**BACHELOR THESIS**

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# Space curves with Pythagorean Hodograph

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Abstract: The works discusses the basics of motions along curves in 3D space and especially those that are both rational and whose frames have the least rotation during their movement. These rotation minimizing rational frames open up avenues in both computer modelling and adjecent fields. We also introduce an alternative way of defining what a Pythagorean hodograph is, which could allow a new path for research.

Keywords: Rational curves, Rotation-minimizing curves,
Pythagorean Hodographs, Frames, Spherical motions

Titul: Prostorové křivky s Pythagorejským hodografem

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Abstrakt: Práce se věnuje základům pohybů podél křivek v třídimenzionálním prostoru a obzvláštěm těm, které jsou racionální a mají repér, jehož rotace je nejmenší možná. Tyto křivky a soustavy s minimem rotace během pohybu umožňují hlubší výzkum v počítačovém modelování a spojených polích. Také předvedeme alternativní definici takzvaných Pythagorejských hodografů, která může otevřít dveře k novému výzkumu.

Hlavní slova: Racionální křivky, Minimalizace rotace,
Pythagorejský hodograf, Repér, Sférický pohyb

# Contents

# 1. Preface

In this work we will discuss some of the work done in researching curves in 3D space and their associated frames. Specifically we will look at some of the work done regarding curves that are both rational and can have a frame defined in a way that prevents it unnecessarily rotating along the tangent - these are called rotation minimizing motions.
The second chapter is mostly an introduction of the necessary topics. How quaternions work and are related to rotation in three dimensional space is the most important part. We also present a lemma we will use to create various corollaries of later results. While the second chapter introduces Spherical motions, we discuss them more deeply in the third chapter. There we present both rotation minimization and an alternative definition of the Pythagorean hodograph, which is a curve with a rational tangent length. In this same chapter we reference "Über zwangläufige rationale Bewegungsvorgänge" [1] by Bert Jüttler for the second part of a proof.
In the fourth chapter we discuss a way of determing whether a quaternion polynomial already represents of rotation minized motion, which was first shown in an article called "A comprehensive characterization of the set of polynomial curves with rational rotation-minimizing frames" [3] by Rida T. Farouki, Graziano Gentili, Carlotta Giannelli, Alessandra Sestini and Caterina Stoppato. While mostly a reproduction of the original article, we have included some collorаries in the Bernstein basis and mention some possible avenues for future research.
In the fifth chapter we discuss a theorem, which was first presented in an article called "Quaternion and Hopf map characterizations for the existence of rational rotation-minimizing frames on quintic space curves" [4] by Rida T. Farouki as Proposition 3 regarding sparial quintics. We present one of the implications from it and a proof, which is significantly simpler, since the original work sought to prove an equality and required a setup using Hopf maps. It was also done through complex numbers, but here we use quaternions.
The final chapter present examples of some hodograph curves and their appearence in 3D space. Some of those curves are example taken from an article called "A complete classification of quintic space curves with rational rotation-minimizing frames" by Rida T. Farouki and Takis Sakkalis [2].
We also used a book called "Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable" by Rida T. Farouki as a study resource, but do not quote from it in our work. Part V of that book is the most relevant to our work, since it also discusses Pythagorean Hodographs in 3D.

# 2. Preliminaries

## 2.1 Spatial Curves

If we take three real $\mathcal{C}^3$ continuos functions $x(t)$, $y(t)$ and $z(t)$, then the triplet $(x(t), y(t), z(t))$ defines a curve in $\mathbf{R}^3$. Now should we for some reason wish to create a triplet of orthonormal vectors at any point of the curve, where one is alligned with the tangent - this is called a frame, we commonly choose the Frenet–Serret frame. This is defined as follows:

**Definition 1.** *Let $c(t) = (x(t), y(t), z(t))$ be a curve in $\mathbf{R}^3$. Then the Frenet-Serret frame at a point $c(t)$, which is not an inflection point, is defined as the triplet*

1. $T(t) := \frac{c'(t)}{||c'(t)||}$*, called the tangent*
2. $N(t) := \frac{T'(t)}{||T'(t)||}$*, called the normal*
3. $B(t) := T(t) \times N(t)$*, caled the binormal*

In 3D the curvature is equal to $||T'(t)||$ and since we don't want division by zero, we can't define the normal and therefore also the binormal at inflection points, which have zero curvature. Limits as we approach might not be equal either. Since the frame is compossed of three unit vectors, then as the parameter $t$ changes, the frame can be though of as turning inside a unit sphere in $\mathbf{R}^3$.
If we move the frame from vector space and display it at $c(t)$, it is unavoidable that tangent for any curve but a straight line changes direction and normal and binormal defined by it must follow. However often the Frenet–Serret frame has rotation of the normal and binormal greater than what is necessary to maintain orthonormality.

**Example 2.** *Let's take a Torus knot defined for instance as $x(t) = rcos(2t)$, $y(t) = rsin(2t)$ and $z(t) = -sin(3t)$ to see what we mean.*

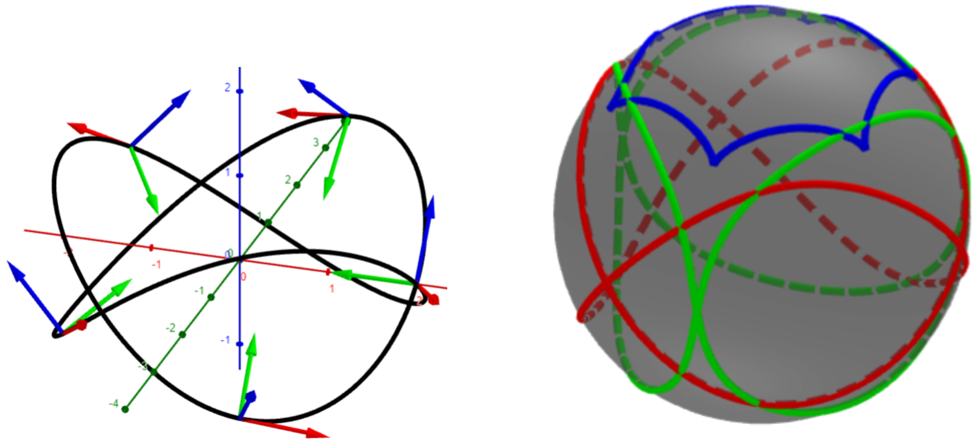

**Figure 2.1:** Example 2:On the left we have a have the Torus knot with five example frames. On the right we see the Unit Sphere and the curves traced by frames

Finding a way to alter the Frent-Serret frame to get rid of this unnecessary rotation can be useful for multiple practical purposes like the movement of robotic arms. We must however first find a good approach to rotations in $\mathbf{R}^3$, since we are in essence creating a function, whose output for all $t$ is a rotation that turns the normal and binormal of the base frame around the tangent. We will call this a correction of the frame. When we speak of rotation minimization, we imply an existence of some measure that compares different frames. This quantity is defined as the amount by which the normal and binormal rotate around the tangent. That is $T(t) \times N'(t)$ or $T(t) \times B'(t)$. Since the tangent doesn't rotate around itself and the binormal is defined by it and the normal, these are equivalent.

## 2.2 Quaternion Calculus

Research into RMFs comes from multiple angles. Some uses complex numbers to interpret rotation in $\mathbf{R}^3$, but this work follows other studies using quaternions and their natural relationship with rotations.

**Definition 3.** *Quaternions are a four dimensional vector space over the real numbers, whose elements of the same name are built using basis vectors* $1, \mathbf{i}, \mathbf{j}$ *and* $\mathbf{k}$. *These vectors take the form* $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, *where* $a, b, c$ *and* $d$ *are real numbers. The basis vectors represent "imaginary numbers" that satisfy the following condition*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

*Further multiplication is not commutative unless one of the elements is* $1$. *These six exceptions to commutativity are:*

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$$

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

*We call $a$ the quaternion's scalar part and $b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ the quaternion's vector part.*

Addition and scalar multiplication are defined as would be expected of vectors. Multiplication is dependent on the order due to noncommunicativity and therefore

$$(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) =$$

$$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i}+$$
$$+(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}.$$

We call the quaternion $a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ the conjugate of $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and it will also be written as the original marker by $*$ as is usually done for conjugation of matricies. All non-zero quaternion have a multiplicative inverse, because

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) = a^2 + b^2 + c^2 + d^2$$

and that result is a purely real number.
Since quaternion are based on vector, we can define both the cross product and the dot product as if we were dealing with 4D Eucleidian space. Notice that $\langle Q, Q\rangle = ||Q||^2$ is satisfied and in fact in general

$$\langle P, Q\rangle = scal(PQ^*) = scal(Q^*P) = scal(P^*Q) = scal(QP^*),$$

due to the squares of vector bases being $-1$. The dot product has a further interesting property that is a subject of the following lemma:

**Lemma 4.** *For any quaternions $P$ and $Q$, we get that $\langle P\mathbf{i}, Q\rangle = -\langle P, \mathbf{i}Q\rangle$.*

