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Properties of function spaces and operators acting on them

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Title: Properties of function spaces and operators acting on them

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Abstract:

The present thesis is focused on the study of properties of function spaces containing measurable functions, and operators acting on them. It consists of four papers.

In the first paper, we establish a new characterization of the set of Sobolev functions with zero traces via the distance function from the boundary of a domain. This characterization is innovative in that it is based on the space $L^{1,\infty}_a$ of functions having absolutely continuous quasinorms in $L^{1,\infty}$.

In the second paper, we investigate properties of certain new scale of spaces governed by a functional involving the maximal nonincreasing rearrangement and powers. Motivation for studying such structures stems from a recent research of sharp Sobolev embeddings into spaces furnished with Ahlfors measures.

In the third paper, we extend discretization techniques for Lorentz norms by eliminating nondegeneracy restrictions on weights. We apply the method to characterize general embeddings between classical Lorentz spaces.

In the fourth paper, we characterize triples of weights for which an inequality involving the superposition of two integral operators holds. We apply results from the third paper to avoid duality and to obtain thereby a general result.

Keywords: Banach function spaces, rearrangement-invariant spaces, weighted inequalities, Sobolev spaces, zero traces

Contents

Introduction

Mathematicians, and scientists in general, have for long time been interested in functions and their properties. Among the most frequently investigated properties of functions are their integrability and their smoothness. Basic integrability properties, which in a certain sense describe the size of a function, can be quantified in terms of the membership to an appropriate Lebesgue space $L^p(\mathscr{R})$, defined as the collection of all measurable functions on a nonatomic σ-finite measure space $\mathscr{R}$, whose absolute value, raised to the power $p \in [1, \infty)$, is integrable, or which are essentially bounded if $p = \infty$. The smoothness depends on the existence and properties of derivatives. By combining these two approaches we arrive at an interesting phenomenon, the so-called *Sobolev spaces*. Sobolev spaces have found a wide range of applications, and they constitute one of the main tools in the analysis of solutions to partial differential equations.

The classical Sobolev space $W^{m,p}(\Omega)$ consists of those m-times weakly differentiable functions acting on an open subset Ω of $\mathbb{R}^n$, which, together with all their weak derivatives of order not exceeding m, belong to the Lebesgue space $L^p(\Omega)$. The membership to a Sobolev space reflects a higher degree of regularity than just the p-integrability. A Sobolev embedding shows that there is a gain in the degree of local integrability when transferring from a derivative to a function. Moreover, if $p > \frac{n}{m}$, then we even obtain the existence of the classical derivative almost everywhere.

Sobolev spaces have been investigated from many points of view, and even their very definition has been several times extended. As already pointed out, in their most classical definition, they are built upon Lebesgue spaces. Even though this definition is sufficient for solving many specific problems, examples show that the Lebesgue spaces are not delicate enough to provide answers to certain subtle questions. Thus, finer scales of function spaces based on integrability properties have to be called into the play.

To express this phenomenon, let us come back to the almost-everywhere existence of the classical derivative. As already mentioned, for $p > \frac{n}{m}$ it is guaranteed, but for $p = \frac{n}{m}$ this is not so. The set of p's having this property is not sharp. More precisely, we cannot find the smallest Lebesgue space such that each function belonging to the corresponding Sobolev space has the classical derivative almost everywhere. However, when we settle for working outside of Lebesgue spaces, we can recall the result of Stein [80], which nails down a sharp function space enjoying this property. It happens to be the Lorentz space $L^{\frac{n}{m},1}(\Omega)$.

The last mentioned example naturally leads us to the family of Lorentz spaces. These spaces are finer than Lebesgue spaces and can be useful for certain delicate questions that have no answers in Lebesgue spaces. Other examples would force us to working with yet other classes of function spaces such as Orlicz spaces, Lorentz–Zygmund spaces, and more. Our approach however will be based on the idea that instead of grappling with intrinsic difficulties specific to each class, we rather develop a wide axiomatic environment of function spaces, the so-called Banach function spaces, which shelters most of the important function spaces, helps us to better understand the concept of Sobolev spaces, and, at the same time, definitely is of independent interest.

1 Banach function spaces

In this section, we will introduce the concept of Banach function spaces and, more generally, of quasi-Banach function spaces. We will present fundamental tools to define quasi-Banach function spaces, and we will define several families of quasi-Banach function spaces, such as Lebesgue spaces, Orlicz spaces, Lorentz spaces and their useful generalizations. A more detailed construction of Banach function spaces can be found in [6] and [68].

Let $(\mathscr{R}, \mu)$ be a nonatomic σ-finite measure space with $\mu(\mathscr{R}) \in (0, \infty]$. We denote by $\mathscr{M}(\mathscr{R}, \mu)$ the set of all μ-measurable functions on $\mathscr{R}$, by $\mathscr{M}_+(\mathscr{R}, \mu)$ the set of all functions in $\mathscr{M}(\mathscr{R}, \mu)$ having nonnegative values.

For $f \in \mathscr{M}(\mathscr{R}, \mu)$, we define the *distribution function* $f_*\colon [0, \infty) \to [0, \mu(\mathscr{R})]$ as

$$f_*(\lambda) = \mu(\{x \in \mathscr{R} : |f(x)| > \lambda\}),$$

the *nonincreasing rearrangement* $f^*\colon [0, \mu(\mathscr{R})) \to [0, \infty]$, as

$$f^*(t) = \inf\{\lambda \geq 0 : f_*(\lambda) \leq t\},$$

and the *maximal nonincreasing rearrangement* $f^{**}\colon (0, \mu(\mathscr{R})) \to [0, \infty]$ by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

Note that f^* is, in a certain sense, a generalized inverse of f_*.

These tools only focus on the size of function, more precisely on the measure of its level sets, but not really on its shape. Now we shall collect properties of functions spaces which determine rearrangement-invariant Banach function spaces through an axiomatic approach.

