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Elastic strings in general relativity

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Abstract: We study a simple model of a one dimensional extended body in a gravitational field, consisting of two point particles connected by an elastic string, and derive equations of motion for this system in both classical mechanics and general relativity. We study the motion of the system with a focus on the relationship between these models, and the difference of its motion from geodesic.

Keywords: motion of extended bodies gravitational field geodesic string

Contents

Introduction

The string is a system of long-standing relevance to theoretical physics, providing simple examples of continuum dynamics, wave phenomena, and even field theory. In this thesis, we use the string as a model of interaction between two point masses falling radially in a spherically symmetric gravitational field, and examine the effects of this interaction on their motion. With this setting, we obtain a one-dimensional model of an extended body in free fall that is simple to describe in both classical and relativistic contexts.

Solving this problem is part of a wider effort to understand the motion of extended bodies in gravitational fields, which is an area of both theoretical and practical interest, for example in the orbital control of satellites Misra and Modi [1986]. Previous research has identified two effects by which the motion differs from geodesic, *swinging* and *swimming* Guéron et al. [2006].

The swinging effect is present in both Newtonian and relativistic gravity, and it is most easily seen in bodies with a controlled inner oscillation. On an intuitive level, the body swings by changing its shape to take advantage of the inhomogeneity of the field. The effect is usually quantified by measuring the position shift between the center of mass of the extended body and a comparable point test particle. These shifts are typically seen to be negative, and the body can thus only accelerate its fall. The position shift—oscillation frequency relationship has also been a subject of several similar previous works Guéron and Mosna [2007].

Swimming on the other hand is a purely relativistic effect, first proposed in Wisdom [2003] as a geometric consequence of moving on a manifold with curvature. In a controlled "glider" model Guéron and Mosna [2007], swimming has been theorised to lead to slower fall than geodesic, but this result has been disputed Veselý and Žofka [2019].

An issue that is common among many of the sample systems in this research area is the usage of a controlled Lagrangian, with constraints between particles or parts of the body being artificially prescribed instead of consequent to a physical interaction. This leads to fundamental problems in the context of relativity, since it contradicts the limit on speed of information exchange between parts of the body. To remedy this, in Veselý and Žofka [2021] a model of interaction via exchange of discrete particles was examined instead, with the conclusion that swimming is observed, but it only accelerates the fall. The reason for adjoining the particles via a continuous string is then that such a model automatically fulfills the information speed limit, and it provides a much broader scope of possibilities for the motion.

The thesis is divided into two chapters. In the first, we develop a general Lagrangian description of a classical elastic string confined to one-dimensional motion, and then use it to describe the desired system of two interacting particles in a Newtonian gravitational field. We solve the equations of motion numerically and observe the resulting swinging effect. The second chapter attempts to replicate these results in the framework of general relativity, studying an elastic string with the speed of sound equal to speed of light. We derive the equations of motion for a particle-string-particle system from first principles, and discuss their possible solution and the implications of our formalism.

1. The Classical String

In classical mechanics, the string is essentially a one-dimensional continuum, and it therefore needs to be described using continuum concepts. In this chapter, we will develop and discuss the Lagrangian formalism for the string, expressed in terms of the displacement field, and then use it to describe our target dumbbell system.

1.1 The Free String

The physical state of a string at any given time is determined by the position of all of its constituent points. To translate this general description to mathematics, we first need to choose a reference configuration - to this end, consider the undeformed string with uniform linear density λ lying at rest along the x-axis, and let $x = 0$ and $x = l$ be its endpoints. Since its motion is limited to the x-direction, the state of the string is fully specified by the displacement function $u(x,t) = r(x,t) - x$, where $r(x,t)$ is the position at time t of the point starting at x. The velocity of this point is then:

$$v(x,t) = \frac{\partial u(x,t)}{\partial t}. \tag{1.1}$$

This is also the velocity of an infinitesimal segment dx at the reference coordinate x. Because the mass of such a segment is $dm = \lambda dx$, we immediately have the kinetic energy of the string:

$$T = \frac{1}{2}\lambda \int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx. \tag{1.2}$$

To derive the potential energy, we need to define more precisely the notion of an elastic string. Denote by $\tau(x,t)$ the force of tension at each reference point x in the deformed configuration, i.e., the force acting on the left half of the string when divided at reference point x. We now suppose that the string obeys a constitutive relation of the form

$$\tau = \kappa \frac{\partial u}{\partial x} \tag{1.3}$$

analogous to the equations of linear elasticity. The interpretation of κ is postponed to the next section.

The definition of u and this constitutive relation is all we need to derive the string dynamics. Suppose the string moves, in a small time δt, from the position $u(x)$ to $u(x)+\delta u(x)$. This must have been due to the combined influence of the tension force and additional external forces. Neglecting the possibility of a force on the ends of the string other than tension, we can define a mass density of the external forces, $f(x)$, such that the total force on the segment dx

is $f(x)dm = f(x)\lambda dx$. The work done during the move $u \to u + \delta u$ is then:

$$\delta W = -\tau(0)\delta u(0) + \tau(l)\delta u(l) + \int_0^l \lambda f \delta u dx \tag{1.4}$$

$$= \int_0^l \left(\frac{\partial}{\partial x} (\tau \, \delta u) + \lambda f \delta u \right) dx \tag{1.5}$$

$$= \int_0^l \left(\frac{\partial \tau}{\partial x} + \lambda f \right) \delta u dx + \kappa \int_0^l \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} dx \tag{1.6}$$

We can now use the continuum equation of motion

$$\frac{\partial \tau}{\partial x} + \lambda f = \lambda \frac{\partial v}{\partial t} \tag{1.7}$$

along with the fact that $\delta u / \delta t$ is just the velocity, to simplify the first term

$$\int_0^l \left(\frac{\partial \tau}{\partial x} + \lambda f \right) \delta u dx = \lambda \int_0^l \frac{\partial v}{\partial t} v \delta t dx \tag{1.8}$$

$$= \frac{1}{2} \lambda \frac{\partial}{\partial t} \left(\int_0^l v^2 dx \right) \delta t \tag{1.9}$$

$$= \delta \left(\frac{1}{2} \lambda \int_0^l v^2 dx \right) \tag{1.10}$$

We recognize this as the change in kinetic energy δT. The remaining term can be similarly manipulated:

$$\delta W - \delta T = \kappa \int_0^l \frac{\partial u}{\partial x} \delta \frac{\partial u}{\partial x} dx \tag{1.11}$$

$$= \delta \left(\frac{1}{2} \kappa \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dx \right) \tag{1.12}$$

and by conservation of energy, the natural interpretation for this term is the change in the elastic potential energy δV_{el}. Therefore, we have:

$$V_{\text{el}} = \frac{1}{2} \kappa \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dx \tag{1.13}$$

and the Lagrangian for our string is:

$$L = \frac{1}{2} \lambda \int_0^l \left(\frac{\partial u}{\partial t} \right)^2 dx - \frac{1}{2} \kappa \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dx - V_f \tag{1.14}$$

where V_f is the potential energy of all the external forces f. If the string is free ($f = 0$), the equation of motion is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.15}$$

with phase velocity:

$$c = \sqrt{\frac{\kappa}{\lambda}} \tag{1.16}$$

Lastly, we need the boundary conditions. By the elastic law, the force exerted on the string at the end $x = 0$ is $\kappa(\partial u(0,t)/\partial x)$; the situation is the same at $x = l$. Therefore, if the string is to be truly free, we can specify the Neumann conditions:

$$\frac{\partial u(0,t)}{\partial t} = \frac{\partial u(l,t)}{\partial t} = 0 \tag{1.17}$$

1.2 Three-dimensional Strings

One problem that section 1.1 did not address is the nature of the constant κ and the validity of equation (1.3). Here, we present two different models for the string, which both reproduce the relation, but give different values of κ.