*Proof.* Let

$$P = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$$
$$Q = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}.$$

Then

$$P\mathbf{i} = -b_1 + a_1\mathbf{i} + d_1\mathbf{j} - c_1\mathbf{k}$$
$$Q\mathbf{i} = -b_2 + a_2\mathbf{i} - d_2\mathbf{j} + c_2\mathbf{k},$$

which gives us

$$\langle P\mathbf{i}, Q\rangle = -b_1a_2 + a_1b_2 + d_1c_2 - c_1d_2$$
$$-\langle P, \mathbf{i}Q\rangle = a_1b_2 - b_1a_2 + c_1d_2 - d_1c_2,$$

which is just a reordering of the previous terms. □

Quaternion are especially useful due to easily representing rotations in 3D space. Interpreting the vector $(x, y, z)$ as a vector part of a quaternion

$$Q := 0 + x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

we can apply a rotation around the axis represented by a vector $(p, q, r)$ by an angle of $\theta$ by conjugating $Q$ with another quaternion

$$R := cos\frac{\theta}{2} + psin\frac{\theta}{2}\mathbf{i} + qsin\frac{\theta}{2}\mathbf{j} + rsin\frac{\theta}{2}\mathbf{k}$$

and the new rotated vector can then be interpreted as the vector part of the quaternion $RQR^{-1}$.
This is a way to express Rodrigues' rotation formula, which for our case would look like

$$(x,y,z)cos\theta + \left(\frac{(p,q,r)}{||(p,q,r)||} \times (x,y,z)\right) sin\theta +$$

$$+\frac{(p,q,r)}{||(p,q,r)||}\left(\frac{(p,q,r)}{||(p,q,r)||} \cdot (x,y,z)\right)(1-cos\theta).$$

We will find it useful to make $R$ a quaternion polynomial dependent on t. This leads us to quaternions calculus, which is the natural extension of polynomials to the quaternion numbers using the most natural definitions - replacing the real numbers $a, b, c$ and $d$ in our quaternion definition, with real polynomials. If we let $a(t), b(t), c(t), d(t), e(t), f(t), g(t)$ and $h(t)$ be eight real polynomials, then we can keep our previous definitions and get:

1. $[a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}] + [e(t) + f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] =$

$$= [a(t) + e(t)] + [b(t) + f(t)]\mathbf{i} + [c(t) + g(t)]\mathbf{j} + [d(t) + h(t)]\mathbf{k}$$

2. $[a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}][e(t) + f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] =$

$$= [a(t)e(t) - b(t)f(t) - c(t)g(t) - d(t)h(t)]$$

$$+[a(t)f(t) + b(t)e(t) + c(t)h(t) - d(t)g(t)]\mathbf{i}$$

$$+[a(t)g(t) - b(t)h(t) + c(t)e(t) + d(t)f(t)]\mathbf{j}$$

$$+[a(t)h(t) + b(t)g(t) - c(t)f(t) + d(t)e(t)]\mathbf{k}$$

3. $[a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}]^* = a(t) - b(t)\mathbf{i} - c(t)\mathbf{j} - d(t)\mathbf{k}$
4. $[a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}][a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}]^*$

$$= |a(t) + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}| = a(t)^2 + b(t)^2 + c(t)^2 + d(t)^2$$

## 2.3 The form of a quaternion polynomial

Since some theorem demand a quaternion polynomial $\mathcal{C}(t)$ be in a specific basis, we will also note that any quaternion polynomial $\mathcal{C}(t)$ of degree N can be written out in two equally valid forms - the monomial and Bernstein bases. Certain aspects of our study are much easier in one of those forms rather than the other.

**Lemma 5.** *If we take the Bernstein basis expansion $\sum_{i=0}^{N} \mathcal{B}_i \binom{N}{i} t^i (1-t)^{N-i}$ and set it equal to monomial series expansion $\sum_{i=0}^{N} \mathcal{A}_i t^i$, then*

$$\begin{bmatrix} \mathcal{A}_N \\ \vdots \\ \mathcal{A}_0 \end{bmatrix} = T \begin{bmatrix} \mathcal{B}_N \\ \vdots \\ \mathcal{B}_0 \end{bmatrix},$$

*where $T$ is an invertible triangular matrix.*
*These matrix multiplication can be thought of as a change of coordinate system we use to view our polynomials and the elements of the matricies are*

$$t_{ij} = (-1)^{i+j}\binom{N}{j-i}\binom{N-j+1}{i-j}$$

$$t_{ij}^{-1} = \frac{\binom{i-1}{j-1}}{\binom{N}{j-1}}.$$

*Proof.* By studying the coefficients that appear when we multiply out the Bernstein basis into the monomial basis we get the equalities

$$\mathcal{A}_n = \sum_{i=0}^{n} \mathcal{B}_i (-1)^{n+i}\binom{n}{i}\binom{N}{n}.$$

Thus we get $t_{ij}$ for the diagonal and above, because we start indices at 1 and not 0 and count the other way. Because elements below the diagonal include binomial coefficients where the bottom number is the greater of the pair, they are zero and so we can extend the definition to the whole matrix. Outside those elements, we have products of a power of -1 and two binomial coefficients that are non-zero along the diagonal, so the determinant is not zero and the matrix is invertable. Observe that the $ik^{th}$ element of the matrix resulting from multiplying matricies with elements $t_{ij}$ and $t_{jk}^{-1}$ is equal to

$$\sum_{j=i}^{k}(-1)^{i+j}\frac{\binom{N}{j-i}\binom{N-j+1}{i-j}\binom{j-1}{k-1}}{\binom{N}{k-1}}.$$

This can be written as

$$\frac{(-1)^i(N-k+1)!}{(N-i+1)!}\sum_{j=i}^{k}\frac{(-1)^j}{(i-j)!(j-k)!},$$

if we are willing to define an inverse of a negative integer's factorial to be 0 through the Gamma function, because $\lim_{x\to y\in\mathcal{Z}^-}\frac{1}{\Gamma(x+1)} = 0$. Now for $k > i$ we have a series full of zeroes. For $k < i$, we have an empty sum and for $k = i$ we have one term equal to one. We can therefore see that our inverse matrix is correct. □

**Example 6.** *If $N = 2$ and we have a quaternion polynomial in the form $\mathcal{B}_2 t^2 + \mathcal{B}_1 2t(1-t) + \mathcal{B}_0(1-t)^2$, then the matrix $T = \begin{bmatrix}1 & -2 & 1\\ 0 & 2 & -2\\ 0 & 0 & 1\end{bmatrix}$ and we can rewrite it as $(\mathcal{B}_2 - 2\mathcal{B}_1 + \mathcal{B}_0)t^2 + (2\mathcal{B}_1 - 2\mathcal{B}_0)t + \mathcal{B}_0$. On the other hand if the polynomial is in the form $\mathcal{A}_2 t^2 + \mathcal{A}_1 t + \mathcal{A}_0$, then the inverse matrix $T^{-1} = \begin{bmatrix}1 & 1 & 1\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & 1\end{bmatrix}$ and we can also write it as $(\mathcal{A}_2 + \mathcal{A}_1 + \mathcal{A}_0)t^2 + (\frac{1}{2}\mathcal{A}_1 + \mathcal{A}_0)2t(1-t) + \mathcal{A}_0(1-t)$.*

## 2.4 Spherical Motions

**Definition 7.** *A 3x3 matrix* $\mathcal{M}(t) = (\mathbf{e}_1(t)|\mathbf{e}_2(t)|\mathbf{e}_3(t))$*, which is orthonormal, is called spherical motion. The trajectory of a point* $x$ *on the unit sphere* $\mathcal{S}_2$ *undergoing the motion* $\mathcal{M}(t)$ *is the set of all points for which a real number* $t$ *exists so that the point can be interpreted as* $\mathcal{M}(t)x$*. Each such point is usually referred to as* $x(t)$*. If the trajectory in question is a rational curve for all inputs* $x$*, then the motion* $\mathcal{M}(t)$ *is called a rational spherical motion (RSM). Often it is assumed* $\mathcal{M}(0) = \mathcal{I}_3$*, so* $x(0) = x$ *is the case for all* $x$*.*

Note that the tangent, normal and binormal from 13 are an example of a spherical motion, since they are an triplet of orthogonal vectors. However since they require a curve without inflection point, they may not be defined for all $t$.

