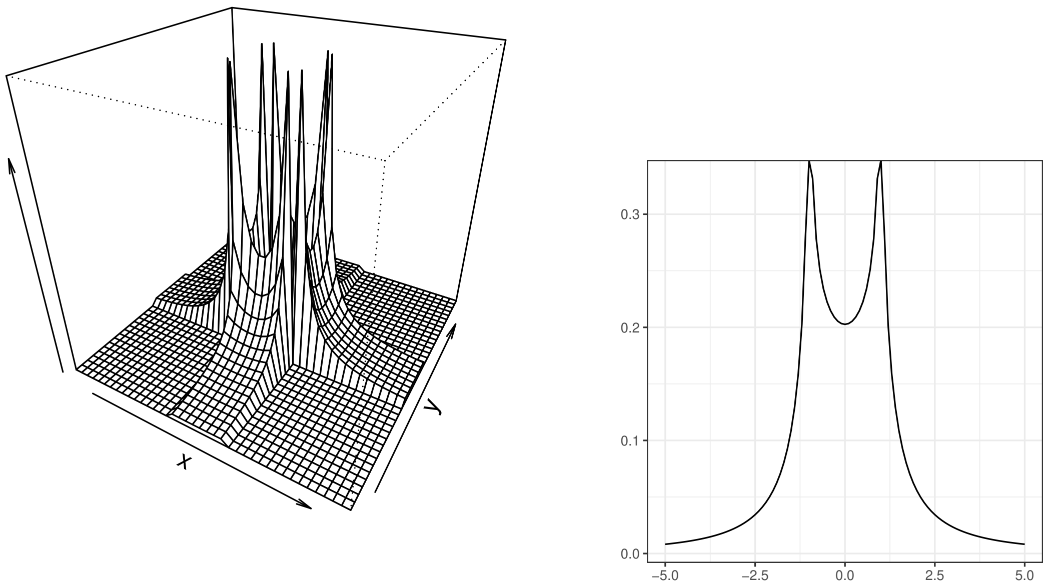


Figure 3.1: The density of the primitive distribution (3.2) and its one-dimensional marginal density (3.3).

since $\int_0^\infty \sin(v)/v \; dv = \pi/2$.

□

An alternative proof of (3.5) can be found in Mazurkiewicz [2007]. Figure 3.1 shows the joint and marginal density for the primitive.

The stochastic decomposition of two-dimensional 1-symmetric distribution is shown in the following theorem. Assuming the integrability of the characteristic function, we implement Theorem 20 and find the two-dimensional 1-Bessel function $J_{2,1}(x_1, x_2)$ from Definition 13. It also provides a direct relationship between the distribution function F of the mixing random variable R and the characteristic generator ψ.

Theorem 25. *The random vector $(X_1, X_2)'$ with a characteristic generator ψ is 1-symmetric if and only if*

$$\psi(t) = \int_{[0,\infty)} \psi_0(tr) \; dF(r)$$

where F is a distribution function of a non-negative random variable and ψ_0 is the primitive characteristic generator (3.1)*.*

Equivalently, $(X_1, X_2)'$ is 1-symmetric if and only if

$$(X_1, X_2)' \stackrel{d}{=} R \left(\frac{U_1}{\sqrt{B}}, \frac{U_2}{\sqrt{1-B}} \right)' \tag{3.6}$$

where R is a non-negative random variable with a distribution function F, the random vector $(U_1, U_2)'$ has a uniform distribution on the unit circle and $B \sim \mathsf{Beta}(\frac{1}{2}, \frac{1}{2})$, variables R, B and $(U_1, U_2)'$ are independent.

The distribution of $(X_1, X_2)'$ is 1-symmetric if and only if it is absolutely *continuous on* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *and with a density*

$$f(x_1,x_2) = \int_0^\infty r^{-2}\ f_0\left(\frac{x_1}{r},\frac{x_2}{r}\right)\ dF(r) \tag{3.7}$$
$$= \frac{F(|x_1|)-F(|x_2|)}{\pi^2(x_1^2-x_2^2)}, \quad x_1^2 \neq x_2^2, \tag{3.8}$$

where f_0 *is the primitive density* (3.2) *and* F *is the distribution function as above (the distribution of* $(X_1,X_2)'$ *has an atom of size* $F(0)$ *at the origin).*

Proof. The proof is structured in the following order. Assuming the integrability of the characteristic generator ψ, the form of the density (3.8) is shown for some F which defines the distribution of R. The form (3.7) implies the stochastic decomposition (3.6) which implies the form of the characteristic function based on Lemma 24.

Without the assumption of integrability (of the characteristic functions) we may modify the random vector $(X_1,X_2)'$ into $(X_1+\frac{1}{n}Z_1, X_2+\frac{1}{n}Z_2)'$ where Z_1, Z_2 are i.i.d. Cauchy distributed random variables independent of $(X_1,X_2)'$ and $(Z_1,Z_2)'$ is 1-symmetric with an integrable characteristic function $e^{-|t_1|-|t_2|}$, $(t_1,t_2)' \in \mathbb{R}^2$. Then the 1-symmetric random vector $(X_1+\frac{1}{n}Z_1, X_2+\frac{1}{n}Z_2)'$ has a characteristic function

$$\psi(|t_1|+|t_2|)\cdot e^{-(|t_1|+|t_2|)/n}, \quad (t_1,t_2)' \in \mathbb{R}^2,$$

which is integrable as $e^{-(|t_1|+|t_2|)/n}$ is integrable and $\psi(|t_1|+|t_2|)$ is bounded by one (Lemma 3). If the random vector $(X_1+\frac{1}{n}Z_1, X_2+\frac{1}{n}Z_2)'$ has a stochastic decomposition $R_n(Y_1,Y_2)'$ as in (3.6), then R (non-negative) can be defined as a limit of R_n, $n\to\infty$. Conversely, any of the conditions (characteristic generator, stochastic decomposition, or density) implies 1-symmetry.

First, assume the vector $(X_1,X_2)'$ is 1-symmetric and its characteristic generator is integrable:

$$\int_{\mathbb{R}^2} |\psi(|s|+|t|)|\ ds\ dt \overset{u=|s|+|t|}{=} 4\int_0^\infty\int_0^u |\psi(u)|\ dt\ du = 4\int_0^\infty u|\psi(u)|\ du < \infty.$$

Then the density of $(X_1,X_2)'$ on $\mathbb{R}^2$ exists and the inversion formula (Theorem 2) may be used: First, 1-symmetry implies $X_1+X_2 \overset{d}{=} -X_1-X_2 \overset{d}{=} X_1-X_2 \overset{d}{=} -X_1+X_2$, let us continue with $x_1,x_2>0$ and the final form will include absolute values. We may use the 1-symmetry

$$\begin{aligned} f(x_1,x_2) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i(t_1x_1+t_2x_2)}\ \psi(|t_1|+|t_2|)\ dt_1\ dt_2 \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \cos(t_1x_1)\cos(t_2x_2)\ \psi(t_1+t_2)\ dt_1\ dt_2 \\ &\overset{u=t_1+t_2}{=} \frac{1}{\pi^2}\int_0^\infty\int_0^u \cos(t_1x_1)\cos((u-t)x_2)\ \psi(u)\ dt\ du. \end{aligned}$$

Let us now use the identity $\cos(tx_1)\cos((u-t)x_2) = \frac{1}{2}\cos(x_2u - t(x_2-x_1)) +$

$\frac{1}{2}\cos(x_2u - t(x_2 + x_1))$ for the inner integral. We get

$$
\begin{aligned}
f(x_1,x_2) &= \frac{1}{2\pi^2}\int_0^\infty \left[\frac{-\sin(ux_1)}{x_2-x_1} + \frac{\sin(ux_2)}{x_2-x_1} + \frac{\sin(ux_1)}{x_2+x_1} + \frac{\sin(ux_2)}{x_2+x_1}\right]\psi(u)\ du \\
&= \frac{1}{2\pi^2}\int_0^\infty \Bigg(\sin(ux_1)\left[\frac{1}{x_1+x_2} + \frac{1}{x_1-x_2}\right] \\
&\qquad + \sin(ux_2)\left[\frac{1}{x_2-x_1} + \frac{1}{x_1+x_2}\right]\Bigg)\psi(u)\ du \\
&= \frac{1}{\pi^2}\int_0^\infty \frac{x_1\sin(ux_1) - x_2\sin(ux_2)}{x_1^2 - x_2^2}\psi(u)\ du \qquad (3.9) \\
&= \frac{1}{\pi^2(x_1^2-x_2^2)}\left[x_1\int_0^\infty \sin(ux_1)\psi(u)\ du - x_2\int_0^\infty \sin(ux_2)\psi(u)\ du\right]
\end{aligned}
$$

for $x_1^2 \neq x_2^2$. Denote $F(x) = x\int_0^\infty \sin(ux)\psi(u)\ du$ for $x > 0$. Let us prove that F is a distribution function on $(0,\infty)$ (non-negative, non-decreasing, and $\int_0^\infty dF(r) = 1$). The function f is a density ($f \geq 0$) which means $F(x_1) \geq F(x_2)$ for $x_1 \geq x_2 > 0$ and $F(0) = 0$. Since the density function $f(x_1,x_2)$ can be expressed via the density f_0 as in (3.4), for $x_1,x_2 \in \mathbb{R}$ we have

$$
\begin{aligned}
f(x_1,x_2) &= \frac{F(|x_1|) - F(|x_2|)}{\pi^2(x_1^2-x_2^2)} \\
&= \frac{1}{\pi^2(x_1^2-x_2^2)}\left[\int_0^{|x_1|} dF(r) - \int_0^{|x_2|} dF(r)\right] \\
&= \int_0^\infty \frac{\mathbb{1}_{[1,\infty)}\left(\frac{|x_1|}{r}\right) - \mathbb{1}_{[1,\infty)}\left(\frac{|x_2|}{r}\right)}{\pi^2(x_1^2-x_2^2)}\ dF(r) \\
&= \int_0^\infty r^{-2} f_0\left(\frac{x_1}{r},\frac{x_2}{r}\right)\ dF(r).
\end{aligned}
$$

Moreover, since f and f_0 are densities (Lemma 24)

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^2} f(x_1,x_2)\ dx_1dx_2 \\
&= \int_{\mathbb{R}^2}\int_0^\infty r^{-2} f_0\left(\frac{x_1}{r},\frac{x_2}{r}\right)\ dF(r)\ dx_1dx_2 \\
&= \int_0^\infty\int_{\mathbb{R}^2} r^{-2} f_0\left(\frac{x_1}{r},\frac{x_2}{r}\right)\ dx_1dx_2\ dF(r) \\
&= \int_0^\infty\int_{\mathbb{R}^2} f_0\left(\tilde{x}_1,\tilde{x}_2\right)\ d\tilde{x}_1d\tilde{x}_2\ dF(r) \\
&= \int_0^\infty 1\ dF(r),
\end{aligned}
$$

we see that F is a distribution function on $(0,\infty)$. The proof for absolutely continuous random vectors is concluded.

□

Other properties of 1-symmetric distributions are proven for general $n \in \mathbb{N}$ in Section 3.2. During the proof in (3.9) we have shown that

$$
J_{2,1}(x_1,x_2) = 4\frac{x_1\sin(x_1) - x_2\sin(x_2)}{x_1^2-x_2^2}, \quad (x_1,x_2)' \in \mathbb{R}^2, \qquad (3.10)
$$

which resembles $J_{2,\infty}$ found in Example 11. If the characteristic generator ψ is integrable, the distribution function of R is equal to

$$F(r) = r \int_0^\infty \sin(ur)\psi(u)\ du, \quad r > 0 \tag{3.11}$$

and the density of R is equal to

$$g(r) = \int_0^\infty (\sin(ur) + ur\cos(ur))\psi(u)\ du, \quad r > 0. \tag{3.12}$$

We may find the stochastic decomposition for some important examples from $S(2,1)$.

Example 13. The cumulative distribution function of R in case of $(X_1, X_2)'$ being i.i.d. Cauchy distributed is equal to

$$F(r) = r \int_0^\infty \sin(ur)e^{-u}\ du = \frac{r^2}{1+r^2}, \quad r > 0,$$

and the corresponding density $g = F'$ is equal to

$$g(r) = \frac{2r}{(1+r^2)^2}, \quad r > 0,$$

as can be verified through the primitive characteristic generator using Theorem 16 as

$$\begin{aligned}
\int_0^\infty g(r)\psi_0(ru)\ dr &= \int_0^\infty \frac{2r}{(1+r^2)^2} \frac{2}{\pi} \int_{ru}^\infty \frac{\sin v}{v}\ dv\ dr \\
&= \frac{2}{\pi} \int_0^\infty \frac{\sin v}{v} \int_0^{\frac{v}{u}} \frac{2r}{(1+r^2)^2}\ dr\ dv \\
&= \frac{2}{\pi} \int_0^\infty \sin(us) \frac{s}{1+s^2}\ ds \\
&= e^{-u},
\end{aligned}$$

where the last equality holds through the residue theorem (Rudin [1987], Theorem 10.42).

Example 14. Let $(X_1, X_2)'$ be a 1-symmetric random vector with a characteristic generator $(1-u)_+^3 = \varphi_{1,3}$ as defined in Example 8.[2] Then $(X_1, X_2)' \stackrel{d}{=} R(Y_1, Y_2)'$ where R and $(Y_1, Y_2)'$ are independent and $(Y_1, Y_2)'$ is the primitive distribution and the density of R is equal to

$$g(r) = \frac{6(\cos(r)r + 2r - 3\sin(r))}{r^4} \mathbb{1}_{(0,\infty)}(r)$$

since

$$F(r) = r \int_0^1 \sin(ur)(1-u)^3\ du = 1 + \frac{6\sin(r) - 6r}{r^3}, \quad r > 0.$$

The moments $\mathsf{E}\ R^m$ exist for $m < 2$ and can be combined with Theorem 30 from Section 3.2 to obtain moments of $(X_1, X_2)'$.

We may look at other random vectors with a characteristic generator $(1-u)_+^\delta$, e.g. for integer $\delta \geq 3$ as finding the density of R requires finding only the integrals of type $\int_0^1 \sin(ur)u^k\ du$ and $\int_0^1 \cos(ur)u^k\ du$ for $k \leq \delta$. Example 8 shows an alternative way how to generate variables with $\delta \geq 3$. Densities of for $\delta \in \{3, 4, 5\}$ are shown in Figure 3.2.

[2] Theorem 33 clarifies the choice of $\delta = 3 = 2 \cdot 2 - 1$.

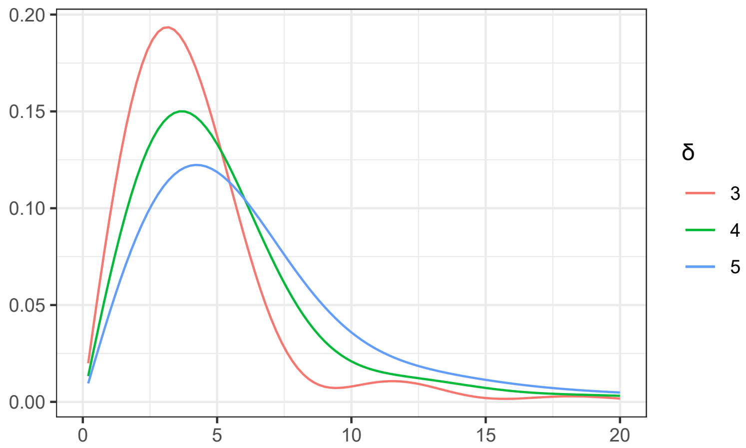


Figure 3.2: Densities of R from a stochastic decomposition (3.6) for a bivariate 1-symmetric distribution with a characteristic generator $(1-u)_+^\delta$ for $\delta \in \{3, 4, 5\}$.

3.2 Higher-Dimensional 1-Symmetric Measures

For higher-dimensional 1-symmetric measures first, we need to find an analogue to the primitive distribution – the Beta distribution of B and $1-B$ is replaced by the Dirichlet distribution. Throughout the thesis, a random vector $(D_1, \ldots, D_n)'$ is said to have a *Dirichlet distribution*[3] $\mathsf{Dir}_n(\beta_1, \ldots, \beta_n)$ with parameters $\beta_1, \ldots, \beta_n > 0$ if the joint density of $(D_1, \ldots, D_n)'$ is equal to

$$f_{(D_1,\ldots,D_n)'}(d_1, \ldots, d_n) = \frac{\Gamma\left(\sum_{i=1}^n \beta_i\right)}{\prod_{i=1}^n \Gamma(\beta_i)} \prod_{i=1}^n d_i^{\beta_i - 1} \mathbb{1}\{(d_1, \ldots, d_n)' \in \Sigma^n\} \tag{3.13}$$

where Σ^n denotes the simplex $\Sigma^n = \{(d_1, \ldots, d_n)' \in [0,1]^n : \sum_{i=1}^n d_i = 1\}$. Furthermore, divided differences of functions are used: For a function $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$ denote

$$f^{|n|}(x_0, \ldots, x_n) = \sum_{k=0}^n \frac{f(x_k)}{\prod_{j \neq k}(x_k - x_j)}. \tag{3.14}$$

the n*-th divided difference* at points $x_0, \ldots, x_n$, we shall write $f^{|n|}(\mathbf{x})$ for brevity.

Lemma 26 defines the multivariate primitive distribution and in Theorem 27 it is proven that any 1-symmetric random variable is a mixture of the primitive. These results are taken from Cambanis et al. [1983]. Further results of this section (Theorem 29) by Gneiting [1998] connect 1-symmetric and 2-symmetric random vectors. Subsection 3.2.1 then summarizes several consequences of the stochastic decomposition.

Lemma 26. *Let $(D_1, \ldots, D_n)'$ be a random vector with Dirichlet distribution with parameters $\frac{1}{2}, \ldots, \frac{1}{2}$ and $(U_1, \ldots, U_n)'$ be uniformly distributed on the unit sphere in $\mathbb{R}^n$ independently of $(D_1, \ldots, D_n)'$. Let $(Y_1, \ldots, Y_n)'$ be a random vector defined as*

$$(Y_1, \ldots, Y_n)' = \left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}}\right)'.$$

[3]For $n = 2$ the Dirichlet distribution $(D_1, D_2)'$ can be written as $(B, 1-B)'$ for some Beta-distributed random variable.

Then its characteristic function can be written as $\mathsf{E}\exp\{i\mathbf{t}'\mathbf{Y}\} = \psi_0(\|\mathbf{t}\|_1)$ *where*

$$\psi_0(u) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_1^\infty \Omega_n(vu^2) v^{-\frac{n}{2}} (v-1)^{\frac{n-3}{2}}\, dv, \quad u \geq 0, \tag{3.15}$$

and $\Omega_n(u_1^2 + \cdots + u_n^2) = \Omega_n(\|\mathbf{u}\|_2^2)$ *denotes the characteristic function*[4] *of the uniform distribution on* S^{n-1}*.*

The density function of $(Y_1, \ldots, Y_n)'$ *can be written as*

$$f_0(y_1, \ldots, y_n) = \frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-2)!\pi^n} \sum_{k=1}^n \frac{(y_i^2 - 1)_+^{n-2}}{\prod_{j \neq k}(y_k^2 - y_j^2)}, \quad |y_i| \neq |y_j|, \quad i, j = 1, \ldots, n. \tag{3.16}$$

Proof. First, let us prove a similar identity as in Lemma 24: For $(D_1, \ldots, D_n)' \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$ and any $s_1, \ldots, s_n \in \mathbb{R}$:

$$\sum_{i=1}^n \frac{s_i^2}{D_i} \stackrel{d}{=} \frac{\left(\sum_{i=1}^n |s_i|\right)^2}{D_1} \tag{3.17}$$

which can be proven by induction. Both the Dirichlet and the uniform distribution on S^{n-1} can be generated recursively:

$$(D_1, \ldots, D_n)' \stackrel{d}{=} \left((1 - D_n) \cdot \widetilde{\mathbf{D}}_{n-1}, D_n\right)', \tag{3.18}$$

$$(U_1, \ldots, U_n)' \stackrel{d}{=} \left(\sqrt{1 - U_n^2} \cdot \widetilde{\mathbf{U}}_{n-1}, U_n\right)' \tag{3.19}$$

where $\widetilde{\mathbf{D}}_{n-1} \sim \mathsf{Dir}_{n-1}(\frac{1}{2}, \ldots, \frac{1}{2})$ independent of $(D_1, \ldots, D_n)'$ and $\widetilde{\mathbf{U}}_{n-1}$ is uniformly distributed on the unit sphere in $\mathbb{R}^{n-1}$ independent of $(U_1, \ldots, U_n)'$ (Fang et al. [1990], Theorem 1.5). Then by induction (the first equality uses the representation (3.18) and the second the induction step for $n-1$)

$$\sum_{i=1}^{n-1} \frac{s_i^2}{D_i} + \frac{s_n^2}{D_n} \stackrel{d}{=} \frac{1}{1 - D_n} \sum_{i=1}^{n-1} \frac{s_i^2}{\widetilde{D}_i} + \frac{s_n^2}{D_n} \stackrel{d}{=} \frac{\left(\sum_{i=1}^{n-1} |s_i|\right)^2}{\widetilde{D}_1(1 - D_n)} + \frac{s_n^2}{D_n}.$$

