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Ideals of Banach Spaces

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Abstract: We study a particular class of subspaces of Banach spaces called ideals. We show that the notions of ideals, locally complemented subspaces, and the existence of a Hahn–Banach extension operator coincide.

We introduce and develop the method of suitable models, a set-theoretic approach which enables us to write technical proofs in simpler terms. We use the method to prove the existence of an almost isometric ideal. We present applications of almost isometric ideals and the method to the strong and local diameter two properties, and the Daugavet property.

Keywords: Banach Space, Suitable Models, Ideal, Locally Complemented Subspace, Almost Isometric Ideal, Hahn–Banach Extension Operator

Contents

Introduction

This thesis concerns a class of subspaces of Banach spaces called ideals. First, we follow the development of this notion.

A subspace Y of a Banach space X is *complemented* if there is a bounded linear projection $P_Y : X \to X$ such that $P_Y(X) = Y$. A well-known result is that a subspace with finite dimension or codimension is complemented. We say Y is a non-trivial complemented subspace if Y is complemented and its dimension and codimension are infinite. We can ask ourselves the following question: Does every Banach space have a non-trivial complemented subspace? In general, the answer is negative. In [14], Gowers and Maurey constructed a Banach space that only has trivially complemented subspaces. Therefore, finding a condition under which a non-trivial complemented subspace exists is intriguing. For more details, the reader is referred to the survey paper [25].

In [15, Corollary 3.8], Heinrich and Mankiewicz proved that a dual space of a non-separable Banach space contains a non-trivial complemented subspace. The critical observation is that working with *Hahn–Banach extension operators* suffices. For a Banach space X and its subspace Y, a linear operator $E : Y^* \to X^*$ is a Hahn–Banach extension operator if it is norm-one and the restriction of Ey^* to Y is equal to y^* for all $y^* \in Y^*$. Once we have this operator, we can find a linear projection $P : X^* \to X^*$ such that $\|P\| \leq 1$ and the kernel of P is the annihilator of Y. We will see this in Theorem 1.4, implication (iii) to (iv).

Heinrich and Mankiewicz proved the existence of this operator in [15, Proposition 3.4]. The proof presented in [15] is involved and uses results from model theory. Later, Sims and Yost, inspired by Lindenstrauss' finite dimensional lemma [22, Lemma 1], simplified the proof using the notion of a *locally complemented subspace*. A subspace Y is locally complemented in X if for every finite-dimensional subspace $F \subset X$ and every $\varepsilon > 0$ there is a linear operator $T : F \to Y$ such that $\|T\| \leq 1 + \varepsilon$ and $Tx = x$ for all $x \in Y \cap F$.

In [13], the authors realized a connection between locally complemented subspaces and the notion of M-ideals, and therefore, they used the term *ideal* instead of locally complemented subspace.

In recent years, ideals with additional properties have been studied. The notion of an *almost isometric ideal* was introduced in [2] in 2014. We do not state the whole definition here, as it is long. It will be defined in Definition 3.1. At this moment, we will only point out its relation to an ideal. In the definition of a locally complemented subspace (equivalently, an ideal), there is an operator T such that $\|Tx\| \leq (1 + \varepsilon)\|x\|$, as we mentioned above. In an almost isometric ideal, we require the operator to satisfy $(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|$. Hence the name "almost isometric."

An essential result in [1] is Theorem 1.5. It states that for every separable subspace, a separable, almost isometric ideal that contains the subspace exists. This result has further applications; see, e.g., [1, Section 3], [26], [24], and [6]. Almost isometric ideals are also studied in [5].

Brief Content Description

In Chapter 1, we introduce the notions of ideals, locally complemented subspaces, and Hahn–Banach extension operators. We state and prove the Ideal Characterization Theorem 1.4. To this end, we develop the theory of limits with respect to ultrafilters. We also introduce *directed ultrafilters*. Limits with respect to a directed ultrafilter have two advantages over limits with respect to nets which we will exploit. We can take a limit over an abstract directed set, and we do not need to worry about the existence because a limit with respect to an ultrafilter in a compact space always exists.

Chapter 2 is devoted to the method of suitable models. It is a set-theoretical tool that allows us to write technical proofs in simpler terms. This approach proves valuable in propositions where for each separable subspace, we construct a separable superspace with additional properties. The reader can see, e.g., [11], where several constructions of separable spaces using the method of suitable models are presented. In Chapter 2, we introduce it, prove several useful lemmas, and show how we use it. We briefly mention its connection to rich families.

Chapter 3 is devoted to the proof of the existence of an almost isometric ideal using the method of suitable models. It is a subject of Corollary 3.6, which is the main new result in this thesis. We also collect some consequences.

Finally, Chapter 4 is about applications of almost isometric ideals and the method of suitable models. We prove a particular subspace has the local or strong diameter two property or the Daugavet property if and only if the whole space has it. Moreover, we unify both separable and non-separable cases. The method of suitable models allows us to consider both cases simultaneously, unlike in [1], where the presented proofs consider only separable spaces with a note that we can extend the result to non-separable spaces.

Notation

We establish the notation we follow in this text.

If f is a mapping and Y is a subset of some set, we denote $f[Y]$ as the image of Y under the mapping f. We designate $[X]^{\leq\omega}$ at most countable subsets of a set X. Analogously, $[X]^{\text{fin}}$ are finite subsets of X. The powerset of a set X is $\mathcal{P}(X)$. By X^Y we denote the set of all functions from Y to X. The cardinality of X is $|X|$. The set of natural numbers and zero is ω. We put $\mathbb{N} := \omega \setminus \{0\}$. If X is a topological space, the minimal cardinality of a dense set $Y \subset X$ is denoted by $\operatorname{dens} Y$. If X is a normed vector space, the diameter of $Y \subset X$, $Y \neq \emptyset$ is $\operatorname{diam} Y := \sup\{\|x - y\|; x, y \in Y\}$. The set of eventually zero sequences is $c_{00}(X) := \{x \in X^{\mathbb{N}}; \exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}, n \geq n_0 : x_n = 0\}$, where X is a normed vector space. We identify $c_{00} := c_{00}(\mathbb{R})$. The support of a sequence $x = (x_n)_n \in c_{00}(X)$ is the set $\operatorname{supp} x := \{n \in \mathbb{N}; x_n \neq 0\}$.

We assume all vector spaces are over the field of real numbers. For a Banach space X, we denote X^* as the dual space to X. The dual space to X^* will be denoted by X^{**}. Let $Y \subset X$ be a subspace. We define the annihilator $Y^\perp$ of Y as the set $\{x^* \in X^*; x^*(x) = 0 \text{ for all } x \in Y\}$. Bi-annihilator $Y^{\perp\perp}$ denotes $(Y^\perp)^\perp$. The operator Id_X is the identity operator $\mathrm{Id} : X \to X$. If it is clear what the domain is, we drop the subscript.

The canonical isometrical embedding from X into X^{**} is denoted by κ_X. That is, $\kappa_X(x)(x^*) \coloneqq x^*(x)$ for $x \in X$, $x^* \in X^*$. If no confusion can arise, we will omit the subscript in κ_X. The rational linear span of a set $B \subset X$ is

$$\operatorname{span}_{\mathbb{Q}} B \coloneqq \left\{ \sum_{i=1}^{m} \lambda_i b_i ; \lambda_1, \dots, \lambda_m \in \mathbb{Q}, b_1, \dots, b_m \in B, m \in \mathbb{N} \right\}.$$

We denote by ℓ_1^k the space $\mathbb{R}^k$ equipped with the ℓ_1 norm. The symbol S_X denotes the unit sphere $\{x \in X; \|x\| = 1\}$, and the symbol B_X denotes the closed unit ball $\{x \in X; \|x\| \leq 1\}$. If T is a mapping, its range is $\operatorname{Rng} T$, its kernel is $\operatorname{Ker} T$, and its domain is $\operatorname{Dom} T$. The notation $T : X \to Y$ means the domain of T is X and $\operatorname{Rng} T \subset Y$. If $Z \subset X$, the restriction of T to the set Z is $T|_Z$, that is $T|_Z : Z \to Y$.

1. Ideals in Banach Spaces

In this chapter, we introduce the notions of ideals, locally complemented subspaces, and Hahn–Banach extension operators. We state the Ideal Characterization Theorem 1.4. In order to prove it, we develop the theory of ultrafilters and limits with respect to ultrafilters. The last section is devoted to the proof of the theorem.

1.1 Characterization of Ideals

We recall the notions discussed in the introductory chapter.

Definition 1.1. Let X be a Banach space, $Y \subset X$ a subspace. A linear operator $E : Y^* \to X^*$ is called a *Hahn–Banach extension operator* if it is norm-one and $Ey^*|_Y = y^*$ for all $y^* \in Y^*$.

Definition 1.2. Let X be a Banach space. We call a subspace $Y \subset X$ *locally complemented in* X if for every finite-dimensional subspace $F \subset X$ and every $\varepsilon > 0$ there is a linear operator $T : F \to Y$ such that $\|T\| \leq 1 + \varepsilon$ and $Tx = x$ for all $x \in Y \cap F$.

A locally complemented subspace is sometimes called locally 1-complemented or 1-locally complemented. We use the definition from [1]. The origin of this notion can be traced to Kalton's [18, Section 3] from 1984. He was motivated by finding a reasonable analogy of $\mathcal{L}_p$ spaces in the setting of p-Banach spaces for $p \in (0, 1)$.

Definition 1.3. Let X be a Banach space. We say a subspace $Y \subset X$ is an *ideal in* X if $Y^\perp$, the annihilator of Y, is the kernel of a norm-one linear projection on X^*.

The notion of an ideal was introduced by Godefroy, Kalton, and Saphar in 1993 in [13]. The notion of ideals introduced in this paper is a generalization of well-known *M-ideals*. At this point, M-ideals were already known, as they were defined in [4] in the year 1972. If Y is a subspace of a Banach space X, Y is an M-ideal if it is an ideal and $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for all $x^* \in X^*$. This justifies the naming, as an M-ideal is a special case of an ideal.

We postpone the proof of the Ideal Characterization Theorem to develop the theory necessary to complete it. The entire Section 1.3 is devoted to the proof itself.

Theorem 1.4 (Ideal Characterization Theorem)**.** *Let X be a Banach space, $Y \subset X$ a closed subspace. Then the following statements are equivalent.*

(i) *Y is locally complemented in X.*

(ii) *There is a linear operator $T : X \to Y^{**}$ such that $\|T\| \leq 1$ and $T(y) = \kappa_Y(y)$ for $y \in Y$.*

(iii) *There exists a Hahn–Banach extension operator from Y^* to X^*.*

(iv) *Y is an ideal in X.*

The following theorem is often called the principle of local reflexivity, initially due to Lindenstrauss and Rosenthal in 1969 [23, Theorem 3.1]. The proof can also be found in [3, Theorem 11.2.4].

Theorem 1.5 (Principle of Local Reflexivity)**.** *Let X be a Banach space. Then $\kappa(X)$ is locally complemented in X^{**}.*

We can immediately state a simple fact about ideals.

Fact 1.6. *Let X be a Banach space, $Y \subset X$ an ideal in X such that both dimension and codimension of Y are infinite. Then there exists a projection $Q : X^* \to X^*$ such that the range of Q has infinite dimension and codimension.*

Proof. Because Y is an ideal, we have a projection $Q : X^* \to X^*$ such that $Y^\perp = \operatorname{Rng} Q$. We utilize a known fact that we can identify $Y^\perp$ and $(X/Y)^*$ via a linear isometry. From this, $\operatorname{Rng} Q$ has infinite dimension. From the identification of Y^* and $X^*/Y^\perp$, we have the codimension of the range of Q is infinite too.

□

Now we turn our attention away from ideals for a while. We will return to them in Section 1.3. We begin with stating a lemma on projections and an auxiliary lemma which will be helpful later.

Lemma 1.7. *Let X be a Banach space. Let $Q : X \to X$ be a projection and denote $Q^* : X^* \to X^*$ the dual operator to Q. Then $\operatorname{Rng} Q^* = (\operatorname{Ker} Q)^\perp$.*

Proof. The dual operator of a projection is clearly a projection. Let $x^* \in \operatorname{Rng} Q^*$ and $x \in \operatorname{Ker} Q$. Because x^* is in the range of Q^*, we can write $Q^*x^*(x) = x^*(x)$. Then by the definition of a dual operator and the fact that Q vanishes in x

$$x^*(x) = Q^*x^*(x) = x^*(Qx) = x^*(0) = 0.$$

Since $x \in \operatorname{Ker} Q$ was arbitrary, we obtain $x^* \in (\operatorname{Ker} Q)^\perp$. We have shown $\operatorname{Rng} Q^* \subset (\operatorname{Ker} Q)^\perp$.

On the other hand, we assume $x^* \in (\operatorname{Ker} Q)^\perp$ and $x \in X$. Because Q is a projection, $Qx - x \in \operatorname{Ker} Q$. With this in mind, we calculate

$$(Q^*x^* - x^*)(x) = x^*(Qx - x) = 0.$$

From this, we have $Q^*x^*(x) = x^*(x)$. This means $x^* \in \operatorname{Rng} Q^*$. Because $x^* \in (\operatorname{Ker} Q)^\perp$ was arbitrary, we see that $\operatorname{Rng} Q^* \subset (\operatorname{Ker} Q)^\perp$.

□

Lemma 1.8. *Let X be a Banach space, $Y \subset X$ a subspace. Let us define a mapping $I : Y^{\perp\perp} \to Y^{**}$ by the formula $Ix^{**}(y^*) := x^{**}(x^*)$ for $x^{**} \in Y^{\perp\perp}$, $y^* \in Y^*$ and $x^* \in X^*$ an arbitrary extension of y^*. Then I is a linear surjective isometry.*

Moreover, $I\kappa_X(y) = \kappa_Y(y)$ for $y \in Y$.

Proof. We demonstrate that the definition of the mapping is correct. Let us have $x^{**} \in Y^{\perp\perp}$, $y^* \in Y^*$ and $x_1^*, x_2^* \in X^*$ two extensions of y^*. Then $x_1^*|_Y - x_2^*|_Y = y^* - y^* = 0$ which means $x_1^* - x_2^* \in Y^\perp$. Then $x^{**}(x_1^* - x_2^*) = 0 = x^{**}(x_1^*) - x^{**}(x_2^*)$. From this, $x^{**}(x_1^*) = x^{**}(x_2^*)$ so the mapping I is well-defined.

The mapping I is linear. We show it is an isometry. Let us have $x^{**} \in Y^{\perp\perp}$. Then

$$\|Ix^{**}\| = \sup_{y^* \in B_{Y^*}} |Ix^{**}(y^*)| = \sup_{x^* \in B_{X^*}} |x^{**}(x^*)| = \|x^{**}\|.$$

The second equality holds because every element of B_{X^*} is an extension of some element of B_{Y^*}.

To show I is also surjective, we consider $v^{**} \in Y^{**}$. Let us put $f^{**}(x^*) := v^{**}(x^*|_Y)$ for $x^* \in X^*$. Then f^{**} is clearly continuous and linear. We will verify that $f^{**} \in Y^{\perp\perp}$. Let us have $x^* \in Y^\perp$. Then

$$f^{**}(x^*) = v^{**}(x^*|_Y) = v^{**}(0) = 0.$$

Because the choice of $x^* \in Y^\perp$ was arbitrary, it follows $f^{**} \in Y^{\perp\perp}$.

We have

$$If^{**}(x^*|_Y) = f^{**}(x^*) = v^{**}(x^*|_Y),$$

for $x^* \in X^*$. The Hahn–Banach theorem assures that for each $y^* \in Y^*$ we find $x^* \in X^*$ satisfying $y^* = x^*|_Y$. This allows us to conclude $If^{**} = v^{**}$. Because $v^{**} \in Y^{**}$ was arbitrary, I is an onto mapping.

To prove the moreover part of the statement, we pick $y \in Y$, $y^* \in Y^*$. Recall the well-know fact that $\kappa_X[Y] \subset Y^{\perp\perp}$. For $x^* \in X^*$ an extension of y^*, we have

$$I\kappa_X(y)(y^*) = \kappa_X(y)(x^*) = x^*(y) = y^*(y) = \kappa_Y(y)(y^*).$$

From this, $I\kappa_X(y) = \kappa_Y(y)$ for $y \in Y$.

□

1.2 Ultrafilters and Limits with Respect to Ultrafilters

We examine the set-theoretic notion of ultrafilters. The study of them will bear its fruit in the following sections.

Definition 1.9. Given a nonempty set X, a *filter on* X is a nonempty family $\mathcal{U}$ of subsets of X such that for arbitrary $A, B \subset X$

(i) $\emptyset \notin \mathcal{U}$,

(ii) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$,

(iii) if $A \in \mathcal{U}$ and $A \subset B$, then $B \in \mathcal{U}$.