Leaving the comfortable one-dimensional idealisation behind, the string is most easily pictured as a long, thin cylinder with a rigid axis. We will assume that the material is linearly elastic and isotropic:

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) \tag{1.18}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{u}$ is the displacement field.

One natural model then arises by supposing that the radius of the string is fixed, so that the displacement is limited to the x direction:

$$\boldsymbol{u} = u(x)\boldsymbol{e}_z \tag{1.19}$$

In this case, equation (1.18) directly gives:

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} \tag{1.20}$$

This is sufficient to confirm (1.3), since we have $\tau = A\sigma_{xx}$ for the tension force from the previous section, where A is the cross-sectional area of the string. From this model, we get the result:

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{1.21}$$

where ρ is the density of the material. This is precisely the speed of longitudinal seismic P-waves in a large three dimensional body.

Another, and perhaps more physically realistic model accounts for the radius of the string changing with the deformation along its axis. Neglecting the possibility of torsion, this gives a displacement vector:

$$\boldsymbol{u} = u(x)\boldsymbol{e}_x + \upsilon(x,r)\boldsymbol{e}_r \tag{1.22}$$

as in cylindrical coordinates (r, φ, x). However, remembering that the string is thin, the r-dependence of the radial displacement υ can be safely approximated to first order as:

$$\upsilon(x,r) = r\eta(x) \tag{1.23}$$

The constant term is necessarily zero due to the a non-zero radial displacement at $r = 0$ leading to singular behaviour.

A natural boundary condition to impose on the string is zero normal stress on the surface of the cylinder (except at the bases). This condition is expressed as:

$$\sigma_{rr} = 0, \quad r = R \tag{1.24}$$

which given our assumptions yields:

$$\eta(x) = -\frac{\lambda}{2(\lambda + \mu)} \frac{\partial u}{\partial x} \tag{1.25}$$

The constant is the Poisson ratio, and the result is essentially the same as that for a material under constant uniaxial strain.

Calculating the normal axial stress again results in:

$$\sigma_{xx} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \frac{\partial u}{\partial x} \tag{1.26}$$

so this model reproduces the same linear dependence with a different κ. The constant is the Young's modulus E, and calculating the wave speed results in the well-known expression:

$$c = \sqrt{\frac{E}{\rho}} \tag{1.27}$$

1.3 Mediating an Interaction

Now that the Lagrangian for the free string (1.28) has been derived, we can consider the string mediating an interaction between two point masses. The way to accomplish this is by adding their respective kinetic energies to the Lagrangian, so that:

$$L = \frac{1}{2}\lambda \int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx - \frac{1}{2}\kappa \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}m_0 \left(\frac{\partial u(0,t)}{\partial t}\right)^2 + \frac{1}{2}m_1 \left(\frac{\partial u(l,t)}{\partial t}\right)^2 \tag{1.28}$$

The point masses m_0, m_1 and the endpoints of the string are rigidly connected, so we can set their velocities equal. Since the Lagrangian is mixed with both point mass and continuum terms, we will derive the equations of motion from scratch using the variational principle.

$$\delta S = 0, \quad S = \int_{t_0}^{t_1} L dt \tag{1.29}$$

Consider then a variation $\delta u(x,t)$ satisfying $\delta u(x,t_0) = \delta u(x,t_1) = 0$. The resulting δS can be expressed as:

$$\begin{aligned} \delta S = \int_{t_0}^{t_1} \int_0^l \left(-\lambda \frac{\partial^2 u}{\partial t^2} + \kappa \frac{\partial^2 u}{\partial x^2}\right) \delta u dx dt \\ + \int_{t_0}^{t_1} \left(\kappa \frac{\partial u(0,t)}{\partial x} - m_0 \frac{\partial^2 u(0,t)}{\partial t^2}\right) \delta u(0,t) dt \\ + \int_{t_0}^{t_1} \left(\kappa \frac{\partial u(l,t)}{\partial x} - m_0 \frac{\partial^2 u(l,t)}{\partial t^2}\right) \delta u(l,t) dt \end{aligned} \tag{1.30}$$

The boundary variations $\delta u(0,t)$, $\delta u(l,t)$ can essentially be taken as independent from $\delta u(x,t)$, since the integral over x does not depend on behaviour at isolated points. We thus obtain three equations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.31}$$
$$m_0 \frac{\partial^2 u(0,t)}{\partial t^2} = \kappa \frac{\partial u(0,t)}{\partial x} \tag{1.32}$$
$$m_1 \frac{\partial^2 u(l,t)}{\partial t^2} = -\kappa \frac{\partial u(l,t)}{\partial x} \tag{1.33}$$

The equation of motion (1.31) remains unchanged, while the second and third equation now act as its boundary conditions. They are also clearly an expression of Newton's second law for the point masses, acted upon by the tension of the string.

The usual method for solving the one-dimensional wave equation on a bounded interval is separation of variables - applying it here gives a solution of the form:

$$u(x,t) = \sum_{n\in\mathbb{N}_0} \left(\sin(a_n x) - \frac{\lambda}{m_0 a_n} \cos(a_n x) \right) \left(A_n \sin(a_n ct) + B_n \cos(a_n ct) \right) \tag{1.34}$$

where A_n, B_n are Fourier coefficients found from the initial conditions $u(x,0)$, $(\partial u/\partial t)(x,0)$. The only problem with this solution are the constants a_n: while they have simple forms for Dirichlet and von Neumann conditions, in our case they are found by solving the equation

$$a_n l = \arctan \frac{\lambda}{m_0 a_n} + \arctan \frac{\lambda}{m_1 a_n} + \pi n \tag{1.35}$$

which can only be done numerically. This solution is therefore of limited interest.

An interesting simplification occurs if we let $\lambda \to 0$, which is reasonable when the mass of the string λl is much smaller than m_0, m_1, so its kinetic energy is negligible. Equation (1.31) then reduces to:

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.36}$$

and its general solution is:

$$u(x,t) = \left(1 - \frac{x}{l}\right) u_0(t) + \frac{x}{l} u_1(t) \tag{1.37}$$

where the notation $u_0(t) = u(0,t)$, $u_1(t) = u(l,t)$ has been introduced for the displacements of the endpoints. The boundary conditions become:

$$m_0 \frac{d^2 u_0}{dt^2} = \frac{\kappa}{l} \left(u_1(t) - u_0(t) \right) \tag{1.38}$$
$$m_1 \frac{d^2 u_1}{dt^2} = -\frac{\kappa}{l} \left(u_1(t) - u_0(t) \right) \tag{1.39}$$

which we recognize as describing the motion of two points connected by a spring. Therefore, if the mass of the string is negligible, it loses all inner degrees of freedom and acts like a spring with spring constant $k = \kappa/l$.