Now, all marginal random vectors have the same distribution (due to the symmetry of (3.13)) we may replace D_n by $(1 - D_n)\widetilde{D}_2$. Thus, using the initial case

$$\sum_{i=1}^{n-1} \frac{s_i^2}{D_i} + \frac{s_n^2}{D_n} \stackrel{d}{=} \frac{1}{1 - D_n}\left(\frac{\left(\sum_{i=1}^{n-1} |s_i|\right)^2}{\widetilde{D}_1} + \frac{s_n^2}{\widetilde{D}_2}.\right) \stackrel{d}{=} \frac{\left(\sum_{i=1}^n |s_i|\right)^2}{\widetilde{D}_1(1 - D_n)} \stackrel{d}{=} \frac{\left(\sum_{i=1}^n |s_i|\right)^2}{D_1}.$$

Since $D_1 \sim \mathsf{Beta}(\frac{1}{2}, \frac{n-1}{2})$, the stated form of the characteristic function is

[4] See Remark 9.

obtained: For $\mathbf{t} = (t_1, \ldots, t_n)' \in \mathbb{R}^n$

$$\begin{aligned}
\mathsf{E}\, e^{i\mathbf{t}'\mathbf{Y}} &= \mathsf{E}\,\mathsf{E}\left(e^{i\mathbf{t}'\mathbf{Y}} \mid (D_1, \ldots, D_n)'\right) \\
&= \mathsf{E}\left(\Omega_n\left(\frac{t_1^2}{D_1} + \cdots + \frac{t_n^2}{D_n}\right)\right) \\
&= \mathsf{E}\left(\Omega_n\left(\frac{\|\mathbf{t}\|_1^2}{D_1}\right)\right) \\
&= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^1 \Omega_n\left(\frac{\|\mathbf{t}\|_1}{w}\right) w^{-\frac{1}{2}}(1-w)^{\frac{n-3}{2}}\, dw \\
&\overset{v=1/w}{=} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_1^\infty \Omega_n(v \cdot \|\mathbf{t}\|_1^2) v^{-\frac{n}{2}}(v-1)^{\frac{n-3}{2}}\, dv.
\end{aligned}$$

The density can be again found by induction using (3.18) and (3.19). The case $n = 2$ is covered by Lemma 24. The induction step follows the same relationship. Denote again $\widetilde{\mathbf{U}}_{n-1}$, $\mathbf{U}_n$ the uniform distribution on the unit sphere in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, respectively, and $\widetilde{\mathbf{D}}_{n-1} \sim \mathsf{Dir}_{n-1}(\frac{1}{2}, \ldots, \frac{1}{2})$, $\mathbf{D}_n \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$, all independent, thus

$$\widetilde{\mathbf{Y}} = \left(\frac{\widetilde{U}_1}{\sqrt{\widetilde{D}_1}}, \ldots, \frac{\widetilde{U}_{n-1}}{\sqrt{\widetilde{D}_{n-1}}}\right)'$$

is assumed to have the $(n-1)$-dimensional primitive distribution with the density in (3.16) with n replace by $n-1$. We aim to show that (3.16) is the density of

$$\mathbf{Y} = \left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}}\right)' = \left(\frac{\sqrt{1-U_n^2}}{\sqrt{1-D_n}}\widetilde{\mathbf{Y}}, \frac{U_n}{\sqrt{D_n}}\right)'.$$

The transformation is done as $(\widetilde{\mathbf{Y}}, U_n, D_n)' \mapsto (\mathbf{Y}, D_n)'$ and integrating the last variable. The transformation is defined as

$$y_k = \sqrt{\frac{1-u^2}{1-d}}\tilde{y}_k, \quad k = 1, \ldots, n-1, \quad y_n = \sqrt{\frac{u}{d}}$$

which means

$$\tilde{y}_k = \sqrt{\frac{1-d_n}{1-u^2}} y_k = y_k\sqrt{\frac{1-d}{1-dy_n^2}}, \quad u = y_n\sqrt{d}$$

and the Jacobian is equal to

$$d^{-\frac{1}{2}}\left(\frac{1-u^2}{1-d}\right)^{\frac{n-1}{2}} = d^{-\frac{1}{2}}\left(\frac{1-dy_n^2}{1-d}\right)^{\frac{n-1}{2}}.$$

The joint density of $(\widetilde{\mathbf{Y}}, U_n, D_n)'$ is

$$\begin{aligned}
&\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{(n-3)!\pi^{n-1}} \cdot \sum_{k=1}^{n-1} \frac{({\tilde{y}_i}^2 - 1)_+^{n-3}}{\prod_{j\neq k}(\tilde{y}_k^2 - \tilde{y}_j^2)} \cdot \mathbb{1}(|\tilde{y}_i| \neq |\tilde{y}_j|) \\
&\cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot (1-u^2)^{\frac{n-3}{2}} \cdot \mathbb{1}_{(-1,1)}(u) \\
&\cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot d^{-\frac{1}{2}}(1-d)^{\frac{n-3}{2}} \cdot \mathbb{1}_{(0,1)}(d)
\end{aligned}$$

since U_n is a marginal distribution of $\widetilde{\mathbf{U}}_{n-1}$ (Fang et al. [1990], Theorem 1.5) and $D_n \sim \mathsf{Beta}(\frac{1}{2}, \frac{n-1}{2})$. The density of $(\mathbf{Y}, D_n)'$ is therefore equal to

$$\frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-3)!\pi^n} \sum_{k=1}^{n-1} \frac{(y_n^2 - 1 + d(y_n^2 - y_k^2))_+^{n-3}}{\prod_{j\neq k}(y_k^2 - y_j^2)} \cdot \left(\frac{1}{1 - dy_n^2}\right)^{n-3} \left(\frac{1 - dy_n^2}{1-d}\right)^{n-2}$$
$$\cdot (1 - dy_n^2)^{\frac{n-3}{2}} \cdot d^{-\frac{1}{2}} (1-d)^{\frac{n-3}{2}} \cdot d^{\frac{1}{2}} \left(\frac{1 - dy_n^2}{1-d}\right)^{-\frac{n-1}{2}}$$
$$\cdot \mathbb{1}_{(-1,1)}(y_n\sqrt{d}) \cdot \mathbb{1}_{(0,1)}(d) \cdot \mathbb{1}(|y_i| \neq |y_j|)$$

where $1 - dy_n^2$, $1 - d$ and d cancel out which results in

$$\frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-3)!\pi^n} \sum_{k=1}^{n-1} \frac{(y_n^2 - 1 + d(y_n^2 - y_k^2))^{n-3}}{\prod_{j\neq k}(y_k^2 - y_j^2)} \tag{3.20}$$
$$\cdot \mathbb{1}_{(-1,1)}(y_n\sqrt{d}) \cdot \mathbb{1}_{(0,1)}(d) \cdot \mathbb{1}(|y_i| \neq |y_j|) \cdot \mathbb{1}\left(d \geq \frac{1 - y_n^2}{y_n^2 - y_k^2}\right).$$

Integrating d means for $k = 1, \ldots, n-1$ finding the integral

$$\int_{\mathcal{D}} (y_n^2 - 1 + w(y_n^2 - y_k^2))^{n-3}\, dw = \left[\frac{(y_n^2 - 1 + w(y_n^2 - y_k^2))^{n-2}}{(n-2)(y_n^2 - y_k^2)}\right]_L^U$$
$$= \frac{S_k}{(n-2)(y_k^2 - y_n^2)}$$

where the domain $\mathcal{D} = [L, U]$ depends on y_k and y_n (as shown below $\mathcal{D}$ is an interval for all possible options of y_k, y_n) and $S_k = (y_n^2 - 1 + L(y_n^2 - y_k^2))^{n-2} - (y_n^2 - 1 + U(y_n^2 - y_k^2))^{n-2}$. The values L, U, S_k are based on the indicators in (3.20):

	L	U	S_k
$y_k^2,\ y_n^2 > 1$	0	y_n^{-2}	$(y_k^2 - 1)^{n-2} - (y_k/y_n)^{2(n-2)} (y_n^2 - 1)^{n-2}$
$y_n^2 > 1 > y_k^2$	$\frac{1-y_k^2}{y_n^2-y_k^2}$	y_n^{-2}	$-(y_k/y_n)^{2(n-2)}(y_n^2 - 1)^{n-2}$
$y_k^2 > 1 > y_n^2$	0	$\frac{1-y_k^2}{y_k^2-y_n^2}$	$(y_k^2 - 1)^{n-2}$
$1 > y_k^2,\ y_n^2$	$\emptyset$	$\emptyset$	0

Table 3.1: Possible values of S_k based on y_k, y_n in (3.20).

Table 3.1 shows $S_k = (y_k^2 - 1)_+^{n-2} - (y_k/y_n)^{2(n-2)} (y_n^2 - 1)_+^{n-2}$. All results combined give

$$f_0(\mathbf{y}) = \frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-2)!\pi^n} \sum_{k=1}^{n-1} \frac{(y_k^2 - 1)_+^{n-2} - (y_k/y_n)^{2(n-2)} (y_n^2 - 1)_+^{n-2}}{(y_k^2 - y_n^2) \cdot \prod_{j\neq k<n}(y_k^2 - y_j^2)}$$
$$= \frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-2)!\pi^n} \left[\sum_{k=1}^{n-1} \frac{(y_k^2 - 1)_+^{n-2}}{\prod_{j\neq k}(y_k^2 - y_j^2)} - \frac{(y_n^2 - 1)_+^{n-2}}{y_n^{2(n-2)}} \sum_{i=1}^{n-1} \frac{y_i^{2(n-2)}}{\prod_{j\neq k}(y_k^2 - y_j^2)}\right].$$

Finally, let us show

$$\frac{(y_n^2 - 1)_+^{n-2}}{y_n^{2(n-2)}} \sum_{i=1}^{n-1} \frac{y_i^{2(n-2)}}{\prod_{j\neq k}(y_k^2 - y_j^2)} = -\frac{(y_n^2 - 1)_+^{n-2}}{\prod_{j\neq k}(y_k^2 - y_j^2)}$$

by the fact that

$$\sum_{i=1}^{n} \frac{y_i^{2(n-2)}}{\prod_{j\neq k}(y_k^2 - y_j^2)} = 0.$$

This sum is the $(n-1)$-st divided difference (3.14) of the function $t \mapsto t^{n-2}$ at points $y_1^2, \ldots, y_n^2$. The mean-value theorem for divided differences (Atkinson [2008], 3.2.12) states that the $(n-1)$-st divided difference of f is equal to $f^{(n-1)}(\xi)/(n-1)!$ where $\xi \in (\min_i y_i^2, \max_i y_i^2)$. For $f(t) = t^{n-2}$ that derivative is zero for all $t \in \mathbb{R}$.

□

Mazurkiewicz [2007] gives explicit but complicated formulas for densities of the $(n-k)$-dimensional marginal random vectors $(Y_{n,1}, \ldots, Y_{n,n-k})'$. For $n = 3$ the marginal densities of $(Y_{3,1}, Y_{3,2}, Y_{3,3})'$ are equal to

$$\begin{aligned} f_{(Y_{3,1},Y_{3,2})'}(y_1, y_2) &= \frac{1}{4\pi^2(y_1^2 - y_2^2)} \left(\frac{y_1^2 - 1}{|y_1|} \log \left| \frac{1+|y_1|}{1-|y_1|} \right| \right. \\ &\quad \left. - \frac{y_2^2 - 1}{|y_2|} \log \left| \frac{1+|y_2|}{1-|y_2|} \right| \right), \quad |y_1| \neq |y_2|, \qquad (3.21) \\ f_{Y_{3,1}}(y) &= \frac{1 - (1-y^2)_+}{4y^2}, \quad y \in \mathbb{R} \setminus \{0\}, \qquad (3.22) \end{aligned}$$

and are shown in Figure 3.3. Note that for $n \geq 4$ the two-dimensional marginal densities are bounded and for $n \to \infty$ they resemble densities of the Cauchy[5] distribution. The density of $Y_{3,1}$ from (3.22) is constant on $(-1, 1)$ and similarly, the density of $(Y_{4,1}, Y_{4,2})'$ is constant on $(-1, 1)^2$ which is connected to the $(n-2)$-dimensional marginals of the uniform distribution on S_2^{n-1} which are uniform on the unit ball B_2^{n-2} (Fang et al. [1990], Theorem 1.5).

Remark 12. As mentioned in Remark 9 the characteristic function of $(U_1, \ldots, U_n)'$ can be written in terms of Bessel functions (Richards [1986])

$$\mathsf{E}\, e^{i\mathbf{t}'\mathbf{U}} = \Omega_n\left(\|\mathbf{t}\|_2^2\right) = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{\|\mathbf{t}\|_2}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(\|\mathbf{t}\|_2), \quad \mathbf{t} \in \mathbb{R}^n,$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind of order $\frac{n}{2} - 1$.

We may now move to the higher-dimensional version of Theorem 25 which again relies on Theorem 2.

Theorem 27. *The random vector $(X_1, \ldots, X_n)'$ is 1-symmetric if and only if its characteristic generator can be written as*

$$\psi(u) = \int_{[0,\infty)} \psi_0(ur)\, dF(r)$$

where F is a distribution function of a non-negative random variable and ψ_0 is defined by (3.15)*. The random vector can be decomposed as*

$$(X_1, \ldots, X_n)' \overset{d}{=} R \left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}} \right)' \qquad (3.23)$$

[5] Section 3.3 shows that the marginals of the primitive in $S(\infty, 1)$ are Cauchy distributed.

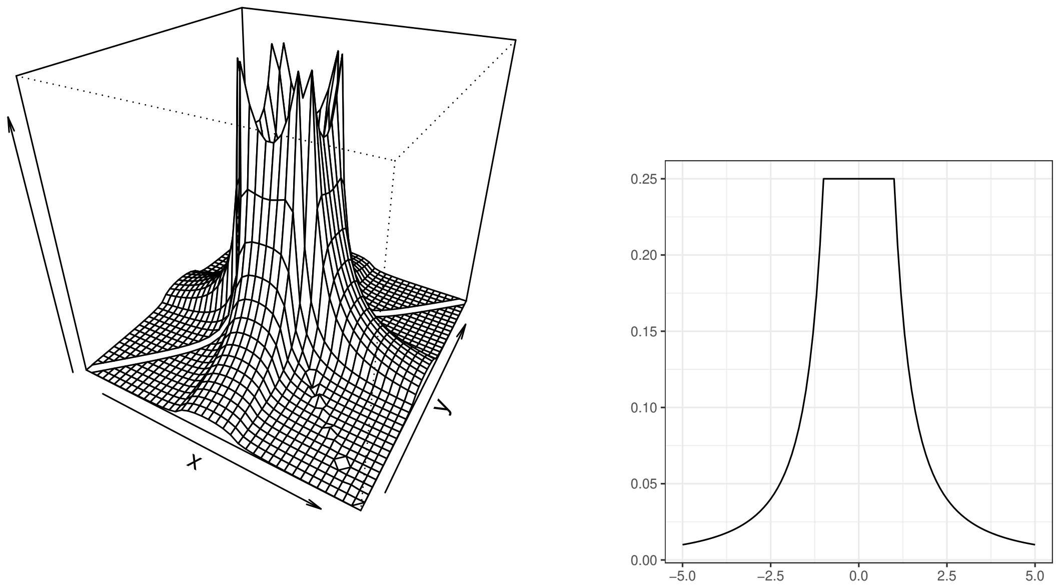


Figure 3.3: The two-dimensional (3.21) and one-dimensional (3.22) marginal densities of the three-dimensional primitive distribution (3.16).

where R has a distribution function F and is independent of the primitive defined by Lemma 26.

The 1-symmetric random variable $\mathbf{X} = (X_1, \ldots, X_n)'$ is absolutely continuous (with the exception of a possible atom at the origin) with a density

$$f(\mathbf{x}) = \int_{[0,\infty)} r^{-n} f_0\left(\frac{\mathbf{x}}{r}\right)\, dF(r) \tag{3.24}$$

$$= \frac{\Gamma^2\left(\frac{n}{2}\right)}{(n-2)!\pi^n} \sum_{k=1}^{n} \frac{\int_0^{|x_k|} (x_k^2 - r^2)^{n-2} r^{2-n}\, dF(r)}{\prod_{j\neq k}(x_k^2 - x_j^2)}, \quad |x_j| \neq |x_k|,\ j,k = 1,\ldots,n.$$

Proof. As in the proof of Theorem 25 the proof will be done only for an integrable characteristic function. Other cases are solved again by modifying $\mathbf{X}$ into $\mathbf{X} + \frac{1}{n}\mathbf{Z}$ where $\mathbf{Z} = (Z_1, \ldots, Z_n)'$ are i.i.d. Cauchy distributed random variables and by taking a weak limit $n \to \infty$. Assuming integrability, the equation (3.24) will be proven which implies the existence of the decomposition (3.23). Converse implications are trivial.

Firstly, assume $\mathbf{X}$ is 1-symmetric with a characteristic function $\psi(|t_1| + \cdots + |t_n|)$ which satisfies

$$\int_{\mathbb{R}^n} \psi(|t_1| + \cdots + |t_n|)\, dt_1 \cdots dt_n = \frac{2^n}{(n-1)!} \int_0^\infty u^{n-1}\psi(u)\, du < \infty.$$

We may use the inversion formula (Theorem 2). Since 1-symmetric distributions are sign-invariant let us define a symmetrisation operator

$$\mathsf{Even} f(x_1, \ldots, x_n) = \frac{1}{2^n} \sum_{\mathcal{C}^n} f(\pm x_1, \ldots, \pm x_n)$$

where all possible 2^n combinations of signs are used. The sum over 2^n possible signs is denoted by $\mathcal{C}^n$. Now for $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$

$$
\begin{aligned}
f(\mathbf{x}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\sum_{i=1}^n t_i x_i} \psi(|t_1| + \cdots + |t_n|)\ dt_1 \cdots dt_n \\
&= \frac{1}{\pi^n} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \cos(t_i x_i) \psi(t_1 + \cdots + t_n)\ dt_1 \cdots dt_n \\
&= \frac{1}{\pi^n} \mathsf{Even} \int_0^\infty \cdots \int_0^\infty e^{i\sum_{i=1}^n t_i x_i} \psi(t_1 + \cdots + t_n)\ dt_1 \cdots dt_n.
\end{aligned}
$$

Let us again use divided differences defined in (3.14). The Hermite-Genocchi formula (Atkinson [2008], Theorem 3.3) states that if the integrals and derivatives below exist then the n-th divided difference is equal to an integral of the n-th derivative over the standard simplex $\Sigma^n = \{(s_1, \ldots, s_n)' \in [0,1]^n : \sum_{i=1}^n s_i = 1\}$. Thus for any functions g, h assuming the integrals and derivatives exist

$$
\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty g\left(\sum_{i=1}^n t_i\right) h^{(n-1)}\left(\sum_{i=1}^n t_i x_i\right)\ dt_1 \cdots dt_n \\
&\overset{T=t_1+\cdots+t_n}{=} \int_0^\infty \int_{\Sigma^n} g(T) T^{n-1} h^{(n-1)}\left(T \sum_{i=1}^n s_i x_i\right)\ ds_1 \cdots ds_{n-1} dT \\
&= \int_0^\infty g(T) T^{n-1} h^{|n-1|}(Tx_1, \ldots, Tx_n)\ dT \\
&= \int_0^\infty g(T) \sum_{k=1}^n \frac{h(Tx_k)}{\prod_{k \neq j}(x_k - x_j)}\ dT
\end{aligned}
$$

which will be used to simplify the integral. For $h^{(n-1)}(t) = e^{it}$ and $g(t) = \psi(t)$

$$
\begin{aligned}
f(\mathbf{x}) &= \frac{1}{\pi^n} \mathsf{Even} \int_0^\infty \cdots \int_0^\infty e^{i\sum_{i=1}^n t_i x_i} \psi(t_1 + \cdots + t_n)\ dt_1 \cdots dt_n \\
&= \frac{1}{\pi^n} \mathsf{Even} \int_0^\infty \psi(u) i^{-n+1} \sum_{k=1}^n \frac{e^{iux_k}}{\prod_{k \neq j}(x_k - x_j)}\ du. \qquad (3.25)
\end{aligned}
$$

The Even operator affects only the divided difference

$$
\begin{aligned}
\mathsf{Even} \frac{e^{iux_k}}{\prod_{k \neq j}(x_k - x_j)} &= \frac{1}{2^n} \sum_{k=1}^n \left[\sum_{\mathcal{C}^{n-1}} \frac{e^{iux_k}}{\prod_{j \neq k}(x_k - \pm x_j)} + \sum_{\mathcal{C}^{n-1}} \frac{e^{-iux_k}}{\prod_{j \neq k}(-x_k - \pm x_j)}\right] \\
&= \frac{1}{2^n} \sum_{k=1}^n \left[\frac{e^{iux_k} 2^{n-1} x_k^{n-1}}{\prod_{j \neq k}(x_k^2 - x_j^2)} + \frac{(-1)^{n-1} e^{-iux_k} 2^{n-1} x_k^{n-1}}{\prod_{j \neq k}(x_k^2 - x_j^2)}\right].
\end{aligned}
$$

And the result of the last formula can be written as

$$
\sum_{k=1}^n \frac{x_k^{n-1} \cos(ux_k)}{\prod_{j \neq k}(x_k^2 - x_j^2)}, \quad \text{for } n \text{ odd}, \quad i \sum_{k=1}^n \frac{x_k^{n-1} \sin(ux_k)}{\prod_{j \neq k}(x_k^2 - x_j^2)}, \quad \text{for } n \text{ even}, \qquad (3.26)
$$

since $\sin(t) = \frac{1}{2i}(e^{it} - e^{-it})$ and $\cos(t) = \frac{1}{2}(e^{it} + e^{-it})$. If we denote the integral part of (3.25) as

$$
B_n(t) = \begin{cases} i^{1-n} t^{\frac{n-1}{2}} \int_0^\infty \cos(u\sqrt{t}) \psi(u)\ du, & n \text{ odd}, \\ i^{2-n} t^{\frac{n-1}{2}} \int_0^\infty \sin(u\sqrt{t}) \psi(u)\ du, & n \text{ even}, \end{cases}
$$

where in both cases B_n is real. The density f of $\mathbf{X}$ can be written as a divided difference at points $x_1^2, \ldots, x_n^2$:

$$f(\mathbf{x}) = \frac{1}{\pi^n} \sum_{k=1}^{n} \frac{B_n(x_k^2)}{\prod_{j \neq k}(x_k^2 - x_j^2)} = \frac{1}{\pi^n} B_n^{|n-1|}(x_1^2, \ldots, x_n^2).$$

Since k-th divided differences of polynomials of order less than k equal zero and B_n is $(n-1)$-times continuously differentiable on $(0, \infty)$, the $(n-1)$-st divided difference of B_n is thus equal to the divided difference of its integral remainder of the Taylor series of B_n at zero (Apostol [1991], Theorem 7.6)

$$\frac{1}{(n-2)!} \int_0^t (w-t)^{n-2} B_n^{(n-1)}(w)\, dw.$$

That means

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{(n-1)!\pi^n} \int_0^\infty \sum_{k=1}^{n} \frac{(x_k^2 - w)_+^{n-2}}{\prod_{j \neq k}(x_k^2 - x_j^2)} B_n^{(n-1)}(w)\, dw \\ &\overset{r=\sqrt{w}}{=} \frac{1}{(n-1)!\pi^n} \int_0^\infty \sum_{k=1}^{n} \frac{(x_k^2 - r^2)_+^{n-2}}{\prod_{j \neq k}(x_k^2 - x_j^2)} 2r B_n^{(n-1)}(r^2)\, dr \\ &= \frac{1}{(n-1)!\pi^n} \int_0^\infty \sum_{k=1}^{n} \frac{(x_k^2/r^2 - 1)_+^{n-2}}{\prod_{j \neq k}(x_k^2/r^2 - x_j^2/r^2)} \frac{2}{r} B_n^{(n-1)}(r^2)\, dr \end{aligned}$$

which is equal to (3.24) if the density of R is set as

$$\tilde{f}(r) = \frac{2}{\Gamma^2\left(\frac{n}{2}\right)} r^{n-1} B_n^{(n-1)}(r^2) \mathbb{1}_{(0,\infty)}(r).$$

That completes the proof of the stochastic decomposition $\mathbf{X} \overset{d}{=} R\mathbf{Y}$.