Definition 1.10. A filter $\mathcal{U}$ on a set X, which is not properly contained in any other filter, is called an *ultrafilter* on the set X. If there is $x \in I$ such that $\{x\} \in \mathcal{U}$, we call $\mathcal{U}$ a *principal ultrafilter*. Otherwise, we call $\mathcal{U}$ a *non-principal ultrafilter*.

The proofs of the following two theorems can be found in [16], Theorem 3.6 and Theorem 3.8, respectively.

Theorem 1.11. *Let X be a set. Then $\mathcal{U}$ is an ultrafilter on X if and only if $\mathcal{U}$ is a filter and for all $A \subset X$ either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Theorem 1.12. *Let X be a set and $\mathcal{A}$ be a subset of $\mathcal{P}(X)$, which has the finite intersection property. Then there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{A} \subset \mathcal{U}$.*

We now define a crucial notion of a limit with respect to an ultrafilter. We use a less general form of the definition from [16, Definition 3.44].

Definition 1.13. Let us have an index set I, an indexed family $(a_i)_{i\in I} \in \mathbb{R}^I$, $a \in \mathbb{R}$ and an ultrafilter $\mathcal{U}$ on the set I. We say a is a *limit of $(a_i)_{i\in I}$ with respect to an ultrafilter $\mathcal{U}$* if for all $\varepsilon > 0$ the set $\{i \in I; |a_i - a| < \varepsilon\} \in \mathcal{U}$. We denote this as

$$\lim_{\mathcal{U}} a_i = a.$$

Informally, $a = \lim_{\mathcal{U}} a_i$ means that "a_i is often close to a." Recall that an ultrafilter can be thought of as a two-valued measure. A given subset of a set either belongs to an ultrafilter or its complement belongs. Now we say that the subset has measure one, "is almost everything," if it is a member of the ultrafilter. We assign zero, "is almost nothing," if the complement is in the ultrafilter.

The following theorem from [16, Theorem 3.52] on the existence of limits with respect to ultrafilters will be essential to us.

Theorem 1.14. *Let us have an indexed family $(a_i)_{i\in I} \in [-C, C]^I$ for some $C > 0$ and an ultrafilter $\mathcal{U}$ on I. Then $\lim_{\mathcal{U}} a_i$ exists and is unique.*

We aim to define directed ultrafilters. We use the definition of a directed set that can be found in [19, p. 65]. Some authors require an additional axiom of antisymmetry.

Definition 1.15. Let I be a set. We say the pair $(I, \leq)$ is a *directed set* if $\leq$ is a binary relation that is reflexive, transitive, and has the property that for each $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$.

The following notion of a directed ultrafilter is inspired by [7, Remark 5].

Definition 1.16. Let $(I, \leq)$ be a directed set. For $i \in I$ we put $[i, \rightarrow) := \{j \in I; i \leq j\}$. We say a non-principal ultrafilter $\mathcal{U}$ on I is a *directed ultrafilter*, if the family $\mathcal{G} := \{[i, \rightarrow); i \in I\} \subset \mathcal{U}$.

Remark 1.17. We observe that a directed ultrafilter does exist. Assume I is infinite and $(I, \leq)$ does not have a maximal element, that is, $[i, \rightarrow) \neq \{i\}$ for every $i \in I$. The system $\mathcal{G} \cup \{I \setminus \{i\}; i \in I\}$ has the finite intersection property. By Theorem 1.12, this system is contained in a directed ultrafilter $\mathcal{U}$.

Notation 1.18. If I is an infinite set such that the directed set $(I, \leq)$ has no maximal element, we say I is an *unbounded directed set*.

We now prove a few properties of limits with respect to ultrafilters which will be important in upcoming theorems and to become acquainted with them.

We begin with a simple observation that limits with respect to a principal ultrafilter are not interesting. This is why we will henceforth concern ourselves only with limits with respect to non-principal ultrafilters.

Lemma 1.19. *Let $\mathcal{U}$ be an ultrafilter on a set I such that there is $i_0 \in I$ satisfying $\{i_0\} \in \mathcal{U}$. Then $\lim_{\mathcal{U}} a_i = a_{i_0}$ for every $(a_i)_{i \in I} \in \mathbb{R}^I$.*

Proof. Choose $\varepsilon > 0$. It holds $\{i_0\} \subset \{i \in I; |a_i - a_{i_0}| < \varepsilon\}$. Because $\mathcal{U}$ is a filter on I, $\{i \in I; |a_i - a_{i_0}| < \varepsilon\} \in \mathcal{U}$. From the definition, $\lim_{\mathcal{U}} a_i = a_{i_0}$.

□

We can summarize the next four lemmas as limit arithmetic. We do need to worry about the existence of a limit. We always work with a compact space $[-C, C]^I$ for some $C > 0$ and by Theorem 1.14, we have the existence and uniqueness.

Lemma 1.20. *Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, $C > 0$ and $(a_i)_{i \in I} \in [-C, C]^I$, $(b_i)_{i \in I} \in [-C, C]^I$. Then*

$$\lim_{\mathcal{U}} (a_i + b_i) = \lim_{\mathcal{U}} a_i + \lim_{\mathcal{U}} b_i.$$

Proof. Denote $A := \lim_{\mathcal{U}} a_i$, $B := \lim_{\mathcal{U}} b_i$. Choose $\varepsilon > 0$. Then $U_1 := \{i \in I; |a_i - A| < \frac{\varepsilon}{2}\} \in \mathcal{U}$ and $U_2 := \{i \in I; |b_i - B| < \frac{\varepsilon}{2}\} \in \mathcal{U}$. Because $\mathcal{U}$ is a filter on I, $U_1 \cap U_2 \in \mathcal{U}$. At the same time, $U_1 \cap U_2 \subset \{i \in I; |(a_i + b_i) - (A + B)| < \varepsilon\}$ from the triangle inequality. By the definition of a limit with respect to $\mathcal{U}$, $\lim_{\mathcal{U}}(a_i + b_i) = A + B$.

□

Lemma 1.21. *Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, $C > 0$. Let $(a_i)_{i \in I} \in [-C, C]^I$, $(b_i)_{i \in I} \in [-C, C]^I$ be such that $\{i \in I : a_i \leq b_i\} \in \mathcal{U}$. Then $\lim_{\mathcal{U}} a_i \leq \lim_{\mathcal{U}} b_i$.*

Proof. Denote $z_i := b_i - a_i$, $\lim_{\mathcal{U}} a_i =: A$, $\lim_{\mathcal{U}} b_i =: B$. By Lemma 1.20, $\lim_{\mathcal{U}} z_i = B - A =: Z$. For any fixed $\varepsilon > 0$, the set $U_1 := \{i \in I; |z_i - Z| < \varepsilon\} \in \mathcal{U}$ and from the assumption $U_2 := \{i \in I; z_i \geq 0\} \in \mathcal{U}$. Then $U_1 \cap U_2 \in \mathcal{U}$. For any index $i \in U_1 \cap U_2$, we obtain $Z \geq z_i - \varepsilon \geq -\varepsilon$. Since ε was arbitrary, we conclude $Z \geq 0$.

□

Lemma 1.22. *Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, $C > 0$ and $(a_i)_{i \in I} \in [-C, C]^I$, $(b_i)_{i \in I} \in [-C, C]^I$. Then*

$$\lim_{\mathcal{U}} (a_i b_i) = \left(\lim_{\mathcal{U}} a_i \right) \left(\lim_{\mathcal{U}} b_i \right).$$

Proof. Denote $A := \lim_{\mathcal{U}} a_i$, $B := \lim_{\mathcal{U}} b_i$. Let us have $\varepsilon > 0$. From the definition of a limit with respect to an ultrafilter, we have sets $U_1 := \{i \in I; |a_i - A| < \frac{\varepsilon}{2C}\} \in \mathcal{U}$ and $U_2 := \{i \in I; |A||b_i - B| < \frac{\varepsilon}{2}\} \in \mathcal{U}$. It follows from the triangle inequality,

$$|a_i b_i - AB| \leq |a_i - A||b_i| + |A||b_i - B| \leq C|a_i - A| + |A||b_i - B| < \varepsilon,$$

that $\mathcal{U} \ni U_1 \cap U_2 \subset \{i \in I; |a_i b_i - AB| < \varepsilon\} \in \mathcal{U}$. From the definition of a limit, we have $\lim_{\mathcal{U}}(a_i b_i) = AB$, as $\varepsilon > 0$ was arbitrary.

□

Lemma 1.23. *Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, $(a_i)_{i\in I} \in [-C, C]^I$ for $C > 0$ and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Then $\lim_{\mathcal{U}} f(a_i) = f(\lim_{\mathcal{U}} a_i)$.*

Proof. Mark $A := \lim_{\mathcal{U}} a_i$. Choose $\varepsilon > 0$. We find $\delta > 0$ such that $f(B(A, \delta)) \subset P(f(A), \varepsilon)$, where P denotes a punctured neighborhood. Then $U := \{i \in I; |A - a_i| < \delta\} \in \mathcal{U}$ and $U \subset \{i \in I; |f(A) - f(a_i)| < \varepsilon\} \in \mathcal{U}$.

□

Let us focus on limits with respect to directed ultrafilters from Definition 1.16. The first result indicates the relationship between them and limits with respect to a net.

Lemma 1.24. *Let $(I, \leq)$ be an unbounded directed set, $\mathcal{U}$ a directed ultrafilter on I, $(a_i)_{i\in I} \in \mathbb{R}^I$ a net such that $\lim_I a_i = a$, $a \in \mathbb{R}$. Then $\lim_{\mathcal{U}} a_i = a$.*

Proof. Assume not. Then, there exists $\varepsilon > 0$ such that the set $A := \{i \in I; |a_i - a| < \varepsilon\} \notin \mathcal{U}$. From Theorem 1.11, $I \setminus A \in \mathcal{U}$. Let us choose $j \in I \setminus A$. Because $\mathcal{U}$ is directed, $[j, \to) \in \mathcal{U}$. Then $(I \setminus A) \cap [j, \to) \in \mathcal{U}$. We show this is a contradiction with $\lim_I a_i = a$. Let us have $i_0 \in I$. Because $\mathcal{U}$ is directed we can find $i \in [i_0, \to) \cap [j, \to)$ which satisfies $|a_i - a| \geq \varepsilon$. This is the contradiction. Hence $\lim_{\mathcal{U}} a_i = a$.

□

A limit of a sequence has the property that if we have two sequences $(a_n)_n$, $(b_n)_n$ and there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n = b_n$, then their limits are the same. Limits with respect to an ultrafilter have an analogous property.

Lemma 1.25. *Let $(I, \leq)$ be an unbounded directed set, $\mathcal{U}$ a directed ultrafilter on I, $C > 0$, $(a_i)_{i\in I} \in [-C, C]^I$, $(b_i)_{i\in I} \in [-C, C]^I$, $i_0 \in I$. If $a_i = b_i$ for any $i \in [i_0, \to)$, then $\lim_{\mathcal{U}} a_i = \lim_{\mathcal{U}} b_i$.*

Proof. The limits exist by Theorem 1.14. Let $A := \lim_{\mathcal{U}} a_i$, $B := \lim_{\mathcal{U}} b_i$. Let us choose $\varepsilon > 0$. We define sets $U_1 := \{i \in I; |A - a_i| < \frac{\varepsilon}{2}\} \in \mathcal{U}$, $U_2 := \{i \in I; |B - b_i| < \frac{\varepsilon}{2}\} \in \mathcal{U}$. Because $\mathcal{U}$ is a directed ultrafilter, we also have $[i_0, \to) \in \mathcal{U}$. Then $U_1 \cap U_2 \cap [i_0, \to) \in \mathcal{U}$. Simultaneously $U_1 \cap U_2 \cap [i_0, \to) \subset \{i \in [i_0, \to); |(a_i + b_i) - (A + B)| < \varepsilon\} \in \mathcal{U}$ by the triangle inequality. From the assumption $a_i = b_i$ for $i \geq i_0$, and by the definition of a limit with respect to the ultrafilter $\mathcal{U}$, it follows $\lim_{\mathcal{U}} 2a_i = A + B$. By Lemma 1.22, $\lim_{\mathcal{U}} 2a_i = 2A$. Thus $A = B$.

□

By Lemma 1.22 and Lemma 1.20, we know that limits with respect to an ultrafilter are linear. We formulate this separate lemma so we can refer to it in the future.

Lemma 1.26. *Let $(I, \leq)$ be an unbounded directed set, $\mathcal{U}$ a directed ultrafilter on I, $C > 0$, $(a_i)_{i\in I} \in [-C, C]^I$, $(b_i)_{i\in I} \in [-C, C]^I$, $\lambda, \mu \in \mathbb{R}$. Then*

$$\lim_{\mathcal{U}} (\lambda a_i + \mu b_i) = \lambda \lim_{\mathcal{U}} a_i + \mu \lim_{\mathcal{U}} b_i.$$

Proof. The limits exist due to Theorem 1.14. By Lemma 1.20 and Lemma 1.22, we have the equality.

□

1.3 Proof of the Ideal Characterization Theorem

We now have the knowledge necessary to prove the Ideal Characterization Theorem 1.4.

Proof of Theorem 1.4. Assume (i) holds. We begin by introducing the system

$$\mathcal{I} := \{I = (F, \varepsilon); F \subset X \text{ is a finite-dimensional subspace}, \varepsilon > 0\},$$

and a binary relation $\leq$ on $\mathcal{I}$. For $I = (F, \varepsilon) \in \mathcal{I}$ and $I' = (F', \varepsilon') \in \mathcal{I}$, we write $I \leq I'$ if $F \subset F'$ and $\varepsilon' \leq \varepsilon$. Then $\leq$ is transitive and reflexive. Given $I = (F, \varepsilon) \in \mathcal{I}$ and $I' = (F', \varepsilon') \in \mathcal{I}$, the pair $J := (F + F', \min\{\varepsilon, \varepsilon'\})$ is a member of $\mathcal{I}$ and satisfies $I \leq J$ and $I' \leq J$. Clearly, $(\mathcal{I}, \leq)$ has no maximal element. Thus $(\mathcal{I}, \leq)$ is an unbounded directed set.

For a given $I = (F, \varepsilon) \in \mathcal{I}$, we denote by $S_I : F \to Y$ the linear bounded mapping such that $\|S_I\| \leq 1 + \varepsilon$ and $S_I|_{F \cap Y} = \mathrm{Id}$. Then we define a mapping $T_I : F \to Y$

$$T_I x = \begin{cases} S_I x & x \in F, \\ 0 & x \notin F. \end{cases}$$

We will show the operator $T : X \to Y^{**}$ defined below satisfies (ii)

$$Tx(x^*) := \lim_{\mathcal{U}} x^*(T_I x) \quad x \in X, x^* \in Y^*.$$

The indexed family $(x^*(T_I x))_{I \in \mathcal{I}}$ is a bounded family of real numbers. By Theorem 1.14, we obtain the existence and uniqueness of the limit with respect to $\mathcal{U}$.

First, we show $Tx \in Y^{**}$. We consider $x \in X$, $x^* \in Y^*$, $\varepsilon > 0$, and the pair $(\mathrm{span}\{x\}, \varepsilon) =: I_0$. Then $I_0 \in \mathcal{I}$ and

$$\begin{aligned} |Tx(x^*)| &= \left|\lim_{\mathcal{U}} x^*(T_I x)\right| \stackrel{(a)}{=} \left|\lim_{\mathcal{U}} x^*(S_I x)\right| \\ &\stackrel{(b)}{=} \lim_{\mathcal{U}} |x^*(S_I x)| \leq \lim_{\mathcal{U}} \|x^*\|\|S_I\|\|x\| \\ &\leq (1+\varepsilon)\|x^*\|\|x\|, \end{aligned} \tag{1.1}$$

where in (a) we used that for any $(F, \delta) \in [I_0, \to)$ the element x belongs to the subspace F. In (b), we employed Lemma 1.23. The mapping Tx is linear by Lemma 1.26. It follows from (1.1) that $Tx \in Y^{**}$. Moreover, since $\varepsilon > 0$ was arbitrary, we have $\|Tx\| \leq \|x\|$ for all $x \in X$.

To show T is linear, we pick $x, y \in X$, $\lambda, \mu \in \mathbb{R}$, and $x^* \in Y^*$. Put $I_0 := (\mathrm{span}\{x, y\}, 1)$. Then, for any $I \in [I_0, \to)$

$$\begin{aligned} x^*(T_I(\lambda x + \mu y)) &= x^*(S_I(\lambda x + \mu y)) \\ &= \lambda x^*(S_I x) + \mu x^*(S_I y) \\ &= \lambda x^*(T_I x) + \mu x^*(T_I y), \end{aligned}$$

because S_I is linear from the assumption. Now we are ready to use Lemma 1.25 in conjunction with limit arithmetic

$$\begin{aligned} T(\lambda x + \mu y)(x^*) &= \lim_{\mathcal{U}} x^*\left(T_I(\lambda x + \mu y)\right) \\ &= \lambda \lim_{\mathcal{U}} x^*\left(T_I x\right) + \mu \lim_{\mathcal{U}} x^*\left(T_I y\right) \\ &= \lambda T(x)(x^*) + \mu T(y)(x^*). \end{aligned}$$

Now it follows from the estimate $\|Tx\| \leq \|x\|$, $x \in X$ that, $\|T\| \leq 1$.