1.4 Radial Fall

The final addition to the Lagrangian are terms describing the gravitational force on both of the point masses and the string. In formulating these, it is convenient to switch from the displacement $u(x,t)$ to the function $r(x,t) = x + u(x,t) + r_0$ describing the actual r-coordinate of the point with reference coordinate x at time t. The Lagrangian (1.28) and the string equation of motion aren't directly influenced by this change; the boundary conditions, however, transform according to:

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x} - 1 \tag{1.40}$$

The classical gravitational potential energy of a point mass m a distance r away from a spherical source with mass M is

$$V_{\text{grav}} = -\frac{GMm}{r} \tag{1.41}$$

and applying the same principle of dividing the string into small pieces with mass $\mathrm{d}m = \lambda \mathrm{d}x$, we get the Lagrangian:

$$L_{\text{grav}} = GM\lambda \int_0^l \frac{1}{r(x,t)} \mathrm{d}x + \frac{GMm_0}{r_0(t)} + \frac{GMm_1}{r_1(t)} \tag{1.42}$$

with $r_0(t) = r(0,t)$ and $r_1(t) = r(l,t)$. Solving for the equations of motion is the same as in the previous section, so we skip to the result:

$$\frac{\partial^2 r}{\partial t^2} - v^2 \frac{\partial^2 r}{\partial x^2} = -\frac{GM}{r^2} \tag{1.43}$$

$$m_0 \frac{\partial^2 r(0,t)}{\partial t^2} = -\frac{GMm_0}{r(0,t)^2} + \kappa \left(\frac{\partial r(0,t)}{\partial x} - 1 \right) \tag{1.44}$$

$$m_1 \frac{\partial^2 r(l,t)}{\partial t^2} = -\frac{GMm_1}{r(l,t)^2} - \kappa \left(\frac{\partial r(l,t)}{\partial x} - 1 \right) \tag{1.45}$$

A key difference from the free string is the non-linearity of the new wave equation, which means that only a numerical solution is possible. Since boundary conditions of this type are not accounted for in pre-programmed computational software like Mathematica, we constructed our own numerical solution in Python. This was done by discretising the wave equation and the boundary conditions as follows

$$r(x, t+\Delta t) \approx 2r(x,t) - r(x, t-\Delta t) - \frac{GM}{r(x,t)^2} + \gamma^2 \left(r(x+\Delta x, t) - 2r(x,t) + r(x-\Delta x, t) \right) \tag{1.46}$$

$$r(0, t+\Delta t) \approx 2r(0,t) - r(0, t-\Delta t) - \frac{GM}{r(0,t)^2} + \frac{\kappa}{m_0} \frac{1}{\Delta x} \left(r(\Delta x, t) - r(0,t) - \Delta x \right) \tag{1.47}$$

$$r(0, t+\Delta t) \approx 2r(0,t) - r(0, t-\Delta t) - \frac{GM}{r(0,t)^2} - \frac{\kappa}{m_1} \frac{1}{\Delta x} \left(r(\Delta x, t) - r(0,t) - \Delta x \right) \tag{1.48}$$

and employing an iterative scheme starting from the initial conditions $r(x,0)$ and $(\partial r/\partial t)(x,0)$, with the conversion

$$r(x, -\Delta t) \approx r(x, 0) - \Delta t \frac{\partial r}{\partial t}(x, 0) \tag{1.49}$$

The constant γ appearing in (1.46) is the Courant parameter

$$\gamma = v \frac{\Delta t}{\Delta x} \tag{1.50}$$

and the condition $\gamma \leq 1$ is the main criterion for numerical stability of this algorithm. This was always ensured by making the time step Δt as small as necessary.

With the numerical solution at hand, we now present a qualitative analysis of some of the features of the system. The choice of units and parameters is not relevant to the results, so they will be chosen somewhat arbitrarily.

To begin, Figure 1.1 and Figure 1.2 show the motion of the string starting from a state of uniform compression and zero initial velocity in two cases, with the mass of the string much smaller than the masses of the endpoints $\lambda l << m_0$, and the reverse. To extract the effect this produces, the gravity is set to zero. The black curves show the motion of the endpoints and the center of mass, while the color shows the deformation of the string $(\partial r/\partial x) - 1$ as a function of both actual position r and time t.

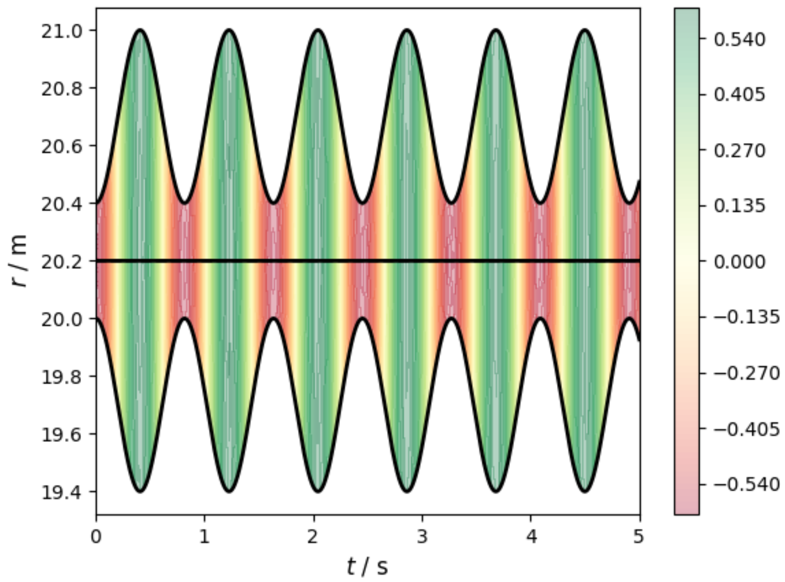

Figure 1.1: Motion of a free string, starting from a state of rest and uniform compression by a factor of 0.4. Low density regime $\lambda l \approx 0.1 \ll 1 \approx m_0$.

Figure 1.1 affirms the low-density limit - the string maintains its uniform tension, and the motion of the endpoints in nearly sinusoidal, which is the exact solution to equations (1.38). In contrast, on Figure 1.2, we see internal compression waves propagating through the string, the tension quickly becomes non-uniform, and the motion of the particles being pulled along is less smooth and more asymmetric.

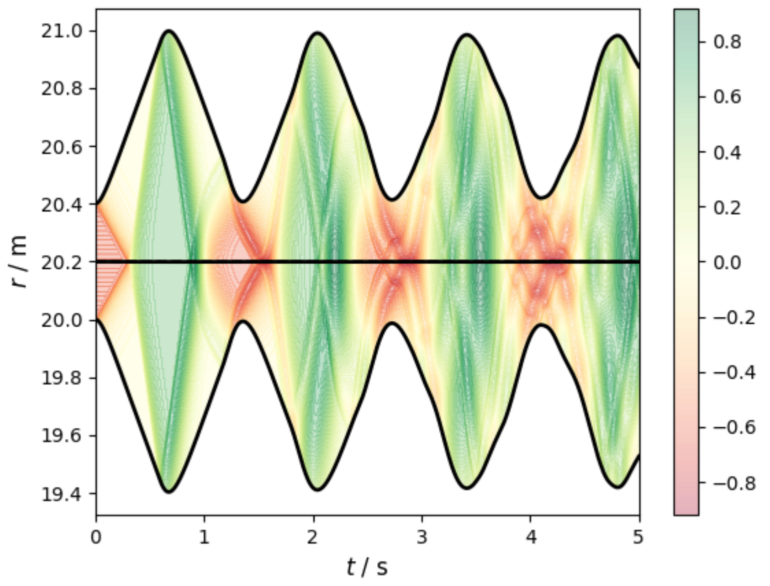

Figure 1.2: The free string with same parameters and initial state as Figure 1.1, but in a high density regime $\lambda l \approx 10 \gg 1 \approx m_0$

A sample solution for the motion in a gravitational field is provided on Figure 1.3, starting again from a state of uniform compression.