**Example 8.** *Taking* $\mathcal{M}(t) = \begin{bmatrix} sin(t+1)sin(t) & cos(t+1) & sin(t+1)cos(t) \\ cos(t+1)sin(t) & -sin(t+1) & cos(t+1)cos(t) \\ cos(t) & 0 & -sin(t) \end{bmatrix}$, *where* $\mathcal{M}(t)\mathcal{M}(t) = \mathcal{I}_3$ *and so* $det(\mathcal{M}(t)) = 1$*, we can see a spherical motion. The projected paths of the canonical basis vectors clearly follow the surface of the sphere, so we can see* $\mathcal{M}(t)$ *is a spherical motion. Since sines and cosines are not rational functions, this is not a rational spherical motion.*

**Example 9.** *This time we will take the more complicated matrix*
$\mathcal{M}(t) = \frac{1}{4t^2+20t+30} \begin{bmatrix} 8t+20 & 4 & 4t^2+20t+22 \\ 4t^2+20t+20 & 4t+10 & -8t-20 \\ -4t-10 & 4t^2+20t+28 & 4 \end{bmatrix}$,
*which is a rational spherical movement as seen from the facts that we used polymials and the four basic operations and that* $det(\mathcal{M}(t)) = 1$ *always.*

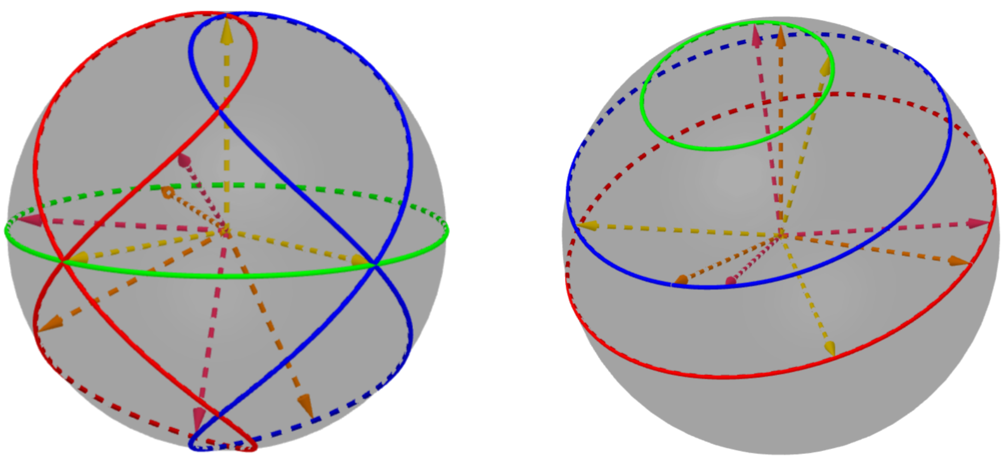

**Figure 2.2:** Example 8 of a spherical motion on the left and Example 9 of a rational spherical motion on the right. Note that the paths of vectros under rational spherical motion need not be circles.

# 3. Rotation-minimizing Motions

**Definition 10.** *We say that a spherical motion* $\mathcal{M}(t) = (\mathbf{e}_1(t)|\mathbf{e}_2(t)|\mathbf{e}_3(t))$ *is Rotation-minimizing if it satisfies*

$$\langle \mathbf{e}_2'(t), \mathbf{e}_3(t) \rangle = 0. \tag{3.1}$$

Notice that once we choose the trajectory of the first basis vector $\mathbf{e}_1$, we are quite constrained. Since $\mathbf{e}_2(t)$ is defined as being of unit length, $\langle \mathbf{e}_2'(t), \mathbf{e}_2(t) \rangle = 0$ too and so to create a rotation-minimizing spherical motion, it is required that

$$\mathbf{e}_2(s) = \frac{\int_{s_0}^{s} \lambda(t)\mathbf{e}_1(t)dt}{|| \int_{s_0}^{s} \lambda(t)\mathbf{e}_1(t)dt||}$$

$$\mathbf{e}_3(t) = \mathbf{e}_1(t) \times \mathbf{e}_2(t),$$

where $\lambda(t)$ is a real function.

**Example 11.** *This is a trivial example of a rotation minimizing motion, which keeps* $\mathbf{e}_2(t)$ *fixed. As a result the other two vector follow circular paths.* $\mathcal{M}(t) = \begin{bmatrix} sin(t) & 0 & cos(t) \\ \frac{4}{5}cos(t) & \frac{3}{5} & \frac{4}{5}sin(t) \\ -\frac{3}{5}cos(t) & \frac{4}{5} & -\frac{3}{5}sin(t) \end{bmatrix}$. *Because* $\mathbf{e}_2'(t)$ *is a zero vector, we get* $\langle \mathbf{e}_2'(t), \mathbf{e}_3(t) \rangle = 0$.

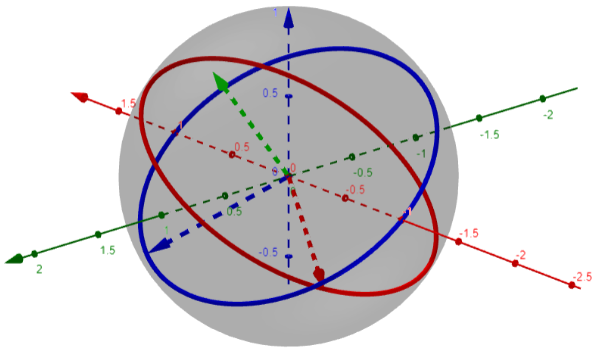

**Figure 3.1:** Example 11: Here we see a Rotation Minimizing Motion

In this work the study of Rotation-minimizing Frames (RMFs) will be done through Quaternion polynomials. First we note the exact way RMFs connect together with quaternion polynomials. This is described by the following theorem:

**Theorem 12.** *Every rational spherical motion* $(\mathbf{e}_1(t)|\mathbf{e}_2(t)|\mathbf{e}_3(t))$ *can be written in terms of a quaternion polynomial* $\mathcal{C}(t)$ *as*

$$(\mathbf{e}_1(t)|\mathbf{e}_2(t)|\mathbf{e}_3(t)) = \frac{\mathcal{C}(t)(\mathbf{i}|\mathbf{j}|\mathbf{k})\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*}. \tag{3.2}$$

*This quaternion polynomial is called the motion polynomial of the RSM. Moreover, the RSM is rotation-minizing if and only if*

$$\langle \mathcal{C}(t)'\mathbf{i}, \mathcal{C}(t) \rangle = 0 \tag{3.3}$$

*Proof.* Recall that we define scalar products as in a 4D Eucleidian vector space. That all spherical motion have a corresponding polynomial was first proven by Bert Jüttler in his work: [1]. Because for any $\mathcal{C}(t)$ as mentioned in the prelude,

$$f[X] = \frac{\mathcal{C}(t)[X]\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*}$$

is a rotational function in 3D space, the result of (3.2) is an orthonormal matrix. The $\mathcal{C}(t)\mathcal{C}(t)^*$ in the divisor is a squared norm of the polynomial. Using direct computation and Lemma 4 we now check that (3.1) is satisfied.

$$\langle \mathbf{e}_2'(t), \mathbf{e}_3(t)\rangle = \langle \left(\frac{\mathcal{C}(t)\mathbf{j}\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*}\right)', e_3(t)\rangle =$$

$$= \frac{\langle \mathcal{C}(t)\mathcal{C}(t)^*\mathcal{C}'(t)\mathbf{j}\mathcal{C}(t)^* + \mathcal{C}(t)\mathcal{C}(t)^*\mathcal{C}(t)\mathbf{j}\mathcal{C}'(t)^*, e_3(t)\rangle}{||\mathcal{C}(t)||^4}$$
$$- \frac{\langle \mathcal{C}(t)\mathbf{j}\mathcal{C}(t)^*\mathcal{C}'(t)\mathcal{C}(t)^* + \mathcal{C}(t)\mathbf{j}\mathcal{C}(t)^*\mathcal{C}(t)\mathcal{C}'(t)^*, e_3(t)\rangle}{||\mathcal{C}(t)||^4} =$$
$$= \frac{\langle \mathcal{C}'(t)\mathcal{C}(t)^*e_2(t) + e_2(t)\mathcal{C}(t)\mathcal{C}'(t)^* - e_2(t)\mathcal{C}'(t)\mathcal{C}(t)^* - e_2(t)\mathcal{C}(t)\mathcal{C}'(t)^*, e_3(t)\rangle}{||\mathcal{C}(t)||^2} =$$
$$= \frac{\langle \mathcal{C}'(t)\mathcal{C}(t)^*e_2(t) - e_2(t)\mathcal{C}'(t)\mathcal{C}(t)^*, e_3(t)\rangle}{||\mathcal{C}(t)||^2} =$$
$$= \frac{\langle \mathcal{C}'(t)\mathcal{C}(t)^*e_2(t), e_3(t)\rangle}{||\mathcal{C}(t)||^2} - \frac{\langle e_2(t)\mathcal{C}'(t)\mathcal{C}(t)^*, e_3(t)\rangle}{||\mathcal{C}(t)||^2} =$$
$$= \frac{\langle \mathcal{C}'(t)\mathbf{i}e_2(t), \mathcal{C}(t)^*e_3(t)\rangle}{||\mathcal{C}(t)||^2} + \frac{\langle e_2(t)\mathcal{C}'(t)\mathbf{i}, e_3(t)\mathcal{C}(t)^*\rangle}{||\mathcal{C}(t)||^2} =$$
$$= \frac{2\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle}{||\mathcal{C}(t)||^2} = \frac{2\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle}{\langle \mathcal{C}(t), \mathcal{C}(t)\rangle}.$$

