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To the Faculty of Mathematics and Physics, Charles University, and the Faculty of Mathematics and Science, University of Jyväskylä

Report on the dissertation of M.Sc. Ondřej Bouchala

The dissertation of M.Sc. Ondřej Bouchala *Geometric Function Theory and Its application in Non-linear Elasticity* belongs to the broad field of geometric analysis. The thesis consists of two published articles and a summary on the results. The first article *Injectivity almost everywhere for weak limits of Sobolev homeomorphisms* (published in the *Journal of Functional Analysis,* joint work with *Stanislav Hencl* and *Anastasia Molchanova*) is on the regularity theory of weak limits of Sobolev homeomorphisms and is connected to the theory of non-linear elasticy. The second article *Existence of quasiconformal mappings in a given Hardy space* (published in the *Proceedings of the AMS,* joint work with Pekka Koskela) belongs to field of quasiconformal geometry and focuses on geometric properties of planar domains. Both articles are published in journals of high academic standing. I will now discuss the main results of these articles.

The motivation for the first article stems from a general question on properties of energy minimal deformations for stored energy functionals. Since energy minimizers need not be injective they may not *a priori* represent a physically relevant deformation. This phenomenon motivates a study of precise properties of weak limits of Sobolev homeomorphisms. In this article, authors consider weak limits of Sobolev homeomorphisms in the space $W^{1,p}$ for $p \geq n-1$ and consider quantitaive questions related to interpeneration of matter and cavitation.

The first main theorem (Theorem 1.1) states that, in the range $p>n-1$ for $n \geq 3$ and $p \geq 1$ for $n=2$, a weak limit of Sobolev homeomorphisms in $W^{1,p}(\Omega,R^n)$, where Ω is a domain in R^n, has a precise representative f for which the fibers $f^{-1}(y)$ are at most singletons for all y in the image of f modulo a set of Hausdorff dimension $n-1$. The proof of this positive result is based on a covering argument and Müller—Spector area formula. Second part of the same theorem shows that this result is sharp. More precisely, they construct a strong limit of homeomorphisms in $W^{1,n-1}$ having the property that fibers over a Cantor set of positive Lebesgue measure are continua. This construction is extremely delicate. It is based on a sequence of carefully constructed contractions of beams, which are organized using the positive measure Cantor set.

Whereas Theorem 1.1 is stated for fibers, i.e. pre-images, authors show in Theorem 1.2 that a similar holds also in the domain for so-called topological image $f^T(x)$ of a point x. The positive result states that the topological image $f^T(x)$ of a point x is a singleton for all points in Ω modulo a set of Hausdorff dimension $n-p$. A similar result was shown previously by Müller and Spector under an additional assumption that the mapping has positive Jacobian. In Theorem 1.2, authors also show that, under the positivity assumption on the Jacobian, the $W^{1,p}$ Sobolev regularity cannot be improved by constructing a strong limit of Sobolev homeomorphisms in $W^{1,n-1}$ violating the property. Due to technical nature of the statement, I do not recall this statement in more detail here, but merely mention that the construction is again a very delicate and similar to the construction in Theorem 1.1. In my opinion, the constructions in Theorems 1.1 and 1.2 may provide a powerful tool to understand optimal regularity of wild constructions in topology such as Bing's wild involution (Ann. Math., 1952).

As mentioned, the second article in the thesis *Existence of quasiconformal mappings in a given Hardy space* belongs to the field of quasiconformal geometry. By a theorem of Hansen (Mich. Math. J., 1970), for a simply connected planar domain Ω, there is an optimal Hardy space H^p containing all Riemann maps from the unit disk to Ω. This gives rise to so-called Hardy number of the domain Ω. In their first main theorem authors show that, quite surprisingly, that there is no counterpart for the Hardy number when we pass from conformal mappings to quasiconformal mappings. More precisely, in Theorem 1.1 authors show that, for any simply connected proper domain Ω and any Hardy space H^p, there exist quasiconformal mappings from the unit disk onto Ω which belongs to that Hardy space. The proof is based on a construction of a quasiconformal self-mapping of the unit disk with a given quasisymmetric boundary data on the disk obtained from the Riemann map of the domain.

In the second main theorem (Theorem 1.2) authors prove an interesting dichotomy for domains Ω: *either there exists a quasiconformal map from the unit disk to Ω which belongs to all Hardy spaces or, for each Hardy space, there exists a quasiconformal mapping onto Ω which does not belong to the given space*. Here the proof reduces to the second case, and to a construction of a quasiconformal mapping, with a controlled distortion and specified growth, of the half-space.

The results in both articles are deep and interesting. The summary due to the candidate presents the results appropriately. Therefore, the manuscript *Geometric Function Theory and its applications in Nonlinear Elasticity* by M.Sc. Ondřej Bouchala **fulfills the requirements of a Ph.D. thesis in mathematics**.

I propose both the Faculty of Mathematics and Physics at the Charles Univeristy and the Faculty of Mathematics and Science at the University of Jyväskylä to give **M.Sc. Ondřej Bouchala a permission to defend his thesis.**

Sincerely yours,

Pekka Pankka
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