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DOCTORAL THESIS

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Geometric Function Theory and its application in Nonlinear Elasticity

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

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In Prague, 25th July 2023

Ondřej Bouchala

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Díky!

Title: Geometric Function Theory and its application in Nonlinear Elasticity

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Abstract: This thesis is divided into two parts. The first part focuses on mappings in $\mathbb{R}^n$ and the weak limits of homeomorphisms in the Sobolev space $W^{1,p}$. Our primary concern is the concept of "injectivity almost everywhere". We demonstrate that when $p \leq n-1$, the weak limit of homeomorphisms can fail to satisfy this condition. Conversely, when $p > n-1$, the weak limit is "injective almost everywhere".

In the second part, we investigate the Hardy spaces in the complex plane. It is established that for a simply connected domain $\Omega \subsetneq \mathbb{C}$, there exists a constant H_Ω such that any conformal mapping from the unit disk in $\mathbb{C}$ onto Ω belongs to the Hardy space H^p for all $p < H_\Omega$. Conversely, for $q > H_\Omega$, no such mapping exists in the space H^q. However, we demonstrate that by allowing quasiconformal mappings instead of conformal ones, a quasiconformal mapping can be found from the unit disk onto Ω that belongs to the Hardy space H^p for every $0 < p < \infty$.

Keywords: Sobolev homeomorphisms, injectivity almost everywhere, quasiconformal mappings, Hardy spaces

Contents

Prologue

Similar to my Ph.D., which was divided into two parts – one conducted in Prague under the supervision of Stanislav Hencl, and the second in Jyväskylä under the supervision of Pekka Koskela – this thesis will also consist of two parts. These parts share common themes, with the most prominent being the examination of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ belonging to some Sobolev space $W^{1,p}$. In both cases, the mappings are expected to be either bijective or, at the very least, "almost bijective".

The first part focuses on weak limits of homeomorphisms, driven by a clear variational motivation to minimize an energy functional. Consequently, we explore the properties of weak limits in this context. In the second part, which delves into Hardy spaces, the variational motivation is less apparent but still present in a more concealed manner. Harmonic maps are closely related to minimization problems, and in this section, we investigate conformal and quasiconformal maps.

1 The injectivity almost everywhere

1.1 Introduction

In the first part of this thesis, our focus lies on studying mappings that can represent deformations within models of nonlinear elasticity. We are particularly interested in the property of injectivity, which prohibits matter from interpenetrating. Therefore, an intuitive candidate would be the class of homeomorphisms. Indeed, under certain conditions, such as assuming that the mapping $f\colon \Omega \to \mathbb{R}^n$ for $\Omega \subseteq \mathbb{R}^n$ has finite energy (where energy is defined by the functional $\int_\Omega W(Df)$, inclusive of special terms like Df, $\operatorname{adj} Df$ or J_f), and reasonable boundary data, the mapping f is a homeomorphism. For further insights into these types of results, refer to e.g. [12, 13, 18, 26].

For a broader notion of injectivity, refer to Ball's paper [2]. However, his approach presumes that the mapping is continuous everywhere. Interestingly, under certain circumstances, such as when stretching an object made of a rubber-like material, cavitations, or small internal cavities, may form. As a result, we seek models that accommodate these discontinuities.

Even though we allow for cavitations, we still strive for injectivity in some manner. One such approach was proposed by Ciarlet and Nečas. They explored mappings that satisfy $f \in W^{1,p}(\Omega)$ for $p > n$, $J_f > 0$ almost everywhere, and

$$\int_\Omega J_f \leq |f(\Omega)|.$$

This last property is nowadays known as the *Ciarlet-Nečas condition*. They demonstrated that such mappings are almost everywhere injective in the image (see below for the definition). The condition $J_f > 0$ is commonplace in nonlinear elasticity models. In the case of real deformations, it wouldn't make sense for the J_f to be negative since we do not alter the orientation, and if J_f tends toward 0 (indicating a substantial material compression), then the energy would rise towards infinity. For additional results following this approach, consult e.g. [3, 4, 5, 6, 8, 19, 27].

The deformations we examine are generally derived as minimizers of certain energy functionals. A relevant class to investigate is the class of weak limits of homeomorphisms, specifically weak limits of homeomorphisms within the Sobolev space $W^{1,p}$. This class can be extended to include mappings that satisfy the (INV) condition, which essentially stipulates that the interior of a sphere is mapped to the interior of the sphere's image (while the exterior is mapped to the exterior). This holds trivially for homeomorphisms, and the (INV) condition remains intact under weak limits. Müller and Spector explored this class in [20]. They demonstrated that if the mapping f satisfies the (INV) condition and $J_f > 0$ almost everywhere, the mapping is injective almost everywhere within the domain. For this, they utilize a topological degree, which requires at least the continuity of the mapping on spheres. Therefore, this approach is valid only for $p > n - 1$.

1.2 Our results

We showed that for $p \leq n-1$ the injectivity almost everywhere can fail, and it can fail even for a strong limit of Sobolev homeomorphisms.

We can consider injectivity almost everywhere in two ways: it could either denote injectivity almost everywhere in the image or in the domain. For the former, the definition is apparent:

Definition 1.1.
Let $f\colon \Omega \to \mathbb{R}^n$ be a mapping. We say that f is *injective almost everywhere in the image* if there exists a set $N \subseteq f(\Omega)$ with $|N| = 0$ such that $f^{-1}(y)$ is a singleton for every $y \in f(\Omega) \setminus N$.

In this case, we proved the following theorem:

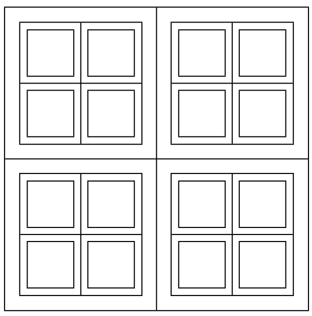
Theorem 1.2.
For every $n \geq 3$, there exists a continuous mapping $f\colon [-1,1]^n \to [-1,1]^n$ with $J_f > 0$ a.e., which is a strong limit of Sobolev homeomorphisms f_k in the space $W^{1,n-1}\big([-1,1]^n, \mathbb{R}^n\big)$ with $f_k(x) = x$ for $x \in \partial[-1,1]^n$ and there is $C_A \subseteq [-1,1]^n$ with $|C_A| > 0$ such that

$$f^{-1}(y) \text{ is a continuum for every } y \in C_A.$$

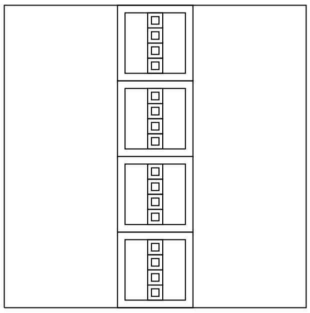
The other possibility is injectivity a.e. in the domain. Here, the definition and statement of the counter-example are somewhat more complicated, so we refer readers to the paper for further details and definitions.

Theorem 1.3.
For every $n \geq 3$ there is $\widetilde{f}\colon [-1,1]^n \to [-1,1]^n$ with $J_{\widetilde{f}} > 0$ a.e. which is a strong limit of Sobolev homeomorphisms $\widetilde{f}_k \in W^{1,n-1}([-1,1]^n, \mathbb{R}^n)$ with $\widetilde{f}_k(x) = x$ for $x \in \partial[-1,1]^n$. The quasicontinuous representative of $\widetilde{f}$ is one-to-one on $[-1,1]^n$ (but $\widetilde{f}([-1,1]^n) \subsetneq [-1,1]^n$). There is a continuous mapping $w\colon [-1,1]^n \to \mathbb{R}^n$ which is a generalized inverse to $\widetilde{f}$, i.e. $w(\widetilde{f}(x)) = x$ for every $x \in [-1,1]^n$ such that there is $C_A \subseteq [-1,1]^n$ with $|C_A| > 0$ and $w^{-1}(x)$ is a continuum for every $x \in C_A$.

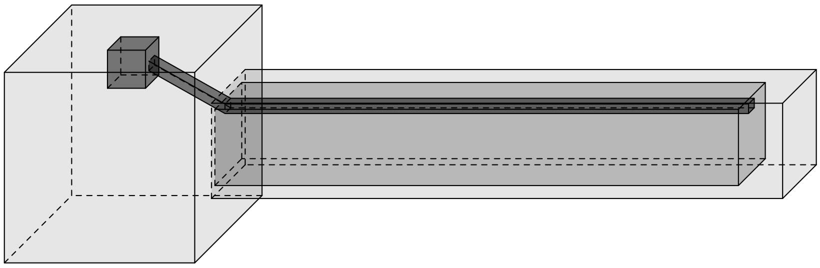
The counter-examples in both instances stem from similar construction. To provide a general understanding, we have included here several illustrative images, for full details please refer to the paper. The final mapping will be composed of several auxiliary mappings. Moving backward, we start with a set with positive measure, a Cantor-type set:



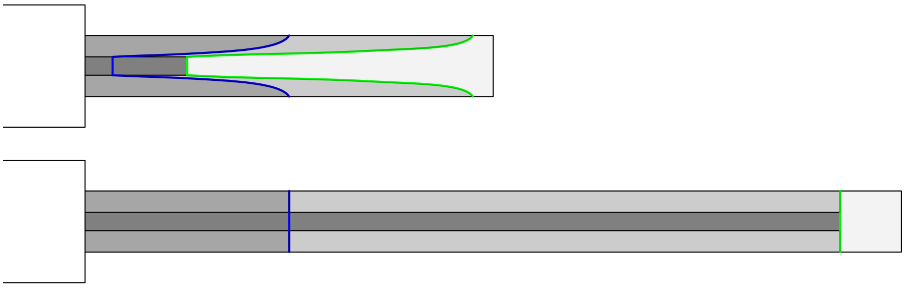
We want a disjoint family of continua, one for each of the points of this set. For that, we need more space around those points, so (in a bi-Lipschitz way) we make our squares (cubes) smaller and rearrange them from a grid-like structure into a tower formation:



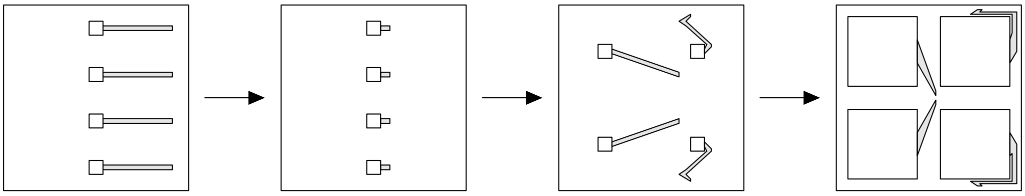
Subsequently, we construct telescopic “tentacles” for each of these squares (or cubes):



These tentacles need to be long, so we need to introduce a map that aids in their elongation:



The final mapping integrates all these constituent parts. We begin with the elongated tentacles, contract them, relocate them from the tower structure back into the grid, and, ultimately, expand the squares (cubes):



If $p > n - 1$, the injectivity a.e. holds. For the case of injectivity a.e. in the image, we have

<u>THEOREM 1.4.</u>
Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f: \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_k \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n - 1$ for $n > 2$ or $p \geq 1$ for $n = 2$. Then there is a precise representative $\hat{f}$ and a set $N_1 \subseteq \mathbb{R}^n$ of Hausdorff dimension $n - 1$ such that the preimage $\hat{f}^{-1}(y)$ consists of only one point for every $y \in \hat{f}(\Omega) \setminus N_1$.

While Müller and Spector in [20] already discovered this result under the additional assumption that $J_f > 0$, their approach could be applied even without this assumption after some technical refinement.

For the case of injectivity a.e. in the domain, we need to use the topological image f^T, which is a set function. For instance, if a cavitation exists, then the topological image of the point from which it sprouted will cover the entire cavity.

<u>THEOREM 1.5.</u>
Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f: \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_k \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n - 1$ for $n > 2$ or $p \geq 1$ for $n = 2$. Then there is a set $N_2 \subseteq \mathbb{R}^n$ of Hausdorff dimension $n - p$ such that the image $f^T(x)$ consists of only one point for every $x \in \Omega \setminus N_2$. If we moreover assume that $J_f > 0$ a.e then there is a set N_3 of zero measure such that $f|_{\Omega \setminus N_3}$ is one-to-one.

In the second part of this theorem, the assumption that $J_f > 0$ a.e. is unavoidable because of locally constant mappings. As before, this result essentially follows from previously known results [5, 20, 21].

2 A mapping in the Hardy space

In this section, our attention is drawn towards the plane. Specifically, we are considering Ω, a simply connected domain that is a subset of $\mathbb{R}^2$, which we identify with the complex plane $\mathbb{C}$. We delve into the boundary behavior of quasiconformal mappings from the unit disk, denoted as $\mathbb{D}$, onto Ω. Our particular interest lies within the Hardy space $H^p(\mathbb{D}, \Omega)$.

2.1 Definition

Hardy spaces were first named by Frigyes Riesz in 1923 in [25], named in honor of G. H. Hardy due to his pioneering work in 1915, [11]. In his paper, Hardy studied the growth of conformal mapping of the unit disk as its boundary was approached. The original definition (using the formula (1)) was for conformal maps, but the same works for quasiconformal mappings as well. Recall that a homeomorphism $f: \mathbb{D} \to \Omega$ is quasiconformal if $f \in W^{1,2}_{loc}(\mathbb{D}, \mathbb{C})$ and if there is a constant K such that $|Df(z)|^2 \leq K \cdot J_f(z)$ holds for almost every $z \in \mathbb{D}$. As conformal mappings map "small circles to circles", the quasiconformal mappings map "small circles to elipses", where the ratio of the semiaxes is bounded by K.

<u>Definition 2.1.</u>
Let $\mathbb{D}$ be the unit disk and $0 < p < \infty$. The Hardy space $H(\mathbb{D}, \Omega)$ is the space of all quasiconformal mappings $f\colon \mathbb{D} \to \Omega$ such that

$$\sup_{0 \leq r < 1} \left(\int_0^{2\pi} \left| f\left(r \cdot e^{i\varphi}\right) \right|^p \, d\varphi \right)^{\frac{1}{p}} < \infty. \tag{1}$$

From (1) we can immediately see that the Hölder inequality implies that $H^p \subseteq H^q$ for $q \leq p$.

To see how this connects to the boundary behavior of the mapping, it is useful to consider an equivalent definition. For this, we must first define what we mean by boundary values for a quasiconformal mapping, given it is only defined at the interior of the disk.

<u>Definition 2.2.</u>
The theorem [24, Theorem 1.7.] allows us to define for each conformal $g\colon \mathbb{D} \to \Omega$ and for almost every ω in S^1

$$g(\omega) := \lim_{r \to 1^-} g(r\omega).$$

According to [17, Theorem 2.], this limit also exists for almost every $\omega \in S^1$ when g is quasiconformal, thereby permitting us to use the same definition.

Utilizing the aforementioned definition, we invoke the following theorem obtained by Zinsmeister [28], which provides an equivalent definition for Hardy spaces:

<u>Theorem 2.3.</u> (Zinsmeister)
Let f be a quasiconformal mapping of $\mathbb{D}$ and let $0 < p < \infty$. Then $f \in H^p$ if and only if $f(\omega) \in L^p(S^1)$, where $S^1 = \partial\mathbb{D}$ is the unit circle in $\mathbb{C}$.

2.2 Questions

Let Ω be a non-empty simply connected domain. Our interest is in the following questions:

$$\begin{aligned}&\textit{Is there a quasiconformal mapping } f\colon \mathbb{D}\to\Omega \textit{ that is}\\ &\textit{in the Hardy space } H^p \textit{ for a given } p>0?\\ &\textit{Is there a quasiconformal mapping } f\colon \mathbb{D}\to\Omega \textit{ that is}\\ &\textbf{\textit{not}} \textit{ in the Hardy space } H^p \textit{ for a given } p>0?\end{aligned} \tag{2}$$

It is natural to start with the Riemann mapping theorem, which provides us with a conformal mapping $h\colon \mathbb{D}\to\Omega$. Due to the Prawitz theorem, the mapping h is in the Hardy space H^p for every $p<\frac{1}{2}$. This conclusion is sharp, as illustrated by the Koebe map $f(z)=\frac{z}{(1-z)^2}$, which does not belong to the Hardy space $H^{\frac{1}{2}}$.

In a similar vein, any K-quasiconformal mapping f is in the Hardy space H^p for every $p<\frac{1}{2K}$, as shown in [1]. This is also a sharp result.

The answers to questions (2) in the context of conformal mappings are known. In 1970 in [9], Hansen introduced the concept of the Hardy number of the set Ω:

<u>Definition 2.4.</u>
Let $\Omega\subsetneq\mathbb{C}$. Let h be a conformal mapping from $\mathbb{D}$ onto Ω. The Hardy number of the set Ω is defined as

$$H_\Omega := \sup\{p\in(0,\infty) : h\colon\ \mathbb{D}\to\Omega \text{ is in } H^p\}.$$

For simply connected domain $\emptyset\subsetneq\Omega\subsetneq\mathbb{C}$, this definition does not depend on the choice of h. Hence, for every $p<H_\Omega$, every conformal mapping $f\colon\ \mathbb{D}\to\Omega$ is in the space $H^p(\mathbb{D},\Omega)$, and for $p>H_\Omega$ there is no conformal mapping from $\mathbb{D}$ onto Ω that belongs to the Hardy space $H^p(\mathbb{D},\Omega)$. The connection between the Hardy number and the geometry of the set Ω has been explored in multiple studies, e.g. in [7, 9, 10, 14, 15, 22, 23].

2.3 Answers

The answers to questions (2) are given by the following theorems.

<u>Theorem 2.5.</u>
Let $\Omega\subsetneq\mathbb{C}$ be a non-empty, simply connected domain. Let $p\in(0,\infty)$. Then there is a quasiconformal mapping f from $\mathbb{D}$ onto Ω, which is in the Hardy space $H^p(\mathbb{D},\Omega)$.

The proof starts with a mapping h provided by the Riemann mapping theorem. As it is conformal, the Prawitz theorem guarantees it belongs to the Hardy space H^q for every $0<q<\frac{1}{2}$. Therefore $h\in L^q(S^1)$ (in the sense of the Definition 2.2). We then create a reparametrization g on S^1, such that the composition $h\circ g$ is in the space $L^p(S^1)$. What remains to be shown is that this mapping on the circle can be extended inside in a quasiconformal manner. This extension will belong to the required Hardy space $H^p(\mathbb{D}, \Omega)$.

To demonstrate this, we must employ intrinsic Hardy spaces. In the case of conformal mapping, the definition is equivalent to the definition of Hardy spaces, but it allows us to utilize the intrinsic norm. For more details, refer to [16]. Through this, we can show that the reparametrization g is doubling, and consequently, we can extend it to a quasiconformal mapping on the entire disk.

To answer the second question (the existence of a "bad" mapping), we must exclude cases where the Riemann mapping from $\mathbb{D}$ onto Ω is in the Hardy spaces H^p for all $0 < p < \infty$, i.e., cases where $H_\Omega = \infty$. This happens not only for bounded sets (where the boundary values from Definition 2.2 are obviously in $L^\infty(S^1)$), but for example for strips such as $\mathbb{R} \times (0, 1)$ as well.

<u>THEOREM 2.6.</u>
Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Then we have the following dichotomy:

(1) Either $f \in H^p(\mathbb{D}, \Omega)$ for all $0 < p < \infty$ and every quasiconformal mapping $f\colon \mathbb{D} \to \Omega$,

(2) or for every $q > 0$ there is a quasiconformal mapping $f\colon \mathbb{D} \to \Omega$ such that $f \notin H^q(\mathbb{D}, \Omega)$.

The idea behind the proof can be outlined as follows: Assume we have a quasiconformal mapping f, which does not belong to the Hardy space $H^p(\mathbb{D}, \Omega)$ for a given $p > 0$. That means that f grows "quickly" as we approach the boundary S^1. In fact, there exists a sequence of points z_n in $\mathbb{D}$ converging towards some point $\omega \in S^1$, such that

$$|f(z_n)| \cdot |\omega - z_n|^{\frac{1-\varepsilon}{p}} \geq c.$$

We then construct a quasiconformal mapping g on the disk that moves the points "further away from the boundary", i.e.

$$|\omega - z_n|^{\frac{1-\varepsilon}{p}} = |g^{-1}(\omega) - g^{-1}(z_n)|^{\frac{1+\varepsilon}{q}}.$$

Therefore

$$\left|f \circ g(g^{-1}(z_n))\right| \cdot |g^{-1}(\omega) - g^{-1}(z_n)|^{\frac{1+\varepsilon}{q}} = |f(z_n)| \cdot |\omega - z_n|^{\frac{1-\varepsilon}{p}} \geq c,$$

which ensures that the growth of $f \circ g$ along the points $g^{-1}(z_n)$ (that is as we approach the boundary) is large and therefore $f \circ g$ is not in the space $H^q(\mathbb{D}, \Omega)$.

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Paper I

Injectivity almost everywhere for weak limits of Sobolev homeomorphisms

Injectivity almost everywhere for weak limits of Sobolev homeomorphisms

Ondřej Bouchala[1] Stanislav Hencl[1] Anastasia Molchanova[2]

Abstract

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a weak (sequential) limit of Sobolev homeomorphisms. Then f is injective almost everywhere for $p > n-1$ both in the image and in the domain. For $p \leq n-1$ we construct a strong limit of homeomorphisms such that the preimage of a point is a continuum for every point in a set of positive measure in the image and the topological image of a point is a continuum for every point in a set of positive measure in the domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f\colon \Omega \to \mathbb{R}^n$ be a mapping. In this paper, we study classes of mappings f that might serve as deformations in Nonlinear Elasticity models. Following the pioneering papers of Ball [1] and Ciarlet and Nečas [7] we ask if our mapping is in some sense injective as the physical 'non-interpenetration of the matter' asks a deformation to be one-to-one.

There are several ways how to obtain injectivity or at least injectivity almost everywhere (*a.e.*) of the mapping f. As in [1] we can ask that our mapping has finite energy where the energy functional $\int_\Omega W(Df)$ contains special terms (like ratio of powers of Df, $\operatorname{adj} Df$ and J_f) and any mapping with finite energy and reasonable boundary data is a homeomorphism (the reader is referred to e.g. [16, 20, 22] and [28] for related results).