It remains to show $T(x) = \kappa_Y(x)$ for $x \in Y$. Let us have $x \in Y$, $x^* \in Y^*$. We set $I_0 := (\operatorname{span}\{x\}, 1)$. Then for any $I \in [I_0, \rightarrow)$, we have $S_I(x) = x$. Finally

$$\begin{aligned} Tx(x^*) &= \lim_{\mathcal{U}} x^*\left(T_I x\right) = \lim_{\mathcal{U}} x^*\left(S_I x\right) \\ &= \lim_{\mathcal{U}} x^*\left(x\right) = x^*(x) \\ &= \kappa_Y(x)(x^*). \end{aligned}$$

Since $x \in Y$ was arbitrary, we conclude $Tx = \kappa_Y(x)$ on Y. This finishes the proof of the implication from (i) to (ii).

Let us assume (ii) holds. We will prove (iii). To this end, we define $E : Y^* \to X^*$, $Ey^*(x) := Tx(y^*)$ where $x \in X$, $y^* \in Y^*$ and $T : X \to Y^{**}$ is the operator from (ii). Because the operator T is linear, it follows that Ey^* is linear for every $y^* \in Y^*$. It is clear that E is linear too, and for $y \in Y$ we have

$$Ey^*(y) = Ty(y^*) = \kappa_Y y(y^*),$$

which means $Ey^*|_Y = y^*$. From $\|T\| \leq 1$, it follows $\|E\| \leq 1$.

We have $Ey^*|_Y = y^*$. Thus for $y^* \in S_{Y^*}$, it follows $\|Ey^*\| = \|y^*\| = 1$. Combining this with the norm estimate above, we see $\|E\| = 1$.

Now we show (iv) follows from (iii). Let $E : Y^* \to X^*$ be a Hahn–Banach extension operator from (iii). We are looking for a norm-one linear projection $Q : X^* \to X^*$ such that $\operatorname{Ker} Q = Y^\perp$. We set $Qx^* := E(x^*|_Y)$, $x^* \in X^*$. Then $\|Q\| \leq 1$ and Q is linear. From the property of the Hahn–Banach extension operator $Ey^*|_Y = y^*$ for $y^* \in Y^*$, we have

$$QQx^* = Q(E(x^*|_Y)) = E(E(x^*|_Y)|_Y) = E(x^*|_Y) = Qx^*,$$

thus Q is a projection. It remains to demonstrate that the kernel of Q is the annihilator of Y. First, we assume $x^* \in \operatorname{Ker} Q$ and $y \in Y$. Then

$$0 = Qx^*(y) = (Ex^*|_Y)(y) = x^*(y).$$

From the arbitrariness of y, we obtain $x^* \in Y^\perp$. Since we chose x^* arbitrarily, we have $\operatorname{Ker} Q \subset Y^\perp$. On the other hand, we pick $x^* \in Y^\perp$, $y \in Y$. It follows

$$Qx^*(y) = (Ex^*|_Y)(y) = x^*(y) = 0.$$

Since this holds for any $x^* \in Y^\perp$, we see $Y^\perp \subset \operatorname{Ker} Q$. Thus Y is an ideal in X.

We will prove that (iv) implies (i). Let us have $\varepsilon > 0$ and a finite-dimensional subspace $F \subset X$.

From the assumption, Y is an ideal in X. We find a projection $Q : X^* \to X^*$ such that $\|Q\| \leq 1$ and $\operatorname{Ker} Q = Y^\perp$. Let us denote $P : X^{**} \to X^{**}$ the dual operator to Q. Then P is a projection and has the same norm. From Lemma 1.7, we have $\operatorname{Rng} P = (\operatorname{Ker} Q)^{\perp} = Y^{\perp\perp}$.

By the Principle of Local Reflexivity 1.5, we have a linear operator $\widetilde{T}_S : S \to \kappa_Y(Y)$ for every finite-dimensional subspace $S \subset Y^{**}$ such that $\|\widetilde{T}_S\| \leq 1 + \varepsilon$ and $\widetilde{T}_S|_{S \cap \kappa_Y(Y)} = \mathrm{Id}$. Let $I : Y^{\perp\perp} \to Y^{**}$ be the surjective linear isometry from Lemma 1.8. We consider the following chain of mappings

$$X \xrightarrow{\kappa_X} X^{**} \xrightarrow{P} Y^{\perp\perp} \xrightarrow{I} Y^{**} \xrightarrow{\widetilde{T}_S} \kappa_Y(Y) \xrightarrow{\kappa_Y^{-1}} Y.$$

Now we define the operator $T : F \to Y$ as

$$T := \kappa_Y^{-1} \circ \widetilde{T}_{IP\kappa_X(F)} \circ I \circ P \circ \kappa_X,$$

where $\widetilde{T}_{IP\kappa_X(F)}$ is the operator corresponding to the finite-dimensional subspace $I(P(\kappa_X(F)))$ of Y^{**}. The chain of mappings above implies T is well defined. Next, we estimate the norm of T,

$$\|T\| \leq \|\kappa_Y^{-1}\| \|\widetilde{T}_{IP\kappa_X(F)}\| \|I\| \|P\| \|\kappa_X\| \leq 1 + \varepsilon.$$

It remains to check whether T is the identity on $F \cap Y$. Since $\kappa_X[F \cap Y] \subset \kappa_X(F) \cap Y^{\perp\perp} \subset \operatorname{Rng} P$, it follows $P \circ \kappa_X[F \cap Y] = \kappa_X[F \cap Y]$ because P is a projection. From Lemma 1.8, we have $I\kappa_X(y) = \kappa_Y(y)$ for $y \in Y$. Then

$$I(P\kappa_X[F \cap Y]) = I(\kappa_X[F \cap Y]) = \kappa_Y[F \cap Y].$$

Finally, for $x \in F \cap Y$

$$\begin{aligned}
Tx &= \kappa_Y^{-1} \circ \widetilde{T}_{IP\kappa_X(F)} \circ I \circ P \circ \kappa_X(x) \\
&= \kappa_Y^{-1} \circ \widetilde{T}_{IP\kappa_X(F)} \circ I \circ \kappa_X(x) \\
&= \kappa_Y^{-1} \circ \widetilde{T}_{IP\kappa_X(F)} \circ \kappa_Y(x) \\
&= \kappa_Y^{-1} \circ \kappa_Y(x) \\
&= x.
\end{aligned}$$

We have shown that $T|_{F \cap Y} = \mathrm{Id}$. Thus Y is indeed locally complemented in X, as the operator T witnesses.

□

2. Suitable Models

This chapter introduces the method of suitable models, a set-theoretical approach that enables us to write technical proofs in simpler terms. This approach was used already by Dow [12] in 1988. In 2009, Kubiś [20] applied this method to obtain new results in functional analysis. The method was later refined by Cúth in 2012 [10]. We follow Cúth's approach from [10], [11], and [8]. The reader is referred to Kunen's book [21], especially chapter four, for more details about the involved set-theoretic background.

In the first section, we introduce some elementary notions. We continue with notions important in the area of suitable models. Then we show how we apply this method. In the last section, we briefly mention its connection to rich families.

2.1 Elementary Notions

We review elementary notions from set theory and logic. We work in the Zermelo–Fraenkel set theory framework with the axiom of choice. Our language uses the following basic symbols. The connectives

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow,$$

the parentheses, the universal and existential quantifier, the equality symbol, the membership symbol

$$(\quad) \quad \forall \quad \exists \quad = \quad \in,$$

and the variable symbols v_j for any $j \in \mathbb{N}$.

Definition 2.1. A finite sequence of basic symbols is called an *expression*. An expression is a *formula* if constructed by the following three rules.

1. For any $i, j \in \mathbb{N}$ the expressions $v_i = v_j$ and $v_i \in v_j$ are formulas.
2. If φ, ψ are formulas, then $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are formulas.
3. If φ is a formula, then $(\exists v_i\ \varphi)$ and $(\forall v_i\ \varphi)$ are formulas for all $i \in \mathbb{N}$.

Notation 2.2. To simplify notation, we write

$$\begin{array}{ll} \neg(v_i \in v_j) & v_i \notin v_j \\ \neg(v_i = v_j) & v_i \neq v_j. \end{array}$$

We drop parentheses if it is clear how to put them in from the context. We also use other letters from the English and Greek alphabet as variables.

Definition 2.3. Let φ be a formula. A *subformula* of φ is a consecutive sequence of symbols from φ, which forms a formula.

Example 2.4. The five subformulas of the formula $\varphi \coloneqq (\exists x(x = y)) \vee (\forall z(z \in y))$ are $x = y$, $z \in y$, $\exists x(x = y)$, $\forall z(z \in y)$, and the formula φ itself.

Definition 2.5. We say an occurrence of a variable x in a formula φ is *bound* if x is a part of some subformula of φ in the form $\exists x(\psi)$ or $\forall x(\psi)$. Otherwise, we call the occurrence of x *free.*

Example 2.6. In the formula $\exists y(x \neq y)$, the variable y is bound, and x is free. In $\forall x \exists y(x \neq y)$, both variables x and y are bound.

Notation 2.7. Let ψ be a formula. If all free variables in ψ are among $x_1, \dots, x_n$ we write $\psi(x_1, \dots, x_n)$. If $y_1, \dots, y_n$ are other variables, $\psi(y_1, \dots, y_n)$ denotes the formula which results from substituting y_i for each free occurrence of x_i.

The notation $\psi(x_1, \dots, x_n)$ does not imply that all the listed variables are free. It also does not mean no other variable is in ψ. We only list variables important for our discussion.

2.2 Further Notions

Now we introduce notions crucial to the method of suitable models.

Definition 2.8. Let M be a set. For any formula ψ we define ψ^M, the *relativization of ψ to M*, by the following rules

$$\begin{aligned}
(x = y)^M &:= x = y, \\
(x \in y)^M &:= x \in y, \\
(\psi \wedge \varphi)^M &:= \psi^M \wedge \varphi^M, \\
(\psi \vee \varphi)^M &:= \psi^M \vee \varphi^M, \\
(\neg\psi)^M &:= \neg(\psi^M), \\
(\psi \to \varphi)^M &:= \psi^M \to \varphi^M, \\
(\psi \leftrightarrow \varphi)^M &:= \psi^M \leftrightarrow \varphi^M, \\
(\exists x\, \psi)^M &:= \exists x(x \in M \wedge \psi^M), \\
(\forall x\, \psi)^M &:= \forall x(x \in M \to \psi^M).
\end{aligned}$$

We write $\exists x \in M\ \psi$ instead of $\exists x(x \in M \wedge \psi)$ to ease notation. Analogously, we write $\forall x \in M\ \psi$ in the place of $\forall x(x \in M \to \psi)$.

Essentially, we obtain the relativization of ψ to M by replacing $\exists x$ with $\exists x \in M$ and $\forall x$ with $\forall x \in M$ in ψ.

Example 2.9. Let M be a set. Put $\psi := \exists z\, \forall x\, ((z \in y) \wedge (x \notin z))$. Then the relativization of ψ to M is $\psi^M = \exists z \in M\, \forall x \in M\, ((z \in y) \wedge (x \notin z))$. To verify this, we follow the definition above

$$\begin{aligned}
\psi^M &= [(\exists z\ \forall x\ ((z \in y) \wedge (x \notin z)))]^M \\
&= \exists z\ (z \in M \wedge [\forall x((z \in y) \wedge (x \notin z))]^M) \\
&= \exists z\ (z \in M \wedge (\forall x(x \in M \to [(z \in y) \wedge (x \notin z)]^M))) \\
&= \exists z\ (z \in M \wedge (\forall x(x \in M \to (z \in y) \wedge (x \notin z)))) \\
&= \exists z\ (z \in M \wedge (\forall x \in M((z \in y) \wedge (x \notin z)))) \\
&= \exists z \in M\ \forall x \in M\ ((z \in y) \wedge (x \notin z)),
\end{aligned}$$

where the last two equalities are our simplified notation. The parentheses [,] bear the same meaning as (,). They are merely a visual aid to help the reader.

Definition 2.10. Let M be a set, $\psi(x_1, \ldots, x_n)$ a formula with all free variables shown. We say ψ *is absolute for* M if

$$\forall y_1, \ldots, y_n \in M \quad \left(\psi^M(y_1, \ldots, y_n) \leftrightarrow \psi(y_1, \ldots, y_n)\right).$$

Definition 2.11. Let $\Phi = (\psi_1, \ldots, \psi_n)$ be a finite list of formulas, X be a set. Let $M \supset X$ be a set such that each $\psi_i \in \Phi$ is absolute for M. Then we say M *is a suitable model for* Φ *containing* X. We write $M \prec (\Phi, X)$.

It is sometimes assumed that the set M is countable. This assumption can be found in [11], [10]. We follow [8] where M is not necessarily countable. A similar approach can be found in [9].

The following theorem asserts that suitable models exist. A proof can be found in [21, Theorem IV.7.8].

Theorem 2.12. *Let* Φ *be a finite list of formulas,* X *be any set. Then there exists a set* R *such that* $R \prec (\Phi, X)$ *and* $|R| \leq \max\{\omega, |X|\}$.

2.3 Practical Use of the Method

In this section, we present known results. All but the last two can be found in [10] and [8]. We provide proofs here because we use a slightly different terminology (closer to newer [9]), and we consider uncountable models (unlike in [10]).

To use the absoluteness of the formula $\psi(x_1, \ldots, x_n)$ for a set M we first need to make sure $x_1, \ldots, x_n \in M$. The following lemma allows us to add elements to M.

Lemma 2.13 (Absoluteness Lemma). *Let* X *be a set,* $\psi(y, x_1, \ldots, x_n)$ *be a formula with all free variables shown. Let* $M \prec ((\psi, \exists y\, \psi(y, x_1, \ldots, x_n)), X)$*. Let us assume* $a_1, \ldots, a_n \in M$ *are such that there exists* u *which satisfies* $\psi(u, a_1, \ldots, a_n)$*. Then there is* $v \in M$ *satisfying* $\psi(v, a_1, \ldots, a_n)$*. Moreover, if the set* u *is unique, then* $u \in M$*.*

Proof. The relativization of the formula $\exists y\, \psi(y, x_1, \ldots, x_n)$ to M is simply $\exists y \in M$ $\psi^M(y, x_1, \ldots, x_n)$. The absoluteness of the formula $\exists y\, \psi(y, x_1, \ldots, x_n)$ implies there is $v \in M$ such that $\psi^M(v, a_1, \ldots, a_n)$ holds. It follows from the absoluteness of $\psi(y, x_1, \ldots, x_n)$ that $\psi(v, a_1, \ldots, a_n)$ holds. If the set u is unique, then immediately $u = v$, thus $u \in M$.

□

We introduce useful conventions.

Convention 2.14. When we write,

> "For a suitable model M the following statement holds."

We mean,

> "There is a finite list of formulas Φ and a countable set Z such that the following statement holds for every $M \prec (\Phi, Z)$."

Convention 2.15. When we write,

> "Let us have a suitable model M for the formulas marked with $(*)$."

It is to be understood as,

> "Let us have all the formulas marked with $(*)$ and their subformulas in the preceding proofs in this thesis and number them $\psi_1, \ldots, \psi_m$. Add all the formulas marked with $(*)$ in the proof below and their subformulas to this list. We have a finite list of formulas $\psi_1, \ldots, \psi_n$. Let us have a set Z such that
>
> $$\mathbb{Q} \cup \{\omega, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, +, -, \cdot, /, <, |\cdot|\} \subset Z,$$
>
> where $+, -, \cdot, /, <$ are the common operations and relation on real numbers, and $|\cdot| : \mathbb{R} \to \mathbb{R}$ is the absolute value. Now we fix a suitable model $M \prec ((\psi_1, \ldots, \psi_n), Z)$."

Convention 2.16. To even further ease notation, we write formulas less formally, and we use other symbols besides the ones present in our basic language. For example, the ordered pair (a, b) is the set $\{\{a\}, \{a, b\}\}$. If f is a function, we write $f(x) = y$ instead of $(x, y) \in f$. We also use other common notation and symbols.

The approach with the formulas marked with $(*)$ allows us to use the preceding results. It also improves the readability of the text.

As an illustration, we prove the following lemma. We prove the first statement in great detail to illustrate how we use the established conventions.

