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Random Dynamical Systems and Their Applications

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Abstract: This thesis extends the existing results in the theory of random dynamical systems driven by fractional noise in Hilbert space. In particular, it broadens the scope of applicability of the results presented by Maria J. Garrido-Atienza, Bohdan Maslowski and Jana Snuparkova in Garrido-Atienza et al. [2016] for fractional noise whose sample paths have a Hölder exponent greater than 1/2. The main object of the research is the following stochastic equation:

$$d\,u(t) = (A(t)u(t) + F(u(t)))d\,t + Bu(t)d\,\omega(t), \quad u(0) = u_0 \in V,$$

where $(V, \|\cdot\|_V)$ is a separable Hilbert space, ω is a stochastic process and the stochastic integral is understood in the Zähle sense.

This thesis contains the proof of a Fubini-type theorem for integration in the sense of Zähle. It is shown that the assumption about ergodicity for the underlying fractional noise in Garrido-Atienza et al. [2016] is redundant and the statements about random dynamical systems which are generated by the solution of the equation and its random attractor remain valid. The thesis also contains the proof of the existence and uniqueness of the solution to the equation above.

Keywords: random dynamical systems, random attractors, fractional noise, infinite dimensional stochastic equations,

Contents

Introduction

Dynamical systems are mathematical models used to describe how a system changes over time. They are widely used in various fields, including physics, engineering, biology, economics, and finance, to study the behaviour of complex systems Bianchi et al. [2019], Mellodge [2015]. A random dynamical system is a type of dynamical system that describe behaviour of a process in the presence of elements of uncertainty, making it useful for modeling systems with unpredictable behaviour.

In recent years, there has been growing interest in the study of random dynamical systems, as they provide a powerful tool for understanding and an extension of dynamical system framework for systems which are driven by some random source. L. Arnold's book Arnold [1999] made a significant impact on the field by providing a comprehensive and rigorous treatment of the theory of random dynamical systems and its applications. One more point of interest is long term behaviour, among the pioneers in this field were H. Crauel, A. Debussche, F. Flandoli, who introduced the concept of random attractors and developed the theory for their existence and properties in their influential paper Crauel et al. [1997]. Another mathematician who has made significant contributions to the theory of random attractors is B. Schmalfuss, whose research Schmalfuss and Flandoli [1996], Schmalfuss [2000] focused on the existence and properties of random attractors for non-autonomous and infinite-dimensional random dynamical systems.

Most of the results in the theory of random dynamical systems are known for the finite dimensional case. Although the infinite dimensional case of stochastic equations driven by fractional noise in Hilbert space has been explored in previous works, such as Duncan et al. [2005], Snuparkova [2010], the equations in these articles has only a local solution that is why do not generate a random dynamical system. In the present thesis, we aim to extend the existing results in this field by building upon the work M. J. Garrido-Atienza, B. Maslowski, and J. Snuparkova Garrido-Atienza et al. [2016]. While this article primarily considers the Fractional Brownian motion, our interest lies in verifying its statements for more general types of fractional noises. To accomplish this, we investigate a specific equation, which serves as the primary focus of our research:

$$d\,u(t) = (A(t)u(t) + F(u(t)))d\,t + Bu(t)d\,\omega(t), \quad u(0) = u_0 \in V, \tag{1}$$

where the integral is understood in the Zähle sense. Such type of equation under pre-defined conditions has a solution and generates a random dynamical system. In the present thesis it was checked and shown that most of all results in Garrido-Atienza et al. [2016] hold for more general type of the noise ω with Hölder exponent larger than 1/2.

The first section serves as an introduction to the main definitions and results that are utilized throughout this thesis. A particular emphasis is given on the fractional integration results in stochastic calculus, which were introduced by M. Zähle in the seminal works Zähle [1998, 2001]. Additionally, the chapter includes relevant definitions and basic results from the theory of semigroups, which are used extensively in the subsequent sections.

In the second section, the results established in part 1-2 in Garrido-Atienza et al. [2016] are extended. The scope of applicability has been broadened to any fractional noise whose sample paths have a Hölder exponent greater than $1/2$, and not just the fractional Brownian motion. Additionally, the Fubini-type theorem (Theorem 21) has been formulated for the integral in the sense of Zähle (1.1), and the solution to equation (1) has been found in Theorems 17, 19, serving as a further extension of the previous work.

The third section provides a concise introduction to random dynamical systems, drawn from the seminal work of L. Arnold in this area (Arnold [1999]). Additionally, it contains definitions of random attractors from various sources, including Crauel et al. [1997], Schmalfuss [2000], Crauel et al. [2008], which propose different definitions and criteria for random attractors (Remark 9). Additionally, the results presented in part of the article Garrido-Atienza et al. [2016] about random attractors were confirmed for a more general source of randomness. One of the main achievement is omitting the assumption of ergodicity for fractional noise (Remark 10).

The focus of the fourth section is an exposition of various examples of noise sources, showing that there are many processes which satisfy the requirements in the previous sections. Examples of the equations are presented and it is shown that a solution of a linear equation which is studied in Duncan et al. [2005] has a global solution and generates a random dynamical system whereas in the article Duncan et al. [2005] this equation has only local solution due to the use of a Skorokhod-type integral instead of a Zähle-type integral.

1. Preliminaries

This chapter covers main definitions and results which are used across the whole thesis. It is devoted to the results of the fractional integration in stochastic calculus introduced by Zähle [1998, 2001], as well as to the definitions and basic results from the theory of semigroups which are used in the further sections.

1.1 Fractional integration

In the present section results and definitions are taken from Samko et al. [1993], Zähle [1998]. Through the course of this section, L_p will denote the $L_p(a,b)$ space on the interval $[a,b]$, where $a<b$, H^λ denotes the space of Hölder continuous functions on the interval $[a,b]$ with Hölder exponent λ and $(-1)^\alpha$ is understood as $e^{i\pi\alpha}$, where i is the imaginary unit. Let us start from the definition of the fractional Riemann-Liouville integral.

Definition 1. *For $f \in L_1$ and $\alpha > 0$ the left- and right-sided fractional Riemann-Liouville integrals of f of order α on (a,b) are given at almost all x by*

$$[I^\alpha_{a+}f](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,dy$$

and

$$[I^\alpha_{b-}f](x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\,dy\ ,$$

respectively, where Γ denotes the Gamma function.

Fractional differentiation may be introduced as an inverse operation. It is sufficient to work with a class of functions where this inverse is well-determined and the Riemann-Liouville derivatives agree with the (more general) version of the derivative in the sense of Weyl (Lemma 19.3 Samko et al. [1993]). The Riemann-Liouville derivatives are defined:

$$[D^\alpha_{a+}f](x) = \mathbb{1}_{(a,b)}(x)\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)}{(x-y)^\alpha}\,dy,$$

and

$$[D^\alpha_{b-}f](x) = \mathbb{1}_{(a,b)}(x)\frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(y)}{(y-x)^\alpha}\,dy.$$

Let $p \geq 1$ and let $I^\alpha_{\substack{a+\\(b-)}}(L_p)$ be the class of functions f which may be represented as an $I^\alpha_{\substack{a+\\(b-)}}$-integral of some L_p-function φ. The function φ in the above representation $f = I^\alpha_{\substack{a+\\(b-)}}\varphi$ is unique in L_p and for $0<\alpha<1$ it agrees a.e. with the *Weyl representation* of the derivative:

$$[D^\alpha_{a+}f](x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}\,dy\right)\mathbb{1}_{(a,b)}(x)$$

and

$$[D^{\alpha}_{b-}f](x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}d\,y\right)\mathbb{1}_{(a,b)}(x).$$

By the construction it follows from Theorem 2.4 Samko et al. [1993], that

$$[I^{\alpha}_{\substack{a+\\(b-)}}[D^{\alpha}_{\substack{a+\\(b-)}}f]] = f, \;\; f \in I^{\alpha}_{\substack{a+\\(b-)}}(L_p)$$

and

$$[D^{\alpha}_{\substack{a+\\(b-)}}[I^{\alpha}_{\substack{a+\\(b-)}}f]] = f, \;\; f \in L_1.$$

If f is continuously differentiable in a neighborhood of $x \in (a,b)$, then we have Samko et al. [1993]:

$$\lim_{\alpha\to 1-}[D^{\alpha}_{\substack{a+\\(b-)}}f](x) = f'(x), \;\; x \in (a,b)$$

Let us define:

$$f_{a+}(x) = \mathbb{1}_{(a,b)}(x)(f(x) - f(a+)), \;\; x \in \mathbb{R}$$

and

$$g_{b-}(x) = \mathbb{1}_{(a,b)}(x)(g(x) - g(b-)), \;\; x \in \mathbb{R}$$

Zähle [1998] defined an integral in the following way.

Definition 2. *The (fractional) integral of f with respect to g is defined by*

$$\begin{aligned}\int_a^b f(x)d\,g(x) = (-1)^{\alpha}\int_a^b [D^{\alpha}_{a+}f_{a+}](x)[D^{1-\alpha}_{b-}g_{b-}](x)d\,x \\ + f(a+)(g(b-) - g(a+))\end{aligned} \tag{1.1}$$

provided that $f_{a+} \in I^{\alpha}_{a+}(L_p)$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q)$ for some $1/p + 1/q \le 1$, $0 \le \alpha \le 1$.

By Proposition 2.1 Zähle [1998] Definition 1.1 independent of the choice of α.

If both the integrand f and g are nice enough, the definition of integral (1.1) coincides with the Lebesgue-Stieltjes integral **(L-S)** and in some cases with the Riemann-Stieltjes integral **(R-S)** Theorem 2.4 Zähle [1998]:

Theorem 1. *Suppose that g has bounded variation with variation measure μ and f and g satisfy the conditions of Definition 1.1 and one of the conditions below is satisfied:*

(i) If $\int_a^b I^{\alpha}_{a+}[|[D^{\alpha}_{a+}f_{a+}]|](x)\mu(d\,x) < \infty$.

(ii) If f is bounded and right-(or left-) continuous at μ-a.a. points

Then

$$\int_a^b f(x)d\,g(x) = (\mathbf{L}-\mathbf{S})\int_a^b f(x)d\,g(x)$$

Remark 1. *As a special case for any continuous function f in (ii) it holds:*

$$\int_a^b f(x) d\, g(x) = (\mathbf{R} - \mathbf{S}) \int_a^b f(x) d\, g(x)$$

Hölder continuous functions require special attention since the fractional noise, which is studied in the further chapters, will have Hölder continuous sample paths of order 1/2 or higher. Theorem 4.2.1 and Proposition 4.4.1 Zähle [1998] give properties of the integral in the case of Hölder functions f and g.

Theorem 2. *If $f \in H^\lambda$, $g \in H^\mu$ for some $\lambda + \mu > 1$, the Riemann-Stieltjes integral **(R-S)** $\int_a^b f d\, g$ exists and agrees with the integral $\int_a^b f d\, g$ in the sense of (1.1).*

Proposition 3. *If $f \in H^\lambda$, $g \in H^\mu$ for some $\lambda + \mu > 1$, $1 - \mu < \alpha < \lambda$ and the derivatives in (1.1) are bounded, then*

$$\mathbb{1}_{(a,b)}(\cdot) \int_a^{(\cdot)} f d\, g \in H^\mu$$

and

$$\mathbb{1}_{(a,b)}(\cdot) \int_{(\cdot)}^b f d\, g \in H^\mu$$

1.2 Fractional integration with respect to fractional noise

The following section complements the results of the previous section with an extension to stochastic calculus. Results from this section are also taken from Zähle [1998, 2001].

Let us study integration with respect to fractional Brownian motion B^H on $(\Omega, \mathcal{F}, \mathbb{P})$. The fractional Brownian motion is a real valued centered Gaussian process on $[0, \infty)$ with stationary increments and variance $\mathbb{E}(B^H(t+s) - B^H(t))^2 = s^{2H}$, where $s, t > 0$ and $H \in (0, 1)$ is a parameter of the process. In Chentsov [1956] it was proved that B^H has a version with sample paths of Hölder continuity of all orders $\lambda < H$ on any finite interval $[0, T]$ $\mathbb{P}$ a.s.. Further, consider the set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ on which every sample path of B^H is Hölder continuous.

The Hölder continuity of sample paths of B^H guarantees the existence of the integral in the sense (1.1)

$$\int_0^t f(s, \omega) d\, B^H(s, \omega),\ 0 < t \leq T,\ \omega \in \Omega_0, \tag{1.2}$$

for any random measurable function f such that $f : [0, T] \times \Omega \to \mathbb{R}$ is measurable and $f_{0+}(\cdot, \omega) \in I^\alpha_{0+}(L_1(0, T))$ for some $\alpha > 1 - H$. For $H > 1/2$ it allows to solve stochastic differential equations path-wise. It is important to highlight that there is no requirements for adaptedness of random function f.

All considerations above might be transferred to any stochastic process which has Hölder continuous sample paths with exponent greater 1/2 with probability 1.

To study integral as an operator it is necessary to introduce some notation of Besov- (or Slobodeckij-) type spaces W_2^α for $0<\alpha<1$ (with modification) given by the (semi) norms:

$$\begin{aligned}
\|f\|_{\widetilde{W}_2^\alpha} &= \left(\int_a^b \int_a^b \frac{(f(x)-f(y))^2}{|x-y|^{2\alpha+1}} dx dy\right)^{1/2} \\
\|f\|_{W_2^\alpha} &= \|f\|_{L_2} + \|f\|_{\widetilde{W}_2^\alpha} \\
\|f\|_{W_{2,\infty}^\alpha} &= \|f\|_{L_\infty} + \|f\|_{\widetilde{W}_2^\alpha} \\
\|f\|_{W_2^\alpha(a+)} &= \left(\int_a^b \frac{f(x)^2}{(x-a)^{2\alpha}} dx\right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha} \\
\|f\|_{W_2^\alpha(b-)} &= \left(\int_a^b \frac{f(x)^2}{(b-x)^{2\alpha}} dx\right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha}
\end{aligned}$$

The following theorems provide a connection between spaces of functions and continuity properties of integrals Theorem 1.1 and Theorem 1.2 Zähle [2001].

Theorem 4. *Suppose $0<\alpha<1$ then the following compact embeddings hold:*

(i) *For $\alpha p>1$ we have $I^\alpha_{\substack{a+\\(b-)}}(L_p) \hookrightarrow H^{\alpha-1/p}$.*

(ii) $W_2^{\alpha+\delta}\,{\substack{(a+)\\(b-)}} \hookrightarrow I^\alpha_{\substack{a+\\(b-)}}(L_2)$, $\delta>0$.

(iii) $I^{\alpha+\delta}_{\substack{a+\\(b-)}}(L_2) \hookrightarrow \widetilde{W}_2^\alpha$, $0<\delta<1-\alpha$.

(iv) *$g\in\widetilde{W}_2^\alpha$ implies $g_{y-}\in W_2^\alpha(y-)(x,y)$ for any $x\in[a,b)$ and Lebesgue almost all $y\in(x,b)$.*

Theorem 5. *Suppose $0<\alpha<1/2$. If $f\in I^\alpha_{a+}(L_2)$, $g_{b-}^{1-\alpha}\in I_{b-}^{1-\alpha}(L_2)$ and f is bounded, then*

(i)

$$\int_a^x f dg \quad \text{and} \quad \int_x^b f dg$$

are continuous functions in $x\in(a,b)$; and

(ii) *it holds that*

$$\|\int_a^{(\cdot)} f dg\|_{W_{2,\infty}^\beta} \le const(\beta)\|f\|_{W_{2,\infty}^\beta}\|g_{b-}\|_{W_2^\beta(b-)}$$

provided that $\beta>1/2$.