The approach motivated by Ball [1] is fine if our mapping is continuous everywhere but in some deformations the cavitation or even fractures may occur. To

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model these phenomena we need conditions which guarantee that our mapping is injective a.e. but on some small set bad things may happen. Ciarlet and Nečas [7] studied the class of mappings that satisfies

$$\int_\Omega J_f \leq |f(\Omega)| \tag{1.1}$$

together with $J_f > 0$ a.e. and they showed that mappings of this class are injective a.e. in the image, see e.g. [2, 3, 4, 5, 12, 27, 31] for further results in this direction or [21, 23] for numerical treatment. The inequality (1.1) is called the Ciarlet–Nečas condition nowadays. Note that the constraint $J_f > 0$ a.e. is usually assumed in models of Nonlinear Elasticity as the 'real deformation' cannot change its orientation and the energy density $W(Df(x))$ should tend to ∞ when $J_f(x) \to 0$, i.e. when we compress too much.

Another approach can be traced to Müller and Spector [25] where they studied a class of mappings that satisfy $J_f > 0$ a.e. together with the (INV) condition (see e.g. [4, 8, 17, 26, 29, 30]). They showed that mappings in their class are one-to-one a.e. (see Section 5 for more information). Informally speaking, the (INV) condition means that the ball $B(x,r)$ is mapped inside the image of the sphere $f(S(a,r))$ and the complement $\Omega \setminus \overline{B(x,r)}$ is mapped outside $f(S(a,r))$ (see Preliminaries for the formal definition).

In all results in the previous paragraph the authors assume that $f \in W^{1,p}(\Omega)$ for some $p > n-1$. We show that injectivity a.e. may fail horribly for $p \leq n-1$ even though the mapping f is even a strong limit of homeomorphisms. We would like to stress that it fails even in the limiting case $p = n-1$ which is technically more involved. The class of mappings that we study in our project consists of weak (sequential) limits of Sobolev homeomorphisms. Homeomorphisms clearly satisfy the (INV) condition and so their weak limit must as well if $p > n-1$, since in this case the (INV) condition is closed under weak convergence (see [25, Lemma 3.3]). Therefore the class of weak limits of Sobolev homeomorphisms is a suitable class for variational models and one could expect that nice properties of homeomorphisms (like invertibility) could be carried to their weak limit.

The class of weak limits of Sobolev homeomorphisms was recently characterized in the planar case by Iwaniec and Onninen [18, 19] and De Philippis and Pratelli [9]. Moreover, one can study the orientation of mappings in this class [15] or even investigate planar BV weak limits and characterize their set of cavities and fractures [6]. In [24] Molchanova and Vodopyanov studied invertibility a.e. of a special subclass of weak limits of homeomorphisms. We generalize some of their results and we show the sharpness of the assumption $p > n-1$. Our first result is about the invertibility a.e. in the image. By a continuum we mean the image of the segment $[0,1]$ in $\mathbb{R}^n$ by a continuous one-to-one mapping. See Preliminaries for the definition of a precise representative of a Sobolev mapping.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f\colon \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_k \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$ for $n > 2$ or $p \geq 1$ for $n = 2$. Then there is a precise representative $\widehat{f}$ and a set $N_1 \subset \mathbb{R}^n$ of Hausdorff* *dimension $n-1$ such that the preimage $\widehat{f}^{-1}(y)$ consists of only one point for every $y \in \widehat{f}(\Omega) \setminus N_1$.*

On the other hand for every integer $n \geq 3$ there is a continuous mapping $f\colon [-1,1]^n \to [-1,1]^n$ with $J_f > 0$ a.e. which is a strong limit of Sobolev homeomorphisms $f_k \in W^{1,n-1}([-1,1]^n, \mathbb{R}^n)$ with $f_k(x) = x$ for $x \in \partial[-1,1]^n$ such that

$$\text{there is } C_A \subset [-1,1]^n \text{ with } |C_A| > 0$$
$$\text{and } f^{-1}(y) \text{ is a continuum for every } y \in C_A.$$

Let us point out that the positive part of the statement essentially follows from the known results and techniques ([4, 25, 26]) while the counterexample is entirely new and it is our main contribution. In the positive direction we only remove the assumption $J_f > 0$ a.e. from [25] to have a mathematically complete theory. It is interesting that the Hausdorff dimension of the critical set N_1 suddenly jumps from $n-1$ to n as p changes from $p > n-1$ to $p = n-1$. Note that the bound of dimension $n-1$ for N_1 for $p > n-1$ is sharp as the mapping $[x_1, x_2, \dots, x_n] \to [0, x_2, \dots, x_n]$ shows. In [25, Section 11] there is a counterexample (in case $p < n = 2$), which shows that the weak limit of a sequence of one-to-one a.e. mappings might be two-to-one in a set of positive measure if (INV) is not satisfied. Our counterexample is entirely different as it is ∞-to-one and it is in some sense 'monotone' as a strong limit of homeomorphisms, which is definitely not the case for a mapping from [25].

Our second result is about the invertibility a.e. in the domain. See Preliminaries for the definition of the topological image $f^T(x)$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f\colon \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_k \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$ for $n > 2$ or $p \geq 1$ for $n = 2$. Then there is a set $N_2 \subset \mathbb{R}^n$ of Hausdorff dimension $n-p$ such that the image $f^T(x)$ consists of only one point for every $x \in \Omega \setminus N_2$. If we moreover assume that $J_f > 0$ a.e then there is a set N_3 of zero measure such that $f|_{\Omega \setminus N_3}$ is one-to-one.*

On the other hand for every $n \geq 3$ there is $\widetilde{f}\colon [-1,1]^n \to [-1,1]^n$ with $J_{\widetilde{f}} > 0$ a.e. which is a strong limit of Sobolev homeomorphisms $\widetilde{f}_k \in W^{1,n-1}([-1,1]^n, \mathbb{R}^n)$ with $\widetilde{f}_k(x) = x$ for $x \in \partial[-1,1]^n$. The quasicontinuous representative of $\widetilde{f}$ is one-to-one on $[-1,1]^n$ (but $\widetilde{f}([-1,1]^n) \subsetneq [-1,1]^n$). There is a continuous mapping $w\colon [-1,1]^n \to \mathbb{R}^n$ which is a generalized inverse to $\widetilde{f}$, i.e. $w(\widetilde{f}(x)) = x$ for every $x \in [-1,1]^n$ such that

$$\text{there is } C_A \subset [-1,1]^n \text{ with } |C_A| > 0$$
$$\text{and } w^{-1}(x) \text{ is a continuum for every } x \in C_A.$$

Locally constant mapping shows that the assumption $J_f > 0$ a.e. is needed for the conclusion that $f|_{\Omega \setminus N_3}$ is one-to-one. Moreover, there is no bound for the Hausdorff dimension of N_3 as there is a Lipschitz mapping f which maps a set of dimension n to a single point (see Example 4.3 below).

As in Theorem 1.1 the positive result essentially follows from the known results ([4, 25, 26]) while the counterexample is entirely new. As above the counterexample exists also for the critical exponent $p = n - 1$ and there is again a sudden jump in the dimension of the critical set N_2 from $n - p \leq 1$ to n.

2 Preliminaries

By $B(c, r)$ we denote the euclidean ball with center $c \in \mathbb{R}^n$ and radius $r > 0$, and $S(c, r)$ stands for the corresponding sphere.

2.1 Precise representative of a Sobolev mapping

Recall the following result from [32, Theorem 3.3.3 and Theorem 2.6.16].

Theorem 2.1. *Let $1 \leq p \leq n$ and let $f \in W^{1,p}(\mathbb{R}^n)$ be a p-quasicontinuous representative and set*

$$E_p = \{x \in \mathbb{R}^n : \ x \text{ is not a Lebesgue point of } f\} .$$

Then $\dim_{\mathcal{H}}(E_p) \leq n - p$.

We put

$$f^*(x) = \begin{cases} \lim\limits_{r \to 0^+} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} f(y)\, dy & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Note, that the representative f^* is p-quasicontinuous (see remarks after [25, Proposition 2.8]). We define a *precise representative* of $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ as any representative which is equal to f^* up to a set of p-capacity 0 (see e.g. [32, Section 2.6] for the definition of capacity).

Here is a useful observation [25, Lemma 2.9] about the representative f^*.

Lemma 2.2. *Let $f_k \to f$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $a \in \Omega$ and $r_a := \operatorname{dist}(a, \partial\Omega)$. Then there is an $\mathcal{L}^1$-null set N_a such that for any $r \in (0, r_a) \setminus N_a$ there exists a subsequence f_j such that $f_j^* \to f^*$ weakly in $W^{1,p}(S(a,r), \mathbb{R}^n)$. Furthermore, if $p > n - 1$ then $f_j^* \to f^*$ uniformly on $S(a, r)$.*

2.2 Topological degree

Given a smooth map f from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ we can define the *topological degree* as

$$\deg(f, \Omega, y_0) = \sum_{\{x \in \Omega : f(x) = y\}} \operatorname{sgn}(J_f(x))$$

if $J_f(x) \neq 0$ for each $x \in f^{-1}(y)$. This definition can be extended to arbitrary continuous mappings and each point, see e.g. [10].

The value of the degree of a continuous mapping $f\colon \overline{B(a,r)} \to \mathbb{R}^n$ depends only on its values on the boundary $S(a,r)$. Thus, given a continuous mapping $f\colon S(a,r) \to \mathbb{R}^n$ we use the notation $\deg(f, S(a,r), y)$ for $\deg(\widehat{f}, B(a,r), y)$, where $\widehat{f}\colon \overline{B(a,r)} \to \mathbb{R}^n$ is any continuous extension of $f\colon S(a,r) \to \mathbb{R}^n$.

The degree is known to be stable under uniform convergence (see e.g. [10, Theorem 2.3 (1)]), i.e.

$$\begin{gathered} f_k \rightrightarrows f \text{ on } S(b,s) \text{ and } y \notin f(S(b,s)) \\ \Downarrow \\ \lim_{k\to\infty} \deg(f_k, S(b,s), y) = \deg(f, S(b,s), y). \end{gathered} \tag{2.2}$$

It is also well-known that for a homeomorphism f and $y \notin f(S(a,r))$ we have

$$\deg(f, S(a,r), y) \neq 0 \Leftrightarrow y \in B(a,r). \tag{2.3}$$

2.3 (INV) condition

Suppose that $f\colon S(a,r) \to \mathbb{R}^n$ is continuous, following [25] we define the *topological image* of $B(a,r)$ as

$$f^T(B(a,r)) := \big\{ y \in \mathbb{R}^n \setminus f(S(a,r)) : \ \deg(f, S(a,r), y) \neq 0 \big\}.$$

Denote

$$E(f, B(a,r)) := f^T(B(a,r)) \cup f(S(a,r)).$$

Definition 2.3 ((INV) condition)**.** We say that $f\colon \Omega \to \mathbb{R}^n$ satisfies the condition (INV), provided that for every $a \in \Omega$ there exists an $\mathcal{L}^1$-null set N_a such that for all $r \in (0, \operatorname{dist}(a, \partial\Omega)) \setminus N_a$ the mapping $f|_{S(a,r)}$ is continuous,

(i) $f(x) \in f^T(B(a,r)) \cup f(S(a,r))$ for $\mathcal{L}^n$-a.e. $x \in \overline{B(a,r)}$ and

(ii) $f(x) \in \mathbb{R}^n \setminus f^T(B(a,r))$ for $\mathcal{L}^n$-a.e. $x \in \Omega \setminus B(a,r)$.

Moreover, we define the multifunction which describes the topological image $f^T(x)$ of a point as

$$f^T(x) := \bigcap_{r>0,\ r\notin N_x} E(f^*, B(x,r)),$$

where f^* is given by (2.1). Let us recall that a quasicontinuous representative of $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, is continuous for every x on almost every sphere $S(x,r)$.

2.4 Cantor-set construction

Following [14, Section 4.3] we consider a Cantor-set construction in $(-1,1)^n$.

Denote the cube with center at a and edge $2r$ by $Q(a,r) = (a_1 - r, a_1 + r) \times \cdots \times (a_n - r, a_n + r)$. Let $\mathbb{V}$ be the set of 2^n vertices of the cube $[-1,1]^n \subset \mathbb{R}^n$

and $\mathbb{V}^k = \mathbb{V} \times \cdots \times \mathbb{V}$, $k \in \mathbb{N}$. Consider a decreasing sequence $\{\alpha_k\}_{k=0}^{\infty}$ such that $\alpha_k \approx \alpha_{k+1}$, $1 = \alpha_0 \geq \alpha_1 \geq \cdots > 0$,

$$r_k = 2^{-k}\alpha_k \text{ and } r'_k = 2^{-k}\alpha_{k-1}.$$

Set $z_0 = 0$, then $Q(z_0, r_0) = (-1, 1)^n$ and we proceed by induction. For

$$\mathbf{v}(k) = (v_1, \dots, v_k) \in \mathbb{V}^k$$

we denote

$$\mathbf{v}(k-1) = (v_1, \dots, v_{k-1})$$

and define (see Fig. 1)

$$z_{\mathbf{v}(k)} = z_{\mathbf{v}(k-1)} + \frac{1}{2} r_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^{k} r_{j-1} v_j,$$

$$Q'_{\mathbf{v}(k)} = Q(z_{\mathbf{v}(k)}, r'_k) \quad \text{and} \quad Q_{\mathbf{v}(k)} = Q(z_{\mathbf{v}(k)}, r_k).$$

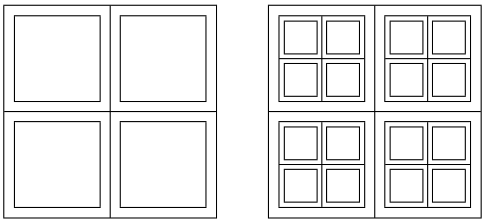

Figure 1: Cubes $Q_{\boldsymbol{v}(k)}$ and $Q'_{\boldsymbol{v}(k)}$ for $k = 1, 2$.

The measure of the k-th frame $Q'_{\mathbf{v}(k)} \setminus Q_{\mathbf{v}(k)}$ is

$$\mathcal{L}^n(Q'_{\mathbf{v}(k)} \setminus Q_{\mathbf{v}(k)}) = (2r'_k)^n - (2r_k)^n \approx 2^{-nk}(\alpha_{k-1} - \alpha_k)\alpha_k^{n-1}, \tag{2.4}$$

and we have 2^{nk} such frames.

Denote $A := \{\alpha_k\}_{k=0}^{\infty}$, the resulting Cantor set

$$C_A := \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{v}(k) \in \mathbb{V}^k} Q_{\mathbf{v}(k)}$$

is a product of n Cantor sets $\mathcal{C}_\alpha$ in $\mathbb{R}$

$$C_A = \mathcal{C}_\alpha \times \cdots \times \mathcal{C}_\alpha,$$

and the number of cubes in $\{Q_{\mathbf{v}(k)} : \mathbf{v}(k) \in \mathbb{V}^k\}$ is 2^{nk}. Hence,

$$\mathcal{L}^n(C_A) = \lim_{k \to \infty} 2^{nk}(2\alpha_k 2^{-k})^n = \lim_{k \to \infty} 2^n \alpha_k^n.$$

2.5 Homeomorphism that maps a Cantor set onto another one

Consider two sequences $A = \{\alpha_k\}_{k=0}^{\infty}$ and $B = \{\beta_k\}_{k=0}^{\infty}$, and two Cantor sets C_A and C_B are designed according Section 2.4. We also define

$$\begin{aligned}
&\widetilde{r}_k = 2^{-k}\beta_k, \quad \widetilde{r}'_k = 2^{-k}\beta_{k-1},\\
&\widetilde{z}_{\mathbf{v}(k)} = \widetilde{z}_{\mathbf{v}(k-1)} + \frac{1}{2}\widetilde{r}_{k-1}v_k = \widetilde{z}_0 + \frac{1}{2}\sum_{j=1}^{k}\widetilde{r}_{j-1}v_j,\\
&\widetilde{Q}'_{\mathbf{v}(k)} = Q(\widetilde{z}_{\mathbf{v}(k)}, \widetilde{r}'_k), \quad \widetilde{Q}_{\mathbf{v}(k)} = Q(\widetilde{z}_{\mathbf{v}(k)}, \widetilde{r}_k).
\end{aligned}$$

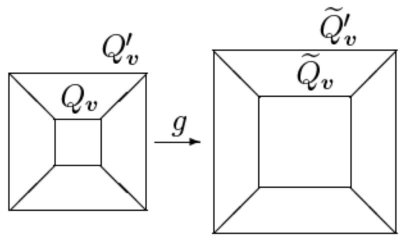

Figure 2: The transformation of $Q'_v \setminus Q_v$ onto $\widehat{Q}'_v \setminus \widehat{Q}_v$ for $n = 2$

There exists a homeomorphism g which maps C_A onto C_B (see Fig. 2). Moreover, in $Q'_{\mathbf{v}(k)} \setminus Q_{\mathbf{v}(k)}$ we have analogously to [14, proof of Theorem 4.10]

$$|Dg(x)| \approx \max\left\{\frac{\widetilde{r}_k}{r_k}, \frac{\frac{\widetilde{r}_{k-1}}{2} - \widetilde{r}_k}{\frac{r_{k-1}}{2} - r_k}\right\} = \max\left\{\frac{\beta_k}{\alpha_k}, \frac{\beta_{k-1} - \beta_k}{\alpha_{k-1} - \alpha_k}\right\} \tag{2.5}$$

and

$$J_g(x) \sim \frac{\frac{\widetilde{r}_{k-1}}{2} - \widetilde{r}_k}{\frac{r_{k-1}}{2} - r_k}\left(\frac{\widetilde{r}_k}{r_k}\right)^{n-1}.$$

Likewise, for $y \in \widetilde{Q}'_{\mathbf{v}(k)} \setminus \widetilde{Q}_{\mathbf{v}(k)}$ we have

$$|Dg^{-1}(y)| \approx \max\left\{\frac{\alpha_k}{\beta_k}, \frac{\alpha_{k-1} - \alpha_k}{\beta_{k-1} - \beta_k}\right\} \text{ and } J_{g^{-1}}(y) \sim \frac{\frac{r_{k-1}}{2} - r_k}{\frac{\widetilde{r}_{k-1}}{2} - \widetilde{r}_k}\left(\frac{r_k}{\widetilde{r}_k}\right)^{n-1}. \tag{2.6}$$

More precisely we define this g as a uniform limit of bilipschitz mappings g_k which map the k-th iteration of the Cantor set C_A onto the k-th iteration of C_B. That is

$$g_k(x) = g(x) \text{ for } x \notin \bigcup_{\mathbf{v}(k) \in \mathbb{V}^k} Q_{\mathbf{v}(k)} \tag{2.7}$$

and

$$g_k \text{ maps } Q_{\mathbf{v}(k)} \text{ onto } \widetilde{Q}_{\mathbf{v}(k)} \text{ linearly for } \mathbf{v}(k) \in \mathbb{V}^k.$$

2.6 Constructing a Cantor tower

We build a Cantor tower as in [13].

Suppose $n \geq 2$ and denote by $\hat{\mathbb{V}}$ the set of points

$$\left(0, 0, \dots, 0, -1 + \tfrac{2j-1}{2^n}\right)$$

where $j = 1, 2, \dots, 2^n$. Sets

$$\hat{\mathbb{V}}^k := \hat{\mathbb{V}} \times \cdots \times \hat{\mathbb{V}}, \quad k \in \mathbb{N},$$

serve as sets of indices in the construction of a Cantor tower.

Suppose that $\{\beta_k\}_{k=0}^{\infty}$ is a decreasing sequence as before with $1 = \beta_0$ and $\beta_i > 2^n \beta_{i+1}$, and define

$$\hat{r}_k := 2^{-k}\beta_k \text{ and } \hat{r}'_k := 2^{-k}\beta_{k-1}. \tag{2.8}$$

Set $\hat{z}_0 = 0$. Then it follows that $Q(\hat{z}_0, \hat{r}_0) = (-1, 1)^n$ and we proceed further by induction. For $\hat{\boldsymbol{v}}(k) := (\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k) \in \hat{\mathbb{V}}^k$ we denote $\hat{\boldsymbol{v}}(k-1) := (\hat{v}_1, \hat{v}_2, \dots, \hat{v}_{k-1})$ and define (see Fig. 3)

$$\begin{aligned} \hat{z}_{\hat{\boldsymbol{v}}(k)} &:= \hat{z}_{\hat{\boldsymbol{v}}(k-1)} + \hat{r}_{k-1}\hat{v}_k = \hat{z}_0 + \sum_{j=1}^{k} \hat{r}_{j-1}\hat{v}_j \\ \widehat{Q}'_{\hat{\boldsymbol{v}}(k)} &:= Q(\hat{z}_{\hat{\boldsymbol{v}}(k)}, \hat{r}'_k) \text{ and } \widehat{Q}_{\hat{\boldsymbol{v}}(k)} := Q(\hat{z}_{\hat{\boldsymbol{v}}(k)}, \hat{r}_k) \end{aligned} \tag{2.9}$$

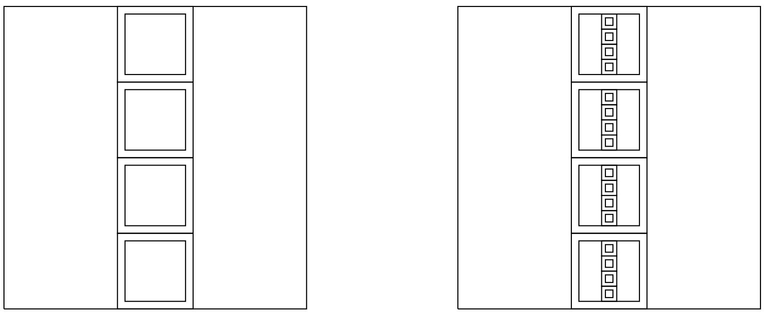

Figure 3: Cubes $\widehat{Q}_{\hat{\boldsymbol{v}}(k)}$ and $\widehat{Q}'_{\hat{\boldsymbol{v}}(k)}$ for $k = 1, 2$ in the construction of the Cantor tower.