Lemma 2.17. *For a suitable model M, the following holds. Let f be a function such that $f \in M$. Then*

(i) $\operatorname{Dom} f \in M$,

(ii) $\operatorname{Rng} f \in M$,

(iii) $f(x) \in M$ *for all* $x \in M \cap \operatorname{Dom} f$,

(iv) *if f is injective,* $f^{-1} \in M$.

Proof. Let us have a suitable model M for the formulas marked with $(*)$. Let us pick a function $f \in M$. The formula

$$(*) \quad \exists D\ \forall x\ (x \in D \leftrightarrow (\exists y\ f(x) = y)),$$

defines the domain of f. The Absoluteness Lemma 2.13 suggests $\operatorname{Dom} f \in M$.

In more detail, the suitable model M contains all the subformulas of this formula. Hence it also contains $\psi(D, f) :\equiv \forall x\ (x \in D \leftrightarrow (\exists y\ f(x) = y))$. We can rewrite the original formula as $\exists D\ \psi(D, f)$. Now M is absolute for both ψ and $\exists D\ \psi(D)$, and $\operatorname{Dom} f$ is uniquely defined by the formula $\exists D\ \psi(D)$. Now it is clear from the Absoluteness Lemma 2.13 that $\operatorname{Dom} f \in M$.

The range of f is defined by the formula

$$(*) \quad \exists R\ \forall y\ (y \in R \leftrightarrow (\exists x\ f(x) = y)).$$

Again from Lemma 2.13, $\operatorname{Rng} f \in M$. The formula

$$(*) \quad \forall x \in D\ (\exists y\ f(x) = y)$$

is absolute for M. From the absoluteness, we conclude that for any $x \in M \cap \operatorname{Dom} f$, we have $f(x) \in M$. To prove the last point, we use the following formula

$$(*) \quad (\exists g : \operatorname{Rng} f \to \operatorname{Dom} f)\ (\forall y \in \operatorname{Rng} f)\ (\exists x \in \operatorname{Dom} f)\ \ (g(y) = x \wedge f(x) = y)\,.$$

Then the Absoluteness Lemma 2.13 implies $f^{-1} \in M$.

□

The following lemma is a collection of fundamental results we will need later.

Lemma 2.18. *For a suitable model M, the following holds.*

(i) $\emptyset \in M$.

(ii) *Let S be a countable set. If $S \in M$, then $S \subset M$.*

(iii) *Let S be a finite set. Then $S \in M$ if and only if $S \subset M$.*

(iv) *If $A, B \in M$, then $A \cap B \in M$, $A \cup B \in M$, $B \setminus A \in M$.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. The first item follows from Lemma 2.13 and the formula

$$(*) \quad \exists x\ \forall z\ (z \in x \leftrightarrow z \neq z).$$

Let us have $S \in M$ countable and infinite. We introduce the formula

$$(*) \quad \exists f\ \ (f : S \to \omega \text{ is a bijective function}).$$

From Lemma 2.13, we have $f \in M$ such that f is a bijection between S and ω and from Lemma 2.17 the inverse function $f^{-1} \in M$. For any $n \in \omega$, it holds $f^{-1}(n) \in M$ from Lemma 2.17. Then $S = \{f^{-1}(n); n \in \omega\} \subset M$.

Now we assume $S \in M$ is a nonempty finite set. Then there is a bijection $g : S \to N$ for some $N \in \omega$. The absoluteness of the formula

$$(*) \quad \exists g\ (g : S \to N \text{ is a bijective function}),$$

and Lemma 2.13 allow us to find $g \in M$ such that g is bijection between S and N. Now the argument is analogous to the case where S is countable. By Lemma 2.17, the inverse $g^{-1} \in M$ and by Lemma 2.17, we have $g^{-1}(n) \in M$ for all $n \in N$. Then $S = \{g^{-1}(n); n \in N\} \subset M$. This proves (ii).

Now we will prove (iii). The implication from the left to the right follows from what we have just proved. We prove the converse implication. Let S be a subset of M. First, we show that if $u, v \in M$, then $u \cup \{v\} \in M$. To this end, we use the following formula

$$(*) \quad \exists x\ \forall z\ (z \in x \leftrightarrow (z \in u\ \vee\ z = v)).$$

The claim $u \cup \{v\} \in M$ immediately follows from Lemma 2.13 because M is a suitable model for this formula and all its subformulas. We proceed with a subsequent claim. For all $n \in \omega$ and any $V \subset M$ such that $|V| = n$, it holds $V \in M$. This claim ensues from the previous one by induction on the cardinality of V. The set S is finite. We have $|S| = m \in \omega$. The last claim allows us to conclude $S \in M$.

We pick $A, B \in M$ to prove the last point. We routinely use the Absoluteness Lemma 2.13 and the absoluteness of the following three formulas,

$$\begin{aligned}
&(*) \quad \exists P\, \forall a\, (a \in P \leftrightarrow (a \in A \ \wedge\ a \in B)),\\
&(*) \quad \exists S\, \forall a\, (a \in S \leftrightarrow (a \in A \ \vee\ a \in B)),\\
&(*) \quad \exists R\, \forall a\, (a \in R \leftrightarrow (a \notin A \ \wedge\ a \in B)).
\end{aligned}$$

We get $A \cap B \in M$, $A \cup B \in M$ and $B \setminus A \in M$.

□

So far, we have stated very general results where we considered only sets. Now we focus on sets with an additional structure of normed vector spaces.

Notation 2.19. Let $(X, +, \cdot, \|\cdot\|)$ be a normed vector space. We say *a suitable model M contains X*, or *X is contained in M*, if $\{X, +, \cdot, \|\cdot\|\} \subset M$. If M is a suitable model which contains X, we denote the set $\overline{X \cap M}$ by X_M.

The following lemma states that the set X_M is, in fact, a subspace. Its existence, and later, other properties, will be crucial to us.

Lemma 2.20. *For a suitable model M, the following holds. If M contains a normed vector space X, then X_M is a closed subspace of X.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. The suitable model M contains the vector addition map $+ : X \times X \to X$. From Lemma 2.17, we have $x + y \in M \cap X$ for any $x, y \in X \cap M$.

Due to our Convention 2.15, we have $\mathbb{R}, \mathbb{Q} \in M$. However, the set $\mathbb{R}$ is uncountable, so $\mathbb{R} \not\subset M$. We turn to $\mathbb{Q}$, a dense countable subset of $\mathbb{R}$ to deal with this issue. Again by Convention 2.15, we have $\mathbb{Q} \subset M$. Now we pick any $\lambda \in \mathbb{Q}$. Then, of course, $\lambda \in M$. The suitable model M contains the scalar multiplication map $\cdot : \mathbb{R} \times X \to X$. From this, for any $\lambda \in \mathbb{Q}$ and $x \in X \cap M$ it holds $\lambda \cdot x = \lambda x \in X \cap M$ by Lemma 2.17. We have just proved that the set $X \cap M$ is a $\mathbb{Q}$-linear subspace of X. Thus the closure $\overline{X \cap M} = X_M$ is a closed subspace of X.

□

The first interesting property of the subspace X_M is that the restriction of a functional to this subspaces does not change its norm. We will see use of this lemma in Chapter 4.

Lemma 2.21. *For a suitable model M, the following holds. Let X be a Banach space contained in M. If $x^* \in X^* \cap M$, then $\|x^*|_{X_M}\| = \|x^*\|$.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. Let us have $x^* \in X^* \cap M$. From the definition of the operator norm, it is clear that $\|x^*|_{X_M}\| \leq \|x^*\|$. We will show the inequality $\|x^*|_{X_M}\| \geq \|x^*\|$ also holds. Let us have $n \in \mathbb{N}$. From the absoluteness of the following formula (and its subformulas) and Lemma 2.13

$$(*) \quad \exists x \in B_X \; \|x^*\| - \frac{1}{n} \leq x^*(x),$$

for each $n \in \mathbb{N}$ we can find some $x \in B_X \cap M$ such that $x^*(x) \geq \|x^*\| - \frac{1}{n}$. It follows $\|x^*|_{X_M}\| \geq \|x^*\|$. Thus we have $\|x^*|_{X_M}\| = \|x^*\|$.

□

The next lemma is technical. We will need it in the proof of Proposition 4.7. In that proof, we will need to write down one formula which encodes linear combinations of arbitrary length. The mapping W introduced below will allow us to do that. We will comment on the issue in more detail in the proof of the aforementioned proposition.

Lemma 2.22. *For a suitable model M, the following holds. Let X be a Banach space contained in M. Then there is a mapping $W : c_{00} \times c_{00}(X) \to X$, $W \in M$ such that for all $\lambda \in c_{00} \cap M$ and for all $x \in c_{00}(X) \cap M$ we have*

$$W(\lambda, x) = \sum_{i \in \operatorname{supp} \lambda \cup \operatorname{supp} x} \lambda(i)x(i) \in M.$$

In particular, there is a mapping $W_{\mathbb{R}} : c_{00} \times c_{00} \to \mathbb{R}$ such that

$$W_{\mathbb{R}}(\lambda, \sigma) = \sum_{i \in \operatorname{supp} \lambda \cup \operatorname{supp} \sigma} \lambda(i)\sigma(i) \in M,$$

for all λ, $\sigma \in c_{00} \cap M$.

Proof. Let us have a suitable model M for the formulas marked with $(*)$. We inductively define a mapping $W' : \mathbb{N} \times c_{00} \times c_{00}(X) \to X$ which sums elements

$$(*) \quad \exists W' : \mathbb{N} \times c_{00} \times c_{00}(X) \to X \; \forall \lambda \in c_{00} \; \forall x \in c_{00}(X) \; \forall n \in \mathbb{N}$$
$$W'(1, \lambda, x) = \lambda(1)x(1) \wedge W'(n+1, \lambda, x) = \lambda(n+1)x(n+1) + W'(n, \lambda, x).$$

By the Absoluteness Lemma 2.13, we have $W' \in M$. To finish the proof, we put $W(\lambda, x) \coloneqq W'(\max\{\operatorname{supp} \lambda, \operatorname{supp} x\}, \lambda, x)$. By the absoluteness of

$$(*) \quad \exists W : c_{00} \times c_{00}(X) \; \forall \lambda \in c_{00} \; \forall x \in c_{00}(X)$$
$$W(\lambda, x) = W'(\max\{\operatorname{supp} x, \operatorname{supp} \lambda\}, \lambda, x)$$

and Lemma 2.13, we have $W \in M$.

To prove the in particular part, we put $X = \mathbb{R}$.

□

2.4 Suitable Models and Rich Families

We point out how the method of suitable models relates to the concept of rich families. The reader can see [11] for more details. We will utilize this notion to summarize propositions proved in Chapter 4. We formulate the following definitions only for Banach spaces, although it is possible to define these concepts for a general topological space too.

Definition 2.23. Let X be a Banach space. The family of all closed separable subspaces of X is $\mathcal{S}(X)$. A family $\mathcal{F} \subset \mathcal{S}(X)$ is *rich* if both following conditions hold.

(i) Each separable subspace of X is contained in an element of $\mathcal{F}$.

(ii) For every sequence $(F_i)_{i=1}^{\infty} \subset \mathcal{F}$ such that $F_1 \subset F_2 \subset F_3 \subset \ldots$, the closure of the union $\overline{\bigcup_{i=1}^{\infty} F_i}$ belongs to $\mathcal{F}$.

Definition 2.24. Let X be a Banach space. We say a family $\mathcal{F} \subset \mathcal{S}(X)$ is *large in the sense of suitable models* if there exists a finite list of formulas Φ and a countable set Y such that

$$\mathcal{F} = \{X_M; M \prec (\Phi, Y) \text{ contains } X\}.$$

The following theorem states that the two approaches coincide if we deal with Banach spaces. The proof can be found in [11, Theorem 4].

Theorem 2.25. *Let X be a Banach space, $\mathcal{F} \subset \mathcal{S}(X)$.*

(i) *If $\mathcal{F}$ is a rich family, then there is $\mathcal{F}' \subset \mathcal{F}$ which is large in the sense of suitable models.*

(ii) *If $\mathcal{F}$ is large in the sense of suitable models, then there is $\mathcal{F}' \subset \mathcal{F}$ which is rich.*

3. Existence of Almost Isometric Ideals Using the Method of Suitable Models

In this chapter, we define almost isometric ideals. We use the method of suitable models to prove their existence. We devote an entire section to Key Lemma 3.4, which is crucial in the proof of the Main Theorem 3.5, which is used to prove the existence of an almost isometric ideal. In the last section, we collect several corollaries.

The notion of an almost isometric ideal was first introduced in [2] in 2014. The introduction is a result of research in *diameter two properties* of Banach spaces. An almost isometric ideal inherits both local and strong diameter two property. We will see this in Chapter 4.

Definition 3.1. Let X be a Banach space, $Y \subset X$ a subspace. We say Y is an *almost isometric ideal in* X if there exists a Hahn–Banach extension operator $E : Y^* \to X^*$ such that for every $\varepsilon > 0$, every finite-dimensional subspace $F \subset X$ and every finite-dimensional subspace $F_* \subset Y^*$ there is a linear operator $T : F \to Y$ which satisfies the following conditions

(i) $Tx = x$ for all $x \in F \cap Y$,

(ii) $(Ey^*)x = y^*(Tx)$ for all $x \in F$, $y^* \in F_*$,

(iii) $(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|$ for all $x \in F$.

The set of all Hahn–Banach extension operators from Y^* to X^* which satisfy the conditions above is denoted by $\mathbb{HB}_{\mathrm{ai}}(Y^*, X^*)$.

Remark 3.2. In [2, Theorem 1.4], it was shown that we could omit (ii) in the definition of an almost isometric ideal. We use the "bulkier" Definition 3.1 because we will later provide proof of the existence of an almost isometric ideal that satisfies all the points in Definition 3.1.

3.1 Key Lemma

The following preliminary lemma essentially states that the linear projection associated with a finite-dimensional, hence complemented, subspace belongs to a suitable model.

Lemma 3.3. *For a suitable model M the following holds. Let X be a Banach space which is contained in M, $B \in M \cap [X]^{\leq\omega}$ such that $E := \overline{\operatorname{span}}\, B$ has finite dimension. Then there is a linear projection $P_B : X \to X$ such that $P_B[X] = \overline{\operatorname{span}}\, B$, $P_B \in M$ and $\operatorname{Ker} P_B \in M$.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$.

The subspace E is a finite-dimensional subspace of a Banach space X, hence it is complemented. There exists a continuous linear projection $P_B : X \to X$ such that $P_B[X] = E$. The absoluteness of the formula

$$(*)\ \exists S\ \forall x\ \left(x \in S \leftrightarrow x \in \bigcap \{P \subset X; P \text{ is a closed subspace}, B \subset P\}\right)$$

and Lemma 2.13 imply $E = \overline{\text{span}} B \in M$. From the formula

$$(*) \quad \exists P\ \ (P : X \to X \text{ is a projection such that } P[X] = E)$$

and the Absoluteness Lemma 2.13, we deduce $P_B \in M$.

Analogously, the formula

$$(*) \quad (\exists K \subset X)\ (\forall x)\ \ (x \in K \leftrightarrow P_B(x) = 0)$$

and Lemma 2.13 imply $\operatorname{Ker} P_B \in M$.

□

The Key Lemma is a vital result. We will utilize it in the proof of the Main Theorem 3.5.

Lemma 3.4 (Key Lemma)**.** *For a suitable model M, the following holds. Let X be a Banach space which is contained in M, $\varepsilon > 0$, $D \in M \cap [X^*]^{fin}$, $B \in M \cap [X]^{fin}$ and E, $E \supset B$ is a finite-dimensional subspace. Then there is a continuous linear mapping $T : E \to X_M$ such that*

(Ka) $Tx = x$ *for all* $x \in B$,

(Kb) $(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|$ *for all* $x \in E$,

(Kc) $\|(d \circ T - d)|_E\| \leq \varepsilon \|d\|$ *for all* $d \in D$,

Proof. Let us have a suitable model M for the formulas marked with $(*)$. Let us have $\varepsilon > 0$, $D \in M \cap [X^*]^{\text{fin}}$, $B \in M \cap [X]^{\text{fin}}$ and $E \supset B$ a finite-dimensional subspace.

Since $\overline{\text{span}} B$ is finite-dimensional, by Lemma 3.3, there is a projection $P_B : X \to X$ satisfying $P_B[X] = \overline{\text{span}} B$, $P_B \in M$, and $U := \operatorname{Ker} P_B \in M$.