□

Again the expression for the n-dimensional 1-Bessel function $J_{n,1}(\mathbf{x})$ (Definition 13) was derived in the process in (3.26): For n odd we have

$$J_{n,1}(\mathbf{x}) = (-1)^{\frac{n-1}{2}} 2^n \sum_{k=1}^{n} \frac{x_k^{n-1} \cos(x_k)}{\prod_{j \neq k}(x_k^2 - x_j^2)} \tag{3.27}$$

and for even n the cosines and sines are switched

$$J_{n,1}(\mathbf{x}) = (-1)^{\frac{n-2}{2}} 2^n \sum_{k=1}^{n} \frac{x_k^{n-1} \sin(x_k)}{\prod_{j \neq k}(x_k^2 - x_j^2)}. \tag{3.28}$$

Misiewicz [1996] (Example II.4.1) lists $J_{n,1}$ only for $n = 2, 3$.

Example 15. Let us find a stochastic decomposition for a random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)'$ where Z_i are i.i.d. Cauchy distributed. The stochastic decomposition uses the properties of the spherically symmetric random vectors (Fang et al. [1990], Section 2.6). We may generate $\mathbf{Z}$ using two i.i.d. random vectors $\mathbf{M}, \mathbf{N} \sim \mathsf{N}_n(0, I_n)$ as

$$(Z_1, \ldots, Z_n)' \overset{d}{=} \left(\frac{M_1}{|N_1|}, \ldots, \frac{M_n}{|N_n|} \right)'.$$

Each normal random vector has a spherical stochastic decomposition $\mathbf{M} \stackrel{d}{=} R_M \mathbf{U}$ and $\mathbf{N} \stackrel{d}{=} R_N \mathbf{U}$ where $R_M \stackrel{d}{=} \|\mathbf{M}\|_2, R_N \stackrel{d}{=} \|\mathbf{N}\|_2 \geq 0$ have χ-distribution with n degrees of freedom and $\mathbf{U} = (U_1, \ldots, U_n)'$ is uniformly distributed on S^{n-1}. Since $(U_1^2, \ldots, U_n^2)' \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$ the random vector $\mathbf{Z} \stackrel{d}{=} R_0 \mathbf{Y}$ where $\mathbf{Y}$ has the primitive distribution, $R_0 \stackrel{d}{=} R_M/R_N$, R_M and R_N are independent with the χ-distribution with n degrees of freedom, which means $R_0^2 \sim \mathsf{F}(n, n)$.

The density function of R is then equal to

$$g(r) = \frac{2\Gamma(n)}{\Gamma^2\left(\frac{n}{2}\right)} \frac{r^{n-1}}{(1+r^2)^n}, \quad r > 0.$$

The density of R will be derived in a different way in Example 16, an alternative approach using the function B_n from the proof of Theorem 27 can be found in Fang et al. [1990] (Lemma 7.2).

Furthermore, Gneiting [1998] found another connection between the (spherically) 2-symmetric and 1-symmetric distributions using repeated integration.

Definition 14. *For a function $f : [0, \infty) \to \mathbb{R}$ such that*

$$\int_0^\infty f(x)\, dx$$

exists and is finite denote $If(t)$ the integral operator

$$If(t) = \int_t^\infty f(x)\, dx, \quad t \geq 0. \tag{3.29}$$

The following lemma connects the Fourier transforms (i.e. finding the density given a characteristic function) of functions $\psi(\|\cdot\|_1)$, $\psi^{(n-1)}(|\cdot|)$ and $\psi^{(n-1)}(\|\cdot\|_2)$ for a differentiable function $\psi : [0, \infty) \to \mathbb{R}$. The relationship follows the proof of Theorem 27 and uses the properties of 2-symmetric distributions (Fang et al. [1990]), and more generally functions depending on the Euclidean norm. The Fourier transform[6] of a function can be again written as a one-dimensional norm-dependent function (Grafakos and Teschl [2012]).

Lemma 28. *Let $\psi : [0, \infty) \to \mathbb{R}$ be a function with $n-1$ bounded and continuous derivatives such that for $k = 0, \ldots, n-1$ we have*

$$\int_0^\infty t^{n+k-1} |\psi^{(k)}(t)|\, dt < \infty.$$

Let $g(x_1, \ldots, x_n)$ be the Fourier transform of $\psi(\|\mathbf{u}\|_1)$, $\mathbf{u} \in \mathbb{R}^n$, $h_1(x)$ be the Fourier transform of $\psi^{(n-1)}(|u|)$, $u \in \mathbb{R}$, and h_{2n-1} be the Fourier transform of $\psi^{(n-1)}(\|\mathbf{u}\|_2)$, $\mathbf{u} \in \mathbb{R}^{2n-1}$. Then g, h_1, h_{2n-1} are bounded and continuous and

$$\begin{aligned} g(x_1, \ldots, x_n) &= \frac{1}{\pi^{n-1}} \sum_{k=1}^n \frac{h_1(x_k)}{\prod_{j \neq k}(x_k^2 - x_j^2)} \\ &= \frac{(-1)^{n-1}}{(n-1)!} h_{2n-1}(\xi), \end{aligned}$$

for some $\xi \in (\min_{1 \leq i \leq n} |x_i|, \max_{1 \leq i \leq n} |x_i|)$.

[6] For a given random vector, its characteristic function and density are each others' Fourier transforms (up to scale constants).

Lemma 28 by Gneiting [1998] relies on the theory of spherically symmetric distributions by Fang et al. [1990] (Section 2.2), 1-symmetric distributions by Cambanis et al. [1983] and divided differences (3.14).

Theorem 29. *Let $\psi \in S(n,1)$ have $n-1$ bounded and continuous derivatives and $\lim_{t\to\infty}\psi(t)=0$. Then $(-1)^{n-1}\psi^{(n-1)}(0)>0$ and if we denote function*

$$\varphi(t)=\frac{\psi^{(n-1)}(t)}{\psi^{(n-1)}(0)},\quad t\geq 0,$$

then $\varphi \in S(2n-1,2)$.

Let $\mathbf{X}$ be a 1-symmetric n-dimensional random vector with a characteristic generator ψ and a stochastic decomposition $\mathbf{X}\stackrel{d}{=}R\mathbf{Y}$ from (3.23) where R has a cumulative distribution function F. Furthermore, let $\mathbf{V}$ be a spherically symmetric $(2n-1)$-dimensional random vector with a characteristic generator φ and a stochastic decomposition[7] $\mathbf{V}\stackrel{d}{=}S\mathbf{U}$ and denote G the cumulative distribution function of S. Then $\lim_{t\to 0+}F(t)=\lim_{t\to 0+}G(t)=0$ and for any Borel subset $B\subset[0,\infty)$ we have

$$\int_B(-1)^{n-1}\psi(0)\;dG(r)=\int_B\frac{\Gamma^2\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{2n-1}{2}\right)}r^{n-1}\;dF(r). \tag{3.30}$$

Conversely, if $\varphi \in S(2n-1,2)$ and $I^{n-1}\varphi(0)$ exists where I is defined by (3.29) applied $(n-1)$-times. Then

$$\psi(t)=\frac{I^{n-1}\varphi(t)}{I^{n-1}\varphi(0)},\quad t\geq 0$$

and (3.30) holds.

Corollary 6. The primitive characteristic generator ψ_0 (3.15) of $S(n,1)$ and the function[8] $\omega_{2n-1}(t)=\Omega_{2n-1}(t^2)$ are related in a following way

$$\psi_0(t)=\frac{I^{n-1}\omega_{2n-1}(t)}{I^{n-1}\omega_{2n-1}(0)},\quad t\geq 0.$$

The integrability of ω_{2n-1} is checked in Gneiting [1998] and the equality holds because, in the respective stochastic decompositions, the mixing random variables (in Theorem 29 denoted as R and S, respectively) have a Dirac δ_1 distribution (Fang et al. [1990]).

Theorem 29 will be further used to establish some sufficient conditions in Theorem 33.

Example 16. Denote $\psi(t)=e^{-t}\in S(n,1)$. Then its $(n-1)$-st derivative equals $(-1)^{n-1}e^{-t}$ and as we know $e^{-t}\in S(2n-1,2)$, since $e^{-\|\mathbf{t}\|_2}$, $\mathbf{t}\in\mathbb{R}^{2n-1}$, is the characteristic function of the spherically symmetric Cauchy distribution.[9]

[7] See Example 5.
[8] See Remark 12.
[9] Remark 11 specifies the two-dimensional density.

Denote $\mathbf{X}^{(2)}$ a random vector with a characteristic function $e^{-\|\mathbf{t}\|_2}$, $\mathbf{t} \in \mathbb{R}^{2n-1}$. Fang et al. [1990] (Chapter 3) derived the distribution of $\|\mathbf{X}^{(2)}\|_2$ as having the density

$$\frac{2}{B(\frac{2n-1}{2}, \frac{1}{2})} \frac{r^{2n-2}}{(1+r^2)^n}, \quad r > 0.$$

The density of R where R is the variable from the stochastic decomposition of $\mathbf{X}^{(1)} \stackrel{d}{=} R\mathbf{Y}$ where $\mathbf{X}^{(1)}$ has a characteristic function $e^{-\|\mathbf{t}\|_1}$, $\mathbf{t} \in \mathbb{R}^n$ can be found using Theorem 29 as

$$\frac{2}{B(\frac{2n-1}{2}, \frac{1}{2})} \frac{r^{2n-2}}{(1+r^2)^n} \frac{\sqrt{\pi}\Gamma\left(\frac{2n-1}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)} r^{1-n} = \frac{2\Gamma(n)}{\Gamma^2\left(\frac{n}{2}\right)} \frac{r^{2n-1}}{(1+r^2)^n}$$

as was derived in a different way in Example 15.

3.2.1 Properties of 1-Symmetric Distributions

We may now utilize the stochastic decomposition to derive some properties of 1-symmetric distributions using the results by Cambanis et al. [1983] and Fang et al. [1990] (Chapter 7). Theorem 30 uses results by Cambanis et al. [1983] but seems to be new as discussed in Remark 13. Some other properties (either by Cambanis et al. [1983] or by Fang et al. [1990], Chapter 7) were omitted for brevity.

Theorem 30. *Let $(X_1, \ldots, X_n)'$ be a 1-symmetric random vector with a stochastic decomposition $R\mathbf{Y}$ based on Theorem 27. Then for $m_1, \ldots, m_n \in (-1, 1)$, $m_1 + \cdots m_n = m$, the mixed moments are equal to*

$$\begin{aligned} \mathsf{E}\left(\prod_{i=1}^n X_i^{m_i}\right) &= \mathsf{E}\, R^m\, \mathsf{E}\left(\prod_{i=1}^n U_i^{m_i}\right) \mathsf{E}\left(\prod_{i=1}^n D_i^{-\frac{m_i}{2}}\right) \\ &= \mathsf{E}\, R^m \cdot i^m \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right)}, \\ \mathsf{E}\left(\prod_{i=1}^n |X_i|^{m_i}\right) &= \mathsf{E}\, R^m \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)\Gamma\left(\frac{n-m}{2}\right)} \cdot \frac{1}{\prod_{i=1}^n \cos(m_i\pi/2)} \end{aligned}$$

if the moment $\mathsf{E}\, R^m$ is finite.

Proof. The stochastic decomposition by independent R, $\mathbf{U}$ (uniform on S^{n-1}) and $\mathbf{D} \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$ is proven in Theorem 27. Since the negative fractional moment of the Dirichlet distribution is included, m_i must be less than 1. Similarly, $m_i > -1$ to guarantee the existence of moments of $\mathbf{U}$.

The fractional moments are found separately for $\mathbf{U}$ and $\mathbf{D}$. First, using the density (3.13):

$$\mathsf{E}\left(\prod_{i=1}^{n} D_i^{-\frac{m_i}{2}}\right) = \mathsf{E}\left(\prod_{i=1}^{n} |D_i|^{-\frac{m_i}{2}}\right)$$
$$= \int_{\Sigma^n} \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} t_i^{-\frac{m_i}{2}+\frac{1}{2}-1}\, dt_1 \cdots dt_n$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \cdot \frac{\prod_{i=1}^{n} \Gamma\left(\frac{1-m_i}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}$$

where Σ^n is the standard simplex from (3.13) and the integrand is up to a constant equal to the density of $\mathsf{Dir}_n(\frac{1-m_1}{2}, \ldots, \frac{1-m_n}{2})$.

The moments of $\mathbf{U}$ are found using the standard normal distribution as in Fang et al. [1990] with the fractional moments of normal distribution in Winkelbauer [2012]. Let $\mathbf{Z} \sim \mathsf{N}_n(0, I_n)$, then $\mathbf{U} \overset{d}{=} \mathbf{Z}/\|\mathbf{Z}\|_2$ which is independent of $\|\mathbf{Z}\|_2$. Moreover, $\|\mathbf{Z}\|_2^2$ is χ^2-distributed with n degrees of freedom. By Winkelbauer [2012] for any $p > -1$ and a random variable $Z \sim \mathsf{N}(0,1)$

$$\mathsf{E}\ Z^p = \frac{i^p\ 2^{\frac{p}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)}, \quad E|Z|^p = \frac{2^{\frac{p}{2}}\ \Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}} = \frac{2^{\frac{p}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)\cos\left(\frac{\pi p}{2}\right)}$$

by the Euler reflection formula.[10] Then the mixed moment is equal to

$$\mathsf{E}\left(\prod_{i=1}^{n} U_i^{m_i}\right) = \mathsf{E}\left(\prod_{i=1}^{n} Z_i^{m_i}\right)(\mathsf{E}\ \|\mathbf{Z}\|_2^m)^{-1}$$
$$= \prod_{i=1}^{n} \mathsf{E}\ Z_i^{m_i} \left(\mathsf{E}\,(\|\mathbf{Z}\|_2^2)^{m/2}\right)^{-1}$$
$$= \frac{i^m \pi^{\frac{n}{2}}}{\prod_{i=1}^{n} \Gamma\left(\frac{1-m_i}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)}.$$

Similarly, for the mixed absolute moments we have

$$\mathsf{E}\left(\prod_{i=1}^{n} |U_i|^{m_1}\right) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \Gamma\left(\frac{m_i+1}{2}\right)$$

which completes the proof.

□

Remark 13. The general form of the mixed moments based on the stochastic decomposition was known to Cambanis et al. [1983] (Section 4) and is mentioned also in Fang et al. [1990] (Section 7.3). However, the moments of the uniform and Dirichlet distribution are not found there (presumably because integer moments do not exist) and the fact that raw mixed moments depend only on the sum of the exponents was left unnoticed by them. Our result seems to be original.

[10] See Example 2.

Example 15 can be used to derive sufficient and necessary conditions for characteristic functions from $S(n,1)$. The main idea is to shift the problem to completely monotone functions which resembles Corollary 7.

Theorem 31. *A function* $\psi : [0,\infty) \to \mathbb{R}$ *belongs to* $S(n,1)$ *if and only if*

$$\phi(s) = \int_0^\infty \psi(rs)g(r)\ dr \tag{3.31}$$

is completely monotone[11] *on* $[0,\infty)$*.The function* g *used in* (3.31) *is defined in Example 15 and takes the form*

$$g(r) = \frac{2\Gamma(n)}{\Gamma^2\left(\frac{n}{2}\right)} \frac{r^{n-1}}{(1+r^2)^n}, \quad r > 0.$$

Proof. Using Theorem 7 completely monotone functions are equivalent to Laplace transforms of non-negative random variables. As in Example 15 denote $\mathbf{Z} \stackrel{d}{=} R_0\mathbf{Y}$ a stochastic decomposition of a vector $\mathbf{Z}$ of i.i.d. Cauchy random variables and g is the density of R_0 and suppose $\mathbf{X} \sim S(n,1,\psi)$ with a stochastic decomposition $\mathbf{X} \stackrel{d}{=} R\mathbf{Y}$ is independent of $\mathbf{Z}$, i.e. $\psi \in S(n,1)$. Then $R_0\mathbf{X} \stackrel{d}{=} R_0R\mathbf{Y} \stackrel{d}{=} R\mathbf{Z}$ and $R_0\mathbf{X} \in S(n,1)$ with the characteristic generator ϕ. Similarly, we may write the characteristic function of $R\mathbf{Z}$ as

$$\int_0^\infty e^{-u\|\mathbf{t}\|_1}\ dF(u), \quad \mathbf{t} \in \mathbb{R}^n,$$

where F is the distribution function of R. Since $R_0\mathbf{X} \stackrel{d}{=} R\mathbf{Z}$ we obtain

$$\phi(s) = \int_0^\infty e^{-su}\ dF(u)$$

and ϕ is a Laplace transform of the random variable R.

Conversely, let $\psi : [0,\infty) \to \mathbb{R}$ be a function such that

$$\phi(s) = \int_0^\infty \psi(rs)g(r)\ dr, \quad s \geq 0,$$

is a completely monotone function. Using Theorem 7 we may find a non-negative random variable R such that ϕ can be rewritten as

$$\phi(s) = \int_0^\infty e^{-rs}\ dF(r), \quad s \geq 0,$$

where F is a cumulative distribution function of R. Thus we may write

$$\int_0^\infty \psi(rs)g(r)\ dr = \int_0^\infty e^{-sr}\ dF(r), \quad s \geq 0. \tag{3.32}$$

The right-hand side of (3.32) is a characteristic generator of a random vector $R\mathbf{Z}$ where R and $\mathbf{Z}$ are independent and $Z_1, \ldots, Z_n$ are i.i.d. Cauchy. If we apply the stochastic decomposition from Theorem 27 then $R\mathbf{Z} \stackrel{d}{=} RR_0\mathbf{Y}$. The function g is the density of the random variable R_0 (Example 15). We know that $RR_0\mathbf{Y}$ is

[11] See Definition 5.

1-symmetric and by Theorem 16 we have that $R\mathbf{Y}$ is 1-symmetric since $\mathsf{E}\ R_0^{it} \neq 0$ for almost all t. The characteristic generator ψ of $R\mathbf{Y}$ must satisfy (3.32).

□

The discussion following Lemma 13 underlines the inclusion $S(n,1) \subset S(m,1)$ for $m < n$. The inclusion can be further characterized by the respective stochastic decompositions in both dimensions.

Theorem 32. *Let $\mathbf{X}^{(n)} \stackrel{d}{=} R^{(n)}\mathbf{Y}^{(n)}$ be an n-dimensional 1-symmetric random vector with a stochastic decomposition from Theorem 27 with an m-dimensional subvector $\mathbf{X}^{(m)}$ which is also 1-symmetric and possesses a stochastic decomposition $R^{(m)}\mathbf{Y}^{(m)}$. Then*

$$R^{(m)} \stackrel{d}{=} R^{(n)}\sqrt{\frac{V_1}{V_2}}$$

where $V_1, V_2 \sim \mathsf{Beta}(\frac{m}{2}, \frac{n-m}{2})$ are independent of $R^{(n)}$.