We say that a functional $\varrho\colon \mathscr{M}_+(\mathscr{R}, \mu) \to [0, \infty]$ is a *Banach function norm* if, for all f, g and $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{M}_+(\mathscr{R}, \mu)$, for every $\lambda \in [0, \infty)$ and for every μ-measurable subset E of $\mathscr{R}$, the following five properties are satisfied:

(P1) $\varrho(f) = 0 \Leftrightarrow f = 0$ μ-a.e. on $\mathscr{R}$; $\varrho(\lambda f) = \lambda \varrho(f)$; $\varrho(f + g) \leq \varrho(f) + \varrho(g)$;
(P2) $g \leq f$ μ-a.e. on $\mathscr{R} \Rightarrow \varrho(g) \leq \varrho(f)$;
(P3) $f_n \nearrow f$ μ-a.e. on $\mathscr{R} \Rightarrow \varrho(f_n) \nearrow \varrho(f)$;
(P4) $\mu(E) < \infty \Rightarrow \varrho(\chi_E) < \infty$;
(P5) $\mu(E) < \infty \Rightarrow \int_E f\, d\mu \leq C_E \varrho(f)$ for some constant $C_E \in (0, \infty)$ possibly depending on E and ϱ but independent of f.

We say that $\varrho\colon \mathscr{M}_+(\mathscr{R}, \mu) \to [0, \infty]$ is a *Banach function quasinorm* if it satisfies (P2), (P3), (P4), and (P1) replaced by its weakened modification (Q1), where

(Q1) $\varrho(f) = 0 \Leftrightarrow f = 0$ μ-a.e. on $\mathscr{R}$, $\varrho(\lambda f) = \lambda \varrho(f)$ and there exists $C \in (0, \infty)$ such that $\varrho(f + g) \leq C(\varrho(f) + \varrho(g))$ for every $f, g \in \mathscr{M}_+(\mathscr{R}, \mu)$.

For a Banach function quasinorm ϱ and $X(\mathscr{R}, \mu) = \{f \in \mathscr{M}(\mathscr{R}, \mu), \varrho(|f|) < \infty\}$, we denote $\|f\|_{X(\mathscr{R},\mu)} = \varrho(|f|)$ for $f \in \mathscr{M}(\mathscr{R}, \mu)$, and we then say that $X(\mathscr{R}, \mu)$ is a *quasi-Banach function space* over $(\mathscr{R}, \mu)$. If there is no risk of confusion, we write X or $X(\mathscr{R})$ instead of $X(\mathscr{R}, \mu)$. In the case ϱ is a Banach function norm, we call $X(\mathscr{R}, \mu)$ a *Banach function space* over $(\mathscr{R}, \mu)$.

If we add moreover the condition

(P6) $\varrho(f)=\varrho(g)$ whenever $f^*=g^*$,

we get *rearrangement-invariant Banach function norm* and *rearrangement-invariant Banach function space*, abbreviated as an *r.i. space*. We use an analogous terminology for quasinorms.

An important property of rearrangement-invariant Banach function spaces is the existence of the so-called representation space containing functions acting on $(0,\mu(\mathscr{R}))$. A norm $\overline{\varrho}$ of the representation space $\overline{X}(0,\mu(\mathscr{R}))$ corresponding to the space $X(\mathscr{R},\mu)$ with norm ϱ is characterized by the relation $\overline{\varrho}(f^*)=\varrho(f)$ for every $f\in\mathscr{M}_+(\mathscr{R},\mu)$. Such norm always exists, and it is uniquely determined for each r.i. space $X(\mathscr{R},\mu)$, see [6, Chapter 2, Theorem 4.10].

The description of relations between individual spaces constitutes a topic of great interest in theory of Banach function spaces. We say that a space X is *continuously embedded* into another space, Y, a fact denoted by $X\hookrightarrow Y$, if there exists a constant $C>0$ such that for all $f\in\mathscr{M}(\mathscr{R},\mu)$ we have $\|f\|_Y\le C\|f\|_X$. The smallest possible constant C that renders the inequality true is called the *norm of the embedding*. If $X\hookrightarrow Y$ and simultaneously $Y\hookrightarrow X$, then $X=Y$ in set theoretical sense, and we say that (quasi)norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent.

A pivotal set of examples of rearrangement-invariant quasi-Banach function spaces is provided by the class of *Lebesgue spaces*. For $p\in(0,\infty]$, the Lebesgue space $L^p(\mathscr{R})=L^p(\mathscr{R},\mu)$ is defined as a collection of all functions $f\in\mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{L^p(\mathscr{R})}<\infty$, where

$$\|f\|_{L^p(\mathscr{R})}=\begin{cases}\left(\int_{\mathscr{R}}|f|^p d\mu\right)^{\frac{1}{p}}, & p\in(0,\infty),\\ \operatorname{ess\,sup}_{\mathscr{R}}|f|, & p=\infty,\end{cases}$$

Recall that if $p\in[1,\infty]$, then $L^p(\mathscr{R})$ is an r.i. space, and if $(\mathscr{R},\mu)$ is of finite measure, then one has $L^{p_1}(\mathscr{R})\hookrightarrow L^{p_2}(\mathscr{R})$ whenever $0<p_2\le p_1\le\infty$, with the norm of the embedding equal to $\mu(\mathscr{R})^{1/p_2-1/p_1}$.

As mentioned above, Lebesgue spaces play a primary role both in the theory and in applications. At the same time, they constitute one of the best known families of function spaces, and quite often they are sufficient for a task at hand, but in many different situations more general function spaces are needed. We shall now introduce a finer class of function spaces, a one already mentioned, namely that of the Lorentz spaces.