For $k := \dim E \cap U$, the space $E \cap U$ is isomorphic to ℓ_1^k, that is, $\mathbb{R}^k$ equipped with the finite ℓ_1 norm. We can find a basis $(e_i)_{i \leq k}$, $\|e_i\| = 1$ of this space and $C > 0$ such that for all $a \in \ell_1^k$ the following inequality holds

$$\left\| \sum_{i=1}^{k} a_i e_i \right\|_X \geq C \|a\|_{\ell_1^k}. \tag{3.1}$$

For all $i \in \mathbb{N} \cup \{\mathbb{N}\}$, we can find a mapping $\Lambda_i : \mathbb{N} \to \ell_1^i$ such that the set $\{\Lambda_i(n); n \in \mathbb{N}\}$ is dense in ℓ_1^i. To ensure the mappings are in the suitable model M, we introduce the following formula for $i \in \mathbb{N}$

$$(*) \quad (\exists \Lambda_i : \mathbb{N} \to \ell_1^i)\ \left(\{\Lambda_i(n); n \in \mathbb{N}\} \text{ is dense in } \ell_1^i\right).$$

Then $\{\Lambda_i; i \in \mathbb{N} \cup \{\mathbb{N}\}\} \in M$ by the Absoluteness Lemma 2.13.

Now we choose $\delta \in \mathbb{Q}$, $\delta > 0$ such that $\delta < \varepsilon$, $\delta < \frac{C\varepsilon}{(3+2k)\|I-P_B\|}$ and $1+\delta < \frac{1}{1-\varepsilon}$. We also pick $G \in \mathbb{N}$ which satisfies $G > \frac{\delta+2}{\delta}$.

We write $GB_{\overline{\operatorname{span}}B} \coloneqq \{x \in \overline{\operatorname{span}}B; \|x\| \leq G\}$. Because $\overline{\operatorname{span}}B$ is finite-dimensional, the ball $GB_{\overline{\operatorname{span}}B}$ is compact. There exists a finite δ-net of $GB_{\overline{\operatorname{span}}B}$. From the absoluteness of the formula

$$(*) \quad \exists S \ (S \text{ is a finite } \delta\text{-net of } GB_{\overline{\operatorname{span}}B}, S \subset GB_{\overline{\operatorname{span}}B}),$$

and Lemma 2.13, we have a finite δ-net B' of $GB_{\overline{\operatorname{span}}B}$ such that $B' \in M$.

For the given δ, we can find $P \subset \mathbb{N}$ such that $W \coloneqq \{\Lambda_k(i); i \in P\}$ is a finite δ-net of the unit sphere in ℓ_1^k. We denote the members of W by $\Lambda_{k,\delta}(i)$ to emphasize their dependence on δ. Now we define a mapping $\phi : (B_U)^k \to \mathbb{R}^{B' \times k} \times \mathbb{R}^{D \times k}$, where B_U is the unit ball in U, by

$$\phi((u_n)) \coloneqq \left(\left(\left\| b + \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n u_n \right\| \right)_{b \in B', i \leq k}, (d(u_n))_{d \in D, n \leq k} \right), \tag{3.2}$$

for $(u_n)_{n=1}^k \in (B_U)^k$. The absoluteness of the formula

$$(*) \quad \exists \varphi \ (\varphi : B_U^k \to \mathbb{R}^{B' \times k} \times \mathbb{R}^{D \times k} \text{ is a mapping which satisfies (3.2)})$$

and Lemma 2.13 assure $\phi \in M$. Because the range of ϕ is separable, the absoluteness of the formula

$$(*) \quad \exists F \in [(B_U)^k]^{\leq \omega} : \phi[F] \text{ is dense in } \phi[(B_U)^k],$$

and Lemma 2.13 imply the existence of a countable set $F \subset (B_U)^k$, $F \in M$ such that $\phi[F]$ is dense in $\phi[(B_U)^k]$. Because F is countable, $F \subset M$ by Lemma 2.18. We use this to approximate the basis $(e_n)_{n \leq k}$, $\|e_n\| = 1$, of $E \cap U$. That is, for $\phi((e_n))$ we find $(f_n)_{n \leq k} \subset F$ such that $\|\phi((e_n)) - \phi((f_n))\| < \delta$ in the norm of the space $\mathbb{R}^{B' \times k} \times \mathbb{R}^{D \times k}$. More precisely

$$\left| \left\| b + \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n e_n \right\| - \left\| b + \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n f_n \right\| \right| < \delta, \quad b \in B', i \leq k, \tag{3.3}$$

and

$$|d(e_n) - d(f_n)| < \delta, \quad d \in D, n \leq k. \tag{3.4}$$

We decompose the space E as the topological sum of $E \cap \overline{\operatorname{span}}B$ and $E \cap U$. Further, we consider the unique linear mapping $S : E \cap U \to X$ which satisfies $S(e_i) = f_i$ for $i = 1, \ldots, k$, where $(e_i)_{i \leq k}$ is the norm-one basis satisfying the estimate (3.1). Next, we introduce $T : E \to X_M$, $T \coloneqq P_B + S \circ (\mathrm{Id}_X - P_B)$. Then T is evidently a continuous linear operator. The operator T can also be written more directly, which is better suited for our needs. For $y \in E \cap \overline{\operatorname{span}}B$ and $a \in \mathbb{R}^k$ we have

$$T\left(y + \sum_{n=1}^{k} a_n e_n\right) = y + \sum_{n=1}^{k} a_n f_n.$$

The mapping T is basically two mappings combined. Each of them acts on one summand of E. On the set $E \cap \overline{\operatorname{span}}B$, the respective mapping is the identity. On $E \cap U$, the corresponding mapping perturbs the vector $\sum_{n=1}^k a_n e_n$ by mapping it to $\sum_{n=1}^k a_n f_n$.

We prove (Ka). If $x \in B \setminus \{0\}$, then x is not a member of $\operatorname{Ker} P_B = U$, as U is the topological complement of B. This means T is the identity on B.

First, we prove (Kb) holds for all y in the form $y = x + \sum_{n=1}^{k} a_n e_n$ where $a \in \ell_1^k$ is from the unit sphere and $x \in GB_{\overline{\operatorname{span} B}}$. Let us have such $x + \sum_{n=1}^{k} a_n e_n \in E$. For this x, we find $b \in B'$ such that $\|x - b\|_X < \delta$, where B' is the finite δ-net of $GB_{\overline{\operatorname{span} B}}$. We can also find $\Lambda_{k,\delta}(i) \in W$ such that $\|\Lambda_{k,\delta}(i) - a\|_{\ell_1^k} < \delta$, where W is the finite δ-net of the unit sphere in ℓ_1^k.

Now we perform two auxiliary calculations,

$$\left\| \sum_{n=1}^{k} a_n e_n - \sum_{n=1}^{k} \left(\Lambda_{k,\delta}(i)\right)_n e_n \right\| \leq \sum_{n=1}^{k} \left| a_n - \left(\Lambda_{k,\delta}(i)\right)_n \right| \|e_n\| < \delta k, \tag{3.5}$$

$$\left\| \sum_{n=1}^{k} a_n f_n - \sum_{n=1}^{k} \left(\Lambda_{k,\delta}(i)\right)_n f_n \right\| \leq \sum_{n=1}^{k} \left| a_n - \left(\Lambda_{k,\delta}(i)\right)_n \right| \underbrace{\|f_n\|}_{\leq 1} < \delta k. \tag{3.6}$$

From this, we deduce

$$\begin{aligned} &\left| \left\| x + \sum_{n=1}^{k} a_n e_n \right\| - \left\| x + \sum_{n=1}^{k} a_n f_n \right\| \right| \\ &\qquad \leq \|x - b\| + \left\| \sum_{n=1}^{k} a_n e_n - \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n e_n \right\| \\ &\qquad + \|x - b\| + \left\| \sum_{n=1}^{k} a_n f_n - \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n f_n \right\| \\ &\qquad + \left| \left\| b + \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n e_n \right\| - \left\| b + \sum_{n=1}^{k} (\Lambda_{k,\delta}(i))_n f_n \right\| \right| \\ &\qquad < \delta + k\delta + \delta + k\delta + \delta \\ &\qquad = \delta(3 + 2k), \end{aligned} \tag{3.7}$$

where we used (3.5), (3.6), and the choice of f_n (3.3) in the last inequality.

Since

$$\sum_{n=1}^{k} a_n e_n = (I - P_B) \left(x + \sum_{n=1}^{k} a_n e_n \right),$$

the following holds

$$\begin{aligned} \left\| x + \sum_{n=1}^{k} a_n e_n \right\| &\geq \frac{\left\| (I - P_B) \left(x + \sum_{n=1}^{k} a_n e_n \right) \right\|}{\|I - P_B\|} \\ &= \frac{1}{\|I - P_B\|} \left\| \sum_{n=1}^{k} a_n e_n \right\| \\ &\overset{(3.1)}{\geq} \frac{C \|a\|_{\ell_1^k}}{\|I - P_B\|} \\ &= \frac{C}{\|I - P_B\|}. \end{aligned}$$

Which means

$$\varepsilon \left\| x + \sum_{n=1}^{k} a_n e_n \right\| \geq \varepsilon \frac{C}{\|I - P_B\|} > \delta(3 + 2k), \tag{3.8}$$

due to our choice of δ. Combining (3.7) and (3.8) yields

$$\left| \left\| x + \sum_{n=1}^{k} a_n e_n \right\| - \left\| x + \sum_{n=1}^{k} a_n f_n \right\| \right| < \varepsilon \left\| x + \sum_{n=1}^{k} a_n e_n \right\|. \tag{3.9}$$

Thus the mapping T satisfies

$$(1-\varepsilon) \left\| x + \sum_{n=1}^{k} a_n e_n \right\| \le \left\| x + \sum_{n=1}^{k} a_n f_n \right\| \le (1+\varepsilon) \left\| x + \sum_{n=1}^{k} a_n e_n \right\|, \tag{3.10}$$

for $a \in \ell_1^k$, $\|a\| = 1$ and $x \in GB_{\overline{\operatorname{span} B}}$.

Now we prove (3.10) also holds for all $a \in S_{\ell_1^k}$ and $x \in X \setminus GB_{\overline{\operatorname{span} B}}$. Let us pick arbitrary $a \in S_{\ell_1^k}$ and $x \in X \setminus GB_{\overline{\operatorname{span} B}}$. We observe

$$\|x\| - 1 \le \|x\| - \left\| \sum_{n=1}^{k} a_n e_n \right\| \le \left\| x + \sum_{n=1}^{k} a_n e_n \right\| \le \|x\| + \sum_{n=1}^{k} \|a_n e_n\| \le \|x\| + 1.$$

The same argument yields

$$\|x\| - 1 \le \left\| x + \sum_{n=1}^{k} a_n f_n \right\| \le \|x\| + 1.$$

This means

$$\frac{\left\| x + \sum_{n=1}^{k} a_n f_n \right\|}{\left\| x + \sum_{n=1}^{k} a_n e_n \right\|} \le \frac{\|x\| + 1}{\|x\| - 1} \text{ and } \frac{\left\| x + \sum_{n=1}^{k} a_n e_n \right\|}{\left\| x + \sum_{n=1}^{k} a_n f_n \right\|} \le \frac{\|x\| + 1}{\|x\| - 1}. \tag{3.11}$$

We consider the real function $f : z \mapsto \frac{z+1}{z-1}$. Because $\|x\| > G > \frac{\delta+2}{\delta}$ and the function f is decreasing, we calculate

$$\frac{\|x\| + 1}{\|x\| - 1} = f(\|x\|) < f\left(\frac{\delta+2}{\delta}\right) = \delta + 1.$$

From this, the first inequality in (3.11) and $\delta < \varepsilon$

$$\left\| x + \sum_{n=1}^{k} a_n f_n \right\| \le (1+\varepsilon) \left\| x + \sum_{n=1}^{k} a_n e_n \right\|. \tag{3.12}$$

From the other inequality in (3.11) and $1 + \delta < \frac{1}{1-\varepsilon}$

$$(1-\varepsilon) \left\| x + \sum_{n=1}^{k} a_n e_n \right\| \le \left\| x + \sum_{n=1}^{k} a_n f_n \right\|. \tag{3.13}$$

This means that (3.10) holds for $a \in \ell_1^k$, $\|a\| = 1$ and $x \in X \setminus GB_{\overline{\operatorname{span} B}}$.

Now we justify why it is enough to work with $a \in S_{\ell_1^k}$. Let us have $x \in \overline{\operatorname{span}} B$ and $a \in \ell_1^k \setminus \{0\}$. We have

$$\left\| x + \sum_{n=1}^{k} a_n f_n \right\| = \|a\|_{\ell_1^k} \left\| \frac{x}{\|a\|_{\ell_1^k}} + \sum_{n=1}^{k} \frac{a_n}{\|a\|_{\ell_1^k}} f_n \right\|.$$

Then, we estimate

$$\|a\|_{\ell_1^k}\left\|\frac{x}{\|a\|_{\ell_1^k}}+\sum_{n=1}^{k}\frac{a_n}{\|a\|_{\ell_1^k}}f_n\right\| \le \|a\|_{\ell_1^k}(1+\varepsilon)\left\|\frac{x}{\|a\|_{\ell_1^k}}+\sum_{n=1}^{k}\frac{a_n}{\|a\|_{\ell_1^k}}e_n\right\|,$$

where we used (3.10) and (3.12). Analogously, we use (3.10) and (3.13) to obtain

$$\|a\|_{\ell_1^k}\left\|\frac{x}{\|a\|_{\ell_1^k}}+\sum_{n=1}^{k}\frac{a_n}{\|a\|_{\ell_1^k}}f_n\right\| \ge (1-\varepsilon)\|a\|_{\ell_1^k}\left\|\frac{x}{\|a\|_{\ell_1^k}}+\sum_{n=1}^{k}\frac{a_n}{\|a\|_{\ell_1^k}}e_n\right\|.$$

We conclude the inequality (3.10) holds for all $x \in \overline{\operatorname{span}}B$ and all $a \in \ell_1^k$. Thus (Kb) holds.

Next, we prove (Kc). Let us denote $y = x + \sum_{n=1}^{k} a_n e_n$, $\|a\|_{\ell_1^k} = 1$ and pick $d \in D \cap [X^*]^{\text{fin}}$. Then from definition (3.4) of f_n, $|d(e_n) - d(f_n)| < \delta$. Also

$$\begin{aligned}
|d(Ty)-d(y)| &= \left|d\left(x+\sum_{n=1}^{k}a_nf_n\right)-d\left(x+\sum_{n=1}^{k}a_ne_n\right)\right| \\
&= \left|\sum_{n=1}^{k}a_nd(e_n-f_n)\right| < \sum_{n=1}^{k}|a_n|\delta = \delta\|a\|_{\ell_1^k} \\
&\le \frac{\delta}{C}\left\|\sum_{n=1}^{k}a_ne_n\right\| \le \frac{\delta}{C}\|(I-P_B)(y)\| \\
&\le \frac{\delta}{C}\|I-P_B\|\|y\| < \varepsilon\|y\|.
\end{aligned}$$

To obtain the result for any $a \in \ell_1^k$ we, as in the previous paragraph, divide by the norm $\|a\|_{\ell_1^k}$ and utilize what we have just proved.

□

3.2 Main Theorem

This is a central result in this text. We will collect several corollaries of this theorem in the section below. In particular, it will allow us to prove the main new result in this text, namely, that X_M is an almost isometric ideal.

Theorem 3.5. *For every suitable model M, the following holds. Let X be a Banach space which is contained in M. Then there is a linear operator $R : X \to (X_M)^{**}$, $\|R\| \le 1$ which satisfies*

(Ra) *$R(x) = \kappa_{X_M}(x)$ for all $x \in X_M$.*

(Rb) *$Rx(d|_{X_M}) = d(x)$ for all $x \in X$ and $d \in \overline{X^* \cap M}$.*

(Rc) *For all $\varepsilon > 0$ and all finite-dimensional subspaces $E \subset X$, $F \subset (X_M)^*$ there exists a linear operator $T : E \to X_M$ such that*

- (Rc1) *$Tx = x$ for all $x \in E \cap X_M$,*
- (Rc2) *$Rx(x^*) = x^*(Tx)$ for all $x \in E$ and $x^* \in F$,*

(Rc3) $(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|$ *for all* $x \in E$.

Proof. Let us have a suitable model M for the formulas marked with $(*)$. We introduce the system

$$\mathcal{I} := \{I = (E, B, D, \varepsilon);\ E \subset X, E \text{ is a finite-dimensional subspace},\\ B \subset X \cap M \text{ is finite}, B \subset E, D \subset X^* \cap M \text{ is finite}, \varepsilon > 0\}.$$

and a binary relation $\leq$ on $\mathcal{I}$. For $I, I' \in \mathcal{I}$ we write $(E, B, D, \varepsilon) = I \leq I' = (E', B', D', \varepsilon')$ if $E \subset E'$, $B \subset B'$, $D \subset D'$ and $\varepsilon' \leq \varepsilon$. This relation is undoubtedly transitive and reflexive. Let us have arbitrary $I = (E, B, D, \varepsilon) \in \mathcal{I}$ and $I' = (E', B', D', \varepsilon') \in \mathcal{I}$. The four-tuple $K := (E + E', B \cup B', D \cup D', \min\{\varepsilon, \varepsilon'\})$ belongs to $\mathcal{I}$ and satisfies $I \leq K$ and $I' \leq K$. It is clear that $(\mathcal{I}, \leq)$ has no maximal element. Thus the pair $(\mathcal{I}, \leq)$ is an unbounded directed set.