Proof. The proof of the theorem uses the relationship for subvectors of the uniform and Dirichlet distribution. If $(U_1, \ldots, U_n)'$ has a uniform distribution on the unit sphere, then $(U_1^2, \ldots, U_n^2)' \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$ (Fang et al. [1990], Theorem 1.5).

Let $\mathbf{Z} = (Z_1, \ldots, Z_n)' = (\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(k)})' \sim \mathsf{N}_n(0, I_n)$ where $\mathbf{Z}^{(1)}, \ \ldots, \ \mathbf{Z}^{(k)}$ are subvectors of dimensions $n_1, \ldots, n_k$, $n_1 + \cdots + n_k = n$. Then $\mathbf{Z}/\|\mathbf{Z}\|_2$ has a uniform distribution on the unit sphere and we may rewrite it as

$$\left(\frac{\mathbf{Z}^{(1)}}{\|\mathbf{Z}\|_2}, \ldots, \frac{\mathbf{Z}^{(k)}}{\|\mathbf{Z}\|_2}\right)' = \left(\frac{\|\mathbf{Z}^{(1)}\|_2}{\|\mathbf{Z}\|_2}\frac{\mathbf{Z}^{(1)}}{\|\mathbf{Z}^{(1)}\|_2}, \ldots, \frac{\|\mathbf{Z}^{(k)}\|_2}{\|\mathbf{Z}\|_2}\frac{\mathbf{Z}^{(k)}}{\|\mathbf{Z}^{(k)}\|_2}\right)'.$$

Since $\mathbf{Z}^{(1)}, \ \ldots, \ \mathbf{Z}^{(k)}$ are independent and $\|\mathbf{Z}^{(i)}\|_2$ is independent of $\mathbf{Z}^{(i)}/\|\mathbf{Z}^{(i)}\|_2$, each vector $\mathbf{Z}^{(i)}/\|\mathbf{Z}^{(i)}\|_2$ has a uniform distribution on the unit sphere in $\mathbb{R}^{n_i}$ (Fang et al. [1990], Theorem 2.6). Moreover, we have

$$\left(\frac{\|\mathbf{Z}^{(1)}\|_2^2}{\|\mathbf{Z}\|_2^2}, \ldots, \frac{\|\mathbf{Z}^{(k)}\|_2^2}{\|\mathbf{Z}\|_2^2}\right)' \sim \mathsf{Dir}_k\left(\frac{n_1}{2}, \ldots, \frac{n_k}{2}\right)$$

as $\|\mathbf{Z}^{(1)}\|_2^2, \ldots, \|\mathbf{Z}^{(k)}\|_2^2$ are independent χ^2-distributed. Therefore, let $\mathbf{Y}^{(1)}, \ldots,$ $\mathbf{Y}^{(k)}$ be independent primitives of $S(n_1, 1), \ \ldots, \ S(n_k, 1)$, respectively. Moreover, denote $\mathbf{V}^{(1)}, \mathbf{V}^{(2)} \sim \mathsf{Dir}_k(\frac{n_1}{2}, \ldots, \frac{n_k}{2})$ independent of all $\mathbf{Y}^{(i)}$. Then the vector

$$\left(\sqrt{\frac{V_1^{(1)}}{V_1^{(2)}}}\mathbf{Y}^{(1)}, \ldots, \sqrt{\frac{V_k^{(1)}}{V_k^{(2)}}}\mathbf{Y}^{(k)}\right)'$$

has a primitive distribution on $S(n, 1)$. Since $V_1^{(1)}, V_1^{(2)} \sim \mathsf{Beta}(\frac{n_1}{2}, \frac{n-n_1}{2})$ the proof is completed.

□

Remark 14. Fang et al. [1990] (Lemma 7.6) derived the density of $\sqrt{V_1/V_2}$ where $V_1, V_2 \sim \mathsf{Beta}(\frac{m}{2}, \frac{n-m}{2})$ are independent.

The result by Gneiting [1998] can be used to derive sufficient conditions for $S(n,1)$ based on the sufficient conditions for $S(n,2)$ which are known as Askey's theorem – let $\phi : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying $\phi(0) = 1$, $\lim_{t\to\infty} \phi(t) = 0$ and $(-1)^k \phi^{(k)}(t) \geq 0$ is convex for $k = \lfloor \frac{n}{2} \rfloor$. Then $\phi \in S(n,2)$. In the one-dimensional case, $n = 1$ the criterion is known as Pólya's theorem.

Theorem 33. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a continuous function such that $\psi(0) = 1$, $\lim_{t\to\infty} \psi(t) = 0$ and $\psi^{(2n-2)}(t)$ is convex. Then $\psi \in S(n,1)$.*

Proof. Our aim is to use Theorem 29 in combination with the representation of the m-times monotone functions from Theorem 8. If ψ satisfies the assumptions of Theorem 33 it is $(2n-1)$-times monotone (Remark 2) and we may find a non-negative random variable R such that

$$\psi(s) = \int_0^\infty (1 - sr)_+^{2n-1} \, dF(r), \quad s \geq 0,$$

where F is the cumulative distribution function of R. Thus, we will apply Theorem 29 only to Kuttner-Golubov functions $(1-t)_+^\delta$ (Example 8).

A function $(1-t)_+^\delta \in S(n,2)$ if and only if $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$ (Gneiting [1998]) which means $(1-t)_+^\delta \in S(2n-1,2)$ if and only if $\delta \geq n-1$. Furthermore, since the anti-derivative of $(1-t)_+^\delta$ is equal to $(1-t)_+^{\delta+1}$ up to a constant, Theorem 29 implies that $(1-t)^\delta \in S(n,1)$ if and only if $\delta \geq 2n-1$.

□

Zastavnyi [2000] further expanded Askey's and Gneiting's sufficient conditions into general α-symmetry, i.e. $(\mathbb{R}^n, \|\cdot\|_\alpha)$ or other quasi-normed spaces, more in Section 3.4.

3.3 Class $S(n,\alpha)$ for $n \geq 3$

Non-triviality of $S(2,\alpha)$ for any $0 < \alpha \leq \infty$ can be established through symmetric 1-stable distributions and vectors of i.i.d. symmetric α-stable random variables (Section 3.1). Higher-dimensional α-symmetric random vectors were known only for $\alpha \leq 2$. Firstly, we characterize $S(\infty,\alpha) = \cap_{n=1}^\infty S(n,\alpha)$ using complete monotonocity. Then we shall focus on classes $S(3,\alpha)$, $\alpha > 2$, as it is the smallest trivial class (in terms of dimensionality).

The results by Bretagnolle et al. [1966] for $S(\infty,\alpha)$ show that the stable distributions are a natural extension of the normal distribution not just in terms of independence (Example 6 and Theorem 15) but also in terms of dimensionality. Schoenberg proved that $\psi \in S(\infty,2)$ if and only if there exists a non-negative distribution λ such that

$$\psi(u) = \int_0^\infty e^{-tu^2} \, d\lambda(t)$$

which for any $n \in \mathbb{N}$ corresponds to the characteristic generator of a random vector $\sqrt{W}\mathbf{Z}$ where W has a distribution λ, $\mathbf{Z}$ is n-dimensional standard normal independent of W (Fang et al. [1990], Section 2.6). The following theorem is an analogue by Bretagnolle et al. [1966] formulated in terms of random variables.

Theorem 34. *The function ψ belongs to $S(\infty, \alpha)$, $\alpha \le 2$, if and only if there exists a non-negative random variable R with a cumulative distribution function F such that*

$$\psi(u) = \int_0^\infty e^{-ru^\alpha} \, dF(r), \quad u \ge 0.$$

If $\alpha > 2$ then $S(\infty, \alpha) = \{1\}$.

Corollary 7. We have a nice characterization of $S(\infty, \alpha)$, $\alpha \in (0, 2]$, from Subsection 1.1.2. By Theorem 7, a function $\psi(u) \in S(\infty, \alpha)$ if and only if $\psi(u^{\frac{1}{\alpha}})$ is completely monotone and $\psi(0) = 1$. Completely monotone functions have by Theorem 7 an integral representation through a non-negative random variable with a cumulative distribution F as

$$s \mapsto \int_0^\infty e^{-sr} \, F(r), \quad s \ge 0.$$

Theorem 34 replaces s by u^α.

Example 17. A non-trivial example (one which is not stable with a characteristic generator equal to e^{-z^α}) of a completely monotone function and therefore a member of $S(\infty, \alpha)$, $\alpha \in (0, 2]$, is the so called *generalized α-symmetric Linnik distribution* (one-dimensional version is defined in Devroye [1990]). Denote for $\alpha \in (0, 2]$ and $\beta \in (0, \infty)$ its characteristic generator

$$\psi_{\alpha,\beta}(u) = \frac{1}{(1 + u^\alpha)^\beta}, \quad u \ge 0.$$

Since $\psi_{1,\beta}$ is completely monotone (Miller and Samko [2001]) and $\psi_{1,\beta}(0) = 1$ we have $\psi_{\alpha,\beta} \in S(\infty, \alpha)$.

Denote $\mathbf{Y}_\alpha$ the random vector with a characteristic function $e^{-\|\mathbf{t}\|_\alpha^\alpha}$, $\mathbf{t} \in \mathbb{R}^n$, and let $V_\beta \sim \mathsf{Gamma}(\beta, 1)$ be a random variable with a density

$$\frac{1}{\Gamma(\beta)} v^{\beta-1} e^{-v}, \quad v > 0.$$

Then $\mathbf{X} = V_\beta^{\frac{1}{\alpha}} \mathbf{Y}_\alpha$ is α-symmetric with a characteristic generator (Theorem 16) $\psi_{\alpha,\beta}$ since for $\mathbf{t} \in \mathbb{R}^n$

$$\begin{aligned}
\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} &= \mathsf{E}\,\mathsf{E} \left[\exp\left\{ i\mathbf{t}' \left(V_\beta^{\frac{1}{\alpha}} \mathbf{Y}_\alpha \right) \right\} \middle| V_\beta \right] \\
&= \mathsf{E} \exp\{-V_\beta \|\mathbf{t}\|_\alpha^\alpha\} \\
&= \int_0^\infty \exp\{-(1 + \|\mathbf{t}\|_\alpha^\alpha) u\} \frac{1}{\Gamma(\beta)} u^{\beta-1} \, du \\
&= \frac{1}{(1 + \|\mathbf{t}\|_\alpha^\alpha)^\beta}.
\end{aligned}$$

The moments of the Gamma distribution are equal to

$$\mathsf{E}\, V_\beta^{\frac{m}{\alpha}} = \int_0^\infty \frac{1}{\Gamma(\beta)} v^{\frac{m}{\alpha}+\beta-1} e^{-v} \, dv = \frac{\Gamma(\frac{m}{\alpha} + \beta)}{\Gamma(\beta)}$$

which may be combined with Example 2. Using Lemma 11 we get that $(1 + u^\lambda)^{-\beta} \in S(\infty, \alpha)$ for $0 < \lambda \le \alpha \le 2$ as we may take $\mathbf{Y}_\lambda$ with a characteristic function $e^{-\|\mathbf{t}\|_\alpha^\lambda}$, $\mathbf{t} \in \mathbb{R}^n$, and $V_\beta^{\frac{1}{\lambda}}$ as above.

For a random vector $\mathbf{X} \sim S(n, \alpha, (1+u^\lambda)^{-\beta})$ there is a stochastic decomposition

$$\mathbf{X} \stackrel{d}{=} V^{\frac{1}{\lambda}} W^{\frac{1}{\alpha}} \mathbf{Z}$$

where $V \sim \mathsf{Gamma}(\beta, 1)$, W is non-negative $\frac{\lambda}{\alpha}$-stable (1.2) and $\mathbf{Y}$ is composed of i.i.d. symmetric α-stable random variables, all jointly independent.

Khokhlov et al. [2020] linked the univariate mixing variable $V^{\frac{1}{\lambda}} W^{\frac{1}{\alpha}}$ to the *generalized Mittag-Leffler distribution.* For $\beta = 1$, $\alpha = \lambda = 2$ the distribution of $\mathbf{X}$ is known as the *multivariate Laplace distribution.*

The moments of $\mathbf{X}$ are for $m_1, \dots, m_n \in (0, \alpha)$, $m_1 + \dots + m_n = m < \lambda$ equal to

$$\begin{aligned}
\mathsf{E}\left(\prod_{k=1}^n |X_k|^{m_k}\right) &= \mathsf{E}\, V^{\frac{m}{\lambda}}\, \mathsf{E}\, W^{\frac{m}{\alpha}} \prod_{k=1}^n \mathsf{E}\left(|Z_k|^{m_k}\right) \\
&= \frac{\Gamma(\frac{m}{\lambda} + \beta)}{\Gamma(\beta)} \frac{\Gamma\left(1 - \frac{m}{\lambda}\right)}{\Gamma\left(1 - \frac{m}{\alpha}\right)} \prod_{k=1}^n \frac{\Gamma(1 - \frac{m_k}{\alpha})}{\cos(m_k \frac{\pi}{2}) \Gamma(1 - m_k)}.
\end{aligned}$$

This example is a new extension to the one-dimensional Linnik distribution defined by Devroye [1990] and the multivariate spherical Linnik distribution defined by Khokhlov et al. [2020].

Triviality of $S(3, \alpha)$, $\alpha > 2$, was shown independently by Zastavnyi [1992] and Lisitkii. The following formulation (in terms of norm dependent positive-definite functions) is due to Zastavnyi. The criterion set by Zastavnyi [1992] holds not only for $(\mathbb{R}^n, \alpha)$, $n \geq 3$, $\alpha > 2$, but for even for $\mathcal{C}(0,1)$ and L_α, $\alpha > 2$.

Let us first state two lemmas which will be helpful to prove Zastavnyi's main result (Theorem 37). Contrary to Zastavnyi the theory of random variables will be used directly for clarity of some arguments.

Lemma 35. *Let ϕ be a real characteristic function such that $\phi'(t)$ exists for each $t \neq 0$ and $\lim_{t\to 0} \frac{\phi'(t)}{t} = 0$. Then ϕ is constant on $\mathbb{R}$.*

Proof. Since $\lim_{t\to 0} \frac{\phi'(t)}{t} = 0$, the limit $\lim_{t\to 0} \phi'(t) = 0$ which means $\phi'(0) = \lim_{t\to 0} \frac{\phi(t) - \phi(0)}{t} = \lim_{t\to 0} \phi'(\xi(t)) = 0$ by the mean-value theorem (Rudin [1976], Theorem 5.10) and $\xi(t)$ lies between 0 and t, i.e. $\lim_{t\to 0} \xi(t) = 0$. Thus we may find the second derivative:

$$\phi''(0) = \lim_{t\to 0} \frac{\phi'(t) - \phi'(0)}{t} = \lim_{t\to 0} \frac{\phi'(t)}{t} = 0.$$

By Theorem 4 the variance of the corresponding random variable is equal to 0 and the random variable must be constant and almost surely zero. The characteristic function is thus constant.

□

Lemma 36. *Let $(E, \|\cdot\|)$ be a real normed space, $f(\|\cdot\|)$ be a characteristic function on $(E, \|\cdot\|)$ and let $g : [0, \infty) \to [0, \infty)$ be a continuously differentiable function with a compact support satisfying $\int_0^\infty g(t)\, dt = 1$. Denote*

$$F(t) = \int_0^\infty f(ts) g(s)\, ds, \quad t > 0.$$

Then

(i) $F(\|\cdot\|)$ *is a characteristic function,*

(ii) F *has continuous derivatives on* $(0,\infty)$*,*

(iii) $\lim_{t\to 0+} tF'(t) = 0$ *and there exists* $c > 0$ *such that* $|tF'(t)| < c$ *on* $(0,\infty)$*,*

(iv) F *being constant implies that* f *is also constant,*

(v) if $\lim_{t\to 0+} f'(t) = 0$*, then* $\lim_{t\to 0+} F'(t) = 0$ *and there exists* $c_1 > 0$ *such that* $|F'(t)| < c_1$ *for* $t > 0$*.*

Proof. For properties (i) and (v), denote $\mathbf{X}$ a random vector such that $f(\|\cdot\|)$ is its characteristic function and let Y be an independent real random variable with a density g. The characteristic function of $Y\mathbf{X}$ is $F(\|\cdot\|)$ (the procedure is similar as in Theorem 16). If F is constant (equal to 1), then $Y\mathbf{X}$ is almost surely zero and since Y is independent and absolutely continuous the random vector $\mathbf{X}$ is almost surely zero, and f is constant. That concludes (i) and (v).

Now for the derivatives in (iii)

$$F(t) = \int_0^\infty f(ts)g(s)\ ds \overset{y=ts}{=} \frac{1}{t}\int_0^\infty f(y)g\left(\frac{y}{t}\right)\ dy$$

which for $t > 0$ by the differentiation of a product satisfy:

$$F'(t) = -\frac{1}{t^2}\int_0^\infty f(y)g\left(\frac{y}{t}\right)\ dy - \frac{1}{t^3}\int_0^\infty f(y)g'\left(\frac{y}{t}\right) y\ dy$$

which is continuous (proving part (ii) of this theorem). Rewriting again

$$tF'(t) = \int_0^\infty f(ts)\left[g(s) + g'(s)s\right]\ ds \overset{t\to 0+}{\to} f(0)\int_0^\infty g(s) + g'(s)s\ ds.$$

The last integral is equal to 0 since $(sg(s))' = g(s) + sg'(s)$ and g has a compact support, i.e. $sg(s) = 0$ for some s sufficiently large and $sg(s) \to 0$ as $s \to 0+$ since $\int_0^\infty g(s)\ ds = 1$. That concludes the proof of (ii) and the first half of (iii).

The boundedness of $tF'(t)$ in (iii) is shown by setting $c = f(0)\int_0^\infty |g(s) + g'(s)s|\ ds$ as

$$|F'(t)t| \le \int_0^\infty |f(ts)||g(s) + g'(s)s|\ ds$$

and $f(0) \ge |f(t)|$ since f is a continuous positive definite function (Lemma 3). For the derivative of F' assume that the compact support of g is covered by the interval $(0,a]$. Then

$$\begin{aligned}|F'(t)| &= \left|\int_0^\infty \frac{f(0)-f(ts)}{ts} s\left[g(s)+g'(s)s\right]\ ds\right| \\ &\le \sup_{x\in[0,at]}\left|\frac{f(x)-f(0)}{x}\right| \int_0^\infty s|g(s)+g'(s)s|\ ds\end{aligned}$$

which tends to zero as $t \to 0+$ if $f'(0) = 0$ since the integral is finite. Moreover, since F' is continuous, $F'(t)t$ is bounded and $\lim_{t\to 0+} F'(t) = 0$ (if $\lim_{t\to 0+} f'(t) = 0$) then F' is also bounded for $t > 0$. That concludes (v).

□

For the main theorem let us first state some properties of the norm:

Since $\|\cdot\|$ is a norm (in particular a positive homogeneous function of order 1), its partial derivative $G(t, y_1, y_2) = \frac{\partial}{\partial t}\|t\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|$ satisfies

$$\begin{aligned} G(kt, ky_1, ky_2) &= \lim_{\varepsilon \to 0} \frac{\|(kt+\varepsilon)\mathbf{a}_3 + ky_1\mathbf{a}_1 + ky_2\mathbf{a}_2\| - \|kt\mathbf{a}_3 + ky_1\mathbf{a}_1 + ky_2\mathbf{a}_2\|}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} |k| \frac{\|(t+\frac{\varepsilon}{k})\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\| - \|t\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|}{k \cdot \frac{\varepsilon}{k}} \\ &= \operatorname{sgn} k \cdot G(t, y_1, y_2) \end{aligned}$$

for $k \neq 0$. Furthermore, $G(t, y_1, y_2)$ is defined for almost all $(t, y_1, y_2)' \in \mathbb{R}^3$ since the norm is convex. The function is also bounded $|G(t, y_1, y_2)| \leq \|\mathbf{a}_3\|$ for any $(t, y_1, y_2)'$ in the domain of G by a combination of convexity and the triangular inequality. Assumptions of Theorem 37 will be checked for the α-norms in Corollary 8.

Theorem 37. *Let $(E, \|\cdot\|)$ be a normed space with dimension at least 3 and assume there exist three linearly independent $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in E$. Define the functions*

$$H(y_1, y_2) = \frac{G(1, y_1, y_2)}{\|\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|}, \quad G(t, y_1, y_2) = \frac{\partial}{\partial t}\|t\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|.$$

Furthermore, assume that $H(y_1, y_2)$ is integrable on $\mathbb{R}^2$ and let $\psi(\|\cdot\|) : E \to \mathbb{R}$ be a continuous positive definite function. Then ψ is constant.

Proof. The proof is done in several steps:

Take any positive definite continuous function $\psi(\|\cdot\|) : E \to \mathbb{R}$, then

$$\tilde{\psi} : (t, y_1, y_2)' \mapsto \psi(\|t\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|), \quad (t, y_1, y_2)' \in \mathbb{R}^3,$$

is a continuous positive definite continuous function on $\mathbb{R}^3$ which depends on the norm $(t, y_1, y_2)' \mapsto \|t\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|$. Denote $\mathbf{X}$ the random vector corresponding to $\tilde{\psi}$. Using Lemma 36 with some g (a density of some random variable Y independent of $\mathbf{X}$), denote $\Psi(\|\cdot\|)$ the characteristic function of $Y\mathbf{X}$.