Given $p,q\in(0,\infty]$, the collection $L^{p,q}(\mathscr{R})$ of all functions $f\in\mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{L^{p,q}(\mathscr{R})}<\infty$, is called the *Lorentz space*, where

$$\|f\|_{L^{p,q}(\mathscr{R})}=\left\|t^{\frac{1}{p}-\frac{1}{q}}f^*(t)\right\|_{L^q(0,\mu(\mathscr{R}))},$$

or, alternatively,

$$\|f\|_{L^{p,q}(\mathscr{R})}=p^{\frac{1}{q}}\left\|t^{1-\frac{1}{q}}f_*(t)^{\frac{1}{p}}\right\|_{L^q(0,\infty)}.$$

Moreover, we define the collection $L^{(p,q)}(\mathscr{R})$ of all functions $f\in\mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{L^{(p,q)}(\mathscr{R})}<\infty$, where

$$\|f\|_{L^{(p,q)}(\mathscr{R})}=\left\|t^{\frac{1}{p}-\frac{1}{q}}f^{**}(t)\right\|_{L^q(0,\mu(\mathscr{R}))}.$$

This set is also called the Lorentz space. It is well known that $L^{(p,q)}(\mathscr{R})\hookrightarrow L^{p,q}(\mathscr{R})$ for $p,q\in(0,\infty]$. Moreover, if $p\in(1,\infty]$, then the spaces $L^{p,q}(\mathscr{R})$

and $L^{(p,q)}(\mathscr{R})$ coincide in the set-theoretical sense, and their (quasi)norms are equivalent. Recall that $L^{p,p}(\mathscr{R}) = L^p(\mathscr{R})$ for every $p \in (0,\infty]$ and that $L^{p,q}(\mathscr{R}) \hookrightarrow L^{p,r}(\mathscr{R})$ whenever $p \in (0,\infty]$ and $0 < q \le r \le \infty$. If either $p \in (0,\infty)$ and $q \in (0,\infty]$ or $p = q = \infty$, then $L^{p,q}(\mathscr{R})$ is a quasi-r.i. space. If one of the conditions

$$\begin{cases} p \in (1,\infty),\ q \in [1,\infty], \\ p = q = 1, \\ p = q = \infty, \end{cases}$$

holds, then $L^{p,q}(\mathscr{R})$ is equivalent to an r.i. space.

We shall introduce some further generalizations of Lebesgue spaces, in a similar direction as Lorentz spaces, that are of use in the thesis. We begin with the so-called *Lorentz–Zygmund spaces.*

Let $\mu(\mathscr{R}) < \infty$, let $p,q \in (0,\infty]$ and let $\alpha \in \mathbb{R}$. We define the collections $L^{p,q;\alpha}(\mathscr{R})$ and $L^{(p,q;\alpha)}(\mathscr{R})$ of functions $f \in \mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{L^{p,q;\alpha}(\mathscr{R})} < \infty$ and $\|f\|_{L^{(p,q;\alpha)}(\mathscr{R})} < \infty$, respectively, where

$$\begin{aligned} \|f\|_{L^{p,q;\alpha}(\mathscr{R})} &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\tfrac{e\mu(\mathscr{R})}{t}\right) f^*(t) \right\|_{L^q(0,\mu(\mathscr{R}))}, \\ \|f\|_{L^{(p,q;\alpha)}(\mathscr{R})} &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\tfrac{e\mu(\mathscr{R})}{t}\right) f^{**}(t) \right\|_{L^q(0,\mu(\mathscr{R}))}. \end{aligned}$$

Again, both structures are called the Lorentz–Zygmund spaces. Note that the choice $\alpha = 0$ yields Lorentz spaces $L^{p,q}(\mathscr{R})$ and $L^{(p,q)}(\mathscr{R})$, respectively. For the sake of brevity, we do not specify here the scale of parameters for Lorentz–Zygmund spaces satisfying axioms of r.i. spaces. Instead, we refer an interested reader to more specific literature, see e.g. [5, 32, 66] for more details.

The next structure which we would like to mention is so far the most general one. In particular, it covers all Lorentz–Zygmund spaces, and more. Given $p \in (0,\infty]$ and a weight w on $(0,\mu(\mathscr{R}))$ (that is, $w \in \mathscr{M}_+(0,\mu(\mathscr{R}))$) the *classical Lorentz spaces* $\Lambda^p(w)(\mathscr{R})$ and $\Gamma^p(w)(\mathscr{R})$ consist of all functions $f \in \mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{\Lambda^p(w)(\mathscr{R})} < \infty$ and $\|f\|_{\Gamma^p(w)(\mathscr{R})} < \infty$, respectively, where

$$\|f\|_{\Lambda^p(w)(\mathscr{R})} = \begin{cases} \left(\int_0^{\mu(\mathscr{R})} f^*(t)^p w(t)\,dt\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty), \\ \operatorname{ess\,sup}_{t\in(0,\mu(\mathscr{R}))} f^*(t)w(t) & \text{if } p = \infty \end{cases}$$

and

$$\|f\|_{\Gamma^p(w)(\mathscr{R})} = \begin{cases} \left(\int_0^{\mu(\mathscr{R})} f^{**}(t)^p w(t)\,dt\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty), \\ \operatorname{ess\,sup}_{t\in(0,\mu(\mathscr{R}))} f^{**}(t)w(t) & \text{if } p = \infty. \end{cases}$$

These spaces were introduced by Lorentz in [52] and have been extensively investigated ever since. Indeed, the Paper III in this thesis is devoted to the study of the spaces of this kind. The question of the (quasi)normability of classical Lorentz spaces is rather complicated, details can be found either scattered in literature or surveyed in [68, Section 10.2]. Note that $\Lambda^q(t^{\frac{q}{p}-1})(\mathscr{R}) = L^{p,q}(\mathscr{R})$ and $\Gamma^q(t^{\frac{q}{p}-1})(\mathscr{R}) = L^{(p,q)}(\mathscr{R})$.

Now we turn to our attention to a class of function spaces which also generalize Lebesgue spaces, but through a completely different approach and in another direction than Lorentz spaces, namely to *Orlicz spaces*. The definition of an Orlicz space slightly varies in the literature. We adopt the one departing from the *Young function* $A\colon [0,\infty) \to [0,\infty]$, that is, a left-continuous convex function satisfying $A(0) = 0$ and such that A is not constant in $(0,\infty)$. The corresponding Orlicz space $L^A(\mathscr{R})$ is then given as the collection of all functions in $\mathscr{M}(\mathscr{R},\mu)$ such that $\|f\|_{L^A(\mathscr{R})} < \infty$, where

$$\|f\|_{L^A(\mathscr{R})} = \inf\left\{\lambda \in (0,\infty) : \int_{\mathscr{R}} A\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\}.$$

In particular, $L^A(\mathscr{R}) = L^p(\mathscr{R})$ if $A(t) = t^p$ for $p \in [1,\infty)$, and $L^A(\mathscr{R}) = L^\infty(\mathscr{R})$ if $A = \infty\chi_{(1,\infty)}$.

