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Report on the Ondrej Týbl Ph.D. thesis "Stochastic Equations with Correlated Noise and their Applicattions"

The dissertation is based on the following two publications:

B. Maslowski and O. Týbl, *Invariant measures and boundedness in the mean for stochastic equations driven by Lévy noise*, Stochastics and Dynamics 22 (2022), Paper No. 2240019, 25 pp.

J. Seidler and O. Týbl, *Stochastic approximation procedures for Lévy-driven SDEs*, J. Optimization Theory Appl. 197)2023), 817–837.

The papers were published in good mathematical journals. The dissertation shows knowledge of Lévy processes, Stochastic Differential Equations, and Markov processes. It is concerned with two natural and important questions: the existence of invariant measure of SDEs driven by Lévy noise and stochastic approximation procedure for Lévy-driven SDEs. The first problem is reduced by the Krylov–Bogolyubov theorem to the boundedness in probability of solutions. The second topic is related to very fashionable theory of Randomized Algorithms. Therefore my impression is favourable.

Invariant measures and boundedness in probability. The main idea, based on the existence of Lyapunov function V, is not original yet natural. In the case of Wiener noise or in the absence of large jumps, a standard is to use as a Lyapunov function a square of the norm. However, even for a large class of linear equations with multiplicative noise one has to consider powers strictly less than 2. In fact, in the thesis, large jumps are allowed and $V(x)$ is like $\|x\|^p$ with $p \in (0,1)$. The main results; Theorem 2.1.1. provides criteria for boundedness in probability of solutions $X^x(t)$, $t \geq 0$, to SDEs with general noise. Then, see Theorems 2.2.1 and 2.3.1, the cases of equations driven by noises with only small jumps and only large jumps are treated. The results are correct, new and interesting. Their proofs are based on the Itô formula and some elementary but clever calculations. Two examples are provided: Example 2.2.1 on $\mathbb{R}^2$ with linear diffusion being a rotation

$$\mathrm{d}X(t) = q\int_{\{|y|\leq c\}} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} X(t-)\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

and, see Example 2.3.1, linear one dimensional equation

$$\mathrm{d}X(t) = bX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t) + qX(t-)\mathrm{d}P(t).$$

In particular it is shown that the noise can stabilize the solution, and this is interesting but to some extend expected situation. In my opinion Chapter 2 provides an important contribution to the topic.

I have the following remarks and questions concerning this part ot the thesis:

(1) Examples are interesting but I wonder if the author can treat the case of infinite Lévy measures. Also in the case of linear equations with additive noise there are if and only if conditions for the existence of an invariant measure, in short the Lévy measure has to have finite logarithmic moment. The author should compare this with conditions obtained from Theorems 2.1.1, 2.2.1, and 2.3.1.
(2) I wonder if the author considered logarithmic Lyapunov function like in the last chapter?
(3) In my opinion, the definition of the generator $\mathcal{L}$ from p. 28 is not 100 % correct; it is stated $\mathcal{L}\colon Dom(\mathcal{L}) \mapsto B_b(\mathbb{R}^m)$. Then the author is checking if $V_p \in Dom(\mathcal{L})$. But in fact, see Lemma 2.1.1, he proves that $\mathcal{L}V_p \in B_{b,loc}(\mathbb{R}^m)$, and then in the proof of the main theorem, that $\mathcal{L}V_p \leq -CV_p$ with $C > 0$. On lines -3, -4 page 32 the powers p are missed (the proof is correct). On lines 15,16 page 34 I do not like (and understand) the sentence "Moreover, the condition ...."

Stochastic approximation procedures. The starting point is the Robbins and Munro stochastic approximation of a root of a randomly perturbed mapping R. Originaly it is discrete time procedure. Its continuous time version was considered by Nevel'son and Khas'minski in 1972. In Nevel'son and Khas'minsk the perturbation was of gaussian white noise type, whereas in the thesis the perturbation allowes also jump parts (large and small jumps). In fact the main result (Theorem 3.1.1.) provides criterion under which the solution X_t to SDE driven by general Lévy noise converge as $t \to +\infty$ to a single point x_0. Then the continuous Lévy perturbation version of Robbins and Munro approximation is considered as a consequence of the general result.

I have the following remarks:

(1) I like the fact that the proof of Theorem 3.1.1 is given is such details.

(2) From pedagogical reasons it would be a good idea to study in details the case of

$$R(x) = \begin{cases} 1 & \text{for } x \leq -1, \\ -x & \text{for } x \in [-1, 1], \\ -1 & \text{for } x \leq -1. \end{cases}$$

Besides of my complaints I think that the thesis is well written, the proofs and its organization is logical and transparent. In my opinion topic is important and the thesis contains original and difficult results.

Summing up, in my opinion the presented dissertation meets the standard of a good Ph.D. thesis in mathematics.