2.7 Bilipschitz mapping which takes a Cantor set onto a Cantor tower

Let us now define the Cantor set C_B as in Section 2.4 by choosing

$$\beta_k = 2^{-k\beta}, \tag{2.10}$$

where $\beta \geq n+1$. Using this sequence we also define the Cantor tower C_B^T as in Section 2.6.

As $\beta \geq n+1$, we see that

$$\widehat{Q}_{\hat{\boldsymbol{v}}(k)} = Q(\hat{z}_{\hat{\boldsymbol{v}}(k)}, 2^{-k}\beta_k) \subsetneq Q(\hat{z}_{\hat{\boldsymbol{v}}(k)}, 2^{-1-k}\beta_{k-1}) = \frac{1}{2}\widehat{Q}'_{\hat{\boldsymbol{v}}(k)}$$

and thus we have enough empty space in $\widehat{Q}'_{\hat{\boldsymbol{v}}(k)} \setminus \widehat{Q}_{\hat{\boldsymbol{v}}(k)}$ to move the cubes of the next generation into a tower formation.

The following theorem from [13, Proposition 2.4] gives us a bilipschitz mapping $L\colon \mathbb{R}^n \to \mathbb{R}^n$ which maps the Cantor set C_B onto the Cantor tower C_B^T. We refer to this mapping as a *tower mapping*.

Theorem 2.4. *Suppose that C_B is the Cantor set and C_B^T is the Cantor tower in $\mathbb{R}^n$ defined by the sequence*

$$\beta_k = 2^{-k\beta}\,,$$

where $\beta \geq n+1$. Then there is a bilipschitz mapping $L\colon \mathbb{R}^n \to \mathbb{R}^n$ which takes C_B onto C_B^T. Moreover,

$$\text{for every } \hat{\boldsymbol{v}}(i) \in \hat{\mathbb{V}}^i \quad L^{-1}(\widehat{Q}_{\hat{\boldsymbol{v}}(i)}) = \widetilde{Q}_{\boldsymbol{v}(i)} \text{ for some } \boldsymbol{v}(i) \in \mathbb{V}^i. \tag{2.11}$$

2.8 Piecewise linear mappings

We define an auxiliary piecewise linear mapping and estimate its derivative.

Let

$$t_1 < t_2 < t_3 < t_4 \text{ and } s_1 < s_2 < s_3 < s_4. \tag{2.12}$$

We consider a piecewise linear mapping $h\colon [t_1, t_4] \to \mathbb{R}$ with $h(t_i) = s_i$, $i = 1, 2, 3, 4$, i.e.

$$h(t; [t_1, s_1], [t_2, s_2], [t_3, s_3], [t_4, s_4]) = \begin{cases} \frac{s_2 - s_1}{t_2 - t_1}(t - t_1) + s_1, & \text{if } t_1 \leq t \leq t_2, \\ \frac{s_3 - s_2}{t_3 - t_2}(t - t_2) + s_2, & \text{if } t_2 < t \leq t_3, \\ \frac{s_4 - s_3}{t_4 - t_3}(t - t_3) + s_3, & \text{if } t_3 < t \leq t_4. \end{cases} \tag{2.13}$$

Clearly

$$|Dh(t)| = \begin{cases} \frac{s_2 - s_1}{t_2 - t_1}, & \text{if } t_1 < t < t_2, \\ \frac{s_3 - s_2}{t_3 - t_2}, & \text{if } t_2 < t < t_3, \\ \frac{s_4 - s_3}{t_4 - t_3}, & \text{if } t_3 < t < t_4. \end{cases} \tag{2.14}$$

3 Injectivity in the image: counterexample in the Theorem 1.1

3.1 Definition of tentacles

We start with a Cantor tower C_B^T and for each point $y \in C_A$ we find a corresponding point $x \in C_B^T$ (see (3.1)). We want to have a continuum l_x (with the end point x) which goes onto y by our mapping. For better visualization we first map C_B^T on itself to squeeze this l_x onto x by mapping h. Then, with the help of a bilipschitz mapping L^{-1} (Theorem 2.4) we transform C_B^T to C_B and finally we map homeomorphically C_B onto C_A by g^{-1} (see Subsection 2.5), i.e. the final mapping

$$f = g^{-1} \circ L^{-1} \circ h$$

squeezes l_x onto y (see Fig. 7).

For any $x \in C_B^T$ we find sequence $\hat{\boldsymbol{v}}(k) \in \hat{\mathbb{V}}^k$ such that

$$x = \bigcap_{k=1}^{\infty} \widehat{Q}_{\hat{\boldsymbol{v}}(k)} \text{ and this corresponds to } y = \bigcap_{k=1}^{\infty} Q_{\boldsymbol{v}(k)} \in C_A, \tag{3.1}$$

where the mapping $\hat{\boldsymbol{v}}(k) \to \boldsymbol{v}(k)$ is given by $L^{-1}(\widehat{Q}_{\hat{\boldsymbol{v}}_k}) = \widetilde{Q}_{\boldsymbol{v}(k)}$ from (2.11). Now for each $\widehat{Q}_{\hat{\boldsymbol{v}}(k)}$ we define a tentacle $T_{\hat{\boldsymbol{v}}(k)}$ (a long and thin polyhedron) which contains $\widehat{Q}_{\hat{\boldsymbol{v}}(k)}$ and we set

$$l_x := \bigcap_{k=1}^{\infty} T_{\hat{\boldsymbol{v}}(k)}. \tag{3.2}$$

First we define a 'straight' tentacle $T^S_{\hat{\boldsymbol{v}}(k)}$ and then we adjust it in the next subsection so that

$$T_{\hat{\boldsymbol{v}}(k+1)} \subset T_{\hat{\boldsymbol{v}}(k)} \text{ whenever } \hat{\boldsymbol{v}}(k+1) \text{ is a continuation of } \hat{\boldsymbol{v}}(k),$$

i.e. first k terms of $\hat{\boldsymbol{v}}(k+1)$ are exactly $\hat{\boldsymbol{v}}(k)$ (see Fig. 4).

Take the parameter β from (2.10) and recall (2.8), that is $\hat{r}_k = 2^{-k}\beta_k = 2^{-k(\beta+1)}$. We define for $k \in \mathbb{N}$

$$a_k = 1 - \sum_{i=0}^{k} \hat{r}_{i+2} \approx 1, \quad c_k = 1 - \sum_{i=0}^{k-1} \hat{r}_{i+2} \approx 1,$$

and further we fix decreasing sequences $0 < b_{k+1} < b_k < \frac{1}{e}$ and $0 < d_{k+1} < d_k < \frac{1}{e}$

$$\text{such that } b_k < d_k < a_k < c_k \text{ and } d_{k+1} < 4^n b_k \tag{3.3}$$

whose exact values we find by induction using Lemma 3.2 below.

For $r > 0$ and $\rho_1 < \rho_2$ we define a parallelepiped

$$P(r, \varrho_1, \varrho_2) := [\varrho_1, \varrho_2) \times (-r, r) \times \cdots \times (-r, r).$$

For each k we also define

$$P'_k := P(d_k, \hat{r}_k, c_k) \quad \text{and} \quad P_k := P(b_k, \hat{r}_k, a_k).$$

Now we define 'straight' tentacles as

$$T'^S_k := Q(0, \hat{r}_k) \cup P'_k \quad \text{and} \quad T^S_k := Q(0, \hat{r}_k) \cup P_k,$$
$$T'^S_{\hat{\boldsymbol{v}}(k)} := \hat{z}_{\hat{\boldsymbol{v}}(k)} + T'^S_k \quad \text{and} \quad T^S_{\hat{\boldsymbol{v}}(k)} := \hat{z}_{\hat{\boldsymbol{v}}(k)} + T^S_k.$$

Both $T_k'^S$ and T_k^S clearly contain $Q(0,\hat{r}_k)$ and note that $T_k^S \subset T_k'^S$ as $c_k > a_k$ and $d_k > b_k$. Moreover, P_k' and $Q(0,\hat{r}_k)$ have one common side and thus $T_k'^S$ is connected. Furthermore, the length of each tentacle $T_k'^S$ is bigger than $a_k > 1 - \frac{1}{1-2^{-\beta-1}}$ and hence

$$l^S := \bigcap_{k=1}^{\infty} T_k^S \text{ is a nontrivial segment.} \tag{3.4}$$

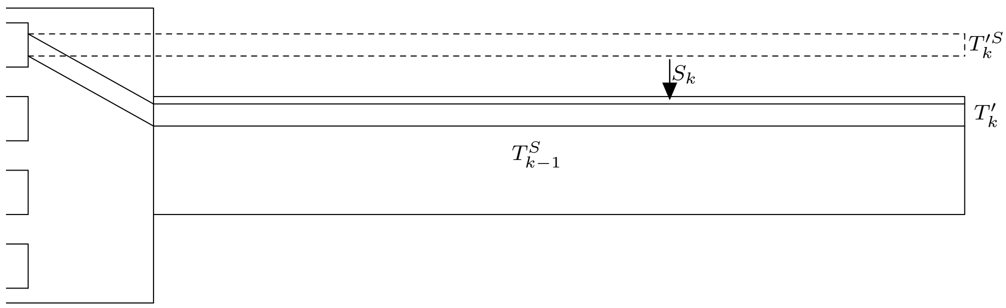

Figure 4: Two generations of tentacles.

Let us estimate

$$|P_k| \approx a_k \cdot (2b_k)^{n-1} \approx b_k^{n-1} \text{ and } |P_k'| \approx c_k \cdot (2d_k)^{n-1} \approx d_k^{n-1}. \tag{3.5}$$

3.2 Shifting of tentacles into previous tentacles

In this section we want to shift the 'straight' tentacles into 'real' tentacles $T_{\hat{\boldsymbol{v}}(k)}$ so that

$$T'_{\hat{\boldsymbol{v}}(k+1)} \subset T_{\hat{\boldsymbol{v}}(k)} \text{ whenever } \hat{\boldsymbol{v}}(k+1) \text{ is a continuation of } \hat{\boldsymbol{v}}(k). \tag{3.6}$$

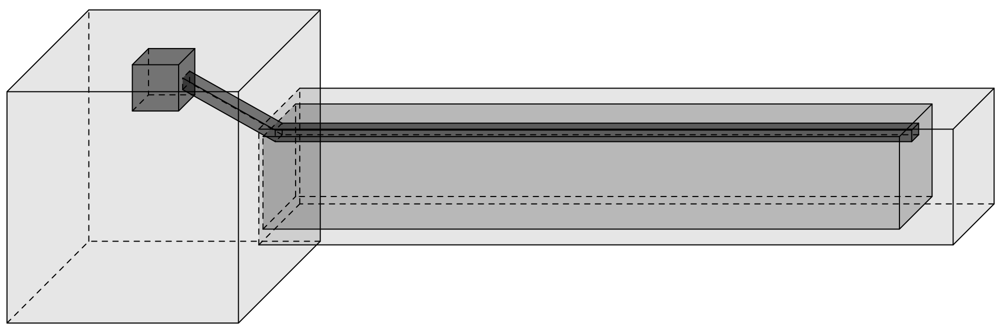

Figure 5: Tentacles T_1', T_1 and T_2'

Set $T_1' = T_1'^S$, $T_1 = T_1^S$. We need a *shifting* mapping

$$(s_k)_{\hat{\boldsymbol{v}}(k)}(t,s) = \begin{cases} s - (\hat{r}_{k-1} - b_{k-1})(\hat{v}_k)_n, & \text{if } t \in (\hat{r}_{k-1}, c_k + \hat{r}_k], \\ s - \frac{(\hat{v}_k)_n(\hat{r}_{k-1} - b_{k-1})}{\hat{r}_k - \hat{r}_{k-1}}(\hat{r}_k - t), & \text{if } t \in (\hat{r}_k, \hat{r}_{k-1}], \\ s, & \text{if } t \in (0, \hat{r}_k]. \end{cases}$$

For $x \in T_k'^S$ define

$$(S_k)_{\hat{\boldsymbol{v}}(k)}(x_1, \dots, x_n) := \big(x_1, \dots, x_{n-1}, (s_k)_{\hat{\boldsymbol{v}}(k)}(x_1, x_n)\big).$$

Note that we have shifted the x_n coordinate by $(\hat{z}_{\boldsymbol{v}(k)} - \hat{z}_{\boldsymbol{v}(k-1)})_n = \hat{r}_{k-1}(\hat{v}_k)_n$ down (see (2.9)), i.e. we have moved the right part of k-tentacle $T_k'^S$ to the height of $(k-1)$-tentacle T_{k-1}^S (see Fig. 4), and then we moved it by $b_{k-1}(\hat{v}_k)_n$ up so that the position of different $T_k'^S$ is different and they are again above each other in the $(k-1)$-tentacle T_{k-1}^S (of height $2b_{k-1}$).

It is easy to see that the Jacobian of this mapping is equal to 1 and hence it does not change the measure of the tentacles. We can estimate its derivative as

$$|D(S_k)_{\hat{\boldsymbol{v}}(k)}| \approx \max\left\{1, \frac{(\hat{v}_k)_n(\hat{r}_{k-1} - b_{k-1})}{\hat{r}_k - \hat{r}_{k-1}}\right\} \approx 1$$

and moreover

$$|D(S_k)^{-1}_{\hat{\boldsymbol{v}}(k)}| \approx 1.$$

For $\hat{\boldsymbol{v}}(k) = (v_1, \dots, v_k)$ we denote $\hat{\boldsymbol{v}}(j) = (v_1, \dots, v_j)$ and we define

$$S_{\hat{\boldsymbol{v}}(k)} := (S_1)_{\hat{\boldsymbol{v}}(1)} \circ \dots \circ (S_k)_{\hat{\boldsymbol{v}}(k)}.$$

Remark 3.1. Note that for each $x \in T_k'^S \cap (Q(0, \hat{r}_{j-1}) \setminus Q(0, \hat{r}_j))$ mapping $S_{\hat{\boldsymbol{v}}(k)}(x)$ is a composition of $k-1$ translations and one bending $(S_j)_{\hat{\boldsymbol{v}}(j)}$ with $|D(S_j)^{-1}_{\hat{\boldsymbol{v}}(j)}| \approx |D(S_j)_{\hat{\boldsymbol{v}}(j)}| \approx 1$. Hence, this composition is also bilipschitz with a constant that does not depend on k.

Let us define the k-th generation as

$$T'_{\hat{\boldsymbol{v}}(k)} := S_{\hat{\boldsymbol{v}}(k)}(T'^S_{\hat{\boldsymbol{v}}(k)}) \text{ and } T_{\hat{\boldsymbol{v}}(k)} := S_{\hat{\boldsymbol{v}}(k)}(T^S_{\hat{\boldsymbol{v}}(k)}),$$

$$P'_{\hat{\boldsymbol{v}}(k)} := T'_{\hat{\boldsymbol{v}}(k)} \setminus Q(\hat{z}_{\hat{v}(k)}, \hat{r}_k) \text{ and } P_{\hat{\boldsymbol{v}}(k)} := T_{\hat{\boldsymbol{v}}(k)} \setminus Q(\hat{z}_{\hat{v}(k)}, \hat{r}_k).$$

This definition ensures (3.6) as $b_k > 2^n d_{k+1}$ (see Fig. 4) by (3.3). Since the shifting map does not change the volume, we obtain from (3.5) that

$$|P_{\hat{\boldsymbol{v}}(k)}| \approx b_k^{n-1} \text{ and } |P'_{\hat{\boldsymbol{v}}(k)}| \approx d_k^{n-1}. \tag{3.7}$$

It is clear that the diameter of l_x, which is defined by (3.2), is bigger than the diameter of l^S (see (3.2) and (3.4)) and hence l_x is a nontrivial continuum. Moreover,

$$\mathcal{L}_n\Big(\bigcup_{x \in C_B^T} l_x\Big) = 0, \text{ since } \mathcal{L}_n\Big(\bigcup_{\boldsymbol{v}(k) \in \hat{\mathbb{V}}^k} T'_{\boldsymbol{v}(k)}\Big) \le 2^{nk}\big(d_k^{n-1} + \hat{r}_k^n\big) \xrightarrow[k \to \infty]{} 0. \tag{3.8}$$

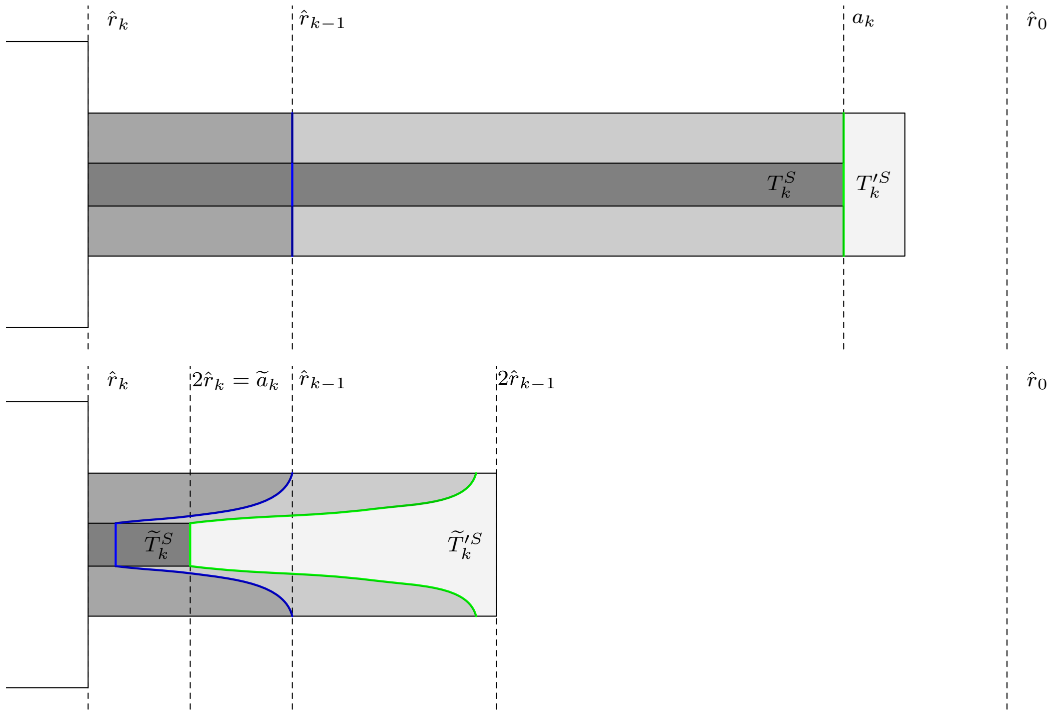

Figure 6: Tentacle squeezing H_k.

3.3 Squeezing inside tentacles

The aim of this section is to obtain a mapping which is identity outside the tentacles and squeezes each continuum l_x onto x for every $x \in C_B^T$.

Analogously to Section 3.1 we define parameters which describe the sizes of squeezed tentacles. We set

$$\begin{aligned}\widetilde{a}_k &= 2\hat{r}_k \approx 2^{-k(\beta+1)},\\ \widetilde{c}_k &= \widetilde{a}_{k-1} = 2\hat{r}_{k-1} \approx 2^{-k(\beta+1)}.\end{aligned}$$

With these parameters we consider for each $k \in \mathbb{N}$

$$\widetilde{P}_k' := P(d_k, \hat{r}_k, \widetilde{c}_k) \quad \text{and} \quad \widetilde{P}_k := P(b_k, \hat{r}_k, \widetilde{a}_k).$$

Now the ‘squeezed’ tentacles (see Fig. 6) are defined by

$$\begin{aligned}\widetilde{T}_k'^S &:= Q(0, \hat{r}_k) \cup \widetilde{P}_k' \quad \text{and} \quad \widetilde{T}_k^S := Q(0, \hat{r}_k) \cup \widetilde{P}_k,\\ \widetilde{T}_{\hat{\boldsymbol{v}}(k)}'^S &:= \hat{z}_{\hat{\boldsymbol{v}}(k)} + \widetilde{T}_k'^S \quad \text{and} \quad \widetilde{T}_{\hat{\boldsymbol{v}}(k)}^S := \hat{z}_{\hat{\boldsymbol{v}}(k)} + \widetilde{T}_k^S.\end{aligned}$$

With the help of piecewise linear mapping from Section 2.8 we can squeeze the ‘straight’ tentacles. The main idea of this construction is that points have zero capacity in $W^{1,n-1}(\mathbb{R}^{n-1})$, i.e. the correct truncation of the function $\log\log\frac{1}{|x|}$ has small support, value 1 at 0 and arbitrarily small norm in $W^{1,n-1}$. For $p < n-1$ it would be enough to work with piecewise affine mappings instead of $\log\log\frac{1}{|x|}$.

Lemma 3.2. *Let $n \geq 3$, $\delta_k > 0$, $\beta \geq n+1$ and $k \in \mathbb{N}$. Then we can find small enough $d_k > b_k > 0$ and a bilipschitz mapping $H_k^S\colon Q(0,1) \to Q(0,1)$ such that $H_0^S(x) = x$ for every $x \in Q(0,1)$*

$$H_k^S(x) = H_{k-1}^S(x) \text{ for each } x \notin P_k',\ H_k^S(x) = x \text{ for } x \in Q(0, \hat{r}_k)$$

$$\text{and } H_k^S \text{ maps } P_k \text{ onto } \widetilde{P}_k \text{ linearly.}$$

Furthermore, $|DH_k^S(x)| \leq 1$ for $x \in P_k$ and

$$\int_{P_k'} |DH_k^S(x)|^{n-1}\, dx \leq \delta_k. \tag{3.9}$$

Proof. Set $H_0^S(x) = x$ and proceed by induction. We define

$$H_k^S(x) = H_{k-1}^S(x) \text{ for } x \notin P_k' \tag{3.10}$$

and it remains to define it on P_k'. Since H_{k-1}^S is the identity on $\{x_1 \leq \hat{r}_{k-1}\}$ and $P_k' \cap \{x_1 \geq \hat{r}_{k-1}\} \subset P_{k-1}$ (see (3.3)) where H_{k-1}^S is linear we obtain that on $\partial P_k'$ we have

$$\begin{aligned} &H_k^S(x) = [l_{k-1}(x_1), x_2, \ldots, x_n], \text{ where } l_{k-1}(x_1) = x_1 \text{ for } x_1 \leq \hat{r}_{k-1} \text{ and} \\ &\text{for } x_1 \in [\hat{r}_{k-1}, c_k] \text{ it is linear with } l_{k-1}(\hat{r}_{k-1}) = \hat{r}_{k-1} \text{ and } l_{k-1}(a_{k-1}) = \widetilde{a}_{k-1}. \end{aligned} \tag{3.11}$$

As $\widetilde{a}_{k-1} < a_{k-1}$ we know that the derivative $|l_{k-1}'| \leq 1$ there.