We can present two more examples if we restrict ourselves to $\mu(\mathscr{R}) < \infty$. If either $p \in (1,\infty)$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \in [0,\infty)$, then we denote by $L^p(\log L)^\alpha(\mathscr{R})$ the Orlicz space associated with a Young function equivalent to $t^p(\log t)^{\alpha p}$ near infinity. If $\beta \in (0,\infty)$, then we denote by $\exp L^\beta(\mathscr{R})$ the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity. Note that some Orlicz spaces coincide with appropriate Lorentz–Zygmund spaces, for example $L^p(\log L)^\alpha(\mathscr{R}) = L^{p,p;\alpha}(\mathscr{R})$ and $\exp L^\beta(\mathscr{R}) = L^{\infty,\infty,-\frac{1}{\beta}}(\mathscr{R})$. A comprehensive study of Orlicz spaces is the subject of the monograph [70], some further details can also be found in [68].

All function spaces that have been mentioned so far possess the property (P6) given above. In other words, they are rearrangement invariant in the sense that their (quasi)norms, applied to a function f, depend only on the measure of level set of $|f|$, or, equivalently, on its nonincreasing rearrangement f^*. We shall now give an example of spaces which are, in most cases, Banach function spaces, but they do not obey the property (P6) except for certain trivial cases. Given $p \in (0,\infty)$ and a weight w, the *weighted Lebesgue space* $L^p_w(\mathscr{R})$ is a collection of all functions in $\mathscr{M}(\mathscr{R},\mu)$ with $\|f\|_{L^p_w(\mathscr{R})} < \infty$, where

$$\|f\|_{L^p_w(\mathscr{R})} = \left(\int_{\mathscr{R}} |f|^p w \, d\mu\right)^{\frac{1}{p}}.$$

For $p \in (1,\infty)$, the weighted Lebesgue space is a Banach function space if and only if both w and $w^{-\frac{1}{p-1}}$ are integrable over all finite measure subsets of $\mathscr{R}$, and it is an r.i. space if and only if the weight is a constant function. Weighted Lebesgue spaces and their generalizations are treated by many authors, a comprehensive overview is given for example in [72].

Although the theory of rearrangement-invariant quasi-Banach function spaces itself is a beautiful piece of mathematics, and we could easily write many more pages about it (interested readers are kindly referred to Paper II, where one scale of function spaces is thoroughly treated from absolute basics), we will now shift our attention to the direct application of structures of rearrangement-invariant (quasi)-Banach function spaces in the theory of Sobolev spaces.

2 Sobolev spaces and their relations to Banach function spaces

In this section, we will describe several advanced applications of (quasi)-Banach function spaces in the theory of Sobolev spaces. We obviously cannot cover all the directions of the current research in this area, hence we focus mainly on those that are directly connected with papers included in this thesis. Let us start with the classical definition of Sobolev spaces.

For $m, n \in \mathbb{N}$, $p \in [1, \infty]$ and an open subset Ω of the ambient Euclidean space $\mathbb{R}^n$ endowed with the Lebesgue measure, the classical Sobolev space $W^{m,p}(\Omega)$ is given as the collection of all m-times weakly differentiable functions $u \colon \Omega \to \mathbb{R}$ such that $D^\alpha u \in L^p(\Omega)$ for every multiindex α of height not exceeding m. This set is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty. \end{cases}$$

We define the space $W_0^{m,p}(\Omega)$ as a closure of all smooth functions having compact support in Ω in the space $W^{m,p}(\Omega)$.

The case $p = \infty$ differs from cases $p \in [1, \infty)$ in many ways. From now on we will focus only on $p \in [1, \infty)$.

2.1 Sobolev functions vanishing at the boundary

The functions belonging to the space $W_0^{m,p}(\Omega)$ are sometimes called "vanishing at the boundary". The cases when $\Omega = \mathbb{R}^n$ and $\Omega \subsetneq \mathbb{R}^n$ differ substantially from one another, in particular, in the former case, $W_0^{m,p}(\mathbb{R}^n)$ coincides with $W^{m,p}(\mathbb{R}^n)$, which is not so in the latter. If $\Omega \subsetneq \mathbb{R}^n$, then we can safely say that the domain Ω has the boundary $\partial\Omega$, and we can also quantify the distance of a point $x \in \mathbb{R}^n$ from the boundary of Ω as

$$d(x) = \operatorname{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} \operatorname{dist}(x, y).$$

The function d will be called here the *distance function from the boundary.* It depends on Ω, and we will thus always presume that Ω is fixed.

Because the definition of $W_0^{m,p}(\Omega)$ via the closure of compactly supported smooth functions in a Sobolev space is not very useful in practice, the functions from $W_0^{m,p}(\Omega)$ were investigated from many other points of view, and many equivalent descriptions of the space $W_0^{m,p}(\Omega)$ have been obtained. To give an example, a well-known characterization of the first-order Sobolev space $W_0^{1,p}(\Omega)$ upon a Lipschitz domain is formulated via the trace operator T (see e.g. [51, Section 6.4]),

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega), Tu = 0 \text{ a.e. in } \partial\Omega\}.$$

The question of the regularity of the domain is important here. Another description of functions from $W_0^{m,p}(\Omega)$ can be given using the point of view of capacitary conditions on the domain such as is the Havin–Bagby theorem, see for example the monograph [41].