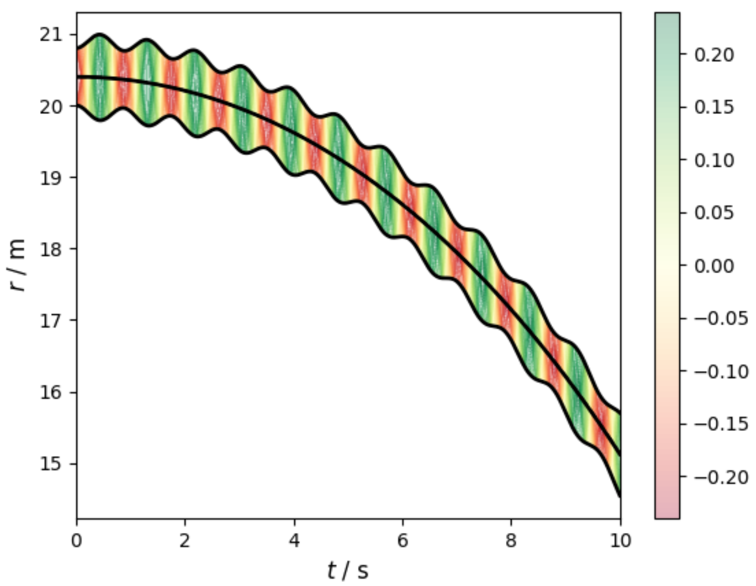

Figure 1.3: The classical string in a Newtonian field, with a low density of the string, $\lambda l \approx 0.1 \ll 1 \approx m_0$. The string starts in a state of uniform compression by a factor of 0.8.

To actually study the swinging effect, we must isolate the center of mass trajectory as plotted on Figure 1.3, and compare it to the trajectory of a free particle with the same initial conditions. One question that is relevant to this field of study is the dependence of the position shift on the characteristic frequency of the oscillation of the system. Working again in the low density limit, this characteristic frequency can be expressed from equation (1.38) as

$$\omega^2 = \frac{\kappa}{l}\left(\frac{1}{m_0} + \frac{1}{m_1}\right) \tag{1.51}$$

In controlled systems, which can use their internal energy to perform the swinging effect through oscillation, there is a well-known relationship between the position shifts and ω. The string-particle system, however, is conservative. On Figure 1.3, the position shift is plotted on a logarithmic scale as a function of time for several values of ω, and otherwise identical parameters and initial conditions; we see that the characteristic frequency is not a factor in the position shifts in our model. This is because our system is conservative, and the frequency of oscillation therefore has no effect on the internal energy expended to work against the field.

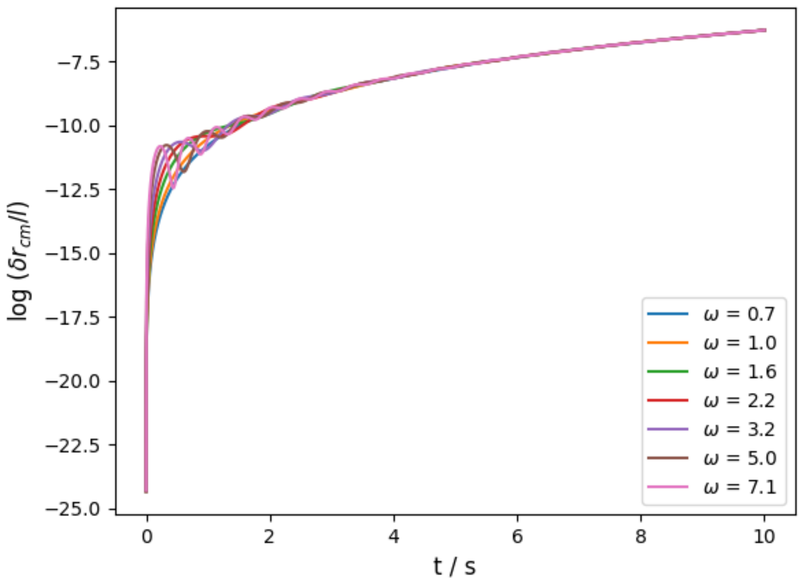

Figure 1.4: The position shift as a function of time for different choices of the elastic constant κ and characteristic frequency ω, on a logarithmic scale with the length unit being the length of the string.

2. The Relativistic String

2.1 The String Lagrangian

To describe the string in general relativity, we will adopt the approach presented in Natário [2014]. The string is represented by a scalar field $\lambda(x^\mu)$ on a two-dimensional spacetime, which gives the worldlines $x^\mu(\tau)$ of individual points as level sets:

$$\lambda(x^\mu(\tau)) = \text{const.} \tag{2.1}$$

We will assume that the two-dimensional spacetime of the string is a totally geodesic submanifold of the universe, which is sufficient to describe free fall in the Schwarzschild metric. The metric and Levi-Civita tensor will be denoted by $g^{\mu\nu}$ and $\varepsilon^{\mu\nu}$ respectively, and we will work in natural units $c = 1$, $G = 1$ throughout.

Given the field λ, we can derive the two-velocity field u^μ by considering the gradient $\nabla_\mu \lambda$. This vector has to be normal to the level set, or equivalently, to the tangent vector along the worldline, which is precisely the direction of u^μ. Since we are in two dimensions, we can take the orthogonal complement with the Levi-Civita tensor, so that

$$u^\mu = \frac{1}{n}\varepsilon^{\mu\nu}\nabla_\nu \lambda \tag{2.2}$$

where n is a normalization factor. From the constraint $u_\mu u^\mu = -1$, we then have

$$n^2 = \nabla_\mu \lambda \nabla^\mu \lambda > 0 \tag{2.3}$$

with the positivity of n^2 ensured by the fact that $\nabla_\mu \lambda$ is spacelike.

The defining relation (2.1) of λ contains a degree of arbitrariness, with the freedom to choose how to "number" the individual worldlines. This can be fixed by demanding that $n = 1$ corresponds to an undeformed string, which at least in a flat spacetime provides a simple interpretation of $1/n$ as the strech factor Natário [2014].

Next, we will derive an expression for the stress energy tensor $T^{\mu\nu}$ of the string. Because the elements of the string are still with respect to the comoving frame with basis vectors $(u^\mu, (1/n)\nabla^\mu\lambda)$, the time-space component of $T^{\mu\nu}$ has to be zero in this frame. Utilizing the two-dimensionality of our problem, however, $T^{\mu\nu}$ then has to have the form of an ideal fluid:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} \tag{2.4}$$

with ρ being the energy density and P the pressure. Since we are in two dimensions, and the space-space component of $T^{\mu\nu}$ represents momentum flux, P actually has the dimensions of force, and represents the tension in the string as observed in the local comoving frame.