Moreover, let us take a random vector $\mathbf{Z}_\varepsilon$, $\varepsilon > 0$, independent of Y and $\mathbf{X}$ with an integrable characteristic function $h(z_1, z_2)e^{-\varepsilon|z_3|}$, where h is an integrable positive definite function. The density f of $Y\mathbf{X} + \mathbf{Z}_\varepsilon$ exists (the characteristic function ϕ_ε of is $Y\mathbf{X} + \mathbf{Z}_\varepsilon$ integrable) and is equal to (Theorem 2)

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\mathbf{x}'\mathbf{t}} \Psi(\|t_3\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_1, t_2) e^{-\varepsilon|t_3|} \, dt_1 dt_2 dt_3, \quad \mathbf{x} \in \mathbb{R}^3.$$

For $\mathbf{x} = (0, 0, s)'$ the function $\tilde{f} : s \mapsto f(0, 0, s)$ is non-negative

$$\tilde{f}(s) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}} e^{-ist_3} e^{-\varepsilon|t_3|} \int_{\mathbb{R}^2} \Psi(\|t_3\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_1, t_2) \, dt_1 dt_2 \, dt_3.$$

If we perceive the function $\tilde{f}$ as a density (up to a constant) of some random vector, the characteristic function of such vector is equal to (using again Theorem 2)

$$t_3 \mapsto e^{-\varepsilon|t_3|} \int_{\mathbb{R}^2} \Psi(\|t_3\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_1, t_2) \, dt_1 dt_2, \quad t_3 \in \mathbb{R} \tag{3.33}$$

up to a constant. As the function (3.33) is positive definite for any $\varepsilon > 0$, we may take a limit $\varepsilon \to 0+$ and the result will also be positive definite. Denote the limit

$$\phi(t) = \int_{\mathbb{R}^2} \Psi(\|t\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|)h(t_1,t_2)\; dt_1 dt_2, \quad t \in \mathbb{R}. \tag{3.34}$$

Let us find the derivative of ϕ at point $t \neq 0$. Using Lemma 36 the derivative of Ψ exists and $|t\Psi(t)|$ is bounded and G is bounded. Then

$$\begin{aligned}\phi'(t) &= \int_{\mathbb{R}^2} \Psi'(\|t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + t\mathbf{a}_3\|)G(t,t_1,t_2)h(t_1,t_2)\; dt_1 dt_2 \\ &\overset{y_k = t_k/t}{=} \operatorname{sgn} t \cdot t^2 \int_{\mathbb{R}^2} \Psi'(|t| \cdot \|\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|)G(1,y_1,y_2)h(ty_1,ty_2)\; dy_1 dy_2.\end{aligned}$$

We aim to evaluate $|\phi'(t)/t|$ for $t \neq 0$: Denote

$$Q(t,y_1,y_2) = \Psi'(|t| \cdot \|\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|) \cdot |t| \cdot \|\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\|.$$

Then for $r \neq 0$ we shall split $\mathbb{R}^2$ into two sets: $B_r = \{(y_1,y_2)' \in \mathbb{R}^2 : \|y_1\mathbf{a}_1 + y_2\mathbf{a}_2\| \leq r\}$ and its complement $\mathbb{R}^2 \setminus B_r$. The ratio $|\phi'(t)/t|$ is estimated as

$$\begin{aligned}\left|\frac{\phi'(t)}{t}\right| &= \left|\int_{\mathbb{R}^2} Q(t,y_1,y_2)H(y_1,y_2)h(ty_1,ty_2)\; dy_1 dy_2\right| \\ &\leq \int_{B_r} |Q(t,y_1,y_2)H(y_1,y_2)h(ty_1,ty_2)|\; dy_1 dy_2 \\ &+ \int_{\mathbb{R}^2\setminus B_r} |Q(t,y_1,y_2)H(y_1,y_2)h(ty_1,ty_2)|\; dy_1 dy_2.\end{aligned}$$

Each integral will be treated differently. The function h is bounded by $h(0,0) = 1$ and $|Q(t,y_1,y_2)| \leq \sup |s\Psi'(s)|$ where the supremum is taken over s satisfying $0 \leq s = |t| \cdot \|\mathbf{a}_3 + y_1\mathbf{a}_1 + y_2\mathbf{a}_2\| \leq |t|(\|\mathbf{a}_3\| + r)$. The first integral is bounded from above by

$$\sup_{0 \leq s \leq |t|(r + \|\mathbf{a}_3\|)} |\Psi'(s)s| \cdot \int_{B_r} |H(y_1,y_2)|\; dy_1 dy_2.$$

The second can be furthermore bounded by the constant c satisfying $|\Psi'(t)t| \leq c$ for $t \neq 0$ from Lemma 36. That concludes

$$\begin{aligned}\left|\frac{\phi'(t)}{t}\right| \leq & \sup_{0 \leq s \leq |t|(r + \|\mathbf{a}_3\|)]} |\Psi'(s)s| \cdot \int_{B_r} |H(y_1,y_2)|\; dy_1 dy_2 \\ &+ c \int_{\mathbb{R}^2\setminus B_r} |H(y_1,y_2)|\; dy_1 dy_2.\end{aligned} \tag{3.35}$$

As from Lemma 36

$$\limsup_{t\to 0} \sup_{0 \leq s \leq |t|(r + \|\mathbf{a}_3\|)} |\Psi'(s)s| \cdot \int_{B_r} |H(y_1,y_2)|\; dy_1 dy_2 = 0$$

we have that the $\limsup_{t\to 0} \left|\frac{\phi'(t)}{t}\right|$ is bounded only by the second part of (3.35)

$$\limsup_{t\to 0} \left|\frac{\phi'(t)}{t}\right| \leq c \cdot \int_{\mathbb{R}^2\setminus B_r} |H(y_1,y_2)|\; dy_1 dy_2$$

which holds for any $r > 0$. Thus, for $r \to \infty$ and since H is assumed to be integrable, the limit is zero

$$\lim_{t \to 0} \frac{\phi'(t)}{t} = 0.$$

The upper limit of the first integral of (3.35) is equal to zero based on Lemma 36. The inequality holds for all $r > 0$ and we may pass $r \to \infty$ by which (since H is assumed to be integrable) we get the limit $\lim_{t \to 0} \phi'(t)/t = 0$. With the help of Lemma 35 a random variable with a characteristic function ϕ is trivial.

Lemma 35 states that

$$\phi(t) = \int_{\mathbb{R}^2} \Psi(\|t\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_1, t_2) \; dt_1 dt_2, \quad t \in \mathbb{R} \tag{3.36}$$

is thus constant. Our aim is to use a similar approach as in Lemma 36 (especially (iv)) again in order to prove that Ψ is constant. For that set $h_n(t_1, t_2) = n^2(1 - n|t_1|)_+(1 - n|t_2|)_+$ which is a continuous positive definite non-negative function with a support $[0, \frac{1}{n}]^2$ and

$$\int_{\mathbb{R}^2} n^2(1 - n|t_1|)_+(1 - n|t_2|)_+ \; dt_1 dt_2 \overset{x_i = nt_i}{=} \int_{\mathbb{R}^2} (1 - |x_1|)_+(1 - |x_2|)_+ \; dx_1 dx_2 = 1.$$

Using h_n, $n \in \mathbb{N}$, we create ϕ_n, $n \in \mathbb{N}$, as in (3.36). Since all ϕ_n, $n \in \mathbb{N}$, follow (3.36), all are constant. We can rewrite

$$\begin{aligned}
\phi_n(t) &= \int_{\mathbb{R}^2} \Psi(\|t\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h_n(t_1, t_2) \; dt_1 dt_2, \\
&= \int_{\mathbb{R}^2} \Psi(\|t\mathbf{a}_3 + t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) n^2(1 - n|t_1|)_+(1 - n|t_2|)_+ \; dt_1 dt_2 \\
&\overset{x_i = nt_i}{=} \int_{\mathbb{R}^2} \Psi\left(\left\|\frac{x_1\mathbf{a}_1 + x_2\mathbf{a}_2}{n} + t\mathbf{a}_3\right\|\right)(1 - |x_1|)_+(1 - |x_2|)_+ \; dx_1 dx_2 \\
&= \phi_n(0).
\end{aligned}$$

As $n \to \infty$, the last integral tends to

$$\int_{\mathbb{R}^2} \Psi\left(\left\|\frac{x_1\mathbf{a}_1 + x_2\mathbf{a}_2}{n} + t\mathbf{a}_3\right\|\right)(1 - |x_1|)_+(1 - |x_2|)_+ \; dx_1 dx_2 \overset{n \to \infty}{\to} \Psi(\|t\mathbf{a}_3\|).$$

In conclusion, $\Psi(\|t\mathbf{a}_3\|) = \Psi(0)$ as it is a point limit of constant functions. Lemma 36 concludes the proof and finally, both Ψ and ψ are constant functions.

□

Corollary 8. Let us check that $(\mathbb{R}^3, \|\cdot\|_\alpha)$, $\alpha > 2$, satisfies the assumptions of Theorem 37. For α-norms, i.e. for $1 \le \alpha \le \infty$ the function H is equal to

$$H(y_1, y_2) = \frac{1}{1 + |y_1|^\alpha + |y_2|^\alpha}$$

which is integrable over $\mathbb{R}^2$ if $\alpha > 2$.

Now for $\alpha = \infty$ the function G is equal to

$$G(t, y_1, y_2) = \begin{cases} 0, & |t| < \max\{|y_1|, |y_2|\}, \\ \text{sign } t, & |t| > \max\{|y_1|, |y_2|\} \end{cases}$$

which means $H(y_1, y_2) = G(1, y_1, y_2) = \mathbb{1}(\max\{|y_1|, |y_2|\} < 1)$. The function H is integrable and $S(n, \infty)$ for $n \geq 3$ is trivial.

Therefore, the class $S(3, \alpha)$, $2 < \alpha \leq \infty$, is trivial. If we combine this result, the fact that vectors of i.i.d. symmetric α-stable random variables are α-symmetric for $\alpha \leq 2$, and the inclusion property of classes $S(n, \alpha)$ (discussed under Lemma 13), we obtain an important result: for $n \geq 3$ the class $S(n, \alpha)$ is trivial if and only if $\alpha > 2$.

The implications of Theorem 37 will be discussed in terms of pseudo-isotropy in Chapter 4. Now we may look at Theorem 23, its proof repeats some steps of the proof of Theorem 37 until the estimation of (3.38).

Proof of Theorem 23. Let us again take some $\psi : [0, \infty) \to \mathbb{R}$ such that $\psi(\|\cdot\|)$ is positive definite on E and $\lim_{t\to 0+} \psi(t) = 0$. Thus, $(t, y)' \mapsto \psi(\|t\mathbf{a}_1 + y\mathbf{a}_2\|)$ is a positive definite function on $\mathbb{R}^2$ depending on the norm $(t, y)' \mapsto \|t\mathbf{a}_1 + y\mathbf{a}_2\|$ of some random vector $\mathbf{X}$. Using Lemma 36 let us create a characteristic function $\Psi(\|\cdot\|)$ (of a random vector $Y\mathbf{X}$ where Y has a density g). The function Ψ also satisfies $\lim_{t\to 0+} \Psi(t) = 0$ (Lemma 36, part (v)). Further, let $h(z_2)e^{-\varepsilon|z_1|}$, $(z_1, z_2)' \in \mathbb{R}^2$, be a characteristic function of random vector $\mathbf{Z}_\varepsilon$, $\varepsilon > 0$, independent of $\mathbf{X}$ and Y.

The density of $Y\mathbf{X} + \mathbf{Z}_\varepsilon$ is equal to (Theorem 2)

$$f(x_1, x_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(x_1t_1+x_2t_2)} \Psi(\|t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_2) e^{-\varepsilon|t_1|} \, dt_1 dt_2.$$

Then $f(s, 0)$ is equal to

$$s \mapsto \frac{1}{(2\pi)^2} \int_{\mathbb{R}} e^{-st_1} e^{-\varepsilon|t_1|} \int_{\mathbb{R}} \Psi(\|t_1\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_2) \, dt_2 \, dt_1$$

and such function can be viewed as a density of a random variable (up to a constant) whose characteristic function (up to a constant) is equal to

$$t \mapsto e^{-\varepsilon|t|} \int_{\mathbb{R}} \Psi(\|t\mathbf{a}_1 + t_2\mathbf{a}_2\|) h(t_2) \, dt_2$$

due to Theorem 2. By limiting $\varepsilon \to 0+$ the function is still continuous positive definite:

$$\phi(t) = \int_{\mathbb{R}} \Psi(\|t\mathbf{a}_1 + x\mathbf{a}_2\|) h(x) \, dx. \tag{3.37}$$

As in Theorem 37, the derivative of ϕ is equal to

$$\begin{aligned} \phi'(t) &= \int_{\mathbb{R}} \Psi'(\|t\mathbf{a}_1 + x\mathbf{a}_2\|) G(t, x) h(x) \, dx \\ &\overset{y=x/t}{=} \text{sgn } t \cdot |t| \int_{\mathbb{R}} \Psi'(|t| \cdot \|\mathbf{a}_1 + y\mathbf{a}_2\|) G(1, y) h(ty) \, dy. \end{aligned}$$

For the estimation of $|\phi'(t)/t|$ we shall separately consider $0 < y < r/\|\mathbf{a}_2\|$ and $r/\|\mathbf{a}_2\| < y < \infty$ for some $r > 0$. Then since $G(1, y)$ is integrable and $1 = h(0) \geq |h(x)|$ we obtain

$$\left|\frac{\phi'(t)}{t}\right| = \left|\int_{\mathbb{R}} \Psi'(|t| \cdot \|\mathbf{a}_1 + y\mathbf{a}_2\|)G(1,y)h(ty)\ dy\right|$$
$$\leq \sup_{0\leq|t|(r+\|\mathbf{a}_1\|)} |\Psi'(s)| \int_{\mathbb{R}} |G(1,y)|\ dy + c_1 \int_{|y|>\frac{r}{\|\mathbf{a}_2\|}}^{\infty} |G(1,y)|\ dy. \qquad (3.38)$$

Lemma 36 ensures that Ψ' is bounded and $\Psi'(t) \overset{t\to 0+}{\to} 0$. As

$$\limsup_{t\to 0} \left|\frac{\phi'(t)}{t}\right| \leq c_1 \int_{|y|>\frac{r}{\|\mathbf{a}_2\|}}^{\infty} |G(1,y)|\ dy$$

for any $r > 0$, we can take $r \to \infty$ which means $|\phi'(t)/t| \to 0$ as $t \to 0$. Therefore, ϕ must be constant. The rest of the proof follows the same steps as the proof of Theorem 37.

□

Among other results concerning higher-dimensional α-symmetric distributions was the proof of triviality of $S(3, \infty)$ by Misiewicz [1989]. The proof uses the density derived in Theorem 20. This result is covered by Theorem 37.

3.4 Sufficient Conditions for $S(n, \alpha)$

The section follows the generalization of Askey's and Gneiting's sufficient conditions (Theorem 29) for the characteristic functions which are based on Kuttner-Golubov and Schoenberg characteristic generators. The most important result of this section is the connection between the constants $\lambda(n, \alpha)$ and $\sigma(n, \alpha)$. Recall how the constants $\sigma(n, \alpha)$, $\lambda(n, \alpha)$ and $\delta(\lambda; n, \alpha)$ are defined in Example 7 and Example 10:

$$\sigma(n, \alpha) = \sup\{\beta \in [0, 2] : \exp\{-t^{\beta}\} \in S(n, \alpha)\}, \qquad (3.39)$$
$$\lambda(n, \alpha) = \sup\{\lambda \in (0, 2] : (1 - t^{\lambda})_+^{\delta} \in S(n, \alpha) \text{ for some } \delta > 0\}, \qquad (3.40)$$
$$\delta(\lambda; n, \alpha) = \inf\{\delta > 0 : (1 - t^{\lambda})_+^{\delta} \in S(n, \alpha)\}, \qquad (3.41)$$

and $\delta(\lambda) = \delta(\lambda; 1, 2)$ as the one-dimensional case.

Example 20 and Example 21 present additional possibilities of generating β-symmetric random vectors from α-symmetric random vectors. First, let us establish a connection between Schoenberg and Kuttner-Golubov problems (Theorem 46 by Zastavnyi [2000] established a less direct approach shown in Section 4).

Example 18. Let $\lambda(n, \alpha) > 0$ as defined in (3.40), then we can show $\lambda(n, \alpha) \leq \sigma(n, \alpha)$ using the following computation. Let $\mathbf{X}_\delta$ be a random vector with a characteristic function $\varphi_{\lambda,\delta}(\|\mathbf{t}\|_\alpha) = (1 - \|\mathbf{t}\|_\alpha^{\lambda})_+^{\delta}$, $\mathbf{t} \in \mathbb{R}^n$, for $\lambda > 0$ and $\delta > 0$ and let $V \sim \mathsf{Gamma}(\delta + 1, 1)$ be independent of $\mathbf{X}$. Denote g the density of V, then

$V^{-\frac{1}{\lambda}}\mathbf{X}_\delta$ is α-symmetric with a characteristic generator

$$\begin{aligned}\int_0^\infty \varphi_{\lambda,\delta}\left(tv^{-\frac{1}{\lambda}}\right) g(v)\ dv &= \int_{t^\lambda}^\infty \left(1-\frac{t^\lambda}{v}\right)^\delta \frac{1}{\Gamma(\delta+1)} v^\delta e^{-v}\ dv \\ &= \frac{1}{\Gamma(\delta+1)}\int_{t^\lambda}^\infty (v-t^\lambda)^\delta e^{-v}\ dv \\ &= e^{-t^\lambda}\int_0^\infty \frac{1}{\Gamma(\delta+1)} s^\delta e^{-s}\ ds\end{aligned}$$

and $e^{-t^\lambda} \in S(n,\alpha)$ which means if $0 < \lambda(n,\alpha)$ then $0 < \lambda(n,\alpha) \le \sigma(n,\alpha)$.

The result is connected to Example 8 as for a random variable $B_\mu \sim \mathsf{Beta}(\delta+1,\mu)$ the limit of μB_μ as $\mu \to \infty$ has a distribution $\mathsf{Gamma}(\delta+1,1)$ just as V (Hasebe [2014]). As was established in Example 8,

$$\mathbf{X}_{\delta+\mu} \stackrel{d}{=} B_\mu^{-\frac{1}{\lambda}}\mathbf{X}_\delta$$

which means

$$\mu^{-\frac{1}{\lambda}}\mathbf{X}_{\delta+\mu} \stackrel{d}{=} (\mu B_\mu)^{-\frac{1}{\lambda}}\mathbf{X}_\delta \xrightarrow{d} V^{-\frac{1}{\lambda}}\mathbf{X}_\delta.$$

The limit of $\mu^{-\frac{1}{\lambda}}\mathbf{X}_{\delta+\mu}$ where $\mathbf{X}_{\delta+\mu}$ has a characteristic function $(1-\|\mathbf{t}\|_\alpha^\lambda)_+^{\delta+\mu}$, $\mathbf{t} \in \mathbb{R}^n$, can be found as a point-wise limit of the characteristic functions of $\mu^{-\frac{1}{\lambda}}\mathbf{X}_{\delta+\mu}$

$$\lim_{\mu\to\infty}\left(1-\frac{\|\mathbf{t}\|_\alpha^\lambda}{\mu}\right)_+^{\delta+\mu} = e^{-\|\mathbf{t}\|_\alpha^\lambda}$$

which is the characteristic function of $V^{-\frac{1}{\lambda}}\mathbf{X}$. This relationship between Kuttner-Golubov and Schoenberg function was (in a different context) mentioned in Jasiulis [2010] (without the connection to the limit behavior of the Beta distribution).

Let us look in detail at the properties of $\lambda(n,\alpha)$ and $\delta(\lambda;n,\alpha)$ (defined in (3.40) and (3.41)). The following theorem and its proof can be found partly in Zastavnyi [2000]. The univariate properties of $\delta(\lambda) = \delta(\lambda;1,2)$ are taken from Gneiting et al. [2001].

Theorem 38. *Let $\lambda(n,\alpha) > 0$. Then $\lambda \mapsto \delta(\lambda;n,\alpha)$, for $0 < \lambda < \lambda(n,\alpha)$, is a continuous increasing function satisfying:*

(i) $(1-u^\lambda)_+^\delta \in S(n,\alpha)$ *if and only* $\delta \ge \delta(\lambda;n,\alpha)$.

(ii) For $\lambda > \lambda(n,\alpha)$ and any $\delta > 0$ the function $(1-u^\lambda)_+^\delta \notin S(n,\alpha)$.

(iii) Generally, $\lim_{\lambda\to 0+}\delta(\lambda;n,\alpha) > 0$ *and* $\lim_{\lambda\to 0+}\delta(\lambda) > 0.4279$.

(iv) There are only two possible situations for $\lambda_0 = \lambda(n,\alpha)$. Either $(1-u^{\lambda_0})_+^\delta \notin S(n,\alpha)$ for any $\delta > 0$ and

$$\lim_{\lambda\to\lambda_0-}\delta(\lambda;n,\alpha) = \infty.$$

Or the limit is finite and $\lambda \mapsto \delta(\lambda;n,\alpha)$ is continuous on $(0,\lambda_0]$. Namely $\lambda(1,2) = 2$ but $\lim_{\lambda\to 2-}\delta(\lambda) = \infty$.

Remark 15. Several authors established lower and upper bounds for $\delta(\lambda)$, $\lambda \in (0,2)$. Gneiting et al. [2001] found numerical lower bounds and $\delta(2-\frac{1}{k}) \leq k$ for any odd $k \in \mathbb{N}$. Further, $\delta(\lambda) \leq 1$ for $\lambda \in (0,1]$.