We focus on the description via the distance function from the boundary. The first step in this research was given in the paper by Kadlec and Kufner [45]. They established an equivalence between the space $W_0^{m,p}(\Omega)$ and certain other space, namely the set

$$\{u \in \mathscr{M}(\Omega) : d^{-(m-|\alpha|)} D^\alpha u \in L^p, |\alpha| \leq m\}$$

endowed with the corresponding norm, where Ω is a Lipschitz domain. Since the proof heavily relies on the Hardy inequality, the result is restricted to $p > 1$. Later, in the monograph by Edmunds and Evans, [27], a proof was given of the fact that, for $p \geq 1$ and $\Omega \neq \mathbb{R}^n$, a function $u \in W^{m,p}(\Omega)$ with $\frac{u}{d^m} \in L^p(\Omega)$ is an element of $W_0^{m,p}(\Omega)$. These results can be summarized in the form of the equivalence

$$u \in W_0^{m,p}(\Omega) \qquad \Leftrightarrow \qquad u \in W^{m,p}(\Omega) \quad \text{and} \quad \frac{u}{d^m} \in L^p(\Omega)$$

for $p > 1$ and a Lipschitz domain Ω. During the years that followed, the result has been improved several times in the sense that weaker and weaker versions of the condition $\frac{u}{d^m} \in L^p(\Omega)$ have been appearing, and the Lebesgue space $L^p(\Omega)$ has consecutively been replaced by essentially larger and larger structures. First came the Lorentz space $L^{p,\infty}(\Omega)$ (in [48], where the result is restricted to $m = 1$), and later the (yet larger) Lebesgue space $L^1(\Omega)$ (in [30] for $m = 1$, and in [31] for higher-order cases). It should be noticed that the regularity of the domain varied in the mentioned results. This feature is of independent interest, but let us for the time being focus solely on the question of spaces. So far the weakest condition known to guarantee the equivalence is the one which has been given in the Paper I of this thesis. It states that, for the first-order case, one has

$$u \in W_0^{1,p}(\Omega) \qquad \Leftrightarrow \qquad |\nabla u| \in L^p(\Omega) \quad \text{and} \quad \frac{u}{d} \in L_a^{1,\infty}(\Omega), \tag{1}$$

where $L_a^{1,\infty}(\Omega)$ is the collection of all functions from $L^{1,\infty}(\Omega)$ which have absolutely continuous $L^{1,\infty}$-quasinorms. Moreover, it is showed in the same paper that when $\frac{u}{d} \in L_a^{1,\infty}(\Omega)$ is replaced by the natural, and yet weaker, condition $\frac{u}{d} \in L^{1,\infty}(\Omega)$, then the corresponding statement is no more sufficient. It seems, therefore, that we discovered a certain type of threshold.

To summarize the content of this subsection, let us recall that we formulated here an interesting problem on characterizing the spaces $W_0^{m,p}(\Omega)$ by conditions involving the distance function from the boundary. We surveyed the pursuit, which lasted for several decades, of the largest possible structure $X(\Omega)$ enjoying the property that $\frac{u}{d^m} \in X(\Omega)$ together with $u \in W^{m,p}(\Omega)$ ensure $u \in W_0^{m,p}(\Omega)$. We developed a new approach to the problem and obtained a new candidate for such a structure, the best one known so far. This time, we abandoned the environment of quasi-r.i. spaces as the question is quite interesting from the (wider) set-theoretical point of view. Moreover, we showed that the nearest larger commonly known quasi-r.i. space no longer works.

Now let us turn to another well-known problem, where the optimality in rearrangement-invariant Banach function spaces has been already reached, but which still serves as a good supply of important open problems.

2.2 Sobolev embeddings and the reduction principle

Sobolev inequalities and Sobolev embeddings constitute important cornerstones of the study of Sobolev spaces and their applications to partial differential equations. They give us an information about the degree of integrability of a function from a Sobolev space. In cases when the underlying domain is of finite measure and of sufficient regularity, the degree of integrability of the function is always better then that of its gradient (of any order), in other words, the integrability parameter in the target space is higher than the one which appears in the definition of the Sobolev space.

Let $\Omega \in \mathbb{R}^n$ be a regular domain (for simplicity, say, a Lipschitz domain). In the range of Lebesgue spaces, the following embeddings are well known:

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega) \qquad \text{for } 1 \leq p < n, \tag{2}$$

$$W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega) \qquad \text{for } p = n, \text{ where } q < \infty, \tag{3}$$

$$W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \qquad \text{for } p > n. \tag{4}$$

For the case $p < n$, $L^{\frac{np}{n-p}}(\Omega)$ is the smallest Lebesgue space which renders the first embedding true. For the case $p = n$, there exists no such optimal Lebesgue space. The embeddings in cases $p \leq n$ follow from classical results by Sobolev ([78, 79]), Gagliardo ([34]) and Nirenberg ([63]). In the case when $p > n$, a classical result of Morrey ([58]) states that functions from $W^{1,p}(\Omega)$ are Hölder continuous with the order of continuity $1 - \frac{n}{p}$, and, moreover, we can even get an existence of the derivative in the classical sense almost everywhere (see [80] and the references therein), but as far as integrability is concerned, (4) yields the best possible answer.

In the limiting case $p = n$, the embedding (3) is not very satisfactory as there is no borderline space. Nevertheless, such a space, naturally smaller than every $L^q(\Omega)$, $q < \infty$, and larger then $L^\infty(\Omega)$, can be found as long as we settle to using more general structures than Lebesgue spaces. The first such space was discovered independently by Trudinger [84], Yudovich [87], Peetre [67] and Pokhozhaev [69]. The embedding reads as

$$W^{1,p}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega) \qquad \text{for } p = n,$$

and moreover this embedding is optimal within the scale of Orlicz spaces, as was later pointed out by Hempel, Morris and Trudinger [42]. Let us also note that the embedding (2), which was mentioned to be optimal among all Lebesgue spaces, can be improved neither in the class of Orlicz spaces (see [14]).

However in the setting of r.i. spaces, there is still a room for a possible improvement. More precisely, it turns out that

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p},p}(\Omega) \qquad \text{for } 1 \leq p < n, \tag{5}$$

$$W^{1,p}(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega) \qquad \text{for } p = n. \tag{6}$$

The embedding (5) is due to Peetre [67], see also O'Neil [64] and Hunt [43]. It is a nontrivial improvement of the embedding (2) since $L^{\frac{np}{n-p},p}(\Omega) \subsetneq L^{\frac{np}{n-p}}(\Omega)$. The embedding in the later case $p = n$, (6), was obtained independently by Hansson [38] and Brézis and Wainger [10] (and can be also directly derived from the results in monograph by Maz'ya, see [54]). Moreover, the target spaces in both the embeddings (5) and (6) turn out to be optimal among all rearrangement-invariant Banach function spaces. This was established by Edmunds, Kerman and Pick [28] and also by Ćwikel and Pustylnik [24]. In [28], a new powerful method was employed, based on the transformation of the problem to a one-dimensional task using an equivalence between the Sobolev inequality and a considerably simpler one-dimensional Hardy-type inequality. This method was called a *reduction principle*.