Therefore the definition of being rotation-minimizing implies condition (3.3). □

**Definition 13.** *We say that a rational curve* $\mathbf{r}(t)$ *in* $\mathbb{R}^3$ *is a Pythagorean hodograph (PH) if there exists a rational spherical motion* $\mathbf{M}(t)$ *so that the vectors* $\{\mathbf{r}'(t), \mathbf{e}_1(t)\}$ *are linearily dependent. Moreover, we say that* $\mathbf{r}(t)$ *is a Rotation-minimizing Pythagorean hodograph (RMPH) curve if* $\mathbf{M}(t)$ *is rotation-minimizing.*

**Example 14.** *An example of a Pythagorean Hodograph is the rational curve*

$$x(t) = \frac{1}{4}t^4 - \frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t, y(t) = \frac{1}{2}t^4 + t^3 - \frac{3}{2}t^2 - 2t, z(t) = \frac{1}{2}t^4 + \frac{1}{3}t^3 - t^2 - t$$

*We then have*

$$r'(t) = (t^3 - 2t^2 - t + 2, 2t^3 + 3t^2 - 3t - 2, 2t^3 + t^2 - 2t - 1)$$

*and can use the Frenet-Serret frame*

$$\mathbf{M}(t) = \begin{bmatrix} \frac{-t^3+2t^2+t-2}{3t^3+2t^2-2t-3} & \frac{-8t^2-18t-7}{\sqrt{10}(3t^2+5t+3)^3} & \frac{12t^5+20t^4+3t^3-18t^2-14t-3}{\sqrt{10}(3t^3+2t^2-2t-3)(3t^2+5t+3)^3} \\ \frac{-2t^3-3t^2+3t+2}{3t^3+2t^2-2t-3} & \frac{5t^2-5}{\sqrt{10}(3t^2+5t+3)^3} & \frac{15t^5+40t^4+25t^3-25t^2-40t-15}{\sqrt{10}(3t^3+2t^2-2t-3)(3t^2+5t+3)^3} \\ \frac{-2t^3-t^2+2t+1}{3t^3+2t^2-2t-3} & \frac{-t^2-6t-4}{\sqrt{10}(3t^2+5t+3)^3} & \frac{-21t^5-50t^4-34t^3+29t^2+52t+24}{\sqrt{10}(3t^3+2t^2-2t-3)(3t^2+5t+3)^3} \end{bmatrix}$$

*as an example of the many rational spherical motions that satisfy* $\{\mathbf{r}'(t), \mathbf{e}_1(t)\}$.

**Lemma 15.** *This definion is equivalent to the classical definion of a Pythagorean hodograph, which requires that* $||\mathbf{r}'(t)||$ *is a piecewise rational function.*

*Proof.* Linear dependence of $\{\mathbf{r}'(t), \mathbf{e}_1(t)\}$ is equivalent to the existence of a piecewise rational function $\lambda(t)$, such that $\mathbf{r}'(t) = \mathbf{e}_1(t)\lambda(t)$. Since $\mathbf{e}_1(t)$ is always a unit vector, then

$$|\lambda(t)| = ||\mathbf{e}_1(t)|||\lambda(t)| = ||\mathbf{r}'(t)||$$

and $||\mathbf{r}'(t)||$ is not only piecewise rational, but outright equal to $|\lambda(t)|$. Going the other way we can use (3.2) and set

$$\frac{\mathcal{C}(t)\mathbf{i}\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*} = \mathbf{e}_1(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{\mathbf{r}'(t)}{\lambda(t)},$$

so we have a polynomial $\mathcal{C}$ to use in defining

$$\mathbf{e}_2(t) = \frac{\mathcal{C}(t)\mathbf{j}\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*}$$

$$\mathbf{e}_3(t) = \frac{\mathcal{C}(t)\mathbf{k}\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*}$$

□

**Example 16.** *Taking the previous example with* $x(t) = \frac{1}{4}t^4 - \frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t, y(t) = \frac{1}{2}t^4 + t^3 - \frac{3}{2}t^2 - 2t, z(t) = \frac{1}{2}t^4 + \frac{1}{3}t^3 - t^2 - t$, *we can see that when we use the Frenet-Serret frame we have* $|\lambda(t)| = ||\mathbf{r}'(t)|| = 3t^3 + 2t^2 - 2t - 3$.

We can use a pair of coprime real polynomials $a(t)$ and $b(t)$ to modify a polynomial $\mathcal{C}(t)$ in a way that allows us to create a RMF. The properties of this pair is described by the following lemma, which also includes an observation regarding this modification of our quaternion polynomial:

**Lemma 17.** *For any quaternion polynomial* $\mathcal{C}(t)$ *the curve defined by*

$$\mathbf{r}(s) := \int_{s_0}^{s} \lambda(t)\mathcal{C}(t)\mathbf{i}\mathcal{C}(t)^* dt \tag{3.4}$$

*is a Pythagorean Hodograph if and only if* $\lambda(t)$ *is a rational function, that insures the integral is also rational. Further it is also rotation minimizing if there is a pair of coprime real polynomials* $a(t)$ *and* $b(t)$ *that satisfy*

$$\frac{\langle \mathcal{C}(t)'\mathbf{i}, \mathcal{C}(t)\rangle}{\langle \mathcal{C}(t), \mathcal{C}(t)\rangle} = \frac{2\,(a(t)b'(t) - a'(t)b(t))}{a(t)^2 + b(t)^2} \tag{3.5}$$

*Proof.* The requirements on $\lambda(t)$, are necessary to avoid the appearence of irrational functions like natural logarithms and inverse trigonometric functions after integration, whose derivatives are often real functions themselves.
If (3.5) is satisfied, than we can define $\mathcal{D}(t) := \mathcal{C}(t)(a(t) - b(t)\mathbf{i})$ and by direct computation

$$\langle \mathcal{D}'(t)\mathbf{i}, \mathcal{D}(t)\rangle = \langle (\mathcal{C}(t)(a(t) - b(t)\mathbf{i}))'\,\mathbf{i}, \mathcal{C}(t)(a(t) - b(t)\mathbf{i})\rangle =$$

$$= \langle (\mathcal{C}(t)(a'(t) - b'(t)\mathbf{i}) + \mathcal{C}'(t)(a(t) - b(t)\mathbf{i}))\,\mathbf{i}, \mathcal{C}(t)(a(t) - b(t)\mathbf{i})\rangle =$$

$$= 2\,(a'(t)b(t) - a(t)b'(t))\,\langle \mathcal{C}(t), \mathcal{C}(t)\rangle + (a(t) + b(t)\mathbf{i})(a(t) - b(t)\mathbf{i})\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle =$$

$$= 2\,(a'(t)b(t) - a(t)b'(t))\,\langle \mathcal{C}(t), \mathcal{C}(t)\rangle + (a(t)^2 + b(t)^2)\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle.$$

Going one way we get $\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle = 0$ implies $\langle \mathcal{D}'(t)\mathbf{i}, \mathcal{D}(t)\rangle = 0$. Going the other way, we first assume that the polynomial $\mathcal{D}(t)$ exists and satisfies

$$\frac{\mathcal{C}(t)\mathbf{i}\mathcal{C}(t)^*}{\mathcal{C}(t)\mathcal{C}(t)^*} = \frac{\mathcal{D}(t)\mathbf{i}\mathcal{D}(t)^*}{\mathcal{D}(t)\mathcal{D}(t)^*} = \mathbf{e}_1(t),$$

then there must be a relationship of the form $\mathcal{D}(t) := \mathcal{C}(t)(a(t) - b(t)\mathbf{i})$, for some real polynomials $a(t)$ and $b(t)$. □

**Definition 18.** *If we have curve defined by* (3.4) *from* $\mathcal{C}(t)$ *of degree* $m$ *and its modifying polynomials* $a(t)$ *and* $b(t)$ *satisfy* (3.5) *and are individually of degrees* $n_a$ *and* $n_b$*, we can say that the resulting hodograph is of a class* $(m, max(n_a, n_b))$*.*

**Example 19.** *A hodograph made from* $\mathcal{C}(t) = (7 - 19\boldsymbol{i} - 26\boldsymbol{j} - 2\boldsymbol{k})t^2 + (-22 + 14\boldsymbol{i} + 16\boldsymbol{j} + 12\boldsymbol{k})t + 10$*,* $a(t) = 27t^2 - 22t + 10$ *and* $b(t) = -19t^2 + 14t$ *would thus be considered of class (2,2) as a result of a second degree Quaternion polynomial and two second degree modifying polynomials.*
*Meanwhile making a hodograph from* $\mathcal{C}(t) = 21t^2 + (21 - 21\boldsymbol{i} - 42\boldsymbol{j} - 42\boldsymbol{k})t - 142 - 63\boldsymbol{i} - 34\boldsymbol{j} + 94\boldsymbol{k}$*,* $a(t) = t - 2$ *and* $b(t) = 1$ *results in a class of (2,1), because the modifying polynomial are of classes 0 and 1 and the higher number takes precedence.*