Further, we define it for $x \in \{a_k\} \times [-d_k, d_k]^{n-1}$ as

$$\begin{aligned} &H_k^S(x) = [\varphi_k(x), x_2, \ldots, x_n] \text{ where} \\ &\varphi_k(x) := l_{k-1}(a_k) - \Big(\log\log \frac{1}{\max\{b_k, |[x_2, \ldots, x_n]|_\infty\}} - \log\log \frac{1}{d_k}\Big), \end{aligned}$$

where $|[x_2, \ldots, x_n]|_\infty = \max\{|x_2|, \ldots, |x_n|\}$. Then it is easy to see $(H_k^S(x))_1 = l_{k-1}(a_k)$ when $x_1 = a_k$ and $|[x_2, \ldots, x_n]|_\infty = d_k$ and thus it agrees with (3.11) there. Moreover, we fix d_k small enough in such a way as ($C_{(3.12)}$ is a constant whose exact value we specify later)

$$\frac{2^{(\beta+1)k(n-1)}}{\log^{n-2} \frac{1}{d_k}} < C_{(3.12)} \delta_k \tag{3.12}$$

and we fix $b_k < d_k$ so that (see Fig. 6)

$$\text{for } |[x_2, \ldots, x_n]|_\infty = b_k \text{ we have } \varphi_k(x) = l_{k-1}(a_k) - \Big(\log\log \frac{1}{b_k} - \log\log \frac{1}{d_k}\Big) = \widetilde{a}_k. \tag{3.13}$$

For every $x \in P_k$ we have $|[x_2, \ldots, x_n]|_\infty \leq b_k$ and thus $\varphi_k(x) = \widetilde{a}_k$. Therefore for every $x \in P_k$ we can define

$$H_k^S(x) = \big[l_k(x_1), x_2, \ldots, x_n\big] \text{ where } l_k \text{ is linear with } l_k(\hat{r}_k) = \hat{r}_k \text{ and } l_k(a_k) = \widetilde{a}_k. \tag{3.14}$$

It is easy to see that $|DH_k^S| \leq 1$ there and that this agrees with (3.11) used for $k-1$ before. Finally on the hyperplane $x \in \{\hat{r}_{k-1}\} \times [-d_k, d_k]^{n-1}$ we define it as (see Fig. 6)

$$H_k^S(x) = [\psi_k(x), x_2, \dots, x_n] \text{ where}$$
$$\psi_k(x) := l_{k-1}(\hat{r}_{k-1}) - A_k\Big(\log\log\frac{1}{\max\{b_k, \|[x_2,\dots,x_n]\|_\infty\}} - \log\log\frac{1}{d_k}\Big).$$

As before it agrees with (3.11) for $x_1 = \hat{r}_{k-1}$ and $\|[x_2,\dots,x_n]\|_\infty = d_k$. The constant A_k is chosen so that for $x \in P_k \cap \{x_1 = \hat{r}_{k-1}\}$, i.e. for $\|[x_2,\dots,x_n]\|_\infty \leq b_k$, it goes along with (3.14). By this and (3.13) we obtain

$$\hat{r}_k \leq l_{k-1}(\hat{r}_{k-1}) - A_k\Big(\log\log\frac{1}{b_k} - \log\log\frac{1}{d_k}\Big) = l_{k-1}(\hat{r}_{k-1}) + A_k\Big(\widetilde{a}_k - l_{k-1}(a_k)\Big)$$

and hence

$$A_k \leq \frac{l_{k-1}(\hat{r}_{k-1}) - \hat{r}_k}{l_{k-1}(a_k) - \widetilde{a}_k} \leq \frac{\hat{r}_{k-1} - \hat{r}_k}{\hat{r}_{k-1} - 2\hat{r}_k} \leq C$$

and so A_k is bounded by a constant independent of k.

For every $[x_2,\dots,x_n] \in [-d_k, d_k]^{n-1}$ we use linear interpolation between values on four hyperplanes ($x_1 = \hat{r}_k$, $x_1 = \hat{r}_{k-1}$, $x_1 = a_k$ and $x_1 = c_k$) with the help of the function h from Section 2.8 and for $x \in P_k'$ we define

$$H_k^S(x) = \Big[h\big(x_1; [\hat{r}_k, \hat{r}_k], [\hat{r}_{k-1}, \psi_k(x)], [a_k, \varphi_k(x)], [c_k, l_{k-1}(c_k)]\big), x_2, \dots, x_n\Big].$$

By (3.10) and (3.11) this mapping is continuous. The mapping H_k^S is bilipschitz on all parts (whilst the bilipschitz constant depends on k) and hence it follows immediately that it is bilipschitz on $Q(0,1)$.

It remains to estimate the integrability of the derivative. By (2.14) we obtain that the derivative with respect to the first coordinate can be estimated as

$$|D_1 H_k^S(x)| \leq \begin{cases} \frac{\psi_k(x) - \hat{r}_k}{\hat{r}_{k-1} - \hat{r}_k}, & \text{for } \hat{r}_k < x_1 < \hat{r}_{k-1}, \\ \frac{\varphi_k(x) - \psi_k(x)}{a_k - \hat{r}_{k-1}}, & \text{for } \hat{r}_{k-1} < x_1 < a_k, \\ \frac{l_{k-1}(c_k) - \varphi_k(x)}{c_k - a_k}, & \text{if } a_k < x_1 < c_k. \end{cases}$$

Since $\varphi_k(x)$ takes values between $l_{k-1}(a_k)$ and $\widetilde{a}_k$ (see (3.13)) and $\psi_k(x)$ takes values between $l_{k-1}(\hat{r}_{k-1}) = \hat{r}_{k-1}$ and $\widetilde{a}_k$ we can estimate this by the universal constant C (where C does not depend on k). Furthermore, by (2.13) we know that we can estimate the derivative with respect to other coordinates by the constant multiple of the corresponding derivative of

$$\frac{\psi_k(x) - \hat{r}_k}{\hat{r}_{k-1} - \hat{r}_k} + \frac{\varphi_k(x) - \psi_k(x)}{a_k - \hat{r}_{k-1}} + \frac{l_{k-1}(c_k) - \varphi_k(x)}{c_k - a_k}.$$

Since $A_k \leq C$ we can estimate this by

$$C\max\Big\{\frac{1}{\hat{r}_{k-1} - \hat{r}_k}, \frac{1}{a_k - \hat{r}_{k-1}}, \frac{1}{c_k - a_k}\Big\}\Big|D\Big(\log\log\frac{1}{\max\{b_k, \|[x_2,\dots,x_n]\|_\infty\}}\Big)\Big|$$

The maximum of the three terms can be estimated by $C\frac{1}{\hat{r}_k} \leq C2^{(\beta+1)k}$ and thus we can estimate the derivative with respect to other coordinates as

$$|D_j H_k^S(x)| \leq \begin{cases} \frac{C2^{(\beta+1)k}}{|[x_2,\dots,x_n]|_\infty \log \frac{1}{|[x_2,\dots,x_n]|_\infty}} & \text{for } b_k < |[x_2,\dots,x_n]|_\infty < d_k, \\ 0 & \text{for } |[x_2,\dots,x_n]|_\infty < b_k. \end{cases}$$

Now a simple change to polar/spherical coordinates in $\mathbb{R}^{n-1}$ and (3.12) gives us

$$\begin{aligned} \int_{P_k'} |DH_k^S(x)|^{n-1}\, dx &\leq C2^{(\beta+1)k(n-1)} \int_{P_k'} \frac{1}{|[x_2,\dots,x_n]|_\infty^{n-1} \log^{n-1} \frac{1}{|[x_2,\dots,x_n]|_\infty}}\, dx \\ &\leq C2^{(\beta+1)k(n-1)} \int_0^{d_k} \frac{1}{r^{n-1}\log^{n-1}\frac{1}{r}} r^{n-2}\, dr \\ &\leq C2^{(\beta+1)k(n-1)} \frac{1}{\log^{n-2}\frac{1}{d_k}} < CC_{(3.12)}\delta_k < \delta_k, \end{aligned}$$

where we have chosen $C_{(3.12)}$ in (3.12) so that the last inequality holds. □

Above we have defined 'straight' tentacles $T'^S_{\hat{\boldsymbol{v}}(k)}$ and we have squeezed them by H_k^S onto squeezed 'straight' tentacles $\widetilde{T}'^S_{\hat{\boldsymbol{v}}(k)}$. Analogously we take 'real' (=twisted) tentacles $T'_{\hat{\boldsymbol{v}}(k)}$ and we squeeze inside them by H_k to obtain 'real' squeezed tentacles $\widetilde{T}'_{\hat{\boldsymbol{v}}(k)}$. For $k \geq 1$ we define

$$H_k(x) := S_{\hat{\boldsymbol{v}}(k)} \circ (H_k^S + \hat{z}_{\hat{\boldsymbol{v}}(k)}) \circ (S^{-1}_{\hat{\boldsymbol{v}}(k)}(x) - \hat{z}_{\hat{\boldsymbol{v}}(k)}) \quad \text{for} \quad x \in T'_{\hat{\boldsymbol{v}}(k)}, \tag{3.15}$$

$$\widetilde{T}'_{\hat{\boldsymbol{v}}(k)} := H_k(T'_{\hat{\boldsymbol{v}}(k)}) \quad \text{and} \quad \widetilde{T}_{\hat{\boldsymbol{v}}(k)} := H_k(T_{\hat{\boldsymbol{v}}(k)}).$$

Theorem 3.3. *Let $n \geq 3$, $\widetilde{\delta}_k > 0$, $\beta \geq n+1$ and $k \in \mathbb{N}$. Then we can find small enough $d_k > b_k > 0$ and a bilipschitz mapping $h_k \colon Q(0,1) \to Q(0,1)$ such that $h_0(x) = x$ for every $x \in Q(0,1)$,*

$$\begin{aligned} h_{k-1}(x) = h_k(x) \text{ for } x \notin \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{\boldsymbol{v}}(k)}, \\ h_k(x) = x \text{ for } x \in \widehat{Q}_{\hat{\boldsymbol{v}}(k)} \text{ and } h_k(P_{\hat{\boldsymbol{v}}(k)}) = \widetilde{P}_{\hat{\boldsymbol{v}}(k)}. \end{aligned} \tag{3.16}$$

We can estimate the integral of its derivative as

$$\int_{\bigcup_{\hat{v}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{v}(k)}} |Dh_k(x)|^{n-1}\, dx \leq \widetilde{\delta}_k. \tag{3.17}$$

Moreover, a pointwise limit h of h_k is continuous, $J_h(x) > 0$ a.e., and

$$h(l_x) = x \text{ for every } x \in C_B^T,$$

where l_x is defined by (3.2).

Proof. We set $h_0(x) = x$ and further we define (see (3.15))

$$h_k(x) = \begin{cases} h_{k-1}(x) & \text{for } x \notin \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} T'_{\hat{\boldsymbol{v}}(k)}, \\ H_k(x) & \text{for } x \in T'_{\hat{\boldsymbol{v}}(k)}, \end{cases} \tag{3.18}$$

which clearly fulfills (3.16) since $H_k(x) = h_{k-1}(x) = x$ on all $Q(\hat{z}_{\hat{\boldsymbol{v}}(k)}, \hat{r}_k)$.

We have 2^{nk} different sets $T'_{\hat{\boldsymbol{v}}(k)}$ and all of them are bilipschitz copy of T'^S_k, mappings $S_{\hat{\boldsymbol{v}}(k)}$ are bilipschitz with a constant that does not depend on k, and hence we obtain by Lemma 3.2 that

$$\int_{\bigcup_{\hat{v}(k)\in\hat{\mathbb{V}}^k} T'_{\hat{\boldsymbol{v}}(k)}} |Dh_k(x)|^{n-1}\, dx \leq 2^{nk} C \int_{T'^S_k} |DH^S_k(x)|^{n-1}\, dx \leq 2^{nk} C\delta_k.$$

Given $\widetilde{\delta}_k$ we set $\delta_k = \frac{1}{C}2^{-nk}\widetilde{\delta}_k$ and find b_k and d_k small enough so that (3.9) and thus also (3.17) holds.

Outside of $\bigcup_{\boldsymbol{v}(k)\in\hat{\mathbb{V}}^k} T'_{\hat{\boldsymbol{v}}(k)}$ all mappings h_l, $l \geq k$, are equal to h_{k-1} and they are therefore bilipschitz there and $J_{h_l} > 0$ a.e. It follows that we can define $h = \lim_{k\to\infty} h_k$ and it is defined everywhere outside of (see (3.2) and (3.6))

$$\bigcap_{k=1}^{\infty} \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} T'_{\hat{\boldsymbol{v}}(k)} = \bigcup_{x\in C_B^T} l_x.$$

Moreover, it is continuous and $J_h > 0$ a.e. there. By (3.8) we know that $\mathcal{L}_n(\bigcup_{x\in C_B^T} l_x) = 0$ and then h is defined a.e. Since

$$h_k(T_{\hat{\boldsymbol{v}}(k)}) = \widetilde{T}_{\hat{\boldsymbol{v}}(k)} \text{ and } \operatorname{diam} \widetilde{T}_{\hat{\boldsymbol{v}}(k)} \to 0$$

it is not difficult to see that $h(l_x) = x$ for every $x \in C_B^T$. The continuity of h everywhere follows. □

3.4 Counterexample in Theorem 1.1

Construction of the counterexample in Theorem 1.1. Let us define a Cantor-type set C_A of positive measure by

$$\alpha_k = \frac{1}{2}\left(1 + 2^{-k\beta}\right).$$

We need the sequence of functions g_k, built in Section 2.5, to map C_A onto the a Cantor-type set C_B with small enough 'windows' defined by (2.10), i.e.

$$\beta_k = 2^{-k\beta} \text{ with } \beta \geq n+1.$$

According to (2.6) in the i-th frame $Q'_{\boldsymbol{v}(i)} \setminus Q_{\boldsymbol{v}(i)}$, $i \leq k$, we have

$$|D(g_k)^{-1}(x)| \approx \max\left\{\frac{\alpha_i}{\beta_i}, \frac{\alpha_{i-1} - \alpha_i}{\beta_{i-1} - \beta_i}\right\} \approx 2^{i\beta} \tag{3.19}$$

and on $\widetilde{Q}_{\boldsymbol{v}(k)}$ we have

$$|D(g_k)^{-1}(x)| \approx \frac{\alpha_k}{\beta_k} \approx 2^{k\beta}. \tag{3.20}$$

We also need the bilipschitz mapping L, defined in Section 2.6, to map C_B to a Cantor tower C_B^T, and we have

$$|DL(x)| \leq l, \quad |DL^{-1}(x)| \leq l. \tag{3.21}$$

Let us start from the Cantor tower C_B^T and apply our mapping h_k from Theorem 3.3 to squeeze the inner part of the cube. Then we need a mapping L^{-1} to go from C_B^T to C_B, and $(g_k)^{-1}$ to go to the Cantor set of positive measure C_A. The final mapping f is a pointwise limit of

$$f_k(x) = (g_k)^{-1} \circ L^{-1} \circ h_k(x)$$

almost everywhere (see Fig. 7). Mappings f_k are clearly bilipschitz and below we show that $f_k \to f$ strongly in $W^{1,n-1}$ and hence f is a strong limit of Sobolev homeomorphisms f_k such that $f_k(x) = x$ on $\partial[-1,1]^n$. We know that $g^{-1} = \lim_{k\to\infty}(g_k)^{-1}$ is a homeomorphism which maps C_B onto C_A and that L^{-1} is a homeomorphism which maps C_B^T onto C_B. By Theorem 3.3 we know that $h = \lim_{k\to\infty} h_k$ is continuous and the standard computation shows that

$f(x) = g^{-1} \circ L^{-1} \circ h(x)$ and it is a continuous mapping which maps C_B^T onto C_A.

By Theorem 3.3 we also know that for every $x \in C_B^T$ we have $h(l_x) = x$ and clearly $g^{-1} \circ L^{-1}(x) = y$ where y is the corresponding point in C_A (see (3.1)). It follows that $f^{-1}(y)$ is a continuum l_x for every $y \in C_A$. Finally, $J_{L^{-1}} > 0$ a.e., $J_h > 0$ a.e. by Theorem 3.3, and by the construction we also have $J_{g^{-1}} > 0$ a.e. as it is locally equal to some bilipschitz mapping g_k^{-1} on $[-1,1]^n \setminus C_B$. It is not difficult to see that f is locally bilipschitz on $[-1,1]^n \setminus \bigcup_{x\in C_B^T} l_x$ and hence we can use the composition formula for derivatives to obtain (see (3.8))

$$J_f(x) = J_{g^{-1}}\big(L^{-1}(h(x))\big) J_{L^{-1}}\big(h(x)\big) J_h(x) > 0 \text{ for a.e. } x \in [-1,1]^n.$$

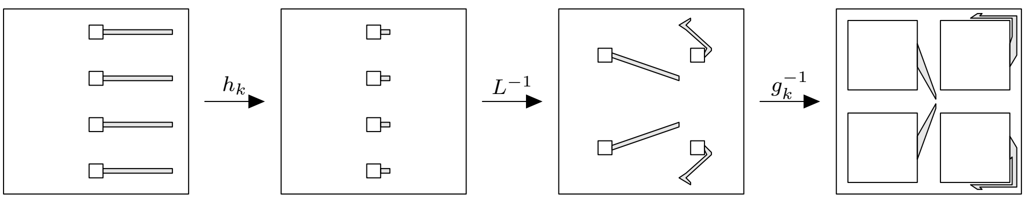

Figure 7: Mapping f_k.

It remains to show that $f \in W^{1,n-1}$. We show that mappings f_k form a Cauchy sequence in $W^{1,n-1}$. Since $f_k \to f$ pointwise, it is easy to see that f_k

converges strongly to f. We have fixed $\beta > n+1$ so that Theorem 2.4 holds and we set

$$\widetilde{\delta}_k = \frac{2^{-k\beta(n-1)}}{k^2}. \tag{3.22}$$

Given this $\widetilde{\delta}_k$ we find $d_k > b_k > 0$ according to Theorem 3.3. Note that all conclusions above (f is continuous, $J_f > 0$ a.e.) are valid, but we need this choice of $d_k > b_k$ to show that $f \in W^{1,n-1}$.

By Theorem 3.3 we know that $h_{k-1}(x) = h_k(x)$ for every $x \notin \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{\boldsymbol{v}}(k)}$ and clearly by (2.7) $g_{k-1}^{-1}(y) = g_k^{-1}(y)$ for $y \notin \bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^{k-1}} \widetilde{Q}_{\boldsymbol{v}(k-1)}$. In view of (2.11) it follows that

$$f_k(x) = f_{k-1}(x) \text{ for } x \notin \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{\boldsymbol{v}}(k)} \cup \bigcup_{\hat{\boldsymbol{v}}(k-1)\in\hat{\mathbb{V}}^{k-1}} \widehat{Q}_{\hat{\boldsymbol{v}}(k-1)} =: M_k.$$

Therefore

$$\begin{aligned}\int_{Q(0,1)} |Df_k - Df_{k-1}|^{n-1} &= \int_{M_k} |Df_k - Df_{k-1}|^{n-1} \\ &\leq C\int_{M_k} |Df_k|^{n-1} + C\int_{M_k} |Df_{k-1}|^{n-1}.\end{aligned} \tag{3.23}$$

Note that f_k is bilipschitz (as a composition of bilipschitz mappings) and hence we can compute its derivative a.e. by the composition of derivatives. With the help of (3.21) we get

$$|Df_k(x)| \leq |Dg_k^{-1}| \cdot |DL^{-1}| \cdot |Dh_k| \leq l\big|Dg_k^{-1}(L^{-1}\circ h_k(x))\big| \cdot \big|Dh_k(x)\big|.$$

By (3.19) and (3.20) we know that everywhere in $Q(0,1)$ we have $|Dg_k^{-1}| \leq C2^{k\beta}$ and $|Dg_{k-1}^{-1}| \leq C2^{k\beta}$. It follows that

$$|Df_k(x)| \leq C2^{k\beta}|Dh_k(x)| \text{ and } |Df_{k-1}(x)| \leq C2^{k\beta}|Dh_{k-1}(x)|.$$

For $x \in \widehat{Q}_{\hat{\boldsymbol{v}}(k-1)} \setminus \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{\boldsymbol{v}}(k)}$ we know that $h_k(x) = h_{k-1}(x) = x$ by Theorem 3.3 and hence

$$\begin{aligned}\int_{\bigcup_{\hat{\boldsymbol{v}}(k-1)\in\mathbb{V}^{k-1}} \widehat{Q}_{\hat{\boldsymbol{v}}(k-1)} \setminus \bigcup_{\hat{\boldsymbol{v}}(k)\in\hat{\mathbb{V}}^k} P'_{\hat{\boldsymbol{v}}(k)}} |Df_k|^{n-1} &\leq C2^{k\beta(n-1)}\mathcal{L}^n\Big(\bigcup_{\hat{\boldsymbol{v}}(k)\in\mathbb{V}^k} \widehat{Q}_{\hat{\boldsymbol{v}}(k)}\Big) \\ &\leq C2^{k\beta(n-1)}2^{nk}(2^{-k}2^{-k\beta})^n \leq C2^{-k\beta}.\end{aligned}$$

With the help of Theorem 3.3 and (3.22) we obtain

$$\int_{\bigcup_{\hat{\boldsymbol{v}}(k)\in\mathbb{V}^k} P'_{\hat{\boldsymbol{v}}(k)}} |Df_k|^{n-1} \leq C2^{k\beta(n-1)} \int_{\bigcup_{\hat{\boldsymbol{v}}(k)\in\mathbb{V}^k} P'_{\hat{\boldsymbol{v}}(k)}} |Dh_k|^{n-1} \leq C2^{k\beta(n-1)}\widetilde{\delta}_k \leq \frac{C}{k^2}.$$

Analogous estimate holds also for Df_{k-1} and hence (3.23) implies that

$$\int_{Q(0,1)} |Df_k - Df_{k-1}|^{n-1} \leq C2^{-k\beta} + \frac{C}{k^2}.$$

Since $2^{-k\beta} + 1/k^2$ is a convergent series it follows immediately that f_k form a Cauchy sequence in $W^{1,n-1}$. It follows that $f \in W^{1,n-1}$. □

4 Injectivity in the domain: counterexample in Theorem 1.2

4.1 Stretching inside tentacles

The following gives us an analog of Lemma 3.2. We can view the mapping $\widetilde{H}_k^S$ as the inverse of H_k^S from Lemma 3.2 but formally we define it otherwise so our estimates are simpler.