It seems reasonable to assume that the energy density and pressure-tension are both functions of coordinates only through n, that is $\rho = \rho(n)$, $P = P(n)$. The energy and tension then only depend on the deformation of the string, and are automatically translation-invariant. It then follows that there is a relation between the two functions

$$P = n\frac{\mathrm{d}\rho}{\mathrm{d}n} - \rho \tag{2.5}$$

The choice of the *elastic law* $\rho(n)$ thus fixes the pressure as well, and is the only constitutive relation that we have to axiomatically assume.

We will primarily work with the choice leading to a "rigid" string, in which the speed of sound is equal to the speed of light (Natário [2014])

$$\rho(n) = \frac{1}{2}\rho_0(n^2+1) \tag{2.6}$$
$$P(n) = \frac{1}{2}\rho_0(n^2-1) \tag{2.7}$$

The rigid string is a limiting case of the more general string with speed of sound v, as $v \to 1$

$$\rho(n) = \frac{\rho_0}{1+v^2}\left(n^{v^2+1}+v^2\right) \tag{2.8}$$
$$P(n) = \frac{\rho_0 v^2}{1+v^2}\left(n^{v^2+1}-1\right) \tag{2.9}$$

In both cases, ρ_0 is the rest mass density of the unstretched string.

For the rigid case, the stress-energy tensor can be rewritten in terms of λ as

$$T_{\mu\nu} = \rho_0\left(\nabla_\mu\lambda\nabla_\nu\lambda - \frac{1}{2}\nabla_\kappa\lambda\nabla^\kappa\lambda g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\right) \tag{2.10}$$

which almost the stress energy for a scalar massless field λ, apart from the covariantly constant last term.

In GR, the Lagrangian formalism rests on the extremalisation of the action

$$S = \int_\Omega \mathrm{d}^4x\sqrt{-g}\,(R - 2\Lambda + 16\pi\mathcal{L}) \tag{2.11}$$

with respect to the metric $g_{\mu\nu}$ and all the fields that occur in the matter Lagrangian $\mathcal{L}$. Variation with respect to the geometry gives the Einstein field equations, the source term being derived from $\delta\mathcal{L}/\delta g^{\mu\nu}$. Variation with respect to the source fields on the other hand gives their dynamic equations, which the geometry of spacetime enters implicitly through the metric.

Identifying the source term as the energy-momentum tensor, with the assumption that $\mathcal{L}$ does not depend on the derivatives of the metric, gives the relation

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L} - 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \tag{2.12}$$

which we will now use to construct the matter Lagrangian of our rigid string.

Since the Lagrangian is a form of energy, and both ρ and P depend on λ only through n^2, it is reasonable to assume that $\mathcal{L} = \mathcal{L}(\zeta)$, where $\zeta = n^2$. From the definition of n,

$$\zeta = g^{\mu\nu}\nabla_\mu\lambda\nabla_\nu\lambda \tag{2.13}$$

and since the variation in (2.12) keeps λ constant, we have

$$\delta\zeta = \nabla_\mu\lambda\nabla_\nu\lambda\,\delta g^{\mu\nu} \tag{2.14}$$

$$\implies \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\zeta}\frac{\delta\zeta}{\delta g^{\mu\nu}} = \nabla_\mu\lambda\nabla_\nu\lambda\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\zeta} \tag{2.15}$$

The energy-momentum tensor then becomes

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}(\zeta) - 2\nabla_\mu\lambda\nabla_\nu\lambda\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\zeta} \tag{2.16}$$

and term-by-term comparison with (2.10) yields the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\rho_0\left(\nabla_\mu\lambda\nabla^\mu\lambda + 1\right) \tag{2.17}$$

In passing, it is interesting to note that the same method works for the string with finite speed of sound, with the resulting Lagrangian:

$$\mathcal{L}_v = -\frac{\rho_0}{1+v^2}\left(\nabla_\mu\lambda\nabla^\mu\lambda\right)^{\frac{1}{2}\left(v^2-1\right)} \tag{2.18}$$

Returning to the rigid string, the equations of motion can be derived from our results in two distinct ways: by varying the action corresponding to $\mathcal{L}$ with respect to λ (keeping the metric constant), or by utilising the conservation of energy and momentum,

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.19}$$

which incidentally also follows from the diffeomorphism invariance of $\mathcal{L}$. For the infinite free rigid string, both approaches lead to the wave equation,

$$\Box\lambda = 0 \tag{2.20}$$

The obvious problem to consider now is the boundary conditions to apply. The variational approach relies on the action

$$S = \int_\Omega \mathrm{d}^2x\sqrt{-g}\mathcal{L} \tag{2.21}$$

where Ω is a given region of spacetime, and $\delta\lambda = 0$ on $\partial\Omega$. This allows for either an infinite string, or a fixed-end problem, neither of which are suited to the radial fall of a dumbbell that we want to describe.

2.2 Attaching the Particles

As in the classical case, the overall system that we want to examine consists of two particles, with rest masses m, m', that are attached to the endpoints of our elastic string, having rest length l and rest mass density ρ_0. In contrast to the classical model, however, how to dynamically attach the particles and the string is rather unclear. In this section, we present an original derivation of the equations of motion.

The action for the complete system has the form

$$S = -\frac{1}{2}\rho_0\int_\Omega \mathrm{d}^2x\sqrt{-g}\left(\nabla_\mu\lambda\nabla^\mu\lambda + 1\right) - m\int_A^B \mathrm{d}\tau - m'\int_{A'}^{B'} \mathrm{d}\tau' \tag{2.22}$$

where $\mathrm{d}\tau$ and $\mathrm{d}\tau'$ are the proper time parametrisations of the particle world-lines $x^\mu = \xi^\mu(\tau)$ and $x^\mu = (\xi')^\mu(\tau)$, and $\lambda(x^\mu)$ once again describes the string. As in the classical case, we would like to connect the string and particles by expressing $\xi^\mu(\tau)$, $(\xi')^\mu(\tau')$ in terms of λ, however, this is only possible implicitly; they satisfy

$$\lambda(\xi^\mu(\tau)) = 0 \tag{2.23}$$

$$\lambda((\xi')^\mu(\tau')) = l \tag{2.24}$$

We therefore need to attach the particles by constraining the variations.

The action S is to be varied with respect to $\delta\xi(s)$, $\delta\xi'(s')$ and $\delta\lambda(x^\mu)$. We will keep $\delta\xi$ and $\delta\xi'$ free apart from the fixed-end conditions $\delta\xi = 0$ at A and B, and similarly for $\delta\xi'$. The points A, B, A', B' will also be constant through the variations. The variation of the particle terms then become simply the relevant two-accelerations:

$$\delta S_{\text{part}} = -m\int_A^B \mathrm{d}\tau g_{\mu\nu}\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)\delta\xi^\nu - m'\int_{A'}^{B'} \mathrm{d}\tau' g_{\mu\nu}\frac{\mathrm{D}}{\mathrm{d}\tau'}\left(\frac{\mathrm{d}(\xi')^\mu}{\mathrm{d}\tau'}\right)\delta(\xi')^\nu \tag{2.25}$$

