Future space-based gravitational-wave detectors will require highly accurate gravitational wave templates for detecting extreme mass ratio inspirals and estimating their parameters. These templates must include the postadiabatic effects like the spin of the secondary body. Therefore, we investigate the influence of the secondary spin on the motion around a Kerr black hole, calculate the corresponding gravitational-wave fluxes to produce flux-driven inspirals and reveal the shifts of the gravitational-wave phases induced by the secondary's spin. In particular, this study begins by considering eccentric equatorial orbits, where we obtain the constants of motion and fundamental frequencies using the Mathisson-Papapetrou-Dixon equations. Next, we derive the linear-in-spin parts of these quantities. We introduce a new Teukolsky equation solver in the frequency domain to calculate the energy and angular momentum fluxes from these trajectories. We use the obtained fluxes to adiabatically evolve the orbital parameters and to find the spin-induced phase shifts. For off-equatorial orbits, a frequency-domain approach is employed to determine the trajectories in the linear-in-spin regime and to compute the respective fluxes. The agreement between the frequency-domain fluxes and those acquired using an existing time-domain solver verifies our findings.