# 4. Quaternions, which are already rotation minimized

**Definition 20.** *$\mathcal{F}_0^{(n)}$ is the set of all polynomials $\mathcal{C}(t) \in \mathbb{H}[t]$ of degree $n$ that satisfy* (3.3)*.*

This allows us to study all RMPHs that exist since all such hodographs are represented by $\mathcal{C}(t)$. We will now define a set of numbers, which we refer to as "Farouki's numbers," though we do not know of such a name existing outside this work and given it is derived from a article with five authors, we may have misattributed it. These appear as a part of Theorem 5.2 in the original paper:

**Definition 21.** *Let $\mathcal{C}(t) = \sum_{i=0}^{N} \mathcal{A}_i t^i$, then we define*

$$c_m^{(n)} := \sum_{k=0}^{m} (k+1)\langle \mathcal{A}_{m-k}, \mathcal{A}_{k+1}\mathbf{i}\rangle \tag{4.1}$$

*for any element of $\mathcal{F}_0^{(N)}$, where $n \in \{0, ..., N\}$ and $m \in \{0, ..., 2n-2\}$. These are the (n,m)-Farouki number of $\mathcal{C}(t)$.*

If we write Farouki's numbers for all $\mathcal{C}(t)$ of degree $N$ in order by first $n$ and them $m$ and allign those belonging to $m = 0$ in a collum, we get a triangle of definitions that is $N+1$ rows tall and $2N-2$ collums wide. Each row of definitions in this "Farouki triangle" then has a relationship with the preceding row that is described by the following lemma:

**Lemma 22.** *For any element of $\mathcal{F}_0^{(N)}$, where $n \in \{1, ..., N\}$ and $m \in \{0, ..., 2n-2\}$ we have*

1. *$c_m^{(n)} = c_m^{(n-1)}$ for $m \in \{0, ..., n-2\}$*
2. *$c_m^{(n)} = c_m^{(n-1)} + (2n-m-1)\langle \mathcal{A}_{m-n+1}, \mathcal{A}_n\mathbf{i}\rangle$ for $m \in \{n-1, ...., 2n-4\}$*
3. *$c_m^{(n)} = (2n-m-1)\langle \mathcal{A}_{m-n+1}, \mathcal{A}_n\mathbf{i}\rangle$ for $m \in \{2n-3, 2n-2\}$*

*Proof.* Because for m between 1 and $n-2$ the definition of $c_m^{(n)}$ doesn't include $\mathcal{A}_n$ in its defining sum we immediately get $c_m^{(n)} = c_m^{(n-1)}$.
Between $n-1$ and $2n-2$ we can observe

$$c_m^{(n)} = c_m^{(n-1)} + n\langle \mathcal{A}_{m-n+1}, \mathcal{A}_n\mathbf{i}\rangle + (m-n+1)\langle \mathcal{A}_n, \mathcal{A}_{m-n+1}\mathbf{i}\rangle =$$

$$= c_m^{(n-1)} + (2n-m-1)\langle \mathcal{A}_{m-n+1}, \mathcal{A}_n\mathbf{i}\rangle,$$

which follows from $\langle \mathcal{A}_{m-n+1}\mathbf{i}, \mathcal{A}_n\rangle = -\langle \mathcal{A}_{m-n+1}, \mathcal{A}_n\mathbf{i}\rangle$.
The last three are the simple equations $c_{2n-1}^{(n)} = n\langle \mathcal{A}_n, \mathcal{A}_n\mathbf{i}\rangle = 0$, $c_{2n-3}^{(n-1)} = (n-1)\langle \mathcal{A}_{n-1}, \mathcal{A}_{n-1}\mathbf{i}\rangle = 0$ and $c_m^{(n-1)} = 0$ for all m greater than $2n-2$. □

This triangle of numbers is important, because its last row can be used to determine whether $\mathcal{C}(t) \in \mathcal{F}_0^{(n)}$, by what values it takes:

**Theorem 23.** *$\mathcal{C}(t) \in \mathcal{F}_0^{(n)}$ is equivalent to*

$$c_m^{(n)} = \sum_{k=0}^{m} (k+1)\langle \mathcal{A}_{m-k}, \mathcal{A}_{k+1}\mathbf{i}\rangle = 0 \tag{4.1}$$

*for all $m \in \{1, ...2n-2\}$.*

*Proof.* Observe that demands of Lemma 17 imposed on

$$\langle \mathcal{C}'(t)\mathbf{i}, \mathcal{C}(t)\rangle = \langle \sum_{j=0}^{N} j\mathcal{A}_j \mathbf{i} t^{j-1}, \sum_{j=0}^{n} \mathcal{A}_j t^j \rangle = \sum_{i=0}^{2n-1} c_i^{(n)} t^i,$$

imply that $\mathcal{C}(t)$ being in the set $\mathcal{F}_0^{(N)}$ is equivalent $c_m^{(N)} = 0$ for all m. □

## 4.1 Degree 3

**Theorem 24.** *The non-trivial elements of $\mathcal{F}_0^{(3)}$ are those polynomials $\mathcal{C}(t) = \mathcal{C}(\mathcal{A}_3 t^3 + \mathcal{A}_2 t^2 + \mathcal{A}_1 t + 1)$, where the following properties are satisfied:*

1. *$\mathcal{C}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in \mathbb{F}$ and $\mathcal{C}$ is nonzero*
2. *the span of 1, $\mathcal{A}_1$ and $\mathcal{A}_2$ is $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$*
3. *the vector part of $\mathcal{A}_3$ is a pure vector, parallel to the vector product $(\mathcal{A}_1\boldsymbol{i}) \times (\mathcal{A}_2\boldsymbol{i})$, whose component along $\boldsymbol{i}$ is $\frac{1}{3}\langle \mathcal{A}_1, \mathcal{A}_2\boldsymbol{i}\rangle$*

*A monic polynomial $t^3 + \mathcal{A}_2 t^2 + \mathcal{A}_1 t + \mathcal{A}_0$, where $\mathcal{A}_0 \in \mathbb{F}$, is a non-trivial elements of $\mathcal{F}_0^{(3)}$ if and only if it satisfies the first two points and $\mathcal{A}_0$ is a pure vector, parallel to the vector product $(\mathcal{A}_1\boldsymbol{i}) \times (\mathcal{A}_2\boldsymbol{i})$, whose component along $\boldsymbol{i}$ is $-\frac{1}{3}\langle \mathcal{A}_1, \mathcal{A}_2\boldsymbol{i}\rangle$.*

*Proof.* We need to make sure Farouki's numbers fullfil $c_0^{(3)} = c_1^{(3)} = c_2^{(3)} = c_3^{(3)} = c_4^{(3)} = 0$ This means that $\langle \mathbf{i}, \mathcal{A}_1\rangle$, $\langle \mathbf{i}, \mathcal{A}_2\rangle$, $\langle \mathcal{A}_1, \mathcal{A}_2\mathbf{i}\rangle - 3\langle \mathbf{i}, \mathcal{A}_3\rangle$, $\langle \mathcal{A}_1\mathbf{i}, \mathcal{A}_3\rangle$ and $\langle \mathcal{A}_2\mathbf{i}, \mathcal{A}_3\rangle$ are all 0. All of them being equal to zero is equivalent to the following three properties, which all memebers of $\mathcal{F}_0^{(3)}$ most therefore satisfy:

1. $\mathcal{A}_1$ and $\mathcal{A}_2$ are orthogonal to $\mathbf{i}$, meaning they are members of $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$
2. the component of $\mathcal{A}_3$ along $\mathbf{i}$ is equal to $\frac{1}{3}\langle \mathcal{A}_1, \mathcal{A}_2\mathbf{i}\rangle$
3. $\mathcal{A}_3$ is orthogonal to both $\mathcal{A}_1\mathbf{i}$ and $\mathcal{A}_2\mathbf{i}$

Simply adding the second property, that the span of 1, $\mathcal{A}_1$ and $\mathcal{A}_2$ is $\mathbb{R}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$, gives us non-triviality. The first property of the theorem is also mainly intended to prevent triviality. The third property of theorem is the fusion of the second and third general properties just proven, and therefore is actually true for all members of $\mathcal{F}_0^{(3)}$ regardless of triviality.
We can use $t^3 + \mathcal{C}_2 t^2 + \mathcal{C}_1 t + \mathcal{C}_0 = \mathcal{C}_0(\mathcal{C}_0^{-1} t^3 + \mathcal{C}_2\mathcal{C}_0^{-1} t^2 + \mathcal{C}_1\mathcal{C}_0^{-1} t + 1)$, to see how the non-triviality condition applies to the monic polynomial. □