Varying the string term is difficult due to the nature of the region of integration Ω. If the ends of the string are fixed, this region is constant with respect to our variation; in our case, however, it changes with $\delta\xi$, $\delta\xi'$. More specifically, Ω is bounded by the curves $\lambda(x^\mu) = 0$ and $\lambda(x^\mu) = l$ from the "left" and "right", and from "above" and "below" by two arbitrary spacelike curves containing the points B, B' and A, A' respectively, where the initial and final conditions for the string are fixed. The variation does not change the spacelike boundaries, and we further impose $\delta\lambda = 0$ there. Define $\delta\Omega$ to be the (signed) region representing the difference between Ω before and after the variation. We then have

$$\delta S_{\text{str}} = \int_{\Omega+\delta\Omega} \mathrm{d}^2x\sqrt{-g}\,(\mathcal{L} + \delta\mathcal{L}) - \int_\Omega \mathrm{d}^2x\sqrt{-g}\mathcal{L} \tag{2.26}$$

$$= \int_\Omega \mathrm{d}^2x\sqrt{-g}\delta\mathcal{L} + \int_{\delta\Omega} \mathrm{d}^2x\sqrt{-g}\mathcal{L} \tag{2.27}$$

where the integral of $\delta\mathcal{L}$ over $\delta\Omega$ has been neglected since it is second order in the variation. The first term can be rewritten as:

$$\int_\Omega \mathrm{d}^2x\sqrt{-g}\delta\mathcal{L} = -\rho_0\int_\Omega \mathrm{d}^2x\sqrt{-g}g^{\mu\nu}\nabla_\mu\lambda\nabla_\nu\,(\delta\lambda) \tag{2.28}$$

$$= -\rho_0\int_\Omega \mathrm{d}^2x\partial_\mu\left(\sqrt{-g}\nabla^\mu\lambda\,\delta\lambda\right) + \rho_0\int_\Omega \mathrm{d}^2x\sqrt{-g}\,\Box\lambda\,\delta\lambda \tag{2.29}$$

The second integral in (2.29) is responsible for the field equation, $\Box\lambda = 0$. We rewrite the first using the divergence theorem,

$$-\rho_0\int_\Omega \mathrm{d}^2x\,\partial_\mu\left(\sqrt{-g}\nabla^\mu\lambda\,\delta\lambda\right) = -\rho_0\int_{\partial\Omega} \mathrm{d}s\,n_\mu\,\nabla^\mu\lambda\,\delta\lambda \tag{2.30}$$

where s is either arc-length or proper time parametrisation of the boundary, and n_μ is the outward pointing normal. This integral can be futher split into two non-zero parts, corresponding to the ξ and ξ' boundary. For both, the approach is similar - for example, on the ξ boundary, we have $l = \tau$, the proper time of the first particle, and $n_\mu = -(1/n)\nabla_\mu\lambda$ from the condition (2.23). From this condition we also obtain the relation

$$\delta\lambda = -\nabla_\nu\lambda\,\delta\xi^\nu \;\text{ on } x^\mu = \xi^\mu(\tau) \tag{2.31}$$

The integral then becomes:

$$-\rho_0 \int_A^B \mathrm{d}\tau\, n_\mu\, \nabla^\mu \lambda\, \delta\lambda = -\rho_0 \int_A^B \mathrm{d}\tau\, n\, \nabla_\nu \lambda\, \delta\xi^\nu \tag{2.32}$$

$$= -\rho_0 \int_A^B \mathrm{d}\tau\, n^2\, n_\nu\, \delta\xi^\nu \tag{2.33}$$

An analogous term with opposite sign appears for the variation $\delta(\xi')^\nu$.

The second term in (2.27), due to the change in Ω, can be similarly split into parts due to $\delta\xi$ and $\delta\xi'$ - we again show only the ξ part. We simplify the integral with a natural change of coordinates,

$$x^\mu(\tau, w) = \xi^\mu(\tau) + w\delta\xi^\mu(\tau) \tag{2.34}$$

where τ is the proper time of ξ, and $w \in [0, 1]$. The Lagrangian is an invariant, so it stays unchanged; however, we must account for the Jacobian of the transformation within the integral. We have (again to first order in the variation):

$$\det\left(\frac{\partial(x^0, x^1)}{\partial(\tau, w)}\right) = \det\begin{pmatrix} u^0(\tau) + w\delta u^0(\tau) & \delta\xi^0(\tau) \\ u^1(\tau) + w\delta u^1(\tau) & \delta\xi^1(\tau)\end{pmatrix} = [\mu\nu] u^\mu \delta\xi^\nu(\tau) \tag{2.35}$$

where $[\mu\nu]$ stands for the antisymmetric symbol with $[01] = 1$. Using $\varepsilon_{\mu\nu} = \sqrt{-g}[\mu\nu]$, the integral becomes:

$$\int_{\delta\Omega} d^2x \sqrt{-g}\mathcal{L} = \int_A^B \mathrm{d}\tau \int_0^1 \mathrm{d}w\, \varepsilon_{\mu\nu} u^\mu \delta\xi^\nu \mathcal{L}(x^\sigma(\tau, w)) \tag{2.36}$$

We expand the Lagrangian as

$$\mathcal{L}(x^\sigma(\tau, w)) = \mathcal{L}(\xi^\sigma(\tau) + w\delta\xi^\sigma(\tau)) = \mathcal{L}(\xi^\sigma(\tau)) + \mathcal{O}(w\delta^\sigma(\tau)) \tag{2.37}$$

and since in (2.36) it is already multiplied by a variation $\delta\xi^\nu$, we need to keep only the zeroth order:

$$\int_{\delta\Omega} d^2x \sqrt{-g}\mathcal{L} = \int_A^B \mathrm{d}\tau \int_0^1 \mathrm{d}w\, \varepsilon_{\mu\nu} u^\mu \delta\xi^\nu \mathcal{L}(\xi^\sigma(\tau)) \tag{2.38}$$

$$= \int_A^B \mathrm{d}\tau\, \varepsilon_{\mu\nu} u^\mu \delta\xi^\nu \mathcal{L}(\xi^\sigma(\tau)) \tag{2.39}$$

$$= -\int_A^B \mathrm{d}\tau \mathcal{L}(\xi^\sigma(\tau)) n_\nu \delta\xi^\nu \tag{2.40}$$

$$= \frac{1}{2}\rho_0 \int_A^B \mathrm{d}\tau (n^2 + 1) n_\nu \delta\xi^\nu \tag{2.41}$$

Taken together, the results (2.33), (2.41) simplify and the variation of the action (2.22) is

$$\delta S = \rho_0 \int_\Omega \mathrm{d}^2x \sqrt{-g}\, \Box\lambda\, \delta\lambda - \int_A^B \mathrm{d}\tau \left(m \frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau}\right) - \frac{1}{2}\rho_0 (n^2 - 1) n^\mu\right) \delta\xi_\mu$$
$$- \int_A^B \mathrm{d}\tau \left(m' \frac{\mathrm{D}}{\mathrm{d}\tau'}\left(\frac{\mathrm{d}(\xi')^\mu}{\mathrm{d}\tau'}\right) + \frac{1}{2}\rho_0 (n^2 - 1) n^\mu\right) \delta(\xi')_\mu \tag{2.42}$$

yielding the equations of motion

$$\Box\lambda = 0 \tag{2.43}$$

$$m\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau}\right) = \frac{1}{2}\rho_0(\nabla_\nu\lambda\nabla^\nu\lambda - 1)n^\mu \tag{2.44}$$

$$m'\frac{\mathrm{D}}{\mathrm{d}\tau'}\left(\frac{\mathrm{d}(\xi')^\mu}{\mathrm{d}\tau'}\right) = -\frac{1}{2}\rho_0(\nabla_\nu\lambda\nabla^\nu\lambda - 1)n^\mu \tag{2.45}$$

2.3 String-Particle Dynamics

Equations (2.44), (2.45) have an easy interpretation: the particles are pulled by a two-force normal to their two-velocity, with magnitude equal to the pressure in the string (2.7). This two-force also satisfies the key criterion that the particles be free when $n = 1$, since the string is then (at least locally) undeformed. However, the derivation still seems somewhat questionable, particularly in the case of the $\delta\Omega$ term; in this section, then, we present the classical limit of the equations and their interpretation, as well as an alternate way to derive the boundary conditions from the variational problem $\delta S = 0$ with S as given in (2.22).