Using various modifications of the reduction principle, several versions of optimal Sobolev embeddings in many situations were proved later. Let us not restrict ourselves to Sobolev spaces built upon Lebesgue spaces. Let the space $W^1X(\Omega)$ be a collection of all weakly differentiable functions on $\Omega \subset \mathbb{R}^n$ that together with their gradient belong to the r.i. space $X(\Omega)$. We define the subspace $W^1_0X(\Omega)$ of functions from $W^1X(\Omega)$ vanishing at the boundary of Ω in an analogous way as it was done for $W^{1,p}_0(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$. A powerful idea in the proof of a first-order Sobolev inequality for functions from $W^1_0X(\Omega)$ states that it can be deduced via symmetrization from its analogue for radially symmetric functions on balls. This symmetrization technique is based on the fact that a rearrangement-invariant norm of a function vanishing at the boundary of Ω is invariant to replacement of the function by its radially decreasing symmetral, and that a rearrangement-invariant norm of the gradient does not increase under the same operation. The latter property is known as the Pólya–Szegő inequality (also as the Faber–Krahn inequality), and it itself is a consequence of the classical isoperimetric inequality in $\mathbb{R}^n$. This approach has its roots in the work of Moser [59], Aubin [3] and Talenti [82], the close connection between isoperimetric inequality and Sobolev inequality was explored by May'za [53], see also Federer and Fleming [33] and De Giorgi [25].

These ingredients combined finally led to the formulation of a pivotal instance of a reduction principle, which reads as follows: Assume that $\Omega \subset \mathbb{R}^n$ is of finite measure (without loss of generality we can assume that $|\Omega| = 1$) and let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Then the Sobolev inequality

$$\|u\|_{Y(\Omega)} \leq C_1 \, \||\nabla u|\|_{X(\Omega)} \tag{7}$$

holds for some constant C_1 and for every function $u \in W^1_0X(\Omega)$ if and only if the Hardy-type inequality

$$\left\| \int_t^1 f(s) s^{-1+\frac{1}{n}} ds \right\|_{\overline{Y}(0,1)} \leq C_2 \, \|f\|_{\overline{X}(0,1)} \tag{8}$$

holds for some constant C_2 and for every function $f \in \overline{X}(0,1)$, where $\overline{X}(0,1)$ and $\overline{Y}(0,1)$ are the representation spaces of $X(\Omega)$ and $Y(\Omega)$. This reduction principle is due to Kerman and Pick [46], for some related work see also Curbera and Ricker [23]. The reduction principle is a key step in the characterization of the optimal target space $Y(\Omega)$ for a given space $X(\Omega)$ in the inequality (7). This is due to Kerman and Pick [46] and Cianchi, Pick and Slavíková [20].

When we focus on functions from $W^1X(\Omega)$, which do not vanish at the boundary of a domain, then the situation changes a little. The Pólya–Szegő inequality fails here, and thus the symmetrization technique mentioned before can not be used directly. For sufficiently regular domains, one can use an extension technique, and the problem can then be solved using the preceding result. We can derive an inequality analogous to (8), but with the full $W^1X(\Omega)$-norm on the right. It turns out that we can get a similar result even for less regular domains, but we need to use finer techniques. For $\Omega \subset \mathbb{R}^n$, $|\Omega| = 1$, let us denote $I_\Omega : [0,1] \to [0,\infty)$ the *isoperimetric function* of Ω given for $s \in [0, \frac{1}{2}]$ as the infimum of the perimeter relative to Ω among all subsets of Ω whose measure lies in $[s, \frac{1}{2}]$, for $s \in [\frac{1}{2}, 1]$ it is given symmetrically. Under some mild assumption on I_Ω near 0, the Sobolev inequality

$$\|u\|_{Y(\Omega)} \le C_1 \|u\|_{W^1X(\Omega)} \tag{9}$$

holds for some constant C_1 and for every function $u \in W^1X(\Omega)$ if and only if the Hardy-type inequality

$$\left\| \int_t^1 f(s) I_\Omega(s)^{-1} ds \right\|_{\overline{Y}(0,1)} \le C_2 \|f\|_{\overline{X}(0,1)} \tag{10}$$

holds for some constant C_2 and for every function $f \in \overline{X}(0,1)$, [20]. Note that, for a John domain, $I_\Omega(s)$ is equivalent to $s^{1-\frac{1}{n}}$ near zero, and thus (10) coincides with (8) (with possibly different constants).

Thanks to the techniques just described, Sobolev embeddings were obtained in quite a general setting. It should be noticed moreover, that all the results have their counterparts in Sobolev spaces of higher order. The extension to higher order is highly nontrivial, mainly because, once again, the Pólya–Szegő principle cannot be used, but it can be developed thanks to the sharp iteration method established and described in [20].

The reduction principle proved to be a powerful and useful method for nailing down the optimal target r.i. space in a Sobolev embedding. It was developed in several versions for more complicated structures, such as embeddings on unbounded domains ([85, 1, 55, 56]), trace embeddings ([15, 19, 18]), Gaussian–Sobolev embeddings [17, 20]), Sobolev embeddings into spaces endowed with Ahlfors measures ([22, 21]), compact Sobolev embeddings ([47, 76, 77, 12, 13]), embeddings of fractional Sobolev spaces ([2]), Sobolev inequalities in the hyperbolic space ([57]), boundedness of integral operators ([29]), Sobolev embeddings into Morrey, Campanato and Hölder spaces ([16, 11]), embeddings of Sobolev spaces involving symmetric gradients ([9]), or optimal function spaces in weighted Sobolev embeddings with monomial weights ([26]).