Also Theorem 5 (ii) guarantees that for $g_{T-}\in W_2^\beta(T-)$, $f\in W_{2,\infty}^\beta$, $\varphi\in W_{2,\infty}^\beta$ and $a\in C^1(\mathbb{R}\times\mathbb{R};\mathbb{R})$, where $1/2<\beta<1$, the integral operator

$$f\mapsto x_0+\int_0^{(\cdot)} a(f,\varphi)dg$$

for fixed $x_0\in\mathbb{R}$ acts from $W_{2,\infty}^\beta$ into itself. The theorem below gives a local contraction property of the integral operator. For that, denote by $W_{2,\infty}^\beta(t_0,t;x_0,1)$ the set of functions f on (t_0,t) with $f(t_0+)=x_0$ and $\|f_{t_0+}\|_{W_{2,\infty}^\beta(t_0,t)}\le 1$.

Theorem 6. *Let $x_0, y_0 \in \mathbb{R}$, $1/2 < \beta < 1$, $g \in \widetilde{W}_2^{\beta}$, $a \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and assume that the partial derivatives $\frac{\partial a}{\partial x_1}$ and $\frac{\partial a}{\partial x_2}$ are locally Lipschitz as function of the first argument. Then for any $t_0 \in (0,T)$ and $c > 0$ there exists some $t \in (t_0, T)$ such that for any $\varphi \in W_{2,\infty}^{\beta}(t_0, t; y_0, 1)$ the integral operator A defined by*

$$Af = x_0 + \int_{t_0}^{(\cdot)} a(f, \varphi) d\, g$$

maps $W_{2,\infty}^{\beta}(t_0, t; y_0, 1)$ into itself and we have

$$\|Af - Ah\|_{W_{2,\infty}^{\beta}(t_0,t)} \le c\|f - h\|_{W_{2,\infty}^{\beta}(t_0,t)}$$

for all $f, h \in W_{2,\infty}^{\beta}(t_0, t; y_0, 1)$.

Our area of interest in this thesis is the noise whose sample paths are Hölder continuous with an exponent grater than 1/2. The following *change of variable* formula is valid.

Theorem 7. *Let $0 < \alpha < 1/2$, $f \in I_{0+}^{\alpha}(L_2)$ be bounded, $g_{T-} \in I_{T-}^{1-\alpha}(L_2)$ and*

$$h(t) = h(0) + \int_0^t f d\, g, \quad t \in (0, T].$$

Then we get for any C^1-function $F(x,t)$ on $\mathbb{R} \times [0,T]$ such that $\frac{\partial F}{\partial x} \in C^1$ and for any $0 \le t_0 < t \le T$:

$$F(h(t), t) - F(h(t_0), t_0) = \int_{t_0}^t \frac{\partial F}{\partial x}(h(s), s) f(s) d\, g(s) + \int_{t_0}^t \frac{\partial F}{\partial s}(h(s), s) d\, s.$$

1.3 Semigroup theory

This section contains results and definitions which are used in the next sections. Most of results in this section are taken from Pazy [1983] except when results have reference to another source.

Definition 3. *Let X be a Banach space. A one parameter family $T(t)$, $0 \le t < \infty$, of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if*

(i) $T(0) = I$, (I is the identity operator on X)

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \ge 0$ (the semigroup property).

Definition 4. *Let $\{T(t)\}_{t \in \mathbb{R}^+}$ be a semigroup and the linear operator A defined by*

$$D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0+} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A)$$

is called the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is called the domain of A.

Definition 5. *A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \to 0+} T(t)x = x \quad \text{for } \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X is also called a semigroup of class C_0 or simply a C_0-semigroup.

Definition 6. *A one-parameter family $S(t)$, $-\infty < t < \infty$, of bounded linear operators on a Banach space X is a C_0 group of bounded operators if it satisfies:*

(i) $S(0) = I$, *(I is the identity operator on X)*

(ii) $S(t+s) = S(t)S(s)$ *for* $-\infty < t, s < \infty$.

(iii) $\lim_{t \to 0} S(t)x = x$ *for* $x \in X$.

Since $\{T(t)\}_{t \in \mathbb{R}^+}$ is a collection of bounded operators the bound is provided by the theorem below.

Theorem 8. *Let $\{T(t)\}_{t \in \mathbb{R}^+}$ be a C_0-semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \text{ for } 0 \leq t < \infty.$$

There are some useful properties of C_0-semigroup which will be used later.

Theorem 9. *Let $\{T(t)\}_{t \in \mathbb{R}^+}$ be a C_0 semigroup and let A be its infinitesimal generator with the domain $D(A)$ in X. Then the following holds:*

(i) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

(ii) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau$$

Let us introduce the following notation. The resolvent set $\rho(A)$ of linear operator A in X is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A. One of the question in semigroup theory is how to recover a semigroup from the infinitesimal generator.

Theorem 10. *Let A be a densely defined operator in X satisfying the following conditions:*

(i) For some $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$.

(ii) There exists a constant M such that

$$\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \Sigma_\delta,\ \lambda \neq 0.$$

Then A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\in\mathbb{R}^+}$, which satisfies $\|T(t)\|_{\mathcal{L}(X)} \leq C$. Moreover,

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda : A) d\,\lambda,$$

where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\pi/2 < \theta < \pi/2+\delta$. The integral above converges for $t > 0$ in the uniform operator topology.

Earlier, we introduced semigroups whose domain was the real nonnegative axis. But to introduce fractional power of operator it is necessary to introduce some extensions.

Definition 7. *Let $\Delta = \{z : \varphi_1 < \arg z < \varphi_2,\ \text{for some } \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$, is an analytic semigroup in Δ if*

(i) $z \mapsto T(z)$ is analytic in Δ,

(ii) $T(0) = I$ and $\lim_{\substack{z\to 0\\ z\in\Delta}} T(z)x = x$, for every $x \in X$,

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

A semigroup $T(t)$ is called analytic if it is analytic in some sector Δ containing the nonnegative axis.

To proceed further it is necessary to impose the following assumption.

Assumption 1. *Let A be a densely defined closed linear operator for which*

$$\rho(A) \supset \Sigma^+ = \{\lambda : 0 < \omega < |\arg\lambda| \leq \pi\} \cup V$$

where V is a neighborhood of zero, and

$$\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq \frac{M}{1+|\lambda|} \quad \text{for} \quad \lambda \in \Sigma^+.$$

For an operator A satisfying Assumption 1 and $\alpha > 0$ the following operator is defined:

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C z^{-\alpha}(A - zI)^{-1} d\,z, \tag{1.3}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\omega < \theta < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z. The integral (1.3) converges in the uniform operator topology for every $\alpha > 0$ and thus defines a bounded linear operator $A^{-\alpha}$. Another representation of $A^{-\alpha}$ has the following form:

$$A^{-\alpha} = \frac{sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(tI + A)^{-1} d\,t, \quad 0 < \alpha < 1.$$

The operator $A^{-\alpha}$ has the following property:

Lemma 11. *There exists a constant $C \in (0,\infty)$ such that*

$$\|A^{-\alpha}\|_{\mathcal{L}(X)} \leq C \quad \text{for } 0 \leq \alpha \leq 1.$$

Now it is possible to introduce the definition below.

Definition 8. *Let A satisfy Assumption 1 with $\omega < \pi/2$. For every $\alpha > 0$ we define*

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

For $\alpha = 0$, we set $A^{\alpha} = I$.

Then the operator from the definition above possesses the representation below.

Theorem 12. *Let $0 < \alpha < 1$. If $x \in D(A)$, then*

$$A^{\alpha}x = \frac{sin(\pi\alpha)}{\pi}\int_0^{\infty} t^{\alpha-1}A(tI + A)^{-1}x dt.$$

Theorem 13. *Let $-A$ be infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$ then, the following holds:*

(i) For every $x \in D(A^{\alpha})$ we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.

(ii) For every $t > 0$ the operator $A^{\alpha}T(t)$ is bounded and for some constants $M_{\alpha}, \delta > 0$ it holds:

$$\|A^{\alpha}T(t)\|_{\mathcal{L}(X)} \leq M_{\alpha}t^{-\alpha}e^{-\delta t}, \quad t > 0.$$

(iii) Let $0 < \alpha \leq 1$, $x \in D(A^{\alpha})$. Then it holds, for some constant $C_{\alpha} > 0$, that:

$$\|T(t)x - x\|_{\mathcal{L}(X)} \leq C_{\alpha}t^{\alpha}\|A^{\alpha}x\|, \quad t > 0.$$

1.4 Equations in Hilbert spaces

Before we introduce the solution of a stochastic equation in Hilbert space, let us introduce the definition of solutions to deterministic equations.

We start with the abstract Cauchy problem. Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ into X. We wish to find a solution $u(t)$ to the initial value problem Pazy [1983]

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{1.4}$$

where by a solution we mean an X valued function $u(t)$ such that $u(t)$ is continuous for $t \geq 0$, continuously differentiable and $u(t) \in D(A)$ for $t > 0$ and equation (1.4) is satisfied. If A is an infinitesimal generator of C_0-semigroup $S(t)$, then, for any $x \in D(A)$, the abstract Cauchy problem for A has a (strong) solution $u(t) = S(t)x$ for every $x \in D(A)$, according to Theorem 9.

Sometimes it is possible to come to a situation when a solution in the sense above is too limited and it is necessary to find some generalization of the solution. Let us consider the inhomogeneous initial value problem for $f \in L^1([0,T], X)$ i.e, a Bochner-integrable function on the interval $[0,T]$ with values in the space X:

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \quad t > 0, \\ u(0) = x, \quad x \in X. \end{cases} . \tag{1.5}$$

Definition 9. *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\in\mathbb{R}^+}$. Let $x \in X$ and $f \in L^1([0,T],X)$. The function $u \in C([0,T],X)$ is given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \le t \le T,$$

is called the mild solution of the initial value problem (1.5) on $[0,T]$.

Another type of a solution might be presented in the following way.

Definition 10. *A function $u \in C([0,T],V)$, where V is Hilbert space, is called a weak solution of (1.5) on $[0,T]$ if for every $v \in D(A^*)$ the function $\langle u(t),v\rangle$ is absolutely continuous on $[0,T]$ and*

$$\langle u(t),v\rangle_V = \langle x,v\rangle_V + \int_0^t [\langle u(s),A^*v\rangle_V + \langle f(s),v\rangle_V]ds \quad \text{a.e. on } [0,T].$$

1.5 Evolution systems

The results in this section are taken from Pazy [1983]. One possible extension of the semigroup structure by introducing a new parameter is covered in this section.

Definition 11. *A two-parameter family of bounded linear operators $U(t,s)$, $0 \le s \le t \le T$, on Banach space X is called an evolution system if the following two conditions are satisfied:*

(i) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ *for* $0 \le s \le r \le t \le T$.

(ii) $(t,s) \mapsto U(t,s)$ *is strongly continuous for* $0 \le s \le t \le T$.

Such systems arise during the study of the following non-autonomous homogeneous equations:

$$\begin{cases} \frac{du(t)}{dt} + A(t)u(t) = 0, & s < t \le T, \\ u(s) = x, \end{cases} \tag{1.6}$$

where $\{A(t)\}_{t\in[0,T]}$ is the family of linear operators in X such that $A : D(A(t)) \subset X \to X$ for every $t \in [0,T]$. An X-valued function $u : [s,T] \to X$ is a classical solution of (1.6) if u is continuous on $[s,T]$, $u \in D(A(t))$ for $s < t \le T$, u is continuously differentiable on $s < t \le T$ and satisfies (1.6).

Let us add some assumptions about the operator family $\{A(t)\}_{t\in[0,T]}$:

Assumption 2. *The problem (1.6) which satisfies the assumptions below is called the parabolic initial value problem:*

(A.1) *The domain $D(A(t)) = D$ of $A(t)$, $0 \le t \le T$ is dense in X and independent of t.*

(A.2) *For $t \in [0,T]$, the resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\lambda \le 0$ *and there is a constant M such that*

$$\|R(\lambda : A(t))\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|+1} \quad \text{for } \mathrm{Re}\lambda \le 0,\ t \in [0,T].$$

(A.3) $-A(t)A(0)^{-1}$ *is a Hölder continuous function in* $\mathcal{L}(X)$*, or equivalently the inequality*

$$\|A(t) - A(s)\|_{\mathcal{L}(D,X)} \leq K|t - s|^{\alpha}$$

is satisfied for every $s, t \in [0, T]$*, for some constants* $K > 0$ *and* $\alpha \in (0, 1]$*, where* $\|x\|_D = \|A(0)x\|_X$*.*

Theorem 14. *Under the assumptions (A.1)-(A.3) there exists a unique evolution system* $U(t, s)$ *on* $0 \leq s \leq t \leq T$*, satisfying:*

(i) *For* $0 \leq s \leq t \leq T$ *exists constant* C *such that:*

$$\|U(t, s)\|_{\mathcal{L}(X)} \leq C. \tag{1.7}$$

(ii) *For* $0 \leq s < t \leq T$*,* $U(t, s) : X \to D$ *and* $t \to U(t, s)$ *is strongly differentiable in* X*. The derivative* $(\partial/\partial t)U(t, s)$ *is a bounded operator from* X *to* X *and it is strongly continuous on* $0 \leq s < t \leq T$*. Moreover,*

$$\frac{\partial}{\partial t}U(t, s) + A(t)U(t, s) = 0 \quad \text{for } 0 \leq s < t \leq T,$$

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\|_{\mathcal{L}(X)} = \|A(t)U(t, s)\| \leq \frac{C'}{t - s}$$

and

$$\|A(t)U(t, s)A(s)^{-1}\|_{\mathcal{L}(D,X)} \leq C'' \quad \text{for } 0 \leq s \leq t \leq T.$$

(iii) *For every* $v \in D$ *and* $t \in (0, T]$*,* $U(t, s)v$ *is differentiable with respect to* s *on* $0 \leq s < t \leq T$ *and*

$$\frac{\partial}{\partial s}U(t, s)v = U(t, s)A(s)v.$$

Assumptions (A.1)-(A.3) guarantee that $-A(t)$ is the infinitesimal generator of an analytic semigroup $S_t(s)$, $s \geq 0$, satisfying

$$\|S_t(s)\| \leq C \quad \text{for } s \geq 0,$$

$$\|A(t)S_t(s)\|_{\mathcal{L}(X)} \leq \frac{C}{s} \quad \text{for } s > 0,$$

where $C > 0$ is some constant which depends on T. The existence and uniqueness of the solution of equation (1.6) was shown in Theorem 5.6.8 Pazy [1983] and has the following form:

$$U(t, s) = S_s(t - s) + \int_s^t S_{\tau}(t - \tau)R(\tau, s)d\tau, \tag{1.8}$$

where

$$
\begin{aligned}
R_1(t,s) &= (A(s)-A(t))S_s(t-s), \\
R_{m+1}(t,s) &= \int_s^t R_1(t,\tau)R_m(\tau,s)d\tau, \\
R(t,s) &= \sum_{i=0}^{\infty} R_i(t,s)
\end{aligned}
\tag{1.9}
$$

or

$$
R(t,s) = R_1(t,s) + \int_s^t R_1(t,\tau)R(\tau,s)d\tau.
$$

Calculation (5.6.26) from Pazy [1983] gives the estimation of $R(t,s)$:

$$
\|R(t,s)\|_{\mathcal{L}(X)} \leq C(t-s)^{\alpha-1}. \tag{1.10}
$$

One more useful estimation is given by equation (5.17) Tanabe [1979]:

$$
\left\| \int_s^t S_\tau(t-\tau)R(\tau,s)d\tau \right\| \leq C_W(t-s)^{\alpha}, \tag{1.11}
$$

where C_W depends only on T, see (5.16) Tanabe [1979].

Remark 2. *By Definition 11 of an evolution system $U(t,s)$ $0 \leq s \leq t \leq T$ we have that:*

$$
\lim_{t\to s} \|U(t,s)\|_{\mathcal{L}(X)} = 1
$$

and since $U(t,s)$ is strongly continuous we can estimate its norm on the compact interval:

$$
\sup_{(t,s)\in[0,T]\times[0,t]} \|U(t,s)\|_{\mathcal{L}(X)} = K(T) < \infty.
$$

We may notice that $K(T)$ is non-decreasing in T. Thus we have:

$$
\lim_{T\to s,\, t\to s} (T-s)\|U(t,s)\|_{\mathcal{L}(X)} = 0.
$$

This guarantees that the inequality:

$$
K(T)(T-s) < 1
$$

always holds true for $T \in [s,a)$, where $a > s$.