We have established that $\lambda(n,\alpha) \leq \sigma(n,\alpha)$ if $\lambda(n,\alpha) > 0$. From Theorem 37 we have $\lambda(n,\alpha) = \sigma(n,\alpha) = 0$ for $n \geq 3$ and $\alpha > 2$ as $S(n,\alpha)$ is trivial. Theorem 23 establishes $\lambda(2,\alpha) \leq 1$ and $\sigma(2,\alpha) = 1$ for $\alpha > 2$.

The goal of the next theorem is the following: to show $\lambda(n,\alpha) = \alpha$ for $n \geq 2$ and $\alpha \leq 2$ in order to prove $\lambda(n,\alpha) = \sigma(n,\alpha)$ for any $n \in \mathbb{N}$ and $0 < \alpha \leq \infty$.[12] Further, we aim to obtain simplification of $\delta(\alpha; n, \alpha)$ using $\delta(\alpha)$ by Zastavnyi [2000].

Theorem 39. *Let $(\mathbb{R}^n, \rho)$ be a quasi-normed space, $\lambda, \delta > 0$ and $D \subset \mathbb{R}^n$ such that $\{u\mathbf{x} : u \geq 0, \mathbf{x} \in D\} = \mathbb{R}^n$. For $\mathbf{x} \in D$ denote*

$$f_{\mathbf{x}}(u) = \frac{1}{\Gamma(\delta+1)u^{\delta+1}} \int_{\mathbb{R}^n} e^{-u\rho^\lambda(\mathbf{t})} e^{i\mathbf{t}'\mathbf{x}}\, d\mathbf{t}, \quad u > 0.$$

Then $(1-\rho(\mathbf{t})^\lambda)_+^\delta$, $\mathbf{t} \in \mathbb{R}^n$, is positive definite if and only if $f_{\mathbf{x}}$ is completely monotone for each $\mathbf{x} \in D$.

Proof. Denote for $\mathbf{x} \in D$

$$g_{\mathbf{x}}(v) = v^{\frac{n}{\lambda}+\delta} \int_{\mathbb{R}^n} (1-\rho^\lambda(\mathbf{s}))_+^\delta e^{iv^{\frac{1}{\lambda}}\mathbf{s}'\mathbf{x}}\, d\mathbf{s} \overset{t_i = v^{-\frac{1}{\lambda}} s_i}{=} \int_{\mathbb{R}^n} (v-\rho^\lambda(\mathbf{t}))_+^\delta e^{i\mathbf{t}'\mathbf{x}}\, d\mathbf{t}, \quad v > 0.$$

Then Theorem 1 and Theorem 7 outline the equivalent conditions for positive definite and completely monotone functions. The following statements are equivalent:

(i) $(1-\rho(\mathbf{t})^\lambda)_+^\delta$, $\mathbf{t} \in \mathbb{R}^n$, is positive definite;

(ii) $\tilde{g}(\mathbf{x}) = \int_{\mathbb{R}^n} (1-\rho(\mathbf{t})^\lambda)_+^\delta\, e^{i\mathbf{x}'\mathbf{t}}\, d\mathbf{t}$ is non-negative for any $\mathbf{x} \in \mathbb{R}^n$;

(iii) the function $g_{\mathbf{x}}(v) = \tilde{g}(v^{\frac{1}{\lambda}}\mathbf{x}) \geq 0$ for any $v > 0$ and $\mathbf{x} \in D$;

(iv) the Laplace transform of $g_{\mathbf{x}}$ is completely monotone for any $\mathbf{x} \in D$.

Equivalence of the first and second statements is a consequence of Theorem 2 relating densities and characteristic functions. Equivalence of (ii) and (iii) is due to the definition of D. Statements (iii) and (iv) are equivalent via Corollary 2 which connects non-negative functions and completely monotone functions. We shall further use the equivalence of (i) and (iv).

[12] The proof of $\lambda(2,\alpha) \geq 1$ for $\alpha \geq 1$ is omitted for brevity but $\sigma(2,\alpha) = \lambda(2,\alpha) = 1$ for $\alpha > 2$ (and can be found in Zastavnyi [1992]).

Fix any $\mathbf{x} \in D$. Let us now find a Laplace transform of the function $g_{\mathbf{x}}$ then for $u > 0$

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-vu} g_{\mathbf{x}}(v)\, dv &= \int_0^\infty e^{-vu} \int_{\mathbb{R}^n} (v - \rho^\lambda(\mathbf{t}))_+^\delta e^{i\mathbf{t}'\mathbf{x}}\, d\mathbf{t}\, dv \\
&\overset{s=v-\rho^\lambda(\mathbf{t})}{=} \int_0^\infty e^{-u(s+\rho^\lambda(\mathbf{t}))} \int_{\mathbb{R}^n} s^\delta e^{i\mathbf{t}'\mathbf{x}}\, d\mathbf{t}\, ds \\
&= \int_{\mathbb{R}^n} e^{i\mathbf{t}'\mathbf{x}} e^{-u\rho^\lambda(\mathbf{t})} \int_0^\infty s^\delta e^{-su}\, ds\, d\mathbf{t} \\
&= \frac{1}{\Gamma(\delta+1) u^{\delta+1}} \int_{\mathbb{R}^n} e^{i\mathbf{t}'\mathbf{x}} e^{-u\rho^\lambda(\mathbf{t})}\, d\mathbf{t} \\
&= f_{\mathbf{x}}(u)
\end{aligned}
$$

and the statement is proven through equivalence of (iv) and (i).

□

Corollary 9. Theorem 39 can be used to derive the already established Theorem 33 which sets sufficient conditions for the class $S(n,1)$ through the function $(1-u)_+^\delta$.

We aim to use it for $(\mathbb{R}^n, \|\cdot\|_\alpha)$, $\alpha < 2$. For that denote

$$
h_{\lambda,x}(u) = \int_{\mathbb{R}^n} e^{-u|x|^\lambda} e^{itx}\, dt, \quad u > 0,
$$

and $(1-|u|^\lambda)_+^\delta$ is positive definite if and only if $u^{-\delta-1} h_{\lambda,x}(u)$ is completely monotone for any $x \in \mathbb{R}$, i.e. $\delta \geq \delta(\lambda)$ if and only if $u^{-\delta-1} h_{\lambda,x}(u)$ is completely monotone for any $x \in \mathbb{R}$. Now for $\rho = \|\cdot\|_\alpha$ and $\lambda = \alpha$ the function $f_{\mathbf{x}}$ defined by Theorem 39 is equal to

$$
\begin{aligned}
f_{\mathbf{x}}(u) &= \frac{1}{\Gamma(\delta+1) u^{\delta+1}} \int_{\mathbb{R}^n} e^{-u\|\mathbf{t}\|_\alpha^\alpha} e^{i\mathbf{t}'\mathbf{x}}\, d\mathbf{t} \\
&= \frac{1}{\Gamma(\delta+1) u^{\delta+1}} \prod_{k=1}^n \int_{\mathbb{R}^n} e^{-u|t_k|^\alpha} e^{it_k x_k} dt_k \\
&= \frac{1}{\Gamma(\delta+1) u^{\delta+1}} \prod_{k=1}^n h_{\alpha,x_k}(u) \\
&= \frac{1}{\Gamma(\delta+1)} \prod_{k=1}^n u^{-\frac{\delta+1-n}{n}-1} h_{\alpha,x_k}(u)
\end{aligned}
$$

therefore $(1-u^\alpha)_+^\delta \in S(n,\alpha)$, $\alpha \in (0,2)$ if and only if $\frac{\delta+1-n}{n} \geq \delta(\alpha)$, i.e. $\delta \geq n\delta(\alpha) + n - 1$.

The Kuttner's function $\delta(\lambda)$ (2.8) thus gives bounds for sufficient conditions for non-trivial $S(n,\alpha)$ in terms of m-times monotone functions for some $m \geq 0$. Theorem 40 excludes $\alpha = 2$ as $\delta(2)$ is not suitable for Theorem 40 (Theorem 39, part (iv)). Sufficient conditions for $S(n,2)$ were formulated differently (e.g. as mentioned in the discussion above Theorem 33).

Theorem 40. *Let $0 < \lambda \leq \alpha < 2$, $m = n\delta(\alpha) + n - 1$ and $f : [0,\infty) \to \mathbb{R}$ be an m-times monotone function such that $f(0) = 1$. Then $f(t^\lambda) \in S(n,\alpha)$.*

Proof. First, let us simplify the proof: we may only work with Kuttner-Golubov functions $(1-u^\lambda)_+^\delta$, $u \geq 0$ (Example 8). For any m-times monotone function $f:[0,\infty)\to\mathbb{R}$ we have an integral representation as

$$f(u)=\int_0^\infty (1-ur)_+^m\, dF(r) \tag{3.42}$$

where F is a cumulative distribution function of some random variable R (Theorem 8). If we prove the statement for $(1-u)_+^m$, the representation from Theorem 8 will solve any other m-times monotone function.

Corollary 9 states that $(1-u^\alpha)^m \in S(n,\alpha)$ if and only if $m \geq n\delta(\alpha)+n-1$. By Example 10 if $(1-u^\lambda)_+^\delta \in S(n,\alpha)$, then also $(1-u^\mu)_+^\delta \in S(n,\alpha)$ for $\mu \leq \lambda$. Thus, we may implement $m = n\delta(\alpha)+n-1$ for any $\lambda \in (0,\alpha]$, since Corollary 9 ensures that $(1-u^\alpha)^m \in S(n,\alpha)$ if and only if $m \geq n\delta(\alpha)+n-1$. Moreover, Example 18 implies $\lambda(n,\alpha) \leq \sigma(n,\alpha)$ and for $n \geq 2$ and $\alpha < 2$ the Schoenberg constant is equal to $\sigma(n,\alpha)=\alpha$. However, $\lambda(n,\alpha) \geq \alpha$ as from Corollary 9 we may find δ so that $(1-u^\alpha)_+^\delta \in S(n,\alpha)$. Thus, $\sigma(n,\alpha)=\lambda(n,\alpha)=\alpha$ for $n \geq 2$ and $\alpha < 2$.

To put it together, Corollary 9 implies that for $m = n\delta(\alpha)+n-1$ the function $(1-u^\alpha)_+^m \in S(n,\alpha)$ for $\alpha < 2$ and $n \geq 2$. Example 10 shows that $(1-u^\lambda)_+^m \in S(n,\alpha)$ for any $\lambda \in (0,\alpha]$. As any m-times monotone function with $f(0)=1$ can be represented through (3.42), the function

$$f(u^\lambda)=\int_0^\infty (1-ru^\lambda)_+^m = dF(r)$$

is a characteristic generator from $S(n,\alpha)$ where the last step is done using Theorem 16.

□

Example 19. We may compare two random vectors with Kuttner-Golubov characteristic generators for $\alpha < 1$ and $\alpha \in (1,2)$, other bivariate random vectors with Kuttner-Golubov characteristic generators are discussed in Example 24. Univariate moments are computed in Example 9.

Let $(X_1,X_2)'$ be a $\frac{1}{3}$-symmetric random vector with a characteristic function $(1-|t_1|^{\frac{1}{3}}-|t_2|^{\frac{1}{3}})_+^3$, $(t_1,t_2)' \in \mathbb{R}^2$, the parameters are set in order to satisfy Theorem 40. Figure 3.4 shows the density of $(X_1,X_2)'$.

Let $(Z_1,Z_2)'$ be a $\frac{5}{3}$-symmetric random vector with a characteristic function $(1-|t_1|^{\frac{5}{3}}-|t_2|^{\frac{5}{3}})_+^7$, $(t_1,t_2)' \in \mathbb{R}^2$ where the parameters where chosen so that $7 = 2\cdot 3+2-1 \geq 2\cdot\delta(2-\frac{1}{3})+2-1$ and that Remark 15 is satisfied. The density is shown in Figure 3.5.

The following examples show how to generate α-symmetric random vectors which are not mixtures of i.i.d. α-stable random variables, both are mentioned in Misiewicz [1996], Section II.4.

Example 20. Let $\mathbf{X}$ be an n-dimensional α-symmetric random vector with a characteristic generator φ. Denote

$$\mathbf{Z}_1=\left(D_1^{-\frac{1}{\alpha}}X_1,\dots,D_n^{-\frac{1}{\alpha}}X_n\right)',$$
$$\mathbf{Z}_2=\left(D_1^{-\frac{1}{\alpha}}X_1,\dots,D_1^{-\frac{1}{\alpha}}X_n\right)'$$

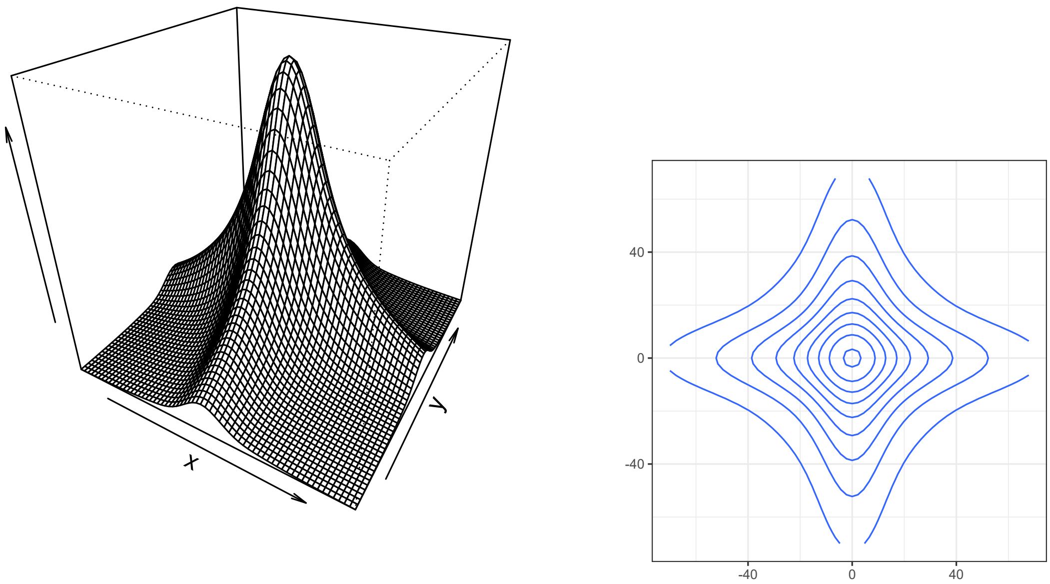


Figure 3.4: The density of $(1 - |t_1|^{\frac{1}{3}} - |t_2|^{\frac{1}{3}})_+^3$, $(t_1, t_2)' \in \mathbb{R}^2$, and its contours.

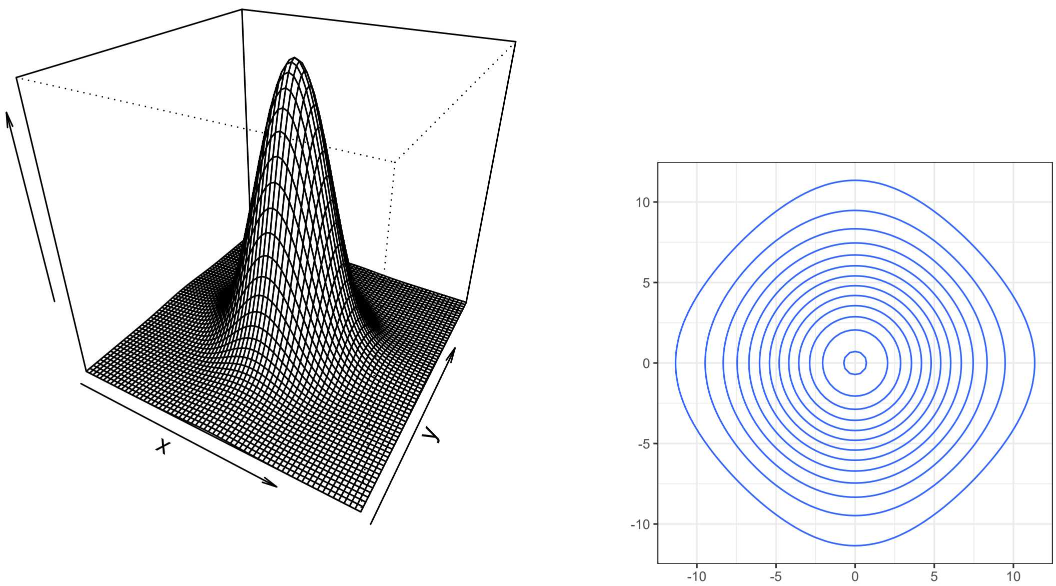


Figure 3.5: The density of $(1 - |t_1|^{\frac{5}{3}} - |t_2|^{\frac{5}{3}})_+^7$, $(t_1, t_2)' \in \mathbb{R}^2$, and its contours.

where $\mathbf{D} = (D_1, \ldots, D_n)' \sim \mathsf{Dir}_n(\frac{1}{2}, \ldots, \frac{1}{2})$ independent of $\mathbf{X}$. Since $\mathbf{Z}_2$ is α-symmetric (Theorem 16), denote its characteristic generator ψ. Let us show that $\psi \in S(n, \frac{\alpha}{2})$. For $\mathbf{t} \in \mathbb{R}^n$ by the law of total probability

$$\begin{aligned}\psi(\|\mathbf{t}\|_\alpha) &= \mathsf{E}\ e^{i\mathbf{t}'\mathbf{Z}_2} \\ &= \mathsf{E}\ \mathsf{E}\left[e^{iD_1^{-\frac{1}{\alpha}}\mathbf{t}'\mathbf{X}}|D_1\right] \\ &= \mathsf{E}\ \varphi\left(D_1^{-\frac{1}{\alpha}}\|\mathbf{t}\|_\alpha\right).\end{aligned}$$

Similarly, for $\mathbf{t} \in \mathbb{R}^n$ using the law of total probability and (3.17)

$$\begin{aligned}
\mathsf{E}\, e^{i\mathbf{t}'\mathbf{Z}_1} &= \mathsf{E}\,\mathsf{E}\left[\exp\left\{i\sum_{k=1}^{n} t_k D_k^{-\frac{1}{\alpha}} X_k\right\} \middle| \mathbf{D}\right] \\
&= \mathsf{E}\,\varphi\left(\left(\sum_{k=1}^{n} \frac{|t_k|^\alpha}{D_k}\right)^{\frac{1}{\alpha}}\right) \\
&= \mathsf{E}\,\varphi\left(D_1^{-\frac{1}{\alpha}}\left(\sum_{k=1}^{n} |t_k|^{\frac{\alpha}{2}}\right)^{\frac{2}{\alpha}}\right) \\
&= \mathsf{E}\,\varphi\left(D_1^{-\frac{1}{\alpha}}\|\mathbf{t}\|_{\frac{\alpha}{2}}\right) \\
&= \psi\left(\|\mathbf{t}\|_{\frac{\alpha}{2}}\right)
\end{aligned}$$

which means $\psi \in S(n, \frac{\alpha}{2})$.

Remark 16. Since the uniform distribution on the unit Euclidean sphere S^{n-1} is the primitive distribution in $S(n, 2)$ (Example 5), Example 20 applied to the uniform distribution the unit sphere in $\mathbb{R}^n$ resembles the primitive distribution in $S(n, 1)$. Cambanis et al. [1983] assumed that if we apply the technique of Example 20 we may obtain a primitive in $S(2, \frac{1}{2})$ which would be

$$\left(\frac{U_1}{B_1\sqrt{B_2}}, \frac{U_2}{(1-B_1)\sqrt{1-B_2}},\right)' \tag{3.43}$$

where $(U_1, U_2)'$ is uniformly distributed on the unit circle in $\mathbb{R}^2$, independent of $B_1, B_2 \sim \mathsf{Beta}(\frac{1}{2}, \frac{1}{2})$. In the same article, they disproved that (3.43) is the primitive of $S(2, \frac{1}{2})$ and were not able to find any factorization of (3.43), i.e. a two-dimensional distribution such that (3.43) is a scale mixture of such distribution with a non-constant scaling variable.

The next example follows similar arguments as above but combines it with Lemma 11.

Example 21. Let $\mathbf{X}$ be an n-dimensional α-symmetric random vector with a characteristic generator φ. Denote

$$\begin{aligned}
\mathbf{V}_1 &= \left(Z_1^{\frac{1}{\alpha}} X_1, \ldots, Z_n^{\frac{1}{\alpha}} X_n\right),' \\
\mathbf{V}_2 &= \left(Z_1^{\frac{1}{\alpha}} X_1, \ldots, Z_1^{\frac{1}{\alpha}} X_n\right)'
\end{aligned}$$

where $Z_1, \ldots, Z_n$ are i.i.d. non-negative $\frac{\beta}{\alpha}$-stable random variables independent of $\mathbf{X}$ with a Laplace transform (1.2), $\beta < \alpha$. The stability of random variables implies

$$\sum_{k=1}^{n} t_k Z_k \stackrel{d}{=} \|\mathbf{t}\|_{\frac{\beta}{\alpha}} Z_1, \quad t_1, \ldots, t_n > 0.$$

Again the vector $\mathbf{V}_2$ is α-symmetric and its characteristic function is equal to

$$\begin{aligned}
\psi(\|\mathbf{t}\|_\alpha) &= \mathsf{E}\, e^{i\mathbf{t}'\mathbf{V}_2} \\
&= \mathsf{E}\,\varphi\left(Z_1^{\frac{1}{\alpha}}\|\mathbf{t}\|_\alpha\right)
\end{aligned}$$

and the characteristic function of $\mathbf{V}_1$ is equal to

$$\begin{aligned}
\mathsf{E}\, e^{i\mathbf{t}'\mathbf{V}_1} &= \mathsf{E}\,\mathsf{E}\left[\exp\left\{i\sum_{k=1}^{n} t_k Z_k^{\frac{1}{\alpha}} X_k\right\}\Big| Z_1,\ldots,Z_n\right] \\
&= \mathsf{E}\,\varphi\left(\left(\sum_{k=1}^{n}\left|t_k Z_k^{\frac{1}{\alpha}}\right|^{\alpha}\right)^{\frac{1}{\alpha}}\right) \\
&= \mathsf{E}\,\varphi\left(Z_1^{\frac{1}{\alpha}}\left(\|\mathbf{t}\|_\beta^\alpha\right)^{\frac{1}{\alpha}}\right) \\
&= \mathsf{E}\,\varphi\left(Z_1^{\frac{1}{\alpha}}\|\mathbf{t}\|_\beta\right) \\
&= \psi(\|\mathbf{t}\|_\beta)
\end{aligned}$$

and $\psi \in S(n,\beta)$.