Lemma 4.1. *Let $n \geq 3$, $\delta_k > 0$, $\beta \geq n+1$ and $k \in \mathbb{N}$. Then we can find small enough $d_k > b_k > 0$ and a bilipschitz mapping $\widetilde{H}_k^S \colon Q(0,1) \to Q(0,1)$ such that $\widetilde{H}_0^S(x) = x$ for every $x \in Q(0,1)$*

$$\widetilde{H}_k^S(x) = \widetilde{H}_{k-1}^S(x) \text{ for } x \notin \widetilde{P}'_k,\ \widetilde{H}_k^S(x) = x \text{ for } x \in Q(0, \hat{r}_k)$$

$$\text{and } \widetilde{H}_k^S \text{ maps } \widetilde{P}_k \text{ onto } P_k \text{ linearly.}$$

Furthermore,

$$\int_{\widetilde{P}_k'^S} |D\widetilde{H}_k^S(x)|^{n-1}\, dx \leq \delta_k. \tag{4.1}$$

Proof. This proof is analogous to the proof of Lemma 3.2 and hence we skip some details. We set $\widetilde{H}_0^S(x) = x$ and we define

$$\widetilde{H}_k^S(x) = \widetilde{H}_{k-1}^S(x) \text{ for } x \notin \widetilde{P}'_k. \tag{4.2}$$

Then on $\partial \widetilde{P}'_k$ we have

$$\begin{aligned} &\widetilde{H}_k^S(x) = [\widetilde{l}_{k-1}(x_1), x_2, \dots, x_n], \text{ where } \widetilde{l}_{k-1}(x) = x_1 \text{ for } x_1 \leq \hat{r}_{k-1} \text{ and} \\ &\text{for } x_1 \in [\hat{r}_{k-1}, \widetilde{c}_k] \text{ it is linear with } \widetilde{l}_{k-1}(\hat{r}_{k-1}) = \hat{r}_{k-1} \text{ and } \widetilde{l}_{k-1}(\widetilde{a}_{k-1}) = a_{k-1}. \end{aligned} \tag{4.3}$$

Further, we define it for $x \in \{\widetilde{a}_k\} \times [-d_k, d_k]^{n-1}$ as

$$\begin{aligned} &\widetilde{H}_k^S(x) = [\widetilde{\varphi}_k(x), x_2, \dots, x_n] \text{ where} \\ &\widetilde{\varphi}_k(x) := \widetilde{l}_{k-1}(\widetilde{a}_k) + \Big(\log\log \frac{1}{\max\{b_k, |[x_2, \dots, x_n]|_\infty\}} - \log\log \frac{1}{d_k}\Big). \end{aligned}$$

We fix d_k small enough so that ($C_{(4.4)}$ is a constant whose exact value we specify later)

$$\frac{2^{(\beta+1)k(n-1)}}{\log^{n-2} \frac{1}{d_k}} < C_{(4.4)} \delta_k \tag{4.4}$$

and we fix $b_k < d_k$ so that

$$\text{for } |[x_2, \dots, x_n]|_\infty = b_k \text{ we have } \widetilde{\varphi}_k(x) = \widetilde{l}_{k-1}(\widetilde{a}_k) + \Big(\log\log \frac{1}{b_k} - \log\log \frac{1}{d_k}\Big) = a_k. \tag{4.5}$$

For every $x \in \widetilde{P}_k$ we can now define

$$\widetilde{H}_k^S(x) = \big[\widetilde{l}_k(x_1), x_2, \dots, x_n\big] \text{ where } \widetilde{l}_k \text{ is linear with } \widetilde{l}_k(\hat{r}_k) = \hat{r}_k \text{ and } \widetilde{l}_k(\widetilde{a}_k) = a_k. \tag{4.6}$$

Finally on the hyperplane $x \in \{\hat{r}_{k-1}\} \times [-d_k, d_k]^{n-1}$ we define it as

$$\begin{aligned}\widetilde{H}_k^S(x) &= [\widetilde{\psi}_k(x), x_2, \dots, x_n] \text{ where}\\ \widetilde{\psi}_k(x) &:= \widetilde{l}_{k-1}(\hat{r}_{k-1}) + \widetilde{A}_k\Big(\log\log\frac{1}{\max\{b_k, |[x_2,\dots,x_n]|_\infty\}} - \log\log\frac{1}{d_k}\Big).\end{aligned}$$

The constant $\widetilde{A}_k$ is chosen so that for $x \in \widetilde{P}_k \cap \{x_1 = \hat{r}_{k-1}\}$, i.e. for every $|[x_2,\dots,x_n]|_\infty \le b_k$ we have

$$\widetilde{l}_{k-1}(\hat{r}_{k-1}) + \widetilde{A}_k\Big(\log\log\frac{1}{b_k} - \log\log\frac{1}{d_k}\Big) = \frac{a_k + c_k}{2}.$$

By this and (4.5) we obtain

$$1 \ge \widetilde{l}_{k-1}(\hat{r}_{k-1}) + \widetilde{A}_k\Big(\log\log\frac{1}{b_k} - \log\log\frac{1}{d_k}\Big) = \widetilde{l}_{k-1}(\hat{r}_{k-1}) + \widetilde{A}_k\Big(a_k - \widetilde{l}_{k-1}(\widetilde{a}_k)\Big)$$

and hence

$$\widetilde{A}_k \le \frac{1}{a_k - \widetilde{l}_{k-1}(\widetilde{a}_k)} = \frac{1}{a_k - \widetilde{a}_k} \le C.$$

For every $x \in [\hat{r}_k, \widetilde{c}_k] \times [-d_k, d_k]^{n-1}$ we define for $x \in \widetilde{P}'_k$

$$\widetilde{H}_k^S(x) = \Big[h\big(x_1; [\hat{r}_k, \hat{r}_k]; [\hat{r}_{k-1}, \widetilde{\psi}_k(x)], [\widetilde{a}_k, \widetilde{\varphi}_k(x)], [\widetilde{c}_k, \widetilde{l}_{k-1}(\widetilde{c}_k)]\big), x_2, \dots, x_n\Big].$$

Again $\widetilde{H}_k^S$ is bilipschitz on $Q(0,1)$. By (2.14) we estimate the derivative with respect to first coordinate

$$|D_1\widetilde{H}_k^S(x)| \le \begin{cases} \frac{\widetilde{\psi}_k(x) - \hat{r}_k}{\hat{r}_{k-1} - \hat{r}_k}, & \text{for } \hat{r}_k < x_1 < \hat{r}_{k-1},\\ \frac{\widetilde{\varphi}_k(x) - \widetilde{\psi}_k(x)}{\widetilde{a}_k - \hat{r}_{k-1}}, & \text{for } \hat{r}_{k-1} < x_1 < \widetilde{a}_k,\\ \frac{l_{k-1}(\widetilde{c}_k) - \widetilde{\varphi}_k(x)}{\widetilde{c}_k - \widetilde{a}_k}, & \text{if } \widetilde{a}_k < x_1 < \widetilde{c}_k. \end{cases}$$

and this is clearly bounded by $C2^{(\beta+1)k}$. Furthermore, by (2.13) and $\widetilde{A}_k \le C$ we know that we can estimate the derivative with respect to other coordinates by

$$C\max\Big\{\frac{1}{\hat{r}_{k-1} - \hat{r}_k}, \frac{1}{\widetilde{a}_k - \hat{r}_{k-1}}, \frac{1}{\widetilde{c}_k - \widetilde{a}_k}\Big\}\Big|D\Big(\log\log\frac{1}{\max\{b_k, |[x_2,\dots,x_n]|_\infty\}}\Big)\Big|$$

The maximum of the three terms can be estimated by $C\frac{1}{\hat{r}_k} \le C2^{(\beta+1)k}$ an a simple change to polar/spherical coordinates in $\mathbb{R}^{n-1}$ and (4.4) gives us

$$\begin{aligned}\int_{\widetilde{P}'_k} |D\widetilde{H}_k^S(x)|^{n-1}\,dx &\le C2^{(\beta+1)k(n-1)}\int_{\widetilde{P}'_k}\frac{1}{|[x_2,\dots,x_n]|_\infty^{n-1}\log^{n-1}\frac{1}{|[x_2,\dots,x_n]|_\infty}}\,dx\\ &\le C2^{(\beta+1)k(n-1)}\int_0^{d_k}\frac{1}{r^{n-1}\log^{n-1}\frac{1}{r}}r^{n-2}\,dr\\ &\le C2^{(\beta+1)k(n-1)}\frac{1}{\log^{n-2}\frac{1}{d_k}} < CC_{(4.4)}\delta_k < \delta_k,\end{aligned}$$

where we have chosen $C_{(4.4)}$ in (4.4) so that the last inequality holds. □

Analogously to Theorem 3.3 we now obtain:

Theorem 4.2. *Let $n \geq 3$, $\widetilde{\delta}_k > 0$, $\beta \geq n+1$ and $k \in \mathbb{N}$. Then we can find small enough $d_k > b_k > 0$ and a bilipschitz mapping $\widetilde{h}_k \colon Q(0,1) \to Q(0,1)$ such that $\widetilde{h}_0(x) = x$ for every $x \in Q(0,1)$,*

$$\begin{aligned} \widetilde{h}_{k+1}(x) &= \widetilde{h}_k(x) \quad \text{for } x \notin \bigcup_{\hat{\boldsymbol{v}}(k) \in \hat{\mathbb{V}}^k} \widetilde{P}'_{\hat{\boldsymbol{v}}(k)}, \\ \widetilde{h}_k(x) &= x \qquad\quad \text{for } x \in \widehat{Q}_{\hat{\boldsymbol{v}}(k)} \text{ and } \widetilde{h}_k(\widetilde{P}_{\hat{\boldsymbol{v}}(k)}) = P_{\hat{\boldsymbol{v}}(k)}. \end{aligned} \tag{4.7}$$

We can estimate the integral of its derivative as

$$\int_{\bigcup_{\hat{v}(k) \in \hat{\mathbb{V}}^k} \widetilde{P}'_{\hat{\boldsymbol{v}}(k)}} |D\widetilde{h}_k(x)|^{n-1}\, dx \leq \widetilde{\delta}_k.$$

Moreover, a pointwise limit $\widetilde{h}$ of $\widetilde{h}_k$ is continuous and one-to-one on $Q(0,1)$ and $J_{\widetilde{h}}(x) > 0$ a.e. And, there is a continuous $\widetilde{t} : Q(0,1) \to Q(0,1)$ which is a generalized inverse to $\widetilde{h}$, i.e. $\widetilde{t}(\widetilde{h}(x)) = x$ for every $x \in [-1,1]^n$. On the other hand,

$$\widetilde{t}(l_x) = x \text{ for every } x \in C_B^T, \tag{4.8}$$

where l_x is defined by (3.2).

Proof. The proof of this theorem is analogous to the proof of Theorem 3.3 and therefore we skip it. We only explain why (4.8) holds.

Outside of $\bigcup_{\boldsymbol{v}(k) \in \hat{\mathbb{V}}^k} \widetilde{T}'_{\hat{\boldsymbol{v}}(k)}$ all mappings $\widetilde{h}_l$, $l \geq k$, are equal to $\widetilde{h}_{k-1}$ and hence they are bilipschitz there and $J_{\widetilde{h}_l} > 0$ a.e. It follows that we can define $\widetilde{h} = \lim_{k\to\infty} \widetilde{h}_k$ everywhere outside of

$$\bigcap_{k=1}^{\infty} \bigcup_{\hat{\boldsymbol{v}}(k) \in \hat{\mathbb{V}}^k} \widetilde{T}'_{\hat{\boldsymbol{v}}(k)} = C_B^T$$

and it is one-to-one and continuous there with $J_{\widetilde{h}} > 0$ a.e. For $x \in C_B^T$ we define $\widetilde{h}(x) = x$ and notice that now $\widetilde{h}$ is one-to-one everywhere.

We define $\widetilde{t} = \widetilde{h}^{-1}$ on $Q(0,1) \setminus \widetilde{h}(C_B^T)$ and notice that $\widetilde{t}$ is continuous there. Since

$$h_k^{-1}(T_{\hat{\boldsymbol{v}}(k)}) = \widetilde{T}_{\hat{\boldsymbol{v}}(k)} \text{ and } \operatorname{diam} \widetilde{T}_{\hat{\boldsymbol{v}}(k)} \to 0$$

it is not difficult to see that for every $a \in l_x := \bigcap_{k=1}^{\infty} T_{\hat{\boldsymbol{v}}(k)}$ we can define $\widetilde{t}(a) = x$ and now $\widetilde{t}$ is continuous everywhere. For $x \in C_B^T$ we have $\widetilde{t}(\widetilde{h}(x)) = \widetilde{t}(x) = x$ and hence $\widetilde{t}$ is a generalized inverse to $\widetilde{h}$. □

4.2 Counterexample in Theorem 1.2

Construction of the counterexample in Theorem 1.2. Again we use the same sequences

$$\alpha_k = \frac{1}{2}\left(1 + 2^{-k\beta}\right) \text{ and } \beta_k = 2^{-k\beta} \text{ with } \beta \geq n+1$$

to define Cantor type sets C_A, C_B and C_B^T. As in the proof of Theorem 1.1 we have the estimates of the derivatives (3.19) and (3.20). We set

$$\widetilde{\delta}_k = \frac{2^{-k\beta(2n-1)}}{k^2}. \tag{4.9}$$

Given this $\widetilde{\delta}_k$ we find $d_k > b_k > 0$ so that we have Theorem 4.2.

Consider the mapping $\widetilde{f}$ as a pointwise limit of

$$\widetilde{f}_k(y) = g_k^{-1} \circ L^{-1} \circ \widetilde{h}_k \circ L \circ g_k(y)$$

almost everywhere (see Fig. 8). For $y \in C_A$ we know that $L \circ g(y) \in C_B^T$ where $\widetilde{h}_k(x) = x$ and hence it is easy to see that the pointwise limit is equal to $\widetilde{f}(y) = y$ for $y \in C_A$. Therefore, we see at once that the pointwise limit of $\widetilde{f}_k$ is

$$\widetilde{f}(y) = g^{-1} \circ L^{-1} \circ \widetilde{h} \circ L \circ g(y) \text{ everywhere.}$$

Since g and L are homeomorphisms and $\widetilde{h}$ is one-to-one we obtain that $\widetilde{f}$ is one-to-one on $Q(0,1)$. It is not difficult to see that $\widetilde{f}$ is locally bilipschitz on $[-1,1]^n \setminus C_A$ and hence we can use the composition formula for derivatives to obtain

$$J_{\widetilde{f}}(y) = J_{g^{-1}} J_{L^{-1}} J_{\widetilde{h}} J_L J_g > 0 \text{ for a.e. } x \in [-1,1]^n \setminus C_A.$$

For $y \in C_A$ we know that $\widetilde{f}(y) = y$ and hence $J_{\widetilde{f}} = 1$ for a.e. $x \in C_A$ once we show that $\widetilde{f} \in W^{1,1}$ since the weak derivative is equal to the approximative derivative a.e.

With the help of Theorem 4.2 we obtain that the continuous mapping

$$w(y) = g^{-1} \circ L^{-1} \circ \widetilde{t} \circ L \circ g(y)$$

is a generalized inverse to $\widetilde{f}$. Moreover, for every $y \in C_A$ we know that $x = L \circ g(y) \in C_B^T$. Therefore, the standard arguments show that for $\widetilde{l}_x = (L \circ g)^{-1}(l_x)$ we have by (4.8)

$$w(\widetilde{l}_x) = g^{-1} \circ L^{-1} \circ \widetilde{t} \circ L \circ g(\widetilde{l}_x) = g^{-1} \circ L^{-1} \circ \widetilde{t}(l_x) = g^{-1} \circ L^{-1}(x) = y.$$

Now $\widetilde{l}_x$ is a continuum and so is $w^{-1}(y)$ for every $y \in C_A$.

By Theorem 4.2 we know that $\widetilde{h}_{k-1} = \widetilde{h}_k$ for every y for which $L(g_k(y)) \notin \bigcup_{\hat{\boldsymbol{v}}(k) \in \hat{\mathbb{V}}^k} \widetilde{P}'_{\hat{\boldsymbol{v}}(k)}$ and $g_{k-1}(y) = g_k(y)$ for $y \notin \bigcup_{\boldsymbol{v}(k-1) \in \mathbb{V}^{k-1}} Q_{\boldsymbol{v}(k-1)}$ by (2.7). In view of (2.11) it follows that

$$\widetilde{f}_k(y) = \widetilde{f}_{k-1}(y) \text{ for } L(g_k(y)) \notin \bigcup_{\hat{\boldsymbol{v}}(k) \in \hat{\mathbb{V}}^k} \widetilde{P}'_{\hat{\boldsymbol{v}}(k)} =: \widetilde{M}_k \text{ and } y \notin \bigcup_{\boldsymbol{v}(k-1) \in \mathbb{V}^{k-1}} Q_{\boldsymbol{v}(k-1)}.$$

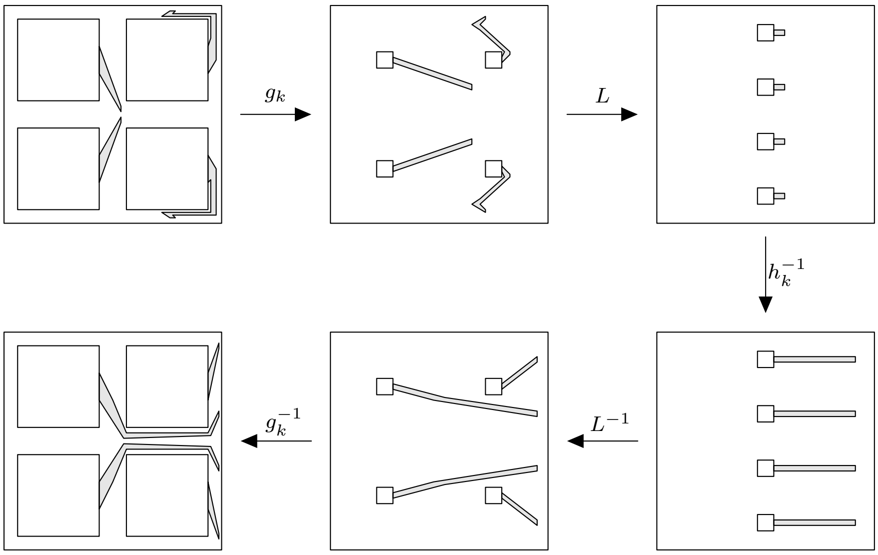

Figure 8: Mapping $\widetilde{f}$.