The main feature of this technique is the transformation of a difficult problem involving functions of several variables to a possibly easier problem involving functions acting on $\mathbb{R}$. The problem in dimension one is then to characterize the target space in a Hardy-type inequality. This step requires deep knowledge on properties of integral operators of the Hardy or the Copson type and various their disguises, and it is of key importance for the ultimate solution of the original problem. For this reason, we shall devote the subsequent section to this fundamental tool.

3 Hardy-type inequalities

The Hardy inequality appeared for the first time in the classical papers by Hardy in 1920's, and the early results were later collected in the book [39]. It has been investigated in many forms ever since. One of the simplest integral versions tells us that

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(t)^p dt \tag{11}$$

for all nonnegative measurable functions f on $(0,\infty)$ and every $p \in (1,\infty)$. The statement is equivalent to the boundedness of the integral averaging operator, also called the Hardy operator, on $L^p(0,\infty)$.

The Hardy inequality proved to be very useful in the last century, and it has been generalized and extended in many ways. Note that even in the preceding section of the introduction to this thesis it was mentioned in connection with two unrelated topics employing different versions of the Hardy inequality as a fundamental tool in the corresponding research. A Hardy-type inequality can often be interpreted as a certain relation between specific function spaces. For this reason, the validity of a particular type of the Hardy inequality, usually depending on some intrinsic parameters or properties of weights, needs to be characterized in order to obtain information of the relations between function spaces. Many researchers devoted plenty of their effort to this task during the last decades.

An important extension of the inequality (11) is obtained when we add weights and extend the range of parameters. Then the inequality reads as

$$\left(\int_0^\infty \left(\int_0^t f\right)^q w(t)\,dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \tag{12}$$

in which C is a positive constant independent of a nonnegative measurable function f on $(0,\infty)$, v and w are weights, $p \in [1,\infty)$, and $q \in (0,\infty)$. The pursuit of characterizing conditions for (12) has a long history. Serious investigation begun in 1950s in the paper by Kac and Krein [44], who characterized it for $p = q = 2$ and $v = 1$. In 1950s and 1960s, several results were proved by Beesack, see e.g. [4]. In late 1960s and early 1970s, the case $p \le q$ was treated extensively. For $p = q$, independently from one another, Tomaselli [83], Talenti [81] and Muckenhoupt [60] established a characterization. Its extension to $p \le q$ was done by Bradley [8], see also [49] and the unpublished manuscript by Artola. The authors of [71] cite a paper by Boyd and Erdős, which however at the end was not published since the authors withdrew it after learning that the same result had been obtained by other authors. In summary, (12) is true if and only if

$$\sup_{t\in(0,\infty)} \left(\int_t^\infty w\right)^{\frac{1}{q}} \left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}} < \infty \quad \text{for } 1 < p \le q$$

and

$$\sup_{t\in(0,\infty)} \left(\int_t^\infty w\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{s\in(0,t)} \frac{1}{v(s)} < \infty \quad \text{for } 1 = p \le q.$$

Here p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

The case $p > q$ is more subtle. Its characterization for $1 \le q < p < \infty$ was established by Maz'ya and Rozin, see [54], the characterizing condition being

$$\int_0^\infty \left(\int_t^\infty w\right)^{\frac{r}{q}} \left(\int_0^t v^{1-p'}\right)^{\frac{r}{q'}} v(t)^{1-p'}\, dt < \infty,$$

where $r = \frac{pq}{p-q}$. A more versatile characterization was given by Sawyer [73], partly in a discretized form. Sinnamon [74] gave a characterization in the case $0 < q < 1 < p < \infty$. His proof was based on working with Halperin's level function. The case $0 < q < p = 1$ appears in the work of Sinnamon and Stepanov [75]. In [7], the inequality (12) is studied when restricted to monotone functions. Recently, a new elementary and universal proof was given by Gogatishvili and Pick [35].

The above-mentioned discretization technique received its antidiscretisation counterpart in the paper by Gogatishvili and Pick [36]. It became a very useful tool for characterization of weighted Hardy-type inequalities. It has been used also in order to prove duality theorems, see [61]. This technique was further improved in the Paper III of this thesis by excluding any restrictions on the weights. This new tool was used also in the Paper IV.

Many more Hardy-type inequalities based on boundedness of the Hardy operator or the Copson operator between r.i. spaces can be seen. As an example we can recall the inequality (8) from the previous section, which also depicts an importance of research of these inequalities.

Another view on a Hardy inequality can be taken when the inner integral on the left-hand side of (11) is understood as a primitive function of the function f. Then we can reformulate the inequality using the notion of the derivative as

$$\int_0^\infty \frac{1}{t^p}|f(t)|^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f'(t)|^p dt$$

for all absolutely continuous functions f on $[0, \infty)$ such that $f(0) = 0$ and every $p \in (1, \infty)$. Based on this, we can obtain useful Hardy-type inequalities for Sobolev functions. An instance of a higher-dimensional version for Sobolev function $f \in W_0^{1,p}(\Omega)$ is the following:

$$\int_\Omega \left(\frac{|u(x)|}{d(x)}\right)^p dx \le c(p, n, \Omega) \int_\Omega |\nabla u(x)|^p dx,$$

where Ω is a sufficiently regular domain and $d(x)$ denotes, once again, the distance function from the boundary of Ω. This formulation of the Hardy inequality is due to Nečas [62], who established it for Lipschitz domains. Later, it was rederived in [50, 86, 65, 54] under weaker assumptions on domains. Finally, so far the most comprehensive assumptions on the regularity of Ω were introduced in the paper by Kinnunen and Martio [48], with roots in a work by Hedberg [40], and independently also in paper by Hajłasz [37].

4 The description of the papers contained in the thesis

In this section, we summarize the main scientific contribution of the papers included in this thesis.

4.1 Paper I: Characterization of functions with zero traces via the distance function and Lorentz spaces

This paper is devoted to the improvement of the condition that characterizes when a function belongs to the space $W_0^{1,p}(\Omega)$ via the distance function from the boundary, see Subsection 2.1. More precisely, the main aim of the paper is to replace the Lebesgue space $L^1(\Omega)$ in [30] (or the Lorentz space $L^{1,q}(\Omega)$, $q \in [1, \infty)$, used in the Master thesis by the author) by a larger space $L_a^{1,\infty}(\Omega)$, given as the collection of functions from $L^{1,\infty}(\Omega)$ with absolutely continuous $L^{1,\infty}$-quasinorms. We thereby get the equivalence (1). The proof of the principal result is achieved through a new approach to the problem which is quite different from the techniques applied in earlier work. The space $L_a^{1,\infty}(\Omega)$ arises in the proof as a natural structure suitable for this problem. This structure had been known before, but its properties have been described in a somewhat unsatisfactory manner, hence we add an appendix at the end of the paper which surveys its function spaces properties.