Remark 17. For each $\alpha > 0$ and $n \geq 2$ the set $\bigcap_{\beta<\alpha} S(n,\beta)$ is trivial. If we take any $\psi \in \bigcap_{\beta<\alpha} S(n,\beta)$, then also $\psi \in S(2,\beta)$ for any β. Then for any $t, s \in \mathbb{R}$ the limit is

$$\lim_{\beta\to 0+} \psi(\|(t,s)\|_\beta) = \begin{cases} \psi(|t|) & s = 0 \\ \psi(|s|) & t = 0 \\ \psi(\infty) & s,t \neq 0 \end{cases}$$

which is continuous positive definite if and only if ψ is constant.

4. Pseudo-Isotropic Distributions

This chapter aims to derive a generalization to α-symmetry where the α-norm may be replaced by a general quasi-norm (Definition 9). Several results of the previous chapters may be reviewed and the α-norm may be replaced by a general quasi-norm without any change in the proof. Theorem 37 and Theorem 23 were already formulated in terms of norm-dependent positive definite continuous functions and directly relate to pseudo-isotropy. The topic is divided into three parts: general properties of pseudo-isotropic distributions, two-dimensional cases, and higher-dimensional distributions. Several simple properties of pseudo-isotropy are shown, and more complicated results are omitted for brevity.

Let us first use a general definition of pseudo-isotropy as a reformulation of the Eaton problem (2.1)

$$\mathbf{t}'\mathbf{X} \stackrel{d}{=} \gamma(\mathbf{t})X_1, \quad \mathbf{t} \in \mathbb{R}^n, \tag{4.1}$$

where $\mathbf{X}$ is non-degenerate, i.e. the linear span of the support of $\mathbf{X}$ is the whole space. By (4.1) we automatically assume $\gamma(\mathbf{e}_1) = \gamma((1, \ldots, 0)') = 1$. We already know (Theorem 10) that if γ permits a Blaschke-Lévy representation (1.8)

$$\gamma^\alpha(\mathbf{t}) = \int_{S^{n-1}} |\mathbf{t}'\mathbf{x}|^\alpha \, d\mu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n$$

for some $\alpha \in (0, 2]$, then γ is a suitable standard for some symmetric α-stable random vector $\mathbf{X}$ with a characteristic function $e^{-\gamma^\alpha(\mathbf{t})}$, $\mathbf{t} \in \mathbb{R}^n$. We can show that $\mathbf{X}$ is pseudo-isotropic, since for any $\mathbf{t} \in \mathbb{R}^n$ the characteristic function of the random variable $Y = \mathbf{t}'\mathbf{X}$ is

$$\mathsf{E}\, e^{iuY} = \mathsf{E}\, e^{iu\mathbf{t}'\mathbf{X}} = e^{-\gamma^\alpha(u\mathbf{t})}, \quad u \in \mathbb{R}. \tag{4.2}$$

On the other hand the characteristic function of $\gamma(\mathbf{t})X_1$ is equal to

$$\mathsf{E}\, e^{iu\gamma(\mathbf{t})X_1} = \mathsf{E}\, e^{iu\gamma(\mathbf{t})\mathbf{e}_1'\mathbf{X}} = e^{-|u|^\alpha \gamma^\alpha(\mathbf{t})\gamma(\mathbf{e}_1)^\alpha}, \quad u \in \mathbb{R}. \tag{4.3}$$

The characteristic functions (4.2) and (4.3) are equal since $\mathbf{e}_1 = (1, \ldots, 0)'$ and $\gamma(\mathbf{e}_1) = 1$ so that the Blaschke-Lévy representation implies positive homogeneity of γ. As in Example 7 we can define a *generalized Schoenberg constant* for a quasi-normed space $(\mathbb{R}^n, \gamma)$ as

$$\sigma(n, \gamma) = \sup\left\{\beta \in (0, 2] : e^{-\gamma^\beta(\mathbf{t})}, \mathbf{t} \in \mathbb{R}^n, \text{is positive definite}\right\}. \tag{4.4}$$

If $\mathbf{X}$ is pseudo-isotropic and ψ is the characteristic function of X_1 then

$$\mathsf{E}\, e^{i\mathbf{t}'\mathbf{X}} = \mathsf{E}\, e^{i\gamma(\mathbf{t})X_1} = \psi(\gamma(\mathbf{t})), \quad \mathbf{t} \in \mathbb{R}^n. \tag{4.5}$$

We may extend the definition of characteristic generators $S(n, \alpha)$ into $S(n, \gamma)$ as functions $\psi : [0, \infty) \to \mathbb{R}$ such that $\psi(\gamma(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$, is a characteristic function of some n-dimensional random vector. In terms of the notation from Chapter 2 we have $S(n, \alpha) \equiv S(n, \|\cdot\|_\alpha)$.

Conversely, assume that $\gamma : \mathbb{R}^n \to [0, \infty)$ is a non-trivial function satisfying $\gamma(y\mathbf{t}) = y\gamma(\mathbf{t})$ for any $\mathbf{t} \in \mathbb{R}^n$, $y \geq 0$, and $\psi : [0, \infty) \to \mathbb{C}$ is a continuous non-constant function such that $\psi(\gamma(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$, is a positive definite function. Under these assumptions Zastavnyi [2000] showed that γ must be even and $\psi(|\cdot|)$ is a positive definite function. The following properties of standards are found in Misiewicz [1996], Section II.2, and Kuritsyn [1992].

Theorem 41. *Let $\gamma : \mathbb{R}^n \to [0, \infty)$ satisfy (4.1) for a non-degenerate random vector $\mathbf{X}$ then*

(i) γ is an even continuous function.

(ii) $\gamma(\mathbf{t}) = 0$ if and only if $\mathbf{t} = \mathbf{0}$.

(iii) The function is positive homogeneous, i.e. $\gamma(y\mathbf{t}) = |y|\gamma(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$, $y \in \mathbb{R}$.

(iv) For any n-dimensional norm ρ there exist constants $m, M, K > 0$ such that for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n$

$$m\rho(\mathbf{t}_1) \leq \gamma(\mathbf{t}_1) \leq M\rho(\mathbf{t}_1), \tag{4.6}$$
$$\gamma(\mathbf{t}_1 + \mathbf{t}_2) \leq K(\gamma(\mathbf{t}_1) + \gamma(\mathbf{t}_2)). \tag{4.7}$$

Proof. For any $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{t} \neq \mathbf{0}$, pseudo-isotropy implies

$$\gamma(\mathbf{t})X_1 \stackrel{d}{=} \mathbf{t}'\mathbf{X} = -(-\mathbf{t}'\mathbf{X}) \stackrel{d}{=} -\gamma(-\mathbf{t})X_1$$

which means $|\gamma(\mathbf{t})| = |\gamma(-\mathbf{t})|$ and since $\gamma > 0$ the function is even. The continuity of the function γ can be proven by contradiction. Assume γ is not continuous at $\mathbf{t}_0 \in \mathbb{R}^n$ and there exists $\mathbf{t}_n \stackrel{n\to\infty}{\to} \mathbf{t}_0$ such that $\gamma(\mathbf{t}_n)$ does not converge to $\gamma(\mathbf{t}_0)$. However, $\mathbf{t}_n'\mathbf{X} \stackrel{d}{\to} \mathbf{t}_0'\mathbf{X}$ which means $\gamma(\mathbf{t}_n)X_1 \stackrel{d}{\to} \gamma(\mathbf{t}_0)X_1$ and by the assumption of contradiction $\gamma(\mathbf{t}_n) \not\to \gamma(\mathbf{t}_0)$. The sequence $\{\gamma(\mathbf{t}_n)\}$ is either bounded (then there is a subsequence with a finite limit and the limit must be equal to $\gamma(\mathbf{t}_0)$ as $\gamma(\mathbf{t}_n), \gamma(\mathbf{t}_0) > 0$) or there is a subsequence with an infinite limit (which also contradicts $\gamma(\mathbf{t}_n)X_1 \stackrel{d}{\to} \gamma(\mathbf{t}_0)X_1$). That concludes (i).

Clearly, $\gamma(\mathbf{0}) = 0$ and by contradiction if $\gamma(\mathbf{t}) = 0$, $\mathbf{t} \neq \mathbf{0}$, then $\mathbf{t}'\mathbf{X} \stackrel{d}{=} 0$ and the random vector is degenerate since $\mathbf{X} \in \{\mathbf{x} \in \mathbb{R}^n : \mathbf{t}'\mathbf{x} = 0\}$ almost surely. For any $\mathbf{t} \in \mathbb{R}^n$ and $y \geq 0$ we have

$$\gamma(y\mathbf{t})X_1 \stackrel{d}{=} (y\mathbf{t})'\mathbf{X} = y\ \mathbf{t}'\mathbf{X} \stackrel{d}{=} y\ \gamma(\mathbf{t})X_1.$$

In combination with evenness of γ this implies positive homogeneity from (ii).

For the last statement denote $S_\rho = \{\mathbf{x} \in \mathbb{R}^n : \rho(\mathbf{x}) = 1\}$ the unit sphere in $(\mathbb{R}^n, \rho)$ and

$$m = \min_{\mathbf{t} \in S_\rho} \gamma(\mathbf{t}), \quad M = \max_{\mathbf{t} \in S_\rho} \gamma(\mathbf{t})$$

which exist and are positive since γ is continuous and positive on S_ρ which is compact and $\mathbf{0} \notin S_\rho$. Then for any $\mathbf{t} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\gamma(\mathbf{t}) = \rho(\mathbf{t})\gamma\left(\frac{\mathbf{t}}{\rho(\mathbf{t})}\right) = \rho(\mathbf{t})\gamma(\mathbf{x})$$

for some $\mathbf{t}/\rho(\mathbf{t}) = \mathbf{x} \in S_\rho$. The inequality (4.7) is proven by setting $K = \frac{M}{m} \in (0, \infty)$ since $m > 0$ and $M < \infty$. Then for $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n$

$$\begin{aligned}\gamma(\mathbf{t}_1 + \mathbf{t}_2) &\leq M\rho(\mathbf{t}_1 + \mathbf{t}_2) \leq M\rho(\mathbf{t}_1) + M\rho(\mathbf{t}_2) \\ &\leq mK\rho(\mathbf{t}_1) + mK\rho(\mathbf{t}_2) \leq K(\gamma(\mathbf{t}_1) + \gamma(\mathbf{t}_2)).\end{aligned}$$

□

Remark 18. Continuity of the standard is usually skipped as trivial and not shown. Kuritsyn [1992] in his proof of continuity stated that since the standard γ is defined on $\mathbb{R}^n$ it is bounded on compact sets. His argument is enhanced in the proof above.

Thus, any suitable standard γ is a quasi-norm. We may revisit isometric embedding from Subsection 1.2.1. Lemma 42 is taken from Zastavnyi [2000].

Lemma 42. *Let $(\mathbb{R}^n, \gamma_1)$ and $(\mathbb{R}^n, \gamma_2)$ be two quasi-normed spaces which are isometric. Then $S(n, \gamma_1) = S(n, \gamma_2)$.*

We have already stated $S(2, 1) = S(2, \infty)$, another example of isometry between norms utilizes elliptical distributions (whose unit balls are ellipsoids). Such norms in $\mathbb{R}^n$ can be written as $\gamma(\mathbf{t}) = \sqrt{\mathbf{t}' A \mathbf{t}}$ for some positive definite $n \times n$ matrix A and we have $S(n, 2) = S(n, \gamma)$. Elliptical distributions are discussed in Fang et al. [1990].

The following two statements show that the standard is (almost) uniquely determined by the random vector and its sum (Misiewicz [1996], Section II.2).

Theorem 43. *Let $\mathbf{X}$ be a pseudo-isotropic random vector. Then its standard γ is uniquely determined.*

Proof. Let $\gamma_1, \gamma_2 : \mathbb{R}^n \to [0, \infty)$ be two standards of the random vector $\mathbf{X}$ which satisfy $\gamma_1(\mathbf{e}_1) = \gamma_2(\mathbf{e}_1) = 1$. If $\gamma_1(\mathbf{t}) \neq \gamma_2(\mathbf{t})$ for some $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{t} \neq \mathbf{0}$, then $\gamma_1(\mathbf{t})\mathbf{X}$ and $\gamma_2(\mathbf{t})\mathbf{X}$ do not have the same distribution (since the standard is positive). However, $\gamma_1(\mathbf{t})\mathbf{X} \overset{d}{=} \mathbf{t}'\mathbf{X} \overset{d}{=} \gamma_2(\mathbf{t})\mathbf{X}$ which contradicts $\gamma_1(\mathbf{t}) \neq \gamma_2(\mathbf{t})$ as $\gamma_1(\mathbf{t}), \gamma_2(\mathbf{t}) > 0$.

□

Theorem 44. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional pseudo-isotropic random vectors such that $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$ is also pseudo-isotropic. Denote the respective characteristic functions of $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}$ as $\psi_1(\gamma_1(\mathbf{t}))$, $\psi_2(\gamma_2(\mathbf{t}))$, and $\psi(\gamma(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$, and the standards are γ_1, γ_2, and γ, respectively. Then either $\gamma_1 = \gamma_2 = \gamma$ or there exist constants $0 < m_1 < m_2 < \infty$ and a function $\tilde{\gamma} : (0, \infty)^2 \to (0, \infty)$ such that*

$$\psi_1(pu)\psi_2(qu) = \psi(\tilde{\gamma}(p, q)u), \quad u > 0,$$

for any $m_1 \leq \frac{p}{q} \leq m_2$.

Proof. If the standards are equal we can rewrite the characteristic generator of the sum as $\psi_1(t)\psi_2(t)$ with γ as the standard. From Theorem 43 the standard is unique since we have normalized the standards and $\gamma_1(\mathbf{e}_1) = \gamma_2(\mathbf{e}_1) = \gamma(\mathbf{e}_1) = 1$.

Otherwise set $g(\mathbf{t}) = \gamma_1(\mathbf{t})/\gamma_2(\mathbf{t})$ which is continuous and positive for $\mathbf{t} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Denote $m_1 = \min\{g(\mathbf{t}) : \mathbf{t} \in S^{n-1}\}$ and $m_2 = \max\{g(\mathbf{t}) : \mathbf{t} \in S^{n-1}\}$. Set $p, q > 0$ such that $m_1 \le \frac{p}{q} \le m_2$ and find $\mathbf{t} \in \mathbb{R}^n$ such that $g(\mathbf{t}) = \frac{p}{q}$ and $\tilde{\gamma}(p,q) = \frac{q\gamma(\mathbf{t})}{\gamma_2(\mathbf{t})}$. Then for any $u > 0$ we have

$$\begin{aligned}
\psi_1(pu)\psi_2(qu) &= \psi_1\left(uq \cdot g(\mathbf{t})\right)\psi_2\left(uq\right) \\
&= \psi_1\left(qu\frac{\gamma_1(\mathbf{t})}{\gamma_2(\mathbf{t})}\right)\psi_2\left(qu\frac{\gamma_2(\mathbf{t})}{\gamma_2(\mathbf{t})}\right) \\
&= \psi_1\left(\gamma_1\left(\frac{qu}{\gamma_2(\mathbf{t})}\mathbf{t}\right)\right)\psi_2\left(\gamma_2\left(\frac{qu}{\gamma_2(\mathbf{t})}\mathbf{t}\right)\right), \\
\psi(\tilde{\gamma}(p,q)u) &= \psi\left(\gamma\left(\frac{qu}{\gamma_2(\mathbf{t})}\mathbf{t}\right)\right)
\end{aligned}$$

where the right-hand sides are equal from pseudo-isotropy since they both represent a characteristic function of $\mathbf{X}_1 + \mathbf{X}_2$ at point $\frac{qu}{\gamma_2(\mathbf{t})}\mathbf{t}$.

□

Similarly, as in Theorem 21 the pseudo-isotropic random vector is bounded if and only if it is elliptical (i.e. $\gamma(\mathbf{t}) = \sqrt{\mathbf{t}'A\mathbf{t}}$ for some positive definite matrix A, more in Fang et al. [1990], Definition 2.2) and assuming no atom at origin each hyperplane has a zero probability (see Lemma 17). The proofs in both theorems are analogous to the ones presented for α-symmetric vectors.

4.1 Two-Dimensional Pseudo-Isotropy

Theorem 12 already stated that any two-dimensional norm $\rho(t_1, t_2)$ is a standard of a symmetric 1-stable random vector with a characteristic function $e^{-\rho(t_1,t_2)}$, i.e. $\sigma(2,\rho) \ge 1$ for any norm where $\sigma(2,\rho)$ is defined in (4.4). Quasi-norms which are not norms (their unit ball are not convex) must satisfy $\sigma(2,\rho) < 1$ (in line with Theorem 12). Theorem 23 by Zastavnyi [1992] gives some conditions for the two-dimensional pseudo-isotropy. Integrability of

$$H(y) = \frac{\partial}{\partial t}\rho(t\mathbf{a}_1 + y\mathbf{a}_2)\,\Big|_{t=1}$$

can be checked for other than α-norms.

Example 22. The assumption of integrability is satisfied e.g. for norms whose unit ball is not strictly convex (Zastavnyi [1992]). For that we have to evaluate the condition of integrability outside the origin. Since the unit ball is not strictly convex we can find two linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2$ such that $\rho(u\mathbf{b}_1 + (1-u)\mathbf{b}_2) = 1$ for each $u \in (0,1)$. Denote $\mathbf{a}_1 = \mathbf{b}_1 + \mathbf{b}_2$ and $\mathbf{a}_2 = \mathbf{b}_2 - \mathbf{b}_1$. Then for any $y \in (1, \infty)$ and $\varepsilon \in (0, y-1)$ we have

$$\begin{aligned}
\rho((1+\varepsilon)\mathbf{a}_2 + y\mathbf{a}_1) &= \rho((y-1-\varepsilon)\mathbf{b}_1 + (y+1+\varepsilon)\mathbf{b}_2) \\
&= 2y \cdot \rho\left(\frac{y-1-\varepsilon}{2y}\mathbf{b}_1 + \frac{y+1+\varepsilon}{2y}\mathbf{b}_2\right) \\
&= 2y \\
&= 2y \cdot \rho\left(\frac{y-1}{2y}\mathbf{b}_1 + \frac{y+1}{2y}\mathbf{b}_2\right) \\
&= \rho(\mathbf{a}_2 + y\mathbf{a}_1)
\end{aligned}$$

which means

$$\begin{aligned}
H(y) &= \frac{\partial}{\partial t}\rho(t\mathbf{a}_1 + y\mathbf{a}_2)\,\Big|_{t=1} \\
&= \lim_{\varepsilon \to 0+} \frac{\rho((1+\varepsilon)\mathbf{a}_2 + y\mathbf{a}_1) - \rho(\mathbf{a}_2 + y\mathbf{a}_1)}{\varepsilon} \\
&= 2y - 2y = 0.
\end{aligned}$$

Thus, if ρ is a two-dimensional norm which is not strictly convex we have

$$\sigma(2, \rho) = 1.$$

Theorem 23 ensures that non-strictly convex norms satisfy $\sigma(2, \rho) \le 1$ and Theorem 12 implies $\sigma(2, \rho) \ge 1$ for any norm.

Remark 19. Let $(X_1, X_2)'$ be a two-dimensional pseudo-isotropic random vector such that $\mathsf{E}\,|X_1| < \infty$, i.e. $\mathsf{E}\,X_1 = 0$ since X_1 is symmetric, then its characteristic generator satisfies $\psi'(0) = 0$.[1] Then norms that satisfy the conditions of Theorem 23 thus cannot be norms of pseudo-isotropic random vectors with finite first moments.

Example 23. As shown in Theorem 12 two-dimensional norms are characterized by $\sigma(2, \rho) \ge 1$ and the following statements are equivalent (as mentioned above Theorem 39)

(i) $(\mathbb{R}^2, \rho)$ is a normed space,

(ii) $\sigma(2, \rho) \ge 1$,

(iii) $(1-u)_+^\delta \in S(2, \rho)$ for $\delta \ge 3$.