Note that for $x \in \bigcup_{\hat{\boldsymbol{v}}(k-1)\in\hat{\mathbb{V}}^{k-1}} \widehat{Q}_{\hat{\boldsymbol{v}}(k-1)} \setminus \widetilde{M}_k$ we have by Theorem 4.2

$$\widetilde{h}_k(x) = \widetilde{h}_{k-1}(x) = x.$$

In view of $g_k(Q_{\boldsymbol{v}(k-1)}) = g_{k-1}(Q_{\boldsymbol{v}(k-1)}) = \widetilde{Q}_{\boldsymbol{v}(k-1)}$ and $L(\widetilde{Q}_{\boldsymbol{v}(k-1)}) = \widehat{Q}_{\hat{\boldsymbol{v}}(k-1)}$ we obtain by Theorem 2.4 for $y \in Q_{\boldsymbol{v}(k-1)} \setminus g_k^{-1}(L^{-1}(\widetilde{M}_k))$ that

$$\widetilde{f}_{k-1}(y) = g_{k-1}^{-1} \circ L^{-1} \circ x \circ L \circ g_{k-1}(y) = y \text{ and similarly } \widetilde{f}_k(y) = y.$$

Therefore

$$\begin{aligned}\int_{Q(0,1)} |D\widetilde{f}_k - D\widetilde{f}_{k-1}|^{n-1} &= \int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |D\widetilde{f}_k - D\widetilde{f}_{k-1}|^{n-1} \\ &\leq C\int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |D\widetilde{f}_k|^{n-1} + C\int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |D\widetilde{f}_{k-1}|^{n-1}.\end{aligned} \tag{4.10}$$

Note that $\widetilde{f}_k$ is bilipschitz (as a composition of bilipschitz mappings) and hence we can compute its derivative a.e. by the composition of derivatives. With the help of (3.21) we get

$$\begin{aligned}|D\widetilde{f}_k(y)| &\leq |Dg_k^{-1}| \cdot |DL^{-1}| \cdot |D\widetilde{h}_k| \cdot |DL| \cdot |Dg_k| \\ &\leq C\big|Dg_k^{-1}(L^{-1} \circ \widetilde{h}_k \circ L \circ g_k(y))\big| \cdot \big|D\widetilde{h}_k(L \circ g_k(y)))\big|.\end{aligned}$$

By the change of variables

$$\begin{aligned}\int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |D\widetilde{f}_k(y)|^{n-1}\, dy &\leq C \int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |Dg_k^{-1}|^{n-1} |D\widetilde{h}_k|^{n-1} \frac{J_L}{J_L} \frac{J_{g_k}}{J_{g_k}}\, dy \\ &\leq C \int_{\widetilde{M}_k} |Dg_k^{-1}(L^{-1} \circ \widetilde{h}_k(x))|^{n-1} |D\widetilde{h}_k(x)|^{n-1} \frac{1}{J_{g_k}((L \circ g_k)^{-1}(x))}\, dx.\end{aligned} \tag{4.11}$$

Note that for every $x \in \widetilde{P}'_{\hat{\boldsymbol{v}}(k)} \subset \widetilde{M}_k$ we know that $L^{-1} \circ \widetilde{h}_k(x)$ lies outside of $\bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} \widetilde{Q}_{\boldsymbol{v}(k)}$ and hence we can use (2.6) to estimate

$$|Dg_k^{-1}(L^{-1} \circ \widetilde{h}_k(x))| \leq C \max_{i=1,\dots,k} 2^{\beta i} = C 2^{\beta k}$$

and

$$\frac{1}{J_{g_k}((L \circ g_k)^{-1}(x))} \leq C 2^{\beta k n}.$$

Now (4.9) and (4.11) imply that

$$\int_{g_k^{-1}(L^{-1}(\widetilde{M}_k))} |D\widetilde{f}_k(y)|^{n-1}\, dy \leq C 2^{k\beta(2n-1)} \int_{\widetilde{M}_k} |D\widetilde{h}_k(x)|^{n-1}\, dx \leq \frac{C}{k^2}.$$

The similar estimate holds also for $D\widetilde{f}_{k-1}$ and hence (4.10) implies that

$$\int_{Q(0,1)} |D\widetilde{f}_k - D\widetilde{f}_{k-1}|^{n-1} \leq \frac{C}{k^2}.$$

Since $1/k^2$ is a convergent series, f_k form a Cauchy sequence in $W^{1,n-1}$ and hence $f \in W^{1,n-1}$. □

Example 4.3. *For every $n \geq 2$ there is a set C_A of Hausdorff dimension n and a Lipschitz mapping $f_L\colon [-1,1]^n \to [-1,1]^n$ with $J_{f_L} > 0$ a.e. which is a strong limit of Sobolev homeomorphisms $f_k \in W^{1,n-1}([-1,1]^n, \mathbb{R}^n)$ with $f_k(x) = x$ for $x \in \partial[-1,1]^n$ such that*

$$f(C_A) \text{ is a point.}$$

Proof. We only briefly sketch the construction. We set $\alpha_k = \frac{1}{k}$ in the construction of a Cantor type set C_A (see Section 2.4). Then it is easy to see that the measure of C_A is zero but its Hausdorff dimension is n. We map this by g from Section 2.5 to a Cantor type set C_B given by sequence $\beta_k = 2^{-\beta k}$, $\beta \geq n+1$, as usual. Note that by (2.5),

$$\frac{\beta_k}{\alpha_k} \leq C \text{ and } \frac{\beta_{k-1} - \beta_k}{\alpha_{k-1} - \alpha_k} \leq C$$

we obtain that g is a Lipschitz mapping.

Then we map C_B by the Lipschitz mapping L from Theorem 2.4 to the Cantor tower C_B^T. Now $C_B^T \subset \{0\}^{n-1} \times (-1,1)$ and it is easy to find a Lipschitz mapping S which squeezes a segment containing C_B^T to a single point, it is one-to-one outside of this segment and equals to identity on $\partial[-1,1]^n$. We can choose S to be

$$S(x) = \left[x_1, x_2, \dots, x_{n-1}, x_n\sqrt{x_1^2 + \dots + x_{n-1}^2}\right]$$

on $Q(0, 1-\delta)$ (fix $\delta > 0$ so that $C_B^T \subset Q(0, 1-\delta)$) and extend it in a Lipschitz way so that $S(x) = x$ on $\partial Q(0,1)$. Finally the mapping $f_L := S \circ L \circ g$ is a mapping for which

$$f_L(C_A) = S(C_B^T) \text{ is a point}$$

and we can obtain it as a weak limit of homeomorphisms in $W^{1,\infty}$ (or even strong limit in $W^{1,p}$ for any $p < \infty$). □

5 Positive statements: the case $p > n-1$

To study the injectivity a.e. with respect to the image we define slightly better (INV) condition, see Corollary 5.3 below. We need the following generalization of [25, Lemma 7.3] for the case with no additional assumptions on J_f.

Lemma 5.1. *Let $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, be a weak limit of homeomorphisms f_k in $W^{1,p}(\Omega, \mathbb{R}^n)$, and a, $b \in \Omega$. Then there exist $\mathcal{L}^1$-null sets N_a and N_b such that for every $r \in (0, r_a) \setminus N_a$ and $s \in (0, r_b) \setminus N_b$ (where $r_x := \operatorname{dist}(x, \partial\Omega)$) the following holds:*

(i) If $B(a,r) \subset B(b,s)$, then

$$E(f^*, B(a,r)) \subset E(f^*, B(b,s)).$$

(ii) If $B(a,r) \cap B(b,s) = \emptyset$, then

$$f^{*T}(B(a,r)) \cap f^{*T}(B(b,s)) = \emptyset.$$

Proof. We may assume that f equals to the representative f^*. By Lemma 2.2, there are $\mathcal{L}^1$-null sets N_a and N_b such that for every $r \in (0, r_a) \setminus N_a$ and $s \in (0, r_b) \setminus N_b$ one has $f_k \to f$ (up to subsequence) uniformly on $S(a,r)$ and $S(b,s)$.

To establish (i), we show that $\deg(f, S(b,s), y) \neq 0$ for $y \in E(f, B(a,r)) \setminus f(S(b,s))$. Let us firstly suppose that $y = f(x)$ for $x \in S(a,r)$. Since $f(S(b,s))$ is compact and f_k converge uniformly on the sphere $S(b,s)$ there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for every $k > k_0$ we have $B(y,\varepsilon) \cap f_k(S(b,s)) = \emptyset$. Moreover, $x \in S(a,r)$ yields $y = \lim_{k\to\infty} f_k(x)$, and we may assume that $f_k(x) \in B(y,\varepsilon)$ for all big enough k. Therefore,

$$\deg(f_k, S(b,s), y) = \deg(f_k, S(b,s), f_k(x)) \tag{5.1}$$

for $k > k_0$. Then the continuity of the degree under uniform convergence (2.2) yields

$$\deg(f, S(b,s), y) = \lim_{k\to\infty} \deg(f_k, S(b,s), y). \tag{5.2}$$

Because f_k are homeomorphisms and $x \in (S(a,r) \setminus S(b,r)) \subset B(b,s)$ we obtain, that

$$\deg(f_k, S(b,s), f_k(x)) \neq 0.$$

Hence, (5.1) and (5.2), as well as the fact that the degree is integer valued, give

$$\begin{aligned}\deg(f, S(b,s), y) &= \lim_{k\to\infty} \deg(f_k, S(b,s), y)\\ &= \lim_{k\to\infty} \deg(f_k, S(b,s), f_k(x)) \neq 0.\end{aligned}$$

It remains to prove the case when $y \notin f(S(a,r))$ (so $\deg(f, S(a,r), y) \neq 0$). As before, the uniform convergence on spheres $S(a,r)$ and $S(b,s)$ and the continuity of the degree ensure

$$\begin{aligned}\deg(f, S(a,r), y) &= \lim_{m\to\infty} \deg(f_m, S(a,r), y) = \deg(f_k, S(a,r), y),\\ \deg(f, S(b,s), y) &= \lim_{m\to\infty} \deg(f_m, S(b,s), y) = \deg(f_k, S(b,s), y),\end{aligned}$$

for some big $k \in \mathbb{N}$. Since $y \in E(f, B(a,r)) \setminus f(S(a,r))$, we have

$$\deg(f, S(a,r), y) \neq 0,$$

and so $\deg(f_k, S(a,r), y) \neq 0$. Further, f_k is a homeomorphism and $B(a,r) \subset B(b,s)$, therefore $\deg(f_k, S(a,r), y) \neq 0$ implies $\deg(f_k, S(b,s), y) \neq 0$ by (2.3). So $\deg(f, S(b,s), y) \neq 0$ and this completes the proof of (i).

To prove (ii) we assume, on the contrary, that $y \in f^T(B(a,r)) \cap f^T(B(b,s))$. Then the uniform convergence and continuity of the degree ensure that there is $k \in \mathbb{N}$

$$\begin{aligned}0 \neq \deg(f, S(a,r), y) &= \lim_{m\to\infty} \deg(f_m, S(a,r), y) = \deg(f_k, S(a,r), y),\\ 0 \neq \deg(f, S(b,s), y) &= \lim_{m\to\infty} \deg(f_m, S(b,s), y) = \deg(f_k, S(b,s), y).\end{aligned}$$

Since f_k is a homeomorphism, $\deg(f_k, S(a,r), y)$ and $\deg(f_k, S(b,s), y)$ cannot both differ from zero, which is a contradiction.

□

Based on Lemma 5.1 we follow [25] and [28] to define the set-valued image

$$f^T(a) := \bigcap_{r>0, r\notin N_a} E(f^*, B(a,r)).$$

Note that $f^T(a)$ is non-empty and compact, as an intersection of a decreasing sequence of non-empty compact sets.

Theorem 5.2. *Let f be a weak limit of homeomorphisms f_k in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$ for $n > 2$ or $p \geq 1$ for $n = 2$. Then there exists an $\mathcal{H}^{n-p}$ null set $NC \subset \Omega$ and a representative $\widehat{f}$ of f such that $\widehat{f}$ is continuous at every $x \in \Omega \setminus NC$. Furthermore $f^T(x)$ is a singletone for every $x \in \Omega \setminus NC$, $\widehat{f} = f^*$ cap_p-a.e. and $\widehat{f}$ can be chosen so that $\widehat{f}(x) \in f^T(x)$ for every $x \in \Omega$.*

Proof. Assume $p > n-1$. The theorem follows from [25, Theorem 7.4] considering the fact that the weak limits of homeomorphisms satisfy the (INV) condition and Lemma 5.1 instead of [25, Lemma 7.3]. Note that the condition $J_f \neq 0$ a.e. comes from [25, Lemma 7.3] and plays no part in the rest of the proof.

The fact that $f^T(x)$ is a singletone follows from the proof of [25, Theorem 7.4] as we have there

$$NC := \big\{x: \ \operatorname{diam}(f^T(x)) > 0\big\}.$$

In the case $n = 2$, $p = 1$ we know that weak limit of homeomorphisms satisfy the (INV) condition thanks to the [9, Lemma 2.6]. And we can use the proof of [25, Theorem 7.4] with [9, Remark 2.9] instead of [25, Lemma 7.3]. □

Proof of the positive part of Theorem 1.2. This follows from Theorem 5.2. The 'moreover' part with the additional assumption that $J_f > 0$ a.e. was known before, see [25, Lemma 3.4]. Note that this lemma holds even in the case $p = 1$, $n = 2$. □

Corollary 5.3. *The representative $\widehat{f}$ from Theorem 5.2 satisfies a strengthened version of condition (INV), that is for every $a \in \Omega$ and $\mathcal{L}^1$-a.e. $r \in (0, r_a)$*

(i) $\widehat{f}(x) \in \widehat{f}^T(B(a,r)) \cup \widehat{f}(S(a,r))$ for every $x \in \overline{B(a,r)}$ and

(ii) $\widehat{f}(x) \in \mathbb{R}^n \setminus \widehat{f}(B(a,r))$ for every $x \in \Omega \setminus B(a,r)$.

Proof. The proof follows from [25, Corollary 7.5] with regard for Lemma 5.1 (or [9, Remark 2.9] for $n = 2$, $p = 1$) and Theorem 5.2. □

Proof of the positive part of Theorem 1.1. We assume that $f = \widehat{f}$, where $\widehat{f}$ is from Corollary 5.3. Suppose, by contradiction, that there is $\delta > 0$ such that for

$$F = \{y \in \mathbb{R}^n: \ \operatorname{diam}(f^{-1}(\{y\})) > 0\}$$

we have $\mathcal{H}^{n-1+\delta}(F) > 0$. Clearly, $F = \bigcup_{k\in\mathbb{N}} F_k$, where

$$F_k = \Big\{y \in \mathbb{R}^n: \ \operatorname{diam}(f^{-1}(\{y\})) > \frac{1}{k}\Big\}.$$

Hence we can fix $k \in \mathbb{N}$ such that $\mathcal{H}^{n-1+\delta}(F_k) > 0$.

For each $x \in \Omega$ there is a radius $r_x < \frac{1}{2k}$, such that

$$f|_{S(x,r)} \in W^{1,p}(S(x,r), \mathbb{R}^n) \cap \mathcal{C}^0(S(x,r), \mathbb{R}^n)$$

(see Lemma 2.2) and the assertion of Corollary 5.3 holds. Choosing a countable covering of Ω with balls $\{B(x_i, r_{x_i})\}_{i=1}^{\infty}$, due to the area formula [25, Proposition 2.7], we know that $\mathcal{H}^{n-1}(f(S(x_i, r_{x_i}))) < \infty$, so $\mathcal{H}^{n-1+\delta}(f(S(x_i, r_{x_i}))) = 0$. Therefore, even for

$$E := \bigcup_{i=1}^{\infty} f(S(x_i, r_{x_i}))$$

we have $\mathcal{H}^{n-1+\delta}(E)=0$. We now claim, that $F_k\subset E$, which is the contradiction with $\mathcal{H}^{n-1+\delta}(F_k)>0$.

Indeed, assume that $y\in F_k\setminus E$. Then there must be points z_1 and z_2 in Ω, such that $f(z_1)=f(z_2)=y$ and $\operatorname{dist}(z_1,z_2)>\frac{1}{k}$. Fix i for which $z_1\in B(x_i,r_{x_i})$, $z_2\notin B(x_i,r_{x_i})$ with the balls $B(x_i,r_{x_i})$ covering Ω and $r_{x_i}<\frac{1}{2k}$. Because $y\notin E$ we know that $y\notin S(x_i,r_{x_i})$. Therefore, Corollary 5.3 (i) states

$$y=f(z_1)\in f^T(B(x_i,r_{x_i}))$$

and the assertion (ii) holds

$$y=f(z_2)\in\mathbb{R}^n\setminus f^T(B(x_i,r_{x_i})),$$

which is a contradiction. □

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Paper II

Existence of quasiconformal mappings in a given Hardy space

Existence of quasiconformal mappings in a given Hardy space

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Abstract

Let Ω be a simply connected domain in $\mathbb{C}$ and let $0 < p < \infty$. We show that there is a quasiconformal mapping f from the unit disk $\mathbb{D}$ onto Ω which is in the Hardy space H^p.

We furthermore show that either all quasiconformal mappings from $\mathbb{D}$ onto Ω are in H^p for every p, or for every $0 < p < \infty$ there is a quasiconformal mapping $f\colon \mathbb{D} \to \Omega$ with $f \notin H^p$.

1 Introduction

The classical definition of H^p declares that an analytic function f belongs to H^p, $0 < p < \infty$ when

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{S^1} |f(r\omega)|^p \,\mathrm{d}\omega \right)^{\frac{1}{p}} < \infty. \tag{1}$$

If f is additionally univalent (i.e. a conformal map), then we have $f \in H^p$ for all $0 < p < \frac{1}{2}$ by the Prawitz theorem and the Koebe map $f(z) = \frac{z}{(1-z)^2}$ shows that this estimate is sharp. In 1970, Hansen [11] introduced the concept of a Hardy number. In the case of a nonempty, simply connected domain Ω, this is defined as the supremum of the exponents $p > 0$ for which $f \in H^p$ for a Riemann map from the unit disk $\mathbb{D}$ onto Ω. Notice that if $p < q$, then $H^q \subseteq H^p$. Given a simply connected domain $\emptyset \subsetneq \Omega \subsetneq \mathbb{C}$, the Hardy number of Ω is independent of the choice of conformal map from $\mathbb{D}$ to Ω. The relation of the Hardy number with the geometry of Ω has been investigated in [5], [11], [12], [13], [14], [21] and [22], and one can find different interpretations of it in [5], [21] and [22].

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In this paper we consider the case of a quasiconformal map of the unit disk $\mathbb{D}$ onto a simply connected domain Ω. Recall that a homeomorphism f is quasiconformal if $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}, \mathbb{C})$ and there is a constant K so that $|Df(z)|^2 \leq KJ_f(z)$ for almost every $z \in \mathbb{D}$. Analogously to the case of an analytic function, we write $f \in H^p$ if f satisfies (1). The analog of the above result for univalent functions from [4, Theorem 3.2] states that f in H^p for all $p < \frac{1}{2K}$ if f is K-quasiconformal.

Various interpretations for the membership in H^p for a quasiconformal map f can be found in [4], [6] and [27].

Our first result shows that the analog of a Hardy number of a domain as the supremum of the Hardy exponents over the entire class of quasiconformal maps is not a meaningful concept.

Theorem 1.1. *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Let $p \in (0, \infty)$. Then there is a quasiconformal map f from $\mathbb{D}$ onto Ω, which is in the Hardy space H^p.*

Analogously to the case of conformal maps (see [5, Theorem A.]), membership in H^p for quasiconformal maps is determined (see [4]) by the maximal growth order of f, that is $f \in H^p$ if and only if $\int_0^1 \sup\{|f(x)| : |z| = r\}^p \, \mathrm{d}r < \infty$.

This suggests that one should prove Theorem 1.1 by trying to improve on the growth order of a given map onto Ω. This is what we do: once Ω is fixed, we consider a Riemann map and improve its maximal growth order by composing with a suitable quasiconformal self-map of the disk. The difficulty is that we also need to temper down the growth in many unknown directions. This is done by constructing a suitable quasisymmetric map on $\partial\mathbb{D}$ via the boundary values of the conformal map and by eventually applying the Beurling-Ahlfors extension procedure.

Theorem 1.1 allows us to prove the following somewhat surprising result. For a quasiconformal map f, write $\ell\big(f(\partial B(0,t))\big)$ for the length of the curve $f(\partial B(0,t))$.

Corollary 1.2. *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Let $p < 2$. Then there is a quasiconformal map $f\colon \mathbb{D} \to \Omega$ such that*

$$\int_0^1 \ell\Big(f\big(\partial B(0,t)\big)\Big)^p \, \mathrm{d}t < \infty.$$

In the case of a conformal map, the convergence of the above integral is actually equivalent with membership in H^p when $0 < p < 2 + \varepsilon$, where the precise value of ε is not known, see [10]. This, together with Theorem 1.1 and the results in [7, Section 8], suggest that the claim of Corollary 1.2 might actually also hold for $p = 2$ or for all $0 < p < 2 + \varepsilon$ for some positive ε. We would like to know if that is the case.

Theorem 1.1 shows that we can find "arbitrarily good" quasiconformal maps from the H^p-perspective ("arbitrarily good" refers to p large). Our third result shows that either all quasiconformal maps onto Ω belong to H^p for all finite p or we can find "arbitrarily bad" quasiconformal maps.

Theorem 1.3. *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Then we have the following dichotomy:*

(1) Either $f \in H^p$ for all $0 < p < \infty$ and each quasiconformal map $f : \mathbb{D} \to \Omega$,

(2) or for each $q > 0$ there is a quasiconformal map $f : \mathbb{D} \to \Omega$ such that $f \notin H^q$.

Our proof of Theorem 1.3 is based on a similar idea as described above, but in this case we do not rely on the Beurling-Ahlfors extension, but construct suitable quasiconformal self-maps of the disk directly using an iteration procedure.

The above theorems deal with the class of all quasiconformal maps. Our proofs give some estimates for the constants of quasiconformality, but it would be interesting to know the optimal estimates. Furthermore, it would be interesting to know sharp ranges for the possible exponents p for which each quasiconformal map $f : \mathbb{D} \to \Omega$ belongs to H^p in terms of geometric data of Ω. This paper deals with the planar setting, but one could also consider the higher dimensional case. Here the natural setting would be to assume that we are given a quasiconformal map $f : B^n(0,1) \to \Omega$. Then necessarily f in H^p for some p by results in [4]. Suppose that f does not belong to H^p for all $p < \infty$ and let p_0 be the supremum of the p for which f in H^p. It is natural to pose the following problem. Given $0 < q < \infty$, can one find quasiconformal self-maps f_1 and f_2 of B^n so that $f \circ f_1 \in H^q$ and $f \circ f_2 \notin H^q$?

The proof of Theorem 1.1 can be found in Chapter 3. Theorem 1.3 is proven in Chapter 4 and the corollaries are proven in Chapter 5.

2 Preliminaries

We write $\mathbb{D} = B(0,1)$ for the open unit ball in $\mathbb{C}$. We denote the unit circle as $S^1 := \partial B(0,1)$ and the upper half-plane as $\mathbb{H} := \{x + iy : y > 0\}$. For an arc $J \subsetneq S^1$, we denote its midpoint as e^{it_J}. Furthermore, by $2J$ and $\frac{J}{2}$ we denote the arc with the same midpoint as J and double and half the length of J respectively.

By $\ell(\gamma)$ we denote the length of a curve γ. Given $u,\ v \in \Omega$ the intrinsic path distance between u and v in Ω is

$$d_I(u,v) := \inf\{\ell(\gamma) : \gamma \text{ is a curve connecting } u \text{ and } v \text{ in } \Omega\}.$$

Furthermore we use the intrinsic "norm" defined as $|f(z)|_I := d_I\big(f(0), f(z)\big)$.

Given $x \in \mathbb{D}$ let

$$B_x := B\left(x, \frac{1-|x|}{2}\right), \qquad S_x := \left\{\frac{y}{|y|} : y \in B_x\right\}.$$

We call B_x a Whitney ball and refer to S_x as its shadow.