For a given $(E, B, D, \varepsilon) = I \in \mathcal{I}$, there is an operator $T_I : E \to X_M$ from the Key Lemma 3.4 which satisfies (Ka)-(Kc). We define the operator $R_I : X \to X_M$ in the following way

$$R_I x := \begin{cases} T_I x & \text{for } x \in E, \\ 0 & \text{for } x \notin E. \end{cases}$$

We will show the sought-after operator $R : X \to (X_M)^{**}$ is the limit of R_I with respect to a directed ultrafilter $\mathcal{U}$ on $(\mathcal{I}, \leq)$

$$Rx(x^*) := \lim_{\mathcal{U}} x^*(R_I x), \quad x \in X,\ x^* \in (X_M)^*.$$

Informally, we want the operator R to be "something like a limit with respect to the w^* topology."

Because $(x^*(R_I x))_{I \in \mathcal{I}}$ is a bounded indexed family of real numbers, the limit with respect to the ultrafilter $\mathcal{U}$ exists and is unique by Theorem 1.14.

To show $Rx \in (X_M)^{**}$, we consider $x \in X$, $x^* \in (X_M)^*$. Choose $\varepsilon \in (0,1)$. Mark $J_0 := (\operatorname{span}\{x\}, \emptyset, \emptyset, \varepsilon)$. Then

$$\begin{aligned} |Rx(x^*)| &= \left|\lim_{\mathcal{U}} x^*(R_J x)\right| \\ &\overset{(a)}{=} \left|\lim_{\mathcal{U}} x^*(T_J x)\right| \\ &\overset{(b)}{=} \lim_{\mathcal{U}} |x^*(T_J x)| \\ &\leq \lim_{\mathcal{U}} \|x^*\| \|T_J\| \|x\| \\ &< (1+\varepsilon)\|x\|\|x^*\|. \end{aligned}$$

In (a), we used for any $(\hat{E}, \hat{B}, \hat{D}, \hat{\varepsilon}) = J \in [J_0, \to)$ the vector x belongs to $\hat{E}$. In (b), we used Lemma 1.23. In the very last inequality, we made good use of the binary relation $\leq$ on $\mathcal{I}$. For any $(E', B', D', \varepsilon') = J \in [J_0, \to)$, we have the estimate $\|T_J\| \leq (1+\varepsilon') \leq (1+\varepsilon)$. Since Rx is linear by the linearity of limits with respect to the ultrafilter $\mathcal{U}$, Lemma 1.26, the computation above shows that $Rx \in (X_M)^{**}$. Since $\varepsilon > 0$ was arbitrarily small, we have

$$\|Rx\| \leq \|x\| \tag{3.14}$$

for every $x \in X$.

Now we show the linearity of R. Pick $x, y \in X$, $\lambda, \mu \in \mathbb{R}$ and $x^* \in (X_M)^*$. We can find a finite-dimensional subspace E_0 which contains both x and y, for example, $\operatorname{span}\{x, y\}$. Put $I_0 \coloneqq (E_0, \emptyset, \emptyset, 1)$. Then for any $I \in [I_0, \to)$

$$\begin{aligned} x^* \left(R_I(\lambda x + \mu y)\right) &= x^* \left(T_I(\lambda x + \mu y)\right) \\ &= \lambda x^*(T_I x) + \mu x^*(T_I y) \\ &= \lambda x^*(R_I x) + \mu x^*(R_I y), \end{aligned}$$

because T_I is linear. This allows us to use Lemma 1.25. Together with Lemma 1.26, we have

$$\begin{aligned} R(\lambda x + \mu y)(x^*) &= \lim_{\mathcal{U}} x^*(R_I(\lambda x + \mu y)) \\ &= \lambda \lim_{\mathcal{U}} x^*(R_I x) + \mu \lim_{\mathcal{U}} x^*(R_I y) \\ &= \lambda R x(x^*) + \mu R y(x^*). \end{aligned}$$

We have just proved that $R : X \to (X_M)^{**}$ is a well-defined operator. It follows from (3.14) that $\|R\| \leq 1$.

To prove (Ra), we consider $x \in X \cap M$. We put $I_0 \coloneqq (\operatorname{span}\{x\}, \{x\}, \emptyset, 1)$. Then for any $I \in [I_0, \to)$, we have $x = T_I x = R_I x$ from (Ka). Hence

$$\kappa_{X_M}(x)(x^*) = x^*(x) = x^*(R_I x)$$

for $x^* \in (X_M)^*$. It follows

$$\kappa_{X_M}(x)(x^*) = \lim_{\mathcal{U}} \kappa_{X_M}(x)(x^*) = \lim_{\mathcal{U}} x^*(R_I x) = R x(x^*),$$

by Lemma 1.25. From the continuity of R, the formula above also holds for $x \in \overline{X \cap M} = X_M$.

To show (Rb) holds, we pick $x \in X$, $d \in X^* \cap M$ and $\delta \in (0, 1)$. Put $I_0 \coloneqq \left(\operatorname{span}\{x\}, \emptyset, \{d\}, \frac{\delta}{\|d\|}\right)$. Let us have $(E, B, D, \varepsilon) = I \in [I_0, \to)$. Using (Kc), we have $\|(d \circ T_I - d)|_E\| \leq \varepsilon \|d\|$. In particular, because $x \in E$, we have $|d(T_I x) - d(x)| \leq \varepsilon \|d\|$ and $T_I x \in X_M$. Then by Lemma 1.21, we have

$$\lim_{\mathcal{U}} |d(T_I x) - d(x)| \leq \lim_{\mathcal{U}} \varepsilon \|d\| = \varepsilon \|d\|.$$

Then from Lemma 1.23 and Lemma 1.25, we have

$$\begin{aligned} |R x(d|_{X_M}) - d(x)| &= \lim_{\mathcal{U}} |d|_{X_M}(R_I x) - d(x)| \\ &= \lim_{\mathcal{U}} |d(T_I x) - d(x)| \\ &\leq \varepsilon \|d\| \\ &\leq \delta \frac{\|d\|}{\|d\|} = \delta, \end{aligned}$$

where in the last inequality we used $\varepsilon \leq \frac{\delta}{\|d\|}$ because $I \in [I_0, \to)$. The choice of δ was arbitrary, thus $R x(d|_{X_M}) = d(x)$ for $d \in X^* \cap M$. By the continuity of R, the equality holds for $d \in \overline{X^* \cap M}$ as well. This concludes the proof of (Rb).

In order to prove (Rc), we consider the mapping $P_M : (X_M)^* \to X^*$ defined by

$$P_M x^*(x) := Rx(x^*), \quad x^* \in (X_M)^*,\ x \in X.$$

Since R is linear and $\|R\| \leq 1$, we have $P_M x^* \in X^*$ and $\|P_M x^*\| \leq \|x^*\|$ for every $x^* \in (X_M)^*$, so P_M is well-defined, linear and $\|P_M\| \leq 1$. Since $R = \kappa_{X_M}$ on X_M, we have $P_M x^*|_{X_M} = x^*|_{X_M}$ for every $x^* \in X^*$.

Now we pick $\varepsilon > 0$ and finite-dimensional subspaces $E \subset X$ and $F \subset (X_M)^*$. Let $\{x_1^*, \dots, x_{\dim F}^*\} \subset S_F$ be a basis of $F \subset (X_M)^*$ and let $\{x_1, \dots, x_{\dim F}\} \subset X_M$ be points which satisfy $x_i^*(x_j) = \delta_{ji}$, where δ_{ji} is the Kronecker delta. Now we consider the operator $Q : X^* \to X^*$ given by the following formula

$$Qx^* := \sum_{i=1}^{\dim F} x^*(x_i) P_M x_i^*, \quad x^* \in X^*.$$

Then Q is clearly bounded and linear. We will show it is a projection as well. Let us begin with an auxiliary calculation. Since $x_i \in X_M$, we use (Ra) together with the biorthogonality of the basis $(x_i^*, x_i)_{i=1}^{\dim F}$

$$P_M x_i^*(x_j) = Rx_j(x_i^*) = \kappa_{X_M}(x_j)(x_i^*) = x_i^*(x_j) = \delta_{ji}.$$

Then

$$\begin{aligned} QQx^* &= Q\left(\sum_{i=1}^{\dim F} x^*(x_i) P_M x_i^*\right) \\ &= \sum_{j=1}^{\dim F} \sum_{i=1}^{\dim F} x^*(x_i) P_M x_i^*(x_j) P_M x_j^* \\ &= \sum_{j=1}^{\dim F} \sum_{i=1}^{\dim F} x^*(x_i) \delta_{ji} P_M x_j^* \\ &= \sum_{i=1}^{\dim F} x^*(x_i) P_M x_i^* \\ &= Qx^*. \end{aligned}$$

So we have Q a projection with $Q[X^*] = P_M[F]$. The dual operator $Q^* : X^{**} \to X^{**}$ is a bounded linear projection. We will relax notation in the rest of this proof. We identify points from X_M with their isometric images in $(X_M)^{**}$. That is, for $x \in X_M$ we write x in the place of $\kappa_{X_M}(x)$. Let us have $x^* \in X^*$, $x^{**} \in X^{**}$. It follows from

$$Q^* x^{**}(x^*) = x^{**}(Qx^*) = \sum_{i=1}^{\dim F} x^*(x_i) x^{**}(P_M x_i^*),$$

that

$$Q^* x^{**} = \sum_{i=1}^{\dim F} x^{**}(P_M x_i^*) x_i, \quad x^{**} \in X^{**},$$

so Q^* satisfies $Q^*[X^{**}] = \operatorname{span}\{x_1, \dots, x_{\dim F}\} \subset X_M$.

We define $K := \max_{i \leq \dim F}\{\|x_i\|, \|x_i^*\|\}$. Pick $\delta > 0$ such that

$$\delta < \frac{\varepsilon}{2K(1 + \dim E)(\dim F)(3 + \varepsilon)}$$

and a finite δ-net N in S_E. Put $\eta := \frac{\varepsilon}{2(1+\dim E)}$. From the compactness of $B_{E\cap X_M}$, we find a finite set $\widetilde{B} \subset B_{E\cap X_M}$ such that for all $x \in B_{E\cap X_M}$ there is $\tilde{b} \in \widetilde{B}$ which satisfies $\|x - \tilde{b}\| < \frac{\eta}{2(\eta+2)}$. There is a mapping $\varphi : \widetilde{B} \to M$ such that for $\tilde{b} \in \widetilde{B}$ it holds $\|\varphi(\tilde{b}) - \tilde{b}\| < \frac{\eta}{2(\eta+2)}$. Finally, we mark the set $\varphi[\widetilde{B}]$ as B. Then by Lemma 2.17, $B \in M$ and for any $x \in B_{E\cap X_M}$ we can find $b \in B$ satisfying $\|x - b\| < \frac{\eta}{\eta+2}$.

Since $\lim_{\mathcal{U}} x^*(R_I x) = Rx(x^*)$ for every $x \in N$ and $x^* \in (X_M)^*$, there exists $I = (E', B', D', \eta) \in \mathcal{I}$ such that $E' \supset \operatorname{span}(E \cup B)$, $B' \supset B$, and

$$|x_i^*(R_I x) - Rx(x_i^*)| < \delta, \quad i \leq \dim F,\ x \in N.$$

We will show the operator $T_I : E' \to X_M$ associated with the four-tuple $(E', B', D', \eta) = I$ satisfies $\|(T_I - \mathrm{Id})|_{E\cap X_M}\| \leq \eta$. Let us choose arbitrary $x \in B_{E\cap X_M}$. We can find $b \in B$ such that $\|x - b\| < \frac{\eta}{\eta+2}$. Then

$$\begin{aligned}
\|T_I x - x\| &\leq \|T_I x - T_I b\| + \|T_I b - b\| + \|x - b\| \\
&< \|T_I(x - b)\| + 0 + \frac{\eta}{\eta+2} \\
&< \|T_I\| \frac{\eta}{\eta+2} + \frac{\eta}{\eta+2} \\
&\leq (1+\eta)\frac{\eta}{\eta+2} + \frac{\eta}{\eta+2} = \eta,
\end{aligned}$$

where we utilized T_I is the identity on B in the second inequality. Thus, $\|(T_I - \mathrm{Id}_X)|_{E\cap X_M}\| < \eta$.

Also $R_I|_{E'} = T_I|_{E'}$ by the definition of R_I and $P_M x_i^*|_{X_M} = x_i^*|_{X_M}$. From all this, we get

$$|P_M x_i^*(T_I - \mathrm{Id})(x)| = |P_M x_i^*(R_I x - x)| = |x_i^*(R_I x) - Rx(x_i^*)| < \delta, \tag{3.15}$$

for $i \in \{1, \ldots, \dim F\}$, $x \in N$.

Now we pick a projection $P : E \to E$ with $\|P\| \leq \dim E$, $P[E] = E \cap X_M$ and consider the operator $S_I : E \to X_M$ defined as

$$\begin{aligned}
S_I &:= P + T_I(\mathrm{Id}_E - P) - Q^*(T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P) \\
&= P + (T_I - Q^*(T_I - \mathrm{Id}_E))(\mathrm{Id}_E - P) \\
&= P + ((T_I - \mathrm{Id}_E) + \mathrm{Id}_E - Q^*(T_I - \mathrm{Id}_E))(\mathrm{Id}_E - P) \\
&= P + (\mathrm{Id}_{X^{**}} - Q^*)(T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P) + \mathrm{Id}_E - \mathrm{Id}_E P \\
&= \mathrm{Id}_E + (\mathrm{Id}_{X^{**}} - Q^*)(T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P).
\end{aligned} \tag{3.16}$$

Then for any $e \in E \cap X_M$, we have $S_I(e) = e + T_I(0) - Q^*(T_I - \mathrm{Id}_E)(0) = e$, e is in the range of P, and so $S_I = \mathrm{Id}$ on $E \cap X_M$. This proves (Rc1). Now we prove (Rc2). For every $x^* \in F$ and $x \in E$ using that $S_I[E] \subset X_M$, we obtain

$$\begin{aligned}
x^*(S_I x) &= P_M x^*(S_I x) \\
&\overset{(3.16)}{=} P_M x^*(x) + \Big((\mathrm{Id}_{X^{**}} - Q^*)(T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P)x\Big)(P_M x^*) \\
&= P_M x^*(x) + \Big((T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P)x\Big)\Big((\mathrm{Id}_{X^*} - Q)P_M x^*\Big) \\
&= P_M x^*(x) + \Big((T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P)x\Big)(0) \\
&= P_M x^*(x) \\
&= Rx(x^*).
\end{aligned}$$

It remains to verify S_I satisfies (Rc3). We begin with a helpful inequality. For arbitrary $e \in S_E$, we find $x \in N$ such that $\|e - x\| \leq \delta$. Recall that $K = \max_{i \leq \dim F}\{\|x_i\|, \|x_i^*\|\}$. Lastly, we will employ $\|P_M\| \leq 1$. Then we have

$$\begin{aligned}
\|Q^*(T_I - \mathrm{Id}_E)\| &\leq \sum_{i=1}^{\dim F} \|x_i\| \sup_{e \in S_E} |P_M x_i^*(T_I e - e)| \\
&\leq \sum_{i=1}^{\dim F} \|x_i\| \sup_{e \in S_E} |P_M x_i^*((T_I e - T_I x) + (x - e) + (T_I x - x))| \\
&\leq (\dim F) K \left(\|T_I\|\delta + \delta + \sup_{x \in N} |P_M x_i^*(T_I x - x)| \right) \\
&\overset{(3.15)}{\leq} (\dim F) K (3 + \varepsilon)\delta.
\end{aligned} \tag{3.17}$$

It follows

$$\begin{aligned}
\|S_I - T_I\| &\overset{(3.16)}{=} \|(\mathrm{Id}_E - T_I)P - Q^*(T_I - \mathrm{Id}_E)(\mathrm{Id}_E - P)\| \\
&\leq \|(\mathrm{Id}_E - T_I)|_{E \cap X_M}\| \|P\| + \|Q^*(T_I - \mathrm{Id}_E)\|(1 + \|P\|) \\
&\overset{(3.17)}{\leq} \eta \dim E + (1 + \dim E)(\dim F) K (3 + \varepsilon)\delta + \eta - \eta \\
&\leq (1 + \dim E)\eta + (1 + \dim E)(\dim F) K (3 + \varepsilon)\delta - \eta \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - \eta = \varepsilon - \eta.
\end{aligned}$$

Now we make two final estimates

$$\begin{aligned}
\|S_I x\| - \|x\| &\leq \|(S_I - T_I)x\| + \|T_I x\| - \|x\| \\
&\leq (\varepsilon - \eta)\|x\| + (1 + \eta)\|x\| - \|x\| \\
&= \varepsilon \|x\|,
\end{aligned}$$

if $\|S_I x\| - \|x\| \geq 0$. The remaining case follows

$$\begin{aligned}
\|S_I x\| - \|x\| &\geq \|T_I x\| - \|(S_I - T_I)x\| - \|x\| \\
&\geq (1 - \eta)\|x\| + (\eta - \varepsilon)\|x\| - \|x\| \\
&= -\varepsilon \|x\|.
\end{aligned}$$

We have shown $(1 - \varepsilon)\|x\| \leq \|S_I x\| \leq (1 + \varepsilon)\|x\|$. The sought-after linear operator T from (Rc) is precisely the operator S_I.