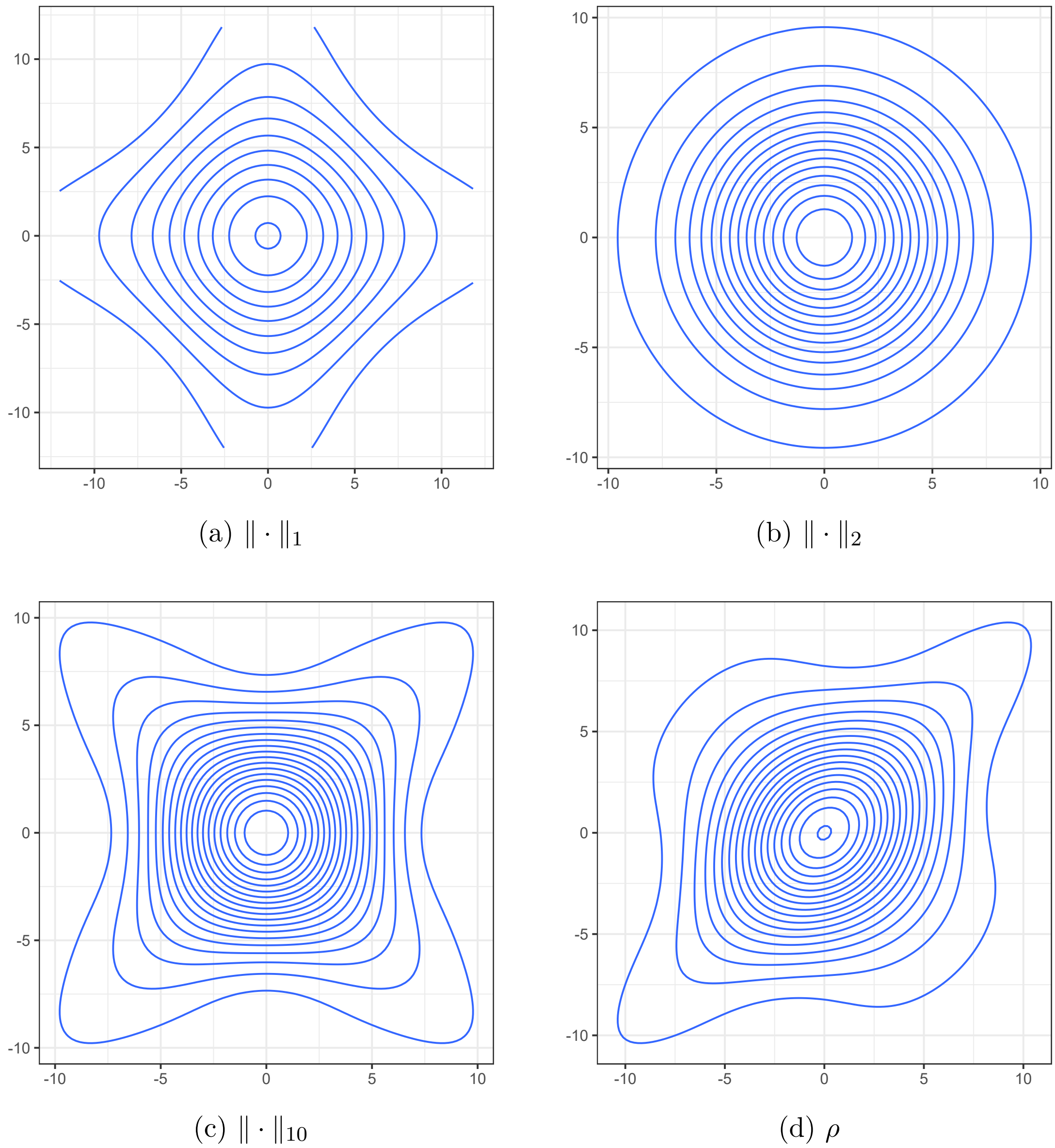
Example 24. Let us compare the densities corresponding to characteristic generators $(1-u)_+^3 \in S(2, \alpha)$ for different values of $\alpha \ge 1$. We may notice in Figure 4.1 that in the case of $\alpha = 1$, the distribution is more concentrated on the axes although for higher α the distribution is concentrated on the diagonals (due to the isometry from Example 3). Spherically symmetric ($\alpha = 2$) characteristic functions define spherically symmetric densities which is not the case for other α.

We may also implement $(1-u)_+^3$ for other norms, e.g.

$$\rho(t_1, t_2) = \begin{cases} \|(t_1, t_2)'\|_2 & t_1 \cdot t_2 \ge 0, \\ \|(t_1, t_2)'\|_\infty & t_1 \cdot t_2 < 0. \end{cases} \tag{4.8}$$

[1] See Theorem 4.

The unit ball of ρ resembles an eye. The marginal density corresponding to all these distributions is equal to $\frac{3(-2+t^2+2\cos(t))}{\pi t^4}$, $t \in \mathbb{R}$, and moments are found in Example 9.



(a) $\|\cdot\|_1$ (b) $\|\cdot\|_2$

(c) $\|\cdot\|_{10}$ (d) ρ

Figure 4.1: Contours of densities of a bivariate pseudo-isotropic vector with a characteristic generator $(1-t)_+^3$, $\mathbf{t} \in \mathbb{R}^2$, for $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_{10}$ and ρ as defined in (4.8).

4.2 Higher-Dimensional Pseudo-Isotropy

Pseudo-isotropy in more than two dimensions can be again perceived in terms of isometric embedding (essentially creating stable pseudo-isotropic random vectors) or with Theorem 37 which was formulated for norms. Zastavnyi [1992] offered an extension of Theorem 37 for some quasi-norms.

Corollary 10. Let $\gamma : \mathbb{R}^{n-1} \to [0, \infty)$, $n \geq 2$ be a quasi-norm, and denote

$$\gamma_\alpha(x_1, \ldots, x_n) = (|x_1|^\alpha + \gamma^\alpha(x_2, \ldots, x_n))^{\frac{1}{\alpha}}, \quad (x_1, \ldots, x_n)' \in \mathbb{R}^n$$

for $\alpha > 2$. By setting $\mathbf{a}_3 = \mathbf{e}_1$, $\mathbf{a}_1 = \mathbf{e}_2$, and $\mathbf{a}_2 = \mathbf{e}_3$, the function $H(y_1, y_2)$ from Theorem 37 is equal to

$$H(y_1, y_2) = \frac{1}{1 + \gamma^\alpha(y_1, y_2, 0, \ldots, 0)}$$

which is integrable for $\alpha > 2$ as $\gamma(y_2, y_3, 0, \ldots, 0)$ is a two-dimensional quasi-norm.

Lemma 42 states that for isometric quasi-normed spaces, the classes of suitable characteristic generators coincide. By that, we may characterize all norms in $\mathbb{R}^n$ induced by an inner product.

Example 25. Let $(\mathbb{R}^n, \rho)$ be a normed space which satisfies the parallelogram rule

$$2(\rho^2(\mathbf{x}) + \rho^2(\mathbf{y})) = \rho^2(\mathbf{x} + \mathbf{y}) + \rho^2(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

then $(\mathbb{R}^n, \rho)$ is isometric with the space $(\mathbb{R}^n, \|\cdot\|_2)$ and thus $S(n, \rho) = S(n, 2)$ and in conclusion $\sigma(n, \rho) = \sigma(n, 2) = 2$ and $\lambda(n, \rho) = \lambda(n, 2) = 2$.

From Zastavnyi [2000] we have the converse implication and the statements

(i) $\lambda(n, \rho) = 2$,

(ii) $\sigma(n, \rho) = 2$,

(iii) $(\mathbb{R}^n, \rho)$ and $(\mathbb{R}^n, \|\cdot\|_2)$ are isometric

are equivalent.

If for a pseudo-isotropic vector there exists some fractional moment of the one-dimensional marginal variable, the situation is much simpler since the isometric embedding of $(\mathbb{R}^n, \gamma)$ into L_α can be used.[2]

Theorem 45. *Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a non-trivial pseudo-isotropic random vector with a standard γ vector such that $\mathsf{E}\,|X_1|^\varepsilon < \infty$ for some $\varepsilon > 0$. Then there exists a maximal $\alpha \in (0, 2]$ and a finite symmetric measure μ on S^{n-1} such that*

$$\gamma^\alpha(\mathbf{t}) = \int_{S^{n-1}} |\mathbf{t}'\mathbf{x}|^\alpha \, d\mu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n.$$

Proof. We can take $\mathsf{E}\,|X_1|^p < \infty$ for $0 < p \leq \min\{\varepsilon, 2\}$ into

$$\mathsf{E}\,|\mathbf{t}'\mathbf{X}|^p = \gamma^p(\mathbf{t})\,\mathsf{E}\,|X_1|^p < \infty$$

for any $\mathbf{t} \in \mathbb{R}^n$. Denote $c = \mathsf{E}\,|X_1|^p \in (0, \infty)$ then

$$\begin{aligned} \gamma^p(\mathbf{t}) &= \frac{1}{c}\,\mathsf{E}\,|\mathbf{t}'\mathbf{X}|^p \\ &= \int_{\mathbb{R}^n} |\mathbf{t}'\mathbf{x}|^p \frac{1}{c}\, dP_{\mathbf{X}}(\mathbf{x}) \end{aligned}$$

[2] See Subsection 1.2.1.

where $P_{\mathbf{X}}$ is the distribution of $\mathbf{X}$. Theorem 9 implies that $e^{-\gamma^p(\mathbf{t})}$, $\mathbf{t} \in \mathbb{R}^n$, is a characteristic function of some symmetric p-stable random vector. Denote $\alpha = \sup\{p \in (0,2] : e^{-\gamma^p(\mathbf{t})}$ is positive definite$\}$, then $e^{-\gamma^\alpha(\mathbf{t})}$ is a positive definite continuous function (it is a limit of positive definite functions and also continuous) and γ has a Blaschke-Lévy representation (1.8) for $\alpha \in (0,2]$ which is by its definition maximal.

□

The previous theorem seems simple but has several consequences:

Theorem 45 for example implies that it is impossible to construct a non-trivial pseudo-isotropic random vector $(X_1, \ldots, X_n)'$ with a finite expectation and a standard γ with a non-convex contours (unit balls). That is because Theorem 45 implies the space $(\mathbb{R}^n, \gamma)$ embeds to some L_1-space which is a normed space. The triangle inequality can be checked with the Blaschke-Lévy representation (1.8): If a function γ can be represented as

$$\gamma(\mathbf{t}) = \int_{S^{n-1}} |\mathbf{t}'\mathbf{x}| \; d\mu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n,$$

for some finite symmetric Borel measure μ on S^{n-1}, then for $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n$ we have

$$\begin{aligned}\gamma(\mathbf{t}_1 + \mathbf{t}_2) &= \int_{S^{n-1}} |(\mathbf{t}_1 + \mathbf{t}_2)'\mathbf{x}| \; d\mu(\mathbf{x}) \\ &\leq \int_{S^{n-1}} |\mathbf{t}_1'\mathbf{x}| + |\mathbf{t}_2'\mathbf{x}| \; d\mu(\mathbf{x}) \\ &= \gamma(\mathbf{t}_1) + \gamma(\mathbf{t}_2)\end{aligned}$$

and γ is a norm (by Definition 9). There are several corollaries of Theorem 45 in terms of stability.

Corollary 11. The Blaschke-Lévy representation which was obtained in the previous theorem implies that under the assumptions of Theorem 45 the function $e^{-\gamma^\alpha(\mathbf{t})}$ is a characteristic function of a symmetric α-stable random vector.

The theorem bears a resemblance to the generalized central limit theorem. Kuritsyn [1992] showed that under the assumption that X_1 lies in the region of attraction of some α-stable random variable and $\mathbf{X}$ is a non-trivial pseudo-isotropic random vector with a standard γ, then γ possesses the Blaschke-Lévy representation with said α. The connection between moments and the region of attraction of stable distributions is discussed in Tucker [1975].

On the other hand by a contrapositive of Theorem 45 if γ does not have a Blaschke-Lévy representation for any $\alpha \in (0,2]$ then it cannot have any moment finite.

Even though the assumption of the finiteness of $\mathsf{E}\,|X_1|^\varepsilon$ for some $\varepsilon > 0$ in Theorem 45 seems weak, it was attempted to find a general proof of the existence of a Blaschke-Lévy representation for any suitable standard. This problem (i.e. the question if any suitable standard has a Blaschke-Lévy representation) remains open.[3]

Corollary 12. The existence of fractional moments of X_1 is linked to the integral of the characteristic function.[4] Thus, let $\psi(\gamma(\cdot))$ be a characteristic function

[3] As stated in the most recent article Koldobsky [2011].

[4] See Corollary 1.

of a pseudo-isotropic random vector $\mathbf{X}$ with a standard γ such that $\psi(|\cdot|)$ is a characteristic function of X_1. Thus, $\mathsf{E}\,|X_1|^\varepsilon < \infty$ for some $\varepsilon \in (0,2]$ holds if and only if (Corollary 1)

$$\int_0^\infty \frac{1-\psi(t)}{t^{1+\varepsilon}}\,dt < \infty. \tag{4.9}$$

Since characteristic functions are uniformly continuous and bounded,[5] the integral condition (4.9) can be reformulated in terms of the behavior of ψ in the neighborhood of origin, i.e. the moment of order $\varepsilon > 0$ exists if for some $u > 0$, $C > 0$ and $\delta \in (\varepsilon, 2]$

$$|1-\psi(t)| \le Ct^\delta, \quad t \in (0,u).$$

This condition was known to Koldobsky [1991] without the connection to moments but Koldobsky used it only to derive the finiteness of (4.9). By that the results are essentially equivalent to Misiewicz [1996], Theorem II.2.6.

The following theorem does not assume the existence of a pseudo-isotropic random vector with a given standard however, it assumes a certain limit behavior of some characteristic function near the origin. The limit bears resemblance to Corollary 1 of the existence of moments $\mathsf{E}\,|\mathbf{t}'\mathbf{X}|^\mu$.

Theorem 46. *Let $\mathbf{X}$ be a random vector on a quasi-normed space $(\mathbb{R}^n, \rho)$ with a characteristic function φ which for some $\mu > 0$ and $\beta > 0$ satisfies*

$$\lim_{u\to 0+} \frac{1-\varphi(u\mathbf{t})}{u^\mu} = \beta\rho^\mu(\mathbf{t}) \tag{4.10}$$

for each $\mathbf{t} \in \mathbb{R}^n$. Then $e^{-u^\mu} \in S(n,\rho)$ and $\mu \le \sigma(n,\rho)$.

The proof of Theorem 46 uses several properties of positive definite functions and can be found in Zastavnyi [2000].

Corollary 13. Theorem 46 implies a similar result for a general quasi-normed space $(\mathbb{R}^n, \rho)$ as it was derived directly in Example 18 such that

$$\lambda(n,\rho) \le \sigma(n,\rho) \le 2$$

where $\lambda(n,\rho)$ is defined analogously to $\lambda(n,\alpha)$ in (2.8).

Koldobsky and Lonke [1999] implemented the proof of Theorem 37 by Zastavnyi [1992] in order to obtain a second-derivative test for the isometric embedding of three-dimensional normed spaces.

Theorem 47. *Let (E,ρ) be a three-dimensional normed space with a normalized basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and for fixed $(x_2,x_3)' \in \mathbb{R}^2$ denote $g_{x_2,x_3}(t) = \rho(t\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$.*

Assume that g_{x_2,x_3} has continuous second derivatives on $\mathbb{R}$ and $g'_{x_2,x_3}(0) = g''_{x_2,x_3}(0) = 0$. Moreover, assume that there exists a constant $C > 0$ such that for any $(x_2,x_3)' \in \mathbb{R}^2$, $\rho(x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = 1$, the function g''_{x_2,x_3} is bounded by C on $\mathbb{R}$.

Then the function $e^{-\rho^\beta(\mathbf{t})}$ is not positive definite for any $\beta \in (0,2]$.

The proof of Theorem 47 can be found in Koldobsky and Lonke [1999] and follows similar steps as Theorem 37. Theorems such as Theorem 47 are useful for norms that are defined implicitly.

[5] See Lemma 3.

Example 26. The α-norm can be generalized into *Orlicz norms*. Let $M : [0, \infty) \to [0, \infty)$ a non-decreasing convex function such that $M(0) = 0$ and $M(t) > 0$ for any $t > 0$. The *Orlicz norm* $\| \cdot \|_M$ is defined implicitly as

$$\sum_{k=1}^{n} M\left(\frac{|x_k|}{\|\mathbf{x}\|_M}\right) = 1, \quad \mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

For $M(t) = t^\alpha$ the Orlicz norm is the α-norm (1.4).

Koldobsky [1997b] found that for Orlicz norms satisfying $M'(0) = M''(0) = 0$ all conditions of Theorem 47 are satisfied and $e^{-\|\mathbf{t}\|_M^\beta}$, $\mathbf{t} \in \mathbb{R}^n$, is not a characteristic function for $n \geq 3$ for any $\beta > 0$.

A weaker condition than the existence of moments is $\mathsf{E}\,|\log|X_1|| < \infty$. If $\mathbf{X}$ is a non-trivial pseudo-isotropic random vector with a standard γ and $\mathsf{E}\,|\log|X_1|| < \infty$ then there exist a finite measure ν on S^{n-1} and $C > 0$ such that

$$\log \gamma(\mathbf{t}) = C + \int_{S^{n-1}} \log |\mathbf{t}'\mathbf{x}| \; d\nu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n. \tag{4.11}$$

This integral representation was linked to isometric embedding into L_0 spaces as defined in Kalton et al. [2007]. Koldobsky [2011] found a proof of the following statement without the additional condition $\mathsf{E}\,|\log|X_1|| < \infty$.

Theorem 48. *Let* $(\mathbb{R}^n, \rho)$ *be a quasi-normed space such that there exists a non-constant continuous function* $f : [0, \infty) \to \mathbb{R}$ *such that* $f(\rho(\mathbf{x}))$, $\mathbf{x} \in \mathbb{R}^n$, *is positive definite. Then* ρ *can be written as*

$$\log \gamma(\mathbf{t}) = C + \int_{S^{n-1}} \log |\mathbf{t}'\mathbf{x}| \; d\nu(\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^n. \tag{4.12}$$

for some finite measure ν *and* $C > 0$.

Remark 20. Theorem 48 is necessary but not sufficient since all three-dimensional spaces $(\mathbb{R}^n, \| \cdot \|_\alpha)$, $\alpha > 2$, embed in L_0 (Kalton et al. [2007]). However, the class $S(3, \alpha)$, $\alpha > 2$, is trivial as shown in Theorem 37. The condition (4.12) in Theorem 48 is thus necessary but not sufficient.

Conclusion

This thesis presented the problem of the existence of α-symmetric distributions and their properties. The concept of α-symmetry naturally generalizes spherically symmetric (also called radial or isotropic) distributions, expanding the characterization through projections. Thus, the α-symmetric distributions are defined through their characteristic functions. Only a few examples are widely known, among them are the vectors of i.i.d. symmetric α-stable random variables for $\alpha \leq 2$. It is known that stable distributions are power-tail heavy and their moments exist only for orders less than α. The thesis derives a connection between α-symmetric distributions and their moments and presents several examples of α-symmetry, including several new ones.

Chapter 1 summarized the two tools mentioned above: the relationship between characteristic functions and moments of non-integer order. Further, it defined symmetric stable distributions in $\mathbb{R}^n$. Symmetric stable distributions and the link between stable distributions and isometric embedding of quasi-normed spaces are presented. The definition and basic properties of α-symmetric distributions can be found in Chapter 2, where the theory concerning the density of α-symmetric distributions is presented. Specific classes of n-dimensional α-symmetric distributions for pairs of $n \in \mathbb{N}$ and $0 < \alpha \leq \infty$ are discussed in Chapter 3.

Non-trivial two-dimensional α-symmetric distributions exist for any α. Contrary, higher-dimensional α-symmetric distributions are non-trivial if and only if $\alpha \leq 2$. Section 3.2 is dedicated to 1-symmetric distributions with a special place among α-symmetric distributions. For other pairs of dimension n and index α, only sufficient conditions for characteristic functions are available.

Similarly, as α-symmetry generalizes spherical symmetry by replacing the Euclidean norm in the definition with an α-norm, we may further generalize the problem to pseudo-isotropic distributions where the α-norm is replaced by a general quasi-norm in Chapter 4. The necessary conditions for pseudo-isotropy are formulated in terms of isometric embeddings.

Stable multivariate distributions have numerous applications (many of them mentioned in Uchaikin and Zolotarev [1999]). Stable α-symmetric distributions and their mixtures can be used in limit theorems: Khokhlov et al. [2020] used elliptically symmetric Linnik distribution as a limit of a random sum of random variables where the number of summands has a particular negative binomial distribution. Pseudo-isotropy (and α-symmetry in particular) can be used in any setting where we are interested in the distribution of linear combinations. A particular application of pseudo-isotropy in portfolio theory was developed by Framstad [2015]. Spherically and elliptically symmetric distributions are also widely used e.g. in regression.

Misiewicz and Ryll-Nardzewski [1987] explored a generalization of pseudo-isotropy to general Banach spaces and Jasiulis and Misiewicz [2008] further connected pseudo-isotropy to weak stability (a generalization of stability). The most recent result is by Misiewicz and Volkovich [2020].

There are several open problems concerning pseudo-isotropy in $\mathbb{R}^n$:

1. Is there a stochastic decomposition for multivariate α-symmetric distribu-

tions other than $\alpha = 1, 2$?

2. Is there a stochastic decomposition for multivariate pseudo-isotropic distribution for a given quasi-norm?

3. Does it generally hold $\lambda(n, \gamma) = \sigma(n, \gamma)$ for any quasi-norm γ in finite-dimensional quasi-normed spaces?

4. Does any pseudo-isotropic standard in $\mathbb{R}^n$ have a Blaschke-Lévy representation?

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