To sum up all the properties described above, it follows that:

$$
\begin{aligned}
\|U(t,s)\|_{\mathcal{L}(X)} &\leq \|S_s(t-s)\|_{\mathcal{L}(X)} + \| \int_s^t S_\tau(t-\tau)R(\tau,s)d\tau\|_{\mathcal{L}(X)} \\
&\leq C_1 + C_W(t-s)^{\alpha} \leq K,
\end{aligned}
\tag{1.12}
$$

where K is some constant which depends on T only.

2. Solutions of semilinear stochastic equations with a bilinear fractional noise

In this section, the following equation will be studied:

$$d\,u(t) = (A(t)u(t) + F(u(t)))d\,t + Bu(t)d\,\omega(t), \quad u(0) = u_0 \in V, \tag{2.1}$$

where $(V, \|\cdot\|_V)$ is a separable Hilbert space, $d\,\omega$ is understood as an extension of Zähle-type integral Chen et al. [2013] and $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators on V. In this chapter and further $(\omega(t), t \geq 0)$ denotes a stochastic process whose sample path are Hölder continuous with exponent $\beta' > 1/2$.

This chapter aims at extending the results of Garrido-Atienza et al. [2016] and to show that the obtained results are valid not only for fractional Brownian motion but for process $(\omega(t), t \geq 0)$ as well. Further in this chapter we will work with the equation (2.1) path-wise and take some particular example of a path ω to work with it. It is possible since nothing except a Hölder continuity of the path is used. Thus all stochastic processes in this chapter are treated as a particular path for given $\omega \in \Omega$. A Fubini-type theorem for the integral (1.1) in the sense of Zähle is proven in this section. As an extension to Garrido-Atienza et al. [2016] the solution to the equation (2.1) is given for $\{A(t)\}_{t\in[0,T]}$ instead of a time independent operator A.

2.1 Solution of linear stochastic equations with a fractional noise

At first, let us consider the autonomous linear problem which is given by

$$d\,v(t) = Av(t)d\,t + Bv(t)d\,\omega(t), \quad v(s) = u_0 \in V, \quad 0 \leq s \leq t. \tag{2.2}$$

All derivations for this type problem are done in Garrido-Atienza et al. [2016] under the following assumptions:

(A) The linear operator $A : D(A) \subset V \to V$ is closed, densely defined and generates an analytic semigroup $\{S_A(t)\}_{t\in\mathbb{R}+}$ on V

(B) The linear operator $B : D(B) \subset V \to V$ is closed, densely defined and generates a strongly continuous group $\{S_B(t)\}_{t\in\mathbb{R}}$

(C) $D(A) \subset D(B)$, $S_B(t)x \in D(A)$ for all $x \in D(A)$ and the operators satisfy the commutativity assumption

$$AS_B(t)x = S_B(t)Ax, \quad x \in D(A),\ t \in \mathbb{R}.$$

Moreover, $D(A^*) \subset D((B^*)^2)$ and $B^*x \in D(A^*)$, for all $x \in D(A^*)$.

Assumption (A) for an analytic semigroup gives the following estimation Garrido-Atienza et al. [2016]:

$$\|S_A(t-s) - S_A(r-s)\|_{\mathcal{L}(V)} \leq K_{\hat{\alpha}} \left(\frac{t-r}{r-s}\right)^{\hat{\alpha}}, \quad \hat{\alpha} \in (0,1],\ 0 \leq s < r \leq t. \quad (2.3)$$

A strongly continuous group $\{S_B(t)\}_{t\in\mathbb{R}}$ might be represented as

$$S_B(t) = \begin{cases} S_{B_+}(t), & t \geq 0, \\ S_{B_-}(-t), & t \leq 0, \end{cases}$$

where $S_{B_+}(t)$ and $S_{B_-}(t)$ are C_0 semigroups, then there exist some constants $M_B > 1$ and $r_b > 0$ such that $\|S_B\|_{\mathcal{L}(V)} \leq M_B e^{r_B|t|}$ holds for all $t \in \mathbb{R}$.

The nonautonomus version of the linear equation (2.2) has the following form:

$$d\,v(t) = A(t)v(t)d\,t + Bv(t)d\,\omega(t), \quad v(s) = u_0 \in V \quad 0 \leq s \leq t. \quad (2.4)$$

The solution of (2.4) was found in the Theorem 17 below under the following assumptions on the family of operators $\{A(t)\}_{t\in[0,T]}$ and operator B:

(C.1) $\{A(t)\}_{t\in[0,T]}$ is the family of closed densely defined operators and both domains $D = \mathrm{D}(A(t))$ and $D^* = \mathrm{D}(A^*(t))$ are independent of t, for $t \in [0,T]$.

(C.2) The closed densely defined linear operator B is defined on a domain $D(B)$ such that $D \subset D(B)$, generates a strongly continuous group $\{S_B(t)\}_{t\in\mathbb{R}}$ with property: $S_B(t)x \in D$ for $x \in D$, and $S_B(t)$ commutes with $A(s)$:

$$A(s)S_B(t)x = S_B(t)A(s)x, \quad \forall s \geq 0,\ \forall t \in \mathbb{R},\ x \in D.$$

Moreover, $D \subset D((B^*)^2)$ and $B^*x \in D^*$, for $x \in D^*$.

(C.3) For each $t \in [0,T]$, the linear operator $A(t)$ is a closed densely defined operator in V whose resolvent set $\rho(A(t))$ contains the half-plane $\mathrm{Re}\lambda \leq 0$ and the resolvent $R(\lambda, A(t))$ satisfies the following inequality:

$$\|R(\lambda, A(t))\|_{\mathcal{L}(V)} \leq \frac{M}{1+|\lambda|}, \quad \forall \lambda \in \mathbb{C},\ \mathrm{Re}\lambda \leq 0$$

for some constant $M > 0$ that does not depend on t. This condition implies that $A(t)$ generates an analytic semigroup for each fixed $t \in [0,T]$ which will be denoted by $S_t(s)$ for $s \geq 0$.

(C.4) There exist constants L and $1/2 < \alpha < 1$ such that

$$\|A(t) - A(s)\|_{\mathcal{L}(D,V)} \leq L|t-s|^{\alpha} \text{ for } s,t \in [0,T],$$

where $\|x\|_D$ is defined by $\|x\|_D = \|A(0)x\|_V$

The equation in the form (1.6) which satisfies the assumptions above is called the parabolic type equation. Theorem 14 guarantees that equation (1.6) generates an evolution system $(S_A(t,s), 0 \leq s \leq t \leq T)$ on V which has the form (1.8).

Remark 3. *The assumption (C.2) gives us that:*

$$S_B(t)S_q(s) = S_q(s)S_B(t), \quad \forall s, q \geq 0, \ \forall t \in \mathbb{R}, \tag{2.5}$$

where $S_q(t)_{t\in\mathbb{R}+}$ is an analytic semigroup generated by $A(q)$.

Remark 4. *The assumption (C.4) gives us that $A(t)$ is a $\mathcal{L}(D, V)$ valued function which is Hölder continuous in the uniform operator topology for $t \in [0, T]$. Thus for each $x \in D$ we have $\sup_{t\in[0,T]} \|A(t)x\|_D = R_x < \infty$.*

For the space H^β, the norm is given by

$$\|u\|_\beta = \|u\|_{\beta,a,b} = \|u\|_{\infty,a,b} + |||u|||_{\beta,a,b},$$

with

$$\|u\|_{\infty,a,b} = \sup_{s\in[a,b]} |u(s)|; \quad |||u|||_{\beta,a,b} = \sup_{a\leq s<t\leq b} \frac{|u(t) - u(s)|}{|t - s|^\beta}.$$

The symbol $C([a, b]; \mathbb{R})$ denotes the space of continuous functions on $[a; b]$ with values in $\mathbb{R}$ with finite supremum norm.

Before we proceed further, it is important to introduce a lemma from Garrido-Atienza et al. [2016].

Lemma 15. *Suppose that $f \in H^\gamma$ and $g \in H^{\gamma'}$ such that $1 - \gamma' < \gamma$. Then*

$$\int_a^b f d g$$

is well-defined in the sense of (1.1). In addition, $\forall a \leq s \leq t \leq b$ there exists a constant c depending only on $b - a$, γ, γ' such that

$$\left| \int_s^t f d g \right| \leq c\|f\|_{\gamma,a,b}|||g|||_{\gamma',a,b}(t - s)^{\gamma'}. \tag{2.6}$$

The solution to the equation (2.1) will be searched in the following forms (assumptions on function F will be stated in the next section):

Definition 12. *Given $T > 0$, a stochastic process $v = \{v(t), t \in [0, T]\}$ is said to be a weak solution to the equation (2.1), if, for any $\zeta \in D^*$,*

$$\langle v(t), \zeta\rangle_V = \langle u_0, \zeta\rangle_V + \int_0^t [\langle v(s), A^*(s)\zeta\rangle_V + \langle F(v(s)), \zeta\rangle_V]d s$$
$$+ \int_0^t \langle v(s), B^*\zeta\rangle_V \, d\omega(s), \quad t \in [0, T]$$

provided that all the integrals above are meaningful.

Definition 13. *A stochastic process $u = \{u(t), t \in [0, T]\}$ is said to be a mild solution to the equation* (2.1)*, if*

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, s)F(u(s))d s, \quad t \in [0, T],$$

where

$$U(t, \omega, s) = S_B(\omega(t) - \omega(s))S_A(t, s), \quad 0 \leq s \leq t \leq T$$

and $(S_A(t, s), 0 \leq s \leq t \leq T)$ is the evolution system, which is the solution to the equation (1.6) with operator family $\{A(t)\}_{t\in[0,T]}$ and $\{S_B(t)\}_{t\in\mathbb{R}}$ is the strong continuous group generated by B.

Remark 5. *If the family of linear operators* $\{A(t)\}_{t\in[0,T]}$ *does not depend on* t*, then the weak solution has the form:*

$$\langle v(t),\zeta\rangle_V = \langle u_0,\zeta\rangle_V + \int_0^t [\langle v(s), A^*\zeta\rangle_V + \langle F(v(s)),\zeta\rangle_V] d\, s \\ + \int_0^t \langle v(s), B^*\zeta\rangle_V \, d\,\omega(s), \quad t\in[0,T],$$

and the mild solution has the form:

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)F(u(s))d\, s, \quad t\in[0,T],$$

where

$$U(t,\omega,s) = S_B(\omega(t)-\omega(s))S_A(t-s), \quad 0\le s\le t\le T,$$

where $\{S_A(t)\}_{t\in\mathbb{R}^+}$*,* $\{S_B(t)\}_{t\in\mathbb{R}}$ *are the analytic semigroup generated by* A *and the strong continuous group generated by* B*, respectively.*

The theorem on the existence of the solution to the equation (2.2) was proven in Garrido-Atienza et al. [2016].

Theorem 16. *Assume that assumptions (A), (B) and (C) hold. Then there exists a weak solution* v *to the linear problem:*

$$dv(t) = Av(t)\, dt + Bv(t)\, d\omega(t), \quad v(s) = u_0 \in V,$$

which is given for $t\ge s\ge 0$ *by*

$$v(t) = U(t,\omega,s)u_0 = S_B(\omega(t)-\omega(s))S_A(t-s)u_0. \tag{2.7}$$

The existence of a weak solution of the equation (2.4) based on the idea of the proof of Theorem 16, but with a new estimation for the evolution system $(S_A(t,s), 0\le s\le t)$.

Theorem 17. *Assume that assumptions (C.1)-(C.4) hold. Then there exists a weak solution* v *to the linear problem:*

$$dv(t) = A(t)v(t)\, dt + Bv(t)\, d\omega(t), \quad v(s) = u_0 \in V,$$

which is given for $t\ge s\ge 0$ *by*

$$v(t) = U(t,\omega,s)u_0 = S_B(\omega(t)-\omega(s))S_A(t,s)u_0. \tag{2.8}$$

Proof. At the beginning let us consider a case when $u_0\in D$. Our goal to apply Theorem 7. Let us consider $\zeta\in D^*$ and define $G:[s,T]\times\mathbb{R}\to\mathbb{R}$ by

$$G(t,x) = \langle S_A(t,s)u_0, S_B^*(x)\zeta\rangle_V,$$

further, it is necessary to define a function $h(t)$. Let $h(s)=0$ and:

$$h(t) = \int_s^t \mathbb{1}_{(s,T)}(z)d\,\omega(z) = \omega(t)-\omega(s), \;\; t\in[s,T].$$

Theorem 9 and properties of C_0-semigroups give us that $G \in C^1([s,T] \times \mathbb{R}; \mathbb{R})$ and $\frac{\partial G}{\partial x}(t,\cdot) \in C^1(\mathbb{R};\mathbb{R})$ for any $t \in [s,T]$. Based on the assumption about $(\omega(t), t \geq 0)$, $\omega_{T-} \in I_{T-}^{1-\delta}(L_2([s,T]))$ and $\mathbb{1}_{(s,T)} \in I_{s+}^{\delta}(L_2([s,T]))$ is bounded for any $\delta \in (1-\beta', 1/2)$ that gives

$$G(t, \omega(t) - \omega(s)) = \langle u_0, \zeta \rangle_V + \int_s^t \langle S_A(r,s)u_0, S_B^*(\omega(r) - \omega(s))B^*\zeta \rangle_V \, d\omega(r) + \int_s^t \langle A(r)S_A(r,s)u_0, S_B^*(\omega(r) - \omega(s))\zeta \rangle_V \, dr$$

and by using the commutativity assumption (C.2), we have:

$$\langle U(t,\omega,s)u_0, \zeta \rangle_V = \langle u_0, \zeta \rangle_V + \int_s^t \langle U(r,\omega,s)u_0, B^*\zeta \rangle_V \, d\omega(r) + \int_s^t \langle U(r,\omega,s)u_0, A^*(r)\zeta \rangle_V \, dr.$$

This finishes the proof for $u_0 \in D$. To continue with $u_0 \in V$ consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ such that $\|x_n - u_0\|_V \xrightarrow[n\to\infty]{} 0$. Such sequence exists since D is dense in V.

