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June 14, 2023

Evaluation of Ph.D. Thesis by Dávid Uhrik

0.1. **Introduction.** The thesis under consideration, entitled "Colorings of Infinite Graphs" deals with the nature of uncountable graphs. The first line of results surrounds the Ramsey-theoretic partition relations $\kappa \to (\lambda, E)^2$ (for κ, λ infinite cardinals, and E a symmetric binary relation on κ) asserting that for all colorings $c : [\kappa]^2 \to 2$, one of the following must hold:

- there exists an order-preserving map $f : \lambda \to \kappa$ such that
$$\{c(f(\alpha), f(\beta)) \mid (\alpha, \beta) \in [\lambda]^2\} = \{0\};$$
- there exists an order-preserving map $g : \kappa \to \kappa$ such that
$$\{c(g(\alpha), g(\beta)) \mid (\alpha, \beta) \in E\} = \{1\}.$$

The main examples of relations E considered in this thesis are the complete graph K_ϵ (namely, $E = [\epsilon]^2$) for some ordinal $\epsilon \leq \kappa$, and the graph $\epsilon_0 : \epsilon_1$ (namely, $E = \{(\alpha, \beta) \mid \alpha < \epsilon_0 \leq \beta < \epsilon_0 + \epsilon_1\}$) for a pair of ordinals $\epsilon_0, \epsilon_1 < \kappa$.

The second line of results surrounds the so-called *Hajnal-Máté graphs*. These graphs are based on a tree $T = \bigcup_{\alpha<\omega_1} T_\alpha$ of height ω_1 and the edge relation E is a subset of $\{\{x, y\} \mid x <_T y\}$ satisfying that for every node $y \in T_\gamma$ of a given level $\gamma < \omega_1$, for every $\beta < \gamma$, the set $\{x \in \bigcup_{\alpha<\beta} T_\alpha \mid \{x, y\} \in E\}$ is finite. The Hajnal-Máté paper from 1975 dealt with the linear case $(T, <_T) = (\omega_1, \in)$, and the general case gradually proved itself useful, most notably in a work of D. Soukup from 2015. The main challenge here is to construct Hajnal-Máté graphs that are uncountably chromatic and omit a prescribed finite family of odd cycles.

The third and last line of results surrounds the infinite version of the Hadwiger conjecture. The Hadwiger conjecture asserts that if G is a simple finite graph of chromatic number n, then the complete graph K_n is a minor of G. Motivated by a counterexample to the infinite version due to Komjáth, the study here mainly focuses on the least size of a graph of chromatic number $\aleph_1$ that does not have $K_{\aleph_1}$ as its minor.

0.2. **Description of the main results.** Theorem 2.5 of the thesis implies that, assuming GCH, for every pair $\theta < \lambda$ of infinite regular cardinals, in the forcing extension for adding λ^+-many Cohen subsets of θ, $\lambda^+ \to (\lambda^+, \epsilon : \epsilon)^2$ holds for every ordinal $\epsilon < \theta^+$. By Proposition 2.7, this cannot be improved to $\lambda^+ \to (\lambda^+, \theta : \theta^+)^2$.

Todorcevic (improving upon a theorem of Hajnal) proved that $\mathfrak{b} = \aleph_1$ entails the failure of $\omega_1 \to (\omega_1, \omega : 2)^2$. He also remarked that the same is true in the forcing extension for adding a single Cohen real. Theorem 2.9 of the thesis provides a proof of this fact. Likewise,

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Proposition 2.14 provides a proof of an unpublished result of Hajnal on the consistency of $\omega_2 \nrightarrow (\omega_1 : 2)^2_\omega$,[1] and Corollary 2.18 proves a result of Galvin asserting that $\omega_2 \to (\omega_1, \omega+2)^2$.

Theorem 2.25 of the thesis (obtained jointly with C. Lambie-Hanson) answers a question of D. Soukup who asked whether the forcing to add a single Cohen real adds an uncountably chromatic Hajnal-Máté graph that is linear (i.e., based on $(T, <_T) = (\omega_1, \in)$) and triangle-free. The theorem shows that starting from a single Cohen real, for every positive integer n, it is possible to construct an uncountably chromatic linear Hajnal-Máté graph G with no odd cycles of length at most $2n+1$. Moreover the graph G does not have $H_{\omega,\omega+2}$ has a subgraph. Then, in Theorem 2.26, a nonlinear variation is constructed on the grounds of ZFC. Finally, after studying another generalization of Hajnal-Máté graphs (δ-HM graphs), the following variation of Galvin's result (mentioned earlier) is discovered, and stated as Proposition 2.35:

$$\omega_2 \to (\omega_1, \epsilon : 2)^2 \text{ holds for every } \epsilon < \omega_1.$$

In subsection 3.2.2, a construction due to Brochet and Diestel of a partition tree T_G of a given graph G is analyzed. This analysis culminates with Theorem 3.32 stating that the least size of a graph of chromatic number $\aleph_1$ that does not have $K_{\aleph_1}$ as its minor is nothing but the least size of a nonspecial tree of height and size $\aleph_1$ that has no branch of size $\aleph_1$. This yields a 2017 result of Komjáth as a corollary.

0.3. **Evaluation.** The thesis examines a family of questions concerning graphs of uncountable chromatic number in a rather comprehensive way. It includes new results together with the details of classical results whose proof are unavailable in the literature. The proofs are correct, the writing style is lucid, and the manuscript demonstrates the author's ability of carrying out creative scientific work.

Theorem 2.6 constitutes a new partition relation for Cohen models. Theorem 2.25 solves an open problem of Soukup and harder generalizations of which. Theorem 3.32 improves a result of Komjáth by establishing a translation of concepts stemming naturally and independently in the studies of infinite graphs and infinite trees. I expect these results to appear in three separate research papers, and to prompt the study of transfinite trees of unconventional nature. Applications of these results are foreseen in set-theoretic topology.

Sincerely,

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[1] We omit the definition of this relation.

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