To perform the classical limit, we will consider the equations of motion in a flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ with coordinates (t, r) and let $c \to \infty$. For the string equation, we immediately have

$$\frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} - \frac{\partial^2\lambda}{\partial r^2} = 0 \xrightarrow{c\to\infty} \frac{\partial^2\lambda}{\partial r^2} = 0 \tag{2.46}$$

Interpreting this result, however, requires transforming from $\lambda(t, r)$ to the classical description $r = r(x, t)$ with x a reference coordinate. From the definition of these functions, it is clear that:

$$\lambda(t, r(x, t)) = r_0(x) \tag{2.47}$$

for some "labelling" function $r_0(x)$. This function can be fixed by the requirement that $n = 1$ corresponds to an unstretched string. Assuming the reference configuration of the string is undeformed, the function $r(x, t)$ has to then be of the form

$$r(x, t) = x + f(t) \tag{2.48}$$

Differentiating (2.47) with respect to x, we obtain

$$\frac{\partial\lambda}{\partial r}\frac{\partial r}{\partial x} = \frac{\mathrm{d}r_0}{\mathrm{d}x} \tag{2.49}$$

But from the classical limit of $n^2 = 1$, $\partial\lambda/\partial r = \pm 1$ and from (2.48), $\partial r/\partial x = 1$, so that:

$$\frac{\mathrm{d}r_0}{\mathrm{d}x} = \pm 1 \tag{2.50}$$

Without loss of generality, we assume $r_0(x) = x$, and therefore

$$\lambda(t, r(x, t)) = x \tag{2.51}$$

Differentiating this equation twice with respect to x gives:

$$\frac{\partial^2 \lambda}{\partial r^2} = -\frac{1}{(\partial r/\partial x)^3}\frac{\partial^2 r}{\partial x^2} \tag{2.52}$$

and the classical limit of the equation of motion for the string is therefore

$$\frac{\partial^2 r}{\partial x^2} = 0 \tag{2.53}$$

This corresponds to the massless ($\lambda \to 0$) limit of the classical elastic string, where it immediately changes its shape in response to changing boundary conditions, and always has uniform tension (1.36).

For the particle equations of motion, we first note that in our limit, the $x = \xi^1$ component of the two-acceleration is just the classical acceleration,

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \xrightarrow{c\to\infty} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \tag{2.54}$$

where the last step follows from $\mathrm{d}t/\mathrm{d}\tau = \gamma \xrightarrow{c\to\infty} 1$. The pressure appearing in (2.44) is

$$\frac{1}{2}\rho_0(n^2-1) \xrightarrow{c\to\infty} \frac{1}{2}\rho_0\left(\left(\frac{\partial\lambda}{\partial r}\right)^2 - 1\right) = \frac{1}{2}\rho_0\left(\left(\frac{1}{\partial r/\partial x}\right)^2 - 1\right) \tag{2.55}$$

and the resulting equation of motion for the left particle (the change in sign is due to the orientation of the normal n^μ)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{1}{2}\rho_0\left(\left(\frac{1}{\partial r/\partial x}\right)^2 - 1\right) \tag{2.56}$$

While this equation is markedly nonlinear in $r(x,t)$ and differs from (1.38), we have to recall that to reach the equations of motion in classical mechanics, we also made the assumption of linear elasticity, which tacitly includes small deformations. Expanding the above around $\frac{\partial r}{\partial x} \approx 1$ to first order, we have

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \approx \rho_0\left(\frac{\partial r}{\partial x} - 1\right) \tag{2.57}$$

In comparison to (1.38), we can therefore make the identification $\rho_0 = \kappa$ - since the string is massless, its rest energy density corresponds to the elastic constant.

Looking back at equation (2.51) suggests an altogether different approach to deriving the boundary conditions - instead of relating the variations $\delta\lambda$ and $\delta\xi$, $\delta\xi'$ as in (2.31), we can try to change coordinates such that the variations are over a fixed region, similarly to the classical case. The reason for not approaching the string this way originally is clear: the formulation in λ provides an elegant form of the field equation and Lagrangian. Formulating the problem in terms of a fixed reference coordinate and time, however, will simplify the boundary problems of section 2.2 immensely. To begin with this new derivation, define the field $\xi^\mu(\chi,\tau)$ by

$$\lambda(\xi^\mu(\tau,\chi)) = \chi \tag{2.58}$$

so that $\xi^\mu(\chi_0,\tau)$ for a fixed χ_0 represents the proper time parametrised world-line of the point with reference coordinate χ_0. We will denote the components of the ξ^μ field $\xi^0 = t(x,\tau)$, $\xi^0 = r(\chi,\tau)$. Differentiating (2.58) with respect to both χ and τ, we obtain the system of equations

$$\frac{\partial\lambda}{\partial t}\frac{\partial t}{\partial\chi} + \frac{\partial\lambda}{\partial r}\frac{\partial r}{\partial\chi} = 1 \tag{2.59a}$$

$$\frac{\partial\lambda}{\partial t}\frac{\partial t}{\partial\tau} + \frac{\partial\lambda}{\partial r}\frac{\partial r}{\partial\tau} = 0 \tag{2.59b}$$

which we can solve for the components of $\nabla_\mu\lambda$,

$$\nabla_\mu\lambda = \left(\frac{\partial\lambda}{\partial t}, \frac{\partial\lambda}{\partial r}\right) = \frac{1}{\Delta}\left(-\frac{\partial r}{\partial\tau}, \frac{\partial t}{\partial\tau}\right) \tag{2.60}$$

where Δ is the determinant:

$$\Delta = \det\left(\frac{\partial(t,r)}{\partial(\tau,\chi)}\right) \tag{2.61}$$

After substituting the expression for $\nabla_\mu\lambda$, and using the normalisation of two-velocity $\partial\xi^\mu/\partial\tau$, the string Lagrangian is in terms of ξ^μ,

$$\mathcal{L} = -\frac{1}{2}\rho_0\left(\frac{1}{-g}\frac{1}{\Delta^2} + 1\right) \tag{2.62}$$

The metric determinant $\sqrt{-g}$ represents here the metric determinant in the old coordinates (t,r) evaluated at the point $\xi^\mu(\tau,\chi)$.