Thanks to assumptions (C.2), (C.3), continuity of ω on $[s,T]$ and (1.12) gives $\|S_A(t,s)\| \leq K$ for $\forall s,t : 0 \leq s \leq t \leq T$ the following holds:

$$\|S_B(\omega(t) - \omega(s))\|_{\mathcal{L}(V)} \leq M_B e^{r_B|\omega(t)-\omega(s)|} \leq M_B e^{2r_B \sup_{0\leq r\leq T}|\omega(r)|} \leq C(\omega). \quad (2.9)$$

Hence,

$$\|U(t,\omega,s)\|_{\mathcal{L}(V)} = \|S_B(\omega(t) - \omega(s))S_A(t,s)\|_{\mathcal{L}(V)} \leq C(\omega)K \leq C_U(\omega), \quad (2.10)$$

where $C(\omega)$ is a constant which depends on ω and $C_U(\omega)$ is a constant which depends on both ω and T. Let us denote:

$$A_1 = |\langle x_n, \zeta \rangle_V - \langle u_0, \zeta \rangle_V|$$
$$A_2 = |\langle U(t,s)x_n, \zeta \rangle_V - \langle U(t,s)u_0, \zeta \rangle_V|$$
$$A_3 = \left| \int_s^t \langle U(r,s)x_n, A^*(r)\zeta \rangle_V \, dr - \int_s^t \langle U(r,s)u_0, A^*(r)\zeta \rangle_V \, dr \right|$$
$$A_4 = \left| \int_s^t \langle U(r,s)x_n, B^*\zeta \rangle_V \, d\omega(r) - \int_s^t \langle U(r,s)u_0, B^*\zeta \rangle_V \, d\omega(r) \right|.$$

We want to show that A_1, A_2, A_3 and A_4 go to zero as $n \to \infty$. It is obvious that $A_1 \xrightarrow[n\to\infty]{} 0$. By using Cauchy–Schwarz inequality and estimation (2.10), we have:

$$\begin{aligned} A_2 &= |\langle U(t,s)x_n, \zeta \rangle_V - \langle U(t,s)u_0, \zeta \rangle_V| \\ &= |\langle U(t,s)(x_n - u_0), \zeta \rangle_V| \\ &\leq C_U(\omega)\|x_n - u_0\|_V \|\zeta\|_V \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

and by using Remark 4:

$$\begin{aligned} A_3 &= \left| \int_s^t \langle U(r,s)x_n, A^*(r)\zeta \rangle_V \, dr - \int_s^t \langle U(r,s)u_0, A^*(r)\zeta \rangle_V \, dr \right| \\ &= \left| \int_s^t \langle U(r,s)(x_n - u_0), A^*(r)\zeta \rangle_V \, dr \right| \\ &\leq C_U(\omega) R_\zeta \|x_n - u_0\|_V (t-s) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

To deal with A_4 we want to apply Lemma 15. It is necessary to show that $|\langle U(\tau,s)x, B^*\zeta\rangle_V|$ for $x \in V$ is a Hölder continuous function of τ with the exponent $\beta \in (1/2, \alpha)$, where α is the constant from assumption (C.4). At first, let us show that the integrand convergences on $[s,T]$:

$$\sup_{\tau\in[s,T]} |\langle U(\tau,s)(x_n - u_0), B^*\zeta\rangle_V| \le C_U(\omega)\|B^*\zeta\|_V\|x_n - u_0\|_V \xrightarrow[n\to\infty]{} 0 \quad (2.11)$$

Now, for $s < \tau_1 < \tau_2 \le T$ it is necessary to show Hölder continuity:

$$\begin{aligned}
&|\langle (U(\tau_2,s) - U(\tau_1,s))x, B^*\zeta\rangle_V| \\
&= |\langle (S_B(\omega(\tau_2)-\omega(s))S_A(\tau_2,s) - S_B(\omega(\tau_1)-\omega(s))S_A(\tau_1,s))x, B^*\zeta\rangle_V| \\
&= |\langle (S_B(\omega(\tau_2)-\omega(s))S_A(\tau_2,s) - S_B(\omega(\tau_1)-\omega(s))S_A(\tau_2,s) \\
&+ S_B(\omega(\tau_1)-\omega(s))S_A(\tau_2,s) - S_B(\omega(\tau_1)-\omega(s))S_A(\tau_1,s))x, B^*\zeta\rangle_V| \\
&\le |\langle S_A(\tau_2,s)(S_B(\omega(\tau_2)-\omega(s)) - S_B(\omega(\tau_1)-\omega(s)))x, B^*\zeta\rangle_V| \\
&+ |\langle S_B(\omega(\tau_1)-\omega(s))(S_A(\tau_2,s) - S_A(\tau_1,s))x, B^*\zeta\rangle_V| = I_1 + I_2
\end{aligned}$$

Let us estimate the first term. From estimation (1.12) we have $\|S_A(t,s)\|_{\mathcal{L}(V)} \le C_A$ for $0 \le s \le t \le T$, where C_A depends on T only and use of Theorem 9 (ii) we have:

$$\begin{aligned}
I_1 &\le |\langle S_A(\tau_2,s)x, (S_B^*(\omega(\tau_2)-\omega(s)) - S_B^*(\omega(\tau_1)-\omega(s)))B^*\zeta\rangle_V| \\
&\le C_A\|x\|_V \left| \int_{\omega(\tau_1)-\omega(s)}^{\omega(\tau_2)-\omega(s)} \|S_B^*(z)(B^*)^2\zeta\|_V \, dz \right| \\
&\le C_A C(\omega)\|x\|_V\|(B^*)^2\zeta\|_V|||\omega|||_{\beta',\tau_1,\tau_2}(\tau_2-\tau_1)^{\beta'}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{s\le\tau_1<\tau_2\le T} \frac{I_1}{(\tau_2-\tau_1)^{\beta}} \\
&\qquad \le C_A C(\omega)\|x\|_V\|S_B^*\|_{\mathcal{L}(V)}\|(B^*)^2\zeta\|_V|||\omega|||_{\beta',s,T}(T-s)^{\beta'-\beta} < \infty.
\end{aligned}$$

The second term might be estimated by using assumptions (C.1)-(C.4) which provide a representation (1.8) for $S_A(t,s)$, the estimations (1.10), (1.11) and the fact that $\{S_r(q)\}_{q\ge0}$ for each $r \in [0,T]$ is the analytic semigroup generated by $A(r)$. We have that:

$$\begin{aligned}
I_2 &\le |\langle (S_A(\tau_2,s) - S_A(\tau_1,s))x, S_B^*(\omega(\tau_1)-\omega(s))B^*\zeta\rangle_V| \\
&= |\langle (S_s(\tau_2-s) - S_s(\tau_1-s))\,x + \left(\int_{\tau_1}^{\tau_2} S_\tau(\tau_2-\tau)R(\tau,s)\,d\tau\right)x \\
&+ \left(\int_s^{\tau_1} (S_\tau(\tau_2-\tau) - S_\tau(\tau_1-\tau))R(\tau,s)\,d\tau\right)x, S_B^*(\omega(\tau_1)-\omega(s))B^*\zeta\rangle_V| \\
&= |\langle (S_s(\tau_2-s) - S_s(\tau_1-s))\,x, S_B^*(\omega(\tau_1)-\omega(s))B^*\zeta\rangle_V| \\
&+ |\langle \left(\int_{\tau_1}^{\tau_2} S_\tau(\tau_2-\tau)R(\tau,s)\,d\tau\right)x, S_B^*(\omega(\tau_1)-\omega(s))B^*\zeta\rangle_V| \\
&+ |\langle \left(\int_s^{\tau_1} (S_\tau(\tau_2-\tau) - S_\tau(\tau_1-\tau))R(\tau,s)\,d\tau\right)x, S_B^*(\omega(\tau_1)-\omega(s))B^*\zeta\rangle_V|
\end{aligned}$$

We will split I_2 into three parts:

$$I_2 = L_1 + L_2 + L_3.$$

According to Remark 3, Corollary 1.10.6 Pazy [1983] and Theorem 9 we have:

$$
\begin{aligned}
L_1 &= |\langle (S_s(\tau_2 - s) - S_s(\tau_1 - s))\, x, S_B^*(\omega(\tau_1) - \omega(s))B^*\zeta\rangle_V| \\
&= |\langle S_B(\omega(\tau_1) - \omega(s))x, (S_s^*(\tau_2 - s) - S_s^*(\tau_1 - s))\, B^*\zeta\rangle_V| \\
&= |\langle S_B(\omega(\tau_1) - \omega(s))x, \int_{\tau_1 - s}^{\tau_2 - s} S_s^*(\tau)A^*B^*\zeta d\tau\rangle_V| \\
&\le \|S_B(\omega(\tau_1) - \omega(s))x\|_V \int_{\tau_1 - s}^{\tau_2 - s} \|S_s^*(\tau)A^*B^*\zeta\|_V d\tau \\
&\le C(\omega)\|x\|_V M_s e^{r_s T}\|A^*B^*\zeta\|_V |\tau_2 - \tau_1|,
\end{aligned}
\tag{2.12}
$$

where $\|S_s^*(\tau)\|_{\mathcal{L}(V)} \le M_s e^{r_s T}$ for some constants $M_s > 1$ and $r_s > 0$.

There is the following bound for the second term by using (1.11):

$$
\begin{aligned}
&\|\int_{\tau_1}^{\tau_2} S_\tau(\tau_2 - \tau)R(\tau, s)\, d\tau\|_{\mathcal{L}(V)} \le C_W(\tau_2 - \tau_1)^\alpha, \\
&L_2 \le C_W(\tau_2 - \tau_1)^\alpha \|x\|_V C(\omega)\|B^*\zeta\|_V,
\end{aligned}
\tag{2.13}
$$

where α is constant from assumption (C.4).

The third term with the help of (2.3), where the constant $\hat{\alpha}$ is set equal to the constant α from (C.4), is estimated by:

$$
\begin{aligned}
&\int_s^{\tau_1} \|S_\tau(\tau_2 - \tau) - S_\tau(\tau_1 - \tau)\|_{\mathcal{L}(V)} \|R(\tau, s)\|_{\mathcal{L}(V)}\, d\tau \\
&\le CK_\alpha \int_s^{\tau_1} \frac{|\tau_2 - \tau_1|^\alpha}{|\tau_1 - \tau|^\alpha} |\tau - s|^{\alpha - 1}\, d\tau \\
&\le CK_\alpha |\tau_2 - \tau_1|^\alpha \int_s^{\tau_1} \frac{1}{|\tau_1 - \tau|^\alpha |\tau - s|^{1-\alpha}}\, d\tau \le CK_\alpha B(1-\alpha, \alpha)|\tau_2 - \tau_1|^\alpha, \\
&L_3 \le CK_\alpha B(1-\alpha, \alpha)|\tau_2 - \tau_1|^\alpha \|x\|_V C(\omega)\|B^*\zeta\|_V
\end{aligned}
\tag{2.14}
$$

where $B(\cdot, \cdot)$ is the Beta function. The following estimation then holds:

$$
\begin{aligned}
I_2 &\le C(\omega)\|x\|_V (M_s e^{r_s T}(\tau_2 - \tau_1)\|A^*B^*\zeta\|_V + C_W(\tau_2 - \tau_1)^\alpha \|B^*\zeta\|_V \\
&+ CK_\alpha B(1-\alpha, \alpha)(\tau_2 - \tau_1)^\alpha \|B^*\zeta\|_V)
\end{aligned}
$$

and having in mind that $\alpha > \beta$:

$$
\begin{aligned}
&\sup_{s \le \tau_1 < \tau_2 \le T} \frac{I_2}{(\tau_2 - \tau_1)^\beta} \\
&\le \sup_{s \le \tau_1 < \tau_2 \le T} C(\omega)\|x\|_V (M_s e^{r_s T}(\tau_2 - \tau_1)^{1-\beta}\|A^*B^*\zeta\|_V + C_W(\tau_2 - \tau_1)^{\alpha - \beta}\|B^*\zeta\|_V \\
&+ CK_\alpha B(1-\alpha, \alpha)(\tau_2 - \tau_1)^{\alpha - \beta}\|B^*\zeta\|_V) \\
&\le C(\omega)\|x\|_V (M_s e^{r_s T}(T - s)^{1-\beta}\|A^*B^*\zeta\|_V + C_W(T - s)^{\alpha - \beta}\|B^*\zeta\|_V \\
&+ CK_\alpha B(1-\alpha, \alpha)(T - s)^{\alpha - \beta}\|B^*\zeta\|_V) < \infty.
\end{aligned}
$$

This gives us that $|\langle U(\cdot, s)x, B^*\zeta\rangle_V| \in H^\beta$. From Lemma 15 we have:

$$
\left| \int_s^t \langle U(r, s)x, B^*\zeta\rangle_V\, d\omega(r) \right| \le c\|\langle U(\cdot, s)x, B^*\zeta\rangle_V\|_{\beta, s, T} |||\omega|||_{\beta', s, T}(T - s)^{\beta'}.
$$

Combination of the estimation above and the result 2.11 gives us that

$$A_4 = \left| \int_s^t \langle U(r,s)x_n, B^*\zeta \rangle_V \, d\omega(r) - \int_s^t \langle U(r,s)u_0, B^*\zeta \rangle_V \, d\omega(r) \right| \xrightarrow[n \to \infty]{} 0$$

and we have

$$A_2 = A_1 + A_3 + A_4 \xrightarrow[n \to \infty]{} 0,$$

which finishes the proof for each $u_0 \in V$. □

Remark 6. *As might be seen from the proof above, only the assumption about the Hölder exponent of the noise sample paths was used. It means that Theorem 16 and Theorem 17 hold for noises whose sample paths have Hölder exponent larger than* $1/2$ *and not only for the fractional Brownian motion.*

2.2 Solutions of semilinear stochastic equations

This part is devoted to the solution of the equation (2.1) for the mapping $F : V \to V$ which is assumed to be Lipschitz continuous, i.e. there exists $L > 0$ such that

$$\|F(u) - F(v)\|_V \le L\|u - v\|_V, \quad u, v \in V. \tag{2.15}$$

The theorem below from Garrido-Atienza et al. [2016] gives us the form of the mild solution to

$$d\,u(t) = (Au(t) + F(u(t)))d\,t + Bu(t)d\,\omega(t), \quad u(0) = u_0 \in V. \tag{2.16}$$

Theorem 18. *Given* $T > 0$*, under the conditions (A), (B), (C) and (2.15) there exists a unique mild solution (cf. Remark 5)* $u \in C([0,T], V)$ *to (2.16) for every* $u_0 \in V$*.*

The theorem below is an extension of the proof of Theorem 2.4 Garrido-Atienza et al. [2016].

Theorem 19. *Given* $T > 0$*, under the conditions (C.1)-(C.4) and (2.15) there exists a unique mild solution (cf. Definition 13)* $u \in C([0,T], V)$ *to (2.1) for every* $u_0 \in V$*.*

Proof. **Step 1.** Assume additionally that F is bounded, i.e. there exists a constant $K > 0$ such that

$$\|F(u)\|_V \le K, \quad u \in V, \tag{2.17}$$

and fix u_0. Define the operator Φ by

$$(\Phi(y))(t) = U(t,0)u_0 + \int_0^t U(t,r)F(y(r))d\,r, \quad t \in [0,T],$$

where $U(t,r)$ is given by (2.8) (ω is omitted here for shorter notation). Our goal is to show that Φ is a continuous contraction from $C([0,T], V)$ into itself. At first, we show that $\Phi : C([0,T],V) \to C([0,T],V)$. Take $y \in C([0,T],V)$ and $0 \le s,t \le T$. Then

$$\|\Phi(y)(t) - \Phi(y)(s)\|_V \le \|U(t,0)u_0 - U(s,0)u_0\|_V + \left\| \int_0^t U(t,r)F(y(r))dr - \int_0^s U(s,r)F(y(r))dr \right\|_V = I_1 + I_2.$$

For I_1 the following holds:

$$\begin{aligned} I_1 &= \|(S_B(\omega(t))S_A(t,0) - S_B(\omega(s))S_A(s,0))u_0\|_V \\ &= \|(S_B(\omega(t))S_A(t,0) - S_B(\omega(s))S_A(t,0) \\ &+ S_B(\omega(s))S_A(t,0) - S_B(\omega(s))S_A(s,0))u_0\| \\ &\le \|(S_B(\omega(t)) - S_B(\omega(s)))S_A(t,0)u_0\|_V \\ &+ \|S_B(\omega(s))(S_A(t,0) - S_A(s,0))u_0\|_V \xrightarrow[s \to t-,\, s \to t+]{} 0 \end{aligned}$$

by the continuity of ω, Corollary 2.2.3 and Theorem 5.6.1 in Pazy [1983]. Let $s < t$, then by estimation (2.10) for $\|U(t,s)\|_{\mathcal{L}(V)} \le C_U$ and (2.17)

$$\begin{aligned} I_2 &\le \left\| \int_0^s (U(t,r) - U(s,r))F(y(r))dr \right\|_V + \left\| \int_s^t U(t,r)F(y(r))dr \right\|_V \\ &\le \left\| \int_0^s \left(S_B(\omega(t) - \omega(r)) - S_B(\omega(s) - \omega(r))\right) S_A(t,r)F(y(r))dr \right\|_V \\ &+ \left\| \int_0^s S_B(\omega(s) - \omega(r))(S_A(t,r) - S_A(s,r))F(y(r))dr \right\|_V + \int_s^t C_U K dr \\ &= I_3 + I_4 + C_U K(t-s). \end{aligned}$$

The last term as $t \to s+$ or $s \to t-$ goes to zero. It remains to estimate I_3 and I_4.