□

3.3 Corollaries

The following corollary is the main new result in this text. We provide proof of the existence of an almost isometric ideal in its broadest definition.

Corollary 3.6. *For every suitable model M, the following holds. Let X be a Banach space which is contained in M. Then X_M is an almost isometric ideal in X.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$ in the proofs above. Let us define the operator $E : (X_M)^* \to X^*$ as

$$Ex^*(x) := Rx(x^*), \quad x^* \in (X_M)^*, \, x \in X,$$

where $R : X \to (X_M)^{**}$ is the operator from Theorem 3.5. We verify that E is a Hahn–Banach extension operator. Because $\|R\| \leq 1$, we immediately have $\|E\| \leq 1$. Now we pick $y^* \in (X_M)^*$ and $y \in X_M$. Then

$$Ey^*(y) = Ry(y^*) = \kappa_{X_M} y(y^*) = y^*(y),$$

where the second equality is due to (Ra) in Theorem 3.5. Thus for all $y^* \in (X_M)^*$, we have $Ey^*|_Y = y^*$. Combining this with the estimate $\|E\| \leq 1$, we have $\|E\| = 1$. Hence, E is a Hahn–Banach extension operator.

Now, we pick $\varepsilon > 0$ and finite-dimensional subspaces $F \subset X$, $F_* \subset (X_M)^*$. From (Rc) in Theorem 3.5, there exists a linear operator $T : F \to X_M$ which satisfies

(i) $Tx = x$ for all $x \in F \cap X_M$,

(ii) $Rx(x^*) = x^*(Tx)$ for all $x \in F$, $x^* \in F_*$,

(iii) $(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|$ for all $x \in F$.

We show that the operator T witnesses X_M is an almost isometric ideal. Because of the listed properties (i) and (iii) of the operator T, we only need to check $Ex^*(x) = x^*(Tx)$ for $x \in F$ and $x^* \in F_*$. This follows from

$$Ex^*(x) = Rx(x^*) \overset{\text{(ii)}}{=} x^*(Tx).$$

Thus X_M is an almost isometric ideal in X according to Definition 3.1. ☐

The corollary below has no mention of suitable models in its statement.

Corollary 3.7. *Let X be a Banach space. Then for each $S \subset X$ there exists an almost isometric ideal Y in X which satisfies $S \subset Y$ and $\operatorname{dens} Y \leq \max\{|S|, \omega\}$.*

Proof. If $|S| = \operatorname{dens} X$, we may put $Y = X$, and there is nothing to prove. We may assume that $|S| < \operatorname{dens} X$. To find the almost isometric ideal Y, we use the results obtained using the method of suitable models. We look at the conventions we established in Convention 2.14, Convention 2.15, and Notation 2.19. According to them, we construct a set that will allow us to prove this corollary. First, the set needs to contain the elementary structures and operations. We define the set A as follows

$$A := S \cup \{X, \|\cdot\|, +, \cdot, \omega, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \overbrace{+, -, /, \cdot}^{\text{operations on } \mathbb{R}}, <, |\cdot|\} \cup \mathbb{Q}.$$

Now we put all formulas (and their subformulas) marked with $(*)$ in the proof of Corollary 3.6 and in the proofs of preceding statements which were used in the proof of Corollary 3.6 to a finite list Φ. From Theorem 2.12, there is a set $M \supset A$ such that $M \prec (\Phi, A)$ and $|M| \leq \max\{\omega, |A|\}$. By Corollary 3.6, X_M is an almost isometric ideal in X. The sought-after almost isometric ideal Y is the subspace X_M.

□

We can find a linear bounded projection such that its kernel is the annihilator of the subspace X_M. This will be useful in the proof of the next Corollary 3.9.

Corollary 3.8. *For every suitable model M, the following holds. Let X be a Banach space which is contained in M. Then there exists a projection $P_M : X^* \to X^*$ which satisfies $\|P_M\| \leq 1$, $\operatorname{Ker} P_M = (X_M)^\perp$ and $\overline{X^* \cap M} \subset P_M[X^*]$.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$ in the proofs above. We define the mapping P_M as follows

$$P_M x^*(x) := Rx(x^*|_{X_M}), \quad x \in X, \; x^* \in X^*,$$

where $R : X \to (X_M)^{**}$ is the mapping from Theorem 3.5. First, we check that P_M is a projection. We begin with an auxiliary calculation. Let us have $x_M \in X_M$ and $x^* \in X^*$. Then from (Ra) in Theorem 3.5

$$P_M x^*(x_M) = Rx_M(x^*|_{X_M}) = \kappa_{X_M} x_M(x^*|_{X_M}) = x^*(x_M). \tag{3.18}$$

With this in mind, we pick $x \in X$ and $x^* \in X^*$. Then

$$P_M P_M x^*(x) = Rx(P_M x^*|_{X_M}) \overset{(3.18)}{=} Rx(x^*|_{X_M}) = P_M x^*(x).$$

Thus, for any $x^* \in X^*$ we have $P_M^2 x^* = P_M x^*$.

Because $\|R\| \leq 1$, it follows $\|P_M\| \leq 1$. To verify $\operatorname{Ker} P_M = (X_M)^\perp$, we first show $(X_M)^\perp \subset \operatorname{Ker} P_M$. To this end, we pick $x^* \in (X_M)^\perp$ and $x \in X$. Then $P_M x^*(x) = Rx(x^*|_{X_M}) = Rx(0) = 0$. To show the other inclusion, we consider $x \in X_M$ and $x^* \in \operatorname{Ker} P_M$. Then from (Ra) in Theorem 3.5

$$0 = P_M x^*(x) = Rx(x^*|_{X_M}) = \kappa_{X_M} x(x^*|_{X_M}) = x^*(x).$$

Thus, for any $x^* \in \operatorname{Ker} P_M$ and all $x \in X_M$ we have $x^*(x) = 0$, so $x^* \in (X_M)^\perp$. Now have $\operatorname{Ker} P_M \subset (X_M)^\perp$.

To show the last point in the statement, we pick $d \in \overline{X^* \cap M}$ and $x \in X$. Then

$$P_M d(x) = Rx(d|_{X_M}) = d(x),$$

where we employed (Rb) from Theorem 3.5 in the last equality. Now we have $P_M d = d$, which means $d \in \operatorname{Rng} P_M$. Thus $\overline{X^* \cap M} \subset \operatorname{Rng} P_M$.

□

Another consequence is that the dual space X^* has a non-trivial complemented subspace.

Corollary 3.9. *Let X be a non-separable Banach space. Then, there is a continuous projection $P : X^* \to X^*$ such that $\operatorname{Rng} P$ and $\operatorname{Ker} P$ are both infinite-dimensional.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. The following formula

$$(*) \quad \exists S\ [|S| = \omega\ \wedge\ (\forall x^*\ x^* \in S \to x^* \notin \operatorname{span}(S \setminus \{x^*\}))]$$

expresses that there is a countable linearly independent set S and the formula holds because the dual space X^* is non-separable. From the absoluteness of the formula and Lemma 2.13, there is a set S which satisfies the formula above and $S \in M$. The set S is countable and by Lemma 2.18, it follows $S \subset M$. Thus, we can find an infinite linearly independent set $\{x_1^*, x_2^*, \dots\} \subset M$.

From Corollary 3.8 there is a continuous projection $P_M : X^* \to X^*$ such that $\overline{X^* \cap M} \subset \operatorname{Rng} P_M$ and $\operatorname{Ker} P_M = (X_M)^\perp$. We have $\{x_1^*, x_2^*, \dots\} \subset \overline{X^* \cap M}$, thus the range of P_M is infinite-dimensional.

We will show $(X_M)^\perp$ is infinite-dimensional too. We find $(y_i)_i \subset X$ which satisfies $y_i \notin \overline{\operatorname{span}}\{y_1, \dots, y_{i-1}\} \cup X_M$ for each $i \in \mathbb{N}$. By one of the corollaries of the Hahn–Banach theorem, we can find $y_i^* \in X^*$ such that $y_i^*(y_i) = 1$ and $y_i^*(y) = 0$ for all $y \in \overline{\operatorname{span}}\{y_1, \dots, y_{i-1}\} \cup X_M$. Then, $\{y_1^*, y_2^*, \dots\}$ is a linearly independent set and each $y_i^* \in (X_M)^\perp$. Thus, $\operatorname{Ker} P_M$ is also infinite-dimensional.

□

4. Applications of Almost Isometric Ideals

In this chapter, we present the usefulness of almost isometric ideals and the method of suitable models. We work with closely related diameter two properties and the Daugavet property.

4.1 Diameter Two Properties

Definition 4.1. Let X be a Banach space. For $\varepsilon > 0$ and $x^* \in S_{X^*}$, a *slice of* B_X is the set $S(x^*, \varepsilon) := \{x \in B_X; x^*(x) > 1 - \varepsilon\}$. We say C is a *finite convex combination of slices of* B_X if it is in the form

$$C = \sum_{i=1}^{n} \lambda_i S(x_i^*, \varepsilon_i),$$

where $n \in \mathbb{N}$, $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$, $x_i^* \in S_{X^*}$, $\varepsilon_i > 0$ for each $i \in \{1, \ldots, n\}$.

Definition 4.2. Let X be a Banach space. We say X has

(i) the *local diameter two property* if every slice of B_X has a diameter of two,

(ii) the *strong diameter two property* if every finite convex combination of slices of B_X has a diameter of two.

We prove an almost isometric ideal inherits local and strong diameter two properties from its superspace. The proofs will be straightforward consequences of the following Proposition 4.4. To formulate it clearly, we introduce the following notation.

Notation 4.3. Let X be a Banach space. For $n \in \mathbb{N}$, we denote the set of all finite convex combination of slices of B_X with n summands by $C_X(n)$. That is, $C \in C_X(n)$ is in the form

$$C = \sum_{i=1}^{n} \lambda_i S(x_i^*, \varepsilon_i),$$

where $\sum_{i=1}^{n} \lambda_i = 1$ and for each $i \in \{1, \ldots, n\}$ we have $\lambda_i \geq 0$, $x_i^* \in S_{X^*}$, $\varepsilon_i > 0$.

The proof below is inspired by [2, Proposition 3.3].

Proposition 4.4. *Let X be a Banach space, $Y \subset X$ an almost isometric ideal in X, $n \in \mathbb{N}$. If every $C \in C_X(n)$ has a diameter of two, then every $C' \in C_Y(n)$ has a diameter of two.*

Proof. Let us have $C' := \sum_{i=1}^{n} \lambda_i S(x_i^*, \varepsilon_i) \in C_Y(n)$, a finite convex combination of slices of B_Y with n summands. The subspace Y is an almost isometric ideal. Thus, a Hahn–Banach extension operator $E \in \mathbb{HB}_{\mathrm{ai}}(Y^*, X^*)$ exists. Then, $S(Ex_i^*, \varepsilon_i)$ is a slice of B_X. We put

$$C := \sum_{i=1}^{n} \lambda_i S(Ex_i^*, \varepsilon_i). \tag{4.1}$$

By the assumption, $C \in C_X(n)$. From this, for any $\delta > 0$ we find x_1, $x_2 \in C$ such that $\|x_1 - x_2\| > 2 - \delta$ and $\max\{\|x_1\|, \|x_2\|\} < 1$

To justify that we can find x_1, x_2 such that $\max\{\|x_1\|, \|x_2\|\} < 1$, we provide the following perturbation argument. For $i \in \{1, 2, \dots, n\}$ we find $y_i \in S(Ex_i^*, \varepsilon_i)$, $w_i \in S(Ex_i^*, \varepsilon_i)$ such that $x_1 = \sum_{i=1}^n \lambda_i y_i$ and $x_2 = \sum_{i=1}^n \lambda_i w_i$. A slice of B_X is an open set. This allows us to find $\eta > 0$ such that $(1-\eta)y_i \in S(Ex_i^*, \varepsilon_i)$, $(1-\eta)w_i \in S(Ex_i^*, \varepsilon_i)$, $i \in \{1, \dots, n\}$, and $\eta < 1 - \frac{2-\delta}{\|x_1 - x_2\|}$. Now we put $\widetilde{x_1} := \sum_{i=1}^n (1-\eta)\lambda_i y_i$ and $\widetilde{x_2} := \sum_{i=1}^n (1-\eta)\lambda_i w_i$. Then $\widetilde{x_1} \in C$, $\widetilde{x_2} \in C$ and $\|\widetilde{x_1}\| < 1$, $\|\widetilde{x_2}\| < 1$. We also have $\|\widetilde{x_1} - \widetilde{x_2}\| = (1-\eta)\|x_1 - x_2\| > 2 - \delta$.

We write x_1, x_2 in the following way

$$x_j = \sum_{i=1}^n \lambda_i s_i^j, \quad j = 1, 2,$$

where $s_i^1, s_i^2 \in S(Ex_i^*, \varepsilon_i)$ and λ_i are the real numbers from (4.1). Now, we put $F := \operatorname{span}\{x_1, x_2, s_1^1, \dots, s_n^1, s_1^2, \dots, s_n^2\} \subset X$ and $F_* := \operatorname{span}\{x_1^*, \dots, x_n^*\} \subset Y^*$. We find $\varepsilon > 0$ such that $(1+\varepsilon)\max\{\|x_1\|, \|x_2\|\} \le 1$.

Because $E \in \mathrm{HB}_{\mathrm{ai}}(Y^*, X^*)$, we find a bounded linear operator $T : F \to Y$ such that

(i) $(Ex^*)x = x^*(Tx)$ for all $x \in F$ and $x^* \in F_*$,

(ii) $(1-\varepsilon)\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$ for $x \in F$.

We will show $Tx_j \in C'$ for $j = 1, 2$. We have $Tx_j = T\left(\sum_{i=1}^n \lambda_i s_i^j\right) = \sum_{i=1}^n \lambda_i T s_i^j$. We also have $\|Tx_j\| \le (1+\varepsilon)\|x_j\| \le 1$ from (ii) and from our choice of ε. It remains to verify $Ts_i^j \in S(x_i^*, \varepsilon_i)$. Because $s_i^j \in S(Ex_i^*, \varepsilon_i)$ and (i) holds, it follows

$$x_i^*(Ts_i^j) = Ex_i^*(s_i^j) > 1 - \varepsilon_i,$$

hence $Ts_i^j \in S(x_i^*, \varepsilon_i)$. Now we see $Tx_j \in C'$ for $j = 1, 2$. Finally $\|Tx_1 - Tx_2\| > (1-\varepsilon)(2-\delta)$, hence $\operatorname{diam} C' \ge (1-\varepsilon)(2-\delta)$. From this, we infer $\operatorname{diam} C' \ge 2$. At the same time $C' \subset B_Y$, so $\operatorname{diam} C' \le 2$. Ultimately, we conclude $\operatorname{diam} C' = 2$.

□

Corollary 4.5. *Let X be a Banach space, $Y \subset X$ an almost isometric ideal in X. If X has the strong diameter two property, then Y has the strong diameter two property.*

Proof. The proof follows immediately from Proposition 4.4 applied to $n \in \mathbb{N}$.

□

Corollary 4.6. *Let X be a Banach space, $Y \subset X$ an almost isometric ideal in X. If X has the local diameter two property, then Y has the local diameter two property.*

Proof. It follows from Proposition 4.4. We take $n = 1$.

□

We use the method of suitable models to prove the strong diameter two property is inherited from the subspace X_M to its superspace. The proof using the method is quite technical.

Proposition 4.7. *For a suitable model M, the following holds. Let X be a Banach space contained in M. If X_M has the strong diameter two property, then X has the strong diameter two property.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. We prove this assertion by contraposition. Let us assume X does not have the strong diameter two property. We aim to write a formula that expresses this statement.

