REPORT ON "COLORINGS OF INFINITE GRAPHS" BY D. UHRIK

This thesis obtains new results in three areas of infinite graph theory: ordinary partition relations, properties of Hajnal-Máté graphs, and the uncountable Hadwiger conjecture. I will describe the most important results from the thesis and explain their significance below.

The study of unbalanced ordinary partition relations has a long history in set theory, going back to the work of Erdős and Rado in 1956. The thesis focuses on relations of the form $\alpha \to (\beta, (\gamma : \delta))^2$, where $\alpha, \beta, \gamma, \delta$ are ordinals. The symbol $\alpha \to (\beta, (\gamma : \delta))^2$ means the following: for any coloring c of the unordered pairs of α into two colors, *either*

- (A) there is a set $X \subseteq \alpha$, such that the order type of X is β and all the unordered pairs from X have the first color; *or*
- (B) there exist sets $A, B \subseteq \alpha$ such that:
  - (1) the order type of A is γ and the order type of B is δ;
  - (2) every element of A is smaller than every element of B;
  - (3) every pair of the form $\{\zeta, \xi\}$, where $\zeta \in A$ and $\xi \in B$ has the second color.

If requirement (B)(2) is dropped, then the resulting relation is denoted by the symbol $\alpha \to (\beta, (\gamma; \delta))^2$. There are several unresolved questions about the relations $\alpha \to (\beta, (\gamma : \delta))^2$ and $\alpha \to (\beta, (\gamma; \delta))^2$, particularly when α is the continuum $2^{\aleph_0}$. For example, a long standing unresolved conjecture of Galvin is that the relation $2^{\aleph_0} \to \left(2^{\aleph_0}, (2^{\aleph_0}; 2^{\aleph_0})\right)^2$ is consistent. Special cases of Galvin's conjecture are known to hold with $2^{\aleph_0}$ being very large. In his thesis, Uhrik shows that the relations $\omega_2 \to (\omega_2, (\omega : \omega))^2$ and $\omega_2 \not\to (\omega_2, (\omega : \omega_1))^2$ hold after adding ω_2 Cohen reals. Actually the result in the thesis is more general; I'm only stating the most important special case here. Since $2^{\aleph_0} = \aleph_2$ after adding ω_2 Cohen reals to a model of CH, the relation $\omega_2 \to (\omega_2, (\omega : \omega))^2$ gives a special case of Galvin's conjecture with $2^{\aleph_0} = \aleph_2$. This is a new result. It is proved using the method of double Δ-systems which was introduced by Todorcevic in the 1980s.

Hajnal–Máté graphs were introduced in the early 1970s. Let $(T, <)$ be a tree of height ω_1 and let δ be a countable ordinal. A graph $G = (T, E)$ is a *T-δ-Hajnal–Máté-graph* if

1. (1) $\{x : xEy\} \subseteq \{x : x \leq y \vee y \leq x\}$;
2. (2) the chromatic number of G is uncountable;
3. (3) for each $y \in T$ and $\alpha < \mathrm{ht}(y)$, $\{x : xEy \wedge \mathrm{ht}(x) < \alpha\}$ has order type less than δ.

If $(T, <) = (\omega_1, \in)$, then T will be omitted, while if $\delta = \omega$, then δ will be omitted. Thus the special case when $(T, <) = (\omega_1, \in)$ and $\delta = \omega$ are known simply as *Hajnal–Máté-graphs* and they were first studied by Hajnal and Máté in the early

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1970s. Further work was done on such graphs and their generalizations by Abraham, Komjáth, Lambie-Hanson, Shelah, D. Soukup, and others. In their original paper, Hajnal and Máté showed that Hajnal–Máté-graphs exist under $\diamondsuit^+$, but that they don't exist under $\mathrm{MA}_{\aleph_1}$. Uhrik's thesis contains several novel results about Hajnal–Máté-graphs and their generalizations. Some of his results answer open problems from the literature. He generalizes the classical theorem of Hajnal and Máté by proving that if $\mathrm{MA}_{\aleph_1}$ holds, then there are no δ-Hajnal–Máté-graphs for any countable δ. He also answers a question of D. Soukup from 2015 by showing that adding a single Cohen real adds a triangle free Hajnal–Máté-graph. Another result from his thesis is that for each $s \in \omega$ there exists a ZFC example of a $\omega^{<\omega_1}$-Hajnal–Máté-graph which contains neither special cycles nor any odd cycles of length less than or equal to $2s + 1$. Here, the property of omitting special cycles is a technical device for ensuring that the constructed graph omits certain types of subgraphs. This theorem of Uhrik extends a theorem of Komjáth and Shelah from 1988. Elementary submodel arguments are used to prove the theorem answering Soukup's question as well as the result extending Komjáth and Shelah's work.

Hadwiger's conjecture is a deep unresolved conjecture in finite graph theory, which states that if the chromatic number of a finite graph G is n, then the complete graph on n vertices K_n is a minor of G. The infinite version of Hadwiger's conjecture is easily seen to be false. For example, it is not hard to construct a countably infinite graph of chromatic number $\aleph_0$ that does not contain K_ω as a minor. The final part of Uhrik's thesis concerns the uncountable version of Hadwiger's conjecture, and more generally, questions about the existence of uncountable connected graphs omitting various complete graph minors. The main innovation here is the use of a technique for translating questions about uncountable graphs into set theoretic questions about trees. This technique is based on a 1994 theorem of Brochet and Diestel that associates a tree T_G with every uncountable graph G. In the final part of his thesis, Uhrik shows how to translate questions about the minors of G into combinatorial questions about T_G which are more familiar to set theorists. For example, he shows that the existence of a κ-Suslin tree is equivalent to the existence of a graph of size κ which has no independent set of size κ and has no K_κ minor. It is worth recalling that many questions in general topology and order theory were historically first translated into combinatorial questions about trees before being treated with set theoretic methods like forcing. The translation procedures discovered in this thesis will likely enable existing set theoretic methods to tackle more problems in uncountable graph theory. Using his analysis of T_G, Uhrik is able to resolve a question about cardinal invariants. Define $\mathfrak{hc}$ to be the minimal cardinality of a graph which does not have a K_{ω_1} minor, but whose chromatic number is ω_1. Define $\mathfrak{st}$ to be the least cardinality of a non-special Aronszajn tree. It was known from earlier work that $\omega_1 \leq \mathfrak{hc} \leq \mathfrak{st} \leq 2^{\aleph_0}$. In Theorem 3.32 of his thesis, Uhrik proves $\mathfrak{hc} = \mathfrak{st}$, showing that these seemingly different definitions yield the same cardinal.

In conclusion, this thesis makes advances in three closely related areas of infinite combinatorics using sophisticated methods. The methods introduced are likely to lead to further advances in these areas in future. Therefore, this thesis meets all international standards and I recommend it warmly.

Sincerely,

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