**Corollary 25.** *By using Lemma 5 we get $\mathcal{A}_3 = \mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1 - 1$, $\mathcal{A}_2 = 3\mathcal{B}_2 - 6\mathcal{B}_1 + 3$, $\mathcal{A}_1 = 3\mathcal{B}_1 - 3$. This allows us to write a version of the Theorem for when the polynomial is written as $\mathcal{C}$ times a polynomial in the Bernstein basis. The three properties become:*

1. $\mathcal{C}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in \mathbb{F}$ *and* $\mathcal{C}$ *is nonzero*
2. *the span of* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *is* $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$
3. *the vector part of* $\mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1$ *is a pure vector, parallel to the vector product* $(\mathcal{B}_1 - 1)\boldsymbol{i} \times (\mathcal{B}_2 - 2\mathcal{B}_1 + 1)\boldsymbol{i}$*, whose component along* $\boldsymbol{i}$ *is* $3\langle(\mathcal{B}_1 - 1), (\mathcal{B}_2 - 2\mathcal{B}_1 + 1)\boldsymbol{i}\rangle$

*For the monic polynomial we can use the lemma again and get* $\mathcal{B}_0$ *must be a pure vector, parallel to the vector product* $((\mathcal{B}_1 - \mathcal{B}_0)\boldsymbol{i} \times (\mathcal{B}_2 - 2\mathcal{B}_1 + \mathcal{B}_0)\boldsymbol{i}$*, whose component along* $\boldsymbol{i}$ *is* $-3\langle(\mathcal{B}_1 - \mathcal{B}_0), (\mathcal{B}_2 - 2\mathcal{B}_1 + \mathcal{B}_0)\boldsymbol{i}\rangle$*.*

## 4.2 Degree 4

Because according to the lemma, some conditions are indentical for degree 4, we find the relevant theorem partly matches. Specifically $c_0^{(4)}, c_1^{(4)}$ and $c_2^{(4)}$ have the exact same definition as $c_0^{(3)}, c_1^{(3)}$ and $c_2^{(3)}$.

**Theorem 26.** *The elements of* $\mathcal{F}_0^{(4)}$ *are those polynomials* $\mathcal{C}(t) = \mathcal{C}(\mathcal{A}_4 t^4 + \mathcal{A}_3 t^3 + \mathcal{A}_2 t^2 + \mathcal{A}_1 t + 1)$*, where the following properties are satisfied:*

1. $\mathcal{C}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \in \mathbb{F}$ *and* $\mathcal{C}$ *is nonzero*
2. $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$
3. *the component of* $\mathcal{A}_3$ *along* $\boldsymbol{i}$ *is* $\frac{1}{3}\langle\mathcal{A}_1, \mathcal{A}_2\boldsymbol{i}\rangle$
4. $\mathcal{A}_4$ *is orthogonal to* $\mathcal{A}_2\boldsymbol{i}$ *and* $\mathcal{A}_3\boldsymbol{i}$ *and its component along* $\boldsymbol{i}$ *is* $\frac{1}{2}\langle\mathcal{A}_1, \mathcal{A}_3\boldsymbol{i}\rangle$
5. $3\langle\mathcal{A}_1, \mathcal{A}_4\boldsymbol{i}\rangle = \langle\mathcal{A}_2, \mathcal{A}_3\boldsymbol{i}\rangle$

*Elements* $\mathcal{C}(t)$ *of* $\mathcal{F}_0^{(4)}$ *are non-trivial if and only if exactly ONE of the following conditions is satisfied:*

1. *the span of 1,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *is* $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$
2. *the span of 1,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *is a plane* $\mathbb{R} + \mathbb{R}u$ *for some unit vector* $u$ *that is orthogonal to* $\boldsymbol{i}$
3. *the span of 1,* $\mathcal{A}_1$*,* $\mathcal{A}_2$ *and* $\mathcal{A}_3$ *is still* $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$

*Proof.* The five properties and the first of the non-trivial triplet come from avoiding obvious triviality and from Farouki's numbers of which there are seven, when studying $\mathcal{F}_0^{(4)}$. Here again the span of 1, $\mathcal{A}_1$ and $\mathcal{A}_2$ being $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ gives us non-triviality.
If the span of 1, $\mathcal{A}_1$ and $\mathcal{A}_2$ is instead included in some plane $\mathbb{R} + \mathbb{R}u$ for some unit vector $u$ that is orthogonal to $\mathbf{i}$, then the second Farouki number, which implies $\mathcal{A}_3$ and $\mathbf{i}$ and orthogonal, also implies $\mathcal{A}_3$ is either an element of this plane or the if it joins our three elements the new span of 1, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ is still $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. If it is the element of the plane, we are forced into non-triviality. However the latter along with the third through sixth Farouki's numbers implies triviality instead. From here we can be the unique form of theorem - a triplet of non-trivialising conditions that destructively intefere. □

**Corollary 27.** *By using Lemma 5 again we get* $\mathcal{A}_4 = \mathcal{B}_4 - 4\mathcal{B}_3 + 6\mathcal{B}_2 - 4\mathcal{B}_1 + 1$*,* $\mathcal{A}_3 = 4\mathcal{B}_3 - 12\mathcal{B}_2 + 12\mathcal{B}_1 - 4$*,* $\mathcal{A}_2 = 6\mathcal{B}_2 - 12\mathcal{B}_1 + 6$*,* $\mathcal{A}_1 = 4\mathcal{B}_1 - 4$*. This allows us* *to write a version of the Theorem for when the polynomial is written as $\mathcal{C}$ times a polynomial in the Bernstein basis. The five properties of the first list become:*

1. $\mathcal{C}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \in \mathbb{F}$ *and* $\mathcal{C}$ *is nonzero*
2. *the span of* $\mathcal{B}_1 - 1$ *and* $\mathcal{B}_2 - 2\mathcal{B}_1 + 1$ *is* $\mathbb{R} + \mathbb{R}j + \mathbb{R}\mathbf{k}$
3. *the component of* $\mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1 - 1$ *along* $\boldsymbol{i}$ *is* $2\langle \mathcal{B}_1 - 1, (\mathcal{B}_2 - 2\mathcal{B}_1 + 1)\boldsymbol{i}\rangle$
4. $\mathcal{B}_4 - 4\mathcal{B}_3 + 6\mathcal{B}_2 - 4\mathcal{B}_1 + 1$ *is orthogonal to* $(\mathcal{B}_2 - 2\mathcal{B}_1 + 1)\boldsymbol{i}$ *and* $(\mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1 - 1)\boldsymbol{i}$ *and its component along* $\boldsymbol{i}$ *is* $8\langle \mathcal{B}_1 - 1, (\mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1 - 1)\boldsymbol{i}\rangle$
5. $\langle \mathcal{B}_1 - 1, (\mathcal{B}_4 - 4\mathcal{B}_3 + 6\mathcal{B}_2 - 4\mathcal{B}_1 + 1)\boldsymbol{i}\rangle = 2\langle \mathcal{B}_2 - 2\mathcal{B}_1 + 1, (\mathcal{B}_3 - 3\mathcal{B}_2 + 3\mathcal{B}_1 - 1)\boldsymbol{i}\rangle$

*The three properties of the second list become:*

1. *the span of 1,* $\mathcal{B}_1 - \mathcal{B}_0$ *and* $\mathcal{B}_2 - \mathcal{B}_0$ *is* $\mathbb{R} + \mathbb{R}j + \mathbb{R}\mathbf{k}$
2. *and span of 1,* $\mathcal{B}_1 - \mathcal{B}_0$ *and* $\mathcal{B}_2 - \mathcal{B}_0$ *is a plane* $\mathbb{R} + \mathbb{R}u$ *for some unit vector* $u$ *that is orthogonal to* $\boldsymbol{i}$
3. *the span of 1,* $\mathcal{B}_1 - \mathcal{B}_0$*,* $\mathcal{B}_2 - \mathcal{B}_0$ *and* $\mathcal{B}_3 - \mathcal{B}_0$ *is still* $\mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$

**Theorem 28.** *For all* $n \geq 5$*,* $\mathcal{F}_0^{(n)}$ *always contains at least one non-trivial element.*

*Proof.* $\mathcal{C}(t) = (n-2)\mathbf{i}t^n + n\mathbf{k}t^{n-1} + \mathbf{j}t + 1$ is an example of one, because we always get $\langle (2-n)nt^{n-1} + n(n-1)\mathbf{j}t^{n-2} - \mathbf{k}, (n-2)\mathbf{i}t^n + n\mathbf{k}t^{n-1} + \mathbf{j}t + 1\rangle = 0$. □

## 4.3 Observations

There are several observation we have, when it comes to finding $\mathcal{C}(t) \in \mathcal{F}_0^{(N)}$ of degree $N$. If we take $\mathcal{C}(t) = \mathcal{C}(\sum_{i=1}^{N} \mathcal{A}_i t^i + 1)$, then we always have the conditions $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ and $\mathcal{A}_N$ is perpendicular to $\mathcal{A}_{N-2}\mathbf{i}$ and $\mathcal{A}_{N-1}\mathbf{i}$. If we wish to find the coefficients $\mathcal{A}_n$, then we probably should not set more then $N - \lceil \frac{N-1}{4} \rceil$, because we have $N+1$ quaternion coefficients and $2N-2$ conditions. This may be an area of potential research.