To use the method of suitable models, we can only work with a finite number of formulas. This poses a problem in this proof. The following formula

$$\exists n \in \mathbb{N}\ \exists \lambda_1, \ldots, \lambda_n,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0\ \exists x_1^*, \ldots, x_n^* \in S_{X^*}$$
$$\exists \varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}\ \operatorname{diam}\left(\sum_{i=1}^{n} \lambda_i S(x_i^*, \varepsilon_i)\right) \leq C,$$

where $C \in (0, 2)$, expresses the existence of a convex combination of slices with a diameter of less than two. It also represents an infinite number of formulas, one for each natural number n. For example, the formulas $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$ are formally different because they have different number of variables. We cannot use the method of suitable models with this formula. To overcome this problem, we will proceed cautiously and utilize Lemma 2.22. This approach will allow us to write a finite number of formulas which encode the same statement as the one we cannot use.

We split the wanted formula into four parts so it is easier to read. We point out only the variables which are important to us, as we agreed on in Notation 2.7. By Lemma 2.22, there are two mappings $W : c_{00} \times c_{00}(X) \to X$, $W \in M$ such that $W(\lambda, x) = \sum_{i \in \operatorname{supp} \lambda \cup \operatorname{supp} x} \lambda(i)x(i) \in M$, for $\lambda \in c_{00}$, $x \in c_{00}(X)$, and $W_{\mathbb{R}} : c_{00} \times c_{00} \to \mathbb{R}$ with analogous properties. We denote the sequence

$$(\underbrace{1, 1, \ldots, 1,}_{k\text{-times}} 0, 0, \ldots) \in c_{00}$$

by $\mathbb{1}_k$. For a fixed $\lambda \in c_{00}$, the sequence $\mathbb{1}_k$ which satisfies $k = \max \operatorname{supp} \lambda$ is denoted by $\mathbb{1}_\lambda$. We write down the first three formulas and comment on them below.

$$\psi_1(\lambda) \coloneqq W_{\mathbb{R}}(\lambda, \mathbb{1}_\lambda) = 1\ \wedge\ \forall n \in \mathbb{N}\ \lambda_i \geq 0,$$
$$\psi_2(x^*, \lambda) \coloneqq \forall i \in \mathbb{N}\ x^*(i) \neq 0 \leftrightarrow \|x^*(i)\| = 1 \leftrightarrow \lambda(i) \neq 0,$$
$$\psi_3(x^*, \varepsilon, P) \coloneqq \forall x\ x \in P \leftrightarrow x \in c_{00}(X)\ \wedge\ \forall i \in \mathbb{N}$$
$$[(x^*(i) = 0 \to x(i) = 0)\ \wedge\ (x^*(i) \neq 0 \to x(i) \in S(x^*(i), \varepsilon(i)))],$$

where $x^* \in c_{00}(X^*)$, λ, $\varepsilon \in c_{00}$ and $S(x^*(i), \varepsilon(i))$ is a slice of B_X. The formula ψ_1 means that $(\lambda_i)_i = \lambda \in c_{00}$ are coefficients of a convex combination. The second formula ψ_2 essentially means "we work with $x^*(1), \ldots, x^*(n) \in S_{X^*}$." The formula ψ_3 says that the set $P \subset c_{00}(X)$ consists of a finite number of points $x(1), \ldots, x(n)$ such that $x(1) \in S(x^*(1), \varepsilon(1))$, $x(2) \in S(x^*(2), \varepsilon(2)), \ldots$ The fourth part of the formula we aim to write is

$$\psi_4(\lambda, x, P) \coloneqq \operatorname{diam}\{W(\lambda, x); x \in P\} \leq C,$$

where $C \in \mathbb{Q} \cap (0,2)$. This last formula encodes that the finite convex combination of slices has a diameter less than two. To better understand it, we can write it down in a more accessible form. We assume $\max\{\operatorname{supp} \lambda, \operatorname{supp} x\} = n_0$ for $\lambda \in c_{00}$, $x \in c_{00}(X)$. Then, we have $W(\lambda, x) = \sum_{i=1}^{n_0} \lambda(i)x(i)$. We recall that the set P consists of finitely many points $x(1), \dots, x(n)$ such that $x(i) \in S(x^*(i), \varepsilon(i))$. The formula ψ_4 can be rewritten as

$$\operatorname{diam} \left\{ \sum_{i=1}^{n_0} \lambda(i)x(i); x(i) \in S(x^*(i), \varepsilon(i)) \right\} \leq C.$$

We adhere to writing formulas with the mapping W. Those formulas are independent of the number of summands and we avoid having infinitely many formulas.

The following formula

$$\begin{aligned} (*) \quad & \exists \lambda \in c_{00}\ \exists \varepsilon \in c_{00}\ \exists x^* \in c_{00}(X^*)\ \exists P \subset c_{00}(X) \\ & \psi_1(\lambda)\ \wedge\ \psi_2(x^*, \lambda)\ \wedge\ \psi_3(x^*, \varepsilon, P)\ \wedge\ \psi_4(\lambda, x, P) \end{aligned}$$

expresses the existence of a finite convex combination of slices with a diameter less than two and holds because X does not have the strong diameter two property.

The absoluteness of this formula and Lemma 2.13 suggest we find $\lambda, \varepsilon \in c_{00} \cap M$, $x^* \in c_{00}(X^*) \cap M$ and $P \in M$ which satisfy $\psi_1(\lambda), \psi_2(x^*, \lambda), \psi_3(x^*, \varepsilon, P)$ and $\psi_4(\lambda, x, P)$. We denote this convex combination of slices by $C_X := \{W(\lambda, x); x \in P\}$. From Lemma 2.21 for each $i \in \mathbb{N}$ such that $x^*(i) \neq 0$, we have $\|x^*(i)|_{X_M}\| = 1$. We use this to define a convex combination of slices of B_{X_M}. We put

$$P' := \{x \in P \cap X_M; x(i) \in S(x^*(i)|_{X_M}, \varepsilon(i)), x^*(i) \neq 0\}$$

Then, $C_{X_M} := \{W(\lambda, x); x \in P'\}$ is a finite convex combination of slices of B_{X_M}. But $C_{X_M} \subset C_X$, thus $\operatorname{diam} C_{X_M} \leq \operatorname{diam} C_X \leq C < 2$. We conclude that the subspace X_M does not have the strong diameter two property.

□

The idea of the proof of the following proposition is the same as in the previous Proposition 4.7. Because we only work with one slice, and not a sum of slices, we do not run into technical difficulties and the proof is quite short.

Proposition 4.8. *For a suitable model M, the following holds. Let X be a Banach space contained in M. If X_M has the local diameter two property, then X has the local diameter two property.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. We prove this proposition by contraposition. Let us assume the space X does not have the local diameter two property. We can find a slice of B_X that witnesses it. That is, the following formula holds

$$(*) \quad \exists x^* \in S_{X^*}\ \exists \varepsilon \in \mathbb{R}\ \operatorname{diam} S(x^*, \varepsilon) \leq C,$$

where $C \in \mathbb{Q} \cap (0,2)$. From the absoluteness of this formula and Lemma 2.13, there are $x^* \in S_{X^*} \cap M$ and $\varepsilon \in \mathbb{R} \cap M$ such that $\operatorname{diam} S(x^*, \varepsilon) \leq C$. It follows by Lemma

2.21 that $\|x^*|_{X_M}\| = 1$. We also have $S(x^*|_{X_M}, \varepsilon) = \{v \in B_{X_M}; x^*(v) > 1 - \varepsilon\} \subset S(x^*, \varepsilon)$. Thus, $\operatorname{diam} S(x^*|_{X_M}, \varepsilon) \leq \operatorname{diam} S(x^*, \varepsilon) \leq C < 2$. We have found a slice of B_{X_M} that does not have a diameter of two. Hence, X_M does not possess the local diameter two property.

□

4.2 The Daugavet Property

Definition 4.9. Let X be a Banach space. We say X has *the Daugavet property* if for every rank-one operator $T : X \to X$ the equality $\|\mathrm{Id}_X + T\| = 1 + \|T\|$ holds.

The following characterization of the Daugavet property was discovered in [17, Lemma 2.2].

Lemma 4.10. *A Banach space X has the Daugavet property if and only if for every $\varepsilon > 0$, $y \in S_X$ and $x^* \in S_{X^*}$ there exists $x \in S_X$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$.*

We prove that the Daugavet property is inherited by almost isometric ideals. The proof is inspired by [2, Proposition 3.8].

Proposition 4.11. *Let X be a Banach space, $Y \subset X$ an almost isometric ideal in X. If X has the Daugavet property, then Y has the Daugavet property.*

Proof. We will use the characterization of the Daugavet property from Lemma 4.10. Let us have $0 < \varepsilon < 2$, $y \in S_Y$ and $y^* \in S_{Y^*}$. Let us also have a slice $S_Y(y^*, \varepsilon)$ of B_Y, that is

$$S_Y(y^*, \varepsilon) = \{w \in B_Y; y^*(w) > 1 - \varepsilon\}.$$

We are looking for a vector $x \in S_Y(y^*, \varepsilon)$ such that $\|x\| = 1$ and $\|x + y\| \geq 2 - \varepsilon$.

The subspace Y is an almost isometric ideal. Thus, a Hahn–Banach extension operator $E \in \mathrm{HB}_{\mathrm{ai}}(Y^*, X^*)$ exists. Let us have $0 < \eta < \frac{\varepsilon}{2}$ and $0 < \delta < \frac{\varepsilon}{2}$ such that $\delta < \frac{\frac{\varepsilon}{2} - \eta}{2 - \eta}$. Next, we consider a slice $S_X(Ey^*, \eta)$ of B_X. Because X has the Daugavet property, we find $z \in X$ which satisfies $\|z\| = 1$, $Ey^*(z) \geq 1 - \eta$, and $\|z + y\| \geq 2 - \eta$. We put $F := \operatorname{span}\{z, y\} \subset X$ and $F_* := \operatorname{span}\{y^*\} \subset Y^*$. We find a linear operator $T : F \to Y$, which satisfies

(i) $Tx = x$ for all $x \in F \cap Y$,

(ii) $(Ey^*)x = y^*(Tx)$ for all $x \in F$, $y^* \in F_*$,

(iii) $(1 - \delta)\|x\| \leq \|Tx\| \leq (1 + \delta)\|x\|$ for all $x \in F$.

We will show the desired vector is $x := \frac{Tz}{\|Tz\|}$. Because $\|z\| = 1$, x is not a zero vector. Immediately, $\|x\| = 1$. We proceed with several estimates,

$$||Tz\| - 1| = \begin{cases} \|Tz\| - 1 \overset{\text{(iii)}}{\leq} (1 + \delta)\|z\| - 1 = \delta \\ 1 - \|Tz\| \overset{\text{(iii)}}{\leq} 1 - (1 - \delta)\|z\| = \delta, \end{cases} \tag{4.2}$$

thus $|\|Tz\|-1| \leq \delta$. Next,

$$\|x - Tz\| = \left\| \frac{Tz}{\|Tz\|} - Tz \right\| = \left| \frac{1}{\|Tz\|} - 1 \right| \|Tz\| = |\|Tz\| - 1| \overset{(4.2)}{\leq} \delta. \tag{4.3}$$

Because $y \in F \cap Y$, we have $Ty = y$,

$$\|Tz + y\| \overset{\text{(i)}}{=} \|T(z+y)\| \overset{\text{(iii)}}{\geq} (1-\delta)\|z+y\| \geq (1-\delta)(2-\eta) > 2 - \frac{\varepsilon}{2}, \tag{4.4}$$

where the last inequality is due to our choice of δ. Because $\|y^*\| = 1$ and the inequality (4.3) holds, we have

$$y^*(x - Tz) \geq -\delta. \tag{4.5}$$

We combine the estimates (4.3) and (4.4),

$$\begin{aligned} \|x+y\| &= \|Tz + y + x - Tz\| \geq \|Tz + y\| - \|x - Tz\| \\ &\overset{(4.3)}{\geq} \|Tz+y\| - \delta \overset{(4.4)}{>} 2 - \frac{\varepsilon}{2} - \delta \\ &> 2 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 2 - \varepsilon. \end{aligned}$$

It remains to verify $x \in S_Y(y^*, \varepsilon)$. To this end, we will utilize $Ey^*(z) \geq 1 - \eta$. We have

$$\begin{aligned} y^*(x) &= y^*(Tz) + y^*(x - Tz) \overset{\text{(ii)}}{=} Ey^*(z) + y^*(x - Tz) \\ &\overset{(4.5)}{\geq} Ey^*(z) - \delta \geq 1 - \eta - \delta > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon. \end{aligned}$$

We have found x satisfying $\|x\| = 1$, $\|x+y\| \geq 2 - \varepsilon$ and $y^*(x) \geq 1 - \varepsilon$. Lemma 4.10 allows us to conclude Y has the Daugavet property.

□

Analogously to the local and strong diameter two properties, we prove the Daugavet property of X_M is inherited to the space X.

Proposition 4.12. *For a suitable model M, the following holds. Let X be a Banach space contained in M. If X_M has the Daugavet property, then X has the Daugavet property.*

Proof. Let us have a suitable model M for the formulas marked with $(*)$. We prove the statement by contraposition. Let us assume X does not have the Daugavet property. From Lemma 4.10, we find witnesses. That is, the following formula holds

$$(*) \quad \exists \varepsilon \in \mathbb{R}\ \exists x^* \in S_{X^*}\ \exists y \in S_X\ \forall x \in S_X\ ((\|x+y\| < 2 - \varepsilon) \vee (x^*(x) < 1 - \varepsilon)).$$

From the absoluteness of this formula and from Lemma 2.13, we find $\varepsilon \in \mathbb{R} \cap M$, $x^* \in S_{X^*} \cap M$ and $y \in S_X \cap M$ such that for all $x \in S_X$ we have $\|x+y\| < 2 - \varepsilon$ or $x^*(x) < 1 - \varepsilon$. We will show ε, $x^*|_{X_M}$ and y witness X_M does not possess the Daugavet property.

Because $y \in M$ and $y \in S_X$, it follows $y \in S_{X_M}$. We also have $x^*|_{X_M} \in (X_M)^*$ and $\|x^*|_{X_M}\| = 1$ from Lemma 2.21. Let us have arbitrary $x \in S_{X_M}$. Then $x^*(x) < 1 - \varepsilon$ or $\|x + y\| < 2 - \varepsilon$. The subspace X_M does not have the Daugavet property by Lemma 4.10.

□

We summarize our results using the notion of rich families introduced in Section 2.4.

Corollary 4.13. *Let X be a Banach space. Then there is a rich family $\mathcal{F} \subset \mathcal{S}(X)$ such that for each $F \in \mathcal{F}$ the following holds.*

(i) *F is an almost isometric ideal.*

(ii) *X has the strong diameter two property if and only if F has the strong diameter two property.*

(iii) *X has the local diameter two property if and only if F has the local diameter two property.*

(iv) *X has the Daugavet property if and only if F has the Daugavet property.*

Proof. It is enough to prove a family $\mathcal{F}$ is large in the sense of suitable models. Then by Theorem 2.25, there is a rich family $\mathcal{F}' \subset \mathcal{F}$.

By our Convention 2.14, if we have a suitable model M there exists a finite list of formulas Φ and a countable set Z such that $M \prec (\Phi, Z)$. Let us put all formulas marked with $(*)$ in the proof of Corollary 3.6, Proposition 4.7, Proposition 4.8, Proposition 4.12, and in the proofs above each of them, into a finite list Φ. We now consider countable sets X_1 from Corollary 3.6, X_2 from Proposition 4.7, X_3 from Proposition 4.8, and X_4 from Proposition 4.12. These countable sets exist due to our Convention 2.14. Put $S \coloneqq X_1 \cup X_2 \cup X_3 \cup X_4$. Finally, we put $\mathcal{F} \coloneqq \{X_M; M \prec (\Phi, S) \text{ contains } X\}$. Thus $\mathcal{F}$ is large in the sense of suitable models.

Let us have $F \in \mathcal{F}$. Then F satisfies (i) by the construction of the family $\mathcal{F}$. The implications from the right to the left in (ii)-(iv) also follow by the construction. The subspace F is an almost isometric ideal by (i). Then, (ii) follows by Corollary 4.5, (iii) by Corollary 4.6, and (iv) by Proposition 4.11.

□

Finally, we compare Corollary 4.13 with similar published results. The Corollary is in a certain sense an improvement of [1, Proposition 3.2] and [1, Proposition 3.3]. They state that a Banach space X has the strong diameter two property (or the local diameter two property or the Daugavet property) if and only if every separable almost isometric ideal Y in X does. We have shown that it suffices to consider a particular rich family $\mathcal{F}$. Moreover, the rich family $\mathcal{F}$ works for all notions.

By [5, Theorem 2.4] the family of all separable almost isometric ideals is a rich family (in [5] the authors use the term "skeleton"). This means that if $\mathcal{F}$ were all almost isometric ideals, proposition (i) and implication from the left to the right in (ii)-(iv) would be satisfied.

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