To handle I_3 it is necessary to re-order it:

$$\begin{aligned} &\left\| \int_0^s \left(S_B(\omega(t) - \omega(r)) - S_B(\omega(s) - \omega(r))\right) S_A(t,r)F(y(r))dr \right\|_V \\ &= \left\| \int_0^s \left(S_B(\omega(t))S_B(-\omega(r)) - S_B(\omega(s))S_B(-\omega(r))\right) S_A(t,r)F(y(r))dr \right\|_V \\ &= \left\| (S_B(\omega(t) - S_B(\omega(s))) \int_0^s S_B(-\omega(r))S_A(t,r)F(y(r))dr \right\|_V \end{aligned}$$

It is worth to notice that if $S_B(-\omega(\cdot))S_A(t,\cdot)F(y(\cdot))$ is a continuous function, then $\int_0^{\cdot} S_B(-\omega(r))S_A(t,r)F(y(r))dr$ is also a continuous mapping. Define

$$\mathcal{K} = \{z \in V; \exists 0 \le s_1 \le t_1 \le T : \; z = \int_0^{s_1} S_B(-\omega(r))S_A(t_1,r)F(y(r))dr\}.$$

Set $\mathcal{K}$ is a compact set since a continuous image of a compact set is compact, which gives us:

$$\lim_{t \to s} \sup_{z \in \mathcal{K}} \|(S_B(\omega(t)) - S_B(\omega(s)))z\|_V = 0$$

because a pointwise convergence becomes a uniform convergence on a compact set. In other words we have for I_3:

$$\begin{aligned} I_3 &= \|(S_B(\omega(t)) - S_B(\omega(s))) \int_0^s S_B(-\omega(r))S_A(t,r)F(y(r))dr\|_V \\ &\le \sup_{z \in \mathcal{K}} \|(S_B(\omega(t)) - S_B(\omega(s)))z\|_V \to 0, \text{ as } t \to s+ \text{ or } s \to t- . \end{aligned}$$

Finally, we turn to I_4, for which with the help of estimations from Theorem 17 (2.9), (2.3),(2.13) and (2.14), we obtain:

$$\begin{aligned}&\|S_B(\omega(s)-\omega(r))(S_A(t,r)-S_A(s,r))F(y(r))\|_V\\&\leq KC(K_\alpha|t-s|^\alpha|s-r|^{-\alpha}\\&+C_W|t-s|^\alpha+CK_\alpha B(1-\alpha,\alpha)|t-s|^\alpha)\end{aligned} \tag{2.18}$$

keeping in mind that $0<\alpha<1$ it gives us the following estimation:

$$\begin{aligned}I_4 &\leq CK(K_\alpha|t-s|^\alpha\int_0^s|s-r|^{-\alpha}dr\\&+C_W|t-s|^\alpha\int_0^s 1dr+CK_\alpha B(1-\alpha,\alpha)|t-s|^\alpha\int_0^s 1dr)<\infty.\end{aligned}$$

Thus $I_4\to 0$ as $t\to s+$ or $s\to t-$.

Let $y_1,y_2\in C([0,T],V)$. Using (2.10) and (2.15) we obtain

$$\begin{aligned}\|\Phi(y_1)-\Phi(y_2)\|_{\infty,0,T}&=\sup_{t\in[0,T]}\left\|\int_0^t U(t,r)(F(y_1(r))-F(y_2(r)))dr\right\|_V\\&\leq C_U LT\|y_1-y_2\|_{\infty,0,T}.\end{aligned}$$

If $T<(C_UL)^{-1}$, then Φ is a contraction. Such interval always exists. Let us remind how constant C_U is defined:

$$C_U(\omega)=C(\omega)K.$$

Using Remark 2 we obtain that the interval $[0,(C_UL)^{-1})$ is not empty. A unique mild solution to the equation (2.1) exists by the Banach fixed point theorem on the interval $[0,(C_UL)^{-1})$. It is possible to extend solution to the closed interval. Let us find $\lim_{t\to(C_UL)^{-1}}u(t)$.

$$u(t)=U(t,0)u_0+\int_0^t U(t,s)F(u(s))ds\quad t\in[0,(C_UL)^{-1})$$

and let

$$A=U(t,0)u_0+\int_0^{(C_UL)^{-1}}U(t,s)F(u(s))ds$$

so we have that:

$$\|A-u(t)\|_V\leq\int_t^{(C_UL)^{-1}}\|U(t,s)F(u(s))\|_V ds\leq C_UK((C_UL)^{-1}-t),$$

which gives us $\lim_{t\to(C_UL)^{-1}}\|A-u(t)\|_V=0$ in other words $u(t)$ has a limit as $t\to(C_UL)^{-1}$. Now we can consider a new initial value problem for $t\in[(C_UL)^{-1},T]$ with initial condition $v=u((C_UL)^{-1})$ and obtain, using the same approach, that a new solution for the problem is $u^1\in C([(C_UL)^{-1},2(C_UL)^{-1}),V)$. Now we can extend the solution u to the interval $[0,2(C_UL)^{-1})$ by defining:

$$u(t)=\begin{cases}u(t), & t\in[0,(C_UL)^{-1}],\\ u^1(t), & t\in[(C_UL)^{-1},2(C_UL)^{-1}].\end{cases} \tag{2.19}$$

$u \in C([0, 2(C_U L)^{-1}, V)$ since we have shown that $\lim_{t \to (C_U L)^{-1}} u(t)$ exists and u^1 is continuous function on $[(C_U L)^{-1}, 2(C_U L)^{-1})$.

The second part of the proof where F is a Lipschitz continuous function is the same as the proof which is given in Garrido-Atienza et al. [2016] in Theorem 2.4 and it will be omitted here.

□

Remark 7. *From the proof of Theorem 2.4 Garrido-Atienza et al. [2016] it follows that there exist constants $C_1(\omega), C_2(\omega) > 0$ such that*

$$\|u(t)\|_V \leq C_1(\omega) + C_2(\omega)\|u_0\|_V, \quad \text{for } t \in [0, T].$$

Remark 8. *As might be seen from the proof of Theorem 19 we use only assumption about the Hölder exponent for the noise sample paths. Which means Theorem 13 and Theorem 19 hold for noises whose sample paths have Hölder exponent larger than 1/2.*

2.3 A mild solution is a weak solution

The next step is to show that a mild solution is a weak solution. The main part of the section is Theorem 2.5 Garrido-Atienza et al. [2016] for which some omitted parts are recovered and the proof of a Fubini-type theorem for the integral in the sense (1.1) is provided. Let us start with Lemma 13.1 from Samko et al. [1993].

Lemma 20. *If $f(x) \in H^\lambda$, $0 < \alpha < \lambda \leq 1$, then*

$$[D^\alpha_{a+} f](x) = \frac{f(a)}{\Gamma(1-\alpha)} \frac{1}{(x-a)^\alpha} + \psi(x),$$

where $\psi \in H^{\lambda-\alpha}$ is such that $\psi(a) = 0$, and satisfies $||\psi||_{H^{\lambda-a}} \leq c||f||_{H^\lambda}$ for some $c \in (0, \infty)$.

Lemma 20 gives us property of the integral:

$$\int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy$$

from the Weyl representation of the fractional derivative $[D^\alpha_{a+} f](x)$. The same result might be achieved for the integral part of the derivative $[D^{1-\alpha}_{b-} f](x)$. Now it is possible to formulate a Fubini-type theorem.

Lemma 21. *Let β and β' be such that $\beta > 1/2$, $\beta' > 1/2$ and $f \in C([0,T] \times [0,T], \mathbb{R})$ be such that for every $r \in [0,T]$, $f(\cdot, r) \in H^\beta([0,T], \mathbb{R})$. Then it holds for the integrals from Definition 1.1:*

$$\int_0^t \int_r^t f(\tau, r)\, d\omega(\tau)\, dr = \int_0^t \int_0^\tau f(\tau, r)\, dr\, d\omega(\tau), \quad 0 < t < T.$$

Proof. Since the definition of the integral (1.1) by Proposition 2.1 Zähle [1998] does not depend on α, let α satisfy $\alpha < \beta$ and $1 - \alpha < \beta'$. Such value is possible because $\beta > 1/2 > 1 - \beta'$. It follows from Zähle [2001] that under assumption of this lemma, the integral in the sense (1.1) takes the following form:

$$\int_a^b f(t)\, d\omega(t) = (-1)^\alpha \int_a^b [D_{a+}^\alpha f](t)[D_{b-}^{1-\alpha}\omega_{b-}](t)\, dt.$$

Then the following estimations hold:

$$\begin{aligned}
&\int_0^t \int_r^t f(\tau, r)\, d\omega(\tau)\, dr = \int_0^t (-1)^\alpha \int_r^t [D_{r+}^\alpha f(\cdot, r)](\tau)[D_{t-}^{1-\alpha}\omega_{t-}](\tau)\, d\tau\, dr \\
&= \int_0^t (-1)^\alpha \int_r^t \left(\frac{f(\tau, r)}{\Gamma(1-\alpha)(\tau - r)^\alpha} + \psi(\tau, r) \right) \left(\frac{\omega(\tau) - \omega(t-)}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} + \psi'(\tau) \right) d\tau\, dr \\
&\leq \int_0^t \int_r^t \left| (-1)^\alpha \left(\frac{f(\tau, r)}{\Gamma(1-\alpha)(\tau - r)^\alpha} + \psi(\tau, r) \right) \left(\frac{\omega(\tau) - \omega(t-)}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} + \psi'(\tau) \right) \right| d\tau\, dr \\
&\leq \int_0^t \left\| \frac{f(\tau, r)}{\Gamma(1-\alpha)(\tau - r)^\alpha} + |\psi(\tau, r)| \right\|_{L^p} \left\| \frac{c|t-\tau|^{\beta'}}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} + |\psi'(\tau)| \right\|_{L^{p'}} dr
\end{aligned}$$

where $p > 1$ and $p' > 1$ satisfy $1/p + 1/p' = 1$. In the second equality, Lemma 20 is used, which guarantees that for each r, $\psi(\cdot, r) \in H^{\beta-\alpha}([0, T], \mathbb{R})$ and $\psi'(\cdot) \in H^{\beta'-1+\alpha}([0, T], \mathbb{R})$. The Hölder property of function ω is used in the last inequality. By the choice of β, β' and α, we have that there exists $p > 1$ such that $\alpha p < 1$ and $\beta' + \alpha - 1 > 0$ so that we have the following estimation with the help of Minkowski inequality:

$$\begin{aligned}
\left\| \frac{f(\tau, r)}{\Gamma(1-\alpha)(\tau - r)^\alpha} + |\psi(\tau, r)| \right\|_{L^p(r,t;d\tau)} &\leq \left\| \frac{f(\tau, r)}{\Gamma(1-\alpha)(\tau - r)^\alpha} \right\|_{L^p(r,t;d\tau)} \\
&\quad + \|\psi(\tau, r)\|_{L^p(r,t;d\tau)} < \infty,
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{c|t-\tau|^{\beta'}}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} + |\psi'(\tau)| \right\|_{L^{p'}(r,t;d\tau)} &\leq \left\| \frac{c|t-\tau|^{\beta'+\alpha-1}}{\Gamma(\alpha)} \right\|_{L^{p'}(r,t;d\tau)} \\
&\quad + \|\psi'(\tau)\|_{L^{p'}(r,t)} < \infty,
\end{aligned}$$

where $\|f(\tau, r)\|_{L^p(r,t;d\tau)}$ defines L^p norm of the function $f(\cdot, r)$ on the interval $[r, t]$ where the value of r is fixed. So we have:

$$\int_0^t \int_r^t |[D_{r+}^\alpha f(\cdot, r)](\tau)[D_{t-}^{1-\alpha}\omega_{t-}](\tau)|\, d\tau\, dr < \infty.$$

To finish the proof we need to show that:

$$\begin{aligned}
&(-1)^\alpha \int_0^t \int_r^t [D_{r+}^\alpha f(\cdot, r)](\tau)[D_{t-}^{1-\alpha}\omega_{t-}](\tau)\, d\tau\, dr \\
&= (-1)^\alpha \int_0^t [D_{0+}^\alpha \left(\int_0^\cdot f(\cdot, r)\, dr \right)](\tau)[D_{t-}^{1-\alpha}\omega_{t-}](\tau)\, d\tau
\end{aligned}$$

Let us start with the proof that:

$$[D_{0+}^\alpha \left(\int_0^\cdot f(\cdot, r)\, dr \right)](\tau) = \int_0^\tau [D_{r+}^\alpha (f(\cdot, r))](\tau)\, dr. \tag{2.20}$$

The left-hand side has the following Weyl representation of the fractional derivative:

$$[D^{\alpha}_{0+}\Big(\int_0^{\cdot} f(\cdot,r)\,dr\Big)](\tau)$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\int_0^{\tau} f(\tau,r)dr}{\tau^{\alpha}} + \alpha\int_0^{\tau}\frac{\int_0^{\tau} f(\tau,r)dr - \int_0^{y} f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right). \tag{2.21}$$

The right-hand side has the following Weyl representation of the fractional derivative:

$$\int_0^{\tau}[D^{\alpha}_{r+}(f(\cdot,r))](\tau)\,dr$$
$$= \int_0^{\tau}\frac{1}{\Gamma(1-\alpha)}\left(\frac{f(\tau,r)}{(\tau-r)^{\alpha}} + \alpha\int_r^{\tau}\frac{f(\tau,r)-f(y,r)}{(\tau-y)^{1+\alpha}}dy\right)dr$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\tau}\frac{f(\tau,r)}{(\tau-r)^{\alpha}}dr + \alpha\int_0^{\tau}\int_r^{\tau}\frac{f(\tau,r)-f(y,r)}{(\tau-y)^{1+\alpha}}dy\,dr\right)$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\tau}\frac{f(\tau,r)}{(\tau-r)^{\alpha}}dr + \alpha\int_0^{\tau}\int_0^{y}\frac{f(\tau,r)-f(y,r)}{(\tau-y)^{1+\alpha}}dr\,dy\right) \tag{2.22}$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\tau}\frac{f(\tau,r)}{(\tau-r)^{\alpha}}dr + \alpha\int_0^{\tau}\frac{\int_0^{y} f(\tau,r)dr - \int_0^{y} f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right)$$
$$= \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\tau}\frac{f(\tau,r)}{(\tau-r)^{\alpha}}dr + \alpha\int_0^{\tau}\frac{\int_0^{\tau} f(\tau,r)dr - \int_0^{y} f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right.$$
$$\left.-\alpha\int_0^{\tau}\frac{\int_y^{\tau} f(\tau,r)dr}{(\tau-y)^{1+\alpha}}dy\right).$$

In the third equality the use of Fubini's theorem is justified by the following estimation:

$$\int_0^{\tau}\int_r^{\tau}\left|\frac{f(\tau,r)-f(y,r)}{(\tau-y)^{1+\alpha}}\right|dy\,dr \le \int_0^{\tau}\int_r^{\tau}\left|\frac{c_r(\tau-y)^{\beta}}{(\tau-y)^{1+\alpha}}\right|dy\,dr$$
$$= \int_0^{\tau}\frac{1}{\beta-\alpha}\left|(\tau-\tau)^{\beta-\alpha}+(\tau-r)^{\beta-\alpha}\right|dr = \int_0^{\tau}\frac{1}{\beta-\alpha}|\tau-r|^{\beta-\alpha}\,dr < \infty.$$