Also if we have $\mathcal{C}(t) \in \mathcal{F}_0^{(N)}$ and wish to find $\mathcal{C}(t) + \mathcal{A}_{N+1}t^{N+1} \in \mathcal{F}_0^{(N+1)}$, then we are always required to satisfy $N+1$ new conditions (per the lemma the $N-1$ others are already satisfied). This seems to imply a possibility of always finding $\mathcal{A}_{N+1}$ for N $\leq 3$, but beyond it this is unlikely in general and possibly at all. This could be another avenue of research into characterizing RMPHs.

Before our work moved in a more general direction, we considered whether what we called Farouki's triangle may have some relationship to functions that seemingly emulate Pascal's triangle or some degenerate form of those. We aren't certain what research into this area could uncover, but it is a possibility for future research as well.

# 5. Characterisation of RMPHs of Class (2,2)

**Theorem 29.** *If we have a quaternion polynomial*

$$\mathcal{C}(t) = \mathcal{B}_2 t^2 + 2\mathcal{B}_1 t(1-t) + \mathcal{B}_0(1-t)^2$$

*with non-zero $\mathcal{B}_0$, then a RMPH of degree (2,2) exists if these Bernstein basis coefficients satisfy*

$$2\mathcal{B}_1 \boldsymbol{i} \mathcal{B}_1^* = \mathcal{B}_2\mathcal{B}_0^* - \mathcal{B}_0\mathcal{B}_2^* = 2vect(\mathcal{B}_2 \boldsymbol{i} \mathcal{B}_0^*). \tag{5.1}$$

*Proof.* Since (5.1) does not change, when we apply multiplication by a quaternion, and $\mathcal{B}_0^{-1} = \frac{\mathcal{B}_0^*}{|\mathcal{B}_0|^2}$ is defined, we can make use of

$$\mathcal{B}_2 t^2 + \mathcal{B}_1 t(1-t) + \mathcal{B}_0(1-t)^2 = \left(\mathcal{B}_2\mathcal{B}_0^{-1}t^2 + \mathcal{B}_1\mathcal{B}_0^{-1}t(1-t) + (1-t)^2\right)\mathcal{B}_0$$

and without loss of generality only study instances of second degree polynomials, where $\mathcal{B}_0 = 1$. Now (5.1) says that $\mathcal{B}_1 \boldsymbol{i} \mathcal{B}_1^*$ equals the vector part of $\mathcal{B}_2\mathbf{i}$. Let's take $\mathcal{B}_1$ freely and real numbers $U, V, P, Q$, such that

$$\mathcal{B}_1 = U + V\mathbf{i} + P\mathbf{j} + Q\mathbf{k}.$$

Now we also take a real number $X$, so that

$$X = \mathcal{B}_1\mathbf{i}\mathcal{B}_1^* - \mathcal{B}_2\mathbf{i}.$$

We can then derive that our condition demands that $\mathcal{B}_2$ must be of the form

$$\mathcal{B}_2 = (X - \mathcal{B}_1\mathbf{i}\mathcal{B}_1^*)\mathbf{i} = (U^2 + V^2 - P^2 - Q^2) + X\mathbf{i} + 2(UP - VQ)\mathbf{j} + 2(UQ - VP)\mathbf{k}$$

and so we need only to find a pair of polynomials that satisfy (3.5) to prove the implication in this direction. The pair that does this is:

$$a(t) = ||\mathcal{B}_1||^2 t^2 + 2U(1-t)t + (1-t)^2$$
$$b(t) = Xt^2 + 2X(1-t)t$$

□

**Corollary 30.** *By using our Lemma 5 on polynomial forms and its inverse matrix for degree 2 we get $\mathcal{B}_2 = \mathcal{A}_2 + \mathcal{A}_1 + \mathcal{A}_0$, $\mathcal{B}_1 = \frac{1}{2}\mathcal{A}_1 + \mathcal{A}_0$ and $\mathcal{B}_0 = \mathcal{A}_0$. This allows us to write a version of the Theorem for $\mathcal{C}(t)$, when written in the monomial basis with a non-zero $\mathcal{A}_0$. After clearing the fraction, expanding and gathering, the condition takes the form*

$$\mathcal{A}_1 \boldsymbol{i} \mathcal{A}_1^* + 2(\mathcal{A}_0 \boldsymbol{i} \mathcal{A}_1^* + \mathcal{A}_1 \boldsymbol{i} \mathcal{A}_0^*) + 4\mathcal{A}_0 \boldsymbol{i} \mathcal{A}_0^* =$$
$$= 2(\mathcal{A}_2\mathcal{A}_0^* - \mathcal{A}_0\mathcal{A}_2^*) + 2(\mathcal{A}_1\mathcal{A}_0^* - \mathcal{A}_0\mathcal{A}_1^*)[+2(\mathcal{A}_0\mathcal{A}_0^* - \mathcal{A}_0^*\mathcal{A}_0)] =$$
$$= 4vect(\mathcal{A}_2 \boldsymbol{i} \mathcal{A}_0^*) + 4vect(\mathcal{A}_1 \boldsymbol{i} \mathcal{A}_0^*)[+4vect(\mathcal{A}_0 \boldsymbol{i} \mathcal{A}_0^*)]$$

*The modifying polynomials are*

$$a(t) = (1-t)^2 + scal[\mathcal{A}_1 + 2\mathcal{A}_0](1-t)t + ||\frac{1}{2}\mathcal{A}_1 + \mathcal{A}_0||t^2$$
$$b(t) = -2scal[(\mathcal{A}_2 + \mathcal{A}_1 + \mathcal{A}_0)\boldsymbol{i}](1-t)t - scal[(\mathcal{A}_2 + \mathcal{A}_1 + \mathcal{A}_0)\boldsymbol{i}]t^2$$

# 6. Gallery of RMFs

These four example were previously presented in works [2] and [4].

**Example 31.** *If we take* $\mathcal{C}(t) = t^2 + (-1 + 2\boldsymbol{i} - 2\boldsymbol{k})t - \frac{64}{41} + \frac{28}{41}\boldsymbol{j} + \frac{50}{41}\boldsymbol{k}$*, we can use the pair* $a(t) = t - 1$ *and* $b(t) = 2$ *can transform this into a hodograph:*

$$x'(t) = t^4 - 2t^3 + \frac{87}{41}t^2 + 8t + \frac{812}{1681}$$

$$y'(t) = -4t^3 + \frac{264}{41}t^2 + \frac{268}{41}t - \frac{6400}{1681}$$

$$z'(t) = -\frac{384}{41}t^2 + \frac{256}{41}t + \frac{3584}{1681}$$

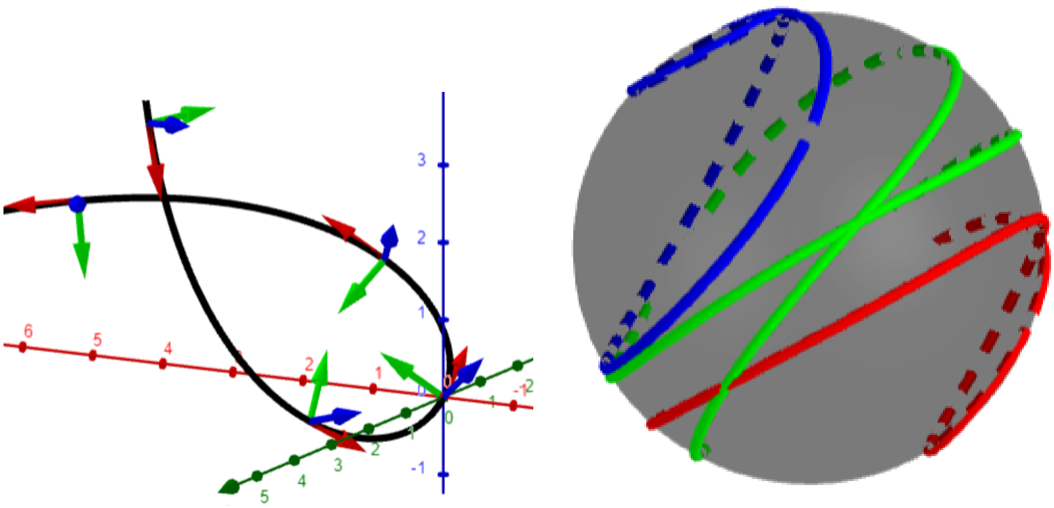

**Figure 6.1:** Example 31: On the left we have the curve with five example frames. On the right we see the Unit Sphere and the curves traced by frames

**Example 32.** *If we take* $\mathcal{C}(t) = t^2 + (-2 + \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})t - \frac{10}{9} - \frac{25}{9}\boldsymbol{j} - \frac{20}{9}\boldsymbol{k}$*, we can use the pair* $a(t) = t - 2$ *and* $b(t) = 1$ *can transform this into a hodograph:*

$$x'(t) = t^4 - 4t^3 - \frac{20}{9}t^2 + 20t + \frac{925}{81}$$

$$y'(t) = 2t^3 - \frac{40}{9}t^2 + \frac{10}{9}t + \frac{400}{81}$$

$$z'(t) = -4t^3 + \frac{140}{9}t^2 - \frac{100}{9}t - \frac{500}{81}$$

.