In previous works [45, 27, 48, 30, 31], various boundary conditions on the domain Ω were assumed. In our paper, considerable effort was spent in order to discuss the best possible boundary condition which we can afford for the proof. We establish some relations between different boundary conditions, and we also present several examples which illustrate their mutual relations.

In the end of the paper, we indicate the necessity of some assumptions that appear in the statements through two fundamental counterexamples. In particular, we show that if we replace $L_a^{1,\infty}(\Omega)$ by the (yet larger) space $L^{1,\infty}(\Omega)$, then the desired characterization is no longer true.

4.2 Paper II: Basic functional properties of certain scale of rearrangement-invariant spaces

This work relates to two papers by the author team Cianchi, Pick and Slavíková, [22, 21], where Sobolev embeddings of arbitrary order have been considered into function spaces on subdomains of $\mathbb{R}^n$ endowed with Ahlfors measures, called sometimes in the literature also Frostman measures, whose decay on balls is dominated by a certain power of their radii. The authors approached the problem from a new angle, combining the classical reduction principles (see Subsection 2.2) with a completely new interpolation technique involving a logarithmically convex combination of two integral operators. Compared to other occurrences of reduction principles that had been used in earlier work, the piece of information obtained from interpolation in [22] turned out to be somewhat mysterious and it took some further technical constructions to nail down the correct target classes in the Sobolev embeddings. In the process, two new structures of functions surfaced in connection of the optimality of target function spaces in general Sobolev embeddings involving Ahlfors measures. One of these had been treated before, but the other one, which we shall denote as $X^{\langle\alpha\rangle}$, apparently was completely new.

Let X be a rearrangement-invariant Banach function space over a nonatomic σ-finite measure space $(\mathscr{R}, \mu)$ and let $\alpha \in (0, \infty)$. We define $X^{\langle\alpha\rangle}$ as the collection of all μ-measurable scalar functions on $(\mathscr{R}, \mu)$ for which the functional $\|f\|_{X^{\langle\alpha\rangle}}$, given by

$$\|f\|_{X^{\langle\alpha\rangle}} = \|((|f|^\alpha)^{**})^{\frac{1}{\alpha}}\|_{\overline{X}(0,\mu(\mathscr{R}))},$$

is finite, where $\overline{X}(0,\mu(\mathscr{R}))$ is the representation space of X. In the paper, a variety of results of these spaces is presented. First, we establish results on basic functional properties of the spaces with the emphasis on the question whether they fulfil the axioms of r.i. spaces. Next we investigate their relations to customary function spaces as well as mutual embeddings, since such information is indispensable when the structures in question appear as target spaces in Sobolev embeddings. We show that while $X^{\langle\alpha\rangle}$ is always continuously embedded into X, the converse is true if and only if the Hardy averaging operator is bounded on an appropriate space. In a particular situation, a characterization of their associate structures is given. In one of the main achievements of the paper, we point out a certain new one-parameter path of function spaces, leading from a Lebesgue space to a Zygmund class, and we undertake a detailed comparison of it to the classical one.

4.3 Paper III: Discretization and antidiscretization of Lorentz norms with no restrictions on weights

In this paper, we develop a new discretization and antidiscretization technique suitable for weighted rearrangement-invariant norms. The main aim is to eliminate "nondegeneracy" restrictions on the involved weights which appear in earlier work available in the literature. The original discretization and antidiscretization technique appeared in the paper [36], where nondegeneracy conditions on appropriate weights were assumed. We develop a new method which enables us to provide two-sided estimates of the optimal constant C that renders the inequality

$$\left(\int_0^L (f^*(t))^q w(t)\,dt\right)^{\frac{1}{q}} \le C\left(\int_0^L \left(\int_0^t u(s)\,ds\right)^{-p}\left(\int_0^t f^*(s)u(s)\,ds\right)^p v(t)\,dt\right)^{\frac{1}{p}}$$

valid for all relevant measurable functions, where $L \in (0,\infty]$, $p, q \in (0,\infty)$ and u, v, w are locally integrable weights, u being strictly positive. In other words, this corresponds to the problem of characterizing the embedding of classical Lorentz spaces $\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$, where $\Gamma^p_u(v)$ is a generalization of $\Gamma^p(v)$ adding the weight u on an appropriate places. In the case of weights that would be otherwise excluded by the restrictions, it is shown that additional limit terms naturally appear in the characterizations of the optimal C. A weak analogue for $p=\infty$ is also presented.

It should be noticed that this paper is furthermore related to [22] and Paper II, as it characterizes certain particular cases of embeddings of the space $X^{\langle\alpha\rangle}$ presented in these papers as a target space in Sobolev embedding involving Ahlfors measures.

4.4 Paper IV: Weighted inequalities for a superposition of the Copson operator and the Hardy operator

In this paper we study a three-weight inequality for the superposition of the Hardy operator and the Copson operator, namely

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)^p v(\tau)\,d\tau\right)^{\frac{q}{p}} u(s)\,ds\right)^{\frac{r}{q}} w(t)\,dt\right)^{\frac{1}{r}} \le C\int_a^b f(t)\,dt,$$

in which (a,b) is any nontrivial interval, q,r are positive real parameters, and $p\in(0,1]$. A simple change of variables can be used to obtain any weighted Lebesgue norm on the right-hand side. Another simple change of variables can be used to equivalently turn this inequality into the one in which the Hardy and Copson operators swap their positions. We focus on characterizing those triples of weight functions (u,v,w) for which this inequality holds for all nonnegative measurable functions f with a constant independent of f. A new method of discretization and antidiscretization presented in Paper III enables us to avoid duality techniques and therefore to remove various restrictions that appear in earlier work.

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