Let us consider the last term separately:

$$-\alpha\int_0^{\tau}\frac{\int_y^{\tau} f(\tau,r)dr}{(\tau-y)^{1+\alpha}}dy = -\alpha\int_0^{\tau}\int_0^{r}\frac{f(\tau,r)}{(\tau-y)^{1+\alpha}}dy\,dr$$
$$-\alpha\int_0^{\tau}f(\tau,r)\int_0^{r}\frac{1}{(\tau-y)^{1+\alpha}}dy\,dr = -\alpha\int_0^{\tau}f(\tau,r)\frac{1}{\alpha}(-\tau^{-\alpha}+(\tau-r)^{-\alpha})dr$$
$$= \int_0^{\tau}\frac{f(\tau,r)}{\tau^{\alpha}}dr - \int_0^{\tau}\frac{f(\tau,r)}{(\tau-r)^{\alpha}}dr = \frac{\int_0^{\tau} f(\tau,r)dr}{\tau^{\alpha}} - \frac{\int_0^{\tau} f(\tau,r)dr}{(\tau-r)^{\alpha}}. \tag{2.23}$$

Fubini's theorem in the first equality was used. It is possible because f is continuous on $[0,T]\times[0,T]$ and we have $\sup_{(\tau,r)\in[0,T]\times[0,T]}|f(\tau,r)| = K < \infty$ and the following estimate holds:

$$\int_0^{\tau}\int_y^{\tau}\frac{|f(\tau,r)|}{(\tau-y)^{1+\alpha}}dr\,dy \le \int_0^{\tau}\int_y^{\tau}\frac{K}{(\tau-y)^{1+\alpha}}dr\,dy = K\int_0^{\tau}\frac{(\tau-y)}{(\tau-y)^{1+\alpha}}dy$$
$$= K\int_0^{\tau}(\tau-y)^{-\alpha}dy = K\frac{1}{1-\alpha}(-(\tau-\tau)^{1-\alpha}+(\tau-0)^{1-\alpha}) = K\frac{\tau^{1-\alpha}}{1-\alpha} < \infty.$$

We obtain the following result by inserting (2.23) into (2.22):

$$\frac{1}{\Gamma(1-\alpha)}\left(\int_0^\tau \frac{f(\tau,r)}{(\tau-r)^\alpha}dr + \alpha\int_0^\tau \frac{\int_0^\tau f(\tau,r)dr - \int_0^y f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right.$$
$$\left.-\alpha\int_0^\tau \frac{\int_y^\tau f(\tau,r)dr}{(\tau-y)^{1+\alpha}}dy\right)$$
$$=\frac{1}{\Gamma(1-\alpha)}\left(\int_0^\tau \frac{f(\tau,r)}{(\tau-r)^\alpha}dr + \alpha\int_0^\tau \frac{\int_0^\tau f(\tau,r)dr - \int_0^y f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right.$$
$$\left.+\frac{\int_0^\tau f(\tau,r)dr}{\tau^\alpha} - \int_0^\tau \frac{f(\tau,r)}{(\tau-r)^\alpha}dr\right)$$
$$=\frac{1}{\Gamma(1-\alpha)}\left(\frac{\int_0^\tau f(\tau,r)dr}{\tau^\alpha} + \alpha\int_0^\tau \frac{\int_0^\tau f(\tau,r)dr - \int_0^y f(y,r)dr}{(\tau-y)^{1+\alpha}}dy\right).$$

We proved equality (2.20). This gives us:

$$(-1)^\alpha \int_0^t [D_{0+}^\alpha\left(\int_0^\cdot f(\cdot,r)\,dr\right)](\tau) D_{t-}^{1-\alpha}\omega_{t-}(\tau)\,d\tau$$
$$= (-1)^\alpha \int_0^t \int_0^\tau [D_{r+}^\alpha(f(\cdot,r))](\tau)\,dr D_{t-}^{1-\alpha}\omega_{t-}(\tau)\,d\tau$$
$$= (-1)^\alpha \int_0^t \int_r^t [D_{r+}^\alpha f(\cdot,r)](\tau)[D_{t-}^{1-\alpha}\omega_{t-}](\tau)\,d\tau\,dr,$$

which finishes the proof that:

$$\int_0^t \int_r^t f(\tau,r)\,d\omega(\tau)\,dr = \int_0^t \int_0^\tau f(\tau,r)\,dr\,d\omega(\tau)$$

□

The theorem below gives a connection between mild and weak solution. The theorem is formulated for nonautonomous case and the proof is presented here to show where Lemma 21 is used to fill the gap in the proof of Theorem 2.5 Garrido-Atienza et al. [2016].

Theorem 22. *Let the assumption of the Theorem 19 be satisfied and let u be the mild solution to the equation (2.1). Then u is also a weak solution to (2.1).*

Proof. Take $t \in [0,T]$, $\zeta \in D^*$. From the expression of the mild solution u and the fact that U is a weak solution to the equation (2.4), we get

$$\langle u(t),\zeta\rangle_V = \langle U(t,0)u_0,\zeta\rangle_V + \int_0^t \langle U(t,r)F(u(r)),\zeta\rangle_V\,dr$$
$$= \langle u_0,\zeta\rangle_V + \int_0^t \langle U(\tau,0)u_0, A^*(r)\zeta\rangle_V\,d\tau + \int_0^t \langle U(\tau,0)u_0, B^*\zeta\rangle_V\,d\omega(\tau)$$
$$+ \int_0^t \langle F(u(r)),\zeta\rangle_V\,dr + \int_0^t\int_r^t \langle U(\tau,r)F(u(r)), A^*(r)\zeta\rangle_V\,d\tau dr$$
$$+ \int_0^t\int_r^t \langle U(\tau,r)F(u(r)), B^*\zeta\rangle_V\,d\omega(\tau)dr$$
$$= \langle u_0,\zeta\rangle_V + \int_0^t \langle F(u(r)),\zeta\rangle_V\,dr + \int_0^t \langle U(\tau,0)u_0, A^*(r)\zeta\rangle_V\,d\tau$$
$$+ \int_0^t\int_0^\tau \langle U(\tau,r)F(u(r)), A^*(r)\zeta\rangle_V\,dr d\tau$$

$$
\begin{aligned}
&+ \int_0^t \langle U(\tau,0)u_0, B^*\zeta\rangle_V \, d\,\omega(\tau) + \int_0^t \int_0^\tau \langle U(\tau,r)F(u(r)), B^*\zeta\rangle_V \, d\,r d\,\omega(\tau) \\
&= \langle u_0, \zeta\rangle_V + \int_0^t \langle u(\tau), A^*(r)\zeta\rangle_V \, d\,\tau + \int_0^t \langle F(u(r)), \zeta\rangle_V \, d\,r \\
&+ \int_0^t \langle u(\tau), B^*\zeta\rangle_V \, d\,\omega(\tau)
\end{aligned}
$$

The use of Fubini's theorem for the equality:

$$
\int_0^t \int_r^t \langle U(\tau,r)F(u(r)), A^*(r)\zeta\rangle_V \, d\,\tau d\,r = \int_0^t \int_0^\tau \langle U(\tau,r)F(u(r)), A^*(r)\zeta\rangle_V \, d\,r d\,\tau
$$

is justified by the following estimation with the help of Remark 4 and Remark 7:

$$
\begin{aligned}
&\left\| \int_0^t \int_r^t \langle U(\tau,r)F(u(r)), A^*(r)\zeta\rangle_V \, d\,\tau d\,r \right\|_V \\
&\leq C_U T^2 R_\zeta (\|F(0)\|_V + L(C_1(\omega) + C_2(\omega)\|u_0\|_V)) < \infty.
\end{aligned}
$$

And Lemma 21 is used to justify:

$$
\int_0^t \int_r^t \langle U(\tau,r)F(u(r)), B^*\zeta\rangle_V \, d\,\omega(\tau) d\,r = \int_0^t \int_0^\tau \langle U(\tau,r)F(u(r)), B^*\zeta\rangle_V \, d\,r d\,\omega(\tau),
$$

where the Hölder continuity of $\langle U(\tau,r)F(u(r)), B^*\zeta\rangle_V$ is known from the proof of Theorem 17. □

3. Random dynamical systems

This section covers short introduction to random dynamical systems from Arnold [1999] and the definition of a random attractor from sources Crauel et al. [1997], Schmalfuss [2000], Crauel et al. [2008] which give a different definition and criteria.

3.1 Random dynamical systems and random attractors

In this section different approaches to define a random dynamical systems are introduced. Let us start with definition from Arnold [1999].

Definition 14. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, be a family of mappings satisfying the following conditions:*

1. *$(\omega, t) \to \theta_t\omega$ is $\mathcal{F} \otimes \mathcal{B}$, $\mathcal{F}$ measurable, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$;*
2. *$\theta_0 = id_\Omega$;*
3. *Flow Property: $\theta_{t+s} = \theta_t \circ \theta_s$, for $s, t \in \mathbb{R}$;*
4. *For each $t \in \mathbb{R}$, θ_t is measure preserving i.e. $\theta_t\mathbb{P} = \mathbb{P}$.*

Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system.

Definition 15. *A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called ergodic if all sets in $\mathcal{I} = \{A \in \mathcal{F} : \theta_t^{-1}A = A,\ t \in \mathbb{R}\}$ have probability 0 or 1.*

Definition 16. *A measurable random dynamical system with time $\mathbb{R}$ on the measurable space $(X, \mathcal{B})$, where X is complete metric space, over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping*

$$\varphi : \mathbb{R} \times \Omega \times X \mapsto X,\ (t, \omega, x) \mapsto \varphi(t, \omega)x$$

with the following properties:

1. *Measurability: φ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}$,$\mathcal{B}$-measurable.*
2. *Cocycle property: For $t \in \mathbb{R}$ and $\omega \in \Omega$ the mappings $\varphi(t, \omega) = \varphi(t, \omega, \cdot) : X \to X$ form a cocycle over $\theta(\cdot)$, i.e. they satisfy*

$$\varphi(0, \omega) = id_X \text{ for all } \omega \in \Omega\ (0 \in \mathbb{R})$$

and composition satisfies:

$$\varphi(t + s, \omega) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega) \text{ for all } \omega \in \Omega,\ s, t \in \mathbb{R}. \tag{3.1}$$

If (3.1) holds identically, then φ is called a *perfect cocycle*. If the cocycle property (3.1) holds only for fixed s and all $t \in \mathbb{R}$, $\mathbb{P}$-a.s. then φ is called a *crude cocycle* (where the exceptional set N_s can depend on s) and when (3.1) holds for fixed s and $t \in \mathbb{R}$, $\mathbb{P}$-a.s. then φ is called a *very crude cocycle* (where the exceptional set $N_{s,t}$ can depend on s and t).

Let us introduce an example of a metric dynamical system. If W is a standard Wiener process in $\mathbb{R}^d$ and $\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}^d) : \omega(0) = 0\}$, $\mathcal{F}^0 =$ (uncompleted!) Borel σ-algebra, $\mathbb{P} =$Wiener measure, $\theta_t\omega = \omega(t + \cdot) - \omega(t)$ describes *Brownian motion as a metric dynamical system.* This also has the name *Wiener Shift.*

One of the theorems represents a particular interest which gives a connection between stochastic differential equations in the Stratonovich sense and random dynamical systems Arnold [1999] under appropriate condition on a probability space and filtration on this space.

Theorem 23. *Let $C^{k,\delta}$ be the Fréchet space of functions $f : \mathbb{R}^d \to \mathbb{R}^d$ whose k-th derivative is locally δ-Hölder continuous. Let $f_0 \in C^{k,\delta}$, $f_1, ..., f_m \in C^{k+1,\delta}$, and $\sum_{j=1}^m \sum_{i=1}^d f_j^i \frac{\partial}{\partial x_i} f_j \in C^{k,\delta}$ for some $k \geq 1$ and $\delta > 0$. Then:*

$$d\,X_t = f_0(X_t)d\,t + \sum_{j=1}^{m} f_j(X_t) \circ d\,W_t^j, \quad t \in \mathbb{R},$$

where W^j for $j \in 1, ..., m$ are independent Wiener processes and $\circ d$ is understood as a Stratonovich-type integral, generates a unique (up to indistinguishability) random dynamical system φ over the dynamical system describing Brownian motion.

The natural question is raised "Does (2.1) generate a random dynamical system?". This question is studied later in further subsection, for now let us introduce different approaches for random attractors and random dynamical systems.

In Garrido-Atienza et al. [2016] the definition and criteria of random attractors were used from Schmalfuss and Flandoli [1996]. Let us introduce them for $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$.

Definition 17. *Let $\mathcal{H}$ be a family of parametrized subsets $D = \{D(\omega)\}_{\omega\in\Omega}$, $D(\omega) \subset V$. We call such a set system $\mathcal{H}$ inclusion closed if it fulfils the properties:*

(i) If $D \in \mathcal{H}$, then for any $\omega \in \Omega$ the set $D(\omega) \subset V$ is non empty.

(ii) If $D \in \mathcal{H}$ and $D' = \{D'(\omega)\}_{\omega\in\Omega}$ such that $\varnothing \neq D'(\omega) \subset D(\omega)$ for any $\omega \in \Omega$, then $D' \in \mathcal{H}$.

Definition 18. *A parameterized set $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{H}$ is called $\mathcal{H}$-absorbing if for any $D \in \mathcal{H}$, $\omega \in \Omega$, there exists a $t_0 = t_0(\omega, D)$ such that*

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \text{for } t \geq t_0.$$

As the distance between two sets the Hausdorff semidistance on metric space $V = (V, d)$ is used, for $A, B \subset V$ distance $d(A, B)$ is defined:

$$d(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y)$$

Definition 19. *The set-valued map $A : \Omega \to \mathcal{F}$, $\omega \mapsto A_\omega$, where A_ω is closed (compact) for all $\omega \in \Omega$, is called a random closed (compact) set if for each $x \in X$ the map $\omega \mapsto d(x, A_\omega)$ is measurable.*

The attractor might be searched and studied across some subsets of all possible subsets of space V. For this purpose in Garrido-Atienza et al. [2016] the following family of sets was used. Let us denote the positive part of logarithm function by $\log_+$.

Definition 20. *(Tempered set Arnold [1999]) A random variable X in V is called tempered if*

$$\lim_{t\to\pm\infty} \frac{\log_+ ||X(\theta_t\omega)||_V}{|t|} = 0, \quad \omega \in \Omega.$$

A random set D is called tempered set if the random variable

$$\Omega \ni \omega \mapsto \sup_{x\in D(\omega)} ||x||_V, \quad \omega \in \Omega$$

is tempered. In particular, the subset of all tempered sets will be denoted by $\mathcal{D}$ for which the convergence relation

$$\lim_{t\to\pm\infty} \frac{\log_+ \sup_{x\in D(\theta_t\omega)} ||x||_V}{|t|} = 0$$

holds for all $\omega \in \Omega$.

Further we will work with the inclusion closed set system $\mathcal{D}$ defined above.

Definition 21. *A random set $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is called a random attractor for the random dynamical system φ if, for any $\omega \in \Omega$, $A(\omega)$ is a compact, $A(\omega)$ is invariant in the sense that*

$$\varphi(t, \omega, A(\omega)) = A(\theta_t\omega), \quad \text{for all } \omega \in \Omega,\ t \geq 0,$$

and moreover satisfies the pullback attractivness property

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}, D(\theta_{-t}\omega)), A(\omega)) = 0, \quad \text{for all } D \in \mathcal{D},\ \omega \in \Omega.$$

The following theorem provides conditions that ensures the existence of global $\mathcal{D}$-attractors Schmalfuss and Flandoli [1996].

Theorem 24. *Let V be a complete metric space and φ is a random dynamical system over metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. The paths $\varphi(t, \omega, \cdot))$, $\omega \in \Omega$, $t \geq 0$, are assumed to be continuous. Moreover, we assume the existence of $\mathcal{D}$-absorbing set B. Each of these sets $B(\omega)$, $\omega \in \Omega$, is supposed to be compact. Then the random dynamical system φ has unique global $\mathcal{D}$-attractor*

$$A(\omega) = \bigcap_{\tau \geq t_0(\omega,B)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}$$

where $t_0(\omega, B)$ is given in the Definition 18.