The advantage of this method is that the worldlines of the boundary particles are now expressed as $\xi^\mu(\tau) = \xi^\mu(\tau,0)$, $\xi^\mu(\tau) = \xi^\mu(\tau,l)$. In applying $\delta S = 0$ with the same action as in section 2.2, we can therefore simply calculate the relevant boundary terms on $\chi = 0$, or $\chi = l$, and these will give the two-force acting on the particles. The string action is expressed in the new coordinates as

$$S_{\text{str}} = -\frac{1}{2}\rho_0\int_0^l \mathrm{d}\chi\int\mathrm{d}\tau\left(\frac{1}{\sqrt{-g}}\frac{1}{\Delta} + \sqrt{-g}\Delta\right) \tag{2.63}$$

We will again show the calculation only for the left particle, with coordinates $\xi^\mu(\tau,0)$. Although the full expression for δS is quite long, the only terms in δS_{str} which contribute to the $\chi = 0$ boundary are:

$$\begin{aligned}\delta S_{\text{str,b}} = {}& \frac{1}{2}\rho_0\int_0^l \mathrm{d}\chi\int\mathrm{d}\tau\frac{1}{\sqrt{-g}}\frac{1}{\Delta^2}\left(\frac{\partial t}{\partial\tau}\frac{\partial(\delta x)}{\partial\chi} - \frac{\partial x}{\partial\tau}\frac{\partial(\delta t)}{\partial\chi}\right) \\ &+ \frac{1}{2}\rho_0\int_0^l \mathrm{d}\chi\int\mathrm{d}\tau\sqrt{-g}\left(\frac{\partial t}{\partial\tau}\frac{\partial(\delta x)}{\partial\chi} - \frac{\partial x}{\partial\tau}\frac{\partial(\delta t)}{\partial\chi}\right)\end{aligned} \tag{2.64}$$

Integrating by parts, and simplifying the resulting boundary term, we have:

$$\delta S_{\text{str,b}} = \frac{1}{2}\rho_0 \int \mathrm{d}\tau \left(\frac{1}{-g}\frac{1}{\Delta^2} - 1\right) \varepsilon_{\mu\nu} \frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau} \delta\xi^\nu(\tau, 0) \tag{2.65}$$

where we again recognize the pressure and the normal vector

$$P = \frac{1}{2}\rho_0 \left(\frac{1}{-g}\frac{1}{\Delta^2} - 1\right) \tag{2.66}$$

$$n_\nu = -\varepsilon_{\mu\nu} u^\mu \tag{2.67}$$

2.4 Falling in a Schwarzschild Spacetime

Although the original aim of this work was to numerically solve the relativistic equations of motion as well as the classical, this has sadly proven too difficult. In this short section, we summarize the formulation of the problem, the approaches that have been attempted and the difficulties encountered.

The main difference between the two problems lies in the domain of numerical integration. The field λ is at any given coordinate time defined only on the region that is occupied by the string, but this region changes dynamically with ξ^μ, $(\xi')^\mu$, making enforcing the boundary conditions seemingly impossible.

In the Schwarzschild coordinates (t, r), with the length unit $2M = 1$, the metric is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{1}{r}\right) & 0 \\ 0 & \left(1 - \frac{1}{r}\right)^{-1} \end{pmatrix} \tag{2.68}$$

and the string equation of motion simplifies to

$$\frac{\partial^2 \lambda}{\partial t^2} - \left(1 - \frac{1}{r}\right)^2 \frac{\partial^2 \lambda}{\partial r^2} = 0 \tag{2.69}$$

and it is therefore a wave equation with coordinate dependent velocity. With regular boundary conditions, this equation is soluble with the same method as presented in section 1.4.

Converting the ξ^μ equation of motion to the r-coordinate of the particle in terms of coordinate time t, we get

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\Gamma^r{}_{tt} + (\Gamma^t{}_{tr} - \Gamma^r{}_{rr}) \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + F_{\text{str}}(r, t) \tag{2.70}$$

with the force from the string given by

$$F_{\text{str}}(r, t) = \frac{\rho_0}{2m}\frac{n^2 - 1}{n} \left(\left(1 - \frac{1}{r}\right)\frac{\partial \lambda}{\partial x} - \left(1 - \frac{1}{r}\right)^{-1} \frac{\partial \lambda}{\partial t}\right) \left(-\left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right) \tag{2.71}$$

with the stretch factor

$$n^2 = -\left(1 - \frac{1}{r}\right)^{-1} \frac{\partial \lambda}{\partial t} + \left(1 - \frac{1}{r}\right) \frac{\partial \lambda}{\partial x} \tag{2.72}$$

and the Christoffel symbols

$$\Gamma^r{}_{tt} = \frac{1}{r^2}\left(1 - \frac{1}{r}\right) \tag{2.73}$$

$$\Gamma^t{}_{tr} = \frac{1}{r^2}\left(1 - \frac{1}{r}\right)^{-1} \tag{2.74}$$

$$\Gamma^r{}_{rr} = -\frac{1}{r^2}\left(1 - \frac{1}{r}\right)^{-1} \tag{2.75}$$

This equation is also solvable by usual numeric methods for a fixed background λ, but solving both (2.69) and (2.70) along with a similar equation for the right particle is much more involved.

The two approaches that were attempted were as follows: firstly, we can try to iterate over the coordinate time t, and keep two fixed length arrays representing the coordinate r and the field λ. At every iteration, we then first update the coordinates of the endpoint particles, and then update the coordinate array r. The next iteration of the λ array is then constructed by the same stencil as in the classical case, but with either linearly or quadratically interpolated values for λ in the previous two iterations, since the same index of the array corresponds to different values of r in different iterations. This approach seemed promising, but turned out to be wildly numerically unstable, at least in the Schwarzschild spacetime.

In the second approach, we choose a fixed r-interval, a compute the values of λ everywhere on it. We assume the string is always contained in this interval, and we try to enforce $\lambda = 0$ and $\lambda = l$ at the positions of the left and right particle respectively at each iteration. This approach fails on a practical level, since there is no way to simultaneously artificially set the value of λ at a given point on the grid, and accurately measure the derivatives of λ at that point.

One interesting aspect of simulating the string to mention is the initial condition, $\lambda(t, r)$. If we want the string to be under zero tension, it has to have the form:

$$\lambda(0, r) = \int_{r_0}^{r} \sqrt{g_{rr}} dr \tag{2.76}$$

where r_0 is the position of the left particle.

Conclusion

The main object of this thesis was the study of a simple system: the dumbbell consisting of two particles connected by an elastic string, subject to a gravitational field. We studied this system in both classical mechanics and general relativity, and obtained a viable description of its dynamics in both frameworks.

In the classical case, we first derived the Lagrangian description of a string from first principles and the assumption of linear elasticity. This assumption was also justified by the more general continuum description of a three-dimensional string confined to one-dimensional displacement. The dynamics was easily modified to account for a bounded string connecting point masses, and we solved the equations of motion numerically. The resulting position shifts were negative and independent of frequency. We have also shown that our model simplifies to a linear spring in the low density limit.

In describing the string in relativity, we followed the definitions and assumptions set out by Natário [2014]. We performed a similar procedure as in the classical case, first deriving the string Lagrangian, and then using a variational principle to derive the equations of motion. The equations, however, turned out to be difficult to solve numerically. Constructing a stable numerical solution is a natural continuation of this work, and we intend to complete it in the future.

The string is a system with a lot of intricacy, and compared to simpler one-dimensional models, it offers many new possibilities for analysing geodesic motion and the deviation of extended bodies from it. We hope to continue our study of this model in the future.

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A. Attachments

A.1 First Attachment