Definition 2.1. The statement of [23, Theorem 1.7.] allows us to define for each conformal $g : \mathbb{D} \to \Omega$ and for almost every ω in S^1

$$g(\omega) := \lim_{r \to 1} g(r\omega).$$

By [18, Theorem 2.] this limit also exists for almost every $\omega \in S^1$ when g is quasiconformal, and hence we can use the same definition.

Definition 2.2. Let f be a quasiconformal map of $\mathbb{D}$ and let $p \in (0, \infty)$. We say that f belongs to the Hardy space H^p if

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{S^1} |f(r\omega)|^p \, \mathrm{d}\omega \right)^{\frac{1}{p}} < \infty.$$

We will also use the intrinsic Hardy space H^p_I. We say that $f \in H^p_I$ if

$$\|f\|_{H^p_I} := \sup_{0<r<1} \left(\int_{S^1} |f(r\omega)|^p_I \, \mathrm{d}\omega \right)^{\frac{1}{p}} < \infty.$$

The following theorem is due to Zinsmeister [27].

Theorem 2.3 (Zinsmeister)**.** *Let f be a quasiconformal map of $\mathbb{D}$ and let $0 < p < \infty$. Then $f \in H^p$ if and only if $f(\omega) \in L^p(S^1)$.*

Theorem 2.4. *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Then there is a conformal map $g\colon \mathbb{D} \to \Omega$ which is in H^q for every $0 < q < \frac{1}{2}$.*

Proof. The existence of such g is due to the Riemann mapping theorem, [20]. The fact that this g is in H^q is due to [24], see [8, Theorem 3.16.]. □

Theorem 2.5. *Let $0 < p < \infty$. Then*

(a) If $g\colon \mathbb{D} \to \mathbb{C}$ is a conformal map, then $g \in H^p$ if and only if $g \in H^p_I$.

(b) If $f\colon \mathbb{D} \to \mathbb{C}$ is a quasiconformal map and $f \in H^p_I$, then $f \in H^p$.

Proof. Part (a) is [16, Theorem 1.1]. To obtain Part (b), it is enough to observe that

$$|f(\omega)|^p \le |f(\omega) - f(0) + f(0)|^p \le c(|f(\omega)|^p_I + |f(0)|^p),$$

where $c = 1$ for $p \le 1$ and $c = 2^{p-1}$ for $p > 1$. □

We will need the concept of quasisymmetry.

Definition 2.6. Let $A, B \subseteq \mathbb{C}$ and let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism $h\colon A \to B$ is η-quasisymmetric, if

$$\frac{|h(a) - h(x)|}{|h(b) - h(x)|} \le \eta \left(\frac{|a - x|}{|b - x|} \right)$$

for all $a \neq x \neq b$ in A. We say that h is quasisymmetric if any such η exists.

Lemma 2.7. *Let h be quasiconformal map of $\mathbb{D}$ onto $\mathbb{D}$. Then h is quasisymmetric.*

Proof. Because of [17, Theorems 8.1. and 8.2.] we know that h extends to a quasiconformal map of the entire $\mathbb{C}$. The claim follows since the [2, Theorem 3.5.3] tells us that each such quasiconformal map is also quasisymmetric. □

We need the fact that the radial limit is the same as the "non-tangential" limit. The following definition and lemma are from [23, Corollary 2.17].

Definition 2.8. We define a Stolz angle at $\omega \in S^1$ as

$$\Delta(\omega, \alpha, \varrho) = \Big\{ z \in \mathbb{D} : |\arg(1 - \overline{\omega} z)| < \alpha, \ |z - \omega| < \varrho \Big\}$$

for $0 < \alpha < \frac{\pi}{2}$ and $\varrho < 2\cos\alpha$.

Lemma 2.9. *Let $g \colon \mathbb{D} \to \mathbb{C}$ be a conformal map. Then for almost every $\omega \in S^1$ and every α and ϱ it holds that*

$$\lim_{z \to \omega, \ z \in \Delta(\omega, \alpha, \varrho)} g(z) = g(\omega).$$

A Borel measure μ is said to be doubling if there is a constant $c > 0$ such that for every x and $r > 0$ it holds that

$$0 < \mu(B(x, 2r)) \leq c \cdot \mu(B(x, r)) < \infty.$$

Theorem 2.10. *Let μ be a doubling measure on S^1, $\mu(S^1) = 2\pi$. Let $h \colon S^1 \to S^1$ be defined as*

$$h(e^{it}) = e^{i\mu\left(J_{0,t}\right)}, \ t \in [0, 2\pi),$$

where $J_{0,t}$ is the arc from 0 to e^{it}. Then h admits a quasiconformal extension to $\mathbb{D}$.

Proof. Because of the doubling property, it follows that h is weakly quasisymmetric, that is for some constant $H > 0$ and for any three points $a, b, x \in S^1$

$$|h(a) - h(x)| \leq H |h(b) - h(x)|$$

whenever $|a - x| \leq |b - x|$. Each such h is quasisymmetric by [25, Theorem 2.16.].

Finally, according to e.g. [2, Section 5.8.1], every quasisymmetric self-map of S^1 can be extended to a quasiconformal self-map of $\mathbb{D}$. □

The following two lemmas are consequences of the Koebe distortion theorem. They can be found in [16, Lemma 3.1 and Lemma 3.2.].

Lemma 2.11. *Let $f \colon \mathbb{D} \to \mathbb{C}$ be a conformal map. Then there exists a constant $c_{2.11}$ such that for any $z \in \mathbb{D}$ and for every $x, y \in B_z$ we have*

$$\frac{1}{c_{2.11}} \leq \frac{|f'(x)|}{|f'(y)|} \leq c_{2.11}.$$

Lemma 2.12. *Let $f \colon \mathbb{D} \to \mathbb{C}$ be a conformal map. Then there is a constant $c_{2.12}$ such that for every $x \in \mathbb{D}$*

$$d(f(x), \partial f(\mathbb{D})) \leq c_{2.12} (1 - |x|) |f'(x)|.$$

The following theorem is due to [9]. This formulation can be found in [16, p. 79].

Theorem 2.13 (Gehring-Hayman theorem). *There is a universal constant $c_{2.13}$ with the following property: Suppose that the map $f\colon \mathbb{D} \to \mathbb{C}$ is conformal and γ is a curve in $\mathbb{D}$ with endpoints 0 and $x \in \overline{\mathbb{D}}$. Then*

$$\ell(f([0,x))) \leq c_{2.13} \cdot \ell(f(\gamma)).$$

We also need the following [16, Lemma 3.4]:

Lemma 2.14. *Let f be a conformal map of $\mathbb{D}$ into $\mathbb{C}$. There is an absolute constant $c_{2.14}$ such that for every $x \in \mathbb{D}$*

$$\frac{\mathscr{H}^1\big(\{\omega \in S_x\colon d_I(f(\omega), f(x)) < c_{2.14} \cdot d(f(x), \partial f(\mathbb{D}))\}\big)}{\mathscr{H}^1(S_x)} > \frac{1}{2},$$

where $\mathscr{H}^1$ is the one-dimensional Hausdorff measure.

3 Proof of Theorem 1.1

Lemma 3.1. *Let g be a conformal map from $\mathbb{D}$ into $\mathbb{C}$. Let J be an arc in S^1 with $\ell(J) < \frac{1}{2}$.*

(a) There exist a constant $c_1 > 0$, such that for almost every $e^{it} \in J$ it holds that

$$|g(e^{it})|_I \geq c_1 \int_0^{1-\frac{\ell(J)}{2}} |g'(s \cdot e^{it_J})| \,\mathrm{d}s. \tag{2}$$

(b) There is a constant $c_2 > 0$, such that given the condition

$$|g(e^{it})|_I \leq c_2 \int_0^{1-\frac{\ell(J)}{2}} |g'(s \cdot e^{it_J})| \,\mathrm{d}s \tag{3}$$

it holds that

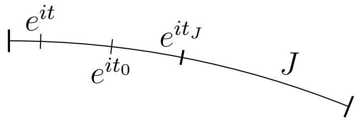
$$\frac{\mathscr{H}^1(\{t \in J : (3) \text{ holds}\})}{\mathscr{H}^1(J)} \geq \frac{1}{8}.$$

Furthermore, there is a $e^{it_0} \in \frac{J}{2}$ for which both (3) for J and (2) for $\frac{J}{2}$ hold.

Proof. To prove part (a) we fix $e^{it} \in J$ such that $|g(e^{it})|_I$ is defined, and we use Theorem 2.13 to observe that

$$|g(e^{it})|_I \geq \frac{1}{c_{2.13}} \int_0^1 |g'(se^{it})| \,\mathrm{d}s \geq \frac{1}{c_{2.13}} \int_0^{1-\frac{\ell(J)}{2}} |g'(se^{it})| \,\mathrm{d}s.$$

Now it is enough to prove that $\forall s \in \left(0, 1 - \frac{\ell(J)}{2}\right) : |g'(se^{it})| \geq \frac{1}{c_{2.11}} |g'(se^{it_J})|$. To show this let us take any such s and let $t_0 = \frac{t+t_J}{2}$. Note that e^{it_J} is the midpoint of J and $e^{it_0} \in J$ is the midpoint between e^{it} and e^{it_J}.



Consider the Whitney ball $B_{se^{it_0}} = B\left(se^{it_0}, \frac{1-s}{2}\right)$. Clearly

$$|se^{it} - se^{it_0}| = s|e^{it} - e^{it_0}| \leq s\frac{\ell(J)}{4} \leq \left(1 - \frac{\ell(J)}{2}\right)\frac{\ell(J)}{4}$$
$$\leq \frac{\ell(J)}{4} = \frac{1-\left(1-\frac{\ell(J)}{2}\right)}{2} \leq \frac{1-s}{2}.$$

Therefore $se^{it} \in B_{se^{it_0}}$. An analogous computation can be done for t_J in the place of t, and so $se^{it_J} \in B_{se^{it_0}}$ as well. Hence we may use Lemma 2.11 to conclude that indeed $|g'(se^{it})| \geq \frac{1}{c_{2.11}}|g'(se^{it_J})|$, which proves (a).

To prove (b), let us first fix $x = \left(1 - \frac{\ell(J)}{2}\right)e^{it_J}$. Obviously $\ell(J) = \mathscr{H}^1(J)$. Then S_x and J have the same midpoint and

$$\ell(J) \geq \ell(S_x) \geq \frac{\ell(J)}{4}. \tag{4}$$

Indeed, $\ell(S_x) = 2\arcsin\left(\frac{1-|x|}{2|x|}\right) = 2\arcsin\left(\frac{\ell(J)}{2(2-\ell(J))}\right)$, and for $\ell(J) \in \left(0, \frac{1}{2}\right)$ we have

$$\ell(J) \geq 2\frac{\ell(J)}{2(2-\ell(J))} \geq 2\arcsin\left(\frac{\ell(J)}{2(2-\ell(J))}\right) \geq \frac{\ell(J)}{2(2-\ell(J))} \geq \frac{\ell(J)}{4}. \tag{5}$$

We want to use Lemma 2.14. Let us consider an e^{it}, for which

$$d_I(g(e^{it}), g(x)) < c_{2.14} \cdot d(g(x), \partial g(\mathbb{D})). \tag{6}$$

Then

$$|g(e^{it})|_I := \inf\{\ell(g \circ \gamma) : \gamma \text{ goes from } 0 \text{ to } e^{it}\}$$
$$\leq \underbrace{\int_0^{1-\frac{\ell(J)}{2}} |g'(se^{it_J})|\,\mathrm{d}s}_{=:(\clubsuit)} + \inf\{\ell(g \circ \gamma_1) : \gamma_1 \text{ goes from } x \text{ to } e^{it}\}.$$

We continue with

$$|g(e^{it})|_I \leq (\clubsuit) + d_I(g(e^{it}), g(x)) \leq (\clubsuit) + c_{2.14} \cdot d(g(x), \partial g(\mathbb{D}))$$
$$\leq (\clubsuit) + c_{2.14} \cdot c_{2.12} \cdot (1-|x|)|g'(x)|,$$

where we used Lemma 2.12. We know that $1 - |x| = \frac{\ell(J)}{2}$, and so

$$|g(e^{it})|_I \leq (\clubsuit) + 2 \cdot c_{2.14} \cdot c_{2.12} \cdot \int_{1-\frac{\ell(J)}{2}-\frac{\ell(J)}{4}}^{1-\frac{\ell(J)}{2}} |g'(x)|\,\mathrm{d}s.$$

Now let us consider the Whitney ball $B_x = B\left(\left(1-\frac{\ell(J)}{2}\right)e^{it_J}, \frac{\ell(J)}{4}\right)$. For every number $s \in \left(1-\frac{\ell(J)}{2}-\frac{\ell(J)}{4}, 1-\frac{\ell(J)}{2}\right)$ the point $s \cdot e^{it_J}$ is in the Whitney ball B_x, as is obviously x. Therefore because of Lemma 2.11 we know that $|g'(x)| \leq c_{2.11} \cdot |g'(se^{it_J})|$. So

$$\begin{aligned} |g(e^{it})|_I &\leq (\clubsuit) + 2 \cdot c_{2.14} \cdot c_{2.12} \cdot c_{2.11} \cdot \int_{1-\frac{\ell(J)}{2}-\frac{\ell(J)}{4}}^{1-\frac{\ell(J)}{2}} |g'(se^{it_J})| \, \mathrm{d}s \\ &\leq \big(1 + 2 \cdot c_{2.14} \cdot c_{2.12} \cdot c_{2.11}\big) \int_0^{1-\frac{\ell(J)}{2}} |g'(se^{it_J})| \, \mathrm{d}s. \end{aligned}$$

Therefore, if for some number t the inequality (6) holds, then, for the same t, (3) holds as well. To finish the proof we observe that $S_x \subseteq J$ (see (4)) and that

$$\begin{aligned} \frac{\mathscr{H}^1(\{t \in J : (3) \text{ holds}\})}{\mathscr{H}^1(J)} &\geq \frac{\mathscr{H}^1(\{t \in S_x : (3) \text{ holds}\})}{4 \cdot \mathscr{H}^1(S_x)} \\ &\geq \frac{\mathscr{H}^1(\{t \in S_x : (6) \text{ holds}\})}{4 \cdot \mathscr{H}^1(S_x)} \geq \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$

because of Lemma 2.14.

To establish the last assertion it is enough to realize that $\frac{S_x}{2} \subseteq \frac{J}{2}$ because of (4). The condition (2) holds for almost all $t \in \frac{J}{2}$. Additionally we know (because of Lemma 2.14) that there is a $t \in \frac{S_x}{2} \subseteq \frac{J}{2}$ for which (6) and therefore (3) holds. □

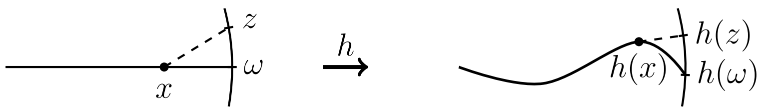
Lemma 3.2. *Let g be conformal map from $\mathbb{D}$ into $\mathbb{C}$ and let h be a quasiconformal map of $\mathbb{D}$ onto itself, homeomorphic up to the boundary. Then for almost all $\omega \in S^1$ it holds that*

$$\lim_{r \to 1^-} g\big(h(r \cdot \omega)\big) = \lim_{r \to 1^-} g\big(r \cdot h(\omega)\big).$$

Proof. Because of Lemma 2.9 it is enough to show that (when $r \to 1^-$) $h(r\omega)$ approaches $h(\omega)$ non-tangentially, that is that there is α and ϱ such that $h(r\omega) \in \Delta(\omega, \alpha, \varrho)$ for r close enough to 1. This follows from [19, Theorem 6], but for the convenience of the reader let us give a proof.

Lemma 2.7 shows that h is in fact quasisymmetric, i.e. there is non-decreasing function $\eta\colon [0,\infty) \to [0,\infty)$ such that for all x, y and z in $\overline{\mathbb{D}}$ it holds that

$$|h(x) - h(y)| \leq \eta\left(\frac{|x-y|}{|x-z|}\right) |h(x) - h(z)|.$$



Let us pick any x on the line form 0 to ω, choose $h(z)$ to be the closest point to $h(x)$ on S^1 and fix $y = \omega$. Then, because z is on the boundary, we know that $\frac{|x-\omega|}{|x-z|} \leq 1$. So

$$|h(x) - h(\omega)| \leq \eta(1) \cdot |h(x) - h(z)| = \eta(1) \cdot d(h(x), S_1).$$

This is precisely the desired non-tangentiality. □

Lemma 3.3. *Let $0 < q < p$ and let $g \in H_I^q$ with $|g|_I \geq K > 0$ on S^1. Suppose that $h\colon \mathbb{C} \to \mathbb{C}$ is a quasiconformal map with $h(0) = 0$, mapping S^1 onto S^1, so that*

$$h^{-1}(e^{it}) := \exp\left(i \left(\int_0^t |g\left(e^{is}\right)|_I^{q-p} \, \mathrm{d}s \right) \cdot \underbrace{2\pi \cdot \left(\int_0^{2\pi} |g\left(e^{is}\right)|_I^{q-p} \, \mathrm{d}s \right)^{-1}}_{=:C} \right)$$

for $t \in [0, 2\pi)$. Then $g \circ h \in H_I^p$.

Proof. Because $p > q$ and $|g|_I > K$ we know that $0 < C < \infty$. Firstly we compute the derivative of $(h|_{\partial\mathbb{D}})^{-1}$:

$$\begin{aligned}|(h^{-1})'(e^{it})| &= \left| \left(\int_0^t |g\left(e^{is}\right)|_I^{q-p} \, \mathrm{d}s \right)' \cdot C \right| = \\ &= |g\left(e^{it}\right)|_I^{q-p} \cdot C.\end{aligned}$$

Let us define for $\omega \in S^1$

$$I(\omega) := d_I(g \circ h(\omega), g \circ h(0)) = d_I(g \circ h(\omega), g(0)).$$

We continue using a change of variables,

$$\int_{S^1} |g \circ h(\omega)|_I^p \, \mathrm{d}\omega = \int_{S^1} I(\omega)^p \, \mathrm{d}\omega = \int_{S^1} I(h^{-1}(\omega))^p \cdot \left|(h^{-1})'(e^{is})\right| \, \mathrm{d}\omega.$$

For almost every $\omega \in S^1$, Lemma 3.2 applied to $h^{-1}(\omega)$ gives

$$g \circ h(h^{-1}(\omega)) := \lim_{r \to 1^-} g\big(h(r \cdot h^{-1}(\omega))\big) = \lim_{r \to 1^-} g\big(r \cdot \omega\big) =: g(\omega).$$

Therefore $I(h^{-1}(\omega)) := d_I(g \circ h(h^{-1}(\omega)), g(0)) = d_I(g(\omega), g(0)) = |g|_I$. So we can continue with

$$\int_{S^1} |g \circ h(\omega)|_I^p \, \mathrm{d}\omega = \int_{S^1} |g(\omega)|_I^p \cdot |g(\omega)|_I^{q-p} \cdot C \, \mathrm{d}\omega = C \cdot \int_{S^1} |g(\omega)|_I^q \, \mathrm{d}\omega < \infty. \; □$$

Proof of Theorem 1.1. We start with the (conformal) Riemann map $g\colon \mathbb{D} \to \Omega$ from Theorem 2.4. By applying an auxiliary translation we may assume that $g(0) = 0$. We know that $g \in H^q$ for all $0 < q < \frac{1}{2}$. Let us fix $q = \frac{1}{3}$. Without loss of generality we may assume that $p > q$. Because of Theorem 2.5 (a) we also know that $g \in H_I^q$. We want to define $f := g \circ h$ where h is as in Lemma 3.3. If we find such an h, then we are done because f is quasiconformal, $f \in H_I^p$, and therefore because of Theorem 2.5 (b) we know that indeed $f \in H^p$.

To find such an h we firstly define $\hat{h}$ on S^1 by setting

$$\hat{h}^{-1}(e^{it}) := \exp\left(i \left(\int_0^t |g\left(e^{is}\right)|_I^{q-p}\, \mathrm{d}s \right) \cdot \underbrace{2\pi \cdot \left(\int_0^{2\pi} |g\left(e^{is}\right)|_I^{q-p}\, \mathrm{d}s \right)^{-1}}_{=:C} \right)$$

for $t \in [0, 2\pi)$. This is a homeomorphism on S^1, and therefore $\hat{h}$ is well-defined on S^1. We want h to be a quasiconformal extension of $\hat{h}$ to $\mathbb{D}$. (Without loss of generality $h(0) = 0$.) The inverse of quasisymmetric map is quasisymmetric (see [2, Lemma 3.2.2.]). Because of this and Theorem 2.10 it is enough to show that

$$\int_{2J} |g(e^{it})|_I^{q-p}\, \mathrm{d}t \le c \cdot \int_J |g(e^{it})|_I^{q-p}\, \mathrm{d}t, \tag{7}$$

whenever $J \subseteq S^1$ is an arc. Without loss of generality we may assume that $\ell(2J) < \frac{1}{2}$.