3.2 Random dynamical systems and random attractors: the second definition

One of the first definitions of random attractor was introduced in Crauel et al. [1997]. Suppose (X, d) is a complete separable metric space and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $S(t, s, \omega) : X \to X$, $-\infty < s \leq t < \infty$, be a parameterized by $\omega \in \Omega$ family of mappings, satisfying the properties:

(D.1) $S(t, r, \omega)S(r, s, \omega)x = S(t, s, \omega)x$ for all $s \leq r \leq t$, $x \in X$ and $\mathbb{P}$-a.a. $\omega \in \Omega$;

(D.2) $S(t, s, \omega)$ is continuous in X, for all $s \leq t$ and $\mathbb{P}$-a.a. $\omega \in \Omega$.

(D.3) for all $s < t$ and $x \in X$, the mapping

$$\omega \mapsto S(t, s, \omega)x$$

is measurable from $(\Omega, \mathcal{F})$ to $(X, \mathcal{B}(X))$

(D.4) The mappings $s \mapsto S(t, s, \omega)x$ is right continuous at any point $s \in (-\infty, t)$, for all $t, x \in X$ and $\mathbb{P}$-a.a. $\omega \in \Omega$.

So the $S(t, s, \omega)x$ is the state at time t of the system whose value at time s is x.

Definition 22. *For given $t \in \mathbb{R}$ and $\omega \in \Omega$, we say that $K(t, \omega) \subset X$ is called an attracting set at time t if, for all bounded sets $B \subset X$,*

$$d(S(t, s, \omega)B, K(t, \omega)) \to 0, \ s \to -\infty,$$

The system $(S(t, s, \omega)_{t \geq s, \, \omega \in \Omega})$ is called asymptotically compact if there exists a measurable set $\Omega_0 \subset \Omega$, with measure one, such that for all $t \in \mathbb{R}$ and $\omega \in \Omega_0$, there exists a compact attracting set $K(t, \omega)$. The set

$$A(B, t, \omega) = \bigcap_{T<t} \overline{\bigcup_{s<T} S(t, s, \omega)B}$$

is called Random Omega-limit set of a bounded set $B \subset X$ at time t. ***Random attractor*** *is the set:*

$$A(t, \omega) = \overline{\bigcup_{B \subset X} A(B, t, \omega)}.$$

Let assume that there exists a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, with property that for all $s < t$ and $x \in X$,

$$S(t, s, \omega)x = S(t - s, 0, \theta_s \omega)x, \ \mathbb{P} - a.s. \tag{3.2}$$

If such a metric dynamical system exists, then it is necessary to show the existence of an absorbing set at time 0.

Remark 9. *It is possible to represent $S(t, s, \omega)$ as a random dynamical system φ cf. Definition 16. Assumption (D.1) gives the following relation:*

$$S(t + s, s, \omega)S(s, 0, \omega)x = S(t + s, 0, \omega)x, \quad s < t, \ x \in X.$$

Now let us use the property (3.2):

$$S(t, 0, \theta_s \omega)S(s, 0, \omega) = S(t + s, 0, \omega) \quad s < t, \ x \in X, \ \omega \in \Omega$$

if we set $\phi(t, \omega) = S(t, 0, \omega)$ then we obtain Definition 16 under assumptions (D.1)-(D.4).

Proposition 25. *Under assumptions (D.1)-(D.4) and (3.2), suppose that for $\mathbb{P}$-a.e. ω there exists a compact attracting set $K(\omega)$ at time 0, i.e., such that for all bounded sets $B \subset X$,*

$$d(S(0, s, \omega)B, K(\omega)) \to 0, \ s \to -\infty$$

Then the random dynamical system $(S(t, s, \omega))_{t \geq s, \omega \in \Omega}$ is asymptotically compact.

If the assumption of Proposition 25 is satisfied, then the random attractor might be defined as:

$$A(\omega) = A(0, \omega)$$

The main result for random attractors from Crauel et al. [1997].

Theorem 26. *Let $(S(t, s, \omega))_{t \geq s, \omega \in \Omega}$ be a random dynamical system satisfying (D.1-D.4). Assume that there exists a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ such that (3.2) holds and that, for a.a. $\omega \in \Omega$, there exists a compact attracting set $K(\omega)$ at time 0. For a.a. $\omega \in \Omega$, we set*

$$A(\omega) = \overline{\bigcup_{B \subset X} A(B, \omega)}$$

where the union is taken over all the bounded subsets of X and $A(B, \omega)$ is given by

$$A(B, \omega) = \bigcap_{T<0} \overline{\bigcup_{s<T} S(0, s, \omega)B}.$$

Then for a.a. $\omega \in \Omega$:

1. *$A(\omega)$ is a nonempty compact subset of X, and if X is connected, it is a connected subset of $K(\omega)$.*

2. *The family $A(\omega)$, $\omega \in \Omega$, is measurable*

3. *$A(\omega)$ is invariant in the sense that*

$$S(t, s, \omega)A(\theta_s \omega) = A(\theta_t \omega), \ s \leq t.$$

4. *$A(\omega)$ is the minimal closed set such that for $t \in \mathbb{R}$, $B \subset X$ bounded*

$$d(S(t, s, \omega)B, A(\theta_t \omega)) \to 0, \text{ when } s \to -\infty.$$

5. *For any bounded set $B \subset X$, $d(S(t, s, \omega)B, A(\theta_t \omega)) \to 0$ in probability when $t \to \infty$.*

And if the time shift θ_t, $t \in \mathbb{R}$, is ergodic, then we also have

6. *There exists a bounded set $B \subset X$ such*

$$A(\omega) = A(B, \omega).$$

7. *$A(\omega)$ is the largest compact measurable set which is invariant in the sense of 3.*

3.3 Random attractors for semilinear stochastic equations with a bilinear fractional noise

Let us return to the equation (2.16) and its solution. The solution of the equation (2.16) is defined by Remark 5 and by Theorem 18.

$$u(t,\omega) = U(t,\omega,0)u_0 + \int_0^t U(t,\omega,s)F(u(s,\omega))d\,s, \quad t \in [0,T],$$

where

$$U(t,\omega,s) = S_B(\omega(t) - \omega(s))S_A(t-s), \quad 0 \le s \le t \le T. \tag{3.3}$$

Let $\mathcal{F} = \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ be a Borel σ-algebra, where $C_0(\mathbb{R},\mathbb{R})$ is a space of continuous functions such that $\omega \in C_0(\mathbb{R},\mathbb{R})$ satisfies $\omega(0) = 0$. Define as Ω the set of β'-Hölder continuous functions on any interval $[-N,N]$ for $N \in \mathbb{N}$ which are zero at zero and satisfies assumption (E) (this will be stated later), moreover $\Omega \in \mathcal{F}$. Let $\mathbb{P}$ be the distribution of stochastic process $(\omega(t), t \in \mathbb{R})$ whose all sample paths belong to Ω. $(\omega(t), t \in \mathbb{R})$ has stationary increments and $\mathbb{P}(\Omega) = 1$. So we can restrict our probability space to $(\Omega, \mathcal{S}, \mathbb{P})$, where $\mathcal{S} = \mathcal{F}_{|\Omega}$ is restriction of $\mathcal{F}$ to Ω. The family of mappings $(\theta_t)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{S}, \mathbb{P})$ are the shift mappings:

$$\theta_t\omega = \omega(t+\cdot) - \omega(t), \;\; \omega \in C_0(\mathbb{R},\mathbb{R}), \; \forall t \in \mathbb{R}.$$

Appendix A.3 Arnold [1999] gives us that $(\theta_t)_{t\in\mathbb{R}}$ is $\mathbb{P}$ measure preserving, thus $(\Omega, \mathcal{S}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Theorem 27. *Under the conditions of Theorem 18, the unique mild solution u of (2.16) generates a random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$ defined by*

$$\varphi(t,\omega,u_0) = U(t,\omega,0)u_0 + \int_0^t U(t,\omega,r)F(u(r,\omega))d\,r.$$

In the study of attractors of the solution to (2.16) in Garrido-Atienza et al. [2016] additional assumptions were added. It is necessary to assume that the mapping F can be represented as $F = aI + G$, where $a \in \mathbb{R}$ and $G : V \to V$ is a bounded Lipschitz continuous function. The bound of G is denoted by C_G. The equation (2.16) might be rewritten as

$$d\,u(t) = (\hat{A}u(t) + G(u(t)))d\,t + Bu(t) \circ d\,\omega(t), \quad u(0) = u_0 \in V, \tag{3.4}$$

where $\hat{A} = aI + A$. This operator again generates the analytic semigroup $S_{\hat{A}} = e^{at}S_A(t)$ for $t \ge 0$. Thanks to Theorem 18, the equation (3.4) has a unique mild solution given by

$$u(t) = \hat{U}(t,0)u_0 + \int_0^t \hat{U}(t,r)G(u(r))d\,r,$$

where $\hat{U}(t,s) = S_B(\omega(t) - \omega(s))S_{\hat{A}}(t-s) = e^{a(t-s)}U(t,s)$. The mild solution of (3.4) generates a random dynamical system which is denoted by φ further. Also let us assume that there are constants $M_A \ge 1$ and $\lambda \in \mathbb{R}$ for which we have:

$$\|S_A(t)\|_{\mathcal{L}(V)} \le M_A e^{\lambda t}, \quad t \ge 0. \tag{3.5}$$

Also, let $M_B \geq 1$ and $\mu \in \mathbb{R}$ be such that

$$\|S_B(t)\|_{\mathcal{L}(V)} \leq M_B e^{\mu|t|}, \quad t \in \mathbb{R}.$$

In Garrido-Atienza et al. [2016] all derivation was done for a fractional Brownian motion for which the law of iterated logarithm exists, but to work with a general process, it is necessary to impose one more assumption:

(E) Stochastic process $(\omega(t), t \in \mathbb{R})$ satisfies $\omega(t) = o(|t|)$ for $t \to \pm\infty$.

The lemma below is a slight modification of the proof of Lemma 4.3 from Garrido-Atienza et al. [2016] where instead of the law of iterated logarithm for fractional Brownian motion, assumption (E) is used.

Lemma 28. *Assume that the stability condition $a + \lambda < 0$ is satisfied, and let $\mathcal{D} = \{D(\omega)\}_{\omega\in\Omega}$ be the family of tempered sets in V. Under the assumptions (A,B,C,E), the ball $B_V(\omega) = B(0, R(\omega))$ with*

$$R(\omega) = 2C_G M_A M_B \int_{-\infty}^{0} e^{\mu|\omega(r)|-(a+\lambda)r} d\,r \tag{3.6}$$

is a $\mathcal{D}$-absorbing set.

Proof. It is necessary to find an estimation of a solution to (3.4) in V, let us consider $\omega \in \Omega$ and its shift $\theta_{-t}\omega$ for $t \geq 0$. Note that

$$\begin{aligned}
&\|\varphi(t, \theta_{-t}\omega, u_0)\|_V \\
&= \left\| S_B(\theta_{-t}\omega(t)) S_{\hat{A}}(t) u_0 + \int_0^t S_B(\theta_{-t}\omega(t) - \theta_{-t}\omega(r)) S_{\hat{A}}(t-r) G(u(r)) d\,r \right\|_V \\
&\leq M_A M_B e^{\mu|\theta_{-t}\omega(t)|+(a+\lambda)t} \|u_0\|_V + C_G M_A M_B \int_0^t e^{\mu|\theta_{-t}\omega(t)-\theta_{-t}\omega(r)|+(a+\lambda)(t-r)} d\,r \\
&= M_A M_B e^{\mu|-\omega(-t)|+(a+\lambda)t} \|u_0\|_V + C_G M_A M_B \int_{-t}^0 e^{\mu|-\omega(y)|-(a+\lambda)y} d\,y.
\end{aligned}$$

From this if $R(\omega)$ is defined by (3.6), in the estimation above take $D \in \mathcal{D}$ and replace $\|u_0\|_V$ by $\sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V + R(\omega)$, then the following inequality holds

$$\|\varphi(t, \theta_{-t}\omega, u_0)\|_V \leq M_A M_B e^{\mu|-\omega(-t)|+(a+\lambda)t} \sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V + R(\omega).$$

Combining assumption (E) with the assumption $a + \lambda < 0$, we get that for $u_0 \in D(\theta_{-t}\omega)$ there exists $t_0 = t_D(\omega)$ such that

$$\|\varphi(t, \theta_{-t}\omega, u_0)\|_V \leq R(\omega), \quad t \geq t_0.$$

This is true in fact, given $\epsilon > 0$ small enough such that $\epsilon\mu + (a + \lambda) < 0$,

$$\lim_{t\to\infty} e^{\mu|-\omega(-t)|+(a+\lambda)t} \sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V \leq \lim_{t\to\infty} e^{(\epsilon\mu+a+\lambda)t} \sup_{u_0 \in D(\theta_{-t}\omega)} \|u_0\|_V = 0$$

It remains to prove that $B_V(\omega)$ is tempered. Choose $0 < \kappa < -(a + \lambda)$, then

$$\begin{aligned}
&\lim_{t\to-\infty} e^{-2\kappa|t|} \int_{-\infty}^0 e^{\mu|\theta_t\omega(r)|-(a+\lambda)r} d\,r \\
&\quad = \lim_{t\to-\infty} e^{\kappa t} \int_{-\infty}^0 e^{-\frac{a+\lambda}{2}r} e^{-\frac{a+\lambda}{2}r} e^{\frac{\kappa}{2}t} e^{\frac{\kappa}{2}t} e^{\mu|\omega(t+r)-\omega(t)|} d\,r \\
&\quad \leq \lim_{t\to-\infty} e^{\kappa t} \int_{-\infty}^0 e^{-\frac{a+\lambda}{2}r} e^{\frac{\kappa}{2}(t+r)+\mu|\omega(t+r)|} e^{\frac{\kappa}{2}(t)+\mu|\omega(t)|} d\,r \\
&\quad \leq \sup_{\tau\in(-\infty,0]} e^{2(\frac{\kappa}{2}t+\mu|\omega(t)|)} \lim_{t\to-\infty} e^{\kappa t} \int_{-\infty}^0 e^{-\frac{a+\lambda}{2}} = 0.
\end{aligned}$$

The supremum above is finite due to assumption (E). Theorem 4.1.3 (i) Arnold [1999] helps to show that the convergence is in place also when $t \to \infty$, hence $B_V(\omega)$ is a tempered ball. $\square$

The assumption (A) guarantees that there exists $\beta_0 \in \mathbb{R}$ such that the operator $(\beta_0 I - A)$ is strictly positive and the fractional power of operator $(\beta_0 I - A)^\delta$ is well defined for every $\delta \in (0, 1]$, see Chapter 2 Pazy [1983]. We need to add one more assumption about the operator A.

(F) $D((\beta_0 I - A)^\gamma) \subset D(B)$ for some $\gamma \in (0, 1)$ and $(\beta_0 I - A)^{-1}$ is a compact operator.

Lemma 29. *Given $\delta \in (0, 1)$, $\beta \in (0, H)$, $0 < \epsilon < T$ and $R > 0$, there exists a random constant $C = C(\omega) > 0$ such that*

$$\|u\|_{\beta,\epsilon,T} \leq C(\omega)$$

and

$$\|(\beta_0 I - A)^\delta u(t)\|_V \leq C(\omega)$$

holds for each $t \in [\epsilon, T]$ and $u_0 \in V$ such that $\|u_0\|_V \leq R$.

Lemma 30. *Under conditions of Lemma 28, there exists a family of compact absorbing sets $C = \{C(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ for the cocycle φ.*

Theorem 24 give us the existence of a random attractor to the equation (3.4). The proof mimics the proof of Corollary 1 in Garrido-Atienza et al. [2016] but the assumption about fractional Brownian motion is replaced by the assumption (E).