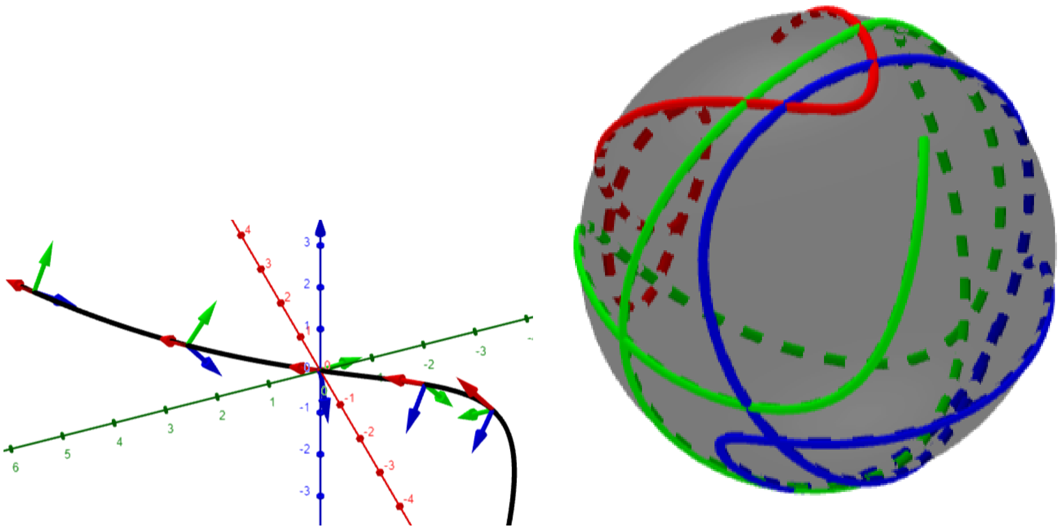

**Figure 6.2:** Example 32: On the left we have the curve with five example frames. On the right we see the Unit Sphere and the curves traced by frames

**Example 33.** *If we take* $\mathcal{C}(t) = 21t^2 + (21 - 21\boldsymbol{i} - 42\boldsymbol{j} - 42\boldsymbol{k})t - 142 - 63\boldsymbol{i} - 34\boldsymbol{j} + 94\boldsymbol{k}$*, we can use the pair* $a(t) = t - 2$ *and* $b(t) = 1$ *can transform this into a hodograph:*

$$x'(t) = 441t^4 + 882t^3 - 8610t^2 + 7434t + 14141$$

$$y'(t) = -3528t^2 - 25032t + 60496$$
$$z'(t) = -1764t^3 + 4956t^2 + 11844t - 21500$$

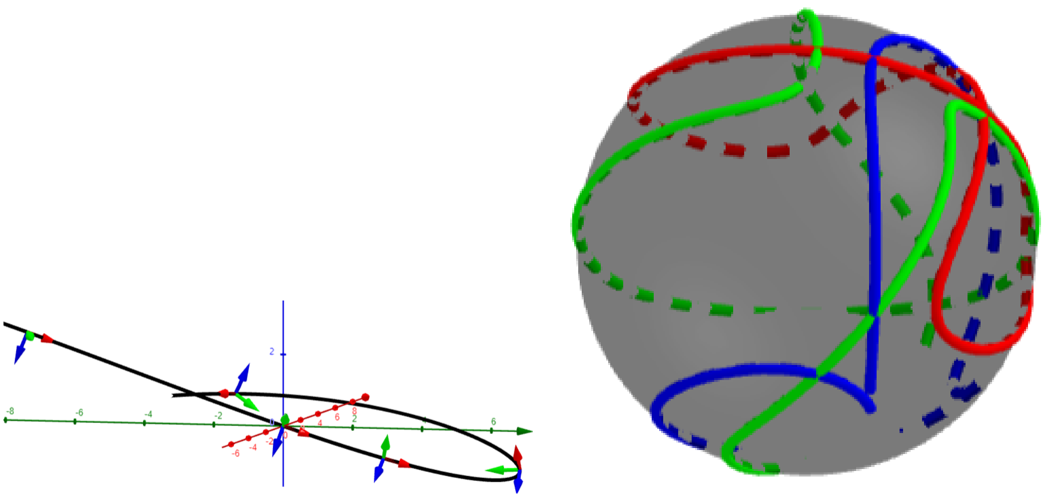

**Figure 6.3:** Example 33: On the left we have the curve with five example frames. The actual hodograph has been scaled down by 10000, so the frames are visible. On the right we see the Unit Sphere and the curves traced by frames

**Example 34.** *If we take* $\mathcal{C}(t) = (7-19\boldsymbol{i}-26\boldsymbol{j}-2\boldsymbol{k})t^2+(-22+14\boldsymbol{i}+16\boldsymbol{j}+12\boldsymbol{k})t+10$, *we can use the pair* $a(t) = 27t^2 - 22t + 10$ *and* $b(t) = -19t^2 + 14t$ *can transform this into a hodograph:*

$$x'(t) = -631t^4 - 260t^3 - 327t^2 - 440t - 156$$

$$y'(t) = 960t^4 - 1080t^3 - 120t^2 + 240t$$

$$z'(t) = 440t^4 - 1880t^3 + 1560t^2 - 320t$$

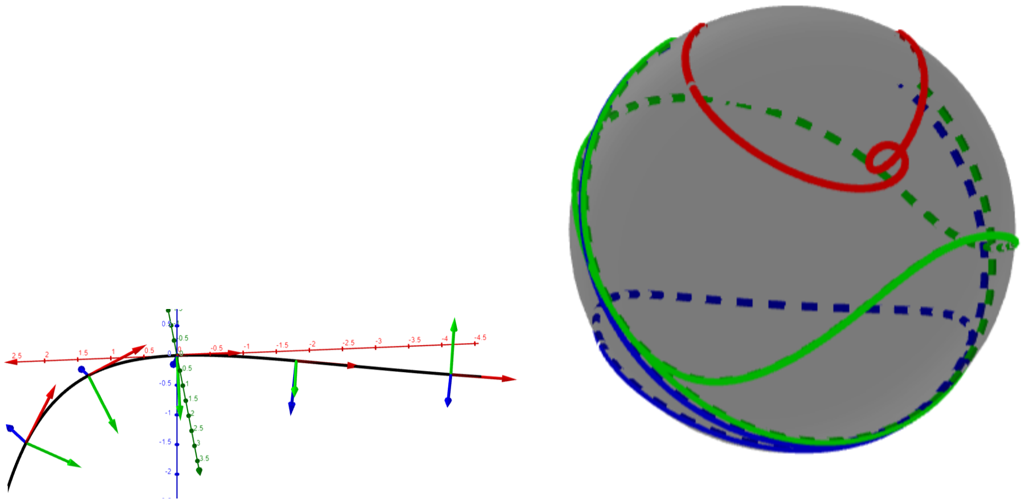

**Figure 6.4:** Example 34: On the left we have the curve with five example frames. Here again the hodograph has been scaled, so the frames are visible. On the right we see the Unit Sphere and the curves traced by frames.

# Conclusion

We have introduced an alternative view of the Pythagorean hodograph in Definition 13 based on the desire for an axis of its frame to always point in the same direction as its tangent, which possibly opens a new way to study how rotation minimizing motions fit into the wider space of spherical motions. We then reviewed a categorization of RMPHs based on Farouki et al's study through what we called "Farouki's numbers," which are defined in (4.1) from the RMPH's quaternion polynomial. During this chapters we also pointed out possible future paths of research.

In the next chapter we presented a theorem about second degree polynomials derived from another work Farouki. We have shown that one implication can be proven relatively simply using quaternions, which is distinct from the original proof using complex numbers. Specifically if a second degree polynomial fullfils (5.1), then it is rotation minimizing. In this and all previous chapters we have also derived collararies to theorems, which have certain assumptions on whether a quaternion polynomial is in Bernstein basis or in the monomial basis, in the other of the two bases.

We would like to undertake further study of categorization of RMPHs and possible relationships between what we have termed "Farouki's numbers" and functions similar to Pascal's triangle to look for a potential simplification of the process discovered by Farouki et al, if such is possible.

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