We claim that there is $0 < c_0 < \infty$ such that

$$\frac{1}{c_0} \int_J |g(e^{it})|_I^{q-p}\, \mathrm{d}t \le \ell(J) \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p} \le c_0 \int_J |g(e^{it})|_I^{q-p}\, \mathrm{d}t. \tag{8}$$

The first inequality follows directly from Lemma 3.1 (a), because $q - p < 0$ and t_J does not depend of t:

$$\begin{aligned}\int_J |g(e^{it})|_I^{q-p}\, \mathrm{d}t &\le c_1^{q-p} \int_J \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p} \mathrm{d}t \\ &= c_1^{q-p} \cdot \ell(J) \cdot \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p}.\end{aligned}$$

To show the second inequality in (8) we use Lemma 3.1 (b). We get

$$\begin{aligned}\int_J |g(e^{it})|_I^{q-p}\, \mathrm{d}t &\ge \int_{\{t\in J:(3)\text{ holds}\}} |g(e^{it})|_I^{q-p}\, \mathrm{d}t \\ &\ge c_2^{q-p} \cdot \int_{\{t\in J:(3)\text{ holds}\}} \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p} \mathrm{d}t \\ &= c_2^{q-p} \cdot \mathscr{H}^1(\{t \in J : (3)\text{ holds}\}) \cdot \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p} \\ &\ge \frac{c_2^{q-p}}{8} \cdot \ell(J) \cdot \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\, \mathrm{d}s \right)^{q-p}.\end{aligned}$$

Obviously $t_J = t_{2J}$. To finish we take $t_0 \in J$ from the "furthermore" of Lemma 3.1 applied for $2J$. We use the estimate (8) twice, firstly for $2J$ and then for J.

$$\begin{aligned}
\int_{2J} |g(e^{it})|_I^{q-p}\,\mathrm{d}t &\overset{(8)}{\leq} c_0 \cdot \ell(2J) \cdot \left(\int_0^{1-\frac{\ell(2I)}{2}} |g'(s \cdot e^{it_{2J}})|\,\mathrm{d}s \right)^{q-p} \\
&\overset{(3)}{\leq} c_0 \cdot \ell(2J) \cdot c_2^{p-q} \cdot |g(e^{it_0})|_I^{q-p} \\
&\overset{(2)}{\leq} (2 \cdot c_0 \cdot c_2^{p-q} \cdot c_1^{q-p}) \cdot \ell(J) \cdot \left(\int_0^{1-\frac{\ell(I)}{2}} |g'(s \cdot e^{it_J})|\,\mathrm{d}s \right)^{q-p} \\
&\overset{(8)}{\leq} \underbrace{\left(2 \cdot c_0^2 \cdot c_1^{q-p} \cdot c_2^{p-q}\right)}_{=:c} \cdot \int_J |g(e^{it})|_I^{q-p}\,\mathrm{d}t.
\end{aligned}$$

This is the asserted inequality (7). □

4 Proof of Theorem 1.3

We will increase the maximal growth order of a given quasiconformal map by pre-composing it with a quasiconformal self-map.

Lemma 4.1. *Let $0 < \alpha < \beta < 2\alpha < \infty$. Let f be K−quasiconformal on the upper half-plane $\mathbb{H}$ and let there be a sequence of points $\{z_n = x_n + iy_n\} \subseteq \mathbb{H}$, $z_n \to 0$ such that*

$$|f(z_n)| \geq c \cdot y_n^{-\alpha}.$$

Then there is a sequence $\{\widehat{z_n} = \widehat{x_n} + i\widehat{y_n}\} \subseteq \mathbb{H}$, $\widehat{z_n} \to 0$ and a $\left(\frac{\beta}{2\alpha-\beta}K\right)$−quasiconformal map $\widehat{f}\colon \mathbb{H} \to \mathbb{C}$ such that $f(\mathbb{H}) = \widehat{f}(\mathbb{H})$ and

$$|\widehat{f}(\widehat{z_n})| \geq \frac{c}{2^{\beta-\alpha}} \cdot \widehat{y_n}^{-\beta}.$$

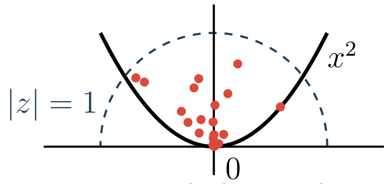
Proof. We want to use radial stretching(s) to move z_n "further away form the boundary". There are two (in principle non-exclusive) options:

(i) There is a subsequence (after relabeling) $z_n = x_n + iy_n \to 0$ with $|f(z_n)| \geq c \cdot y_n^{-\alpha}$ and
$$y_n \geq x_n^2,$$

(ii) or there is a subsequence (after relabeling) $z_n = x_n + iy_n \to 0$ with $|f(z_n)| \geq c \cdot y_n^{-\alpha}$ and
$$y_n < x_n^2.$$

Hence it is sufficient to find $\widehat{f}$ and $\{\widehat{z_n}\}$ in both cases. We will do this below.

<u>The case (i):</u>



In this case it is enough to use one radial stretching in the half-disk $\{x + iy : x^2 + y^2 \leq 1,\ y > 0\}$. Without loss of generality we may assume that $\{z_n\} \subseteq \mathbb{D}$. Set

$$h(z) := z \cdot |z|^{\frac{\beta}{2\alpha-\beta}-1}.$$

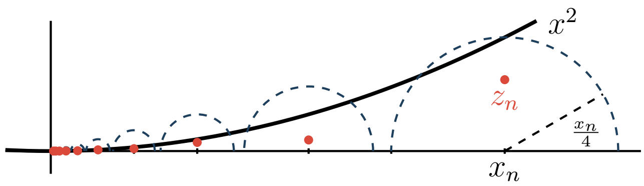
Notice that $\frac{\beta}{2\alpha-\beta} > 1$ and that h is $\frac{\beta}{2\alpha-\beta}$−quasiconformal and identity for $|z| = 1$. Let us furthermore define

$$\widehat{f}(z) = \begin{cases} f(z), & |z| \geq 1, \\ f(h(z)), & |z| < 1, \end{cases} \qquad \widehat{z_n} = z \cdot |z|^{\frac{2\alpha-\beta}{\beta}-1}.$$

Then $f(\mathbb{H}) = \widehat{f}(\mathbb{H})$, $\widehat{f}$ is $\left(\frac{\beta}{2\alpha-\beta} \cdot K\right)$−quasiconformal and $\widehat{f}(\widehat{z_n}) = f(z_n)$. We finish with the computation

$$\begin{aligned} \widehat{y_n}^{-\beta} &= \left(\mathrm{Im}\left(z \cdot |z|^{2\frac{\alpha-\beta}{\beta}}\right)\right)^{-\beta} = y_n^{-\beta}\left(x_n^2 + y_n^2\right)^{\frac{1}{2}\left(2\frac{\beta-\alpha}{\beta}\right)\beta} \\ &\leq y_n^{-\beta}\left(y_n + y_n^2\right)^{\beta-\alpha} \leq 2^{\beta-\alpha} \cdot y_n^{-\alpha} \\ &\leq \frac{2^{\beta-\alpha}}{c}|f(z_n)| = \frac{2^{\beta-\alpha}}{c}|\widehat{f}(\widehat{z_n})|. \end{aligned}$$

<u>The case (ii):</u>



Here we need one radial stretching for each z_n. Without loss of generality we may assume that $x_{n+1} < \frac{3}{5}x_n$. Then the disks $B\left(x_n, \frac{x_n}{4}\right)$ are disjoint. Set

$$h_n(z) := x_n + \left(\frac{x_n}{4}\right)^{1-\frac{\beta}{2\alpha-\beta}}(z - x_n) \cdot |z - x_n|^{\frac{\beta}{2\alpha-\beta}-1}.$$

Then, for each n, the map h_n is $\frac{\beta}{2\alpha-\beta}$-quasiconformal and identity for $|z - x_n| = \frac{x_n}{4}$. Let us furthermore define

$$\widehat{f}(z) := \begin{cases} f(h_n(z)), & |z - x_n| < \frac{x_n}{4}, \\ f(z), & \text{otherwise} \end{cases},$$

$$\widehat{z_n} := x_n + i\left(x_n^{2\frac{\beta-\alpha}{\beta}} \cdot y_n^{\frac{2\alpha-\beta}{\beta}}\right).$$

Then $f(\mathbb{H}) = \widehat{f}(\mathbb{H})$, $\widehat{f}$ is $\left(\frac{\beta}{2\alpha-\beta} \cdot K\right)$-quasiconformal and $\widehat{f}(\widehat{z_n}) = f(z_n)$. Finally

$$\widehat{y_n}^{-\beta} = x_n^{2(\alpha-\beta)} \cdot y_n^{\beta-2\alpha} \leq \left(y_n^{\frac{1}{2}}\right)^{2(\alpha-\beta)} \cdot y_n^{\beta-2\alpha} = y_n^{-\alpha}$$
$$\leq \frac{1}{c} \cdot |f(z_n)| \leq \frac{2^{\beta-\alpha}}{c} \cdot |\widehat{f}(\widehat{z_n})|. \qquad \square$$

Lemma 4.2. *Let $0 < a < b < \infty$. Let f be $K-$quasiconformal on the upper half-plane $\mathbb{H}$ and let there be a sequence of points $\{z_n = x_n + iy_n\} \subseteq \mathbb{H}$, $z_n \to 0$ such that*

$$|f(z_n)| \geq c \cdot y_n^{-a}.$$

Then for every $\varepsilon > 0$ there is a sequence $\{\widehat{z_n} = \widehat{x_n} + i\widehat{y_n}\} \subseteq \mathbb{H}$, $\widehat{z_n} \to 0$, constant $c_{4.2} = c_{4.2}(a, b, c)$ and a $\left((1+\varepsilon) \cdot \frac{b^2}{a^2} \cdot K\right)-$quasiconformal map $\widehat{f} \colon \mathbb{H} \to \mathbb{C}$ such that $\widehat{f}(\mathbb{H}) = f(\mathbb{H})$ and

$$|\widehat{f}(\widehat{z_n})| \geq c_{4.2} \cdot \widehat{y_n}^{-b}.$$

Proof. Let us fix $1 < s < 2$. Let $m = \left\lfloor \frac{\log b - \log a}{\log s} \right\rfloor + 1$, where $\lfloor \bullet \rfloor$ is the floor function, that is $\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\}$. Then $s^m a > b$. We will use the Lemma 4.2 m-times. For $k \in \{0, 1, \dots, m-1\}$ let

$$\alpha_k = s^k a, \qquad \beta_k = s^{k+1} a.$$

Then $0 < \alpha_k < \beta_k < 2\alpha_k < \infty$. Let $\widehat{f_0} = f$. Then $\widehat{f_0}$ is $\left(\frac{s}{2-s}\right)^0 \cdot K-$quasiconformal. Using Lemma 4.2 inductively for α_k, β_k and $\widehat{f_k}$, where $k = 0, 1, \dots, m-1$, we obtain a $\left(\left(\frac{s}{2-s}\right)^{k+1} \cdot K\right)-$quasiconformal f_{k+1} (because $\frac{\beta_k}{2\alpha_k - \beta_k} = \frac{s}{2-s}$). Set $\widehat{f} = \widehat{f_m}$. Then from the inductive process we obtain a sequence $\widehat{z_n}$ and a (possibly small but positive) constant $c_{4.2}$ such that

$$|\widehat{f}(\widehat{z_n})| \geq c_{4.2} \cdot \widehat{y_n}^{-s^m a} \geq c_{4.2} \cdot \widehat{y_n}^{-b}.$$

The map $\widehat{f}$ is $\left(\left(\frac{s}{2-s}\right)^m \cdot K\right)-$quasiconformal, that is $\left(\left(\frac{s}{2-s}\right)^{\left\lfloor \frac{\log b - \log a}{\log s} \right\rfloor + 1} \cdot K\right)-$quasiconformal. It is easy to check that

$$\lim_{s \to 1^+} \left(\left(\frac{s}{2-s}\right)^{\left\lfloor \frac{\log b - \log a}{\log s} \right\rfloor + 1} \cdot K\right) = \frac{b^2}{a^2} K.$$

Therefore we can fix $s > 1$ such that

$$\left(\left(\frac{s}{2-s}\right)^{\left\lfloor \frac{\log b - \log a}{\log s} \right\rfloor + 1} \cdot K\right) < \left((1+\varepsilon) \cdot \frac{b^2}{a^2} \cdot K\right).$$

Then $\widehat{f}$ is $\left((1+\varepsilon) \cdot \frac{b^2}{a^2} \cdot K\right)-$quasiconformal and we are done. $\square$

Lemma 4.3. *Let $f \in H^q(\mathbb{D})$. Then for each $z \in \mathbb{D}$*

$$|f(z)| \leq c_{4.3}(1-|z|)^{-\frac{1}{q}}.$$

Proof. If $f \in H^q$ then $\int_{\mathbb{D}} |f|^{2q} < \infty$ by [4, Theorem 9.1.]. And [15, Theorem 3.1.] implies that in this case there is $c_{4.3} > 0$ such that $|f(z)| \leq c_{4.3}(1-|z|)^{-\frac{2}{2q}}$. □

Theorem 4.4. *Let $0 < q < p < \infty$. Let g be a K –quasiconformal map with $g \notin H^p(\mathbb{D})$. Then for every $\varepsilon > 0$ we can find a $\left((1+\varepsilon)\cdot\frac{p^2}{q^2}\cdot K\right)$ –quasiconformal map $\widehat{g}$ such that $\widehat{g}(\mathbb{D}) = g(\mathbb{D})$ and $\widehat{g} \notin H^q$.*

Proof. Let us fix $\widehat{\varepsilon} < 1$. We know that $g \notin H^p(\mathbb{D})$. Therefore [4, Theorem 3.3.] yields

$$\int_0^1 M(r,g)^p = \infty,$$

where $M(r,g)$ is the maximal function defined as $M(r,g) := \sup\{|g(z)| : |z| = r\}$. Hence we can find a sequence $\{r_n\}$ such that $r_n \to 1$ and $M(r_n,g)^p \geq |1-r_n|^{-1+\widehat{\varepsilon}}$. Consequently we can find a sequence of points $\{u_n\} \subseteq \mathbb{D}$ such that $|g(u_n)| \geq |1-|u_n||^{\frac{-1+\widehat{\varepsilon}}{p}}$ and $u_n \to 1$.

Using a simple Möbius transformation T we obtain a sequence $\{z_n\} \subseteq \mathbb{H}$, $\{z_n\} \to 0$ and K –quasiconformal $f \colon \mathbb{H} \to \mathbb{C}$ such that $g(\mathbb{D}) = f(\mathbb{H})$ and

$$|f(z_n)| \geq c \cdot |\operatorname{Im}(z_n)|^{\frac{-1+\widehat{\varepsilon}}{p}}.$$

Lemma 4.2 gives us $\widehat{z_n}$ and a $\left((1+\widehat{\varepsilon})\cdot\left(\frac{1+\widehat{\varepsilon}}{1-\widehat{\varepsilon}}\right)^2\cdot\frac{p^2}{q^2}\cdot K\right)$ –quasiconformal map $\widehat{f}$, $\widehat{f}(\mathbb{H}) = f(\mathbb{H}) = g(\mathbb{D})$, for which

$$|\widehat{f}(\widehat{z_n})| \geq c_{4.2} \cdot |\operatorname{Im}(\widehat{z_n})|^{\frac{-1-\widehat{\varepsilon}}{q}}.$$

Reversing the transformation T we obtain a sequence of points $\{\widehat{u_n}\} \in \mathbb{D}$ and a $\left((1+\widehat{\varepsilon})\cdot\left(\frac{1+\widehat{\varepsilon}}{1-\widehat{\varepsilon}}\right)^2\cdot\frac{p^2}{q^2}\cdot K\right)$ –quasiconformal map $\widehat{g} \colon \mathbb{D} \to \mathbb{C}$, $\widehat{g}(\mathbb{D}) = g(\mathbb{D})$ such that

$$|\widehat{g}(\widehat{u_n})| \geq c_{4.4} \cdot |1-|\widehat{u_n}||^{\frac{-1-\widehat{\varepsilon}}{q}}.$$

To conclude that $\widehat{g}$ is not in H^q it is enough to use Lemma 4.3.

Finally, choosing a sufficiently small $\widehat{\varepsilon} > 0$ we get the required $\left((1+\varepsilon)\frac{p^2}{q^2}K\right)$– quasiconformality of $\widehat{g}$. □

Proof of Theorem 1.3. Let us assume that the condition (1) is not met, that is there exists $0 < p < \infty$ and a quasiconformal map $g \colon \mathbb{D} \to \Omega$, $g \notin H^p$. Let $q > 0$ be given. Without loss of generality we may assume that $q < p$. Theorem 4.4 applied to g gives the conclusion. □

5 Corollaries

Corollary 5.1. *Let $\Omega \subsetneq \mathbb{C}$ be a non-empty, simply connected domain. Let $p < 2$. Then there is a quasiconformal map $f : \mathbb{D} \to \Omega$ such that*

$$|Df| \in L^p(\mathbb{D}).$$

Proof. Let $q = \frac{2+p}{2-p}$. By Theorem 1.1 we know that there is a quasiconformal map $f : \mathbb{D} \to \Omega$, $f \in H^q$. Then Theorem [4, Theorem 9.1] implies that $f \in L^{2q}(\mathbb{D})$ and Theorem [4, Theorem 9.3] yields that $|Df| \in L^p(\mathbb{D})$. □

Following Astala and Gehring [1] we write

$$a_f(x) := \exp\left(\int_{B_x} \frac{\log J_f(y)}{2|B_x|}\, \mathrm{d}y\right),$$

where $|B_x|$ is the Lebesgue measure of B_x. Notice that for conformal f the mean value property implies that $a_f = |Df|$.

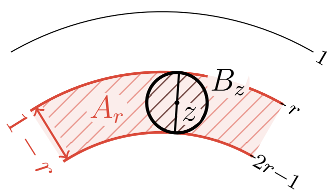
Proof of Corollary 1.2. Let us define the annulus A_r of width $1 - r$ by setting $A_r = \{z : 2r - 1 < |z| < r\}$ for $0 < r < 1$. For the area of A_r it holds that $|A_r| = \pi(1-r)(3r-1) < 2\pi(1-r)$.

Let us consider a_f in the annulus A_r for a given quasiconformal map f. We know that a_f is almost constant on Whitney balls, that is if we consider a Whitney ball B then for all x and y in B we have

$$a_f(x) < c \cdot a_f(y). \tag{9}$$

This is a consequence of the Koebe distortion theorem [3, Theorem 3.2.]. See also the first half of the proof of [3, Theorem 3.3.].

If we consider the Whitney ball covering a radial segment of A_r,



that is $B\left(z, \frac{1-r}{2}\right)$, then from (9) it follows that a_f is on this radial segment (up to the constant c) constant. Therefore, using Hölder inequality with $q > 1$ and $p < q < 2$,

$$\int_0^1 \left(\int_{\partial B(0,r)} a_f\right)^p \mathrm{d}r \le c \int_0^1 \left(\int_{A_r} \frac{a_f}{1-r}\right)^p \mathrm{d}r$$

$$\leq c\int_0^1 (1-r)^{-p}|A_r|^{\frac{q-1}{q}\cdot p}\left(\int_{A_r} a_f^q\right)^{\frac{p}{q}} \mathrm{d}r$$
$$\leq (2\pi)^{\frac{p}{q}} c \underbrace{\left(\int_0^1 (1-r)^{-\frac{p}{q}}\,\mathrm{d}r\right)}_{=:I_1} \cdot \underbrace{\left(\int_{\mathbb{D}} a_f^q\right)^{\frac{p}{q}}}_{=:I_2}.$$

Because $p < q < 2$ we know that $I_1 < \infty$. Corollary 5.1 yields that there is a quasiconformal map $f\colon \mathbb{D} \to \Omega$ such that $|Df| \in L^q(\mathbb{D})$. Moreover, for such an f it holds that $\int_{\mathbb{D}} a_f^q < \infty$ by [4, Lemma 2.5.]. Therefore $I_2 < \infty$, so for this f we have

$$\int_0^1 \left(\int_{\partial B(0,t)} a_f\right)^p \mathrm{d}r < \infty.$$

Finally, the Sullivan-Tukia-Väisälä approximation theorem [26, Corollary 7.12.] provides us with a (locally Lipschitz) quasiconformal map $\widehat{f}\colon D \to \Omega$ for which

$$\int_0^1 \left(\int_{\partial B(0,t)} |\widehat{f}'|\right)^p \mathrm{d}r < \infty.$$ □

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List of publications and citations

This is a list of our publications and their citations as of 24^{th} July 2023 according to Google Sholar.

1. **Measures of Non-Compactness and Sobolev–Lorentz Spaces**
    - O. Bouchala, 2020
    - Zeitschrift für Analysis und ihre Anwendungen, Volume 39, Issue 1, pp. 27–40
    - DOI 10.4171/ZAA/1649
    - Citations:
        - D. E. Edmunds, J. Lang and Z. Mihula. Measure of noncompactness of Sobolev embeddings on strip-like domains.
          Journal of Approximation Theory, 269, 105608, 2021.
        - J. Lang and Z. Mihula. Different degrees of non-compactness for optimal Sobolev embeddings.
          Journal of Functional Analysis, 284(10), 109880, 2023.
        - J. Lang, V. Musil, M. Olšák and L. Pick. Maximal non-compactness of Sobolev embeddings.
          The Journal of Geometric Analysis, 31(9), 9406-9431, 2021.

2. **Injectivity almost everywhere for weak limits of Sobolev homeomorphisms**
    - O. Bouchala, S. Hencl and A. Molchanova, 2020
    - Journal of Functional Analysis, Vol. 279, Issue 7, 108658
    - DOI 10.1016/J.JFA.2020.108658
    - Citations:
        - S. Krömer. Global invertibility for orientation-preserving Sobolev maps via invertibility on or near the boundary.
          Archive for Rational Mechanics and Analysis, 238(3), 1113-1155, 2020.
        - M. Bresciani. Linearized von Kármán theory for incompressible magnetoelastic plates.
          Mathematical Models and Methods in Applied Sciences, 31(10), 1987-2037, 2021.
        - P. Pedregal. A Multiplicative Version of Quasiconvexity for Hyperelasticity.
          Journal of Elasticity, 151(2), 219-236, 2022.
        - S. Almi, E. Davoli and M. Friedrich. Non-interpenetration conditions in the passage from nonlinear to linearized Griffith fracture.
          Journal de Mathématiques Pures et Appliquées, 175, 1-36, 2023.
        - M. Bresciani, E. Davoli and M. Kružík. Existence results in large-strain magnetoelasticity.
          Annales de l'Institut Henri Poincaré C., 2022.

- T. Iwaniec and J. Onninen. The Dirichlet principle for inner variations.
  Mathematische Annalen, 383(1-2), 315-351, 2022.
- A. Schikorra and J. M. Scott. Weak limits of fractional Sobolev homeomorphisms are almost injective.
  Studia Mathematica, 269, 241-260, 2023.

3. **Existence of quasiconformal mappings in a given Hardy space**

- O. Bouchala and P. Koskela, 2023
- To appear in Proceedings of the American Mathematical Society
- DOI 10.1090/PROC/16418