Theorem 31. *Under the condition (E) and (F) and the stability condition $a + \lambda < 0$, the random dynamical system φ has a unique random attractor $A = \{A(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$.*

Remark 10. *The proof of existence of a random attractor of random dynamical system φ in Garrido-Atienza et al. [2016] is done under the assumption that the underlying metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is ergodic. There is only one reference to the ergodic property of $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ in the analogue of Theorem 24, which is taken from Schmalfuss and Flandoli [1996]. The original source of Theorem 24, which is Schmalfuss and Flandoli [1996], does not have the assumption about ergodicity of $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. This gives us that all results above are valid for more general process $(w(t), t \in \mathbb{R})$ and not only for fractional Brownian motion.*

4. Examples

This section is devoted to description of examples for noise and some equations for which theorems above hold.

4.1 Noise: examples

Fractional Brownian Motion

One of the main examples of noise is fractional Brownian motion which originally was used in Garrido-Atienza et al. [2016]. This process has the following representation for $t \in \mathbb{R}$ and Hurst parameter H Picard [2011]:

$$B^H(t) = \frac{1}{\Gamma(1+H)} \int_{\mathbb{R}} \left[(t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] dW_r, \ t \in \mathbb{R}.$$

where W is a standard Wiener process on $\mathbb{R}$. The process has Hölder continuous version of every order $\delta < H$ on any interval $[-N, N]$ for $N \in \mathbb{N}$. Also, it is known that fractional Brownian motion is ergodic with respect to the shift mappings $(\theta_t)_{t\in\mathbb{R}}$ Garrido-Atienza and Schmalfuss [2011].

Rosenblatt Process

Rosenblatt process might be expressed as a miltiple Itô integral Čoupek [2018]:

$$Z(t) = a(H) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_0^t (s-y_1)_+^{-\frac{2-H}{2}} (s-y_2)_+^{-\frac{2-H}{2}} \, ds \right) dW_{y_1} \, dW_{y_2}, \ t \geq 0$$

and

$$Z(t) = a(H) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(- \int_{-t}^0 (s-y_1)_+^{-\frac{2-H}{2}} (s-y_2)_+^{-\frac{2-H}{2}} \, ds \right) dW_{y_1} \, dW_{y_2}, \ t < 0$$

where

$$a(H)^2 = \frac{\sqrt{\frac{H}{2}(2H-1)}}{B(\frac{H}{2}, 1-H)},$$

and W is a standard Wiener process on $\mathbb{R}$. This process has a Hölder continuous version of every order $\delta < H$ on any interval $[-N, N]$ for $N \in \mathbb{N}$. It also was shown in Čoupek [2018] that Rosenblatt process has stationary increments which allows us to show that the shift mappings $(\theta_t)_{t\in\mathbb{R}}$ are measure preserving transformations with respect to $(Z(t), t \in \mathbb{R})$ and the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $Z(t)$ is defined. The law of iterated logarithm for Rosenblatt process is provided by Goodman and Kuelbs [1993]. So all results which are obtained in Garrido-Atienza et al. [2016] hold for Rosenblatt process.

As a generalization for the processes above we consider a process which has stationary increments and for which assumption (E) holds true. It is known from Samoradnitsky and M.Taqqu [2017] that any second-order H-self-similar process

$\{X(t), t \in I\}$, where $I \subset \mathbb{R}$ is interval, with stationary increments must have the following covariance function for some $\sigma > 0,\ H \in (0, 1)$:

$$\mathbb{E}X(t)X(s) = \frac{\sigma^2}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad s, t \in I. \tag{4.1}$$

Certain processes with such covariance function was studied in Mori and Oodaira [1986] for which the law of iterated logarithm exists. All results till the end of the current section are taken from the paper Mori and Oodaira [1986]. The processes of the form are studied:

$$X(t) = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} Q_t(u_1, \dots, u_m) dW_{u_1} \dots dWu_m, \quad t \geq 0, \tag{4.2}$$

where the right side is the multiple Wiener integral with respect to standard Brownian motion W with $W(0) = 0$ and symmetric functions Q_t belong to $L^2(\mathbb{R}^m)$. Q_t is called the kernel. The following assumptions are imposed for the kernel:

$$Q_{ct}(cu_1, \dots, cu_m) = c^{H-m/2} Q_t(u_1, \dots, u_m),\ c > 0,\ t \geq 0, \tag{4.3}$$

where $0 < H < 1$ is a constant, and

$$Q_{t+h}(u_1, \dots, u_m) - Q_t(u_1, \dots, u_m) = Q_h(u_1 - t, \dots, u_m - t),\ t \geq 0,\ h \geq 0. \tag{4.4}$$

Those conditions imply that $Q_0(u_1, \dots, u_m) = 0$. Condition (4.3) implies that X is self-similar with parameter H, and (4.4) implies that X has stationary increments. It is known that a multiple Wiener integral has moments of all orders, therefore we have $\mathbb{E}|X(1)|^r < \infty$ and:

$$\mathbb{E}|X(s) - X(t)|^r = \mathbb{E}|X(|s - t|)|^r = |s - t|^{rH} \mathbb{E}|X(1))|^r$$

This means that the process $(X(t))_{t \geq 0}$ admits a measurable version and moreover $(X(t))_{t \geq 0}$ also admits a version with Hölder continuous sample paths up to the order H by Chentsov [1956]. Let us make kernel Q more specific. Introduce a symmetric function q on $\mathbb{R}^m$ which is homogeneous with degree $-\lambda$, i.e.,

$$q(cu_1, \dots, cu_m) = c^{-\lambda} q(u_1, \dots, u_m),\ c > 0. \tag{4.5}$$

It is supposed that $\lambda \in (m/2, (m+1)/2)$. In addition, suppose that q satisfies

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} |q(u_1, \dots, u_m) q(u_1 + 1, \dots, u_m + 1)| du_1 \dots du_m < \infty. \tag{4.6}$$

Then

$$Q_t(u_1, \dots, u_m) = \int_0^t q(v - u_1, \dots, v - u_m) dv \tag{4.7}$$

defines a kernel satisfying $Q_t \in L^2(\mathbb{R}^m)$, (4.3) and (4.4) with $H = m/2 - \lambda + 1$.

At first it is necessary to introduce the space in which almost every sample path of X is contained. We now assume that $m \geq 1$ and $H \in (1/2, 1)$. Let

$$v(t) = t^{H-m/2}(1 + |\log(t)|)^{m/2},\ t > 0,$$

and define $C_v(\mathbb{R}^+)$ as a space of all continuous functions y on $\mathbb{R}^+$ satisfying

$$\lim_{t\to\infty} y(t)/v(t) = \lim_{t\to 0+} y(t)/v(t) = 0.$$

The norm of this space is defined as:

$$\|y\|_{C_v} = \sup_{t>0} |y(t)|/v(t),$$

$C_v(\mathbb{R}^+)$ becomes a Banach space.

If $\xi \in L^2(\mathbb{R})$ is square integrable function and Q_t satisfies (4.3) and (4.4), then

$$y(t) = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} Q_t(u_1, \dots, u_m)\xi(u_1)\dots\xi(u_m)d\,u_1 \dots d\,u_m,\ t \geq 0, \tag{4.8}$$

describes a function in $C_v(\mathbb{R}^+)$. From Schwartz inequality we have:

$$|y(t)|^2 \leq \|Q_t\|_2^2 \|\xi\|_2^{2m} = t^{2H}\|Q\|_2^2\|\xi\|_2^{2m}$$

where $Q = Q_1$ and $\|\cdot\|_2$ is a norm on a space of L^2 functions. And the continuity follows from the estimate

$$|y(t+h) - y(t)|^2 \leq \|Q_{t+h} - Q_t\|_2^2\|\xi\|_2^{2m} = h^{2H}\|Q\|_2^2\|\xi\|_2^{2m}$$

Let us impose an assumption on the kernel Q

(G) Q_t is represented as in (4.7) with a symmetric function q satisfying (4.5) and (4.6), where $\lambda = m/2 + 1 - H$, $1/2 < H < 1$. K_Q denotes the set of all functions y represented by (4.8) with $\xi \in L^2(\mathbb{R})$ such that $\|\xi\|_2 \leq 1$. Let $\{X_n\}$ be a sequence of random functions defined by

$$X_n(t) = \frac{X(nt)}{n^H(2\log\log(n))^{m/2}},\ t \geq 0,\ n \geq 3.$$

Now it is possible to formulate a theorem Mori and Oodaira [1986].

Theorem 32. *Let X be a self-similar process with stationary increments having representation (4.2) with kernel Q_t satisfying (G). Then with probability one $X_n \in C_v(\mathbb{R}^+)$, $n \geq 1$. Furthermore with probability one the sequence $\{X_n, n \geq 3\}$ is relatively compact in $C_v(\mathbb{R}^+)$ and the set of its points coincides with K_Q.*

The following theorem guarantees that a process which is represented as (4.2) for almost all sample paths has the estimation $X(t) = o(t)$ for $t \to \infty$. This means that we have a big family of stochastic processes which satisfy to condition (E) from Chapter 3. Both Fractional Brownian motion and Rosenblatt process are particular examples of choice of function q and normalizing constants c_1 and c_2 which gives $\mathbb{E}|X_1|^2 = 1$:

$$q(u) = c_1 u_+^{H-3/2}$$

for the fractional Brownian motion and

$$q(u_1, u_2) = c_2 (u_1)_+^{-\frac{2-H}{2}} (u_2)_+^{-\frac{2-H}{2}}$$

for the Rosenblatt process.

4.2 Equation: Examples

Let us consider two examples which were observed in the article Duncan et al. [2005]. At first, consider the following stochastic parabolic equation of $2k$th order and the stochastic process $(\omega(t))_{t\geq 0}$ that satisfies assumptions from chapter above:

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = L(t,\xi)u(t,\xi) + bu\frac{d\omega}{dt} \\ u(0,\xi) = x_0(\xi) \end{cases} \tag{4.9}$$

for $(t,\xi) \in [0,T] \times \mathcal{O}$, with the Dirichlet boundary conditions

$$\left(\frac{\partial u}{\partial \xi}\right)^{\alpha}(t,\xi) = 0, \quad (t,\xi) \in [0,T] \times \partial\mathcal{O}, \ \alpha \in \{0,1,...,k-1\},$$

where $k \in \mathbb{N}$, $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class C^k, $b \in \mathbb{R}/\{0\}$ and

$$L(t,\xi) = \sum_{|\alpha|\leq 2k} a_{\alpha}(t,\xi)D^{\alpha} \tag{4.10}$$

To satisfy assumption (C.1) we require that (4.10) is a strongly elliptic on $\mathcal{O}$, uniformly in $(t,\xi) \in [0,T] \times \bar{\mathcal{O}}$ and $a_{\alpha}(t,\cdot) \in \mathcal{C}^{2k}(\bar{\mathcal{O}})$ for each $t \in [0,T]$. So equation might be rewritten in the form

$$\begin{cases} dX(t) = A(t)X(t)dt + BX(t)d\omega, \\ X(0) = x_0 \in V, \end{cases} \tag{4.11}$$

for $t \in [0,T]$, where $V = L^2(\mathcal{O})$, $(A(t)u)(\xi) = L(t,\xi)u(t,\xi)$, $\mathrm{Dom}(A(t)) = D = H^{2k}(\mathcal{O}) \cap H_0^k(\mathcal{O})$ and $B = bI \in \mathcal{L}(V)$. It is assumed that

$$\sup_{\xi\in\mathcal{O}} |a_{\alpha}(t,\xi) - a_{\alpha}(s,\xi)| \leq M|t-s|^{\gamma}$$

for $|\alpha| \leq 2k$, $s,t \in [0,t]$ and a constant M. Hypotheses (C.3) and (C.4) are satisfied (Tanabe [1979], Theorem 3.8.3). Assumption (C.2) is satisfied. All those assumptions guarantee the existence of a weak solution to the linear problem thanks to Theorem 17. If operator $A(t) = A$ is time independent then assumptions (A), (B) and (C) are satisfied. It means that if equation (4.11) satisfies a stability condition of Corollary 31 ($a = 0$, $\lambda < 0$), all results up to Lemma 30 hold. In other words, the equation (4.11) possesses a random attractor.

The second example is the stochastic Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} a_{ij}(t)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(t,\xi) + \sum_{i=1}^{d} d_i(t)\frac{\partial u}{\partial\xi_i}(t,\xi) \\ +c(t)u(t,\xi) + \sum_{i=1}^{d} b_i\frac{\partial u}{\partial\xi_i}(t,\xi)\frac{d\omega(t)}{dt}, \\ u(0,\xi) = x_0(\xi), \end{cases} \tag{4.12}$$

for $(t,\xi) \in [0,T] \times \mathbb{R}^d$, where a_{ij}, d_i, c are Hölder continuous functions for $\forall i,j \in 1,...,d$ and $b_i \in \mathbb{R}$. To satisfy assumption (C.1) it is assumed that the differential operator

$$L(t) = \sum_{i,j=1}^{d} a_{ij}(t)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d} d_i(t)\frac{\partial}{\partial\xi_i} + c(t)I$$

is uniformly elliptic, which means

$$\sum_{i,j=1}^{d} a_{ij}(t)v_i v_j > 0$$

is satisfied for $v \in \mathbb{R}^d/\{0\}$ and $t \in [0,T]$. Equation (4.12) is rewritten as

$$\begin{cases} dX(t) = A(t)X(t)dt + BX(t)d\omega(t), \\ X(0) = x_0, \end{cases} \tag{4.13}$$

where $X(t), x_0 \in V$, $V = L^2(\mathbb{R}^d)$, $A(t) = L(t)$ with $\mathrm{Dom}(A(t)) = \mathrm{Dom}(A^*(t)) = H^2(\mathbb{R}^d)$, $B = \sum_{i=1}^{d} b_i \frac{\partial}{\partial \xi_i}$, $\mathrm{Dom}(B) = C^1(\mathbb{R}^d)$. Theorem 5.2.1 Tanabe [1979] provides that assumption (C.1),(C.3) and (C.4) are satisfied. Operator B generates strongly continuous groups on V that is:

$$[S(t)x_0](\xi) = x_0(\xi_1 + b_1 t, ..., \xi_d + b_d t)$$

for $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$ and $t \in \mathbb{R}$. So (C.2) is also satisfied. This means that all requirements for Theorem 17 are satisfied and a weak solution to the problem (4.13) exists.

Let us consider one dimensional case of problem (4.12):

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = a\frac{\partial^2 u}{\partial \xi^2}(t,\xi) + b\frac{\partial u}{\partial \xi}(t,\xi)\frac{d\omega}{dt}(t), \\ u(0,\xi) = x_0(\xi), \end{cases} \tag{4.14}$$

for $t > 0$ and $\xi \in \mathbb{R}$ where $a > 0$ and $b \in \mathbb{R}$
$\{0\}$. The ellipticity condition is satisfied since $a > 0$. The solution for the equation is given as

$$X(t) = S(\omega(t))U(t,0)x_0,$$

where $[S(s)x](\xi) = x(\xi + bs)$ and U is the evolution system corresponding to the equation

$$\begin{cases} \frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial \xi^2}, \\ y(0) = x_0. \end{cases}$$

Thus, U is a heat semigroup on $\mathbb{R}$

$$(S_\Delta(t)x_0)(\xi) = \int_{\mathbb{R}} (4\pi t)^{-1/2} \exp\left[-\frac{1}{4}(\xi - \zeta)^2\right] x_0(\zeta)d\zeta$$

and

$$X(t) = S(\omega(t))S_\Delta(at)x_0$$

Despite Example 2.5 in Duncan et al. [2005] for the same problem the solution is defined only for $t \in [0,T]$, where T depends on a. This difference between solutions is explained by integration type in article. In Duncan et al. [2005] Skorokhod integral is used whereas we use the integral in the sense of Defenition 1.1 Zähle [2001], which gives us slightly different version of the solution, which is defined for any $t \geq 0$.

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